Gaussian Mixture MCMC applied to X-ray tomography for segmented imaging

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Summary

- Basic concepts about X-ray attenuation to define the *forward model*
- Bayesian approach for inverse problems
- Bayesian Linearized Inversion (BLI) applied to X-ray tomography
- Statistical model assuming multimodality/multi-minerals
- Markov Chain Monte Carlo methods (MCMC)
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- Results

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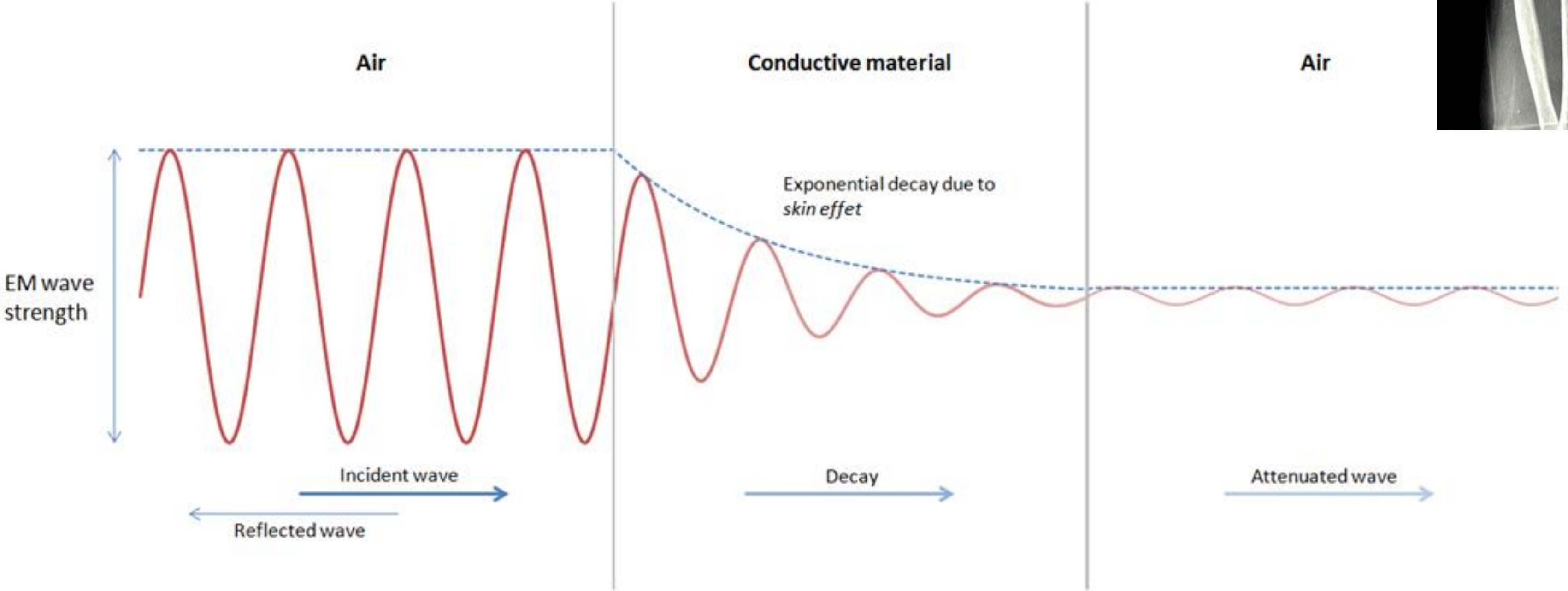
Basic concepts about attenuation

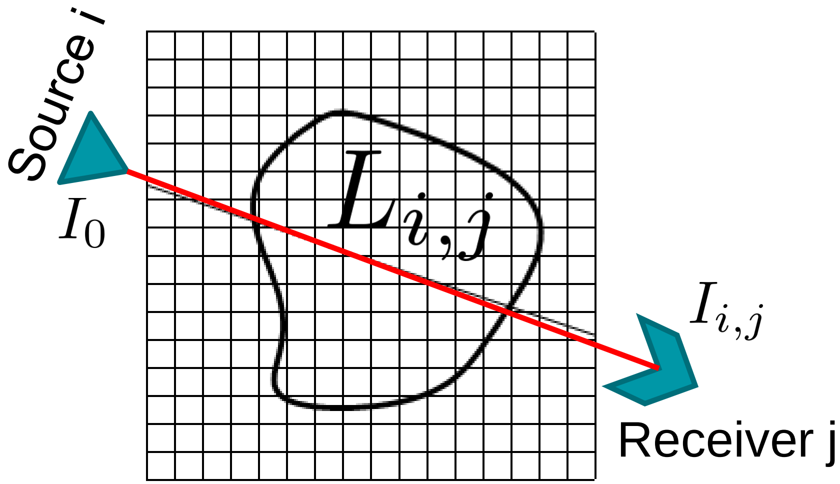
- Exponential attenuation model

$$I(x) = I_0 e^{-x\lambda}$$



Basic concepts about attenuation

- Forward model of the X-ray tomography



  - The observed intensity is modelled by a line integral for each pair source I receptor j

