Coupled Partial Differential Equations in Finance for European Pricing Options

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Research practice 1
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1 Introduction

In asset pricing theory, the representation of derivative prices can be formulated through partial differential equations (PDE's), specifically, parabolic PDE's. In that context, the mathematical finance problems related to volatility and jump components of underlying risk-neutral dynamics translate into option prices, where parabolic PDE's turn out to be effective models that better capture market movement. In fact, in the 70's, the familiar theoretical widespread valuation formula for options was established by Black & Scholes (1973), which later awarded them the 1997 Nobel Price in Economics. In this association, the scientific community has performed innovative work which includes a European option pricing liquidity shocks model by Koleva *et al.* (2017), methods for the coupled PDE's from regime-Switching option pricing by Ma & Zhou (2018), estimation of PDE's and its impact on finance by Kristensen (2008) and by Kristensen & Mele (2011), and others.

The PDE's described in finance have some particular characteristics, namely, they are posed on a bounded domain in time $(0, T)$, $T > 0$, with usually a singular final condition in $t = T$, and very often over unbounded spatial domains, which leads to imposing suitable limit conditions (boundary conditions) at infinity to get well-posed problems (in the sense of Hadamard) and therefore to get appropriate numerical schemes. Involving maximum principle, high order accuracy, and stability in space is a difficult task for numerical methods approximating degenerate parabolic equations (for instance, volatility or transition densities into the model). In the last decades, several finite difference schemes have been proposed to solve evolution problems. In summary, these schemes can be classified into three categories: the explicit, implicit, and implicit–explicit schemes for time discretization.

On the other hand, in the spatial dimension, the finite element method is typically used for discretizing the state variables and their operators over a fixed mesh, for example in the work of Ramirez (1997). There are many papers on numerical methods for option pricing, mostly for one-dimensional models of a single risk factor, using standard, second-order finite difference methods, see, e.g., Achdou & Pironneau (2005), and reference therein. More recently, high-order finite difference schemes that use a compact stencil, three points in space, were proposed by Tangman *et al.* (2008) and by Zhao *et al.* (2007) for linear problems.

A European–style claim offers the holder the right but not the obligation to either buy (Call) or sell (Put) a predetermined contingent payoff at maturity. For example, vanilla call options allow the holder to purchase the underlying security at the strike price if the exercise value of the underlying settles above it at expiration. As recent financial innovation is spanning the market, more exotic options are created and flourished, such as binary call options, whose exercise values are either some fixed amount of cash (cash–or–nothing) or the value of the underlying security itself (asset–or–nothing), if the option expires in the money.

In this study, we complement the literature on option pricing by providing a closed-form extension for European-type option prices, which can offer insight into a variety of models in asset pricing governed by coupled linear parabolic equations.

2 Statement of the problem

2.1 Preliminaries

The convection diffusion equation, or advection-diffusion equation, refers to a type of partial different equation that combines the fenomenas of advection or transport of a given substance and diffusion of random movement of a particle. It describes a certain flow or a stochastic movement.

$$\frac{\partial c}{\partial t} + \nabla \cdot (vc) + \nabla \cdot (D\nabla c) - R = 0 \tag{1}$$

where

- c is the variable of interest, the behavior being predicted.
- D is the diffusion coefficient.
- v is the velocity field, the speed in which the particle or the object is moving with.
- R is the source of c.
- ∇c is the concentration gradient.

If one analyses term by term,

- $\nabla \cdot (vc)$ describes convection or advection.
- $\nabla \cdot (D\nabla c)$ describes difussion.
- R is a destruction or creation of c, depending on the symbol.

2.2 Option pricing theory

An option is a type of financial derivative in the financial market. Option pricing theory describes a model which estimates the value of an *options contract*, an agreement to facilitate the exchange of an underlying security, a stock, at a certain *fair price*, by calculating the probability that it will be valued *in the money (ITM)* at a given expiration time. That model has to account for current market price, volatility, interest rate, strike price, and time to expiration (Ganti, 2021). The Black & Scholes (1973) model, shown in (2), is a famously recognized model, which is a differential equation that requires the five parameters mentioned previously. Other models include binomial option pricing and Monte Carlo simulation.

