The University of New South Wales
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Thesis Report - Part A

BSc Computer Science (Honours)

The Chromatic Derivatives and its Applications

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*Dedicated to memory of my beloved grandfather,
Hai-Xiang Tiao (1918–2014).*

Contents

Chapter 1

Introduction

It is no overstatement that the Nyquist-Shannon sampling theorem is indispensable to the fields of communication and signal processing. It allows us to convert continuous real-world analog signals into discrete signals, represented as a sequence a numbers, to then be processed and manipulated digitally on a computer - it makes digital signal processing possible, both in theory and in practice.

Let f be a signal of finite energy bandlimited by B, i.e. f is a continuous L^2 function whose Fourier transform has support within $[-B, B]$. We denote the class of such functions $\mathbf{BL}(B)$. The interpolation (reconstruction) formula complementary to the sampling theorem, commonly known as the Whittaker–Shannon interpolation formula, is given by

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\mathrm{sinc}(t-n) \tag{1.1}$$

where sinc is the (normalized) cardinal sine function, defined by $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$ for $x \neq 0$ and $\mathrm{sinc}(0) = 1$.

This series expansion in eq. (1.1) can be seen as a special case of the generalized Fourier series, with the complete orthonormal system of univariate functions $\{\mathrm{sinc}(t-n)\}_{n\in\mathbb{Z}}$ as its basis. Roughly speaking, the interpolation formula given by eq. (1.1) and Fourier series in general are "global" in the sense that it requires the samples of the signal at integer points of arbitrarily large absolute value, i.e. values at points spanning a large part of its domain. Additionally, the truncation error of Fourier series is distributed along the domain of the function.

On the other hand, consider a signal $f \in \mathbf{BL}(\pi)$, which is also an analytic function. Its

Taylor series expansion about point $t = t_0$ is given by

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(t_0)\frac{(t-t_0)^n}{n!} \tag{1.2}$$

We mainly consider the expansion about point $t_0 = 0$, which is also known as the Maclaurin series. The Taylor series are said to "local" in the sense that values of the nth order derivatives $f^{(n)}(t_0)$ can be obtains by samples of the signal on an arbitrarily small neighborhood of t_0. It is no surprise then that the truncation error of Taylor series is very small in the neighborhood on which it has been computed and grows dramatically as we move away from this neighborhood.

In stark contrast to the Shannon-Whittaker interpolation, the Taylors series, though instrumental in mathematical analysis, finds relatively limited use in digital signal processing, due to its multitude of shortcomings. We give a brief overview of these problems now, and extend our discussion in section 2.2.

1. numerical evaluation of derivatives, let alone those of higher orders are highly sensitive to noise.
2. the Taylor series expansion of a signal $f \in \mathbf{BL}(\pi)$ does not converge uniformly on $\mathbb{R}$.
3. as mentioned already, truncation error of Taylor series accumulates quite rapidly.
4. the Shannon-Whittaker interpolation of signal $f \in \mathbf{BL}(\pi)$ converges to f in $\mathbf{BL}(\pi)$ while the truncations (approximation by finite sum) of the Taylor series expansion do not even belong to $\mathbf{BL}(\pi)$.
5. For any input signal $f \in \mathbf{BL}(\pi)$ to a continuous, linear, time-invariant system L, its output can be characterized entirely by samples of the signal and the impulse response of the system, $L[\text{sinc}]$. More concisely,

$$L[f](t) = \sum_{n=-\infty}^{\infty} f(n)L[\text{sinc}](t-n) \tag{1.3}$$

   Nothing resembling this holds for the Taylor series expansions.

In light of these problems, it is no wonder we see little application of Taylor series in digital signal processing. This is a pity, because there are a number of (albeit unusual) problems in signal processing that require processing and analyzing signals on an extremely local and almost microscopic level. Examples include designing signal processors that exploit the predictive properties of the signal, methods for image reconstruction problems that specifically depend on information from local regions of the images rather than from the image as a whole, and adaptive filtering applications.

