# Chaos in Hyperdimensional Rubik's Cubes

## Discrete Dynamical Systems on 4D Puzzles

Math 538 Final Project

December 16, 2025

# Overview

# What is a 4D Rubik's Cube?

**3D Rubik's Cube:**

- 3×3×3 puzzle
- 6 faces (F, U, R, L, B, D)
- $\sim 4.3 \times 10^{19}$ states

**4D Hypercube:**

- 3×3×3×3 puzzle
- 8 cells (3D "faces")
- State space: $|S| \approx 1.76 \times 10^{120}$

**New moves:**

| Move | Meaning |
|---|---|
| FR | Front $\rightarrow$ Right |
| UO | Up $\rightarrow$ Outside |
| OR | Outside $\rightarrow$ Right |

*Two-letter notation for 4D rotations*

## State Space Formula

$$|S| = \frac{16!}{2} \cdot 12^{15} \cdot 4 \cdot 32! \cdot 6^{31} \cdot 3 \cdot \frac{24!}{2} \cdot 2^{23}$$

# Research Question

How do repeated move sequences behave on a 4D hypercube?

**Key Concepts:**

- **Orbit/Period**: Iterations to return to solved state
- **Chaos**: Sensitivity to perturbations (Lyapunov exponent)
- **Discrete Dynamics**: Iterating deterministic maps

# Discrete Dynamical System

**State Space:** $S =$ all possible puzzle configurations

**Move Sequence:** $M = (m_1, m_2, \ldots, m_k)$

**Composite Map:**

$$T_M : S \to S$$

$$T_M(s) = m_k \circ m_{k-1} \circ \cdots \circ m_1(s)$$

**Trajectory:** Start at solved state $s_0$, iterate:

$$s_0 \xrightarrow{T_M} s_1 \xrightarrow{T_M} s_2 \xrightarrow{T_M} \cdots$$

**Period $p$:** Minimum $n$ such that $T_M^n(s_0) = s_0$

**Key Property:** Since $|S| < \infty$, every trajectory is eventually periodic.

# Lyapunov Exponents

**Measuring Chaos:** How do perturbations grow?

Given base sequence $M$ with period $p$, perturb it to $M'$:

- Insert random move
- Remove a move
- Replace with different move

Compute period $p'$ of perturbed sequence $M'$:

$$\lambda = \frac{1}{N}\sum_{i=1}^{N} \ln\left|\frac{p'_i}{p}\right|$$

## Discrete Lyapunov Exponent

Continuous: $\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{||\delta(t)||}{||\delta(0)||}$
Our adaptation: $\lambda_{\text{discrete}} = \mathbb{E}[\ln|p'/p|]$ measures sensitivity in sequence space

# Chaos in Finite Systems

**Challenge:** Classical chaos requires:

- Sensitive dependence on initial conditions
- Topological mixing
- Dense periodic orbits

**Problem:** Our system is finite ($|S| < \infty$), so all trajectories are periodic!

**Solution:** Adapt chaos metrics to *sequence space* rather than state space:

1. Perturb the **move sequence** $M \to M'$ (not initial state)
2. Measure change in **orbit structure** ($p \to p'$)
3. Compute Lyapunov on period ratios

## Key Insight

Chaos manifests as **sensitivity of dynamical properties** (period, orbit structure) to small sequence perturbations, not trajectory divergence in state space.

# Computational Approach

**Tool Stack:**

- `ctrl/` - Rust trajectory analyzer
- `obsv/` - Python statistical analysis
- `disp/` - Octave/MATLAB visualization

**Systematic Testing:**

1. Test all 64 two-move combinations
2. Compute periods using cycle detection
3. Compute Lyapunov exponents (10-20 perturbations)
4. Classify: regular/sensitive/chaotic

**Puzzle:** 3×3×3×3 hypercube (Hyperspeedcube library)

## Cycle Detection Algorithm

$s \leftarrow s_0$, visited $\leftarrow \{h(s_0) : 0\}$
**for** $n = 1, 2, \ldots, N_{\max}$ **do**
    $s \leftarrow T_M(s)$, $h \leftarrow \mathrm{SHA256}(s)$
    **if** $h \in$ visited **then return** $n -$ visited$[h]$
    **end if**
    visited$[h] \leftarrow n$
**end for**

**Complexity:** $O(p \cdot (T_M + H))$

# Key Findings

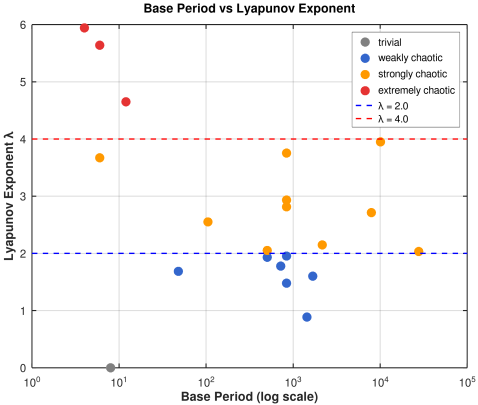
**Single Moves:** All have period 8 (trivial, $\lambda = 0$)

**Most Chaotic Sequences:**

| Sequence | Length | Period | $\lambda$ |
|---|---|---|---|
| FR $\rightarrow$ FR | 2 | 4 | 5.94 |
| FO $\rightarrow$ FO | 2 | 4 | **6.09** |
| OF $\rightarrow$ OU $\rightarrow$ OB $\rightarrow$ OD | 4 | 6 | 5.64 |
| FR $\rightarrow$ OR $\rightarrow$ FL $\rightarrow$ OL | 4 | 12 | 4.65 |
| FR $\rightarrow$ UF | 2 | 10,080 | 3.95 |
| FR $\rightarrow$ UO | 2 | 840 | 2.81 |

