# Homework 3

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## 1 Exercise 3

### Part 1

I will prove that if $[x]_B$ and $[y]_B$ are not disjoint, then they must be identical.

Assume that they are not disjoint. Then there exists some $z \in [x]_B \cap [y]_B$. This means that $xBz$ and $yBz$. By symmetry, I have $xBz$ and $zBy$, which by transitivity implies $xBy$ (and $yBx$ by symmetry).

Now, let an arbitrary $w \in [x]_B$. This implies that $xBw$, and together with $yBx$, implies that $yBw$, which means $w \in [y]_B$.

Similarly, let an arbitrary $w \in [y]_B$. This implies that $yBw$, and together with $xBy$, implies that $xBw$, which means $w \in [x]_B$.

Thus, $[x]_B$ and $[y]_B$ are identical.

### Part 2

#### a.

I will prove that $B$ is reflexive, transitive, and symmetric.

Since $C$ is a preorder, I know that $xCx$ for all $x$, and thus $xBx$ for all $x$, which means $B$ is reflexive.

Let $x, y, z$ be arbitrary elements and suppose $xBy$ and $yBz$. This implies both $xCy \wedge yCx$ and $yCz \wedge zCy$. By transitivity of $C$, this implies $xCz \wedge zCx$, which means $xBz$ and thus $B$ is transitive.

Lastly, $B$ is symmetric by definition. If $xBy$ for arbitrary $x, y$, then $xCy \wedge yCx$, which also implies $yBx$.

#### b.

I will prove that $\preceq$ is a partial order, or equivalent, $\preceq$ is reflexive, transitive, and antisymmetric.

Since $C$ is reflexive, $xCx$ for all $x$, thus $[x]_B \preceq [x]_B$, which means $\preceq$ is reflexive.

Let $x, y, z$ be arbitrary elements and suppose $[x]_B \preceq [y]_B$ and $[y]_B \preceq [z]_B$. This implies $xCy$ and $yCz$, which implies $xCz$ by transitivity of $C$. This means $[x]_B \preceq [z]_B$ and thus, $\preceq$ is transitive.

Let $x, y$ be arbitrary elements and suppose $[x]_B \preceq [y]_B$ and $[y]_B \preceq [x]_B$. This implies $xCy$ and $yCx$, which implies $xBy$ by definition of $B$. This means $[x]_B = [y]_B$ and thus, $\preceq$ is antisymmetric.

# 2 Exercise 4

## 2.1 Part 1

($\Rightarrow$): Assume $B$ is total. Let $x, y$ be arbitrary elements in $S$. then either $xBy$ or $yBx$.

1. $xBy$: then $\langle x, y\rangle \in B$
2. $yBx$: then $\langle x, y\rangle \in B^{-1}$

This implies $S \times S \subseteq B \cup B^{-1}$. Since it's trivial that $B \cup B^{-1} \subseteq S \times S$, $S \times S = B \cup B^{-1}$.

($\Leftarrow$): Assume $S \times S = B \cup B^{-1}$. Let $x, y$ be arbitrary elements in $S$. By the assumption, either $\langle x, y\rangle$ is in $B$ or $B^{-1}$. This implies either $xBy$ or $yBx$, and thus $B$ is total.

## Part 2

($\Rightarrow$): Assume $B$ is a strict partial order, or equivalently, $B$ is the strict part of some partial order $C$.

1. By the strictness of $B$, $\neg(xBx)$ for all $x$, which implies that $B$ is irreflexive.
2. Let $x, y, z$ be arbitrary elements and suppose $xBy$ and $yBz$. This implies that $xCy \wedge x \neq y$ and $yCz \wedge y \neq z$. By the transitivity of $C$, $xCz$. I also know that $x \neq z$ by the following proof by contradiction:
   - Assume $x = z$. This implies that $xCy$ and $yCx$. By the antisymmetry of $C$, $x = y$, which contradicts the original constraint $x \neq y$.

   Thus, $xBz$ and $B$ is transitive.

($\Leftarrow$): Assume $B$ is irreflexive and transitive. I will prove that $B$ is equivalent to being the strict part of some partial order $C$.

Define $C = \{\langle x, y\rangle \mid xBy \vee x = y\}$. It is easy to see that $B$ is equivalent to the strict part of $C$. I will prove that $C$ is reflexive, transitive, and antisymmetric.

Firstly, it is by definition that $C$ is reflexive.