Since Taylor series cannot readily be applied to these problems, we must reconcile the local properties of the Taylor series with the ideal features of the Fourier series that make it so crucial to digital signal processing.

To alleviate some of the problems that render Taylor series ineffective for signal processing, the *chromatic derivatives* were introduced by Dr. Ignjatović in 2000 [8]. Naturally, the theory emerged in the conception of a novel switching amplifier with the ability anticipate (predict) future signal values by exploiting the local signal behavior.

The *chromatic expansions*, complementary to the chromatic derivatives, were later introduced [9] and several applications and novel techniques that harness the theory of chromatic derivatives were subsequently proposed. Some of these include the design of communication channel equalizers [6], digital transceivers, image compression methods [4].

The theory has since been generalized and extended to systems corresponding to several classical families of orthogonal polynomials [5, 7] and cast in to the conventional signal processing framework [12, 14]. More recently, the chromatic derivatives and expansions were generalized to multiple dimensions [11].

We believe (multidimensional) chromatic derivatives and expansions lends itself naturally to many applications in important fields such as image processing. The exploration, conception and implementation of some these applications will be the subject of this thesis.

This report is structured as follows. In section 2.2, we formally define the chromatic derivatives, specifically those associated with the Legendre polynomials. For a concise and more generalized exposition of chromatic derivatives, please consult [10], which also forms the basis for this present report. Continuing in section 2.2, we discuss some of its properties, and contrast chromatic expansions with Taylor/Fourier series expansions with some examples. In section 2.3, we outline some of the image processing problems we intend to tackle and give a brief literature review of the conventional methods used to solve these problems. Finally, in chapter 3 we propose our approach to solving these problems with the machinery of multidimensional chromatic derivatives, and roughly outline the plan for carrying out our approach.

Chapter 2

Background

2.1 Preliminaries

The *Legendre functions* are solutions to the *Legendre differential equation*, which is the second-order ordinary differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}L_n(x)\right]+n(n+1)L_n(x)=0 \tag{2.1}$$

The *Legendre polynomials*, also known as *Legendre functions of the first kind*, satisfy the following recurrence relation:

$$L_0(x)=1 \tag{2.2}$$

$$L_1(x)=x \tag{2.3}$$

$$L_{n+1}(x)=\frac{2n+1}{n+1}xL_n(x)-\frac{n}{n+1}L_{n-1}(x) \tag{2.4}$$

The Legendre polynomials are orthogonal (and complete) over the closed interval $[-1,1]$

$$\frac{1}{2}\int_{-1}^{1}L_n(x)L_m(x)dx=\frac{1}{2n+1}\delta_{mn}$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij}=\begin{cases}0 & \text{for } i\neq j,\\ 1 & \text{for } i=j.\end{cases} \tag{2.5}$$

We can easily normalize and scale to obtain Legendre polynomials that are orthogonal (and complete) over the closed interval $[-\pi, \pi]$

$$P_n^L(x) = \sqrt{2n+1} L_n\left(\frac{x}{\pi}\right) \tag{2.6}$$

so that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(x) P_m^L(x) dx = \delta_{mn}$$

Likewise, it satisfies the recurrence relation

$$P_0^L(x) = 1 \tag{2.7}$$

$$P_1^L(x) = \frac{\sqrt{3}}{\pi}x \tag{2.8}$$

$$P_{n+1}^L(x) = \frac{\sqrt{4(n+1)^2-1}}{(n+1)\pi} x P_n^L(x) - \frac{n\sqrt{4(n+1)^2-1}}{(n+1)\sqrt{4n^2-1}} P_{n-1}^L(x) \tag{2.9}$$

More concisely, we can define this as

$$P_{n+1}^L(x) = \frac{1}{\gamma_n} x P_n(x) - \frac{\gamma_{n-1}}{\gamma_n} P_{n-1}(x) \tag{2.10}$$

with $P_{-1}(x) = 0, P_0(x) = 1$, and

$$\gamma_n = \frac{(n+1)\pi}{\sqrt{4(n+1)^2-1}} \tag{2.11}$$

with $\gamma_{-1} = 1$.