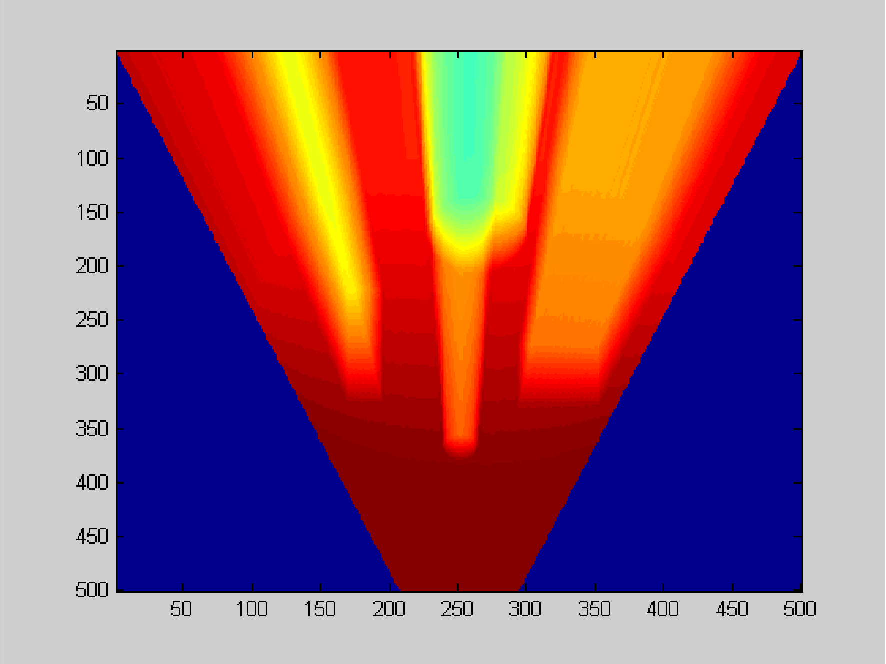
$$I_{i,j} = I_0 exp\left(-\int \lambda(x,y)d\boldsymbol{s}\right)$$

  - Numerically:

$$I_{i,j} = I_0 exp\left(-\sum_{k \in L_{i,j}} \lambda_k\right)$$

Basic concepts about attenuation

- Forward model of the X-ray tomography



  - The observed intensity is modelled by a line integral for each pair source I receptor j

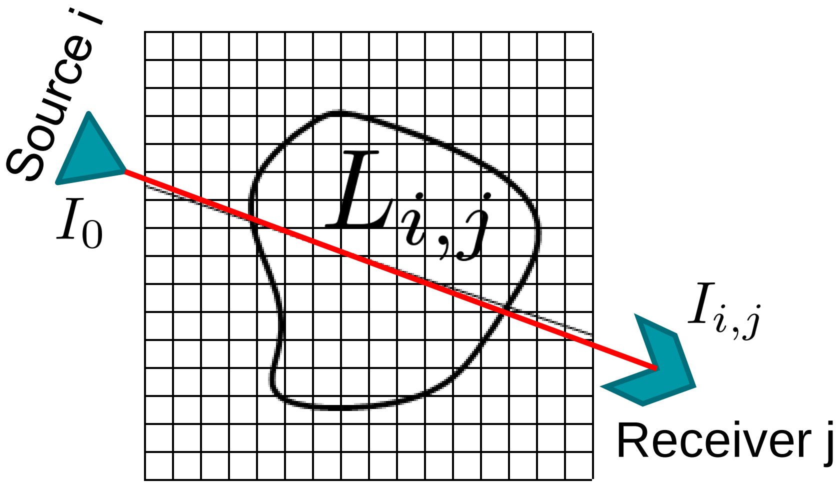
$$I_{i,j} = I_0 exp\left(- \int \lambda(x,y) d\boldsymbol{s} \right)$$

  - Numerically:

