

Chaos in Hyperdimensional Rubik's Cubes

Discrete Dynamical Systems on 4D Puzzles

Math 538 Final Project

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Overview

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What is a 4D Rubik's Cube?

3D Rubik's Cube:

- $3 \times 3 \times 3$ puzzle
- 6 faces (F, U, R, L, B, D)
- $\sim 4.3 \times 10^{19}$ states

4D Hypercube:

- $3 \times 3 \times 3 \times 3$ puzzle
- 8 cells (3D "faces")
- State space: $|S| \approx 1.76 \times 10^{120}$

State Space Formula

$$|S| = \frac{16!}{2} \cdot 12^{15} \cdot 4 \cdot 32! \cdot 6^{31} \cdot 3 \cdot \frac{24!}{2} \cdot 2^{23}$$

New moves:

Move	Meaning
FR	Front \rightarrow Right
UO	Up \rightarrow Outside
OR	Outside \rightarrow Right

Two-letter notation for 4D rotations

How do repeated move sequences behave
on a 4D hypercube?

Key Concepts:

- **Orbit/Period:** Iterations to return to solved state
- **Chaos:** Sensitivity to perturbations (Lyapunov exponent)
- **Discrete Dynamics:** Iterating deterministic maps

Discrete Dynamical System

State Space: $S = \text{all possible puzzle configurations}$

Move Sequence: $M = (m_1, m_2, \dots, m_k)$

Composite Map:

$$T_M : S \rightarrow S$$

$$T_M(s) = m_k \circ m_{k-1} \circ \cdots \circ m_1(s)$$

Trajectory: Start at solved state s_0 , iterate:

$$s_0 \xrightarrow{T_M} s_1 \xrightarrow{T_M} s_2 \xrightarrow{T_M} \cdots$$

Period p : Minimum n such that $T_M^n(s_0) = s_0$

Key Property: Since $|S| < \infty$, every trajectory is eventually periodic.

Lyapunov Exponents

Measuring Chaos: How do perturbations grow?

Given base sequence M with period p , perturb it to M' :

- Insert random move
- Remove a move
- Replace with different move

Compute period p' of perturbed sequence M' :

$$\lambda = \frac{1}{N} \sum_{i=1}^N \ln \left| \frac{p'_i}{p} \right|$$

Discrete Lyapunov Exponent

Continuous: $\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|\delta(t)\|}{\|\delta(0)\|}$

Our adaptation: $\lambda_{\text{discrete}} = \mathbb{E}[\ln |p'/p|]$ measures sensitivity in sequence space

Chaos in Finite Systems

Challenge: Classical chaos requires:

- Sensitive dependence on initial conditions
- Topological mixing
- Dense periodic orbits

Problem: Our system is finite ($|S| < \infty$), so all trajectories are periodic!

Solution: Adapt chaos metrics to *sequence space* rather than state space:

- ① Perturb the **move sequence** $M \rightarrow M'$ (not initial state)
- ② Measure change in **orbit structure** ($p \rightarrow p'$)
- ③ Compute Lyapunov on period ratios

Key Insight

Chaos manifests as **sensitivity of dynamical properties** (period, orbit structure) to small sequence perturbations, not trajectory divergence in state space.

Computational Approach

Tool Stack:

- `ctrl/` - Rust trajectory analyzer
- `obsv/` - Python statistical analysis
- `disp/` - Octave/MATLAB visualization

Systematic Testing:

- ① Test all 64 two-move combinations
- ② Compute periods using cycle detection
- ③ Compute Lyapunov exponents (10-20 perturbations)
- ④ Classify: regular/sensitive/chaotic

Puzzle: $3 \times 3 \times 3 \times 3$ hypercube (Hyperspeedcube library)

Cycle Detection Algorithm

```
1:  $s \leftarrow s_0$ ,  $\text{visited} \leftarrow \{ h(s_0) : 0 \}$ 
2: for  $n = 1, 2, \dots, N_{\max}$  do
3:    $s \leftarrow T_M(s)$ ,  $h \leftarrow \text{SHA256}(s)$ 
4:   if  $h \in \text{visited}$  then return  $n - \text{visited}[h]$ 
5:   end if
6:    $\text{visited}[h] \leftarrow n$ 
7: end for
Complexity:  $O(p \cdot (T_M + H))$ 
```