2.2 Chromatic derivatives

Definition

The chromatic derivatives associated with Legendre polynomials are defined as

$$\mathcal{K}_t^n = (-i)^n P_n^L\left(i\frac{d}{dt}\right) \tag{2.12}$$

When applied to univariate functions, there should be no ambiguity as to which variable we are differentiating with respect to, so we can use Heaviside's notation to denote the differential operator, i.e. $\mathcal{D} = \mathcal{D}_t = \frac{d}{dt}$.

So now we have

$$\mathcal{K}^n = (-i)^n P_n^L(i\mathcal{D}) \tag{2.13}$$

and this of course satisfies the recurrence relation

$$\mathcal{K}^{n+1} = \frac{1}{\gamma_n}\left(\mathcal{D} \circ \mathcal{K}^n\right) + \frac{\gamma_{n-1}}{\gamma_n}\mathcal{K}^{n-1} \tag{2.14}$$

with $\mathcal{K}^{-1} = 0, \mathcal{K}^0 = \mathbb{1}$ where $\mathbb{1}$ is the identity operator and γ_n are as defined as before. Note that $\mathcal{K}^{-1}$ signifies the constant 0 operator.

Now, it is easy to see that

$$\mathcal{K}_t^n[e^{i\omega t}] = i^n P_n^L(\omega)e^{i\omega t} \tag{2.15}$$

since the polynomials $P_n^L(x)$ contains only variables with powers of the same parity as that of n, and the operators $\mathcal{K}_t^n$ only have real coefficients.

Let $x(t) \in \mathbf{BL}(\pi)$. We now have,

$$\begin{aligned}
\mathcal{K}_t^n[x(t)] &= \mathcal{K}_t^n[\mathcal{F}^{-1}[X(\omega)]] \\
&= \mathcal{K}_t^n\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\mathcal{K}_t^n[e^{i\omega t}]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} i^n P_n^L(\omega)X(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} i^n P_n^L(\omega)X(\omega)e^{i\omega t}d\omega
\end{aligned}$$

Remark. We denote the Fourier transform and its inverse respectively as

$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{2.16}$$

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \tag{2.17}$$

Frequency response

Systems (operators) can have a variety of effects on input signals of different frequencies, e.g. they may amplify certain frequency components while attenuating others. The *frequency response* of a system is the way the system output relates to the system input signals for different frequencies in the frequency domain. More precisely, consider the input signal $X(\omega)$ in the frequency domain. Denote the system output as $Y(\omega)$. The frequency response $H(\omega)$ satisfies the relationship

$$Y(\omega) = H(\omega)X(\omega) \tag{2.18}$$

and is therefore given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \tag{2.19}$$

Let us now contrast the frequency response of the normalized nth order differential operator $\frac{1}{\pi^n}\frac{d^n}{dt^n} = \frac{1}{\pi^n}\mathcal{D}_t^n$ with that of the chromatic derivative operator $\mathcal{K}_t^n$.

Let $x(t) \in \mathbf{BL}(\pi)$,

$$\begin{aligned}
y(t) &= \frac{1}{\pi^n}\mathcal{D}_t^n[x(t)] \\
&= \frac{1}{\pi^n}\mathcal{D}_t^n[\mathcal{F}^{-1}[X(\omega)]] \\
&= \frac{1}{\pi^n}\mathcal{D}_t^n\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{1}{\pi^n}\mathcal{D}_t^n[e^{i\omega t}]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} i^n\left(\frac{\omega}{\pi}\right)^n X(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}d\omega
\end{aligned}$$

So now we have

$$Y(\omega) = i^n\left(\frac{\omega}{\pi}\right)^n X(\omega)$$

and therefore

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{i^n\left(\frac{\omega}{\pi}\right)^n X(\omega)}{X(\omega)} = i^n\left(\frac{\omega}{\pi}\right)^n$$

Since the frequency response is a complex function, we can decompose it further into its complex modulus and argument, respectively called the *amplitude response* and *phase response*. We will look specifically at the amplitude response, which represents the system's tendency to amplify or attenuate the signal.