$$I_{i,j} = I_0 exp\left(- \sum_{k \in L_{i,j}} \lambda_k \right)$$

Basic concepts about attenuation

- Matrix form of the forward model

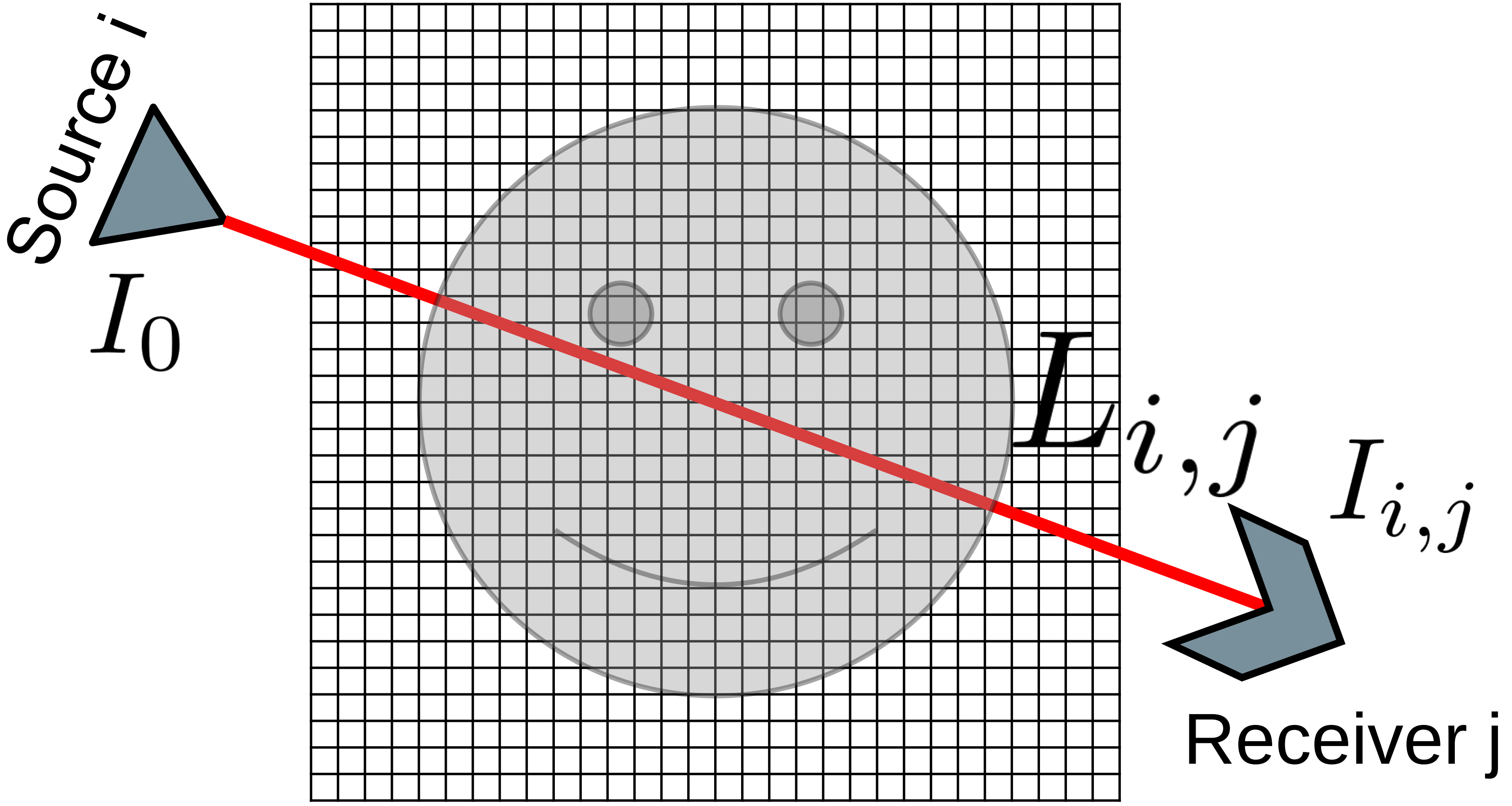


$$ln\left(\frac{I_{I,j}}{I_0}\right) = -\sum_{x_i \in L_{i,j}} \lambda_i$$

$$\boldsymbol{d} = \boldsymbol{G}\boldsymbol{m} + \boldsymbol{e}_d$$

$$\boldsymbol{d} = \begin{pmatrix} ln(I_{1,1}/I_0) \\ \vdots \\ ln(I_{I,J}/I_0) \end{pmatrix} \quad \boldsymbol{m} = \begin{pmatrix} \lambda_{11} \\ \vdots \\ \lambda_{nn} \end{pmatrix}$$

  - G is a sparse matrix with 0s and 1s that computes the summation of the attenuation values (model parameters ***m***)

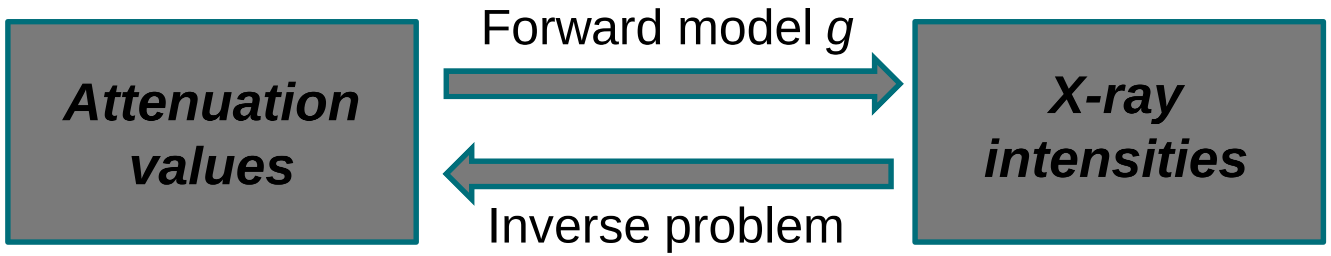


Source i
I_0
$L_{i,j}$
$I_{i,j}$
Receiver j

X-ray tomography as an inverse problem

- Estimating the attenuation values (model parameter **m**) based on X-ray intensity measures of different positions of source receivers (observed data **d**).

$$\boldsymbol{d} = g(\boldsymbol{m}) + \boldsymbol{e}_d$$



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Bayesian approach for inverse problems

- The solution of the inverse problem is given by the Bayesian posterior distribution.
    - More informative than a single solution
    - Accounts for the observed data and the prior info

$$\boldsymbol{d} = g(\boldsymbol{m}) + \boldsymbol{e}_d$$

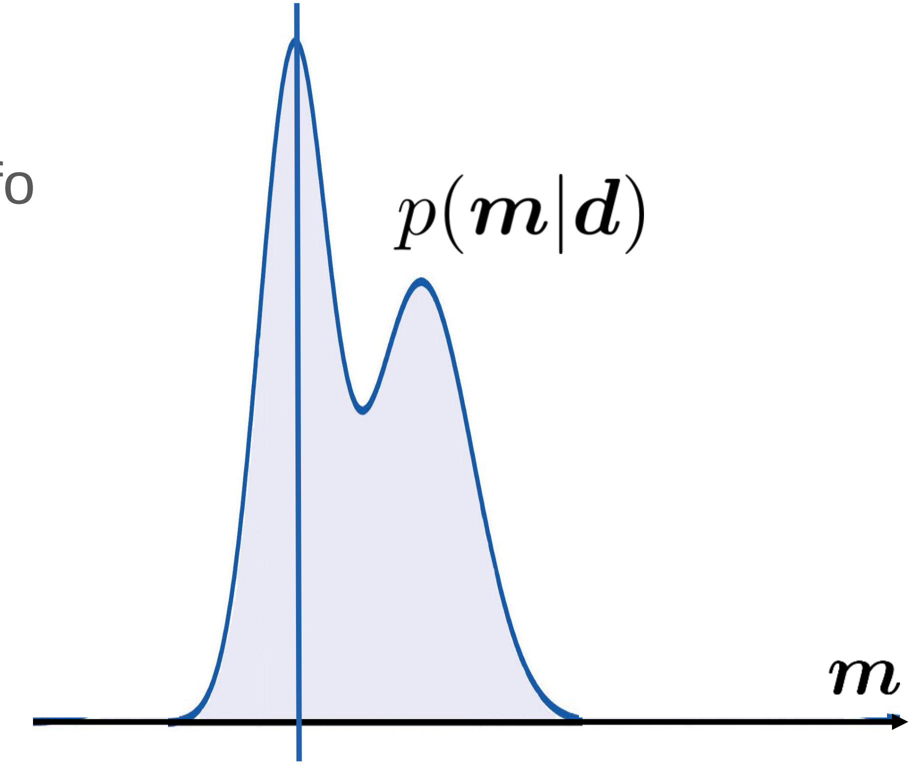
Likelihood

Posterior

$$p(\boldsymbol{m}|\boldsymbol{d}) = \frac{p(\boldsymbol{d}|\boldsymbol{m})p(\boldsymbol{m})}{p(\boldsymbol{d})}$$

