Chromatic Derivatives:
"A New Perspective of Signal Representation"

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Fourier Series as an Example

- For the time domain signal $f(t)$ with period T, We can expand it into coefficients a_n by the Fourier series expansion, which is

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi nt/T}. \quad (1)$$

$$a_n = \frac{1}{T}\int_0^T f(t) e^{-j2\pi nt/T}\, dt, \quad (2)$$

- (1) can be viewed as the linear combination of the bases, $e^{j2\pi nt/T}$.
- $a_n = \left\langle f(t), \frac{1}{T} e^{j2\pi nt/T} \right\rangle$.
- Global approximation: a_0, a_1, a_{-1} contains the rough shape information.

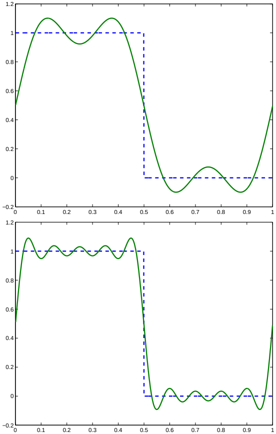

Figure: Reconstruction by 11 a_n's (top) and 21 a_n's (bottom).

Taylor Series as an Example

- In Calculus [4], we can expand function $f(t)$ around t_0 into linear combination of polynomials $(t - t_0)^n/n!$,

$$f(t) = a_0 + a_1(t - t_0) + a_2\frac{(t - t_0)^2}{2!} + \dots + a_n\frac{(t - t_0)^n}{n!} + \dots \quad (3)$$

$$a_n = [\mathcal{D}_t^n f(t)]|_{t=t_0} \quad (4)$$

- Local approximation: a_0, a_1, a_2 contains the shape information around the expansion point.
- Stability problem.

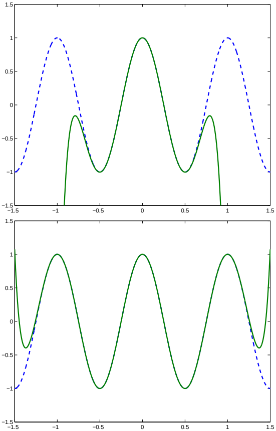

Figure: Reconstruction by 11 a_n's (top) and 21 a_n's (bottom).

Comparison between Fourier-based and Taylor-based Representation

	Fourier-based	Taylor-based
Physical Meaning	The frequency component	The slope component
Bases	Sinusoids	Polynomials
Coeff. by	Integration	High-order derivatives
Approx.	Global	Local
Variants	Fourier Transform, Sine/Cosine Transform	Chromatic Derivatives

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First Look at Chromatic Derivatives

- Aleksandar Ignjatović introduced the concept and coined "Chromatic Derivatives" around 2000. [1, 2].
- Its main concept is to replace the order-n derivative with linear combinations of smaller order derivatives.

$$\mathcal{D}_t^n \quad \Rightarrow \quad \sum_{k=0}^{n} c_k \mathcal{D}_t^k \tag{5}$$

- The coefficients c_k are determined by orthogonal polynomials.
- It is more bounded and noise-resistant than Taylor series.

$$\boxed{f(t) = a_0 + a_1(t - t_0) + a_2 \frac{(t - t_0)^2}{2!} + \ldots, \qquad a_n = \left[\mathcal{D}_t^n f(t)\right]\big|_{t=t_0}}$$

Legendre Polynomials as an Example

- Differential equation:

$$\left[\left(1-x^2\right)\mathcal{D}_x^2 - 2x\mathcal{D}_x\right] P_n(x) = -n\,(n+1)\,P_n(x).$$

- Complete and orthogonal over $[-1,1]$.

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{m,n}.$$

- First four polynomials:

$$P_0(x) = 1, \quad P_1(x) = x,$$
$$P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2},$$
$$P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x.$$

- $P_n(-x) = (-1)^n P_n(x)$.

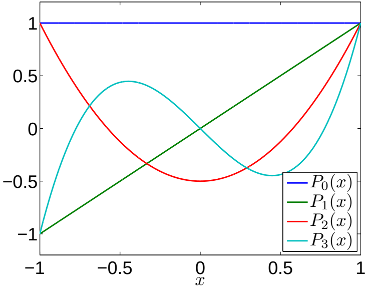


Figure: Legendre polynomials $P_n(x)$ when $n = 0, 1, 2, 3$.