$$C = S_t N(d_1) - Ke^{-rt} N(d_2) \tag{2}$$

$$where:$$

$$d_1 = \frac{\ln \frac{S_t}{K} + \left(r + \frac{\sigma_v^2}{2}\right)t}{\sigma_s \sqrt{t}}$$

$$and$$

$$d_2 = d_1 - \sigma_s \sqrt{t}$$

where C is the call option price, S the current stock price, K the strike price, r the risk-free interest rate, and t the time to maturity. According to Ganti (2021), the longer that an investor has to exercise the option, the greater the likelihood that it will be ITM and therefore, longer-dated options are more profitable. The same can be said about volatility and interest rates. Volatility refers to the likelihood of changes in the market price at a given time, and models which use its value need to use an estimation of the implied volatility, for future market price changes, instead of historical volatility, realized volatility, or statistical volatility(past market price changes).

The Black-Scholes model uses a log-normal distribution to model stock prices since they cannot take negative values. Some assumptions that the model makes are there are no transaction costs, a constant risk-free interest rate, and constant volatility. In reality, some of its assumptions do not hold true, many modifications have been done to the model since its creation. An example of an incorrect assumption is the constant volatility, where it can be observed that it fluctuates with supply and demand, and modifications to the model account for volatility skewness or *volatility smile*.

Furthermore, the model assumes that the options are European style, or executable only at maturity, the set execution time. According to Chen (2022), execution refers to the right for the option holder to sell or buy the asset at the *fair price*, which is specified in the contract and can be determined for a self-financing portfolio, a portfolio where any changes to its assets are done without inflows or outflows of money, on the equation bellow (3):

$$P_t = \mathbb{E}\left(\exp\left(-\int_t^T r ds\right)\phi\left(S_T\right)\right) \tag{3}$$

where P_t is the fair price, $\phi(S_T)$ the payoff function at time t. Meanwhile, American style refers to options where holders may execute the options at any time before or at maturity. This allows the holder to gain profit as soon as it is deemed most favorable, and it makes American options more valuable. European options trade over the counter (OTC) while American options trade on standardized exchanges. Options can be either call options or put options. The former refers to when a holder has the right to buy the underlying security and are profitable when the stock price is well above the strike price. The latter refers to when a holder has the right to sell the underlying security and are profitable in the other case.

2.3 Formulation of the problem

Let

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}_t - rS_t \begin{pmatrix} \frac{\partial}{\partial S_t} & 0 \\ 0 & \frac{\partial}{\partial S_t} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} + \sigma^2 S_t^2 \begin{pmatrix} \frac{\partial^2}{\partial S_t^2} & 0 \\ 0 & \frac{\partial^2}{\partial S_t^2} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} + \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4}$$

be a coupled parabolic partial differential equation describing a self-financing portfolio with two risky assets at time t,

$$\boldsymbol{M} = \begin{pmatrix} \frac{\partial}{\partial S_t} & 0 \\ 0 & \frac{\partial}{\partial S_t} \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} \frac{\partial^2}{\partial S_t^2} & 0 \\ 0 & \frac{\partial^2}{\partial S_t^2} \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}, \quad \vec{P} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

where $\boldsymbol{M}$ is the transport matrix, $\boldsymbol{K}$ the diffusion operator matrix, $\boldsymbol{A}$ the coupled matrix, and $\vec{P}$ the pricing function vector for two assets. Then (4) can be rewritten as:

$$\vec{P}_t + rS_t\boldsymbol{M}\vec{P} + \sigma^2 S_t^2 \boldsymbol{K}\vec{P} - \boldsymbol{A}\vec{P} = \boldsymbol{0} \tag{5}$$

where $\vec{P}$ are the prices of the option as a function of stock price S and time$t$, r is the risk-free interest rate, and σ is the volatility of the stock.

If the value of the pricing function $\vec{P}$ satisfies (5), then P_t (3) is the value of a self-financing portfolio with value $\phi(S_T)$. For the problem to be well-posed, the system needs to have established boundaries when $S = 0$ or $S \to \infty$.
If one separates the terms into two sides, shown in (6):

$$\vec{P}_t + \sigma^2 S_t^2 \boldsymbol{K}\vec{P} = \boldsymbol{A}\vec{P} - rS_t\boldsymbol{M}\vec{P} \tag{6}$$

where the left side represents a *time decay* term, or the change of price in time, and the convexity adjustment value, which is a relationship between the price and the interest rate, and represents the cost of hedging. The right side is the riskless return from a long position in the derivative and a short position consisting of shares of the underlying. For both sides of the equation to balance, they both most be riskless, so the hedging term must account for the lost of the time decay.

