

**Exercise 2.1** Prove the “homogeneous Nullstellensatz,” which says if  $\mathfrak{a} \subseteq S$  is a homogeneous ideal, and if  $f \in S$  is a homogeneous polynomial with  $\deg f > 0$ , such that  $f(P) = 0$  for all  $P \in Z(\mathfrak{a})$  in  $\mathbf{P}^n$ , then  $f^q \in \mathfrak{a}$  for some  $q > 0$ .

*Solution:* Let  $\mathfrak{a}$  be a homogeneous ideal of  $k[x_0, \dots, x_n]$ . Then in  $\mathbf{P}^n$ , we have that

$$Z(\mathfrak{a}) = \left\{ P \in \mathbf{P}^n \mid \text{For all } f \in \mathfrak{a}, f(P) = 0 \right\}$$

while in  $\mathbf{A}^{n+1}$ , we have that

$$Z'(\mathfrak{a}) = \left\{ Q \in \mathbf{A}^{n+1} \mid \text{For all } f \in \mathfrak{a}, f(Q) = 0 \right\}.$$

We can write a surjective map  $\pi : Z'(\mathfrak{a}) \rightarrow Z(\mathfrak{a})$  sending an affine point to its projective equivalence class. In addition, for a projective point  $P \in Z(\mathfrak{a})$ , we can observe that  $\pi^{-1}(P) \subset Z'(\mathfrak{a})$  (the elements of  $P$ 's equivalence class).

Thus, if  $f$  is homogeneous with a nonzero degree and  $f(P) = 0$  for all  $P \in Z(\mathfrak{a})$ , then  $f(Q) = 0$  for all  $Q \in Z'(\mathfrak{a})$ . By the usual Nullstellensatz, this implies that  $f^q \in \mathfrak{a}$  for some  $q > 0$ , which proves the result.  $\square$

**Exercise 2.2** For a homogeneous ideal  $\mathfrak{a} \subseteq S$ , show that the following conditions are equivalent:

- (i.)  $Z(\mathfrak{a}) = \emptyset$  (the empty set);
- (ii.)  $\sqrt{\mathfrak{a}} = S$  or the ideal  $S_+ = \bigoplus_{d>0} S_d$ ;
- (iii.)  $\mathfrak{a} \supseteq S_d$  for some  $d > 0$ .

*Solution:*

- (i)  $\implies$  (ii): Suppose  $\mathfrak{a}$  is a homogeneous ideal and  $Z(\mathfrak{a}) = \emptyset$ . By the homogeneous Nullstellensatz, it is vacuously true that for every  $f \in \bigoplus_{d>0} S_d$ ,  $f(P) = 0$  for all  $P \in Z(\mathfrak{a})$ , and so for every such  $f$  there exists a  $q > 0$  such that  $f^q \in \mathfrak{a}$ . Therefore,  $\bigoplus_{d>0} S_d \subseteq \sqrt{\mathfrak{a}}$ . If  $\mathfrak{a}$  contains a unit, then  $\sqrt{\mathfrak{a}} = S$ . If  $\mathfrak{a}$  does not contain a unit, then  $\sqrt{\mathfrak{a}} \subseteq \bigoplus_{d>0} S_d \implies \bigoplus_{d>0} S_d = \sqrt{\mathfrak{a}}$ .

- (ii)  $\implies$  (iii): Suppose  $\sqrt{\mathfrak{a}} = S$ . Then this implies that  $1 \in \mathfrak{a} \implies \mathfrak{a} = S$ . Hence  $S_d \subset \mathfrak{a}$ .

