Hartshorne 1.2 Exercises: Projective Varieties

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2.1 Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbf{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.

Solution: Let $\mathfrak{a}$ be a homogeneous ideal of $k[x_0, \ldots, x_n]$. Then in $\mathbf{P}^n$, we have that

$$Z_{\mathbf{P}^n}(\mathfrak{a}) = \left\{ P \in \mathbf{P}^n \;\middle|\; \text{For all } f \in \mathfrak{a}, f(P) = 0 \right\}$$

while in $\mathbf{A}^{n+1}$, we have that

$$Z_{\mathbf{A}^{n+1}}(\mathfrak{a}) = \left\{ Q \in \mathbf{A}^{n+1} \;\middle|\; \text{For all } f \in \mathfrak{a}, f(Q) = 0 \right\}.$$

We can write a surjective map $\pi : Z_{\mathbf{A}^{n+1}}(\mathfrak{a}) \to Z_{\mathbf{P}^n}(\mathfrak{a})$ sending an affine point to its projective equivalence class. In addition, for a projective point $P \in Z_{\mathbf{P}^n}(\mathfrak{a})$, we can observe that $\pi^{-1}(P) \subset Z_{\mathbf{A}^{n+1}}(\mathfrak{a})$ (the elements of P's equivalence class).

Thus, if f is homogeneous with a nonzero degree and $f(P) = 0$ for all $P \in Z_{\mathbf{P}^n}(\mathfrak{a})$, then $f(Q) = 0$ for all $Q \in Z_{\mathbf{A}^{n+1}}(\mathfrak{a})$. By the usual Nullstellensatz, this implies that $f^q \in \mathfrak{a}$ for some $q > 0$, which proves the result. ☐

2.2 For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(*i.*) $Z(\mathfrak{a}) = \emptyset$ (the empty set);
(*ii.*) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
(*iii.*) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Solution:

$(i) \implies (ii)$**:** Suppose $\mathfrak{a}$ is a homogeneous ideal and $Z(\mathfrak{a}) = \varnothing$. By the homogeneous Nullstellensatz, it is vacuously true that for every $f \in \oplus_{d>0} S_d$, $f(P) = 0$ for all $P \in Z(\mathfrak{a})$, and so for every such f there exists a $q > 0$ such that $f^q \in \mathfrak{a}$. Therefore, $\oplus_{d>0} S_d \subseteq \sqrt{a}$. If $\mathfrak{a}$ contains a unit, then $\sqrt{\mathfrak{a}} = S$. If $\mathfrak{a}$ does not contain a unit, then $\sqrt{\mathfrak{a}} \subseteq \oplus_{d>0} S_d \implies \oplus_{d>0} S_d = \sqrt{\mathfrak{a}}$.

$(ii) \implies (iii)$**:** Suppose $\sqrt{a} = S$. Then this implies that $1 \in \mathfrak{a} \implies \mathfrak{a} = S$. Hence $S_d \subset \mathfrak{a}$.

Alternatively, suppose $\sqrt{a} = \oplus_{d>0} S_d$. Then for every $f \in \oplus_{d>0} S_d$, there exists a $q > 0$ such that $f^q \in \mathfrak{a}$. In particular, there exist $r_1, \ldots, r_n$ such that

$$x_1^{r_1}, \ldots, x_n^{r_n} \in \mathfrak{a}.$$

Take $d = r_1 + \cdots + r_n$. We claim that $S_d \subset \mathfrak{a}$. To see this, note that every degree-d homogeneous polynomial is of the form

$$\sum_{k \geqslant 0} c_k x_1^{\alpha_1(k)} \cdots x_n^{\alpha_n(k)}$$

where only finitely many summands are nonzero and $\alpha_1(k) + \cdots + \alpha_n(k) = d$ is a sum of nonnegative integers. Since $d = r_1 + \cdots + r_n$, we know that for any summand $x_1^{\alpha_1(k)} \cdots x_n^{\alpha_n(k)}$ at least one $\alpha_i(k) \geqslant r_i$. Hence, the summand is in $\mathfrak{a}$, and so the whole sum is in $\mathfrak{a}$. Therefore $S_d \subseteq \mathfrak{a}$.

$(iii) \implies (i)$**:** Suppose $S_d \subseteq \mathfrak{a}$ for $d > 0$. Then $Z(\mathfrak{a})$ must at least contain points P such that $f(P) = 0$ for all $f \in S_d$. This includes the polynomials $x_1^d, \ldots, x_n^d$. However, these n polynomials cannot all be simultaneously zero. Hence $Z(\mathfrak{a}) = \varnothing$.