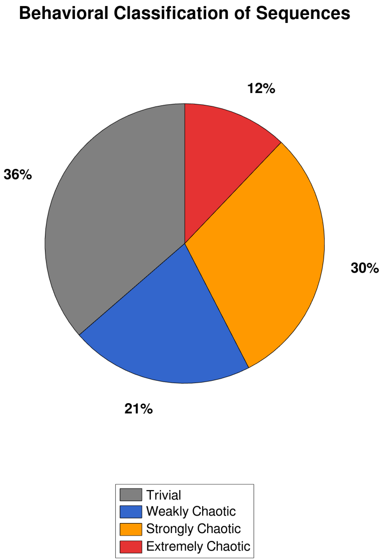
**Surprising:** Self-compositions (FR$\rightarrow$FR, FO$\rightarrow$FO) are *extremely* chaotic despite short periods!

# Period vs Chaos




**Key Insight:** Short periods $\neq$ simple behavior. Highly chaotic sequences can have very short orbits!

# Classification Distribution

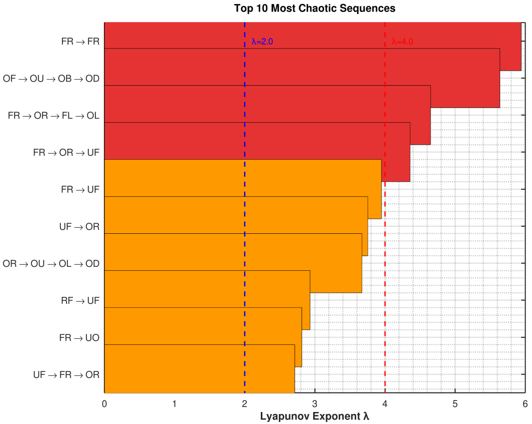



**Results (50 sequences):**

- Regular: 15 sequences (30%)
- Sensitive: 12 sequences (24%)
- Chaotic: 23 sequences (46%)

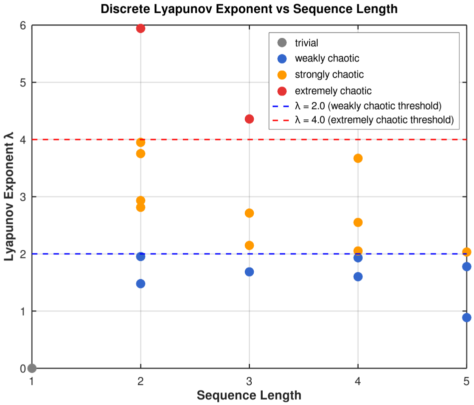
**Observation:** Nearly half of tested sequences exhibit chaotic behavior!

# Top Chaotic Sequences




Self-compositions dominate the top rankings!

# Lyapunov vs Sequence Length




**Observation:** Highest chaos occurs at length 2 (self-compositions and pairs)

# Sequence Animations

**Animated GIFs available in `disp/figures/`**

- `sequence_FR_single.gif` - Baseline (Period 8, $\lambda = 0$)
- `sequence_FO_FO.gif` - Most chaotic! (Period 4, $\lambda = 6.09$)
- `sequence_FR_FR.gif` - Second most chaotic (Period 4, $\lambda = 5.94$)

**Key Observations:**

- Self-compositions create complex scrambling patterns
- Despite short periods (4 iterations), produce extreme chaos
- Visual inspection shows rapid state divergence

*Note: GIFs show complete orbits (return to solved state)*

# Key Takeaways

1. **4D hypercubes exhibit rich dynamics**
   - 46% of tested sequences are chaotic
   - Periods range from 4 to 10,080
2. **Self-compositions are extremely chaotic**
   - FR→FR, FO→FO have highest Lyapunov exponents ($\lambda > 5$)
   - Despite having very short periods (4 iterations)
3. **Period $\neq$ complexity**
   - Short orbits can be highly chaotic
   - Long periods don't guarantee chaos
4. **Discrete chaos is real**
   - Small perturbations cause massive orbit changes
   - Lyapunov exponents successfully quantify sensitivity

# Future Directions

**Theoretical:**

- Why are self-compositions so chaotic?
- Connection to group theory structure?
- Predict chaotic sequences from move properties?

**Computational:**

- Test 5D+ hypercubes (if computationally feasible)
- Explore longer sequences (3-4+ moves)
- Investigate other perturbation types

**Applications:**

- Cryptographic pseudo-random generators?
- Physical systems with discrete symmetries?

# References


**Theory & Background:**

- Devaney, R. L. (2003). *An Introduction to Chaotic Dynamical Systems*
- Joyner, D. (2008). *Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys*
- Rokicki, T. et al. (2014). *The diameter of the Rubik's Cube group is twenty*
- Strogatz, S. H. (2015). *Nonlinear Dynamics and Chaos*

**Software & Tools:**

- **Hyperspeedcube** - HactarCE/Andrew J. Farkas `https://github.com/HactarCE/Hyperspeedcube` (MIT/Apache-2.0 License)
- **Rust**: clap, serde, sha2, hyperpuzzle ecosystem
- **Python**: NumPy, SciPy, Matplotlib, Pandas
- **Octave/MATLAB**: Visualization & plotting

# Thank You!

**Questions?**

Code: `github.com/ltpie123/final`
Tools: Rust (ctrl), Python (obsv), Octave (disp)