Alternatively, suppose  $\sqrt{\mathfrak{a}} = \bigoplus_{d>0} S_d$ . Then for every  $f \in \bigoplus_{d>0} S_d$ , there exists a  $q > 0$  such that  $f^q \in \mathfrak{a}$ . In particular, there exist  $r_1, \dots, r_n$  such that

$$x_1^{r_1}, \dots, x_n^{r_n} \in \mathfrak{a}.$$

Take  $d = r_1 + \dots + r_n$ . We claim that  $S_d \subset \mathfrak{a}$ . To see this, note that every degree- $d$  homogeneous polynomial is of the form

$$\sum_{k \geq 0} c_k x_1^{\alpha_1(k)} \dots x_n^{\alpha_n(k)}$$

where only finitely many summands are nonzero and  $\alpha_1(k) + \dots + \alpha_n(k) = d$  is a sum of nonnegative integers. Since  $d = r_1 + \dots + r_n$ , we know that for any summand  $x_1^{\alpha_1(k)} \dots x_n^{\alpha_n(k)}$  at least one  $\alpha_i(k) \geq r_i$ . Hence, the summand is in  $\mathfrak{a}$ , and so the whole sum is in  $\mathfrak{a}$ . Therefore  $S_d \subseteq \mathfrak{a}$ .

- (iii)  $\implies$  (i): Suppose  $S_d \subseteq \mathfrak{a}$  for  $d > 0$ . Then  $Z(\mathfrak{a})$  must at least contain points  $P$  such that  $f(P) = 0$  for all  $f \in S_d$ . This includes the polynomials  $x_1^d, \dots, x_n^d$ . However, these  $n$  polynomials cannot all be simultaneously zero. Hence  $Z(\mathfrak{a}) = \emptyset$ .

(i)  $\implies$  (ii). Since  $Z(\mathfrak{a}) = \emptyset$ , the zero set of  $\mathfrak{a}$  in affine space Did you mean projective space? is either  $\emptyset$  or  $\{0\}$ . In the first case, we certainly have  $\sqrt{\mathfrak{a}} = S$ . In the second case, we have  $\sqrt{\mathfrak{a}} = \{p \in S : p(0) = 0\} = S_+$ .

(ii)  $\implies$  (iii). It suffices to show that there exists  $d$  such that all monomials of degree  $d$  lies in  $\mathfrak{a}$ . Since  $S_+ \subset \sqrt{\mathfrak{a}}$ , for each  $0 \leq i \leq n$ , there exists  $d_i$  such that  $x_i^{d_i} \in \mathfrak{a}$ . Let  $d = n \sum_{i=0}^n d_i$ . Then if a monomial has degree  $d$ , then there exists  $i$  such that the exponent of  $x_i$  in the monomial is at least  $\sum_{i=0}^n d_i$ , and therefore at least  $d_i$ , which implies that the monomial is in  $\mathfrak{a}$ .

(iii)  $\implies$  (i). If  $S_d \subseteq \mathfrak{a}$  for some  $d > 0$ , then  $x_i \in \sqrt{\mathfrak{a}}$  for every  $0 \leq i \leq n$ , which means that  $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) = \emptyset$ .  $\square$

**Exercise 2.3**

- (a) If  $T_1 \subseteq T_2$  are subsets of  $S^h$ , then  $Z(T_1) \supseteq Z(T_2)$ .
- (b) If  $Y_1 \subseteq Y_2$  are subsets of  $\mathbf{P}^n$ , then  $I(Y_1) \supseteq I(Y_2)$ .

- (c) For any two subsets  $Y_1, Y_2$  of  $\mathbf{P}^n$ ,  $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$ .
- (d) If  $\mathfrak{a} \subseteq S$  is a homogeneous ideal with  $Z(\mathfrak{a}) \neq \emptyset$ , then  $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ .
- (e) For any subset  $Y \subseteq \mathbf{P}^n$ ,  $Z(I(Y)) = \overline{Y}$ .

*Solution:* I did (a) – (c), but I don't feel like  $\text{\TeX}$ -ing them.