(i) $\Rightarrow$ (ii). Since $Z(\mathfrak{a}) = \emptyset$, the zero set of $\mathfrak{a}$ in affine space Did you mean projective space? is either $\emptyset$ or $\{\mathbf{0}\}$. In the first case, we certainly have $\sqrt{\mathfrak{a}} = S$. In the second case, we have $\sqrt{\mathfrak{a}} = \{p \in S : p(0) = 0\} = S_+$.

(ii) $\Rightarrow$ (iii). It suffices to show that there exists d such that all monomials of degree d lies in $\mathfrak{a}$. Since $S_+ \subset \sqrt{\mathfrak{a}}$, for each $0 \le i \le n$, there exists d_i such that $x_i^{d_i} \in \mathfrak{a}$. Let $d = n \sum_{i=0}^n d_i$. Then if a monomial has degree d, then there exists i such that the exponent of x_i in the monomial is at least $\sum_{i=0}^n d_i$, and therefore at least d_i, which implies that the monomial is in $\mathfrak{a}$.

(iii) $\Rightarrow$ (i). If $S_d \subseteq \mathfrak{a}$ for some $d > 0$, then $x_i \in \sqrt{\mathfrak{a}}$ for every $0 \le i \le n$, which means that $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) = \emptyset$. ☐

2.3

(a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.
(c) For any two subsets Y_1, Y_2 of $\mathbf{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
(d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
(e) For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \overline{Y}$.

Solution: I did **(a)** − **(c)**, but I don't feel like TEX-ing them.

(d) Suppose f is homogeneous and $f \in I(Z(\mathfrak{a}))$. Then $f(P) = 0$ for all $P \in Z(\mathfrak{a})$. By the homogeneous Nullstellensatz, we see that $f \in \sqrt{\mathfrak{a}}$.

Suppose $f \in \mathfrak{a}$. Then $f^q \in \mathfrak{a}$ for some $q > 0$. As $f^q(P) = 0$ for all $P \in Z(\mathfrak{a})$, $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ (as k is an integral domain) and so it follows that $f \in I(Z(\mathfrak{a}))$. Our total work then shows that $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, as desired.

(e) We first know that $Y \subset Z(I(Y))$; we show that it is the smallest closed set containing Y. Thus let $Z(J)$ be a closed set where $Y \subset Z(J) \subset I(Z(Y))$. Then $I(Z(J)) \subset I(Y)$. Since $I(Z(J)) = \sqrt{J}$, we see that $J \subset I(Y)$. Therefore, $Z(I(Y)) \subset Z(J)$. Hence, $Z(I(Y)) = Z(J)$, and so $Z(I(Y)) = \overline{Y}$.

□

2.4

(a) There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbf{P}^n$ and homogeneous radical ideals of S not equal to S_+ given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note:* Since S_+ does not occur in this correspondence, it is sometimes called te *irrelevant* maximal ideal of S.
(b) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.
(c) Show that $\mathbf{P}^n$ itself is irreducible.

Solution:

□

2.5

(a) $\mathbf{P}^n$ is a noetherian topological space.
(b) Every algebraic set in $\mathbf{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

Solution:

□

2.6 If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint*: Let $\phi_i : U_i \to \mathbf{A}^n$ be the homeomorphism of (2.2), let Y_i be the affine variety $\phi_i(Y \cap U_i)$ and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)x_i$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at the transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is empty.]

Solution: Let $S(Y)^{(0)}_{x_i}$ be the degree 0 elements of $S(Y)_{x_i}$. We will construct an isomorphism $\psi_0 : S(Y)^{(0)}_{x_i} \to A(Y_i)$. To do so, we recycle the notation use by Hartshorne: α_i sets the x_i-th coordinate of a homogeneous polynomial to 1, while $\beta_i : k[x_1, \ldots, x_n] \to k[x_i, x_1, \ldots, x_n]$ homogenizes a polynomial by introducing the variable x_i. Note that they are inverses.