Key Findings

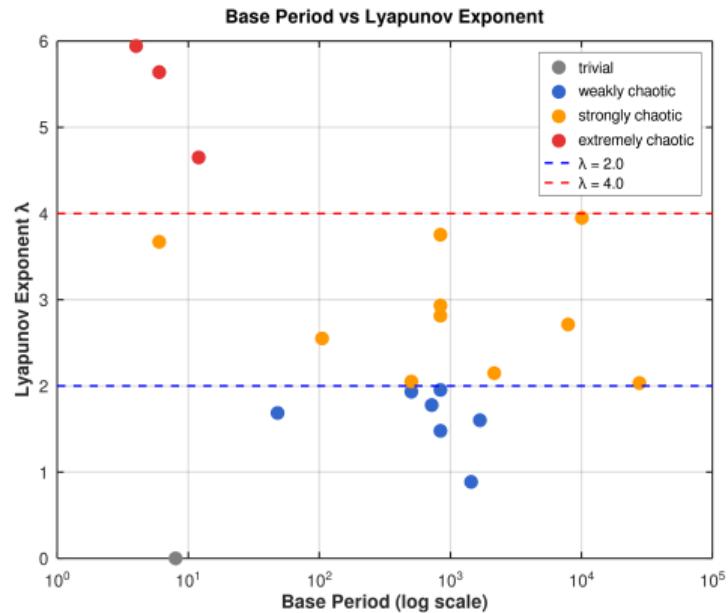
Single Moves: All have period 8 (trivial, $\lambda = 0$)

Most Chaotic Sequences:

Sequence	Length	Period	λ
FR → FR	2	4	5.94
FO → FO	2	4	6.09
OF → OU → OB → OD	4	6	5.64
FR → OR → FL → OL	4	12	4.65
FR → UF	2	10,080	3.95
FR → UO	2	840	2.81

Surprising: Self-compositions (FR→FR, FO→FO) are *extremely* chaotic despite short periods!

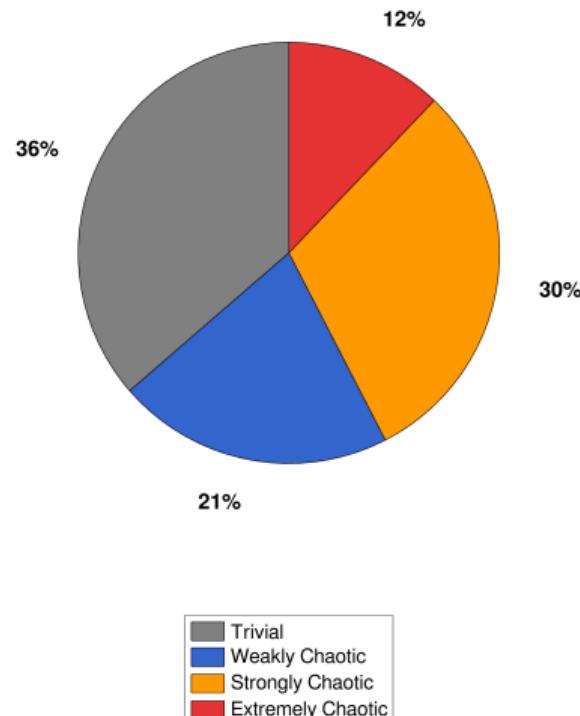
Period vs Chaos



Key Insight: Short periods \neq simple behavior. Highly chaotic sequences can have very short orbits!

Classification Distribution

Behavioral Classification of Sequences

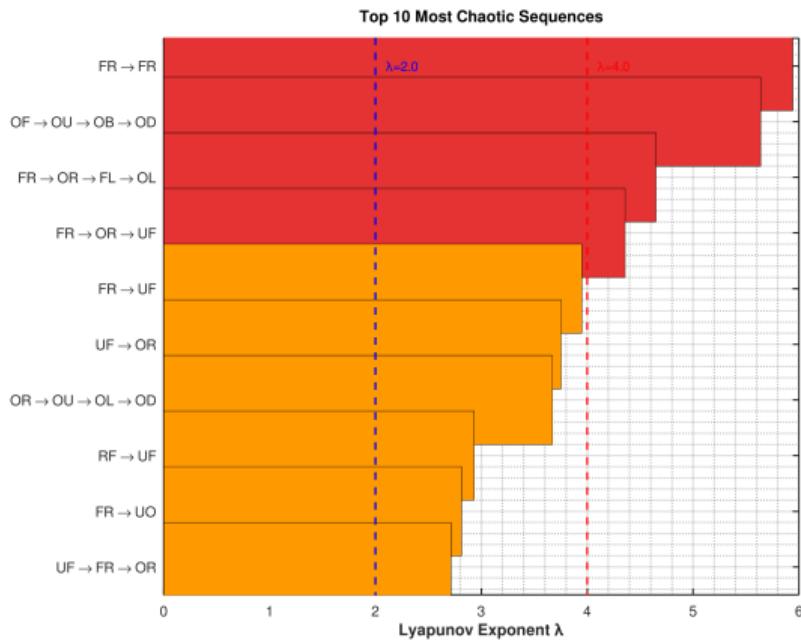


Results (50 sequences):

