线性代数期中参考题

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11-12

1. (子空间, 线性无关, 基, 直和)

   设 U 是 $\mathbb{R}^\infty$ 的一个子集, U 中的元素 v 对所有 i 都满足 $v_i + v_{i+2} = v_{i+1}$.

   (1) 求证: U 是 $\mathbb{R}^\infty$ 的一个子空间.

   (2) 设 $x, y \in U$, 满足: $x = (0, 1, 1, 0, -1, -1, 0, 1, 1, \cdots)$, $y = (1, 0, -1, -1, 0, 1, 1, 0, -1, \cdots)$. 求证: x, y 线性无关.

   (3) 求证: x, y 是 U 的一组基.

   (4) 设 W 是 $\mathbb{R}^\infty$ 的一个子集, W 中的元素满足 $v_1 = 0$ 且 $v_2 = 0$. 求证: $\mathbb{R}^\infty = U \oplus W$.

   Solution:

   (1) It suffices to verify the three properties: (i) $0 \in U$; (ii) closed under vector addition; (iii) closed under scalar multiplication, which is quite easy.

   (2) Suppose not. Then we can find $a, b \in \mathbb{R}$, s.t. $ax + by = 0$. Note that $ax + by = (b, a, \cdots)$. If $ax + by = 0$ then $b = 0$ and $a = 0$ since two sequences are equal iff their terms are all equal. This means that x and y are linearly independent.

   (3) Since we have already shown that (x, y) is a linearly independent set, we just need to show that it spans U. It's easy to see $\operatorname{span}\{x, y\} \subseteq U$, next we show $U \subseteq \operatorname{span}\{x, y\}$. In fact, let $u \in U$. Write $u = (u_1, u_2, \cdots)$. Then we claim that $u = u_1 y + u_2 x$. We can show that all the terms of u and $u_1 y + u_2 x$ match up by induction (omitted).

   (4) $\mathbb{R}^\infty = U \oplus W$ iff $\mathbb{R}^\infty = U + W$ and $U \cap W = \{0\}$. It's easy to see $U + W \subseteq \mathbb{R}^\infty$. we need to show that any sequence can be written as the sum of an element of U and an element of W. Let $x = (x_1, x_2, \cdots) \in \mathbb{R}^\infty$. Let $u = (x_1, x_2, x_2 - x_1, -x_2, x_1 - x_2, x_1, x_2, \cdots)$ be the element of U that starts with x_1 and x_2. Let $w = x - u$. Since u and x have the same first and second term, $w = (0, 0, w_3, w_4, \cdots)$. So $w \in W$. Thus, $\mathbb{R}^\infty = U + W$.
   To show $U \cap W = \{0\}$, suppose $v \in U \cap W$. We will show that $v = 0$ by induction. Write $v = (v_1, v_2, \cdots)$. Since $v \in W$, $v_1 = v_2 = 0$. Suppose $v_{n-1} = v_n = 0$. Then we need to show that $v_{n+1} = 0$. Since $v_{n+1} = v_n - v_{n-1}$, we have that $v_{n+1} = 0$. So by induction, $v = 0$. Therefore, $U \cap W = \{0\}$.

2. (线性变换与多项式空间)

   考虑一个线性变换 $T \in \mathcal{L}(P_2(\mathbb{R}), P_3(\mathbb{R}))$. 假设我们知道 T 的部分信息如下:

$$T(x^2 + 1) = x^2 - x,$$

$$T(1) = 2x + 1.$$

   基于以上信息, 回答问题, 简要给出证明或举出反例.

   (1) T 可能是单射吗?

   (2) T 可能是满射吗?

   (3) 我们能够确定 $T(x^2 + x + 1)$ 吗?

   (4) 我们能够确定 $x^2 + 3x + 2 \in \operatorname{Range}(T)$ 吗?

Solution:

(1) Yes. For example, consider the transformation T defined by the formula $T(ax^2 + bx + c = ax^2 + (b - 3a + 2c)x + (c - a)$. We can easily check: $T(x^2 + 1) = x^2 - x$ and $T(1) = 2x + 1$. To show this map is injective, we note that if $ax^2 + bx + c \in \operatorname{Null} T$, we must have $a = 0, b - 3a + 2c = 0, c - a = 0$. Therefore $\operatorname{Null}(T) = 0$ and T is injective.

(2) No. We know that $\dim P_2(\mathbb{R}) = 3$ and $\dim P_3(\mathbb{R}) = 4$. Corollary 3.6 (on page 46) states that $T : V \to W$ cannot be surjective if $\dim V < \dim W$.

(3) No. For the T defined in (1), we have $T(x^2 + x + 1) = x^2$. But if we define $T(ax^2 + bx + c) = bx^3 + ax^2 + (-3a + 2c)x + (c - a)$ (which can be shown reasonable easily), we have $T(x^2 + x + 1) = x^3 + x^2 - x$.

(4) Yes. $T(x^2 + 2) = T(x^2 + 1 + 2) = x^2 - x + 2(2x + 1) = x^2 + 3x + 1$.

