

# Lecture 10: Frequency estimation – subspace methods

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## Introduction

As we know, a wss process can be modeled as an output of an LTI filter driven by white noise... another important model represents the analyzed sequence  $x_n$  as a sum of complex exponents in white noise.

$$x_n = \sum_{i=1}^p A_i e^{jn\omega_i} + w_n \quad (10.2.1)$$

Where complex amplitudes are assumed:

$$A_i = |A_i| e^{j\phi_i} \quad (10.2.2)$$

with  $\phi_i$  uncorrelated rv uniformly distributed over  $[-\pi, \pi]$ . Although the frequencies  $\omega_i$  and magnitudes  $|A_i|$  are not random, they are unknown. The power spectrum of  $x_n$  consists of a set of  $p$  impulses of area  $2\pi|A_i|$  at frequencies  $\omega_i$  for  $i = 1, 2, \dots, p$ , plus the power spectrum of the additive noise  $w_n$ .

This type of signals are found in sonar applications, speech processing,...

Parameters of complex exponents (amplitude, frequency) are usually of interest rather than the estimation of the power spectrum itself.

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## Eigendecomposition of the autocorrelation matrix

Although it is possible to estimate the frequencies of complex exponents from the peaks of the spectrum estimated by any method, this approach would not fully use the assumed parametric form of the process.

An alternative is to use frequency estimation algorithms taking into account the properties of the process. These algorithms are based on eigendecomposition of the autocorrelation matrix into a signal subspace and a noise subspace...

First, we consider the 1<sup>st</sup>-order process:

$$x_n = A_1 e^{jn\omega_1} + w_n \quad (10.3.1)$$

that consists of a single complex exponential in white noise. The amplitude of the complex exponential is

$$A_1 = |A_1| e^{j\phi_1} \quad (10.3.2)$$

where  $\phi_1$  is uniformly distributed and  $w_n$  is white noise with a variance  $\sigma_w^2$ .

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## Eigendecomposition of the autocorrelation matrix

The autocorrelation sequence of the noisy complex exponential  $x_n$  is

$$r_x(k) = P_1 e^{jk\omega_1} + \sigma_w^2 \delta(k) \quad (10.4.1)$$

where  $P_1 = |A_1|^2$  is the power in the complex exponential. Therefore, the  $M \times M$  autocorrelation matrix for  $x_n$  is a sum of an autocorrelation matrix due to the signal  $\mathbf{R}_s$  and an autocorrelation matrix due to the noise  $\mathbf{R}_n$ :

$$\mathbf{R}_x = \mathbf{R}_s + \mathbf{R}_n \quad (10.4.2)$$

where the signal autocorrelation matrix is

$$\mathbf{R}_s = P_1 \begin{bmatrix} 1 & e^{-j\omega_1} & e^{-j2\omega_1} & \dots & e^{-j(M-1)\omega_1} \\ e^{j\omega_1} & 1 & e^{-j\omega_1} & \dots & e^{j(M-2)\omega_1} \\ e^{j2\omega_1} & e^{j\omega_1} & 1 & \dots & e^{j(M-3)\omega_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{j(M-1)\omega_1} & e^{j(M-2)\omega_1} & e^{j(M-3)\omega_1} & \dots & 1 \end{bmatrix} \quad (10.4.3)$$

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## Eigendecomposition of the autocorrelation matrix

The signal autocorrelation matrix has a rank of one, and the noise autocorrelation matrix is diagonal:

$$\mathbf{R}_n = \sigma_w^2 \mathbf{I} \quad (10.5.1)$$

and has a full rank. Defining

$$\mathbf{e}_1 = \left[ 1, e^{j\omega_1}, e^{j2\omega_1}, \dots, e^{j(M-1)\omega_1} \right]^T \quad (10.5.2)$$

$\mathbf{R}_s$  can be written in terms of  $\mathbf{e}_1$  as follows

$$\mathbf{R}_s = P_1 \mathbf{e}_1 \mathbf{e}_1^H \quad (10.5.3)$$

Since the rank of  $\mathbf{R}_s$  is one, then  $\mathbf{R}_s$  has only one nonzero eigenvalue. With

$$\mathbf{R}_s \mathbf{e}_1 = P_1 (\mathbf{e}_1 \mathbf{e}_1^H) \mathbf{e}_1 = P_1 \mathbf{e}_1 (\mathbf{e}_1^H \mathbf{e}_1) = MP_1 \mathbf{e}_1 \quad (10.5.4)$$

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## Eigendecomposition of the autocorrelation matrix

it follows that the nonzero eigenvalue is equal to  $MP_1$  and that  $\mathbf{e}_1$  is the corresponding eigenvector. In addition, since  $\mathbf{R}_s$  is Hermitian, the remaining eigenvectors  $\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_M$  will be orthogonal to  $\mathbf{e}_1$ :

$$\mathbf{e}_1^H \mathbf{v}_i = 0; \quad i = 2, 3, \dots, M \quad (10.6.1)$$

We notice that if we let  $\lambda_i^s$  be the eigenvalues of  $\mathbf{R}_s$ , then

$$\mathbf{R}_x \mathbf{v}_i = (\mathbf{R}_s + \sigma_w^2 \mathbf{I}) \mathbf{v}_i = \lambda_i^s \mathbf{v}_i + \sigma_w^2 \mathbf{v}_i = (\lambda_i^s + \sigma_w^2) \mathbf{v}_i \quad (10.6.2)$$

Therefore, the eigenvectors of  $\mathbf{R}_x$  are the same as those of  $\mathbf{R}_s$ , and the eigenvalues of  $\mathbf{R}_x$  are

$$\lambda_i = \lambda_i^s + \sigma_w^2 \quad (10.6.3)$$

The largest eigenvalue of  $\mathbf{R}_x$  is

$$\lambda_{\max} = MP_1 + \sigma_w^2 \quad (10.6.4)$$

and the remaining  $M-1$  eigenvalues are equal to  $\sigma_w^2$ .

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## Eigendecomposition of the autocorrelation matrix

Thus, it is possible to extract all of the parameters of interest about  $x_n$  from the eigenvalues and eigenvectors of  $\mathbf{R}_s$  as follows:

1. Perform an eigendecomposition of the autocorrelation matrix  $\mathbf{R}_x$ . The largest eigenvalue will be equal to  $MP_1 + \sigma_w^2$  and the remaining eigenvalues will be  $\sigma_w^2$ .
2. Use the eigenvalues of  $\mathbf{R}_x$  to solve for the power  $P_1$  and the noise variance as

$$\sigma_w^2 = \lambda_{\min} \quad (10.7.1)$$

$$P_1 = \frac{1}{M}(\lambda_{\max} - \lambda_{\min}) \quad (10.7.2)$$

3. Determine the frequency  $\omega_1$  from the eigenvector  $\mathbf{v}_{\max}$  that is associated with the largest eigenvalue, using, for instance, the second coefficient of  $\mathbf{v}_{\max}$ :

$$\omega_1 = \arg\{v_{\max}(1)\} \quad (10.7.3)$$

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## Eigendecomposition of the autocorrelation matrix: Example

### *Eigendecomposition of a complex exponentials in noise.*

Let  $x_n$  be a 1<sup>st</sup>-order harmonic process consisting of a single complex exponential in white noise:

$$x_n = A_1 e^{in\omega_1} + w_n \quad (10.8.1)$$

with a 2 x 2 autocorrelation matrix

$$\mathbf{R}_x = \begin{bmatrix} 3 & 2(1-j) \\ 2(1+j) & 3 \end{bmatrix} \quad (10.8.2)$$

The eigenvalues of  $\mathbf{R}_x$  are

$$\lambda_{1,2} = 3 \pm |2(1+j)| = 3 \pm 2\sqrt{2} \quad (10.8.3)$$

and the eigenvectors are

$$\mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ \pm \frac{\sqrt{2}}{2}(1+j) \end{bmatrix} \quad (10.8.4)$$