Definition of Chromatic Derivatives

The Chromatic Derivatives associated with the Legendre polynomials

$$\mathcal{K}_t^n = (-j)^n P_n^L \left(j\mathcal{D}_t \right), \tag{6}$$

where

- subscript t means derivatives with respect to t.
- superscript n means the order of the polynomials.
- $P_n^L(\cdot) = \sqrt{2n+1} P_n(\cdot/\pi)$ are the normalized and scaled Legendre polynomials, which are orthonormal over $[-\pi, \pi]$.

	Operator	Coefficient
Derivatives	$\mathcal{D}_t^3$	$\left[\mathcal{D}_t^3 f(t)\right]\Big\|_{t=t_0}$.
Chromatic derivatives	$\frac{5}{2}\mathcal{D}_t^3 + \frac{3}{2}\mathcal{D}_t$	$\left[\frac{5}{2}\mathcal{D}_t^3 f(t) + \frac{3}{2}\mathcal{D}_t f(t)\right]\Big\|_{t=t_0}$

Table: Comparison of the 3rd derivative and the 3rd chromatic derivative.

The Analysis Equation of Chromatic Derivatives

- It is easy to verify that

$$\mathcal{K}_t^n e^{j\omega t} = j^n P_n^L(\omega)\, e^{j\omega t}. \tag{7}$$

- We can extend to the Fourier transformable signal $x(t)$ [1],

$$\mathcal{K}_t^n x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j^n P_n^L(\omega)\, X(\omega)\, e^{j\omega t}\, d\omega. \tag{8}$$

- If $x(t)$ is bandlimited in $[-\pi, \pi]$ and $t = 0$, (8) becomes

Analysis equation

$$a_n = \left[\mathcal{K}_t^n x(t)\right]\big|_{t=0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} j^n P_n^L(\omega)\, X(\omega)\, d\omega. \tag{9}$$

  - Filtering + Sampling.
  - Orthogonal polynomial expansion.

[1] $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$

Reconstruction from the Chromatic Derivatives

- Because the Legendre functions are orthogonal bases, from (9), the synthesis equation is

$$X\left(\omega\right) = \sum_{n=0}^{\infty} a_n(-j)^n P_n^L\left(\omega\right). \tag{10}$$

- Taking the inverse Fourier transform on ω yields

Synthesis equation

$$x(t) = \sum_{n=0}^{\infty} a_n\sqrt{2n+1} j_n\left(\pi t\right), \tag{11}$$

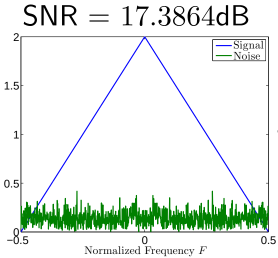
where $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$ is the spherical Bessel function.

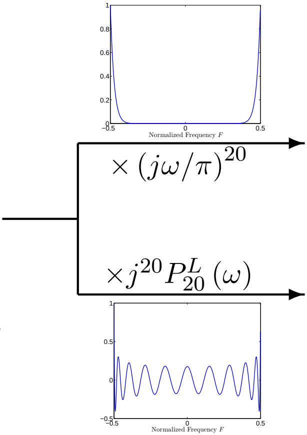
- Note that (11) resembles the "Whittaker-Shannon interpolation formula", which is

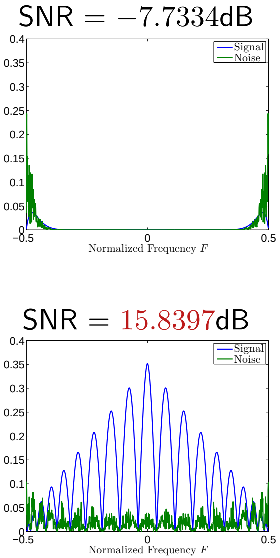
$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}\left(t-n\right). \tag{12}$$