Prior

Normalization factor



Best solution (minimum error)

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BLI - Bayesian Linearized Inversion

- It is successfully applied in the AVO seismic inversion
- For a linear forward model and Gaussian assumption for the prior, likelihood and noise, the posterior distribution can be analytically obtained:

$$p(\boldsymbol{m}|\boldsymbol{d}) = N(\boldsymbol{m}; \boldsymbol{\mu}_{m|d}, \boldsymbol{\Sigma}_{m|d})$$

$$\boldsymbol{\mu}_{m|d} = \boldsymbol{\mu}_m + \boldsymbol{\Sigma}_m \boldsymbol{G}^T \left(\boldsymbol{G} \boldsymbol{\Sigma}_m \boldsymbol{G}^T + \boldsymbol{\Sigma}_d \right)^{-1} (\boldsymbol{d} - \boldsymbol{G} \boldsymbol{\mu}_m)$$

$$\boldsymbol{\Sigma}_{m|d} = \boldsymbol{\Sigma}_m - \boldsymbol{\Sigma}_m \boldsymbol{G}^T \left(\boldsymbol{G} \boldsymbol{\Sigma}_m \boldsymbol{G}^T + \boldsymbol{\Sigma}_d \right)^{-1} \boldsymbol{G} \boldsymbol{\Sigma}_m,$$

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Statistical model

$\boldsymbol{m}$ - Attenuation values
$\boldsymbol{\kappa}$ - Segment configuration
$\boldsymbol{d}$ - X ray intensities

- Prior distribution

$$p(\mathbf{m}) = \sum_{\boldsymbol{\kappa} \in \Omega^n} p(\mathbf{m}|\boldsymbol{\kappa}) p(\boldsymbol{\kappa}).$$

  - Facies prior (first order Makov chain)

$$p(\boldsymbol{\kappa}) = p(\boldsymbol{\kappa}_1) \prod_{t=2}^{n} p(\boldsymbol{\kappa}_t | \boldsymbol{\kappa}_{t-1})$$

Transition matrix
(sedimentology parameter)

  - Prior of properties conditioned to the segments

$$p(\boldsymbol{m}|\boldsymbol{\kappa}) = N(\boldsymbol{m}; \boldsymbol{\mu}_{m|\kappa}, \boldsymbol{\Sigma}_{m|\kappa})$$

Statistical model – Summary

(Buland and Omre, 2003; Grana and Della Rossa, 2010; Grana et al, 2017; Fjeldstad and Grana, 2018; de Figueiredo et al, 2019)

$\boldsymbol{m}$ - Attenuation values
$\boldsymbol{\kappa}$ - Segment configuration
$\boldsymbol{d}$ - X-ray intensities
t - Position
$g(.)$ - Forward model

- Prior distribution

$$p(\mathbf{m}) = \sum_{\boldsymbol{\kappa}\in\Omega^n} N(\boldsymbol{m}; \boldsymbol{\mu}_{m|\kappa}, \boldsymbol{\Sigma}_{m|\kappa})p(\boldsymbol{\kappa}).$$

$$p(\boldsymbol{\kappa}) = p(\boldsymbol{\kappa}_1)\prod_{t=2}^{n} p(\boldsymbol{\kappa}_t|\boldsymbol{\kappa}_{t-1})$$

- Likelihood distribution

$$p(\mathbf{d}|\mathbf{m}) = N_{n_\theta n_d}(\mathbf{d}; g(\mathbf{m}), \boldsymbol{\Sigma}_d),$$

- Linear case - posterior distribution:

$$p(\boldsymbol{m}|\boldsymbol{d}) \propto \sum_{\boldsymbol{\kappa}\in\Omega^n} N(\boldsymbol{m}; \boldsymbol{\mu}_{m|d,\kappa}, \boldsymbol{\Sigma}_{m|d,\kappa})p(\boldsymbol{d}|\boldsymbol{\kappa})p(\boldsymbol{\kappa}),$$

Huge number of modes,
10^15 to 10^25

$$\boldsymbol{\mu}_{m|d,\kappa} = \boldsymbol{\mu}_{m|\kappa} + \boldsymbol{\Sigma}_{m|\kappa}\boldsymbol{G}^T\left(\boldsymbol{G}\boldsymbol{\Sigma}_{m|\kappa}\boldsymbol{G}^T + \boldsymbol{\Sigma}_d\right)^{-1}(\boldsymbol{d} - \boldsymbol{G}\boldsymbol{\mu}_{m|\kappa}),$$

$$\boldsymbol{\Sigma}_{m|d,\kappa} = \boldsymbol{\Sigma}_{m|\kappa} - \boldsymbol{\Sigma}_{m|\kappa}\boldsymbol{G}^T\left(\boldsymbol{G}\boldsymbol{\Sigma}_{m|\kappa}\boldsymbol{G}^T + \boldsymbol{\Sigma}_d\right)^{-1}\boldsymbol{G}\boldsymbol{\Sigma}_{m|\kappa},$$

Posterior components
Is analytically obtained
(Buland & Omre, 2003)

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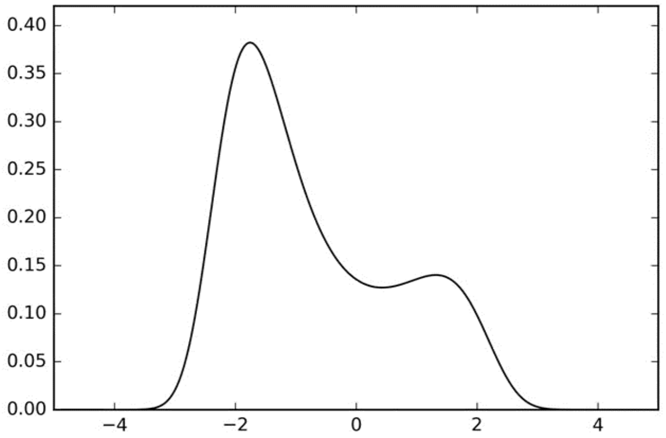
Markov Chain Monte Carlo methods (MCMC)

