CHROMATIC DERIVATIVES AND EXPANSIONS

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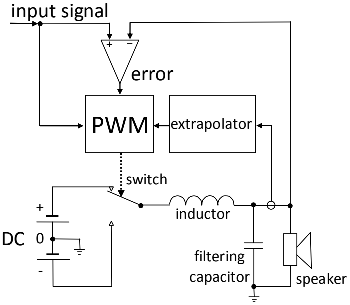
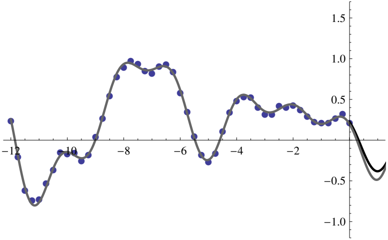
How it started: designing a switching amplifier





- What kind of extrapolation functions should we use?

- This is a **very unusual** signal processing problem: we are interested in **extremely local**, "microscopic" signal behavior.

Representing functions from the Palley-Wiener space

- Let $f \in \mathbf{PW}(\pi)$, i.e., $f \in L^2$ with $\widehat{f(\omega)}$ supported on $[-\pi, \pi]$

Shannon's Expansion:
(Whittaker–Kotelnikov–Nyquist–Shannon)

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}$$

- **global in nature** – requires samples $f(n)$ for all n;
- **fundamental** to signal processing;
- **truncations poorly represent local signal behavior**

Taylor's Expansion:

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{t^n}{n!}$$

- **local in nature** – requires $f^{(n)}(t)$ at a single instant $t = 0$.
- **very little use** in signal processing – **why ?**

Numerical differentiation of band limited signals

Let $f \in \mathbf{PW}(\pi)$; then $\dfrac{f^{(n)}(t)}{\pi^n} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathrm{i}^n \left(\frac{\omega}{\pi}\right)^n \widehat{f(\omega)}\, e^{\mathrm{i}\,\omega t}\, d\omega.$

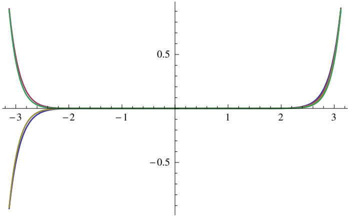


Figure : $(\omega/\pi)^n$ for $n = 15 - 18$

- derivatives of high order **obliterate the spectrum**.
- transfer functions of the (normalized) derivatives **cluster together and are nearly indistinguishable.**
- can we find a better base for the space of linear differential operators? An **orthogonal base**??

Orthogonal base for the space of linear diff. operators

- Start with normalized and re-scaled Legendre polynomials:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega)P_m^L(\omega)\mathrm{d}\omega = \delta(m-n).$$

- Obtain operator polynomials by replacing ω^k with $\mathrm{i}^k\, d^k/dt^k$ and renormalizing with $(-\mathrm{i})^n$:

$$\mathcal{K}_t^n = (-\mathrm{i})^n P_n^L\left(\mathrm{i}\ \frac{d}{dt}\right)$$

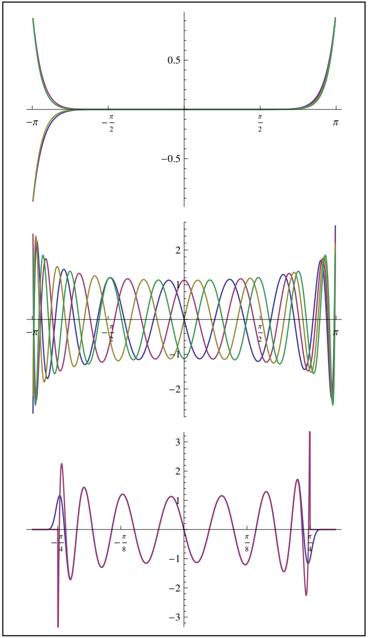
- Definition of $\mathcal{K}^n$ chosen so that

$$\mathcal{K}_t^n[e^{\mathrm{i}\,\omega t}] = \mathrm{i}^n P_n^L(\omega)\, e^{\mathrm{i}\,\omega t}.$$

- Thus, for $f \in \mathbf{PW}(\pi)$,

$$\mathcal{K}^n[f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{i}^n P_n^L(\omega)\widehat{f(\omega)}e^{\mathrm{i}\,\omega t}d\omega.$$

Why are chromatic derivatives a better base?



- Compare the graphs of the transfer functions of $1/\pi^n d^n/dt^n$, i.e., $(\omega/\pi)^n$ (first graph) and of $\mathcal{K}^n$, i.e., $P^L_n(\omega)$ (second graph).