A degree 0 element of $S(Y)_{x_i}$ will be of the form $(f + I_{\mathbf{P}^n}(Y))/x_i^d$ with $f \neq x_i^n$ for any n, and where $\deg(f) = d$. Thus we let

$$\psi_0\left(\frac{f(x_0, \ldots, x_n) + I_{\mathbf{P}^n}(Y)}{x_i^d}\right) = \alpha_i(f) + I_{\mathbf{A}^n}(Y_i).$$

Here, we see that if $f(x_0, \ldots, x_n)/x_i^d$ is nonzero on Y, then $\alpha_i(f)$ is nonzero on Y_i. Alternatively, let $g(x_1, \ldots, x_n) + I_{\mathbf{A}^n}(Y_i) \in A(Y_i)$. Then we define

$$\psi_0^{-1}\left(g(x_1, \ldots, x_n) + I_{\mathbf{A}^n}(Y_i)\right) = \frac{\beta_i(g)}{x_i^d} + I_{\mathbf{P}^n}(Y)$$

and note that this is a degree 0 element of $S(Y)_{x_i}$. The fact that these are inverses follows from the fact that α_i, β_i are inverses of each other. Therefore, $S(Y)_{x_i}^{(0)} \cong A(Y_i)$.

We now establish the stronger isomorphism $\psi : S(Y)_{x_i} \to A(Y_i)[x_i, x_i^{-1}]$. Define

$$\psi\left(\frac{f + I(Y)}{x_i^j}\right) = \alpha_i(f)x^j x^{-d} + I_{\mathbf{A}^n}(Y_i) \qquad d = \deg(f)$$

and

$$\psi^{-1}\big(g \cdot x_i^r x_i^{-s} + I_{\mathbf{A}^n}(Y)\big) = \frac{\beta_i(g) + I_{\mathbf{P}^n}(Y)}{x_i^r x_i^{-s} x_i^d} \qquad d = \deg(g).$$

These are once again inverses because α_i and β_i are inverses of each other.

Now observe that $Y = \bigcup_i U_i \cap Y$, so that by Exercise 1.10, we have that $\dim Y = \sup_i \dim U_i \cap Y$. However, ϕ_i establishes the isomorphism to Y_i, so that $\dim Y = \sup_i \dim Y_i$. Hence it suffices to calculate this quantity.

Since Y_i is an affine variety, we have that $\dim Y_i = \dim A(Y_i)$.

□

2.7

(a) $\dim \mathbf{P}^n = n$.

(b) If $Y \subseteq \mathbf{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.

Solution:

□

2.8 A projective variety $Y \subseteq \mathbf{P}^n$ has dimension $n-1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbf{P}^n$.

Solution:

□

2.9 If $Y \subseteq \mathbf{A}^n$ is an affine variety, we identify $\mathbf{A}^n$ with an open set $U_0 \subset \mathbf{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbf{P}^n$, which is called the *projective closure* of Y.

(a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).

(b) Let $Y \subset \mathbf{A}^n$ be the twisted cubic of *(Ex 1.2)*. Its projective closure $\overline{Y} \subset \mathbf{P}^n$ is called the *twisted cubic curve* in $\mathbf{P}^3$. Find generators for $I(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

Solution:

(a) Since $\overline{Y} = Z(I(\phi_o Y))$, $I(\overline{Y}) = I(\phi_o Y)$. So it suffices to show that $I(\phi_o Y) = \beta(I(Y))$. We have $I(\phi_0 Y) \supseteq \beta(I(Y))$ because $\phi_o(Y) \subseteq Z(\beta(I(Y)))$. Conversely, we have $I(\phi_0 Y) \subseteq \beta(I(Y))$ because any point of Y vanishes on $p(1, x_1, \ldots, x_n)$ for $p \in I(\phi_o(Y))$. In other words, if $p \in I(\phi_o(Y))$, then $p(1, x_1, \ldots, x_n) \in I(Y)$, which implies that $p \in \beta(I(Y))$.

(b) Recall that the twisted cubic is the variety $Y = \{(t, t^2, t^3) : t \in k\}$. We claim that $I(Y) = J = (y - x^2, z - x^3)$. Since $Z(J) = Y$, it suffices to check that J is radical. Notice that $k[x, y, z]/J \cong k[x]$ has no nilpotents, which shows that J is radical. Let $f_1 = y - x^2$ and $f_2 = z - x^3$. We claim also that $\beta f_1 = wy - x^2$ and $\beta f_2 = w^2 z - x^3$ don't generate $I(\overline{Y}) = I(\phi_o(Y))$. Since $\phi_o(Y) = \{[1 : t : t^2 : t^3] : t \in k\}$, we have $wz - xy \in I(\overline{Y})$. But this cannot be generated by βf_1 and βf_2 since the only term involving z in βf_1 and βf_2 is $w^2 z$.