## Eigendecomposition of the autocorrelation matrix: Example

Therefore, the variance of white noise is

$$\sigma_w^2 = \lambda_{\min} = 3 - 2\sqrt{2} \quad (10.9.1)$$

and the power in the complex exponential is

$$P_1 = \frac{1}{M}(\lambda_{\max} - \lambda_{\min}) = 2\sqrt{2} \quad (10.9.2)$$

Finally, the frequency of the complex exponential is

$$\omega_1 = \arg \left\{ \frac{\sqrt{2}}{2}(1 + j) \right\} = \frac{\pi}{4} \quad (10.9.3)$$

We may also notice that finding parameters of a 1<sup>st</sup>-order harmonic process could be simpler:

$$r_x(1) = 2(1 + j) = 2\sqrt{2}e^{j\pi/4} = P_1 e^{j\omega_1} \Rightarrow P_1 = 2\sqrt{2}; \omega_1 = \pi/4 \quad (10.9.4)$$

Once  $P_1$  is known, noise variance is  $\sigma_w^2 = r_x(0) - P_1 = 3 - 2\sqrt{2}$  (10.9.5)

## Eigendecomposition of the autocorrelation matrix

In practice, however, the preceding approach is not very useful since it requires that the autocorrelation matrix must be known exactly. The estimated autocorrelation matrix can be used. However, eigenvalues and eigenvectors can be quite sensitive to small errors in autocorrelation.

Therefore, instead we consider using a weighted average as follows.

Let  $\mathbf{v}_i$  be a noise eigenvector of  $\mathbf{R}_x$ , i.e., one that has an eigenvalue of  $\sigma_w^2$ , and let  $v_i(k)$  be the  $k^{\text{th}}$  component of  $\mathbf{v}_i$ . We compute the DTFT of the coefficients  $\mathbf{v}_i$ :

$$V_i(e^{j\omega}) = \sum_{k=0}^{M-1} v_i(k) e^{-jk\omega} = \mathbf{e}^H \mathbf{v}_i \quad (10.10.1)$$

The orthogonality condition implies that

$$V_i(e^{j\omega}) = 0 \quad \text{at } \omega = \omega_1 \quad (10.10.2)$$

the frequency of the complex exponential.

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## Eigendecomposition of the autocorrelation matrix

Therefore, we form the frequency estimation function:

$$\hat{P}_i(e^{j\omega}) = \frac{1}{\left| \sum_{k=0}^{M-1} v_i(k) e^{-jk\omega} \right|^2} = \frac{1}{|\mathbf{e}^H \mathbf{v}_i|^2} \quad (10.11.1)$$

The frequency estimation function will be large (infinite in theory) at  $\omega = \omega_i$ . Thus, the location of the peak of this frequency estimation function may be used to estimate the frequency of the complex exponential. However, since (10.11.1) uses only a single eigenvector and, therefore, may be sensitive to errors in estimation of  $\mathbf{R}_x$ , we may consider using a weighted average of all noise eigenvectors as follows:

$$\hat{P}_i(e^{j\omega}) = \frac{1}{\sum_{i=2}^M \alpha_i |\mathbf{e}^H \mathbf{v}_i|^2} \quad (10.11.2)$$

where  $\alpha_i$  are appropriately chosen constants.



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## Eigendecomposition of the autocorrelation matrix: Example

### *Eigendecomposition of a complex exponentials in noise.*

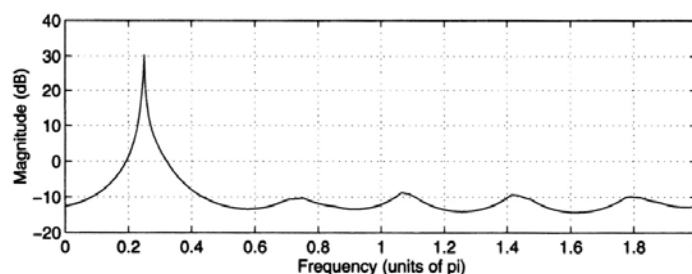
Let  $x_n$  be a wss process consisting of a single complex exponential in unit variance white noise:

$$x_n = 4e^{j(n\pi/4+\phi)} + w_n \quad (10.12.1)$$

where  $\phi$  is a uniformly distributed rv. Using  $N = 64$  values of  $x_n$ , a  $6 \times 6$  autocorrelation matrix was estimated and an eigendecomposition was performed.

Frequency estimation function as in (10.11.2) with  $\alpha_i = 1$ .

The peak at  $\omega = 0.2539\pi$ .

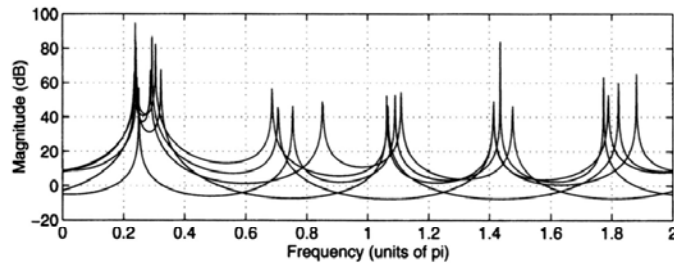


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## Eigendecomposition of the autocorrelation matrix: Example

The minimum eigenvalue was  $\lambda_{min} = 1.08$ , which is close to the variance of the white noise.

Frequency estimation functions as in (10.10.1) estimated for each of the noise eigenvectors.



Although each plot has a peak that is close to  $\omega = 0.25\pi$ , it is difficult to (with a single plot) to distinguish the correct peak from a spurious one.

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## Eigendecomposition of the autocorrelation matrix

Let us consider next what happens in the case of two complex exponentials in white noise:

$$x_n = A_1 e^{jn\omega_1} + A_2 e^{jn\omega_2} + w_n \quad (10.14.1)$$

where the amplitudes of complex exponentials are

$$A_i = |A_i| e^{j\phi_i}; \quad i = 1, 2 \quad (10.14.2)$$

and  $\omega_1$  and  $\omega_2$  are the frequencies with  $\omega_1 \neq \omega_2$ . If the variance of  $w_n$  is  $\sigma_w^2$ , then the autocorrelation of  $x_n$  is

$$r_x(k) = P_1 e^{jk\omega_1} + P_2 e^{jk\omega_2} + \sigma_w^2 \delta(k) \quad (10.14.3)$$

where

$$P_1 = |A_1|^2; \quad P_2 = |A_2|^2 \quad (10.14.4)$$

## Eigendecomposition of the autocorrelation matrix

Therefore, the autocorrelation matrix may again be written as a sum:

$$\mathbf{R}_x = P_1 \mathbf{e}_1 \mathbf{e}_1^H + P_2 \mathbf{e}_2 \mathbf{e}_2^H + \sigma_w^2 \mathbf{I} \quad (10.15.1)$$

where

$$\mathbf{R}_s = P_1 \mathbf{e}_1 \mathbf{e}_1^H + P_2 \mathbf{e}_2 \mathbf{e}_2^H \quad (10.15.2)$$

is a rank two matrix representing the component of  $\mathbf{R}_x$  that is due to the signal, and

$$\mathbf{R}_n = \sigma_w^2 \mathbf{I} \quad (10.15.3)$$

is a diagonal matrix that is due to the noise. Another way to express this decomposition is

$$\mathbf{R}_x = \mathbf{E} \mathbf{P} \mathbf{E}^H + \sigma_w^2 \mathbf{I} \quad (10.15.4)$$

where

$$\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2] \quad (10.15.5)$$

is an  $M \times 2$  matrix containing the two signal vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$ ; and

$$\mathbf{P} = \text{diag} \{P_1, P_2\} \quad (10.15.6)$$

## Eigendecomposition of the autocorrelation matrix

(10.15.6) is a diagonal matrix containing the signal powers.