- Transfer functions of $\mathcal{K}^n$ form a sequence of well separated comb filters which preserve spectral features of the signal, thus we call them the **chromatic derivatives**.

- Third graph: transfer function of the ideal filter $\mathcal{K}^{15}$ (red) vs. transfer function of a transversal filter (blue), (128 taps, $2\times$ oversampling.)

Local representation of the scalar product in $\mathbf{PW}(\pi)$

Proposition: Assume that $f, g \in \mathbf{PW}(\pi)$; then the sums on the left hand side of the following equations **do not depend** on the choice of the instant t, and

$$\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2 = \int_{-\infty}^{\infty} f(x)^2 \, dx = \|f\|^2$$

$$\sum_{n=0}^{\infty} \mathcal{K}^n[f](t) \, \mathcal{K}^n[g](t) = \int_{-\infty}^{\infty} f(x) g(x) \, dx = \langle f, g \rangle$$

$$\sum_{n=0}^{\infty} \mathcal{K}^n[f](t) \; \mathcal{K}_t^n[g(u-t)] = \int_{-\infty}^{\infty} f(x) g(u-x) \, dx = (f * g)(u)$$

- These are the **local equivalents** of the usual, "globally defined" norm, scalar product and convolution!

- **Aim: "maximally localized"** signal processing, suitable for control applications or transient analysis.

Fixing Taylor's Expansion: Chromatic Expansion

Proposition: Let $\text{sinc}\,(t) = \dfrac{\sin t}{t}$ and let $f(t)$ be **any analytic function**. Then,

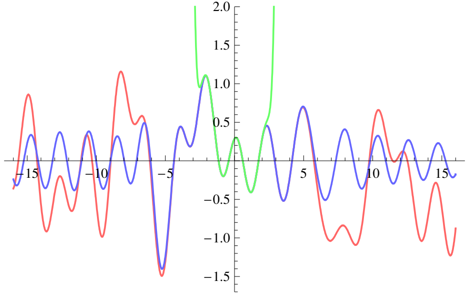
$$f(t) \quad = \quad \sum_{n=0}^{\infty} \mathcal{K}^n[f](u)\, \mathcal{K}^n_u[\text{sinc}\,\pi(t-u)]$$

Compare such **chromatic expansion** with the Taylor expansion:

$$f(t) \quad = \quad \sum_{n=0}^{\infty} f^{(n)}(u)\, \frac{(t-u)^n}{n!}$$

- The truncations of the chromatic series **belong to** $\mathbf{PW}(\pi)$.
- If $f \in \mathbf{PW}(\pi)$ the series converges **uniformly and in** L^2.

Chromatic approximation versus Taylor's approximation



- **red:** the signal; **blue:** the chromatic approximation of order 15; **green:** the Taylor approximation of order 15.
- $f^{(k)}(0) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left[\sum_{m=0}^{n} (-1)^m \, \mathcal{K}^m[f](0) \, \mathcal{K}^m[\mathrm{sinc}\,](t)\right]_{t=0}$
- Thus, chromatic approximations are **local**

General families of chromatic derivatives

- Given a family of orthonormal polynomials $P_n(\omega)$ we can always define differential operators

$$\mathcal{K}_t^n = (-\mathrm{i})^n \, P_n^L\left(\mathrm{i} \, \frac{d}{dt}\right)$$

Question:

What are the families of orthogonal polynomials such that for the corresponding differential operators $\mathcal{K}^n$ and some associated function $\boldsymbol{m}(t)$ we have

$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](u) \mathcal{K}_u^n[\boldsymbol{m}](t-u)$$

for important classes of functions, and when is the convergence uniform?

Examples:
Legendre Polynomials/Spherical Bessel functions

- For the (normalized) **Legendre polynomials**

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega)P_m^L(\omega)\mathrm{d}\omega = \delta(m-n)$$

and for

$$\boldsymbol{m}(t) = \mathrm{sinc}(\pi t)$$

we have

$$\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\sqrt{2n+1}\, j_n(\pi t)$$

and

$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\,\sqrt{2n+1}\, j_n(\pi t)$$

holds for all entire functions;
the convergence is uniform for functions in $\mathbf{PW}(\pi)$.