The amplitude response is given by

$$|H(\omega)| = \left|i^n\left(\frac{\omega}{\pi}\right)^n\right| = |i^n|\cdot\left|\left(\frac{\omega}{\pi}\right)^n\right| = \frac{1}{\pi^n}|\omega|^n$$

and is plotted in fig. 2.1a for $n = 15, \ldots, 18$.

Now let us look at the frequency response of the chromatic derivative operator. We first compute the output of the system as before

$$\begin{aligned} y'(t) &= \mathcal{K}_t^n[x(t)] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} i^n P_n^L(\omega)X(\omega)e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y'(\omega)e^{i\omega t}d\omega \end{aligned}$$

So we have

$$Y'(\omega) = i^n P_n^L(\omega)X(\omega)$$

and therefore

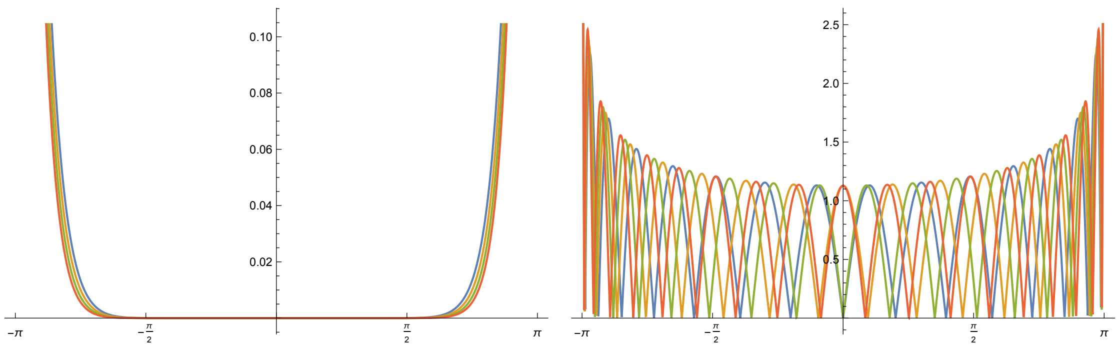
$$H'(\omega) = \frac{Y'(\omega)}{X(\omega)} = \frac{i^n P_n^L(\omega)X(\omega)}{X(\omega)} = i^n P_n^L(\omega)$$

The amplitude response is given by

$$|H'(\omega)| = |P_n^L(\omega)|$$

and is plotted in fig. 2.1b for $n = 15, \ldots, 18$.

Figure 2.1: Amplitude responses of the differential (left) and chromatic derivative (right) operators for $n = 15, \ldots, 18$.



(a) Amplitude response $H(\omega)$

(b) Amplitude response $H'(\omega)$

Just from visually comparing these plots alone, the differences are striking. We see in fig. 2.1a that the frequency response not only attenuates the frequency components of the signal, it practically obliterates its entire spectrum, save maybe for its edges which in practice contains mostly noise anyway. This is the essence of why numerical evaluation of higher order derivatives from signal samples makes no practical sense.

Fortunately, this does not imply numerical evaluation of higher order derivatives is inherently unfeasible. Unlike the set of differential operators $\{\mathcal{D}^n\}_{n\in\mathbb{Z}}$, the set of chromatic derivative operators $\{\mathcal{K}^n\}_{n\in\mathbb{Z}}$ is an orthogonal basis for the vector space of linear differential operators with real coefficients.

A *comb filter* adds a delayed version of a signal to itself, causing constructive and destructive interference. From fig. 2.1b, we see that the amplitude response of the chromatic derivative operators form a family of feedforward comb filters that are well-separated, interleaved, and increasingly refined. Rather than attenuating frequency components, it encodes and augments the frequency components of the signal and overall preserves its spectral features.