In addition to decomposing  $\mathbf{R}_x$  into a sum of two autocorrelation matrices as in (10.15.4), we may also perform an eigendecomposition of  $\mathbf{R}_x$  as follows.

Let  $\mathbf{v}_i$  and  $\lambda_i$  be the eigenvectors and eigenvalues of  $\mathbf{R}_x$  respectively with the eigenvalues arranged in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M \quad (10.16.1)$$

Since

$$\mathbf{R}_x = \mathbf{R}_s + \sigma_w^2 \mathbf{I} \quad (10.16.2)$$

then

$$\lambda_i = \lambda_i^s + \sigma_w^2 \quad (10.16.3)$$

where  $\lambda_i^s$  are the eigenvalues of  $\mathbf{R}_s$ . Since the rank of  $\mathbf{R}_s$  equals two, then  $\mathbf{R}_s$  has only two nonzero eigenvalues, and both of them are greater than zero ( $\mathbf{R}_s$  is nonnegative definite). Therefore, the first two eigenvalues of  $\mathbf{R}_x$  are greater than  $\sigma_w^2$  and the remaining eigenvalues are equal to  $\sigma_w^2$ .



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## Eigendecomposition of the autocorrelation matrix

Therefore, the eigenvalues and eigenvectors of  $\mathbf{R}_x$  may be divided into two groups.

The first group, consisting of the two eigenvectors that have eigenvalues greater than  $\sigma_w^2$ , is referred to as signal eigenvectors that span a 2D subspace called the signal subspace.

The second, consisting of those eigenvectors that have eigenvalues equal to  $\sigma_w^2$ , is referred to as the noise eigenvectors that span an  $M-2$  dimensional subspace called the noise subspace.

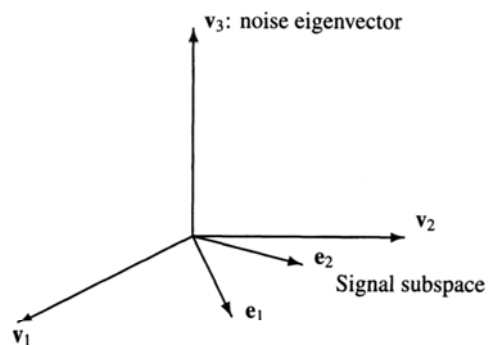
Since  $\mathbf{R}_x$  is Hermitian, the eigenvectors  $\mathbf{v}_i$  form an orthogonal set. Therefore, the signal and noise subspaces are orthogonal. That is to say, for any vector  $\mathbf{u}$  in the signal subspace and for any vector  $\mathbf{v}$  in the noise subspace:

$$\mathbf{u}^H \mathbf{v} = 0 \quad (10.17.1)$$

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## Eigendecomposition of the autocorrelation matrix

Geometrical interpretation of the orthogonality of the signal and noise subspaces. The 1D noise subspace is spanned by the noise eigenvector  $\mathbf{v}_3$  and the signal subspace containing the signal vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  is spanned by the signal eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .



Unlike the case for a single complex exponential, with a sum of two complex exponentials in noise, the signal eigenvectors will generally not be equal to  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . Nevertheless,  $\mathbf{e}_1$  and  $\mathbf{e}_2$  will lie in the signal subspace that is spanned by the signal eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , and, since the signal subspace is orthogonal to the noise subspace,  $\mathbf{e}_1$  and  $\mathbf{e}_2$  will be orthogonal to the noise eigenvectors  $\mathbf{v}_i$ .

## Eigendecomposition of the autocorrelation matrix

That is:

$$\mathbf{e}_1^H \mathbf{v}_i = 0; \quad i = 3, 4, \dots, M \quad (10.19.1)$$

$$\mathbf{e}_2^H \mathbf{v}_i = 0; \quad i = 3, 4, \dots, M \quad (10.19.2)$$

Therefore, as in the case of one complex exponential, the complex exponential frequencies  $\omega_1$  and  $\omega_2$  may be estimated using a frequency estimation function of the following form:

$$\hat{P}(e^{j\omega}) = \frac{1}{\sum_{i=3}^M \alpha_i |\mathbf{e}_i^H \mathbf{v}_i|^2} \quad (10.19.3)$$

For a general case of a wss process consisting of  $p$  distinct complex exponentials in white noise, the  $M \times M$  autocorrelation sequence is

$$r_x(k) = \sum_{i=1}^p P_i e^{jk\omega_i} + \sigma_w^2 \delta(k) \quad (10.19.4)$$

## Eigendecomposition of the autocorrelation matrix

where  $P_i = |A_i|^2$  is the power in the  $i^{\text{th}}$  component. Therefore, the autocorrelation matrix can be written as

$$\mathbf{R}_x = \mathbf{R}_s + \mathbf{R}_n = \sum_{i=1}^p P_i \mathbf{e}_i \mathbf{e}_i^H + \sigma_w^2 \mathbf{I} \quad (10.20.1)$$

where

$$\mathbf{e}_i = \begin{bmatrix} 1 & e^{j\omega_i} & e^{j2\omega_i} & \dots & e^{j(M-1)\omega_i} \end{bmatrix}^T \quad (10.20.2)$$

is a set of  $p$  linearly independent vectors. As for two complex exponentials, (10.20.1) may be written as follows:

$$\mathbf{R}_x = \mathbf{E} \mathbf{P} \mathbf{E}^H + \sigma_w^2 \mathbf{I} \quad (10.20.3)$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_p \end{bmatrix} \quad (10.20.4)$$

is an  $M \times p$  matrix containing the  $p$  signal vectors  $\mathbf{e}_i$  and

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## Eigendecomposition of the autocorrelation matrix

$$\mathbf{P} = \text{diag} \{ P_1 \quad \dots \quad P_p \} \quad (10.21.1)$$

is a diagonal matrix of signal powers. Since the eigenvalues of  $\mathbf{R}_x$  are

$$\lambda_i = \lambda_i^s + \sigma_w^2 \quad (10.21.2)$$

where  $\lambda_i^s$  are the eigenvalues of  $\mathbf{R}_s$ , and since  $\mathbf{R}_s$  is a matrix of rank  $p$ , then the first  $p$  eigenvalues of  $\mathbf{R}_x$  will be greater than  $\sigma_w^2$  and the last  $M-p$  eigenvalues will be equal to  $\sigma_w^2$ . Therefore, the eigenvalues and eigenvectors of  $\mathbf{R}_x$  may be divided into two groups: the signal eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_p$  that have eigenvalues greater than  $\sigma_w^2$  and the noise eigenvectors  $\mathbf{v}_{p+1}, \dots, \mathbf{v}_M$  that have eigenvalues equal to  $\sigma_w^2$ . Assuming that the eigenvectors have been normalized to have unit norm, we can use the spectral theorem to decompose  $\mathbf{R}_x$  as follows:

$$\mathbf{R}_x = \sum_{i=1}^p (\lambda_i^s + \sigma_w^2) \mathbf{v}_i \mathbf{v}_i^H + \sum_{i=p+1}^M \sigma_w^2 \mathbf{v}_i \mathbf{v}_i^H \quad (10.21.3)$$

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## Eigendecomposition of the autocorrelation matrix

The decomposition in (10.21.3) may be written in matrix notation as

$$\mathbf{R}_x = \mathbf{\Lambda}_s \mathbf{V}_s \mathbf{V}_s^H + \mathbf{\Lambda}_n \mathbf{V}_n \mathbf{V}_n^H \quad (10.22.1)$$

where

$$\lambda_i = \lambda_i^s + \sigma_w^2 \quad (10.22.2)$$

is the  $M \times p$  matrix of signal eigenvectors and

$$\mathbf{V}_n = [\mathbf{v}_{p+1}, \mathbf{v}_{p+2}, \dots, \mathbf{v}_M] \quad (10.22.3)$$

is the  $M \times (M-p)$  matrix of noise eigenvectors and where  $\mathbf{\Lambda}_s$  and  $\mathbf{\Lambda}_n$  are diagonal matrices containing the eigenvalues  $\lambda_i = \lambda_i^s + \sigma_w^2$  and  $\lambda_i = \sigma_w^2$  respectively. We may be interested in projecting a vector onto either the signal subspace or the noise subspace. The projection matrices that will perform these projections onto the signal and noise subspaces are

$$\mathbf{P}_s = \mathbf{V}_s \mathbf{V}_s^H \quad \mathbf{P}_n = \mathbf{V}_n \mathbf{V}_n^H \quad (10.22.4)$$

