A Survey On Regularized Wasserstein Barycenter

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Overview

1. Methods to approximate regularized Wasserstein barycenters
   - Method 1: A generalization of the Sinkhorn algorithm (GSA)
   - Method 2: Continuous Regularized Wasserstein Barycenters (CWB)

2. Numerics
   - Case: 1D Gaussians
   - Case: 2D Gaussians

A Wasserstein Barycenter:

$$\underset{b \in \Sigma_n}{\operatorname{argmin}} \sum_{s=1}^{d} \lambda_s W_C^{\varepsilon}(a_s, b) \tag{1}$$

A generalization of the Sinkhorn algorithm:

$$u_s^{(l)} = \frac{a_s}{\mathrm{K} v_s^{(l-1)}}, \tag{2}$$

$$B^{(l)}(A, \lambda) = \prod_{s=1}^{d} \left(\mathrm{K}^T u_s^{(l)} \right)^{\lambda_s}, \tag{3}$$

$$v_s^{(l)} = \frac{B^{(l)}(A, \lambda)}{\mathrm{K}^T a_s^{(l)}}. \tag{4}$$

The regularized barycenter problem:

$$\sup_{\substack{\{f_i\}_{i=1}^n \subset C(\mathcal{X}) \\ \{g_i\}_{i=1}^n \subset C(\mathcal{Y})}} \mathop{\mathbb{E}}_{\substack{X_i \sim \alpha_i \\ Y \sim \beta}} \left[\sum_{i=1}^n \lambda_i \left(f_i(X_i) - R^\star \left(f_i(X_i) + g_i(Y) - \sum_{j=1}^n \lambda_j g_j(Y) - c(X_i, Y) \right) \right) \right] \tag{5}$$

Algorithm 1: Stochastic gradient descent to solve the regularized barycenter problem (5)

Initialize parameterizations $\{(f_{\theta_i}, g_{\phi_i}\}_{i=1}^n$;

for $l \leftarrow 1$ to n_{epochs} **do**

$\forall i \in \{1, \cdots, n\}$: sample $x^{(i)} \sim \alpha_i$; sample $y \sim \nu$;

$$\overline{g} \leftarrow \sum_{i=1}^n \lambda_i g_{\phi_i}(y);$$

$$F \leftarrow \sum_{i=1}^n \lambda_i \left(f_{\theta_i} x^{(i)} - R^\star \left(f_{\theta_i} x^{(i)} + g_{\phi_i}(y) - \overline{g} - c(x^{(i)}, y) \right) \right);$$

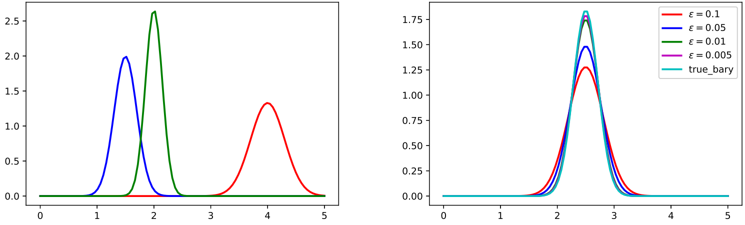
for $i = 1, \cdots, n$: $\texttt{ApplyGradient}(F, \theta_i)$; $\texttt{ApplyGradient}(F, \phi_i)$;

end

return dual potentials $\{(f_{\theta_i}, g_{\phi_i}\}_{i=1}^n$.

Numeric 1D Gaussians

We visualize the true barycenter and approximated solution by GSA method.

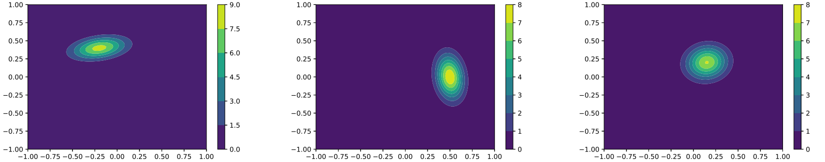


(a) Input distributions (b) The barycenter

Figure: Input Gaussian distributions and barycenter.

Numeric 2D Gaussians

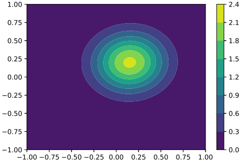
We visualize the true barycenter and approximated solution by GSA and CWB method with both entropic and quadratic regularization.



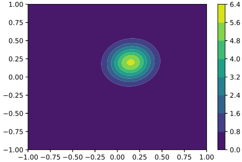
(a) Input 1 (b) Input 2 (c) Ground truth barycenter

Figure: Input Gaussian distributions and ground truth barycenter.

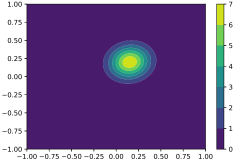
Numeric 2D Gaussians - GSA



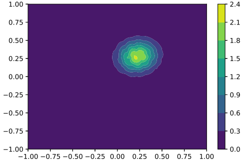
(a) Case $\varepsilon = 10^{-1}$



(b) Case $\varepsilon = 10^{-2}$



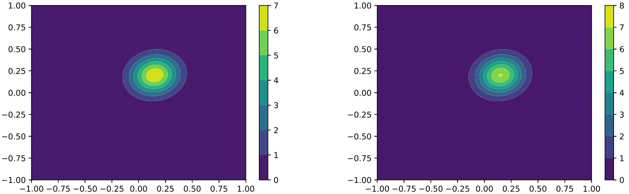
(c) Case $\varepsilon = 10^{-3}$



(d) Case $\varepsilon = 10^{-4}$

Figure: Compute barycenter by Sinkhorn with different values of epsilon.

Numeric 2D Gaussians - CWB



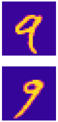
(a) Entropic regularization $\varepsilon = 10^{-3}$

(b) Quadratic regularization $\varepsilon = 10^{-4}$

Figure: Compute barycenter by CWB.

Numeric for images

We visualize the approximated barycenter by GSA and CWB method (using different methods to recover) with quadratic regularization.



(a) Source

(b) Intergrated

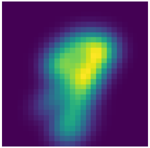
(c) MCMC

(d) Barycentric

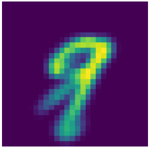
(e) Gradient map

(f) Neural network

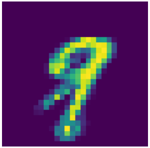
Figure: Several methods to recover barycenter



(a) $\varepsilon = 10^{-2}$



(b) $\varepsilon = 10^{-3}$



(c) $\varepsilon = 10^{-4}$

Figure: Sinkhorn algorithm for images

Conclusion

With 2D Gaussians case, we run the test case with both Sinkhorn and CWB method. We have the ground truth barycenter in Gaussian case and we can compare the approximated solution to the true barycenter. Moreover, we also have a comparison between two regulization, when quadratic regulization gives a better result compared to entropic regularization. Then we conclude that, in that case, the CWB method gives the better performance. With test case for images, virtually, we can see the approximated barycenter with numerical integration is clearest.

The End