3 Objectives

3.1 General objective

Extend the standard Black and Scholes model for European pricing options to coupled parabolic equations.

3.2 Specific objectives

- Review the existing literature related to partial differential equations in pricing options in finance.
- Make a theoretical model of coupled parabolic systems associated with pricing functions for two assets by using a stochastic-deterministic approach.
- Implement the coupled parabolic systems that describe the dynamic behavior of European pricing options throughout the finite element method.
- Illustrate numerical experiments that show the feasibility of our coupled model and its results.

4 Justification

European pricing options issues and their modeling using a single parabolic equation have been intensively studied in the last fifty years giving rise to interesting techniques, new challenges, and open problems. The literature is vast and it is difficult to mention all the intensive studies about this subject.

It is only in the last fifteen years that the challenging issue of modeling coupled parabolic systems has attracted the interest of the financial community. This kind of system appears mathematically, for instance, in optimal control theory as a characterization of the optimal control (with one equation coupled to its adjoint equation) but also appears, for example, in the study of chemical reactions performed by Érdi & Tóth (1989), and a wide variety of mathematical biology and physical situations. However, the study of scalar heat coupled equations and their applications in finance are in an early stage.

Taking into account the results given by Kristensen (2008) on asset pricing models by a finite number of state variables, and a variety of different estimators of the drift and volatility term, here we aim to generalize such results for a parabolic equation modeling the European option pricing problem to a coupled parabolic system formed by two equations with coupling term in cascade form. Additionally, subjects on stability and consistency are treated. In that context, new discrete schemes using the finite element method are developed as well. This aspect is a key point in our work since by having coupling terms for the asset dynamic, the variational formulation must be developed and implemented carefully.

5 Scope

This study is to focus on coupled linear parabolic equations as a way to model European option prices. It will extend previous work where a single PDE was used to model the same behavior. The study is limited to European options where the expiration date is fixed and represents less risk, and will not attempt to model others such as American options which have a higher risk.

This study will not delve deeply into stochastic processes or use stochastic methods due to a lack of expertise in the area. Finite elements may pose a certain level of difficulty to implement and were chosen due to their presence in the literature, but its accuracy for the problem stated is not known.

The main tool that will be used for this study will be FreeFEM, a programming language, and software used for solving partial differential equations using the finite elements method. The expected results are to create a model which accurately describes the behavior of European options, and to prove the stability and consistency of the said model.

6 State of the art

The estimation of partial differential equations (PDE's) has been studied to model dynamic systems such as asset pricing in finance, which uses the solution to the mathematical problem as the price allocated to the asset. A particular subdivision of PDE's, linear parabolic partial differential equations (LPPDE's), are commonly used for pricing financial derivatives, which began with the work of Black & Scholes (1973). It was introduced in their paper *The Pricing of Options and Corporate Liabilities* and later by Merton (1973) in *Theory of Rational Option Pricing* which expanded the applications of the proposed model by using compound Poisson processes to model jumps and started a large strand of option pricing literature. Björk (1998) then generalized the idea for financial derivatives under the assumption that the trading time is continuous. Heston (1993) then created the other main branch by using a mean-reverting square-root process to model stochastic volatility.

However, the Black-Scholes model implies that the volatility is constant and that there is a normal return distribution, which does not always correspond to observed behavior. Many studies have attempted to modify the said model to better explain the skewness to the left and heavier tails for the return distribution, which is referred to as asymmetric leptokurtic features, and a *smile* volatility. Chaos theory, fractal Brownian motion, and stable processes are used to incorporate the changes in the return distribution in asset pricing models by Mandelbrot (1963) and Rogers (1997). Barndorff-Nielsen & Shephard (2001) and Blattberg & Gonedes (1977) used generalized hyperbolic models, such as log t model and log hyperbolic model. For American options, these methods present trouble finding an analytical solution.