## Eigendecomposition of the autocorrelation matrix

As was the case for one and two complex exponentials in white noise, the orthogonality of the signal and noise subspaces may be used to estimate the frequencies of the complex exponentials. Specifically, since each signal vector  $\mathbf{e}_1, \dots, \mathbf{e}_p$  lies in the signal subspace, this orthogonality implies that  $\mathbf{e}_i$  will be orthogonal to each of the noise eigenvectors:

$$\mathbf{e}_i^H \mathbf{v}_k = 0; \quad i = 1, 2, \dots, p; \quad k = p+1, \dots, M \quad (10.23.1)$$

Therefore, the frequencies may be estimated via a frequency estimation function such as

$$\hat{P}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^M \alpha_i |\mathbf{e}^H \mathbf{v}_i|^2} \quad (10.23.2)$$

We will develop several different types of frequency estimation algorithms based on (10.23.2) next...

## Pisarenko Harmonic Decomposition

Considering the problem of estimating the frequencies of a sum of complex exponentials in white noise, Pisarenko has demonstrated that these frequencies could be derived from the eigenvector corresponding to the minimum eigenvalue of the autocorrelation matrix. The resulting technique is limited due to its sensitivity to noise. However, it provided the stimulus for the development of other more robust eigenvalue decomposition methods.

$x_n$  is assumed to be a sum of  $p$  complex exponentials in white noise where  $p$  is known. It is also assumed that  $p+1$  values of the autocorrelation sequence are either known or have been estimated. With a  $(p+1) \times (p+1)$  autocorrelation matrix, the dimension of the noise subspace is one, and it is spanned by the eigenvector corresponding to the minimum eigenvalue

$$\lambda_{\min} = \sigma_w^2 \quad (10.24.1)$$

## Pisarenko Harmonic Decomposition

Denoting this eigenvector by  $\mathbf{v}_{\min}$ , it follows that it will be orthogonal to each of the signal vectors  $\mathbf{e}_i$ :

$$\mathbf{e}_i^H \mathbf{v}_{\min} = \sum_{k=0}^p v_{\min}(k) e^{-jk\omega_i} = 0; \quad i = 1, 2, \dots, p \quad (10.25.1)$$

Therefore:

$$V_{\min}(z) = \sum_{k=0}^p v_{\min}(k) e^{-jk\omega} \quad (10.25.2)$$

is equal to zero at each of the complex exponential frequencies  $\omega_l$  for  $l = 1, 2, \dots, p$ . Consequently, the z-transform of the noise eigenvector, referred to as eigenfilter, has  $p$  zeros on the unit circle:

$$V_{\min}(z) = \sum_{k=0}^p v_{\min}(k) z^{-k} = \prod_{k=1}^p (1 - e^{j\omega_k} z^{-1}) \quad (10.25.3)$$

and the frequencies of the complex exponentials may be extracted from the roots of the eigenfilter.

## Pisarenko Harmonic Decomposition

As an alternative to rooting  $V_{\min}(z)$ , we may also form the **frequency estimation function**:

$$\hat{P}_{PHD}(e^{j\omega}) = \frac{1}{|\mathbf{e}^H \mathbf{v}_{\min}|^2} \quad (10.26.1)$$

which is a special case of (10.23.2) with  $M = p+1$  and  $\alpha_{p+1} = 1$ .

Since  $\hat{P}_{PHD}(e^{j\omega})$  will be large (in theory, infinite) at the frequencies of the complex exponentials, the locations of the peaks in  $\hat{P}_{PHD}(e^{j\omega})$  may be used as frequency estimates. Although written in the form of a power spectrum,  $\hat{P}_{PHD}(e^{j\omega})$  is called a **pseudospectrum** (or eigenspectrum) since it does not contain any information about the power in the complex exponentials, nor does it contain noise components.

Once the frequencies of the complex exponentials are determined, the powers  $P_i$  may be found from the eigenvalues of  $\mathbf{R}_x$  as follows...

## Pisarenko Harmonic Decomposition

Let us assume that the signal subspace eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ , have been normalized so that

$$\mathbf{v}_i^H \mathbf{v}_i = 1 \quad (10.27.1)$$

Since

$$\mathbf{R}_x \mathbf{v}_i = \lambda_i \mathbf{v}_i; \quad i = 1, 2, \dots, p \quad (10.27.2)$$

multiplying on the left both sides of (10.27.2) by  $\mathbf{v}_i^H$ , we obtain

$$\mathbf{v}_i^H \mathbf{R}_x \mathbf{v}_i = \lambda_i \mathbf{v}_i^H \mathbf{v}_i = \lambda_i; \quad i = 1, 2, \dots, p \quad (10.27.3)$$

Incorporating (10.20.1), we derive

$$\mathbf{v}_i^H \mathbf{R}_x \mathbf{v}_i = \mathbf{v}_i^H \left\{ \sum_{k=1}^p P_k \mathbf{e}_k \mathbf{e}_k^H + \sigma_w^2 \mathbf{I} \right\} \mathbf{v}_i = \lambda_i; \quad i = 1, 2, \dots, p \quad (10.27.4)$$

## Pisarenko Harmonic Decomposition

(10.27.4) can be simplified to

$$\sum_{k=1}^p P_k \left| \mathbf{e}_k^H \mathbf{v}_i \right|^2 = \lambda_i - \sigma_w^2; \quad i = 1, 2, \dots, p \quad (10.28.1)$$

We observe that the terms  $\left| \mathbf{e}_k^H \mathbf{v}_i \right|^2$  in the sum correspond to the squared magnitude of the DTFT of the signal subspace eigenvector  $\mathbf{v}_i$  evaluated at frequency  $\omega_k$ :

$$\left| \mathbf{e}_k^H \mathbf{v}_i \right|^2 = \left| V_i(e^{j\omega_k}) \right|^2 \quad (10.28.2)$$

where

$$V_i(e^{j\omega_k}) = \sum_{l=0}^p v_i(l) e^{-jl\omega} \quad (10.28.3)$$

## Pisarenko Harmonic Decomposition

Therefore, (10.28.1) may be rewritten as

$$\sum_{k=1}^p P_k \left| V_i(e^{j\omega_k}) \right|^2 = \lambda_i - \sigma_w^2; \quad i = 1, 2, \dots, p \quad (10.29.1)$$

(10.29.1) is a set of  $p$  linear equations in the  $p$  unknowns  $P_k$ :

$$\begin{bmatrix} |V_1(e^{j\omega_1})|^2 & |V_1(e^{j\omega_2})|^2 & \dots & |V_1(e^{j\omega_p})|^2 \\ |V_2(e^{j\omega_1})|^2 & |V_2(e^{j\omega_2})|^2 & \dots & |V_2(e^{j\omega_p})|^2 \\ \vdots & \vdots & \ddots & \vdots \\ |V_p(e^{j\omega_1})|^2 & |V_p(e^{j\omega_2})|^2 & \dots & |V_p(e^{j\omega_p})|^2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_p \end{bmatrix} = \begin{bmatrix} \lambda_1 - \sigma_w^2 \\ \lambda_2 - \sigma_w^2 \\ \vdots \\ \lambda_p - \sigma_w^2 \end{bmatrix} \quad (10.29.2)$$

that may be solved for powers  $P_k$ . Therefore, the PHD estimates the complex exponential frequencies either from the roots of the eigenfilter  $V_{\min}(z)$  or from the locations of the peaks in the frequency estimation function  $\hat{P}_{PHD}(e^{j\omega})$ , and then solves the linear equations (10.29.2) for powers of the complex exponentials.

