Exercise 17.

Solution: Recall from Exercise 14 that

$$\varinjlim M_i \cong C/D = \left(\bigoplus_{i \in I} M_i \right) / D,$$

where D is the submodule of $C = \bigoplus_{i \in I} M_i$ generated by all elements of the form $x_i - \mu_{ij}(x_i)$. Also recall that $\sum M_i$ is the set of all finite sums. Consider the map

$$\phi : \bigoplus_{i \in I} M_i \to \sum M_i.$$

For the first isomorphism

$$\varinjlim M_i \cong \sum M_i,$$

it suffices to show that $\ker \phi = D$.

(a) Clearly $\phi(x_i - \mu_{ij}(x_i)) = 0$. So $D \subseteq \ker \phi$.

(b) Conversely, let $x = (x_i)_{i \in I} \in C$. Then $x_i \neq 0$ for finitely many $i \in I$. Therefore, there exists $k \in I$ such that if $x_i \neq 0$, then $M_i \subseteq M_k$. Since $\phi(x) = 0$, we must have

$$x_k = \sum_{i \in I, i \neq k, x_i \neq 0} \mu_{ik}(-x_i).$$

Therefore, we may write

$$x = (x_i)_{i \in I} = x_k + \sum_{i \in I, i \neq k, x_i \neq 0} x_i = \sum_{i \in I, i \neq k, x_i \neq 0} (\mu_{ik}(-x_i) + x_i) \in D.$$

The second isomorphism $\sum M_i \cong \bigcup M_i$ follows from the fact that a finite sum of elements in various M_i's may be rewritten as a sum of elements in some M_k.

□

Exercise 21.

Solution: By Exercise 14, A is a $\mathbb{Z}$-module and the mappings $\mu_i : A_i \to A$ are $\mathbb{Z}$-module homomorphisms. It remains to endow A with a multiplicative structure, check that it is compatible with the addition in A, and check that the mappings $\mu_i : A_i \to A$ respect multiplication.

Let $x \in A$. Recall from Exercise 15 that there exists $i \in I$ and $x_i \in A_i$ such that $x = \mu_i(x_i)$. Therefore, given $x, y \in A$, there exists i and $x_i, y_i \in A_i$ such that $x = \mu_i(x_i)$ and $y = \mu_i(y_i)$. Thus, define $xy = \mu_i(x_i y_i)$.

Claim 0.1. This multiplication is well-defined.

Proof. Since the system is directed, it suffices to show that if $i \leq k$ and $x = \mu_i(x_i) = \mu_k(x_k)$, $y = \mu_i(y_i) = \mu_k(y_k)$, then $\mu_i(x_i y_i) = \mu_k(x_k y_k)$.

By Exercise 15, $m \in \ker \mu_k$ if and only if there exists $j \geq k$ such that $\mu_{kj}(m) = 0$. In other words,

$$\ker \mu_k = \bigcup_{k \leq j} \ker \mu_{kj}.$$

Since μ_{kj} is a ring homomorphism, the right hand side is a union of ideals in A_k, and so the left hand side is also an ideal of A_k.

Since $\mu_k(\mu_{ik}(x_i) - x_k) = \mu_i(x_i) - \mu_k(x_k) = x - x = 0$, we have that $\mu_{ik}(x_i) - x_k \in \ker \mu_k$. Similarly, $\mu_{ik}(y_i) - y_k \in \ker \mu_k$. Now we are ready to show the desired equality.

$$\begin{aligned}
\mu_k(x_k y_k) - \mu_i(x_i y_i) &= \mu_k(x_k y_k - \mu_{ik}(x_i)\mu_{ik}(y_i)) \\
&= \mu_k((x_k - \mu_{ik}(x_i))y_k + \mu_{ik}(x_i)(y_k - \mu_{ik}(y_i))) \\
&= \mu_k((x_k - \mu_{ik}(x_i))y_k) + \mu_k(\mu_{ik}(x_i)(y_k - \mu_{ik}(y_i))) \\
&= 0,
\end{aligned}$$

where at the last step, we used the property that $\ker \mu_k$ is an ideal. □

It's not hard to check that this multiplication respects addition, and it follows directly from our definition of the multiplication structure of A that the mappings $\mu_i : A_i \to A$ are ring homomorphisms.

Claim 0.2. If $A = 0$, there exists $i \in I$ such that $A_i = 0$.

Proof. Choose some $j \in I$ and suppose that $A_j \neq 0$. Let 1_j be the identity element of A_j. If $A = 0$, we have $\mu_j(1_j) = 0$. By Exercise 15, there exists $k \geq j$ such that $\mu_{jk}(1_j) = 0$. Since μ_{jk} is a ring homomorphism, this is only possible if $A_k = 0$. □

□

Exercise 22.

Solution:

$\square$