Let $x, y, z$ be arbitrary elements and suppose $xCy$ and $yCz$. This implies that $xBy \vee x = y$ and $yBz \vee y = z$. There are a two cases to consider.

1. Either $x = y$ or $y = z$ or both: this immediately implies $xCz$ by substitution.
2. $xBy$ and $yBz$: by transitivity of B, $xBz$, which implies $xCz$.

Let $x, y$ be arbitrary elements and suppose $xCy$ and $yCx$. This implies that $xBy \vee x = y$ and $yBx \vee y = x$. Since $B$ is irreflexive and transitive, $xBy$ and $yBx$ can not both be true. Thus, $x = y$ and $C$ is antisymmetric.

## Part 3

I will prove that $\preceq$ is reflexive, transitive, and antisymmetric. Note that this is exactly the same argument as the $(\Leftarrow)$ case of Part 2.

It is immediate that $\preceq$ is reflexive, since $x \preceq x \equiv x \prec x \vee x = x$ and the LHS is always true.

Let $x, y, z$ be arbitrary elements and suppose $x \preceq y$ and $y \preceq z$. This implies that $x \prec y \vee x = y$ and $y \prec z \vee y = z$. There are a two cases to consider.

1. Either $x = y$ or $y = z$ or both: this immediately implies $x \preceq z$ by substitution.

2. $x \prec y$ and $y \prec z$: by transitivity of $\prec$ (proved in Part 2), $x \prec z$, which implies $x \preceq z$.

Let $x, y$ be arbitrary elements and suppose $x \preceq y$ and $y \preceq x$. This implies that $x \prec y \vee x = y$ and $y \prec x \vee y = x$. Since $\prec$ is irreflexive and transitive (by Part 2), $x \prec y$ and $y \prec x$ can not both be true. Thus, $x = y$ and $\preceq$ is antisymmetric.

## Part 4

I will prove that $\prec$ is irreflexive and transitive (this is enough by Part 2).

Let $x$ be an arbitrary element. Then $x \prec x$ would be equivalent to $x \preceq x \wedge \neg(x \preceq x)$, which is always false. Thus, $\prec$ is irreflexive.

Let $x, y, z$ be arbitrary elements and suppose $x \prec y$ and $y \prec z$. This implies $x \preceq y \wedge \neg(y \preceq x)$ and $y \preceq z \wedge \neg(z \preceq y)$. By transitivity of $\preceq$, $x \preceq z$. It must also be true that $\neg(z \preceq x)$ since $z \preceq x$ would imply $z \preceq y$ which contradictions the original constraints.

## Part 5

$(\Rightarrow)$: Assume $B$ is a strict total order, or equivalently, $B$ is the strict part of some total order $C$. The proof for (a) and (b) is the same as that in Part 2. I will focus on proving (c).

Let $x, y$ be arbitrary elements. Since $C$ is total, either $xCy$ or $yCx$ or both. This implies either $xBy \wedge x \neq y$ or $yBx \wedge y \neq x$ or $x = y$ (by antisymmetry of $C$). Thus, (c) is true.

$(\Leftarrow)$: Assume $B$ satisfies (a), (b), and (c). Define $C = \{\langle x, y\rangle \mid xBy \vee x = y\}$. By Part 2 and Part 3, $B$ is a strict partial order, or equivalently, the strict part of partial order $C$. I will prove that $C$ is total.

Let $x, y$ be arbitrary elements. By the assumption, at least one of the following claims must be true.

1. $x = y$: then $xCy$ by definition.
2. $xBy$: then $xCy$ by definition.
3. $yBx$: then $yCx$ by definition.

Thus, $xCy \vee yCx$, which implies $C$ is total.

### Part 6

($\Rightarrow$): Assume $B$ is a well order. Then, by definition, $B$ is well-founded and, by Part 5, the other property holds.

($\Leftarrow$): Assume $B$ is well-founded and the other property holds. By Part 5, $B$ is a strict total order. Combining with well-foundedness, $B$ is a well order.

## 3 Exercise 5

### Part 1

($\Rightarrow$): Assume $\langle S, \prec \rangle$ is well-founded. Let $T$ be a non-empty subset of $S$. I will prove that there is a minimal element under $\prec$ in $T$ by contradiction.