Examples:
Chebyshev polynomials /Bessel functions

- For the (normalized) **Chebyshev polynomials** of the first kind:

$$\int_{-\pi}^{\pi} \frac{P_n^T(\omega) P_m^T(\omega)}{\pi^2 \sqrt{1 - \left(\frac{\omega}{\pi}\right)^2}} \mathrm{d}\omega = \delta(n-m).$$

for $\boldsymbol{m}(t) = J_0(\pi t)$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \sqrt{2}\, J_n(\pi t)$ and

$$f(t) = f(u) J_0(\pi t) + \sqrt{2} \sum_{n=1}^{\infty} \mathcal{K}^n[f](0) J_n(\pi t)$$

- The Neumann series - converges for all entire functions;

- Convergence uniform for band limited functions which satisfy

$$\int_{-\pi}^{\pi} |\widehat{f}(\omega)|^2 \sqrt{1 - (\omega/\pi)^2}\, \mathrm{d}\omega < \infty$$

Examples:
Hermite polynomials/Gaussian monomials

- For the (normalized) **Hermite polynomials**

$$\int_{-\infty}^{\infty} P_n^H(\omega) P_m^H(\omega) \, \frac{\mathrm{e}^{-\omega^2}}{\sqrt{\pi}} \, \mathrm{d}\omega = \delta(n-m)$$

and $\boldsymbol{m}(t) = \mathrm{e}^{-t^2/4}$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \, \frac{t^n}{\sqrt{2^n \, n!}} \, \mathrm{e}^{-t^2/4}$

- chromatic expansion converges for entire functions s.t.

$$\limsup_{n\to\infty} \frac{|f^{(n)}(z)|^{1/n}}{\sqrt{n}} < \infty$$

- converges uniformly for all entire functions s.t.

$$\int_{-\infty}^{\infty} |\widehat{f(\omega)}|^2 \, \mathrm{e}^{\omega^2} \mathrm{d}\omega < \infty$$

Examples: the hyperbolic family

Let $\boldsymbol{m}(z) = \operatorname{sech}(z)$ and let $L_n(\omega)$ be such that

$$\frac{1}{2}\int_{-\infty}^{\infty} L_n(\omega)\; L_m(\omega) \operatorname{sech}\left(\frac{\pi\omega}{2}\right) \mathrm{d}\omega = \delta(m-n)$$

then $\boldsymbol{m}(z)$ is analytic on the strip $S = \{z : \operatorname{Im}(z) < \pi/2\}$ and

$$\mathcal{K}^n[\boldsymbol{m}](z) = (-1)^n \tanh^n(z) \operatorname{sech}(z).$$

The chromatic expansion

$$f(z) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0) \tanh^n(z) \operatorname{sech}(z)$$

converges uniformly inside the strip S whenever $f(z)$ analytic inside this strip and its Fourier transform satisfies

$$\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2 \cosh(\omega) \mathrm{d}\omega < \infty$$

General families of chromatic derivatives

Definition: A family of polynomials $P_n(\omega)$ which is orthonormal with respect to a non-decreasing bounded moment distribution function $a(\omega)$, i.e., such that

$$\int_{-\infty}^{\infty} P_n(\omega) P_m(\omega) \mathrm{d}a(\omega) = \delta(m-n)$$

is **chromatic** if the moments μ_n of $a(\omega)$,

$$\mu_n = \int_{-\infty}^{\infty} \omega^n \mathrm{d}a(\omega)$$

satisfy $\mu_{2n+1} = 0$ and

$$\rho = \limsup_{n\to\infty} \left(\frac{\mu_n}{n!}\right)^{1/n} < \infty.$$

Definition: $L^2_{a(\omega)}$ is the space of functions $\phi(\omega)$ satisfying

$$\int_{-\infty}^{\infty} |\phi(\omega)|^2 \mathrm{d}a(\omega) < \infty.$$

General families of chromatic derivatives

Theorem: If $P_n(\omega)$ are a chromatic family of polynomials orthonormal with respect to $a(\omega)$, then they are a complete base of the space $L^2_{a(\omega)}$. (Follows from a theorem of Riesz)

Definition: Λ^2 is the space of functions $f(t)$ analytic on $S_{\rho/2}$ such that for the chromatic derivatives $\mathcal{K}^n$ which correspond to $P_n(\omega)$ we have

$$\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 < \infty.$$

Theorem: Let $P_n(\omega)$ be a chromatic family of polynomials orthonormal with respect to $a(\omega)$, and let

$$\boldsymbol{m}(z) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}a(\omega)$$

Then $\boldsymbol{m}(z)$ is analytic on the strip $S_{\rho/2} = \{z : |\mathrm{Im(z)}| < \rho/2\}$.

- Thus, $\boldsymbol{m}(z)$ belongs to $L^2_{a(\omega)}$.