Chromatic expansions

Let $x(t) \in \mathbf{BL}(\pi)$. We can write its Fourier transform $X(\omega)$ as a generalized Fourier series

$$X(\omega) = \sum_{n=0}^{\infty} a_n (-i)^n P_n^L(\omega)$$

since we can plug this into the orthogonality relationship of Legendre polynomials to obtain

$$\begin{aligned}
\int_{-\pi}^{\pi} X(\omega) i^m P_m^L(\omega) d\omega &= \int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} a_n (-i)^n P_n^L(\omega) \right) i^m P_m^L(\omega) d\omega \\
&= \sum_{n=0}^{\infty} a_n (-i)^n i^m \int_{-\pi}^{\pi} P_n^L(\omega) P_m^L(\omega) d\omega \\
&= 2\pi \sum_{n=0}^{\infty} a_n (-i)^n i^m \delta_{mn} \\
&= 2\pi a_m
\end{aligned}$$

Therefore,

$$a_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) i^m P_m^L(\omega) d\omega = \mathcal{K}_t^m[x(t)]|_{t=0} = \mathcal{K}^m[x](0)$$

Now we take the inverse Fourier transform

$$
\begin{aligned}
x(t) = \mathcal{F}^{-1}[X(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=0}^{\infty} a_n(-i)^n P_n^L(\omega)\right)e^{i\omega t}d\omega \\
&= \sum_{n=0}^{\infty} a_n\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(-i)^n P_n^L(\omega)e^{i\omega t}d\omega\right) \\
&= \sum_{n=0}^{\infty} a_n\sqrt{2n+1}j_n(\pi t) \\
&= \sum_{n=0}^{\infty} \mathcal{K}^m[x](0)\sqrt{2n+1}j_n(\pi t)
\end{aligned}
$$

where $j_n(\omega) = \sqrt{\frac{\pi}{2\omega}}J_{n+\frac{1}{2}}(\omega)$ are the spherical Bessel functions.

Thus, we arrive at the chromatic expansion associated with Legendre polynomials

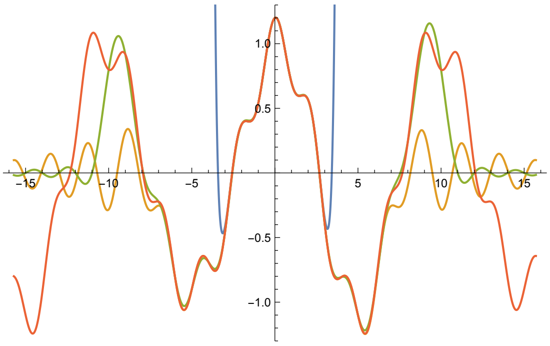
$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^m[f](0)\sqrt{2n+1}j_n(\pi t) \tag{2.20}$$

More generally, let $f(t)$ be any function analytic on $\mathbb{R}$. Then for all $t \in \mathbb{R}$,

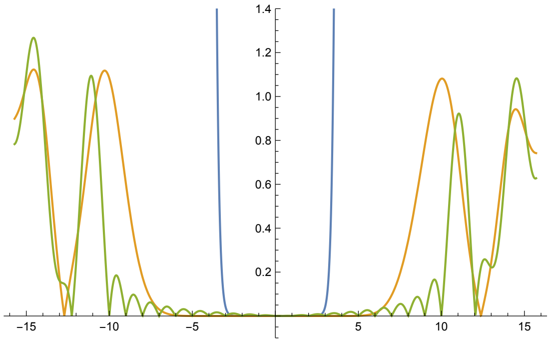
$$
\begin{aligned}
f(t) &= \sum_{n=0}^{\infty} K^n[f](u)K_u^n[\text{sinc}(t-u)] && (2.21) \\
&= \sum_{n=0}^{\infty}(-1)^n K^n[f](u)K^n[\text{sinc}](t-u) && (2.22)
\end{aligned}
$$

and if $f \in \mathbf{BL}(\pi)$, then the series converges uniformly on $\mathbb{R}$ and also in $\mathbf{BL}(\pi)$.