## Pisarenko Harmonic Decomposition

Summary:

1. Given that a process consists of  $p$  complex exponentials in white noise, find the minimum eigenvalue  $\lambda_{\min}$  and the corresponding eigenvector  $\mathbf{v}_{\min}$  of the  $(p+1) \times (p+1)$  autocorrelation matrix  $\mathbf{R}_x$ .
2. Set the white noise power equal to the minimum eigenvalue  $\lambda_{\min} = \sigma_w^2$  and set the frequencies equal to the angles of the roots of the eigenfilter:

$$V_{\min}(z) = \sum_{k=0}^p v_{\min}(k) e^{-jk\omega}$$

or the location of the peaks in the frequency estimation function:

$$\hat{P}_{PHD}(e^{j\omega}) = \frac{1}{|\mathbf{e}^H \mathbf{v}_{\min}|^2}$$

3. Compute the powers of the complex exponentials by solving the linear equations (10.29.2).

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## Pisarenko Harmonic Decomposition

MATLAB implementation:

```
function [vmin,sigma] = phd(x,p)
% Pisarenko Harmonic Decomposition
    x = x(:);
    R = covar(x,p+1);
    [v,d] = eig(R);
    sigma = min(diag(d));
    index = find(diag(d) == sigma);
    vmin = v(:,index);
```

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## Pisarenko Harmonic Decomposition (Example)

Example: two complex exponentials in noise.

Suppose that first three autocorrelation coefficients of a random process consisting of two complex exponentials in white noise are

$$r_x(0) = 6$$

$$r_x(1) = 1.92705 + j4.58522$$

$$r_x(2) = -3.42705 + j3.49541$$

We will use Pisarenko's method to find the frequencies and powers of these exponentials. Since  $p = 2$ , we perform an eigendecomposition of the  $3 \times 3$  autocorrelation matrix

$$\mathbf{R}_x = \begin{bmatrix} 6 & 1.92705 + j4.58522 & -3.42705 + j3.49541 \\ 1.92705 + j4.58522 & 6 & 1.92705 + j4.58522 \\ -3.42705 + j3.49541 & 1.92705 + j4.58522 & 6 \end{bmatrix}$$



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## Pisarenko Harmonic Decomposition (Example)

The eigenvectors are

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 0.5763 - j0.000 & -0.2740 + j0.6518 & -0.2785 + j0.3006 \\ 0.2244 + j0.5342 & 0.0001 + j0.0100 & -0.3209 + j0.7492 \\ -0.4034 + j0.4116 & 0.2830 + j0.6480 & 0.4097 - j0.0058 \end{bmatrix}$$

and the eigenvalues are

$$\lambda_1 = 15.8951; \quad \lambda_2 = 1.1049; \quad \lambda_3 = 1.0000$$

Therefore, the minimum eigenvalue is  $\lambda_{\min} = 1$  and the corresponding eigenvector is

$$\mathbf{v}_{\min} = \begin{bmatrix} -0.2785 - j0.3006 \\ -0.3209 + j0.7492 \\ 0.4097 - j0.0058 \end{bmatrix}$$

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## Pisarenko Harmonic Decomposition (Example)

Since the roots of eigenfilter are

$$z_1 = 0.5 + j0.8660 = e^{j\frac{\pi}{3}}; \quad z_2 = 0.3090 + j0.9511 = e^{j\frac{2\pi}{5}}$$

the frequencies of complex exponentials are

$$\omega_1 = \frac{\pi}{3}; \quad \omega_2 = \frac{2\pi}{5}$$

To find the powers in the complex exponentials, we must compute the squared magnitude of the DTFT of the signal eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  at the complex exponential frequencies  $\omega_1$  and  $\omega_2$ .

With

$$\begin{aligned} |V_1(e^{j\omega_1})|^2 &= 2.9685; & |V_1(e^{j\omega_2})|^2 &= 2.9861 \\ |V_2(e^{j\omega_1})|^2 &= 0.0315; & |V_2(e^{j\omega_2})|^2 &= 0.0139 \end{aligned}$$

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## Pisarenko Harmonic Decomposition (Example)

then the equation (10.29.2) becomes

$$\begin{bmatrix} 2.9685 & 2.9861 \\ 0.0315 & 0.0139 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 - \sigma_w^2 \\ \lambda_2 - \sigma_w^2 \end{bmatrix}$$

where  $\sigma_w^2 = \lambda_{\min} = 1$ . Solving for  $P_1$  and  $P_2$ , we find

$$P_1 = 2; \quad P_2 = 3$$

**Note:** we used the Pisarenko's method to estimate the frequencies of two complex exponentials in white noise. Next, we will consider a single sinusoid in white noise. Although a sinusoid is a sum of two complex exponentials, the two frequencies are constrained to be negatives of each other  $\omega_1 = -\omega_2$ . As a result, an autocorrelation sequence will be real-valued, which forces the eigenvectors to be real and constrains the roots of the eigenfilter to occur in complex conjugate pairs.

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## Pisarenko Harmonic Decomposition (Example)

Example: one sinusoid in noise.

Let  $x_n$  be a random phase sinusoid in white noise

$$x_n = A \cdot \sin(n\omega_0 + \varphi) + w_n$$

with

$$r_x(0) = 2.2; \quad r_x(1) = 1.3; \quad r_x(2) = 0.8$$

The eigenvalues of the 3 x 3 autocorrelation matrix  $\mathbf{R}_x = \text{toep}\{2.2, 1.3, 0.8\}$  are

$$\lambda_1 = 4.4815; \quad \lambda_2 = 1.4; \quad \lambda_3 = 0.7185$$

Therefore, the white noise power is

$$\sigma_w^2 = \lambda_{\min} = 0.7185$$

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## Pisarenko Harmonic Decomposition (Example)

The eigenvectors of  $\mathbf{R}_x$  are

$$\mathbf{v}_1 = \begin{bmatrix} 0.5506 \\ 0.6275 \\ 0.5506 \end{bmatrix}; \quad \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix}; \quad \mathbf{v}_3 = \begin{bmatrix} 0.4437 \\ -0.7787 \\ 0.4437 \end{bmatrix}$$

We observe the symmetry of eigenvectors. To estimate the frequency of the sinusoid, we find the roots of the eigenfilter  $V_{\min}(z)$ , which is the z-transform of  $\mathbf{v}_3$ :

$$V_{\min}(z) = 0.4437(1 - 1.755z^{-1} + z^{-2})$$

We find that  $V_{\min}(z)$  has roots at  $z = e^{\pm j\omega_0}$  where

$$2 \cos \omega_0 = 1.755 \Rightarrow \omega_0 = 0.159\pi$$

Finally, the power in the sinusoid may be estimated using (10.29.2). This computation, however, can be simplified since  $x_n$  contains a single sinusoid.

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## Pisarenko Harmonic Decomposition (Example)

Since the autocorrelation sequence for a single sinusoid in white noise is

$$r_x(k) = \frac{1}{2} A^2 \cos(k\omega_0) + \sigma_w^2 \delta(k)$$

Then for  $k = 0$

$$r_x(0) = \frac{1}{2} A^2 + \sigma_w^2$$

With  $r_x(0) = 2.2$  and  $\sigma_w^2 = 0.7185$  it follows that

$$A^2 = 2.963$$

The Pisarenko harmonic decomposition method is not commonly used in practice for the following reasons:

- 1) The number of complex exponents needs to be known;
- 2) The noise must be white or (for a modified PHD) its power spectrum must be known;
- 3) Finding eigen-decomposition for high order problems may be computationally expensive.

## MUSIC

One improvement to PHD is the Multiple Signal Classification (MUSIC) method that is another frequency estimation technique.