General families of chromatic derivatives

Theorem: A function $f(z)$ is in Λ^2 if and only if there exists a function $\phi_f(\omega) \in L^2_{a(\omega)}$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi_f(\omega) \mathrm{e}^{\mathrm{i}\,\omega z} \mathrm{d}a(\omega)$$

For such $f(z) \in \Lambda^2$ and for the corresponding $\phi_f(\omega) \in L^2_{a(\omega)}$,

$$\phi_f(\omega) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0) P_n(\omega)$$

Corollary: For all $t \in \mathbb{R}$,

$$\begin{aligned} \|f\|^2_\Lambda &= \sum_{n=0}^{\infty} |\mathcal{K}^n[\boldsymbol{m}](0)|^2 = \|\phi_f(\omega)\|^2_{a(\omega)} \\ &= \|\phi_f(\omega)\,\mathrm{e}^{\mathrm{i}\,\omega t}\|^2_{a(\omega)} = \sum_{n=0}^{\infty} |\mathcal{K}^n[\boldsymbol{m}](t)|^2 \end{aligned}$$

General chromatic expansions

Theorem: If $f(z) \in \Lambda^2$, then

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0) \mathcal{K}^n[\boldsymbol{m}](t)$$

with the series converging uniformly on strips $S_{\rho/2-\epsilon}$.

A geometric interpretation

- For every $t \in \mathbb{R}$ the mapping $f \mapsto \langle \mathcal{K}^m[f](t) \rangle_{m \in \mathbb{N}}$ is an isometry between Λ^2 and l_2.
- The orthonormal base $\{\mathcal{K}^n[\boldsymbol{m}](t)\}_{n \in \mathbb{N}}$ of Λ^2 is then mapped

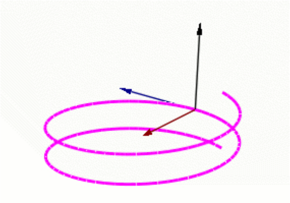
  into an orthonormal base $\{\langle(\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](t)\rangle_{m \in \mathbb{N}}\}_{n \in \mathbb{N}}$.
- For every t,

$$\begin{aligned}
\langle \mathcal{K}^m[f](t) \rangle_{m \in \mathbb{N}} &= \sum_{n=0}^{\infty} \mathcal{K}^n[f](0) \langle (\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](t) \rangle_{m \in \mathbb{N}} \\
&= [(\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](t)]_{m,n \in \mathbb{N}} \; \langle \mathcal{K}^m[f](0) \rangle_{m \in \mathbb{N}}
\end{aligned}$$

A geometric interpretation - cont.

- Let $\vec{e}_{n+1}(t) = \langle (\mathcal{K}^n \circ \mathcal{K}^m)[\boldsymbol{m}](t) \rangle_{m\in\mathbb{N}}$ and let $H(t)$ be an antiderivative of $\vec{e}_1(t) = \langle \mathcal{K}^m[\boldsymbol{m}](t) \rangle_{m\in\mathbb{N}}$. Then

$$\begin{aligned} \vec{e}_1(t) &= \vec{H}'(t) \\ \vec{e}_1{}'(t) &= \gamma_0\, \vec{e}_2(t) \\ \vec{e}_k{}'(t) &= -\gamma_{k-2}\, \vec{e}_{k-1}(t) + \gamma_{k-1}\, \vec{e}_{k+1}(t) \quad (\text{for } k \geq 2). \end{aligned}$$



- γ_n are the recursion coefficients:
  $\gamma_n\, P_{n+1}(x) = x\, P_n(x) - \gamma_{n-1} P_{n-1}(x)$
- $\{\vec{e}_{n+1}(t)\}_{n\in\mathbb{N}}$ is the moving frame of a helix $H(t)$ in l_2;
- Its generalized curvatures are the recursion coefficients γ_n.
- The formulas are the **Frenet–Serret** formulas.

General families of chromatic derivatives

We are now back to the theorem:

Theorem: If $f(z) \in \Lambda^2$, then

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\mathcal{K}^n[\boldsymbol{m}](t)$$

with the series converging uniformly on strips $S_{\rho/2-\epsilon}$.

But how about the **local (non-uniform) convergence** of the chromatic series??

For example, in the case of the Chebyshev polynomials $T_n(\omega)$ and the Bessel functions of the first kind $J_n(\omega)$, we know that the chromatic series is just the Newmann series, and that the above equality holds **for every entire function** $f(z)$!

Weakly bounded families

Recall that a family of polynomials is orthonormal with respect to a moment distribution function $a(\omega)$ with all odd moments $\mu_{2n+1} = 0$ if and only if there exist $\gamma_n > 0$ such that

$$\gamma_n \, P_{n+1}(\omega) = \omega \, P_n(\omega) - \gamma_{n-1} \, P_{n-1}(\omega).$$

Definition: Such family of polynomials $P_n(\omega)$ is:

1. **bounded** if for some $M, m > 0$ and all n we have
$$m < \gamma_n < M;$$
2. **weakly bounded** if for some $0 \le p < 1$ we have
$$m < \gamma_n < M \, n^p \quad \text{and} \quad \frac{\gamma_n}{\gamma_{n+1}} < M$$

- Bounded families are also weakly bounded with $p = 0$.