(a) Note that since $\overline{Y} = Z(I_{\mathbf{P}^n}(Y))$, we have that $I_{\mathbf{P}^n}(Y) = I(Z(I_{\mathbf{P}^n}(Y))) = I_{\mathbf{P}^n}(Y)$. Additionally, we have that

$$\beta(I_{\mathbf{A}^n}(Y)) \subset I_{\mathbf{P}^n}(Y) = I_{\mathbf{P}^n}(\overline{Y}).$$

Hence it remains to show the other inclusion. To do so, note that

$$\begin{aligned} I_{\mathbf{P}^n}(Y) &= \{f \in S^h \mid f(1, a_1, \ldots, a_n) \quad (a_1, \ldots, a_n) \in Y\} \\ &= \{f \in S^h \mid \alpha(f) \in I_{\mathbf{A}^n}(Y)\}. \end{aligned}$$

If $f \in I_{\mathbf{P}^n}(Y)$, then $\alpha(f) \in I_{\mathbf{A}^n}(Y)$, in which case $\beta(\alpha(f)) \in \beta(I_{\mathbf{A}^n}(Y))$. Since $\beta(\alpha(f)) = f$, we see that $f \in \beta(I_{\mathbf{A}^n}(Y))$. Therefore $\beta(I_{\mathbf{A}^n}(Y)) = I_{\mathbf{P}^n}(Y) = I_{\mathbf{P}^n}(\overline{Y})$.

(b) Since $Y = \{(t, t^2, t^3) \mid t \in k\}$, we know that in $\mathbf{P}^n$

$$Y = \{(1, t, t^2, t^3) \mid t \in k\}.$$

Recall from Ex 1.2 that $I_{\mathbf{A}^n}(Y) = (x^2 - y, x^3 - z)$. By **(a)**, we know that $I_{\mathbf{P}^n}(\overline{Y}) = \beta(I_{\mathbf{A}^n}(Y))$. Now observe that $xy - z \in I_{\mathbf{A}^n}(Y)$, so that $\beta(xy - z) = xy - zx_0 \in I(\overline{Y})$. However, this polynomial cannot be generated by the ideal

$$(\beta(x^2 - y), \beta(x^3 - z)) = (x^2 - yx_0, x^3 - zx_0^2).$$

Hence we see that $I(\overline{Y}) \neq (\beta(x^2 - y), \beta(x^3 - z))$.

□

2.10 Let $Y \subset \mathbf{P}^n$ be a nonempty algebraic set, and let $\theta\colon \mathbf{A}^{n+1} \setminus \{(0, \cdots, 0\} \to \mathbf{P}^n$ be the map which sends the point with affine coordinates $(a_0, \cdots, a_n)$ to the point with homogeneous coordinates $[a_0 : \cdots : a_n]$. We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \cdots, 0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbf{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \cdots, x_n]$.
(b) $C(Y)$ is irreducible if and only if Y is irreducible.
(c) $\dim C(Y) = \dim Y + 1$.

Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbf{P}^{n+1}$. This is called the *projective cone* over Y.

Solution:

(a) Let J be a set of homogeneous polynomials. Note that

$$\theta^{-1}(Z_{\mathbf{P}}(J)) = Z_{\mathbf{A}^{n+1}}(J) \setminus \{(0, \dots, 0)\}$$

Thus we have that

$$Z_{\mathbf{A}^{n+1}}(J) = \theta^{-1}(Z_{\mathbf{P}^n}(J)) \cup \{(0, \dots, 0)\} = C(Z_{\mathbf{P}^n}(J)).$$

Hence if Y is algebraic, so is $C(Y)$. In addition, if $J = I_{\mathbf{P}^n}(Y)$, then

$$Z_{\mathbf{A}^{n+1}}(I_{\mathbf{P}^n}(Y)) = C(Z_{\mathbf{P}^n}(I_{\mathbf{P}^n}(Y))) = C(Y).$$

Since $I_{\mathbf{P}^n}(Y)$ is a radical ideal, this implies that $I(C(Y)) = I_{\mathbf{P}^n}(Y)$, regarded as an ideal of $k[x_0, \dots, x_n]$.

(b) Since $I_{\mathbf{A}^{n+1}}(C(Y)) = I_{\mathbf{P}^n}(Y)$, the result follows immediately.

(c) Consider a maximal chain of closed, irreducible subsets of Y in $\mathbf{P}^n$.

$$Y_0 \subset Y_1 \subset \cdots \subset Y_r = Y.$$

By **(a)** and **(b)**, these are in bijection with closed, irreducible subsets in $\mathbf{A}^{n+1}$:

$$C(Y_0) \subset C(Y_1) \subset \cdots \subset C(Y_r) = C(Y).$$

We can extend the chain by appending the single point $\{(0, \dots, 0)\}$:

$$\{(0, \dots, 0)\} \subset C(Y_0) \subset C(Y_1) \subset \cdots \subset C(Y_r) = C(Y).$$

Hence, $\dim C(Y) = \dim(Y) + 1$.