Assume that  $x_n$  is a random process consisting of  $p$  complex exponentials in white noise with a variance  $\sigma_w^2$  and let  $\mathbf{R}_x$  be the  $M \times M$  autocorrelation matrix of  $x_n$  with  $M > p+1$ . If the eigenvalues of  $\mathbf{R}_x$  are arranged in decreasing order  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M$  and if  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M$  are the corresponding eigenvectors, then we may divide these eigenvectors into two groups: the  $p$  signal eigenvectors corresponding to  $p$  largest eigenvalues and  $M-p$  noise eigenvectors that, ideally, have eigenvalues equal to  $\sigma_w^2$ . With inexact autocorrelations, the smallest  $M-p$  eigenvalues will only be approximately equal to  $\sigma_w^2$ .

Although we could consider estimating the white noise variance by averaging the  $M-p$  smallest eigenvalues

$$\hat{\sigma}_w^2 = \frac{1}{M-p} \sum_{k=p+1}^M \lambda_k \quad (10.39.1)$$

estimating the frequencies of complex exponentials is more difficult.

## MUSIC

Since the eigenvectors of  $\mathbf{R}_x$  are of length  $M$ , each of the noise subspace eigenfilters

$$V_i(z) = \sum_{k=0}^{M-1} v_i(k) z^{-k}; \quad i = p+1, \dots, M \quad (10.40.1)$$

will have  $M-1$  roots (zeros). Ideally,  $p$  of these roots will lie on the unit circle at the frequencies of the complex exponentials, and the eigenspectrum

$$\left| V_i(e^{j\omega}) \right|^2 = \frac{1}{\left| \sum_{k=0}^{M-1} v_i(k) e^{-jk\omega} \right|^2} \quad (10.40.2)$$

associated with the noise eigenvector  $\mathbf{v}_i$  will have sharp peaks at the frequencies of the complex exponentials. However, the remaining  $M-p-1$  zeros may lie anywhere and, in fact, some may lie close to the unit circle, giving rise to spurious peaks in the eigenspectrum.

## MUSIC

Furthermore, with inexact autocorrelations, the zeros of  $V(z)$  that are on the unit circle may not remain on the unit circle. Therefore, when only one noise eigenvector is used to estimate the complex exponential frequencies, there may be some ambiguity in distinguishing the desired peaks from the spurious ones. Therefore, in the MUSIC algorithm, the effects of these spurious peaks are reduced by averaging, using the frequency estimation function

$$\hat{P}_{MU}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^M |\mathbf{e}^H \mathbf{v}_i|^2} \quad (10.41.1)$$

The frequencies of the complex exponentials are then estimated as the locations of the  $p$  largest peaks in (10.41.1). Once the frequencies are determined, the power of each complex exponential may be found using (10.29.2).

## MUSIC

Instead of searching for the peaks of  $\hat{P}_{MU}(e^{j\omega})$ , an alternative is to use a method called root MUSIC that involves rooting a polynomial. Since the z-transform equivalent of (10.41.1) is

$$\hat{P}_{MU}(z) = \frac{1}{\sum_{i=p+1}^M V_i(z) V_i^*\left(\frac{1}{z^*}\right)} \quad (10.42.1)$$

then the frequency estimates may be taken to be the angles of the  $p$  roots of the polynomial

$$D(z) = \sum_{i=p+1}^M V_i(z) V_i^*\left(\frac{1}{z^*}\right) \quad (10.42.2)$$

that are closest to the unit circle.

## MUSIC

MATLAB implementation:

```
function Px = music(x,p,M)
% MUSIC algorithm
x = x(:);
if M < p+1 | length(x) < M, error('Size of R is inappropriate'), end
R = covar(x,M);
[v,d] = eig(R);
[y,i] = sort(diag(d));
Px = 0;
for ind = 1:M-p
    Px = Px + abs(fft(v(i(ind)),1024));
end;
Px = -20*log10(Px);
```

## Other eigenvector methods

In addition to the Pisarenko and MUSIC algorithms, a number of other eigenvector methods exists for estimating the frequencies of complex exponentials in noise.

One of them, the **Eigen Vector (EV)** method, is related to the MUSIC. The EV estimates the exponential frequencies from the peaks of eigenspectrum:

$$\hat{P}_{EV}(z) = \frac{1}{\sum_{i=p+1}^M \frac{1}{\lambda_i} |\mathbf{e}^H \mathbf{v}_i|^2} \quad (10.44.1)$$

where  $\lambda_i$  is the eigenvalue associated with the eigenvector  $\mathbf{v}_i$ . If  $w_n$  is white noise and if the autocorrelation sequence  $r_x(k)$  is known exactly for  $k = 0, 1, \dots, M-1$ , then the eigenvalues in (10.44.1) are equal to the white noise variance:

$$\lambda_i = \sigma_v^2 \quad (10.44.2)$$

and the EV eigenspectrum is the same as the MUSIC pseudospectrum to within a constant. However, with estimated autocorrelations, the EV method differs from MUSIC and produces fewer spurious peaks.

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## Other eigenvector methods

EV: MATLAB implementation:

```
function Px = ev(x,p,M)
    % Eigen Vector algorithm
    x = x(:);
    if M < p+1, error('Specified size of R is too small'), end
    R = covar(x,M);
    [v,d] = eig(R);
    [y,i] = sort(diag(d));
    Px = 0;
    for ind = 1:M-p
        Px = Px + abs(fft(v(:i(ind)),1024)).^2/abs(y(ind)));
    end;
    Px = -10*log10(Px);
```

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## Other eigenvector methods

Another interesting eigendecomposition-based method is the **minimum norm** algorithm. Instead of forming an eigenspectrum that uses all of the noise eigenvectors as in MUSIC and EV, the minimum norm algorithm uses a single vector  $\mathbf{a}$  that is constrained to lie in the noise subspace, and the complex exponential frequencies are estimated from the peaks of the frequency estimation function

$$\hat{P}_{MN}(z) = \frac{1}{|\mathbf{e}^H \mathbf{a}|^2} \quad (10.46.1)$$

With  $\mathbf{a}$  constrained to lie in the noise subspace, if the autocorrelation sequence is known exactly, then  $|\mathbf{e}^H \mathbf{a}|^2$  will have nulls at the frequencies of each complex exponential. Therefore, the z-transform of the coefficients in  $\mathbf{a}$  may be factored as:

$$A(z) = \sum_{k=0}^{M-1} a_k z^{-k} = \prod_{k=1}^p (1 - e^{j\omega_k} z^{-1}) \prod_{k=p+1}^{M-1} (1 - z_k z^{-1}) \quad (10.46.2)$$

where  $z_k$  for  $k = p+1, \dots, M-1$  are the spurious roots that do not, in general, lie on the unit circle.

## Other eigenvector methods

The problem, therefore, is to determine which vector in the noise subspace minimizes the effects of the spurious zeros on the peaks of  $\hat{P}_{MN}(e^{j\omega})$ . The approach that is used in the minimum norm algorithm is to find the vector  $\mathbf{a}$  that satisfies the following three constraints:

1. The vector  $\mathbf{a}$  lies in the noise subspace.
2. The vector  $\mathbf{a}$  has minimum norm.
3. The first element of  $\mathbf{a}$  is unity.

The first constraint ensures that  $p$  roots of  $A(z)$  lie on the unit circle. The second constraint ensures that the spurious roots of  $A(z)$  lie inside the unit circle, i.e.,  $|z_k| < 1$ . The third constraint ensures that the minimum norm solution is not the zero vector.