Examples:

- **Bounded families** $(p = 0)$:
  - **Legendre** polynomials: $\gamma_n = \frac{\pi(n+1)}{\sqrt{4(n+1)^2-1}} \to \frac{\pi}{2}$
  - **Chebyshev** polynomials: $\gamma_0 = \frac{\pi}{\sqrt{2}}$ and $\gamma_{n+1} = \frac{\pi}{2}$
- **Weakly bounded family** $(p = 1/2)$:

  Hermite polynomials: $\gamma_n = \sqrt{(n+1)/2}$;
- A family which is **not weakly bounded** $(p = 1)$:

  Hyperbolic family: $\gamma_n = n + 1$;
- The last example shows that if we want $\boldsymbol{m}(z)$ to be entire, then the bound $p < 1$ is sharp.

Lemma: Every weakly bounded family of orthonormal polynomials is chromatic.

Theorem: Let $\{P_n(\omega)\}_{n\in\mathbb{N}}$ be a weakly bounded family and let $f(z)$ be an entire function. If

$$\lim_{n\to\infty}\left|\frac{f^{(n)}(0)}{n!^{1-p}}\right|^{1/n} = 0$$

then for every $z \in \mathbb{C}$

$$f(z) = \sum_{j=0}^{\infty}(-1)^j\,\mathcal{K}^j[f](0)\,\mathcal{K}^j[\boldsymbol{m}](z).$$

The convergence is uniform on every disc of finite radius.

Corollary: If $\{P_n(\omega)\}_{n\in\mathbb{N}}$ is is a bounded family then the chromatic expansion of every entire function $f(z)$ point-wise converges to $f(z)$ for all z.

- It turns out that many of the classical formulas such as

$$\mathrm{e}^{\mathrm{i}\,\omega t} = \sum_{n=0}^{\infty} \mathrm{i}^n\, T_n(\omega) J_n(t)$$
$$J_0(t+u) = J_0(u)J_0(t) + 2\sum_{n=1}^{\infty}(-1)^n J_n(u)J_n(t)$$
$$J_0(t)^2 + 2\sum_{k=1}^{\infty} J_n(t)^2 = 1$$
$$J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) = 1$$

are special cases of **chromatic expansions valid for all weakly bounded families of polynomials** and their associated $\boldsymbol{m}(z)$:

$$\mathrm{e}^{\mathrm{i}\,\omega t} = \sum_{n=0}^{\infty} \mathrm{i}^n P_n(\omega)\mathcal{K}^n[\boldsymbol{m}](t)$$
$$\boldsymbol{m}(t+u) = \sum_{n=0}^{\infty}(-1)^n \mathcal{K}^n[\boldsymbol{m}](u)\mathcal{K}^n[\boldsymbol{m}](t)$$
$$\sum_{k=1}^{\infty} \mathcal{K}^n[\boldsymbol{m}](t)^2 = 1$$
$$\boldsymbol{m}(z) + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} \frac{\gamma_{2k-2}}{\gamma_{2k-1}}\right)\mathcal{K}^{2n}[\boldsymbol{m}](z) = 1$$

Theorem: Assume $P_n(\omega)$ is weakly bounded and let k be an integer such that $p \leq 1 - 1/k$. Then:

(a) there exists $K > 0$ such that

$$|\mathcal{K}^n[\boldsymbol{m}](z)| < \frac{|Kz|^n}{n!^{1-p}} \, \mathrm{e}^{|Kz|^k};$$

(b) for every $f(t) \in \Lambda_2$ there exist $C, L > 0$ such that

$$|f(z)| \leq C\mathrm{e}^{L|z|^k}.$$

In particular, for every bounced family ($p = 0$ and $k = 1$) we get that functions $f \in \Lambda_2$

- are of exponential type;
- have a finitely supported Fourier-Stieltjes transform in $L^2_{a(\omega)}$ (they may not be in L_2).

- **Is the converse also true?**

A mild generalization of the Paley-Wienner Theorem??

Conjecture: Assume that $f(z)$ is an entire function for which there exist a symmetric moment distribution function $a(\omega)$ and a function $\varphi(\omega) \in L^2_{a(\omega)}$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi(\omega) \mathrm{e}^{\mathrm{i}\, z\omega} \mathrm{d}a(\omega).$$

The following are equivalent:

(a) f is of exponential type, i.e., there exist $C, L > 0$ such that

$$|f(z)| < C\mathrm{e}^{L|z|}, \quad (z \in \mathbb{C});$$

(b) $a(\omega)$ can be chosen such that $\mathrm{d}a(\omega)$ is finitely supported.