Figure 2.2: Plot of sinusoidal $\frac{1}{10}\sin\left(\frac{3\pi t}{10}\right)+\cos\left(\frac{\pi t}{5}\right)+\frac{1}{5}\cos\left(\frac{9\pi t}{10}\right)$ (red), its Taylor series approximation (blue), Shannon-Whittaker interpolation (green) and chromatic approximation (yellow), all with 21 terms.



(a) Approximations



(b) Absolute error

As an example, consider the sinusoidal signal $f(t)=\frac{1}{10}\sin\left(\frac{3\pi t}{10}\right)+\cos\left(\frac{\pi t}{5}\right)+\frac{1}{5}\cos\left(\frac{9\pi t}{10}\right)$, which is plotted in fig. 2.2a in red. The chromatic approximation of order 21 is shown in yellow, while the Taylor series of order 21 and Whittaker-Shannon interpolation with 21 terms are shown in blue and green respectively. The same color coding scheme applies to fig. 2.2b, which plots the absolute errors of each approximation. Immediately, we see that the truncation error of Taylor series explodes quite dramatically as we move away from the point of expansion, while the error accumulation of the chromatic approximation is quite gentle - even comparable to that of the Whittaker-Shannon interpolation.

Properties

To contrast the chromatic derivatives and its expansion with Taylor series expansions, we consider some of its properties. Compare these with the issues outlined in chapter 1.

- The chromatic approximations are bounded on $\mathbb{R}$ and belong to $\mathbf{BL}(\pi)$.
- As mentioned already, unlike Taylor series expansions, if $f \in \mathbf{BL}(\pi)$, then the series converges uniformly on $\mathbb{R}$ and also in $\mathbf{BL}(\pi)$.
- The above property implies that if L is a filter, then L commutes with the differential

operators K^n. Therefore, for every $f \in \mathbf{BL}(\pi)$,

$$L[f](t) = \sum_{n=0}^{\infty}(-1)^n \mathcal{K}^n[f](0)\mathcal{K}^n[L[\mathrm{sinc}]](t) \tag{2.23}$$

Despite the above dissimilarities, the chromatic approximation are local, like the Taylor series approximation, since its coefficients are the values of the differential operators $\mathcal{K}^n[f](u)$ at instant u.

So we see that chromatic expansions, while local, still possesses the important properties that make the Whittaker-Shannon interpolation so essential to digital signal processing. These features, coupled with the numerical robustness (noise tolerance) of chromatic derivatives, makes the chromatic approximations important and useful for processing and analyzing empirically sampled data, and promises applications in digital signal processing and related fields such as image processing.

2.3 Image processing applications

We now consider some problems which chromatic derivatives naturally lends itself to, specifically those in the field of image processing. This is of course by no means an exhaustive list of problems the chromatic derivatives can address but rather the problems we are most interested in solving as part of this thesis project.

Padding for neighborhood operations

A linear filter is a type of *neighborhood operator* that uses a weighted combination of the pixel values in the vicinity of a given pixel to determine its final output value.

Denote $f(i,j)$ the original pixel value and let $h(k,l)$ be the *convolution matrix* or *kernel*. The *convolution* between f and h is given by

$$\begin{aligned} g(i,j) = [f * h](i,j) &= \sum_k \sum_l f(i-k, j-l) \cdot h(k,l) \\ &= \sum_k \sum_l f(k,l) \cdot h(i-k, j-l) \end{aligned} \tag{2.24}$$

The linear filter is commonly used to create a wide range of effects, such as adding soft blurs, sharpening details, accentuating edges, etc.