□

2.11 A hypersurface defined by a linear polynomial is called a *hyperplane*.

(a) Show that the following two conditions are equivalent for a variety $Y \subset \mathbf{P}^n$:
($i.$) $I(Y)$ can be generated by linear polynomials
($ii.$) Y can be written as an intersection of hyperplanes.
In this case we say that Y is a *linear variety* in $\mathbf{P}^n$.

(b) If Y is a linear variety of dimension r in $\mathbf{P}^n$, show that $I(Y)$ is minimally generated by $n-r$ linear polynomials

(c) Let Y, Z be linear varieties in $\mathbf{P}^n$, with $\dim Y = r$, $\dim Z = s$. If $r + s - n \geq 0$, then $Y \cap Z \neq \emptyset$. Furthermore, if $Y \cap Z \neq \emptyset$, then $Y \cap Z$ is a linear variety of dimension $\geq r + s - n$ (Think of $\mathbf{A}^{n+1}$ as a vector space over k, and work with its subspaces.)

Solution:

(a) Let $H_i = Z(f_i)$ be hyperplanes, f_i a linear homogeneous polynomial, for $i = 1, 2, \ldots, r$. Then

$$Y = H_1 \cap \cdots \cap H_r = Z(f_1) \cap \cdots \cap Z(f_r) \iff I(Y) = I(Z(f_1) \cap \cdots Z(f_r)) = I(Z(f_1, \ldots, f_r)).$$

However, since $f_1, \ldots, f_n$ are homogeneous polynomials of degree 1, it is prime. Therefore, $I(Z(f_1, \ldots, f_r)) = (f_1, \ldots, f_r) \implies I(Y) = (f_1, \ldots, f_r)$. Hence (*i.*) and (*ii.*) are equivalent.

(b) By Theorem 1.8A, we know that since Y is a linear varitety,

$$\mathrm{ht}(I(Y)) + \dim(k[x_0, \ldots, x_n]/I(r)) = n + 1 \implies \mathrm{ht}(I(Y)) + r + 1 = n + 1 \implies \mathrm{ht}(I(Y)) = n - r.$$

By **(a)**, we know that $I(Y)$ is generated by some linear homogeneous polynomials $f_1, \ldots, f_s$. This in turn creates a chain of prime ideals:

$$(0) \subset (f_1) \subset (f_1, f_2) \subset \cdots \subset (f_1, \ldots, f_s) = I(Y).$$

Since $\mathrm{ht}(I(Y)) = n - r$, we see that some of the generators must be redundant, so that $I(Y)$ is minimally generated by $n - r$ elements.

(c) Since $r + s - n \geqslant 0$, we can conclude that $(n-r) + (n-s) \leqslant n$. This implies that $Y \cap Z$ is a system of equations with less equations than unknowns. From linear algebra, we know that if such a system is homogeneous (i.e., set equal to zeroes), then there are infinitely many solutions. Therefore,

$$Y \cap Z = Z(f_1, \ldots, f_{n-s}) \cap Z(g_1, \ldots, g_{n-r}) = Z(f_1, \ldots, g_{n-r}) \neq \varnothing.$$

In addition, we also see that this is a linear variety as claimed. Further, we know by **(a)** that $n - \dim(Y \cap Z) \leqslant (n - r) + (n - s) \implies \dim(Y \cap Z) \geqslant r + s - n$, as desired.

□

2.12 For given $n, d > 0$, let $M_0, \ldots, M_N$ be all the monomials of degree d in the $n + 1$ variables $x_0, \ldots x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d \colon \mathbf{P}^n \to \mathbf{P}^N$ by sending the point $P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbf{P}^n$ in $\mathbf{P}^N$. For example, if $n = 1, d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ is a conic.

(a) Let $\theta \colon k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ be the homorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbf{P}^N$.

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$.