- Simulate a chain according to an acceptance rule in order to converge to a target distribution
  - Metropolis algorithm
    - Acceptance factor of new configurations $\tilde{\boldsymbol{m}}$ depends on only the previous one $\boldsymbol{m}^{\boldsymbol{i-1}}$:

$$r = \frac{p_t(\boldsymbol{m}^i)}{p_t(\boldsymbol{m}^{i-1})} = \frac{p(\boldsymbol{d}|\boldsymbol{m}^{\boldsymbol{i}})p(\boldsymbol{m}^i)}{p(\boldsymbol{d}|\boldsymbol{m}^{i-1})p(\boldsymbol{m}^{i-1})}$$



Ridlo W. Wibowo
Publicado em 8 de jul de 2016

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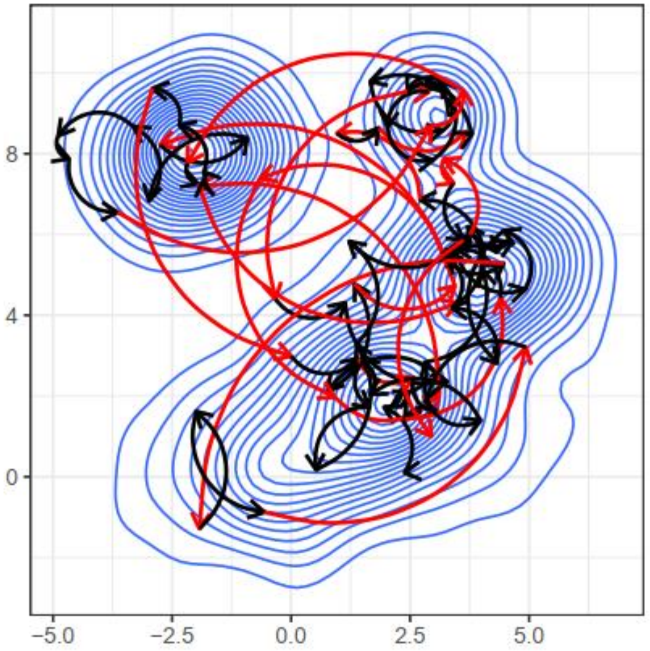
MCMC for multimodal posterior

Based on Holmes, C et al, 2018

- ***Local move*** (continuous properties) e ***Jump move*** *(segments)*



from Holmes, C et al, 2018

Local move

$$\boldsymbol{\kappa}^i = \boldsymbol{\kappa}^{i-1}$$

$$\boldsymbol{m}^i \sim N(\boldsymbol{m}; \boldsymbol{\mu}_{m|d,\kappa}, \boldsymbol{\Sigma}_{m|d,\kappa})$$

Jump move

$$\boldsymbol{\kappa}^i \sim p(\boldsymbol{\kappa}|\boldsymbol{\kappa}^{i-1})$$

$$r = \frac{p(\boldsymbol{d}|\boldsymbol{\kappa^i})}{p(\boldsymbol{d}|\boldsymbol{\kappa^{i-1}})} = \frac{N(\boldsymbol{d}; \boldsymbol{G}\boldsymbol{\mu}_{m|\kappa^i}, \boldsymbol{G}\boldsymbol{\Sigma}_{m|\kappa^i}\boldsymbol{G}^T + \boldsymbol{\Sigma}_d)}{N(\boldsymbol{d}; \boldsymbol{G}\boldsymbol{\mu}_{m|\kappa^{i-1}}, \boldsymbol{G}\boldsymbol{\Sigma}_{m|\kappa^{i-1}}\boldsymbol{G}^T + \boldsymbol{\Sigma}_d)}$$

