Question 1. GRAPH ALGORITHMS. (35%) Let $G = (V, E)$ be a connected undirected graph in which edge (u, v) has positive weight $w[u, v]$.

(i) (30%) Describe the Prim-Dijkstra algorithm for finding the minimal spanning tree.
(ii) (3%) What is the simplest data structure you can think of if you want worst-case complexity $O(|V|^2)$?
(iii) (2%) Which data structure would you recommend if all edge weights are 1, 2 or 3?

Question 2. DYNAMIC PROGRAMMING. (15%) (5%) State the all pairs shortest path problem. (10%) Give the dynamic programming solution commonly attributed to Floyd and Warshall.

Question 3. DEFINITIONS. (14%) Define the following terms.

(i) Cartesian tree.
(ii) Kraft's inequality.
(iii) The knapsack problem.
(iv) Shannon's lower bound for prefix coding.
(v) Suffix array.
(vi) The future event set.
(vii) Strongly connected component.

Question 4. COMPLEXITY. (14%) Give worst-case complexities in Θ notation for the following algorithms (no explanations needed):

(i) An algorithm with time complexity described by $T_n = T_{\lfloor n/3 \rfloor} + T_{n-\lfloor n/3 \rfloor} + 1$ for $n > 3$, and $T_n = 0$ for $n \leq 3$.
(ii) An algorithm with time complexity described by $T_n = T_{\lfloor n/3 \rfloor} + T_{n-\lfloor n/3 \rfloor} + n$ for $n > 3$, and $T_n = 0$ for $n \leq 3$.
(iii) Mergesort of n items [comparison oracle model].
(iv) The best algorithm for creating a red-black tree given a sorted list of n keys.
(v) The standard dynamic programming algorithm for finding the longest common subsequence of two books that each have n symbols.
(vi) The KMP (Knuth-Morris-Pratt) algorithm for finding a pattern of length m in a text string of length n.
(vii) Strassen's method for multiplying two $n \times n$ matrices [RAM model].

Question 5. INDUCTION PROOFS. (10%) Prove by induction that a node of rank r in a red-black tree has at least 2^r external nodes in its subtree.

Question 6. HUFFMAN CODE. (10%) The Huffman code is the optimal prefix code. Design a binary Huffman code for the symbols 1 through 9, where symbol i has probability $i/45$ of occurring in the input. (No explanations needed—just show a code.)

Question 7. FAST FOURIER TRANSFORM. (10%) Consider the FFT as an oracle. As input, the oracle accepts two polynomials (each given as vectors of integer-valued coefficients). Its output is a vector of (again, integer-valued) coefficients of the product polynomial. We can use the oracle once at zero cost. Everything else, including preparing the input for the oracle, and processing its output, is calculated in bit model complexity. Explain how you can now multiply two n-bit integers in bit model time $O(n \log n)$.

Question 8. RANDOM BINARY SEARCH TREE. (10%) Recall that in a random binary search tree, we first randomly shuffle the n keys, and then construct it by standard insertion of the keys into an initially empty tree. What is the probability that the 10-th smallest element is an ancestor of the 100-th smallest element when $n = 1000$?