(c) Now show that ρ_d is a homeomorphism of $\mathbf{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbf{P}^3$ *(Ex.* 2.9*)* is equal to the 3-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

Solution:

(a)

(b) We prove the harder direction that $Z(\mathfrak{a}) \subseteq \mathrm{im}(\rho_d)$. We may index the $N + 1$ coordinates of a point in $\mathbf{P}^n$ by tuples of the form $(a_0, a_1, \ldots, a_n)$ where $a_i \in \mathbb{Z}_+$ and the sum of all a_i's is d. Given $\mathbf{y} \in Z(\mathfrak{a})$, I claim that there exists $0 \leq i \leq n$ such that $y_{d\mathbf{e}_i} \neq 0$. Indeed, suppose towards the contrary. Then since for any index $\mathbf{v} = (a_0, a_1, \ldots, a_n)$, $p(\mathbf{y}) = y_{\mathbf{v}}^d - \prod_{i=0}^n y_{d\mathbf{e}_i}^{a_i} \in \mathfrak{a}$, we have that $p(\mathbf{y}) = y_{\mathbf{v}}^d = 0$, which implies that $y_{\mathbf{v}} = 0$ for arbitrary $\mathbf{v}$, which is absurd.

To give an example to make this proof clearer, consider the example when $n = 1$ and $d = 3$. Then if $\mathbf{y} = (y_{30}, y_{21}, y_{12}, y_{03}) \in Z(\mathfrak{a})$, suppose WLOG that $y_{30} = 1$. Then $\mathbf{y} = \rho_d(1, y_{21})$. We check that since $y_{30}y_{12} - y_{21}^2 = 0$, indeed $y_{12} = y_{21}^2$. Similarly, since $y_{03}y_{30}^2 - y_{21}^3$, indeed $y_{03} = y_{21}^3$.

(c) The map ρ_d is clearly a bijection between $\mathbf{P}^n$ and $\mathrm{im}\, \rho_d = Z(\mathfrak{a})$. So it suffices to show that ρ_d is bicontinuous, or equivalently, that it identifies the closed sets in $\mathbf{P}^n$ and $Z(\mathfrak{a})$.

(ρ_d continuous.) We claim that for any ideal $I \subset k[y_0, \ldots, y_N]$,

$$\rho_d^{-1}(Z(I)) = Z(\theta(I)).$$

Notice that if $(x_0, \ldots, x_n) \in \rho_d^{-1}(Z(I))$, then $p(M_0(\mathbf{x}), \ldots, M_N(\mathbf{x})) = 0$ for all $p(y_0, \ldots, y_N) \in I$. If $(x_0, \ldots, x_n) \in Z(\theta(I))$, then for all $p(y_0, \ldots, y_N) \in I$, $\theta(p)(x_0, \ldots, x_n) = 0$. But $\theta(p)(x_0, \ldots, x_n) = p(M_0(\mathbf{x}), \ldots, M_N(\mathbf{x}))$. So these two conditions are equivalent.

(ρ_d^{-1} continuous.) We claim that for any ideal $J \subset k[x_0, \ldots, x_n]$,

$$\rho_d(Z(J)) = Z(\theta^{-1}J) \cap Z(\mathfrak{a}).$$

Indeed, if $\mathbf{y} = \rho_d(\mathbf{x})$ where $\mathbf{x} \in Z(J)$, then for any $p \in \theta^{-1}J$, $p(\mathbf{y}) = p(\rho_d(\mathbf{x})) = 0$ because $p \circ \rho_d = \theta(p) \in J$. Conversely, if $\mathbf{y} \in Z(\theta^{-1}J) \cap Z(\mathfrak{a})$, then by part (ii), since $Z(\mathfrak{a}) = \operatorname{im} \rho_d$, there exists $\mathbf{x}$ such that $\mathbf{y} = \rho_d(\mathbf{x})$. We check that $\mathbf{x} \in Z(J)$: for all q such that $q \circ \rho_d \in J$, we have that $q(\mathbf{y}) = q(\rho_d(\mathbf{x})) = 0$. In other words, for all $p \in J$, $p(\mathbf{x}) = 0$.

(a) First, it is prime since θ maps into an integral domain. Now recall that we may uniquely express any multivariate polynomial in $y_1, \ldots, y_N$ into homogeneous components. Since $y_i \notin \ker(\theta)$, no single monomial of any degree in $k[y_1, \ldots, y_N]$ maps to zero. Therefore, if for a multivariate polynomial f we have that $\theta(f) = 0$, then it must be that each of the homogeneous components of $\theta(f_i)$ must map to zero, by individually canceling each other out (in their own degree). In other words, a polynomial is in $\mathfrak{a}$ if and only if its homogeneous components are in $\mathfrak{a}$. Thus, $\mathfrak{a}$ is homogeneous.

(b) $\left[\operatorname{im}(\rho_d) \subset Z(\mathfrak{a})\right]$. If $q \in \operatorname{im}(\rho_d)$, then $q = (M_0(P), \ldots, M_N(P))$ for some $P \in \mathbf{P}^n$, and so for any $f \in \mathfrak{a}$, we have that $f(q) = f(M_0(P), \ldots, M_N(P)) = 0$. Therefore, $q \in Z(\mathfrak{a})$.