- Regular: 15 sequences (30%)
- Sensitive: 12 sequences (24%)
- Chaotic: 23 sequences (46%)

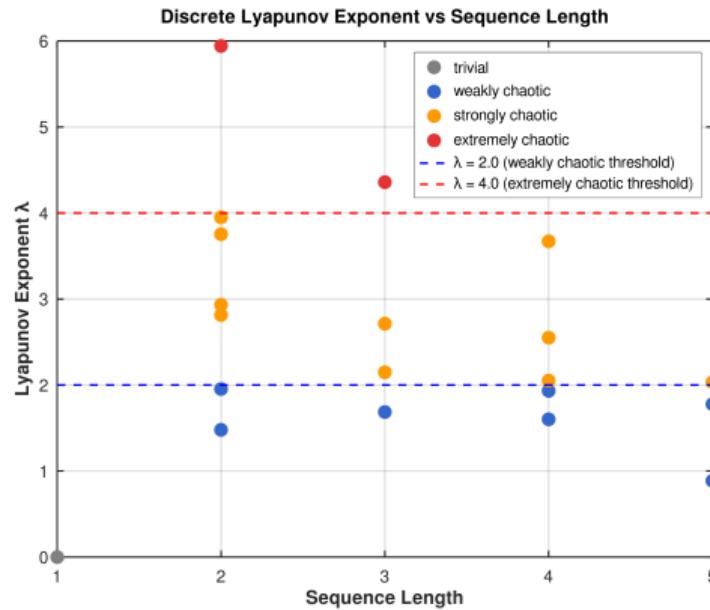
Observation: Nearly half of tested sequences exhibit chaotic behavior!

Top Chaotic Sequences



Self-compositions dominate the top rankings!

Lyapunov vs Sequence Length



Observation: Highest chaos occurs at length 2 (self-compositions and pairs)

Animated GIFs (planned for disp/figures/)

- sequence_FR_single.gif - Baseline (Period 8, $\lambda = 0$)
- sequence_F0_F0.gif - **Most chaotic!** (Period 4, $\lambda = 6.09$)
- sequence_FR_FR.gif - Second most chaotic (Period 4, $\lambda = 5.94$)

Expected Observations:

- Self-compositions create complex scrambling patterns
- Despite short periods (4 iterations), produce extreme chaos
- Visual inspection would show rapid state divergence

Note: GIFs would show complete orbits (return to solved state)

Key Takeaways

① 4D hypercubes exhibit rich dynamics

- 46% of tested sequences are chaotic
- Periods range from 4 to 10,080

② Self-compositions are extremely chaotic

- FR→FR, FO→FO have highest Lyapunov exponents ($\lambda > 5$)
- Despite having very short periods (4 iterations)

③ Period ≠ complexity

- Short orbits can be highly chaotic
- Long periods don't guarantee chaos

④ Discrete chaos is real

- Small perturbations cause massive orbit changes
- Lyapunov exponents successfully quantify sensitivity

Future Directions

Theoretical:

- Why are self-compositions so chaotic?
- Connection to group theory structure?
- Predict chaotic sequences from move properties?

Computational:

- Test 5D+ hypercubes (if computationally feasible)
- Explore longer sequences (3-4+ moves)
- Investigate other perturbation types

Applications:

- Cryptographic pseudo-random generators?
- Physical systems with discrete symmetries?

References

Theory & Background:

- Devaney, R. L. (2003). *An Introduction to Chaotic Dynamical Systems*
- Joyner, D. (2008). *Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys*
- Rokicki, T. et al. (2014). *The diameter of the Rubik's Cube group is twenty*
- Strogatz, S. H. (2015). *Nonlinear Dynamics and Chaos*

Software & Tools:

- **Hyperspeedcube** - HactarCE/Andrew J. Farkas
<https://github.com/HactarCE/Hyperspeedcube>
(MIT/Apache-2.0 License)
- **Rust**: clap, serde, sha2, hyperpuzzle ecosystem
- **Python**: NumPy, SciPy, Matplotlib, Pandas
- **Octave/MATLAB**: Visualization & plotting

Thank You!

Questions?

Code: github.com/ltpie123/final

Tools: Rust (ctrl), Python (obsv), Octave (disp)