Regarding the volatility smile, Hull & White (1987), and Engle (1995) propose stochastic volatility and Auto-Regressive Conditional Heteroskedasticity (ARCH) models, while Lévy processes are used by Geman *et al.* (2001), which are continuous-time stochastic processes with stationary independent increments. Then, a double exponential jump-diffusion model for producing analytical solutions for option pricing is proposed by Kou (2002), which can account for both of the mentioned problems with the Black-Scholes model, and later by Kou & Wang (2004). Then, another inaccuracy appears with the Black-Scholes model, where it is assumed that returns and volatilities are independent, but in reality, they are correlated. Carr & Wu (2004) addresses it and the other two by using time-changed Lévy processes which unifies the two literature branches mentioned previously.

Kristensen (2008) further explores the relationship between diffusion models and LPPDE's and its application to asset-pricing theory with derivative prices as solutions to the PDE's. They demonstrate that derivative prices are consistent and apply them to three leading cases of preliminary estimators: Nonparametric, semiparametric, and fully parametric ones. They also obtain confidence bands and standard errors for implied prices of bonds, and other derivatives. Then, Kristensen & Mele (2011) use closed-form approximations for asset prices in continuous time with Taylor series expansions of conditional expectations.

Current work includes the research of Zhang *et al.* (2016) who use numerical approximate schemes to discretize the fractional Black-Scholes model for European double barrier options with second order accuracy for simulating three models based on Lèvy processes. Koleva *et al.* (2017) study parabolic-ordinary system modeling markets to liquidity shocks with space compact finite difference schemes. Ma & Zhou (2018) use Laplace transform methods to a discretized coupled PDE, by finite difference methods, for regime-switching option pricing. Kirkby *et al.* (2020) models option pricing with systems of stochastic diffusion processes on correlated assets using continuous time Markov Chain approximation techniques.

7 Proposed methodology

After checking the specialized literature for the pricing options by using partial differential equations, we propose to extend the Black and Scholes model to a model involving coupling terms in a parabolic–type equation. Usually, coupled parabolic systems are used for describing interaction among the state variables and their impact along a trajectory, either representing a deterministic or stochastic dynamic. In our context, for European pricing options, the coupled terms are showing the asset dynamic among and their relation.

In the next section, the main numerical methods for developing and implementing European pricing options using coupled parabolic PDE's are briefly described.

7.1 The finite differences method

The finite difference method (FDM) is an approximate method for solving partial differential equations by replacing derivatives with finite differences. It is used to discretize the PDE of interest with respect to S, the spatial dimension, by Taylor series expansions, and then t, the time dimension. It transforms a PDE into a system of linear equations that can be solved using matrix algebra techniques. These linear equations can be computed efficiently which, along with their simplicity, is why they were chosen as a method for the current study.

It is one of the oldest and simplest methods for approximating complex equations which model real-life behavior. Nonetheless, it is often that same simplicity that may cause a poor approximation and simply using smaller differences may not be an appropriate solution to the resulting error. It was first studied by Euler with the name *forward Euler* and was published in his book *Institutionum calculi integralis* and has been studied by many others since then. Then, the method was first applied to option pricing by Brennan *et al.* (1977).

7.2 Finite elements method

Finite elements method (FEM) is used to numerically obtain approximate solutions for PDE's by breaking the bounded spatial domain into simple geometric elements, usually triangles for two-dimensional problems and tetrahedra for three-dimensional problems, which make up a mesh (Ramirez, 1997). Then, simple functions are used to approximate the field in each element, and an error arises, which can be reduced by an increased number of elements in the mesh. However, more elements mean more computational cost, which often represents a disadvantage to the method. It was developed by Hrennikoff (1941) to solve certain structural problems that arouse in aerospace and civil engineering and further explored by Turner *et al.* (1956). Although it was early work, they both focused on mesh discretization of a continuous domain into a set of discrete sub-domains. Since then, FEM has been generalized for problems of different nature than the ones it was proposed for and can be used as a universal tool for solving differential equations numerically.

For the problem at hand, FEM was chosen because it is more flexible for solving PDE's than finite differences. It comes in handy when addressing more geometrically complex problems that may arise when barrier options are greater than one. Finally, FEM is naturally stable as demonstrated by Achdou *et al.* (2007), which is a current interest of the study.