A real generalization of the Paley - Wienner Theorem??

Conjecture: Assume that $f(z)$ is an entire function for which there exist a symmetric moment distribution function $a(\omega)$ and a function $\varphi(\omega) \in L^2_{a(\omega)}$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi(\omega) \mathrm{e}^{\mathrm{i}\, z\omega} \mathrm{d} a(\omega),$$

and let $k \geq 1$ be an integer. Then the following are equivalent:

(c) there exist $C, L > 0$ such that

$$|f(z)| < C \mathrm{e}^{L|z|^k}, \quad (z \in \mathbb{C});$$

(d) $a(\omega)$ can be chosen such that the corresponding γ_n satisfy $\gamma_n < M\, n^{1-1/k}$.

Some more questions:

Recall that a family of polynomials is chromatic if

$$\rho \;=\; \limsup_{n\to\infty} \frac{\mu_n^{1/n}}{n} \;=\frac{1}{\mathrm{e}}\limsup_{n\to\infty}\left(\frac{\mu_n}{n!}\right)^{1/n} < \infty$$

Lemma: $P_n(\omega)$ are chromatic if and only if for every $0 \leq \alpha < \rho$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha|\omega|}\mathrm{d}\,a(\omega) < \infty$$

Qestion: Is it possible to characterize weakly bounded families purely in terms of the properties of the corresponding $a(\omega)$?

Qestion: If not, is it possible to characterize $a(\omega)$ for which

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha|\omega|}\mathrm{d}\,a(\omega) < \infty$$

purely in terms of the asymptotic behavior of the recursion coefficients γ_n of the corresponding family of orthonormal polynomials?

Almost Periodic Functions

- Trigonometric functions do not belong to the spaces Λ_2:

$$\|e^{i\omega t}\|_\Lambda^2 = \sum_{n=0}^{\infty} |\mathcal{K}^n[e^{i\omega t}]|^2 = \sum_{n=0}^{\infty} P_n(\omega)^2 \to \infty$$

Definition: Assume that $P_n(\omega)$ are defined by

$$\gamma_n \, P_{n+1}(\omega) = \omega \, P_n(\omega) - \gamma_{n-1} \, P_{n-1}(\omega).$$

with γ_n such that $\gamma_n \geq m > 0$;

$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{\gamma_k} = \infty.$$

We denote by $\mathcal{C}$ the vector space of analytic functions such that the sequence

$$\nu_n^f(t) = \frac{\sum_{k=0}^n \mathcal{K}^k[f](t)^2}{\sum_{k=0}^n \frac{1}{\gamma_k}}$$

converges uniformly on every finite interval.

Theorem: Let $f, g \in \mathcal{C}$ and

$$\sigma_n^{fg}(t) = \frac{\sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t)}{\sum_{k=0}^{n} \frac{1}{\gamma_k}}$$

then the sequence $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ converges to a constant function.

In particular, also

$$\nu_n^f(t) = \frac{\sum_{k=0}^{n} \mathcal{K}^k[f](t)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_k}}$$

converges to a constant value independent of t.

Corollary: $\lim_{n\to\infty} \nu_n^f(t)$ is a semi-norm.

Definition: Let $\mathcal{C}_0 \subset \mathcal{C}$ consists of $f(t)$ such that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \mathcal{K}^k[f](t)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_k}} = 0.$$

We define $\mathcal{C}_2 = \mathcal{C}/\mathcal{C}_0$.

Definition: For $f, g \in \mathcal{C}$ we define

$$\|f\| = \lim_{n\to\infty} \left(\frac{\sum_{k=0}^{n} \mathcal{K}^k[f](t)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_k}} \right)^{1/2}$$

$$\langle f, g \rangle = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t)}{\sum_{k=0}^{n} \frac{1}{\gamma_k}}$$

What is in $\mathcal{C}_2$: Examples

Do the trigonometric functions belong to $\mathcal{C}_2$??

$$\frac{\sum_{k=0}^{n} |\mathcal{K}^k[\mathrm{e}^{\mathrm{i}\omega t}]|^2(t)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_k}} = \frac{\sum_{k=0}^{n} P_n(\omega)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_k}}$$

Clearly, the limit would be independent of t, but does it exist??