Note that the summations defined eq. (2.24) is over all values of k and l. This is because the values of f and h will usually be 0 for values outside of a defined region. For an image of dimensions, (i.e. height and width respectively) M and N, for all $i < 0$ or $i \geq M$ and $j < 0$ or $j \geq N, f(i,j) = 0$. Similarly, if a kernel h is of size (K, K), then for all $|k| > K$ or $|l| > K, h(k,l) = 0$.

One obvious problem that arises is that the convolution operation requires pixel values that are outside the boundaries of the image. By default these values would be padded by zeros, which basically corresponds to a black frame surrounding the image. Several alternative approaches are taken, such as

constant set all pixel values outside the border to some constant value

clamp replicate the edge pixels indefinitely

wrap repeat/tile the image indefinitely

mirror reflect pixels across image border

crop ignore the positions which would require pixel values outside the border, effectively cropping the image by K on each side of the image.

Please consult Szeliski [13, p. 111-115] for an extended discussion of linear filters and the various padding modes employed in practice.

All of these modes can be formulated mathematically [see 13, p. 114-115]. For example, the clamp mode can be expressed as follows. Let $\tilde{f}(i,j)$ be *extended* pixel values, defined for all i, j. We can express it as a function of the original pixel values $f(k,l)$ and the dimensions (height, width) of the image (M, N).

$$\begin{aligned} \tilde{f}(i,j) &= f(k,l), \\ k &= \max(0, \min(M-1, i)), \\ l &= \max(0, \min(N-1, j)). \end{aligned}$$

The alternative method we propose is to essentially "estimate" the values $f(i,j)$ for $i < 0$ or $i \geq M$ and $j < 0$ or $j \geq N$ and effectively extend the image in a visually plausible manner. Note that we need only extend the image by as far as $\lceil \frac{K}{2} \rceil$ pixels beyond the boundaries.

This problem is well-suited to (multidimensional) chromatic derivatives since we can sample the image and place emphasis on the local boundary sample values in order to approximate the values beyond the boundaries. This can be seen as a special (easier) case of the next problem, so can serve to warm us up and equip us to tackle the next problem.

Digital image inpainting

Inpainting is the process of reconstructing lost or deteriorated portions of an image. The overarching objectives are to restore its unity and visual consistency. Traditionally, it is practiced by professional conservator-restorators to restore and retouch damaged artwork. Nowadays, digitalized methods of inpainting are used on digital images for such things as refilling the gaps left from the removal of undesirable object or artifacts from images, such as blemishes, date stamps, and sometimes even people (see fig. 2.3).

Digital techniques for inpainting are widespread and supported by popular photo/graphics editing tools. However, to achieve desirable results, digital inpainting remains a mostly manual (and often laborious) process.

In recent year, methods to completely automate the process of digital image inpainting have received considerable research interest, and the methods devised thus far have demonstrated convincing results. Of course, when we say automate, some level of user interaction is still required; the user must select the region to be removed.

Figure 2.3: Removing large objects from images (image obtained from [3]).

(a) Original image (b) Object removed and gap inpainted

Many of the current methods are inspired by and seek to replicate the techniques used by professional restorators. Their underlying principle is to make use of the structure surrounding the gap and extend it into the gap itself. Other textural details of the surrounding are then applied.

In particular, the literature is dominated by three classes of techniques [3]:

Textural inpainting

Textural inpainting makes use of texture synthesis to fill large regions with repetitive two-dimensional textural patterns. While these are good at replicating the textures in a consistent manner, they have difficulty filling regions in photographs of real world scenes, which usually consist of linear or geometric structures.

Structural inpainting

To address the previous issue, structural inpainting techniques specifically focus on propagating linear structures (called *isophotes* in the literature) into the target region via diffusion. The method is inspired by partial differential equations that describes the distribution of physical heat and work well for small regions. For big regions, however, they introduce blur and also has the tradeoff of poorly synthesizing textures. The methods proposed in [1, 2] belong to this this class of inpainting techniques.

Combination

The method devised in [3] combines the strengths of both approaches, using techniques called *exemplar-based texture synthesis*, which contains the essential process required to replicate both texture and structure.