Assume that there is no minimal element. This implies that for every $t \in T$, there exists $t' \in T$ where $t' \prec t$ and $t' \neq t$. Then, it is possible to construct an $\omega$-decreasing sequence $\langle t_0, t_1, \ldots \rangle$ in $\prec$ by starting at some element $t_0 \in T$ and, for every $t_i$ where $i > 0$, $t_i$ is such that $t_i \prec t_{i-1}$ and $t_i \neq t_{i-1}$. This contradicts the terminating property of $\prec$.

($\Leftarrow$): Assume that all non-empty subsets of $S$ have a minimal element under $\prec$. I will prove that $\prec$ is terminating by contradiction.

Assume there exists an $\omega$-descreasing sequence $\langle t_0, t_1, \ldots \rangle$. Let $T = \{t_i : i \in \omega\}$ be a non-empty subset of $S$. By the assumption, it has a minimal element $t_m$. But $t_{m+1} \prec t_m$ where $t_{m+1} \in T$, which leads to a contradiction.

### Part 2

($\Rightarrow$): Assume $\langle S, \prec \rangle$ is a woset. Let $T$ be a non-empty subset of $S$. By Part 1, $T$ has a minimal element $t_m$. Let $t$ be an arbitrary element in $T$. For the strict version of least element, it is only necessary to consider cases where $t \neq t_m$. Since $t_m$ is minimal, any arbitrary $t$ must satisfy $t_m \prec t$. Thus, $t_m$ is also the least element.

($\Leftarrow$): Assume all non-empty subsets of $S$ have a least element. Since a least element is also minimal, by Part 1, $\prec$ is well-founded. By Part 6 of Exercise 4, it is sufficient to show that $\langle \forall x, y \in S :: xBy \vee yBx \vee x = y \rangle$. Let $x, y$ be arbitrary elements and let $T = \{x, y\}$. By the assumption, $T$ has a least element, either $x$ or $y$. Thus, $xBy \vee yBx \vee x = y$.

# 4 Exercise 6

### Part 1

Since $U$ contains $\langle x, x\rangle$ for all $x$, $U$ is reflexive. Let $R_0, R_1, \ldots$ be the reflexive relations. Since each $R_i$ is reflexive, each contains $\langle x, x\rangle$ for all $x$. Thus, any arbitrary intersections $\bigcup_{j\in I} R_j$ where $I \subseteq \omega$ contains them too, which means $R_0, R_1, \ldots$ form a closure system.

### Part 2

This is immediate since $U$ is not irreflexive.

### Part 3

It is easy to see that $U$ is symmetric. Let $R_0, R_1, \ldots$ be the symmetric relations. Since each $R_i$ is symmetric, if $\langle x, y\rangle \in R_i$ then $\langle y, x\rangle \in R_i$ for all $x, y$. Let $Q = \bigcap_{j\in I} R_j$, where $I \subseteq \omega$, be an arbitrary intersection, and let $\langle x_Q, y_Q\rangle$ be an arbitrary pair in $Q$. It must be that $\langle x_Q, y_Q\rangle \in R_j$ for all $j$. By symmetry of every $R_j$, $\langle y_Q, x_Q\rangle$ must be in $Q$ as well, which makes $Q$ symmetric. Thus, $R_0, R_1, \ldots$ form a closure system.

### Part 4

This is immediate since $U$ is not asymmetric.

### Part 5

This is immediate since $U$ is not antisymmetric.

### Part 6

It is easy to see that $U$ is transitive. Let $R_0, R_1, \ldots$ be the transitive relations. Since each $R_i$ is transitive, if $\langle x, y\rangle \in R_i$ and $\langle y, z\rangle \in R_i$, then $\langle x, z\rangle \in R_i$, for all $x, y, z$. Let $Q = \bigcap_{i\in I} R_j$, where $I \subseteq \omega$, be an arbitrary intersection. If there exists $x_Q, y_Q, z_Q$ such that $\langle x_Q, y_Q\rangle \in Q$ and $\langle y_Q, z_Q\rangle \in Q$, then both of them must be in $R_j$ for all $j$. By transitivity of every $R_j$, $\langle x_Q, z_Q\rangle$ must be in $Q$ as well, which makes $Q$ transitive. Thus, $R_0, R_1, \ldots$ form a closure system.

# 5 Exercise 9

($\Rightarrow$): Assume that WFI holds. I will prove that $\prec$ is terminating by contradiction.