- (d) Suppose  $f$  is homogeneous and  $f \in I(Z(\mathfrak{a}))$ . Then  $f(P) = 0$  for all  $P \in Z(\mathfrak{a})$ . By the homogeneous Nullstellensatz, we see that  $f \in \sqrt{\mathfrak{a}}$ .  
Suppose  $f \in \mathfrak{a}$ . Then  $f^q \in \mathfrak{a}$  for some  $q > 0$ . As  $f^q(P) = 0$  for all  $P \in Z(\mathfrak{a})$ ,  $f(P) = 0$  for all  $P \in Z(\mathfrak{a})$  (as  $k$  is an integral domain) and so it follows that  $f \in I(Z(\mathfrak{a}))$ . Our total work then shows that  $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ , as desired.
- (e) We first know that  $Y \subset Z(I(Y))$ ; we show that it is the smallest closed set containing  $Y$ . Thus let  $Z(J)$  be a closed set where  $Y \subset Z(J) \subset I(Z(Y))$ . Then  $I(Z(J)) \subset I(Y)$ . Since  $I(Z(J)) = \sqrt{J}$ , we see that  $J \subset I(Y)$ . Therefore,  $Z(I(Y)) \subset Z(J)$ . Hence,  $Z(I(Y)) = Z(J)$ , and so  $Z(I(Y)) = \overline{Y}$ .

□

#### Exercise 2.4

- (a) There is a one-to-one inclusion-reversing correspondence between algebraic sets in  $\mathbf{P}^n$  and homogeneous radical ideals of  $S$  not equal to  $S_+$  given by  $Y \mapsto I(Y)$  and  $\mathfrak{a} \mapsto Z(\mathfrak{a})$ . *Note:* Since  $S_+$  does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of  $S$ .
- (b) An algebraic set  $Y \subseteq \mathbf{P}^n$  is irreducible if and only if  $I(Y)$  is a prime ideal.
- (c) Show that  $\mathbf{P}^n$  itself is irreducible.

*Solution:*

□

#### Exercise 2.5

- (a)  $\mathbf{P}^n$  is a noetherian topological space.
- (b) Every algebraic set in  $\mathbf{P}^n$  can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

*Solution:*

□

**2.6** If  $Y$  is a projective variety with homogeneous coordinate ring  $S(Y)$ , show that  $\dim S(Y) = \dim Y + 1$ . [*Hint:* Let  $\phi_i : U_i \rightarrow \mathbf{A}^n$  be the homeomorphism of (2.2), let  $Y_i$  be the affine variety  $\phi_i(Y \cap U_i)$  and let  $A(Y_i)$  be its affine coordinate ring. Show that  $A(Y_i)$  can be identified with the subring of elements of degree 0 of the localized ring  $S(Y)_{x_i}$ . Then show that  $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$ . Now use (1.7), (1.8A), and (Ex 1.10), and look at the transcendence degrees. Conclude also that  $\dim Y = \dim Y_i$  whenever  $Y_i$  is empty.]

*Solution:*

□

#### 2.7

- (a)  $\dim \mathbf{P}^n = n$ .
- (b) If  $Y \subseteq \mathbf{P}^n$  is a quasi-projective variety, then  $\dim Y = \dim \overline{Y}$ .

*Solution:*

□

**2.8** A projective variety  $Y \subseteq \mathbf{P}^n$  has dimension  $n - 1$  if and only if it is the zero set of a single irreducible homogeneous polynomial  $f$  of positive degree.  $Y$  is called a *hypersurface* in  $\mathbf{P}^n$ .

*Solution:*

□

**2.9** If  $Y \subseteq \mathbf{A}^n$  is an affine variety, we identify  $\mathbf{A}^n$  with an open set  $U_0 \subset \mathbf{P}^n$  by the homeomorphism  $\varphi_0$ . Then we can speak of  $\overline{Y}$ , the closure of  $Y$  in  $\mathbf{P}^n$ , which is called the *projective closure* of  $Y$ .