7.2.1 Discretization

The combination of the finite differences methods and finite elements method to obtain a numeric scheme to approximate the value of the partial differential equation over time. Firstly, the triangulation $\mathcal{T}_h$ of Ω is based on the Delaunay-Voronoi algorithm. Then, the finite element space is a space of polynomial functions on elements. V_h is the space of continuous functions affine in x, y in each triangle in $\mathcal{T}_h$.

canonical basis, hat function ϕ_k

Then:

$$V_h\left(\mathcal{T}_h, P_1\right) = \left\{ w(x,y) \mid w(x,y) = \sum_{k=1}^{M} w_k \phi_k(x,y), w_k \text{ are real numbers} \right\} \tag{7}$$

no me queda muy claro esto

$$u(x,y) \simeq u_h(x,y) = \sum_{k=0}^{M-1} u_k \phi_k(x,y) \tag{8}$$

$$-\int_\Omega v\Delta u \ \mathrm{d}x \ \mathrm{d}y = \int_\Omega vf \ \mathrm{d}x \ \mathrm{d}y$$

Green's formula

$$\int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x \ \mathrm{d}y - \int_\Omega fv \ \mathrm{d}x \ \mathrm{d}y = 0 \quad \forall v \text{ satisfying } v = 0 \text{ on } \partial\Omega$$

formulacion variacional. variational formulation.

$$\begin{array}{rcl} u_t - k(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}) & = & f(x,y,t), \quad \text{for all } (x,y) \text{ in } \Omega \in \mathbb{R} \\ u(x,y,t) & = & 0, \quad \text{for all } (x,y) \text{ in } \partial\Omega \in \mathbb{R} \\ u(x,y,t=0) & = & u_0, \quad \text{for all } (x,y) \text{ in } \partial\Omega \in \mathbb{R} \end{array} \tag{9}$$

Its multiplied by a test function uu and integrated by parts over Ω:

$$\int_0^T \int_\Omega u_t \cdot uu \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t - k\int_0^T \int_\Omega (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}) uu \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t = 0 \tag{10}$$

Then, the Green identity is applied:

$$\int_0^T \int_\Omega u_t \cdot uu \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t + k\int_0^T \int_\Omega (\frac{\partial u}{\partial x}\frac{\partial uu}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial uu}{\partial y}) - k\int_0^T \int_{\partial\Omega} \nabla n \cdot uu = 0 \tag{11}$$

$$\int_0^T \int_\Omega u_t \cdot uu \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t + k\int_0^T \int_\Omega (\frac{\partial u}{\partial x}\frac{\partial uu}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial uu}{\partial y}) - k\int_0^T \int_{\partial\Omega} \alpha(u - uold) \cdot uu = 0 \tag{12}$$

$$\begin{aligned} &\int\int_{\mathcal{T}_h} (u \cdot uu + v \cdot vv) \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t + \Delta t\sigma \int\int_{\mathcal{T}_h} (x^2+y^2)(\frac{\partial u}{\partial x}\frac{\partial uu}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial uu}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial vv}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial vv}{\partial y}) \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t \\ &- \int\int_{\mathcal{T}_h} (u_t \cdot uu + v_t \cdot vv) \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t - \Delta t \int\int_{\mathcal{T}_h} (f_1 \cdot uu + f_2 \cdot vv) \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t \\ &+ \Delta t \int\int_{\mathcal{T}_h} (\alpha \cdot uu \cdot v + \beta \cdot vv \cdot u) \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t + \int\int_{\mathcal{T}_h} (x+y)(\alpha \cdot uu(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}) + \beta \cdot vv(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y})) \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}t \\ &= 0 \end{aligned} \tag{13}$$

with

$$\begin{array}{rcl} uv & = & x(1-x)(1-y) \\ aux1 & = & 2\sigma^2(x^2+y^2)(x(1-x) + y(1-y)) \\ aux2 & = & (1-2x)y(1-y) + (1-2y)x(1-x) \\ f_1 & = & e^(-2t)(-2uv - aux1\alpha aux2) + \alpha uv * t^(-2) \\ f_2 & = & t^(-2)(-2uv - aux1\beta aux2) + \beta uve^(-2t) \end{array} \tag{14}$$

8 Schedule

Table 1: Schedule

Activity	*Weeks*																	
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*
Literature review	■	■	■	■	■													
Theoretical and numerical studies						■	■	■										
Simulation design									■	■	■	■						
Simulation testing													■	■	■			
Article writing																■	■	■