Assume that $\prec$ is not terminating and let $Q = \{a_i\}_{i\in\omega}$ be the $\omega$-decreasing sequence. Let $P.w \equiv w \notin Q$. It is easy to see that this makes the LHS of WFI false, since, for example, $P.a_0$ is false. However, for the RHS, consider two cases:

1. $w \in Q$: then $\langle \forall v \prec w : P.v \rangle$ is false, which means $\langle \forall v \prec w : P.v \rangle \Rightarrow P.w$ is true.

2. $w \notin Q$: then $P.w$ is true, thus making $\langle \forall v \prec w : P.v \rangle \Rightarrow P.w$ true.

This contradicts the LHS.

($\Leftarrow$) Assume that $\prec$ is terminating. I will prove both directions of the equivalent.

1. ($\Rightarrow$): it is easy to see that, if $P.w$ is true for all $w$, then the LHS must also be true.

2. ($\Leftarrow$): assume the LHS is true and, for the sake of contradiction, the RHS is false. Then there exists a non-empty subset $T$ of $W$ such that $T = \{w \mid \neg P.w\}$. By a similar argument to Part 1 of Exercise 5, there exists a minimal element $w_T$ in $T$. This implies that, for all $v \prec w_T$, $v \notin T$, which implies $P.v$ is true. By the LHS, $\langle \forall v \prec w_T : P.v \rangle \Rightarrow P.w_T$ must be true, which means $P.w_T$ is true and leads to a contradiction.

Thus, the WFI holds.

# 6 Exercise 10

($\Rightarrow$): Assume $B$ is well-founded. I will prove that its transitive closure $B^T$ is also well-founded by contradiction.

Assume that there exists an $\alpha$-decreasing sequence $a_0, a_1, \ldots$ in $B^T$ where $\alpha \subseteq \omega$. If every $\langle a_{i+1}, a_i \rangle$ is also in $B$, then the same sequence is an $\alpha$-decreasing sequence in $B$. If there is some $\langle a_{i+1}, a_i \rangle$ not in $B$, then it must exists a $b \in B$ such that $a_{i+1}Bb$ and $bBa_i$, else $B^T$ would not be the least transitive set. Then, an $\alpha'$-decreasing sequence can be constructed by adding $b$ in between $a_i$ and $a_{i+1}$. This leads to a contradiction with the well-foundedness of $B$.

($\Leftarrow$): Assume $B^T$ is well-founded. It is immediate that $B$ is well-founded, since if there is an $\alpha$-decreasing sequence in $B$ then that same sequence is in $B^T$.

# 7 Exercise 11

I will prove this by induction on $n$.

**Base case**: $\prec_1$ is well-founded by assumption.

**Induction step**: Assume that $\prec_n$ is well-founded for all $n \geq 1$. I will prove that $\prec_{n+1}$ is well-founded by contradiction.

Assume that $\prec_{n+1}$ is not well-founded. This implies there is an $\alpha$-decreasing sequence $a_0, a_1, \ldots$ in $\prec_{n+1}$, where $a_i = \langle x^i_{n+1}, x^i_n, \ldots, x^i_1 \rangle$. Since $\prec_1$ is well-founded by assumption, there exists an element $a_k$ in the sequence where for every $j \geq k$, $a^{j+1}_{n+1} = a^j_{n+1}$, which implies that $\langle a^{j+1}_n, \ldots, a^{j+1}_1 \rangle \prec_n \langle a^j_n, \ldots, a^j_1 \rangle$. This means that the sequence $a_k, a_{k+1}, \ldots$ is a decreasing sequence in $\prec_n$, thus contradicting the assumption.

## 8 Exercise 12

The following sequence is infinitely decreasing in the dictionary order.

$$b, ab, aab, aaab, \dots, a^i b, \dots$$

## 9 Exercise 13

Suppose there exists $f$ and $f'$ that both satisfy WFD. I will prove that they must be identical by strong induction on $x$.

**Base case**: for all minimal values $x$, $f.x = g(x, \emptyset)$ and $f'.x = g(x, \emptyset)$, which are identical.

**Induction step**: Assume that for for all $y \prec x$, $f.y$ and $f'.y$ are identical. I will prove that $f.x$ and $f'.x$ are identical. By definition, $f.x = g(x, \{\langle y, f.y\rangle : y \prec x\})$ and $f'.x = g(x, \{\langle y, f'.y\rangle : y \prec x\})$. Since by the assumption, $\{\langle y, f.y\rangle : y \prec x\} = \{\langle y, f'.y\rangle : y \prec x\}$, $f.x$ and $f'.x$ are identical.

