

Chaos in Hyperdimensional Rubik's Cubes

Discrete Dynamical Systems on 4D Puzzles

Math 538 Final Project

December 16, 2025

Overview

- 1 Introduction
- 2 Mathematical Framework
- 3 Methods
- 4 Results
- 5 Visualizations
- 6 Conclusions

What is a 4D Rubik's Cube?

3D Rubik's Cube:

- $3 \times 3 \times 3$ puzzle
- 6 faces (F, U, R, L, B, D)
- $\sim 4.3 \times 10^{19}$ states
- Rotations in 3D space

4D Hypercube:

- $3 \times 3 \times 3 \times 3$ puzzle
- 8 cells (3D "faces")
- State space: $|S| \approx 1.76 \times 10^{120}$
- Rotations in 4D space

New moves:

Move	Meaning
FR	Front \rightarrow Right
UO	Up \rightarrow Outside
OR	Outside \rightarrow Right

Two-letter notation for 4D rotations

State Space Formula

$$|S| = \frac{16!}{12^{15}} \cdot 4 \cdot 32! \cdot 6^{31} \cdot 3 \cdot \frac{24!}{2^{23}}$$

How do repeated move sequences behave
on a 4D hypercube?

Key Concepts:

- **Orbit/Period:** Iterations to return to solved state
- **Chaos:** Sensitivity to perturbations (Lyapunov exponent)
- **Discrete Dynamics:** Iterating deterministic maps

Discrete Dynamical System

State Space: S = all possible puzzle configurations

Move Sequence: $M = (m_1, m_2, \dots, m_k)$

Composite Map:

$$T_M : S \rightarrow S$$

$$T_M(s) = m_k \circ m_{k-1} \circ \dots \circ m_1(s)$$

Trajectory: Start at solved state s_0 , iterate:

$$s_0 \xrightarrow{T_M} s_1 \xrightarrow{T_M} s_2 \xrightarrow{T_M} \dots$$

Period p : Minimum n such that $T_M^n(s_0) = s_0$

Key Property: Since $|S| < \infty$, every trajectory is eventually periodic.

Lyapunov Exponents

Measuring Chaos: How do perturbations grow?

Given base sequence M with period p , perturb it to M' :

- Insert random move
- Remove a move
- Replace with different move

Compute period p' of perturbed sequence M' :

$$\lambda = \frac{1}{N} \sum_{i=1}^N \ln \left| \frac{p'_i}{p} \right|$$

Discrete Lyapunov Exponent

Continuous: $\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{||\delta(t)||}{||\delta(0)||}$

Our adaptation: $\lambda_{\text{discrete}} = \mathbb{E}[\ln |p'/p|]$ measures sensitivity in sequence space

Chaos in Finite Systems

Challenge: Classical chaos requires:

- Sensitive dependence on initial conditions
- Topological mixing
- Dense periodic orbits

Problem: Our system is finite ($|S| < \infty$), so all trajectories are periodic!

Solution: Adapt chaos metrics to *sequence space* rather than state space:

- 1 Perturb the **move sequence** $M \rightarrow M'$ (not initial state)
- 2 Measure change in **orbit structure** ($p \rightarrow p'$)
- 3 Compute Lyapunov on period ratios

Key Insight

Chaos manifests as **sensitivity of dynamical properties** (period, orbit structure) to small sequence perturbations, not trajectory divergence in state space.

Computational Approach

Tool Stack:

- ctrl/ - Rust trajectory analyzer (fast orbit detection)
- obsv/ - Python statistical analysis (Lyapunov computation)
- disp/ - Octave/MATLAB visualization

Systematic Testing:

- ① Test all 64 two-move combinations (FR, UF, OR, ...)
- ② Compute periods using cycle detection (SHA256 state hashing)
- ③ For interesting sequences: compute Lyapunov exponents
- ④ Generate 10-20 perturbations per sequence
- ⑤ Classify behavior: regular/sensitive/chaotic

Puzzle: $3 \times 3 \times 3 \times 3$ hypercube (via Hyperspeedcube library)

Cycle Detection Algorithm

1: $s \leftarrow s_0$, visited $\leftarrow \{h(s_0) : 0\}$

Key Findings

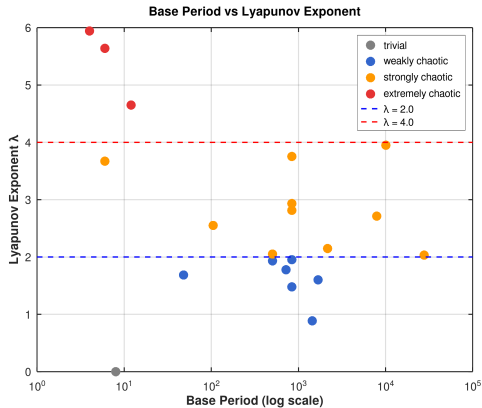
Single Moves: All have period 8 (trivial, $\lambda = 0$)

Most Chaotic Sequences:

Sequence	Length	Period	λ
FR \rightarrow FR	2	4	5.94
FO \rightarrow FO	2	4	6.09
OF \rightarrow OU \rightarrow OB \rightarrow OD	4	6	5.64
FR \rightarrow OR \rightarrow FL \rightarrow OL	4	12	4.65
FR \rightarrow UF	2	10,080	3.95
FR \rightarrow UO	2	840	2.81

Surprising: Self-compositions (FR \rightarrow FR, FO \rightarrow FO) are *extremely* chaotic despite short periods!