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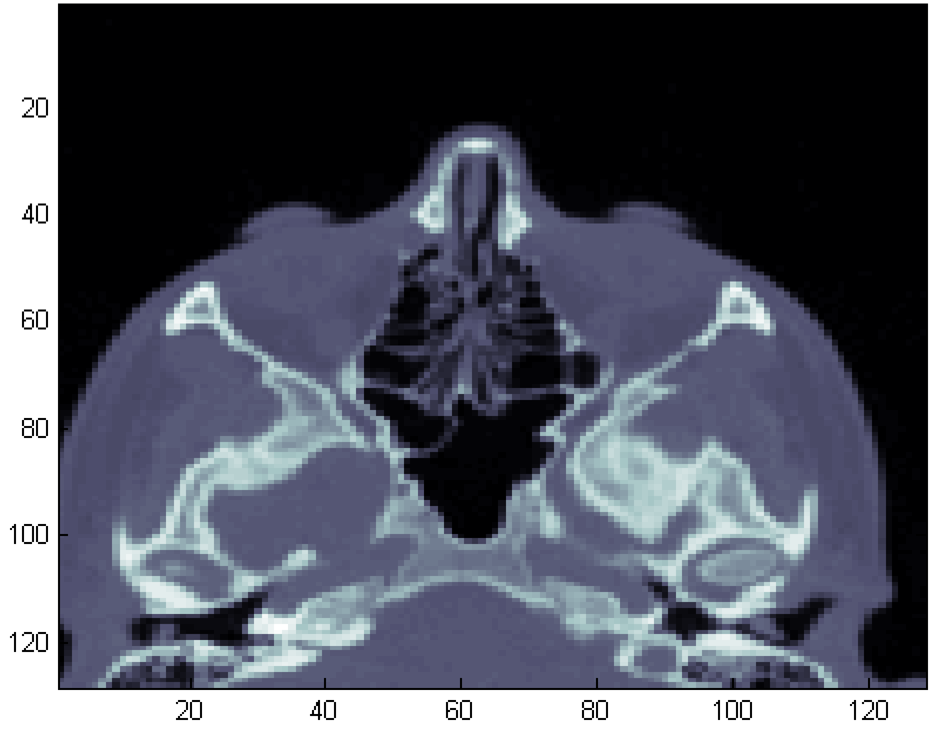
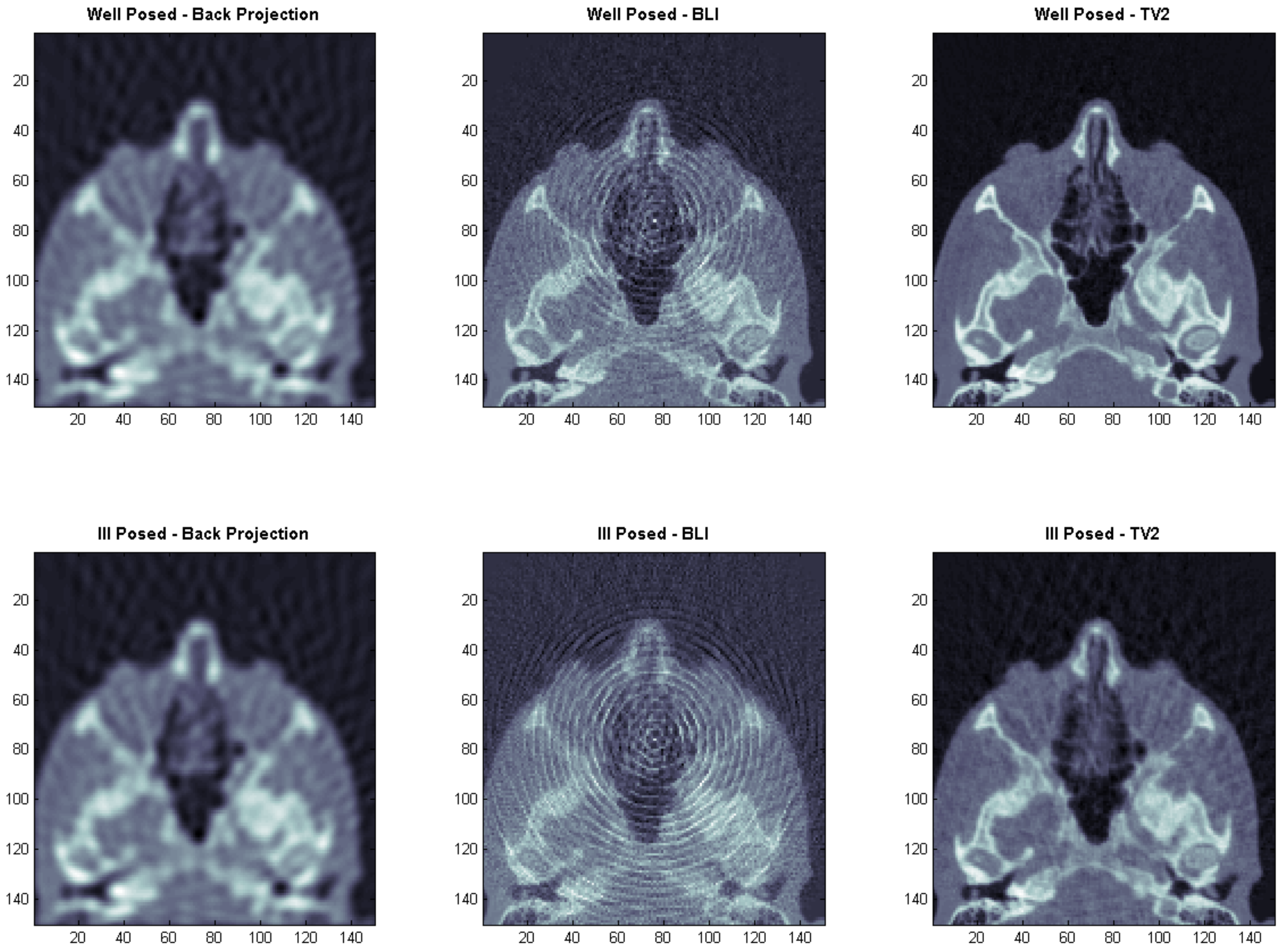
Results – Bayesian Linearized inversion