While we consider these conventional state-of-the-art methods as part of our evaluation framework, we mostly focus on novel techniques involving multidimensional chromatic derivatives, which can also be viewed as a combination of textural and structure inpainting. The development and implementation of these novel techniques forms the major portion of this thesis project. We propose our approach in chapter 3.

Chapter 3

Research Proposal

3.1 Approach

Fourier analysis and other techniques traditionally used in digital signal processing can readily be applied to image processing, since images are just signals in the spatial domain.

As such, we can use the Fourier transform to decompose images into its frequency components, or even represent an image using 2-dimensional Fourier series.

Likewise, we can represent an image using a 2-dimensional chromatic expansion, which will be better suited for local interpolations. Consider fig. 3.1. Here, the white dot points are the signals samples, while the surface is the chromatic expansion.

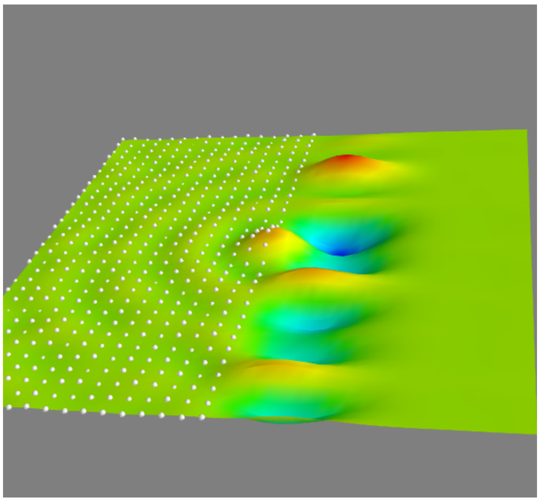


Figure 3.1: Chromatic expansion of a signal in the spatial domain

One thing we notice is that the surface fits the sampled points quite nicely, creating a sort of ripple, while immediately to the right of the boundaries, there are some irregular peaks and valleys. We surmise that this phenomenon is the result of overfitting, which will be one of the first problems we have to overcome, using regularization.

Note that at image borders, we require the texture and structure of the border regions to extend only in an outward direction. For missing values, i.e. gaps within the image, we must preserve textural and structural consistency between all sides of the gap surroundings. For example, if a straight line meets a gap but obviously continues to the other side of the gap, our interpolation must extend the line into the gap and have it join the line extending outside.

In other words, we must satisfy certain constraints imposed by the values at the boundaries of the gap to ensure continuity across gaps. Some work will be required to formulate this precisely as a mathematical programming problem in this context.

This roughly outlines the general approach and the problems we foresee at the present time.

3.2 Plan

Here is an outline of the plan:

1. Complete literature review and background readings
    - on the theory of chromatic derivatives, especially the multidimensional chromatic derivatives and its expansions,
    - state-of-the-art digital image inpainting methods
2. Reproduce (implement) existing state-of-the-art methods as part of the evaluation framework for comparison
3. Develop the necessary framework for using multidimensional chromatic derivatives (expansions) for image representation
4. Devise methods for inpainting based on chromatic expansions
    - Formalize the constraints imposed by values surrounding the gaps in the image, as mentioned in section 3.1.
    - Develop regularization methods to prevent overfitting
5. Implement methods above
6. Experiment with, benchmark and test the implementation empirically
7. Analyze results, revisiting 4 as needed.

8. Prepare and summarize the findings from project, including methods devised, implementations, results obtained, etc. for presentation and final thesis report.

Chapter 4

Conclusion

In this report, we gave the motivation for the chromatic derivatives by demonstrating the shortcomings of Taylor series when processing signals. We defined chromatic derivatives and its expansions, examined its properties in some detail and showed how it can be applied to image processing problems.

In conclusion, we think chromatic derivatives are a powerful tool that promises many interesting and important applications. We look forward to exploring these applications and hope to thereby extend and further enrich the theory of chromatic derivatives.

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