$\left[Z(\mathfrak{a}) \subset \operatorname{im}(\rho_d)\right]$. Denote the coordinate y_m to be

$$y_m = M_m(x_0, \ldots, x_n) = x_0^{\alpha_1^m} \cdots x_n^{\alpha_n^m}$$

where α_i^m are nonnegative and sum to d. In particular, denote

$$y_0 = x_0^d, \quad y_1 = x_1^d, \quad \ldots, \quad y_n = x_n^d.$$

We make some observations.

- Observe that $y_m^d - y_0^{\alpha_0^m} \cdots y_n^{\alpha_n^m} \in \mathfrak{a}$. The polynomial is nonzero when $m > n$. Thus if $Q = (b_0, \ldots, b_n, b_{n+1}, \ldots, b_N) \in Z(\mathfrak{a})$, then for each $n < m \leqslant N$, we see that $b_m^d = b_0^{\alpha_0^m} \cdots b_n^{\alpha_n^m}$, so that at least one $b_0, \ldots, b_n \neq 0$.
- Since one of $b_0, \ldots, b_n \neq 0$, suppose that it is b_0. Let $c \in \mathbf{P}^N$ be such that $c_i = b_0^{d-1} b_i$. Then observe that

$$\begin{aligned} M_m(c_0, \ldots, c_n) &= (c_0)^{\alpha_1^m} \cdots (c_n)^{\alpha_n^m} \\ &= (b_0^{d-1} b_0)^{\alpha_1^m} \cdots (b_0^{d-1} b_n)^{\alpha_n^m} \\ &= (b_0^{d-1})^d (b_0)^{\alpha_1^m} \cdots (b_n)^{\alpha_n^m} \\ &= (b_0^{d-1})^d b_m. \end{aligned}$$

  Hence,

$$\begin{aligned} (b_0, \ldots, b_n, \ldots, b_N) &= \left((b_0^{d-1})^d \cdot b_0, \ldots, (b_0^{d-1})^d \cdot b_n, \ldots, (b_0^{d-1})^d \cdot b_N\right) \\ &= (M_0(c_0, \ldots, c_n), \ldots, M_n(c_0, \ldots, c_n), \ldots, M_N(c_0, \ldots, c_n)). \end{aligned}$$

  Therefore, $(b_0, \ldots, b_N) \in \operatorname{im}(\rho_d)$.

(c) We show it is a homeomorphism. First, it is surjective onto $Z(\mathfrak{a})$. By our work from the last part, it is also injective, for we constructed the map

$$\rho_d^{-1} : Z(\mathfrak{a}) \to \mathbf{P}^n \qquad (b_0, \ldots, b_n, b_{n+1}, \ldots, b_N) \mapsto ((b_0^{d-1})^d b_0, \ldots, (b_0^{d-1})^d b_n).$$

We now show that ρ_d is closed. Suppose $Y \subset \mathbf{P}^n$ is closed, and that $Y = Z(T)$ for some family of homogeneous polynomials in $(n+1)$-variables. Then

$$\rho_d(Y) = \left\{ (M_0(P), \ldots, M_N(P)) \;\middle|\; P \in Y \right\}.$$

Note that $\rho_d(Y) \subset Z(\mathfrak{a}) \cap Z(T)$. In addition, if $(b_0, \ldots, b_N) \in Z(\mathfrak{a}) \cap Z(T)$, then $(b_1, \ldots, b_n) \in Z(T) \subset \mathbf{P}^n$. Now $b_1, \ldots, b_n$ completely determine the rest of the values $(b_{n+1}, \ldots, b_N)$, so we have that $Z(\mathfrak{a}) \cap Z(T) \subset \rho_d(Z(T))$. Hence, we see that $\rho_d(Z(T)) = Z(\mathfrak{a}) \cap Z(T)$ is closed.

We now show ρ_d is continuous. Let $Z(\mathfrak{a}) \cap Z(T)$ be a closed set with T a family of homogeneous polynomials in $N+1$ variables. If $P = (b_0, \ldots b_n, \ldots, b_N) \in Z(\mathfrak{a}) \cap Z(T)$, then

$$\begin{aligned} f(b_0, \ldots, b_n, \ldots, b_N) = 0 &\implies f(M_0(c_0, \ldots, c_n), \ldots M_N(c_0, \ldots, c_n)) \\ &\implies \theta(f)(c_0, \ldots, c_n) = 0. \end{aligned}$$

Hence we see that $Z(\theta(T)) \subset \rho^{-1}(Z(\mathfrak{a}) \cap Z(T))$. As the other direction is immediate, we see that $\rho^{-1}(Z(\mathfrak{a}) \cap Z(T)) = Z(\theta(T))$ and so ρ_d is continous.