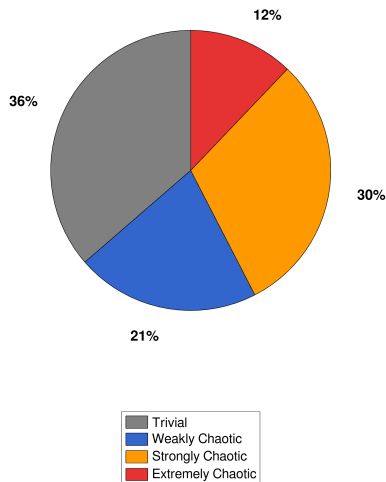
Period vs Chaos



Key Insight: Short periods \neq simple behavior. Highly chaotic sequences can have very short orbits!

Classification Distribution

Behavioral Classification of Sequences

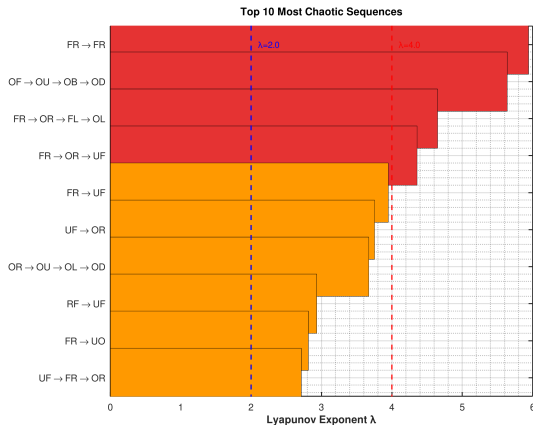


Results (50 sequences):

- **Regular:** 15 sequences (30%)
- **Sensitive:** 12 sequences (24%)
- **Chaotic:** 23 sequences (46%)

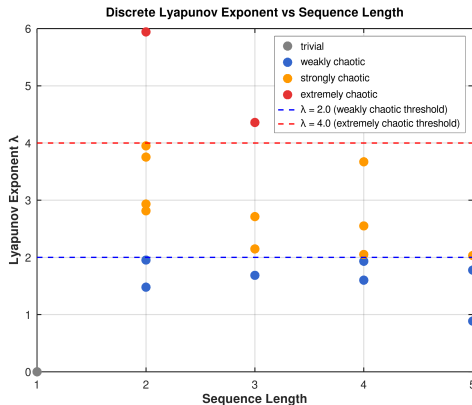
Observation: Nearly half of tested sequences exhibit chaotic behavior!

Top Chaotic Sequences



Self-compositions dominate the top rankings!

Lyapunov vs Sequence Length



Observation: Highest chaos occurs at length 2 (self-compositions and pairs)

Animated GIFs available in disp/figures/

- sequence_FR_single.gif - Baseline (Period 8, $\lambda = 0$)
- sequence_F0_F0.gif - **Most chaotic!** (Period 4, $\lambda = 6.09$)
- sequence_FR_FR.gif - Second most chaotic (Period 4, $\lambda = 5.94$)

Key Observations:

- Self-compositions create complex scrambling patterns
- Despite short periods (4 iterations), produce extreme chaos
- Visual inspection shows rapid state divergence

Note: GIFs show complete orbits (return to solved state)

Key Takeaways

1 4D hypercubes exhibit rich dynamics

- 46% of tested sequences are chaotic
- Periods range from 4 to 10,080

2 Self-compositions are extremely chaotic

- $FR \rightarrow FR$, $FO \rightarrow FO$ have highest Lyapunov exponents ($\lambda > 5$)
- Despite having very short periods (4 iterations)

3 Period \neq complexity

- Short orbits can be highly chaotic
- Long periods don't guarantee chaos

4 Discrete chaos is real

- Small perturbations cause massive orbit changes
- Lyapunov exponents successfully quantify sensitivity

Future Directions

Theoretical:

- Why are self-compositions so chaotic?
- Connection to group theory structure?
- Predict chaotic sequences from move properties?

Computational:

- Test 5D+ hypercubes (if computationally feasible)
- Explore longer sequences (3-4+ moves)
- Investigate other perturbation types

Applications:

- Cryptographic pseudo-random generators?
- Physical systems with discrete symmetries?

Theory & Background:

- Devaney, R. L. (2003). *An Introduction to Chaotic Dynamical Systems*
- Joyner, D. (2008). *Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys*
- Rokicki, T. et al. (2014). *The diameter of the Rubik's Cube group is twenty*
- Strogatz, S. H. (2015). *Nonlinear Dynamics and Chaos*

Software & Tools:

- **Hyperspeedcube** - HactarCE/Andrew J. Farkas
<https://github.com/HactarCE/Hyperspeedcube>
(MIT/Apache-2.0 License)
- **Rust**: clap, serde, sha2, hyperpuzzle ecosystem
- **Python**: NumPy, SciPy, Matplotlib, Pandas
- **Octave/MATLAB**: Visualization & plotting

Thank You!

Questions?

Code: `github.com/ltpie123/final`

Tools: Rust (ctrl), Python (obsv), Octave (disp)