9 Results

Coupling terms	Δt	Total $L^2 error$	u $L^2 error$	v $L^2 error$
$\alpha(x,y) = x, \quad \beta(x,y) = y$	10^-1	$\mathbb{P}1 = 0.061441$	$\mathbb{P}1 = 0.041452$	$\mathbb{P}1 = 0.045352$
		$\mathbb{P}2 = 0.049405$	$\mathbb{P}2 = 0.022564$	$\mathbb{P}2 = 0.043952$
$\alpha(x,y) = x^2, \quad \beta(x,y) = y$		$\mathbb{P}1 = 0.061441$	$\mathbb{P}1 = 0.041452$	$\mathbb{P}1 = 0.045352$
		$\mathbb{P}2 = 0.049405$	$\mathbb{P}2 = 0.022564$	$\mathbb{P}2 = 0.043952$
$\alpha(x,y) = sin(x), \quad \beta(x,y) = cos(x)$		$\mathbb{P}1 = 2.180150\times10^8$	$\mathbb{P}1 = 2.179170\times10^8$	$\mathbb{P}1 = 6.530140 \times10^6$
		$\mathbb{P}2 = 6.517610\times10^8$	$\mathbb{P}2 = 6.514340\times10^8$	$\mathbb{P}2 = 2.063680 \times10^7$

Table 2: Approximation using Lagrange-type polynomials $\mathbb{P}1$, $\mathbb{P}2$ for a self-financing portfolio with two risky assets using different coupling terms. The temporal dimension is $[0, T = 1]$.

No tenemos buenos resultados con funciones sinusoidadles lo cual muestra que el modelo es sensible a los parámetros de acoplamiento y por tnanto una adaptacion del modelo deberia ser considerada al fuutu¡om, teniendo en cuenta kos parametros de acoplamiento, la condicion inicial y los parametros de conductividad termica.(lo que acompaña a las segundas derivadas).

Coupling terms	Δt	Total $L^2 error$	u $L^2 error$	v $L^2 error$
$\alpha(x,y) = x, \quad \beta(x,y) = y$	10^-3	$\mathbb{P}1 = 0.055629$	$\mathbb{P}1 = 0.043626$	$\mathbb{P}1 = 0.034515$
		$\mathbb{P}2 = 0.068180$	$\mathbb{P}2 = 0.035592$	$\mathbb{P}2 = 0.058153$
$\alpha(x,y) = x^2, \quad \beta(x,y) = y$		$\mathbb{P}1 = 0.055629$	$\mathbb{P}1 = 0.043626$	$\mathbb{P}1 = 0.034515$
		$\mathbb{P}2 = 0.068180$	$\mathbb{P}2 = 0.035592$	$\mathbb{P}2 = 0.058153$
$\alpha(x,y) = sin(x), \quad \beta(x,y) = cos(x)$		$\mathbb{P}1 = \infty$	$\mathbb{P}1 = \infty$	$\mathbb{P}1 = \infty$
		$\mathbb{P}2 = \infty$	$\mathbb{P}2 = \infty$	$\mathbb{P}2 = \infty$

Table 3: Approximation using Lagrange-type polynomials $\mathbb{P}1$, $\mathbb{P}2$ for a self-financing portfolio with two risky assets using different coupling terms. The temporal dimension is $[0, T = 1]$.

intro: definir el problema de opciones. usando edp. metodokogia: diferencias finitas y elementos finitos. discretizacion del modelo usando ambas. implementacion del modelo discreto en freefem++. lenguaje de programacion que permite resolver edps en 2d y 3d. Resultados: 1. tabla quitando error total, un plot de los 3. 2. bajo otra configuracion (tiempos). 3. escrito: interpretacion al contexto opciones (hay resultados bueno sy otros no tan buenos segun el acoplamiento). 4. conclusiones y trabajo futuro: implementamos un modelo de finanzas de edps acoplado que descrube opciones de precios en finanzas. Los acoplamientos polinomiales nos permiten considerar una familia de pares de opciones de precio cuyo orden de convergencia es de 10-2. Calibrar o reformular los oarametros del modelo con el porposito de obtener o extender la interaccion entre pares de opciones de precios en finanzas. Usar datos reales para comparar los resultados obtenidos con el modelo planteado.

10 Intellectual property

According to the internal regulation on intellectual property within Universidad EAFIT, the results of this research practice are product of *Luisa Toro Villegas* and *Cristhian David Montoya Zambrano*.

In case further products, beside academic articles, that could be generated from this work, the intellectual property distribution related to them will be directed under the current regulation of this matter determined by Universidad EAFIT (2017).

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