Example: Noise Tolerance

Taylor: $\frac{1}{2\pi}\int_{-\pi}^{\pi}(j\omega/\pi)^n X(\omega)\, d\omega$

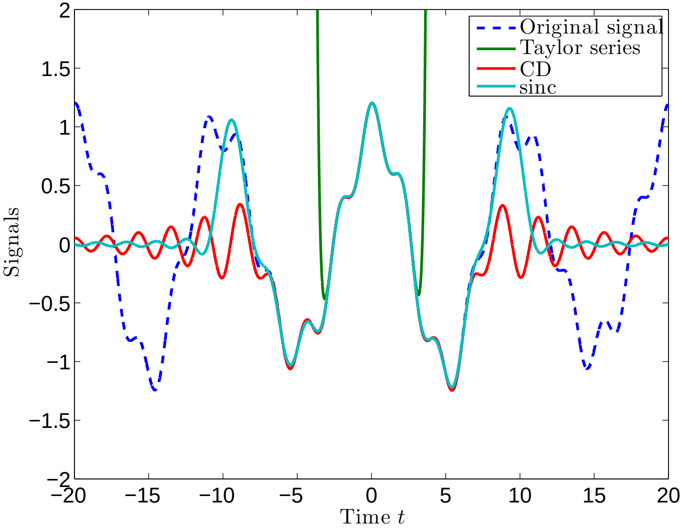






CD: $\frac{1}{2\pi}\int_{-\pi}^{\pi} j^n P_n^L(\omega)\, X(\omega)\, d\omega$

Example: Approximate Sinusoids

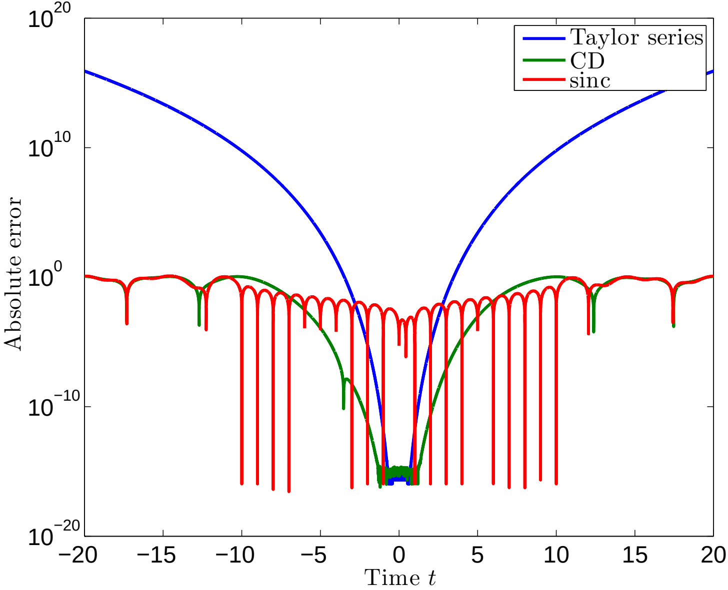


- Input signal

$$\cos\left(\frac{\pi t}{5}\right) + \frac{1}{5}\cos\left(\frac{9\pi t}{10}\right) + \frac{1}{10}\sin\left(\frac{3\pi t}{10}\right)$$

- Synthesis by 21 a_n's.
- Local approximation?
- Bounded?
- Error?

Example: Approximate Sinusoids (Cont'd)



Comparison among Taylor Series/Chromatic Derivatives/Sinc Interpolation

	Taylor series	Chromatic derivatives	sinc interpolation
Analysis	(4)	(9)	$x[n] = x(n)$
Synthesis	(3)	(11)	(12)
Noise Tolerance	Low	High	-
Local approx.	Yes	Yes	Yes
Bounded	No	Yes	Yes
Local approx. accuracy	Best	Good	Poor

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Chromatic Derivative Filter Banks [3]

- Is there a system diagram to implement chromatic derivatives? For example, the analysis equation mentioned before is

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} j^n P_n^L(\omega) X(\omega)\, d\omega.$$