*Solution:*

- (a) Since  $\overline{Y} = Z(I(\phi_o Y))$ ,  $I(\overline{Y}) = I(\phi_o Y)$ . So it suffices to show that  $I(\phi_o Y) = \beta(I(Y))$ . We have  $I(\phi_o Y) \supseteq \beta(I(Y))$  because  $\phi_o(Y) \subseteq Z(\beta(I(Y)))$ . Conversely, we have  $I(\phi_o Y) \subseteq \beta(I(Y))$  because any point of  $Y$  vanishes on  $p(1, x_1, \dots, x_n)$  for  $p \in I(\phi_o(Y))$ . In other words, if  $p \in I(\phi_o(Y))$ , then  $p(1, x_1, \dots, x_n) \in I(Y)$ , which implies that  $p \in \beta(I(Y))$ .
- (b) Recall that the twisted cubic is the variety  $Y = \{(t, t^2, t^3) : t \in k\}$ . We claim that  $I(Y) = J = (y - x^2, z - x^3)$ . Since  $Z(J) = Y$ , it suffices to check that  $J$  is radical. Notice that  $k[x, y, z]/J \cong k[x]$  has no nilpotents, which shows that  $J$  is radical. Let  $f_1 = y - x^2$  and  $f_2 = z - x^3$ . We claim also that  $\beta f_1 = wy - x^2$  and  $\beta f_2 = w^2 z - x^3$  don't generate  $I(\overline{Y}) = I(\phi_o(Y))$ . Since  $\phi_o(Y) = \{[1 : t : t^2 : t^3] : t \in k\}$ , we have  $wz - xy \in I(\overline{Y})$ . But this cannot be generated by  $\beta f_1$  and  $\beta f_2$  since the only term involving  $z$  in  $\beta f_1$  and  $\beta f_2$  is  $w^2 z$ .

□

**2.10** Let  $Y \subset \mathbf{P}^n$  be a nonempty algebraic set, and let  $\theta: \mathbf{A}^{n+1} \setminus \{(0, \dots, 0)\} \rightarrow \mathbf{P}^n$  be the map which sends the point with affine coordinates  $(a_0, \dots, a_n)$  to the point with homogeneous coordinates  $[a_0 : \dots : a_n]$ . We define the *affine cone* over  $Y$  to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \dots, 0)\}.$$

- (a) Show that  $C(Y)$  is an algebraic set in  $\mathbf{A}^{n+1}$ , whose ideal is equal to  $I(Y)$ , considered as an ordinary ideal in  $k[x_0, \dots, x_n]$ .
- (b)  $C(Y)$  is irreducible if and only if  $Y$  is irreducible.
- (c)  $\dim C(Y) = \dim Y + 1$ .

Sometimes we consider the projective closure  $\overline{C(Y)}$  of  $C(Y)$  in  $\mathbf{P}^{n+1}$ . This is called the *projective cone* over  $Y$ .

*Solution:*

□

**2.11** A hypersurface defined by a linear polynomial is called a *hyperplane*.

- (a) Show that the following two conditions are equivalent for a variety  $Y \subset \mathbf{P}^n$ :
  - (i.)  $I(Y)$  can be generated by linear polynomials
  - (ii.)  $Y$  can be written as an intersection of hyperplanes.
 In this case we say that  $Y$  is a *linear variety* in  $\mathbf{P}^n$ .
- (b) If  $Y$  is a linear variety of dimension  $r$  in  $\mathbf{P}^n$ , show that  $I(Y)$  is minimally generated by  $n - r$  linear polynomials
- (c) Let  $Y, Z$  be linear varieties in  $\mathbf{P}^n$ , with  $\dim Y = r$ ,  $\dim Z = s$ . If  $r + s - n \geq 0$ , then  $Y \cap Z \neq \emptyset$ . Furthermore, if  $Y \cap Z \neq \emptyset$ , then  $Y \cap Z$  is a linear variety of dimension  $\geq r + s - n$  (Think of  $\mathbf{A}^{n+1}$  as a vector space over  $k$ , and work with its subspaces.)

*Solution:*

□

**2.12** For given  $n, d > 0$ , let  $M_0, \dots, M_N$  be all the monomials of degree  $d$  in the  $n + 1$  variables  $x_0, \dots, x_n$ , where  $N = \binom{n+d}{d} - 1$ . We define a map  $\rho_d: \mathbf{P}^n \rightarrow \mathbf{P}^N$  by sending the point  $P = (a_0, \dots, a_n)$  to the point  $\rho_d(P) = (M_0(a), \dots, M_N(a))$  obtained by substituting the  $a_i$  in the monomials  $M_j$ . This is called the *d-uple embedding* of  $\mathbf{P}^n$  in  $\mathbf{P}^N$ . For example, if  $n = 1, d = 2$ , then  $N = 2$ , and the image  $Y$  of the 2-uple embedding of  $\mathbf{P}^1$  in  $\mathbf{P}^2$  is a conic.