- **Chebyshev polynomials**: $\gamma_n = \pi/2$ and if $0 < \omega < \pi$ then:

$$\|\mathrm{e}^{\mathrm{i}\omega t}\| = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} P_n^T(\omega)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_k}} = \lim_{n\to\infty} \frac{\pi}{2(n+1)} \sum_{k=0}^{n} P_n^T(\omega)^2 = \frac{\pi}{2}$$

- For all $0 < \sigma, \omega < \pi$, $\sigma \neq \omega$,

$$\langle \mathrm{e}^{\mathrm{i}\sigma t}, \mathrm{e}^{\mathrm{i}\omega t} \rangle = \lim_{n\to\infty} \frac{\pi}{2(n+1)} \sum_{k=0}^{n} P_k^T(\sigma) P_k^T(\omega) = 0$$

- **Hermite polynomials**: $\gamma_n = \sqrt{\frac{n+1}{2}}$; then

$$\sum_{k=0}^{n} \frac{1}{\gamma_k} \sim 2\sqrt{2(n+1)}$$

and for all $0 < \omega, \sigma$ such that $\omega \neq \sigma$,

$$\|e^{i\omega t}\| = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} P_k^H(\omega)^2}{2\sqrt{2(n+1)}} = \frac{e^{\omega^2}}{2\sqrt{\pi}},$$

and

$$\langle e^{i\sigma t}, e^{i\omega t}\rangle = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} P_k^H(\sigma) P_k^H(\omega)}{2\sqrt{2(n+1)}} = 0.$$

Thus, in this space every two pure harmonic oscillations with distinct positive frequencies are mutually orthogonal.

A Conjecture

If for some $0 \leq p < 1$ the recursion coefficients γ_n satisfy

$$0 < \lim_{n\to\infty} \frac{\gamma_n}{n^p} < \infty \tag{1}$$

then $\sum_{k=0}^{n} \frac{1}{\gamma_k} \sim (n+1)^{1-p}$.

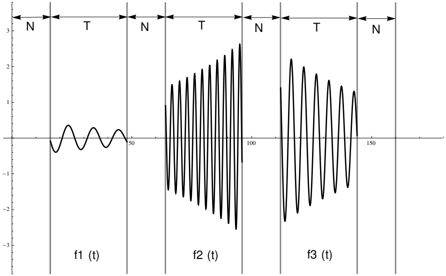
Conjecture (from my EJA paper): If (**??**) holds, then

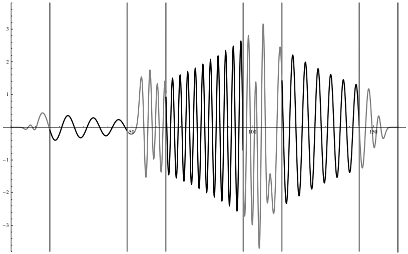
$$0 < \lim_{n\to\infty} \frac{\sum_{k=0}^{n} P_k(\omega)^2}{(n+1)^{1-p}} < \infty$$

for all ω in the support of the corresponding $a(\omega)$.

- It turns out that the special case for $p = 0$ is a **well known, open problem** (as I was told by P. Nevai and V. Totik).

Application: signal interpolation



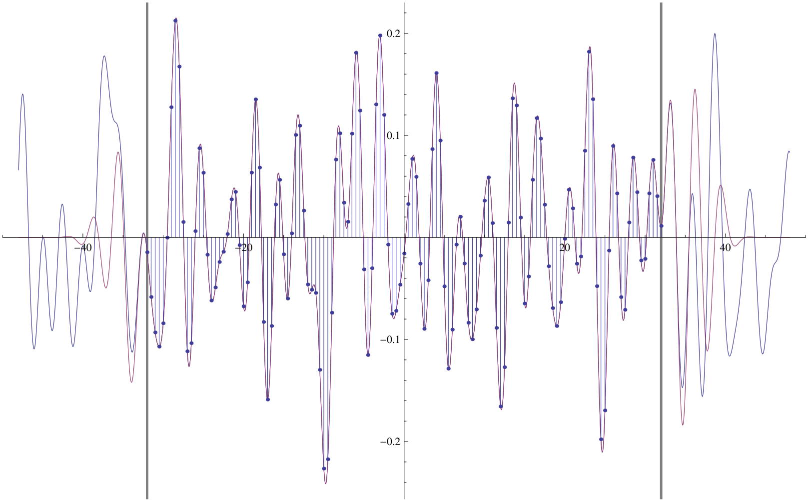


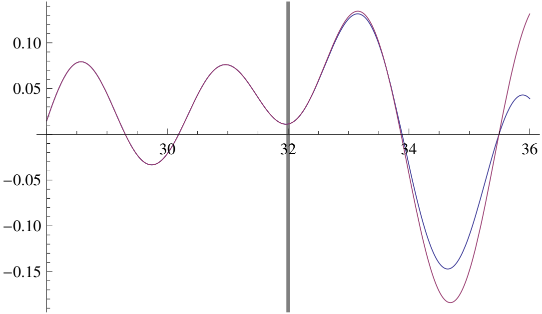
Given pieces of band limited signals join them so that the out of band energy is minimal.