3. (特征值与特征向量, 对角化)
设 V 是一个有限维向量空间, 且 $\dim V = n$. 设 $S \in \mathcal{L}(V)$ 是 V 上的线性算子, 且有 n 个不同的特征值. 设 $T \in \mathcal{L}(V)$ 是另一个线性算子. 求证: 如果 $ST = TS$, 那么 T 可对角化.

Solution: Firstly, we prove:
Lemma 1: If v is an eigenvector for S with eigenvalue λ, then Tv is also an eigenvector for S with eigenvalue λ (or Tv = 0). PROOF: set $w = Tv$, then $Sw = STv = TSv = T(\lambda v) = \lambda Tv = \lambda w$. We now turn to the assumption that S has n distinct eigenvalues. Let $\lambda_1, \cdots, \lambda_n$ be the n eigenvalues of S, and let $v_1, \cdots, v_n$ be the corresponding eigenvectors (so $S(v_k) = \lambda_k v_k$). The vectors $v_1, \cdots, v_n$ are linearly independent, since eigenvectors whose eigenvalues are distinct are linearly independent. Since $\dim V = n$, these n vectors form a basis for V. We next prove:
Lemma 2: If $u \in V$ satisfies $S(u) = \lambda_k u$, then $u = cv_k$ for some $c \in \mathbf{F}$. PROOF: we note that $v_1, \cdots, v_n$ is a basis of V, any $u \in V$ can be written as $u = c_1 v_1 + \cdots + c_n u_n$. We can compute $S(u)$ as $S(u) = c_1 S(v_1) + \cdots c_n S(v_n) = c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n$. If we also assume that $S(u) = \lambda_k u$, then $S(u) = c_1 \lambda_k v_1 + \cdots + c_n \lambda_k v_n$. Subtracting the latter equation from the former gives $0 = c_1(\lambda_1 - \lambda_k)v_1 + \cdots + c_n(\lambda_n - \lambda_k)v_n$. So we conclude that $c_i = 0$ for all i other than k, which is the desired result.
We now prove the claim that T is diagonalizable by showing that $v_1, \cdots, v_n$ is an eigenbasis for T. Consider a single vector v_k from this basis, which is an eigenvector of S with eigenvalue λ_k. By Lemma 1, we know that Tv_k is an eigenvector of S with eigenvalue λ_k, or else $Tv = 0$; in either case, it satisfies $S(Tv_k) = \lambda_k T v_k$. Therefore by Lemma 2, we see that $Tv_k = cv_k$ for some $c \in \mathbf{F}$. In other words, v_k is an eigenvector of T (since it is nonzero). Therefore $v_1, \cdots, v_n$ is an eigenbasis for T, so T is diagonalizable.
<u>Remark</u>: To make the problem much easier, we can divide it into three parts, lemma 1, lemma 2 and the final conclusion.

4. (可逆映射, 对角化)
设 V 是有限维的向量空间, 且 $T \in \mathcal{L}(\mathcal{V})$. 假设 $\operatorname{Range}(T) \neq \operatorname{Range}(T^2)$.

(1) 求证: T 不可以对角化.

(2) 以下说法正确的是?

  (i) T 一定可逆.

  (ii) T 一定不可逆.

  (iii) T 可能可逆也可能不可逆.

  证明你的结论.

Solution:

(1) If T is diagonalizable, then there exists a basis $v_1, \cdots, v_n$ for V s.t. $T(v_i) = \lambda_i v_i$ for all $i = 1, \cdot, n$. For each i, let $c_i = \dfrac{1}{\lambda_i}$ if $\lambda_i \neq 0$ and 0 if $\lambda_i = 0$. Note that in either case we have

$c_i\lambda_i^2 = \lambda_i$. We know that Range$(T^2) \subset$ Range(T). We will prove that Range$(T) \subset$ Range(T^2). Assume that $w \in$ Range(T), so we can write $w = T(v)$ for some $v \in V$. Since $v_1, \cdots, v_n$ is a basis for V, we can write $v = a_1v_1 + \cdots + a_nv_n$. Now we have $w = a_1\lambda_1v_1 + \cdots + a_n\lambda_nv_n$. Now define $u = a_1c_1v_1 + \cdots + a_nc_nv_n$. I claim that $T^2(u) = w$.(omit the proof) We conclude $w \in$ Range(T^2). Since w was an arbitrary element of Range(T), this shows that Range$(T) \subset$ Range(T^2). Combined with Range$(T^2) \subset$ Range(T), this implies that Range$(T) =$ Range(T^2), contradicting the hypothesis of the question. Therefore T must not be diagonalizable.

(2) (ii) is right. If T is invertible, then T must be surjective, so Range$(T) = V$. Separately, if T is invertible, then so is T^2.(Its inverse is given by $(T^{-1})^2$.) But if T^2 is invertible, then it is surjective, and so Range$(T^2) = V$ as well. This contradicts the hypothesis of the question.