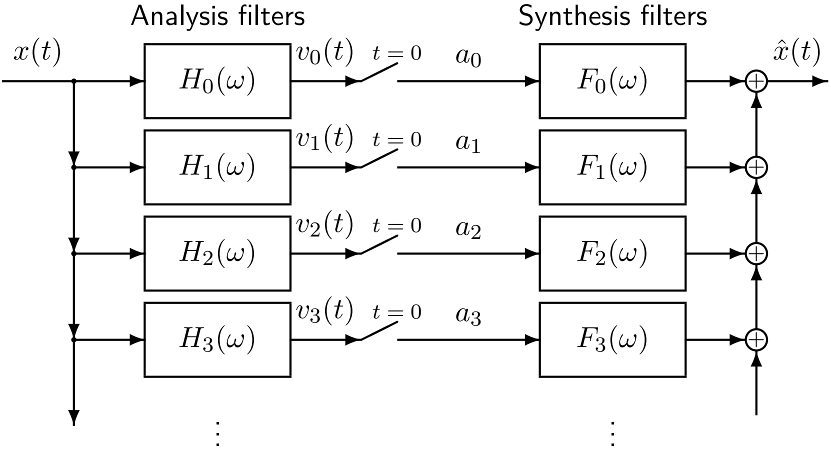
  we can let the analysis filters $H_n(\omega) = j^n P_n^L(\omega)$ and synthesis filters $F_m(\omega)$.

- What is the condition on perfect reconstruction? Biorthogonal condition

$$\int_{-\pi}^{\pi} H_n(\omega) F_m(\omega)\, d\omega = \delta_{n,m}. \tag{13}$$

- How do we design the filters in the system?
- Is there any constraint on the input signal?

Infinite-Channel Chromatic Derivative Filter Banks



$$\boxed{a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_n(\omega)\, X(\omega)\, d\omega, \qquad X(\omega) = \sum_{m=0}^{\infty} a_m F_m(\omega)}$$

Perfect Reconstruction

- Assume that $X(\omega) \in \mathrm{Span}\{F_m(\omega)\}_{m=0}^{\infty}$, i.e.

$$X(\omega) = \sum_{m=0}^{\infty} c_m F_m(\omega). \tag{14}$$

- The output of the nth analysis channel is

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) H_n(\omega)\, d\omega \\ &= \sum_{m=0}^{\infty} c_m \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} F_m(\omega) H_n(\omega)\, d\omega}_{\text{Biorthogonality}} = c_n \end{aligned} \tag{15}$$

- The reconstructed spectrum $\hat{X}(\omega)$ is

$$\hat{X}(\omega) = \sum_{n=0}^{\infty} a_n F_n(\omega) = X(\omega). \tag{16}$$

Filter Design in Chromatic Derivative Filter Banks

Biorthogonal equation

$$\int_{\text{desired band}} H_n(\omega)F_m(\omega)\,d\omega = \delta_{n,m}. \tag{17}$$

- We do not have to design $H_n(\omega)$ and $F_m(\omega)$ in (17) directly.
- Instead, we can choose pre-defined orthogonal polynomials such as
  - Legendre polynomials or Chebyshev polynomials for bandlimited signals.
  - Laguerre polynomials for analytic signals.
  - Hermite polynomials for nonbandlimited signals [5].
- Selecting

$$H_n(\omega) \propto P_n\left(\frac{\omega}{\omega_0}\right),$$

$$F_m(\omega) \propto P_m\left(\frac{\omega}{\omega_0}\right) W\left(\frac{\omega}{\omega_0}\right),$$

satisfies (17) directly.

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Conclusion

- In my presentation, The concept of chromatic derivatives were introduced.
- The features of Fourier-based, Taylor-based, chromatic derivatives and sinc interpolation were discussed.
    - The noise tolerance, the stability and the local approximation ability are very good when it is compared to the other two methods.
- The chromatic derivative filter banks are useful for implementation. The design steps are relatively simple.
- Lots of key properties and applications of chromatic derivatives are still under research.

References

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[3] M. J. Narasimha, A. Ignjatović, and P. P. Vaidyanathan, "Chromatic derivative filter banks," *IEEE Signal Processing Letters*, vol. 9, no. 7, pp. 215–216, 2002.

[4] G. B. Thomas, M. D. Weir, J. Hass, and F. R. Giordano, *Thomas' Calculus*, 11th ed. Addison Wesley.

[5] G. G. Walter and X. Shen, "A sampling expansion for nonbandlimited signals in chromatic derivatives," *IEEE Transactions on Signal Processing*, vol. 53, no. 4, pp. 1291–1298, 2005.