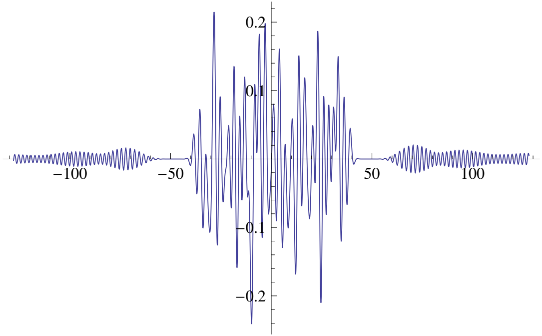
We use chromatic expansions to ensure that the resulting signal is N times continuously differentiable Then

$$|\widehat{f}(\omega)| \leq \frac{|\mathcal{K}^n[f]\widehat{\ }(\omega)|}{|P_n(\omega)|} \leq \frac{M}{|P_n(\omega)|}$$

Extrapolation filter




0.10
0.05
−0.05
−0.10
−0.15
30
32
34
36


0.2
0.1
−0.1
−0.2
−100
−50
50
100

Application: frequency estimation

Idea: *A signal is a sum of at most N shifted and damped sine waves iff it is a solution to a homogeneous linear differential equation with constant coefficients of order at most $2N$.*

A rough sketch of the frequency estimation algorithm:

▶ Choose the chromatic derivatives which are orthogonal with respect to the power spectrum density of the noise:

- take polynomials $P_n(\omega)$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(\omega)P_m(\omega)S(\omega)\,d\omega = \delta(m-n)$$

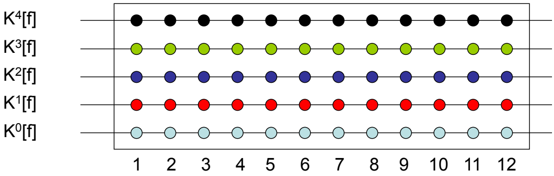
- Let $\mathcal{K}^n$ be the chromatic derivatives corresponding to the polynomials $P_n(\omega)$, i.e., let

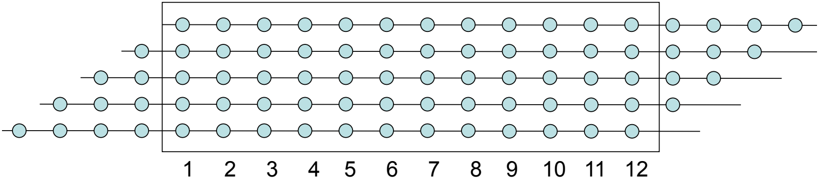
$$\mathcal{K}^n = (-\mathrm{i})^n P_n(-\mathrm{i}\,d/dt).$$

Then, assuming $E[\nu(n)^2] = \rho^2$, we have

$$E\{\mathcal{K}^n[\nu](n)\mathcal{K}^m[\nu](n)\} = \delta(m-n)\rho^2$$

so we can apply the standard SVD or ED methods.





error Cazdow's method: 0.0025;
error Cazdow's method+CD method: 0.0018
(SNR$= -10db$; 10 000 runs)

What if we allow time varying coefficients? We can easily detect chirps, ets. In fact, transients can be classified according to what type of differential eqation they satisfy!

CONJECTURE:

Classification via the minimal degree linear differential equation satisfied by a transient can play the role which the spectrum plays for the "steady state" signals!!

My website

`http://www.cse.unsw.edu.au/~ignjat/diff/`

contains papers on chromatic derivatives as well as some programs. The most complete presentation is in " Chromatic Derivatives, Chromatic Expansions and Associated Spaces", available as

`http://www.cse.unsw.edu.au/~ignjat/diff/ChromaticDerivatives.pdf`

`The programs are mostly an uncommented mess, except perhaps for the tutorial available at the above web page, but I will clean them up and and comment them before the end of the year, hopefully.`

`If you have a slightest interest in this stuff please do get in touch, I'd love to collaborate!`

Details can be found in:

- A. Ignjatovic: Local Approximations Based on Orthogonal Differential Operators, **Journal of Fourier Analysis and Applications**, Vol. 13, Issue 3 (2007).

- — "— : Chromatic derivatives and local approximations, **IEEE Transactions on Signal Processing**, Volume 57, Issue 8 (2009).

- — "— : Chromatic derivatives, chromatic expansions and associated spaces, **East Journal on Approximations**, Volume 15, Number 3 (2009).

- A. Ignjatovic and A. Zayed: Multidimensional chromatic derivatives and series expansions, **Proceedings of the American Mathematical Society**, 139 (2011).

THANK YOU!