## 10 Lemma 5

($\Rightarrow$): Assume $\alpha$ is an ordinal, or equivalently, $\langle \alpha, \prec\rangle$ is well-ordered for some $\prec$ and for all $\beta \in \alpha, \beta = s.\beta$.

By Definition 3, I will prove that $\alpha$ is transitive by proving that for all $\beta \in \alpha, \beta \subset \alpha$. This is immediate by Lemma 4.

Using Definition 3 and Corollary 1, it is also immediate that $\alpha$ is transitive.

($\Leftarrow$): Assume $\alpha$ is transitive and well-ordered by $\in$. Using the well-ordered property, transitivity, and Lemma 1, I know that for every $\beta \in \alpha$, $\beta = \{\gamma \mid \gamma \prec \beta\}$ since $\forall \gamma : \gamma \prec \beta$, $\gamma \in \beta$. This implies $\beta = s.\beta$. Combine this with the trichotomy property from the well-order property, by Corollary 1, $\alpha$ is an ordinal.

## 11 Lemma 8

Using Lemma 7, given two ordinals $\alpha$ and $\beta$, I can construct a new ordinal $\gamma$ that contains them both by setting $\gamma = \{\alpha, \beta\} \cup \alpha \cup \beta$. For the remainder of the proof, WLOG let's assume $\alpha \in \beta$.

To see why $\gamma$ is an ordinal, it is necessary to first see that the trichotomy property holds (or that it is well-ordered). Let $x, y$ be arbitrary elements in $\gamma$. Then, either one of them is $\beta$ or they are both in $\beta$. If they are both in $\beta$, the trichotomy property holds since $\beta$ is an ordinal. If, say $y = \beta$, then $x \in \beta$, which implies the trichotomy also holds.

Next, it is sufficient to show that $\gamma$ is transitive. Let $x$ be an arbitrary element in $\gamma$. Then, either $x = \beta$ or $x \in \beta$. If $x = \beta$, then by definition, $x \subset \gamma$. If $x \in \beta$, then by definition of $\beta$, $x \subset \beta \subset \gamma$.

## 12 Lemma 10

I will prove this by contradiction. Assume that there exists $x \in S$ such that $\langle S, A\rangle \cong \langle pred(x, A), res(A, x)\rangle$. Then, there is a bijection $f : S \to pred(x, A)$ such that for any $x, y \in S$, $xAy$ iff $(f.x)A'(f.y)$, where $A' = res(A, x)$.

Let $T = \{w \in S \mid f(w) \neq w\}$. Since $\langle S, A\rangle$ is well-ordered, there is a least element in $T$, call it $t$. Hence, for every $w \in S : w \prec t$, $w \notin T$ and $f(w) = w$. There are two cases to consider.

1. $t \prec f(t)$: Let $t'$ be such that $f(t') = t \Rightarrow f(t') \prec f(t) \Rightarrow t' \prec t$. But this implies $f(t') = t'$, hence a contradiction.

2. $f(t) \prec t$: This implies $f(t) \notin T \Rightarrow f(f(t)) = f(t)$. By injectivity, $f(t) = t$, hence a contradiction.

## 13 Lemma 15

I will prove the lemma by constructing an isomorphic ordinal for a given woset $\langle S, A\rangle$.

Since $\langle S, A\rangle$ is a woset, there is a least element in $S$, call it $s$. I will define the bijection $f$ as follows.

- $f(s) = \emptyset$, and
- for all $w \in S$, $f(w) = \{f(u) \mid uAw\}$

It is easy to see that $f(w)$ is an ordinal that is well-ordered by $\in$ for all $w \in S$. It is also easy to see that if $xAy$ then $f(x) \in f(y)$.

## 14 Exercise 14

$$
\begin{aligned}
&(\omega^{(\omega+1)^2})(\omega^{(\omega+\omega 12)(\omega^{\omega+1}+\omega^2 9)})\\
=&(\omega^{(\omega+1)(\omega+1)})(\omega^{w13(\omega^{\omega+1}+\omega^2 9)})\\
=&(\omega^{(\omega+1)\omega+\omega+1})(\omega^{\omega 13\omega^{\omega+1}+\omega 13\omega^2 9})\\
=&(\omega^{\omega^2+\omega+1})(\omega^{\omega^{\omega+1}+\omega^3 9})\\
=&\omega^{\omega^{\omega+1}+\omega^3 9}
\end{aligned}
$$