To solve this constrained minimization problem, we begin by observing that the first constraint ( $\mathbf{a}$  lies in the noise subspace) may be written as

$$\mathbf{a} = \mathbf{P}_n \mathbf{v} \quad (10.47.1)$$

## Other eigenvector methods

where

$$\mathbf{P}_n = \mathbf{V}_n \mathbf{V}_n^H \quad (10.48.1)$$

is the projection matrix that projects an arbitrary vector  $\mathbf{v}$  onto the noise subspace (see Eqn. (10.22.4)). The third constraint may be expressed as

$$\mathbf{a}^H \mathbf{u}_1 = 1 \quad (10.48.2)$$

where

$$\mathbf{u}_1 = [1, 0, \dots, 0]^T \quad (10.48.3)$$

This constraint may be combined with the previous one in (10.47.1) as follows

$$\mathbf{v}^H (\mathbf{P}_n^H \mathbf{u}_1) = 1 \quad (10.48.4)$$

Then, using (10.47.1), the norm of  $\mathbf{a}$  may be written as

$$\|\mathbf{a}\|^2 = \|\mathbf{P}_n \mathbf{v}\|^2 = \mathbf{v}^H (\mathbf{P}_n^H \mathbf{P}_n) \mathbf{v} \quad (10.48.5)$$



## Other eigenvector methods

Since  $\mathbf{P}_n$  is the projection matrix, it is Hermitian:  $\mathbf{P}_n = \mathbf{P}_n^H$ , and independent:  $\mathbf{P}_n^2 = \mathbf{P}_n$ . Therefore:

$$\|\mathbf{a}\|^2 = \mathbf{v}^H \mathbf{P}_n \mathbf{v} \quad (10.49.1)$$

and it follows that minimizing the norm of  $\mathbf{a}$  is equivalent to finding the vector  $\mathbf{v}$  that minimizes the quadratic form  $\mathbf{v}^H \mathbf{P}_n \mathbf{v}$ . We may now reformulate the constrained minimization problem as follows:

$$\min \mathbf{v}^H \mathbf{P}_n \mathbf{v} \quad \text{subject to} \quad \mathbf{v}^H (\mathbf{P}_n^H \mathbf{u}_1) = 1 \quad (10.49.2)$$

Once the solution to (10.49.2) is found, the minimum norm solution can be formed by projecting  $\mathbf{v}$  onto the noise subspace using (10.47.1).

The solution to (10.49.2) will be

$$\mathbf{v} = \lambda \mathbf{P}_n^{-1} (\mathbf{P}_n^H \mathbf{u}_1) = \lambda \mathbf{u}_1 \quad (10.49.3)$$

## Other eigenvector methods

where

$$\lambda = \frac{1}{\mathbf{u}_1^H \mathbf{P}_n \mathbf{u}_1} \quad (10.50.1)$$

Therefore, the minimum norm solution is

$$\mathbf{a} = \mathbf{P}_n \mathbf{v} = \lambda \mathbf{P}_n \mathbf{u}_1 = \frac{\mathbf{P}_n \mathbf{u}_1}{\mathbf{u}_1^H \mathbf{P}_n \mathbf{u}_1} \quad (10.50.2)$$

Which is simply the projection of the unit vector onto the noise subspace, normalized so that the first coefficient is one.

In terms of eigenvectors of the autocorrelation matrix, the minimum norm solution may be written as

$$\mathbf{a} = \frac{(\mathbf{V}_n \mathbf{V}_n^H) \mathbf{u}_1}{\mathbf{u}_1^H (\mathbf{V}_n \mathbf{V}_n^H) \mathbf{u}_1} \quad (10.50.3)$$

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## Other eigenvector methods

MN: MATLAB implementation:

```
function Px = min_norm(x,p,M)
% Minimum Norm algorithm
x = x(:);
if M < p+1, error('Specified size of R is too small'), end
R = covar(x,M);
[v,d] = eig(R);
[y,i] = sort(diag(d));
V = [];
for ind = 1:M-p
    V = [V, v(:,i(ind))];
end;
a = V*V(1,:);
Px = -20*log10(abs(fft(a,1024)));
```

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## Noise Subspace methods: Summary

### Table 1.1 Noise Subspace Methods for Frequency Estimation

Pisarenko	$\hat{P}_{PHD}(e^{j\omega}) = \frac{1}{ \mathbf{e}^H \mathbf{v}_{\min} ^2}$
MUSIC	$\hat{P}_{MU}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^M  \mathbf{e}^H \mathbf{v}_i ^2}$
Eigenvector Method	$\hat{P}_{EV}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^M \frac{1}{\lambda_i}  \mathbf{e}^H \mathbf{v}_i ^2}$
Minimum Norm	$\hat{P}_{MN}(e^{j\omega}) = \frac{1}{ \mathbf{e}^H \mathbf{a} ^2} \quad ; \quad \mathbf{a} = \lambda \mathbf{P}_n \mathbf{u}_1$

## Noise Subspace methods: Example

Let  $x_n$  be a process consisting of a sum of four complex exponentials in white noise

$$x_n = \sum_{k=1}^4 A_k e^{j(n\omega_k + \phi_k)} + w_n$$

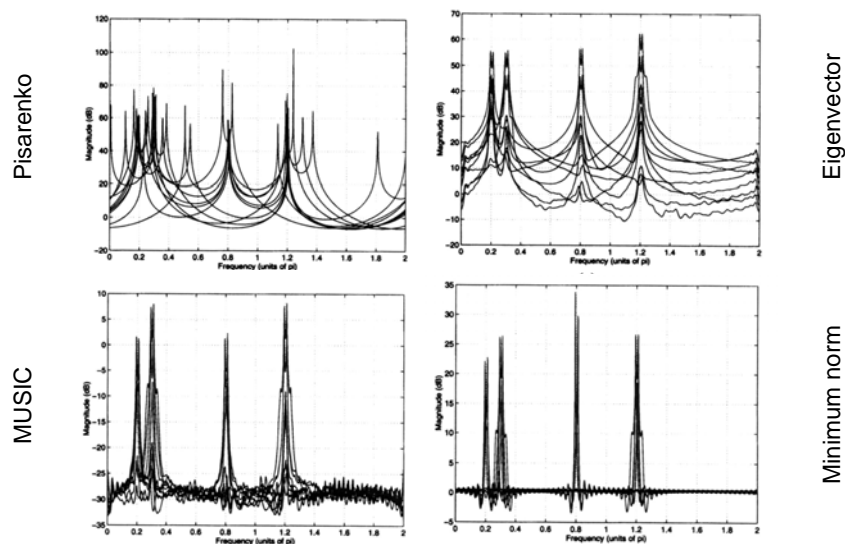
where the amplitudes  $A_k$  are equal to one, the frequencies  $\omega_k$  are  $0.2\pi$ ,  $0.3\pi$ ,  $0.8\pi$ , and  $1.2\pi$ , the phases are uncorrelated random variables that are uniformly distributed over the interval  $[0, 2\pi]$ , and  $w_n$  is white noise with the variance  $\sigma_w^2 = 0.5$ .

Using 10 different realizations of  $x_n$  with  $N = 64$ , the frequency estimation functions were obtained using Pisarenko's method, the MUSIC algorithm, the eigenvector method, and the minimum norm algorithm.

For the Pisarenko's method, the frequency estimation function was derived from the  $5 \times 5$  autocorrelation matrix that was estimated from the 64 values of  $x_n$ .

For the MUSIC, eigenvector, and minimum norm algorithm, the frequency estimation functions were formed from the  $64 \times 64$  autocorrelation matrix that was estimated from  $x_n$ , i.e.,  $M = 64$ .

## Noise Subspace methods: Example



## Noise Subspace methods: Example

Except for the Pisarenko's method, the frequency estimation functions for this process produce accurate estimates for the frequencies of the exponentials.

The most well-defined peaks are produced with the minimum norm algorithm.

However, it is important to point out that not all of the frequency estimation functions shown in the previous slide have all four well-defined peaks. In some cases, for instance, only two or three peaks are observed...