- Synthetic case



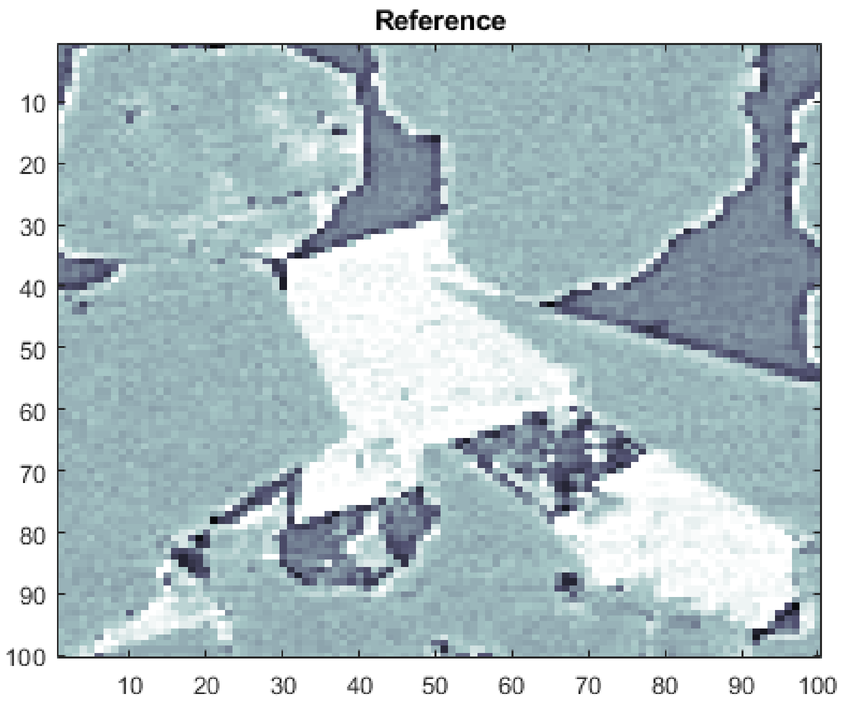
Results – Bayesian Linearized inversion

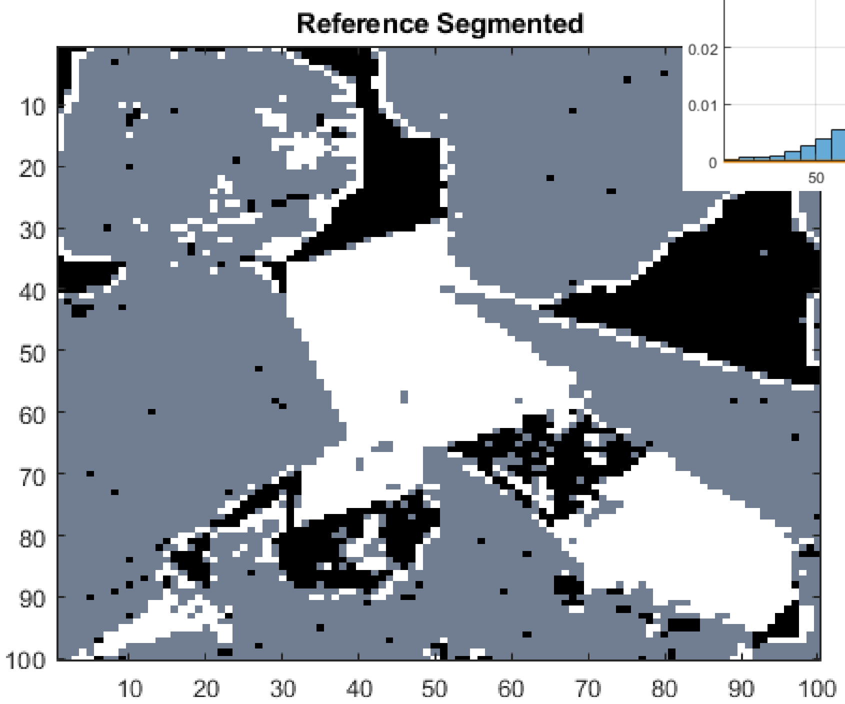
- BLI with and without prior correlation in comparison to the Back projection technique. Well posed and ill posed inverse problem.