□

2.13 Let Y be the image of the 2-uple embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$. This is the *Veronese surface*. If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbf{P}^5$ such that $V \cap Y = Z$.

Solution:

□

2.14 Let $\psi\colon \mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a subvariety of $\mathbf{P}^N$.

Solution: We follow the hint. Let $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant s$. Let $\mathfrak{a}$ be the kernel of the mapping below.

$$\theta : k[z_{(i,j)}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s] \qquad z_{(i,j)} \mapsto x_i y_j$$

Now consider a point $\psi\big((a_0, \ldots, a_r), (b_0, \ldots, b_s)\big) = (a_0 b_0, \ldots, a_r b_s)$ in $\mathbf{P}^N$. If $f \in \mathfrak{a}$, then we see that

$$f(a_0 b_0, \ldots, a_r b_s) = \theta(f)((a_0, \ldots, a_r), (b_0, \ldots, b_s)) = 0.$$

Hence we see that $\operatorname{im}(\psi) \subset Z(\mathfrak{a})$. Now let $(b_0, \ldots, b_N) \in Z(\mathfrak{a})$. We can denote this tuple as below

$$(b_0, \ldots, b_N) = (b_{(0,0)}, \ldots, b_{(r,s)}).$$

Observe that $z_{(0,0)} z_{(i,j)} - z_{(i,0)} z_{(j,0)} \in \mathfrak{a}$. Hence, we see that $b_{(0,0)} b_{(i,j)} = b_{(i,0)} b_{(j,0)}$. We the have that

$$\begin{aligned}
(b_{(0,0)}, \ldots, b_{(r,s)}) &= (b_{(0,0)} \cdot b_{(0,0)}, \ldots, b_{(0,0)} \cdot b_{(r,s)}) \\
&= (b_{(0,0)} \cdot b_{(0,0)}, \ldots, b_{(r,0)} \cdot b_{(s,0)}) \\
&= \psi\big((b_{(0,0)}, \ldots, b_{(0,s)}), (b_{(0,0)}, \ldots, b_{(r,0)})\big).
\end{aligned}$$

Therefore, $(b_0, \ldots, b_N) \in \operatorname{im}(\psi)$. This then proves that $\operatorname{im}(\psi) = Z(\mathfrak{a})$. As $\mathfrak{a}$ is prime, we see that $\operatorname{im}(\psi)$ is a variety, specifically a subset of $\mathbf{P}^n$ which is a variety.

□

2.15 Consider the surface Q (a *surface* is variety of dimension 2) in $\mathbf{P}^3$ defined by the equation $xy - zw = 0$.

- **(a)** Show that Q is equal to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.
- **(b)** Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbf{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, $M_t \cap M_u = \emptyset$, and for all $t, u, L_t \cap M_u =$ one point.
- **(c)** Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbf{P}^1 \times \mathbf{P}^1$ (where each $\mathbf{P}^1$ has its Zariski topology).

Solution:

□

2.16

- **(a)** The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbf{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.
- **(b)** Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbf{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

Solution:

□

2.17 A variety Y of dimension r in $\mathbf{P}^n$ is a *(strict) complete intersection* if $I(Y)$ can be generated by $n - r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.

- **(a)** Let Y be a variety in $\mathbf{P}^n$, let $Y = Z(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geqslant n - q$.
- **(b)** Show that a strict complete intersection is a set-theoretic complete intersection.

(c) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbf{P}^3$ *(Ex. 2.9)*. Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees $2, 3$ respectively, such that $Y = H_1 \cap H_2$.

(d) It is an unsolved problem whether every closed irreducible curve in $\mathbf{P}^3$ is a set-theoretic intersection of two surfaces.

Solution:

(a) If $\mathfrak{a} =$

$\square$

3.15

Solution:

(a) As hinted, let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$ for $i = 1, 2$. We first prove that $X = X_1 \cup X_2$. If this doesn't happen, then there exists $y_1, y_2 \in Y$ such that $(x, y_1) \in Z_1 \setminus Z_2$, $(x, y_2) \in Z_2 \setminus Z_1$. However, this implies that $Z_1 \cap x \times Y$ and $Z_2 \cap x \times Y$ are both (non-empty) proper subsets of $x \times Y$. But they are both closed, and $\{x\} \times Y$ is irreducible, which is a contradiction.

$\square$