## Principal Components SE

Previously, we saw that the orthogonality of signal and noise subspaces could be used to estimate frequencies of  $p$  complex exponentials in white noise. Since these methods only use vectors that lie in the noise subspace, they are referred to as noise subspace methods. We consider next a set of algorithms that use vectors lying in the signal subspace. These methods are based on a principal components analysis of the autocorrelation matrix and are referred to as signal subspace methods.

Let  $\mathbf{R}_x$  be an  $M \times M$  autocorrelation matrix of a process consisting of  $p$  complex exponentials in white noise. Using the eigendecomposition of  $\mathbf{R}_x$ :

$$\mathbf{R}_x = \sum_{i=1}^M \lambda_i \mathbf{v}_i \mathbf{v}_i^H = \sum_{i=1}^p \lambda_i \mathbf{v}_i \mathbf{v}_i^H + \sum_{i=p+1}^M \lambda_i \mathbf{v}_i \mathbf{v}_i^H \quad (10.56.1)$$

where was assumed that the eigenvalues are arranged in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M \quad (10.56.2)$$

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## Principal Components SE

We notice that the second term in (10.56.1) is due only to the noise, we may form a reduced rank to the signal autocorrelation matrix,  $\mathbf{R}_s$ , by keeping only the principal eigenvectors of  $\mathbf{R}_x$ :

$$\hat{\mathbf{R}}_s = \sum_{i=1}^p \lambda_i \mathbf{v}_i \mathbf{v}_i^H \quad (10.57.1)$$

This principal components approximation may then be used instead of  $\mathbf{R}_x$  in a spectral estimator, such as the minimum variance method or the maximum entropy method. The overall effect of this approach is to filter out a portion of the noise, thereby enhancing the estimate of the signal spectral components (complex exponentials).

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## Blackman-Tukey FE

The Blackman-Tukey estimate of the power spectrum is computed as a DTFT of a windowed autocorrelation sequence:

$$\hat{P}_{BT}(e^{j\omega}) = \sum_{k=-M}^M \hat{r}_x(k) W_k e^{-jk\omega} \quad (10.58.1)$$

If  $W_k$  is a Bartlett window, the Blackman-Tukey estimate may be written in terms of  $\mathbf{R}_x$  as follows:

$$\hat{P}_{BT}(e^{j\omega}) = \frac{1}{M} \sum_{k=-M}^M (M - |k|) \hat{r}_x(k) e^{-jk\omega} = \frac{1}{M} \mathbf{e}^H \mathbf{R}_x \mathbf{e} \quad (10.58.2)$$

Using the eigendecomposition of the autocorrelation matrix, (10.58.2) may be written as

$$\hat{P}_{BT}(e^{j\omega}) = \frac{1}{M} \sum_{i=1}^M \lambda_i |\mathbf{e}^H \mathbf{v}_i|^2 \quad (10.58.3)$$

## Blackman-Tukey FE

If  $x_n$  consists of  $p$  complex exponentials in white noise, and if the eigenvalues of  $\mathbf{R}_x$  are arranged in decreasing order as in (10.56.2), the principal components version of Blackman-Tukey spectrum estimate is

$$\hat{P}_{PC-BT}(e^{j\omega}) = \frac{1}{M} \mathbf{e}^H \hat{\mathbf{R}}_s \mathbf{e} = \frac{1}{M} \sum_{i=1}^p \lambda_i |\mathbf{e}^H \mathbf{v}_i|^2 \quad (10.59.1)$$

```

ML      function Px = bt_pc(x,p,M)
implementation: x = x(:);
            if M < p+1, error('Specified size of R is too small'), end
            R = covar(x,M);
            [v,d] = eig(R);
            [y,k] = sort(diag(d));
            Px = 0;
            for ind = M-p+1:M
                Px = Px+abs(fft(v(:,k(ind)),1024))*sqrt(real(y(ind))));
            end;
            Px = 20*log10(Px)-10*log10(M);

```

## Minimum Variance FE

Given the autocorrelation sequence  $r_x(k)$  of a process  $x_n$  for lags  $|k| \leq M$ , the  $M^{\text{th}}$ -order minimum variance spectrum estimate is

$$\hat{P}_{MV}(e^{j\omega}) = \frac{M}{\mathbf{e}^H \mathbf{R}_x^{-1} \mathbf{e}} \quad (10.60.1)$$

With the eigendecomposition of the autocorrelation matrix, its inverse is

$$\mathbf{R}_x^{-1} = \sum_{i=1}^p \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_i^H + \sum_{i=p+1}^M \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_i^H \quad (10.60.2)$$

where  $p$  is the number of complex exponentials. Keeping only the first  $p$  principal components of  $\mathbf{R}_x^{-1}$  leads to the principal components minimum variance estimate:

$$\hat{P}_{PC-MV}(e^{j\omega}) = \frac{M}{\sum_{i=1}^p \frac{1}{\lambda_i} |\mathbf{e}^H \mathbf{v}_i|^2} \quad (10.60.3)$$

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## Autoregressive FE

Autoregressive spectrum estimation using the autocorrelation, covariance, or modified covariance algorithms involves finding the solution to a set of linear equations of the form

$$\mathbf{R}_x \mathbf{a}_M = \varepsilon_m \mathbf{u}_1 \quad (10.61.1)$$

where  $\mathbf{R}_x$  is an  $(M+1) \times (M+1)$  autocorrelation matrix. From the solution to these equations

$$\mathbf{a}_M = \varepsilon_m \mathbf{R}_x^{-1} \mathbf{u}_1 \quad (10.61.2)$$

A spectral estimate is found as follows:

$$\hat{P}_{AR}(e^{j\omega}) = \frac{|b_0|^2}{|\mathbf{e}^H \mathbf{a}_M|^2} \quad (10.61.3)$$

where

$$|b_0|^2 = \varepsilon_m \quad (10.61.4)$$

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## Autoregressive FE

On the other hand, if  $x_n$  consists of  $p$  complex exponentials in noise, we may form a principal components solution as follows:

$$\mathbf{a}_{pc} = \varepsilon_m \left( \sum_{i=1}^p \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_i^H \right) \mathbf{u}_1 \quad (10.62.1)$$

or

$$\mathbf{a}_{pc} = \varepsilon_m \sum_{i=1}^p \frac{v_i^*(0)}{\lambda_i} \mathbf{v}_i = \varepsilon_m \sum_{i=1}^p \alpha_i \mathbf{v}_i \quad (10.62.2)$$

Here  $v_i^*(0)$  is the first element of the normalized eigenvector  $\mathbf{v}_i$  and

$$\alpha_i = \frac{v_i^*(0)}{\lambda_i} \quad (10.62.3)$$

Assuming that

$$|b_0|^2 = \varepsilon_m$$

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## Autoregressive FE

The principal components autoregressive spectrum estimate is

$$\hat{P}_{PC-AR}(e^{j\omega}) = \frac{1}{\left| \sum_{i=1}^p \alpha_i \mathbf{e}^H \mathbf{v}_i \right|^2} \quad (10.63.1)$$

Note that as the number of autocorrelations increases, only  $p$  principal eigenvectors and eigenvalues are used in (10.63.1). This property allows for an increase in the model order without a corresponding increase in the spurious peaks that would be due to the noise eigenvectors of the autocorrelation matrix.

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## Signal Subspace methods: Summary

### *Signal Subspace Methods*

Blackman-Tukey	$\hat{P}_{PC-BT}(e^{j\omega}) = \frac{1}{M} \sum_{i=1}^p \lambda_i  \mathbf{e}^H \mathbf{v}_i ^2$
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Minimum variance	$\hat{P}_{PC-MV}(e^{j\omega}) = \frac{M}{\sum_{i=1}^p \frac{1}{\lambda_i}  \mathbf{e}^H \mathbf{v}_i ^2}$
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Autoregressive	$\hat{P}_{PC-AR}(e^{j\omega}) = \frac{1}{\left  \sum_{i=1}^p \alpha_i \mathbf{e}^H \mathbf{v}_i \right ^2}$
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