- (a) Let  $\theta: k[y_0, \dots, y_N] \rightarrow k[x_0, \dots, x_n]$  be the homomorphism defined by sending  $y_i$  to  $M_i$ , and let  $\mathfrak{a}$  be the kernel of  $\theta$ . Then  $\mathfrak{a}$  is homogeneous prime ideal, and so  $Z(\mathfrak{a})$  is a projective variety in  $\mathbf{P}^N$ .
- (b) Show that the image of  $\rho_d$  is exactly  $Z(\mathfrak{a})$ .

- (c) Now show that  $\rho_d$  is a homeomorphism of  $\mathbf{P}^n$  onto the projective variety  $Z(\mathfrak{a})$ .  
 (d) Show that the twisted cubic curve in  $\mathbf{P}^3$  (Ex. 2.9) is equal to the 3-uple embedding of  $\mathbf{P}^1$  in  $\mathbf{P}^3$ , for suitable choice of coordinates.

*Solution:*

- (a)  
 (b) To give an example to make this proof clearer, consider the example when  $n = 1$  and  $d = 3$ . Then if  $\mathbf{y} = (y_{30}, y_{21}, y_{12}, y_{03}) \in Z(\mathfrak{a})$ , suppose WLOG that  $y_{30} = 1$ . Then  $\mathbf{y} = \rho_d(1, y_{21})$ . We check that since  $y_{30}y_{12} - y_{21}^2 = 0$ , indeed  $y_{12} = y_{21}^2$ . Similarly, since  $y_{03}y_{30} - y_{21}^3$ , indeed  $y_{03} = y_{21}^3$ .  
 (c) The map  $\rho_d$  is clearly a bijection between  $\mathbf{P}^n$  and  $\text{im } \rho_d = Z(\mathfrak{a})$ . So it suffices to show that  $\rho_d$  is bicontinuous, or equivalently, that it identifies the closed sets in  $\mathbf{P}^n$  and  $Z(\mathfrak{a})$ .  
 ( $\rho_d$  continuous.) We claim that for any ideal  $I \subset k[y_0, \dots, y_N]$ ,

$$\rho_d^{-1}(Z(I)) = Z(\theta(I)).$$

Notice that if  $(x_0, \dots, x_n) \in \rho_d^{-1}(Z(I))$ , then  $p(M_0(\mathbf{x}), \dots, M_N(\mathbf{x})) = 0$  for all  $p(y_0, \dots, y_N) \in I$ . If  $(x_0, \dots, x_n) \in Z(\theta(I))$ , then for all  $p(y_0, \dots, y_N) \in I$ ,  $\theta(p)(x_0, \dots, x_n) = 0$ . But  $\theta(p)(x_0, \dots, x_n) = p(M_0(\mathbf{x}), \dots, M_N(\mathbf{x}))$ . So these two conditions are equivalent.

( $\rho_d^{-1}$  continuous.) We claim that for any ideal  $J \subset k[x_0, \dots, x_n]$ ,

$$\rho_d(Z(J)) = Z(\theta^{-1}J) \cap Z(\mathfrak{a}).$$

Indeed, if  $\mathbf{y} = \rho_d(\mathbf{x})$  where  $\mathbf{x} \in Z(J)$ , then for any  $p \in \theta^{-1}J$ ,  $p(\mathbf{y}) = p(\rho_d(\mathbf{x})) = 0$  because  $p \circ \rho_d = \theta(p) \in J$ . Conversely, if  $\mathbf{y} \in Z(\theta^{-1}J) \cap Z(\mathfrak{a})$ , then by part (ii), since  $Z(\mathfrak{a}) = \text{im } \rho_d$ , there exists  $\mathbf{x}$  such that  $\mathbf{y} = \rho_d(\mathbf{x})$ . We check that  $\mathbf{x} \in Z(J)$ : for all  $q$  such that  $q \circ \rho_d \in J$ , we have that  $q(\mathbf{y}) = q(\rho_d(\mathbf{x})) = 0$ . In other words, for all  $p \in J$ ,  $p(\mathbf{x}) = 0$ .