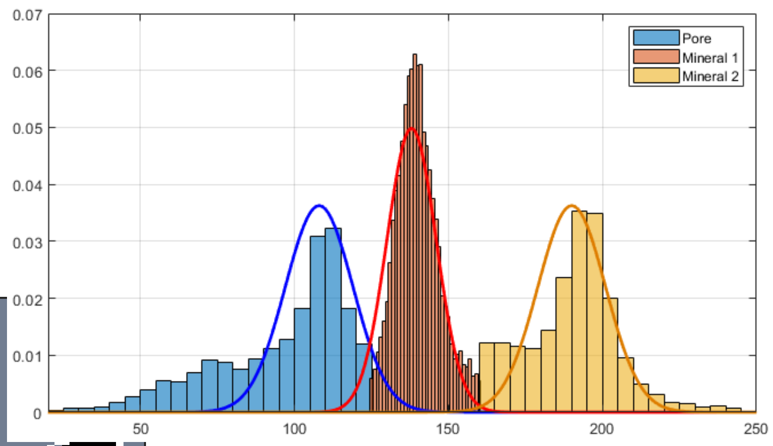


Results – Gaussian Mixture MCMC

- Synthetic case with SNR=100

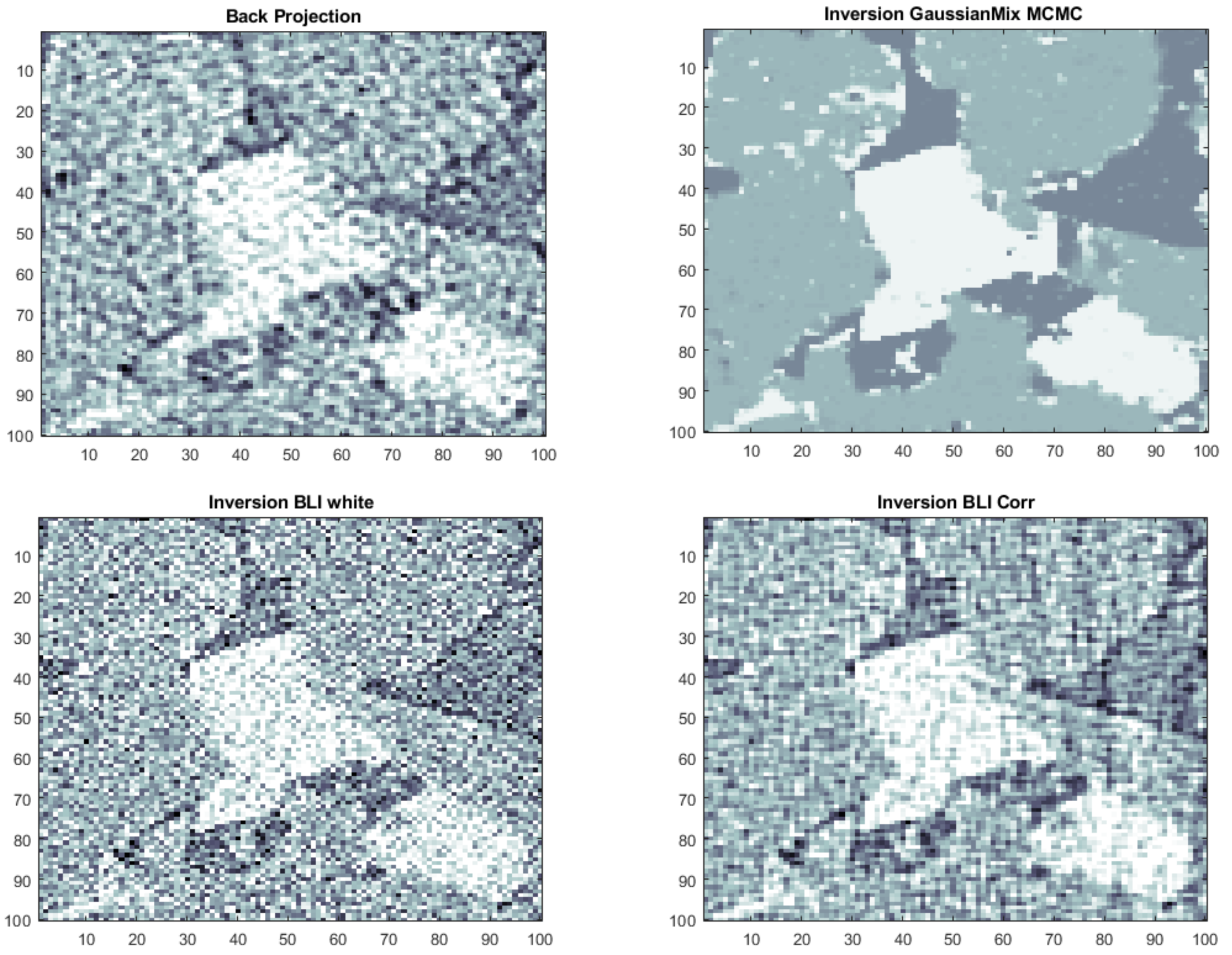






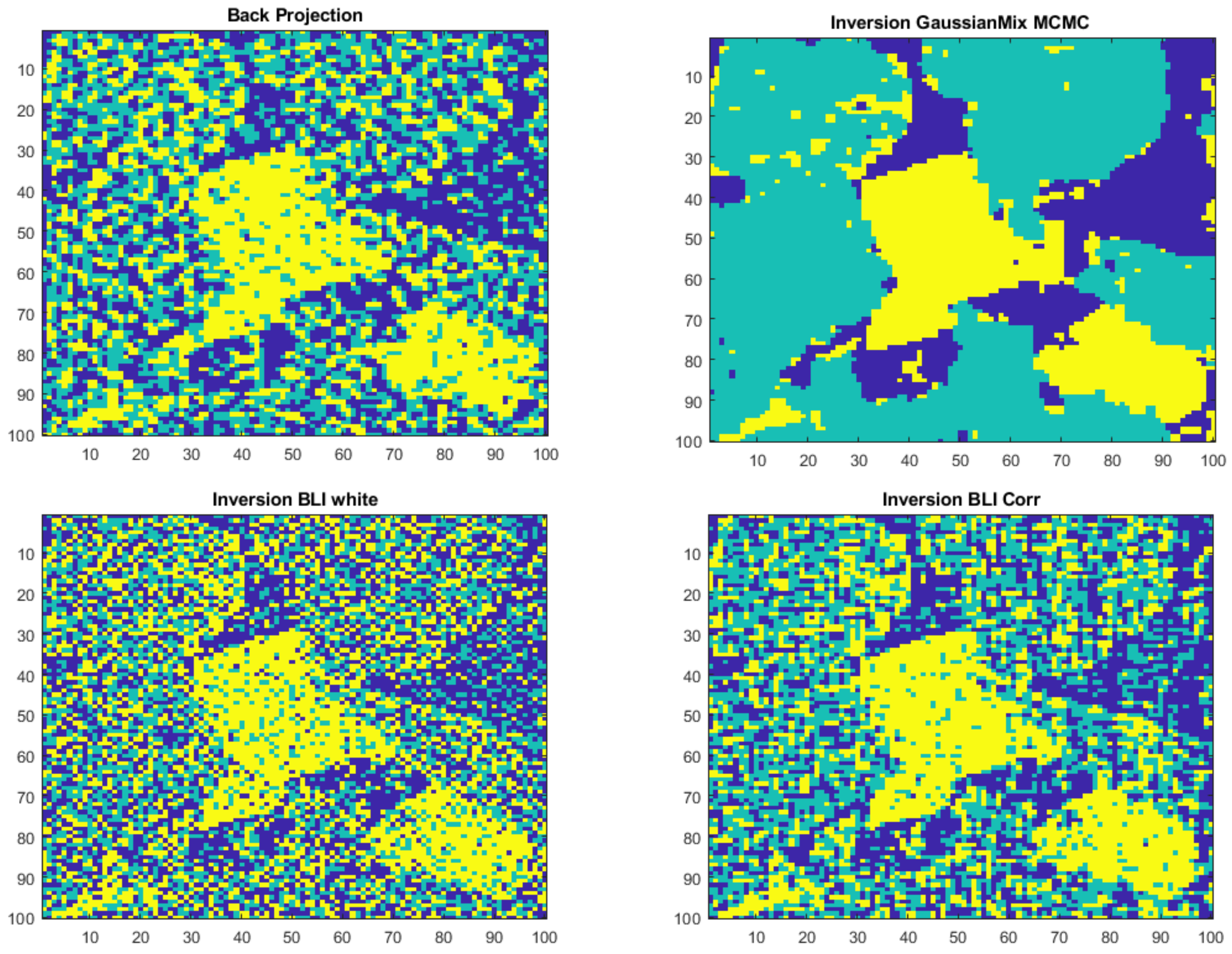
Results – Gaussian Mixture MCMC

- Continuous property (attenuation factor). Comparison to Back-projection and BLI.



Results – Gaussian Mixture MCMC

- Discrete/categorical property, segment. Comparison to Back-projection and BLI with Bayesian inference.



Conclusions and Perspective

- The Gaussian Mixture MCMC provides better results than Back projection and BLI although it requires a higher computation effort
- Despite the synthetic case does not include 2 different segments with similar attenuation distributions and with different textural characteristics. It is expected that it can solve problems this problem by setting different transition probabilities at the transition matrix
- Perspective: Application of the method to dual energy tomography with bivariate local distribution