□

**2.13** Let  $Y$  be the image of the 2-uple embedding of  $\mathbf{P}^2$  in  $\mathbf{P}^5$ . This is the *Veronese surface*. If  $Z \subseteq Y$  is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface  $V \subseteq \mathbf{P}^5$  such that  $V \cap Y = Z$ .

*Solution:*

□

**2.14** Let  $\psi: \mathbf{P}^r \times \mathbf{P}^s \rightarrow \mathbf{P}^N$  be the map defined by sending the ordered pair  $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$  to  $(\dots, a_i b_j, \dots)$  in lexicographic order, where  $N = rs + r + s$ . Note that  $\psi$  is well-defined and injective. It is called the *Segre embedding*. Show that the image of  $\psi$  is a subvariety of  $\mathbf{P}^N$ .

*Solution:*

□

**2.15** Consider the surface  $Q$  (a *surface* is variety of dimension 2) in  $\mathbf{P}^3$  defined by the equation  $xy - zw = 0$ .

- (a) Show that  $Q$  is equal to the Segre embedding of  $\mathbf{P}^1 \times \mathbf{P}^1$  in  $\mathbf{P}^3$ , for suitable choice of coordinates.  
 (b) Show that  $Q$  contains two families of lines (a *line* is a linear variety of dimension 1)  $\{L_t\}, \{M_t\}$ , each parametrized by  $t \in \mathbf{P}^1$ , with the properties that if  $L_t \neq L_u$ , then  $L_t \cap L_u = \emptyset$ ; if  $M_t \neq M_u$ ,  $M_t \cap M_u = \emptyset$ , and for all  $t, u$ ,  $L_t \cap M_u = \text{one point}$ .  
 (c) Show that  $Q$  contains other curves besides these lines, and deduce that the Zariski topology on  $Q$  is not homeomorphic via  $\psi$  to the product topology on  $\mathbf{P}^1 \times \mathbf{P}^1$  (where each  $\mathbf{P}^1$  has its Zariski topology).

*Solution:*

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- (a) The intersection of two varieties need not be a variety. For example, let  $Q_1$  and  $Q_2$  be the quadric surfaces in  $\mathbf{P}^3$  given by the equations  $x^2 - yw = 0$  and  $xy - zw = 0$ , respectively. Show that  $Q_1 \cap Q_2$  is the union of a twisted cubic curve and a line.
- (b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let  $C$  be the conic in  $\mathbf{P}^2$  given by the equation  $x^2 - yz = 0$ . Let  $L$  be the line given by  $y = 0$ . Show that  $C \cap L$  consists of one point  $P$ , but that  $I(C) + I(L) \neq I(P)$ .

*Solution:*

□

**2.17** A variety  $Y$  of dimension  $r$  in  $\mathbf{P}^n$  is a (*strict*) *complete intersection* if  $I(Y)$  can be generated by  $n - r$  elements.  $Y$  is a *set-theoretic complete intersection* if  $Y$  can be written as the intersection of  $n - r$  hypersurfaces.

- (a) Let  $Y$  be a variety in  $\mathbf{P}^n$ , let  $Y = Z(\mathfrak{a})$ ; and suppose that  $\mathfrak{a}$  can be generated by  $q$  elements. Then show that  $\dim Y \geq n - q$ .
- (b) Show that a strict complete intersection is a set-theoretic complete intersection.
- (c) The converse of (b) is false. For example let  $Y$  be the twisted cubic curve in  $\mathbf{P}^3$  (Ex. 2.9). Show that  $I(Y)$  cannot be generated by two elements. On the other hand, find hypersurfaces  $H_1, H_2$  of degrees 2, 3 respectively, such that  $Y = H_1 \cap H_2$ .
- (d) It is an unsolved problem whether every closed irreducible curve in  $\mathbf{P}^3$  is a set-theoretic intersection of two surfaces.

*Solution:*

- (a) If  $\mathfrak{a} =$

□