

## 3.1

- (a) Show that any conic in  $\mathbf{A}^2$  is isomorphic either to  $\mathbf{A}^1$  or  $\mathbf{A}^1 - \{0\}$  (cf. Ex. 1.1).
- (b) Show that  $\mathbf{A}^1$  is *not* isomorphic to any proper open subset of itself. (This result is generalized by (Ex. 6.7) below.)
- (c) Any conic in  $\mathbf{P}^2$  is isomorphic to  $\mathbf{P}^1$ .
- (d) We will see later (Ex. 4.8) that any two curves are homeomorphic. But show now that  $\mathbf{A}^2$  is not even homeomorphic to  $\mathbf{P}^2$ .
- (e) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

3.2 A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

- (a) For example, let  $\varphi : \mathbf{A}^1 \rightarrow \mathbf{A}^2$  be defined by  $t \mapsto (t^2, t^3)$ . Show that  $\varphi$  defines a bijective bicontinuous morphism of  $\mathbf{A}^1$  onto the curve  $y^2 = x^3$ , but that  $\varphi$  is not an isomorphism.
- (b) For another example, let the characteristic of the base field  $k$  be  $p > 0$ , and define a map  $\varphi : \mathbf{A}^1 \rightarrow \mathbf{A}^2$  by  $t \mapsto t^p$ . Show that  $\varphi$  is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism*.

*Solution:*

- (a) **Bijection:** Since  $k$  is an integral domain, this map is injective. In addition, the map is surjective: if  $(x_0, y_0) \in Z(y^2 - x^3)$  are nonzero, we set  $t_0 = \frac{y_0}{x_0}$ . We then have that  $\varphi(t_0) = (t_0^2, t_0^3) = (x_0, y_0)$ .

**Closed:** Let  $Y = Z(T) \in \mathbf{A}^1$  be a closed set,  $T$  some family of polynomials in  $k[x]$ . Establishing notation, write

$$\psi : k[x, y] \rightarrow k[t] \quad x \mapsto t^2, y \mapsto t^3$$

Now observe that

$$\varphi(Y) = \left\{ (z_0^2, z_0^3) \mid z_0 \in Y \right\} = \left\{ (z_0^2, z_0^3) \mid f(z_0) = 0 \quad \forall f \in T \right\} = Z(y^2 - x^3, T')$$

where  $T' = \{g \in k[x, y] \mid \psi(g) \in T\}$ . Hence  $\varphi$  is a closed.

**Continuous:** Now we show this function is continuous. Let  $W = Z(S) \cap Z(y^2 - x^3)$  where  $S$  is a family of polynomials in  $k[x, y]$ . Then

$$\varphi^{-1}(W) = \left\{ t_0 \in \mathbf{A}^1 \mid f(t_0^2, t_0^3) = 0 \quad \forall f \in T \cap (y^2 - x^3) \right\}.$$

If  $f(t_0^2, t_0^3) = 0$ , then  $\psi(f)(t_0) = 0$ . Hence  $\varphi^{-1}(W) \subseteq Z(\psi(T))$ . The alternative direction is immediate, and we have that  $\varphi^{-1}(W) = Z(\psi(T))$ , so that  $\varphi^{-1}(W)$  is closed.

**Morphism:** We show that this function is actually a morphism of varieties. Consider a regular map  $f : V \subset Z(y^2 - x^3) \rightarrow k$  on some open subset  $V \subset Z(y^2 - x^3)$ . Let  $P \in \varphi^{-1}(V)$ . Since  $f : V \rightarrow k$  is regular, there exists an open set  $U \subseteq V$  containing  $\varphi(p)$  such that

$$f(x, y) = \frac{g(x, y)}{h(x, y)} \quad \forall (x, y) \in U.$$

Now observe that  $\varphi^{-1}(U)$  is an open subset of  $\varphi^{-1}(V)$  containing  $P$ , and moreover that

$$f \circ \varphi(t) = f(t^2, t^3) = \frac{g(t^2, t^3)}{h(t^2, t^3)} \quad \forall t \in \varphi^{-1}(U).$$

Hence,  $f \circ \varphi$  is regular whenever  $f$  is, making  $\varphi$  a morphism of varieties.

We finally show that this is not an isomorphism. This is because

$$k[x, y]/(y^2 - x^3) \not\cong k[x] \implies A(Z(y^2 - x^3)) \not\cong A(\mathbf{A}^1).$$

and this is due to the fact that  $\psi$  is not surjective (e.g.,  $x \notin \text{im}(\psi)$ , although  $\ker(\psi) = I(y^2 - x^3)$ ). Hence, these varieties are not isomorphic.

## 3.3

- (a) Let  $\varphi : X \rightarrow Y$  be a morphism. Then for each  $P \in X$ ,  $\varphi$  induces a homomorphism of local rings  $\varphi_P^* : \mathcal{O}_{\varphi(P), Y} \rightarrow \mathcal{O}_{P, X}$ .
- (b) Show that a morphism  $\varphi$  is an isomorphism if and only if  $\varphi$  is a homeomorphism, and the induced map  $\varphi_P^*$  on local rings is an isomorphism, for all  $p \in X$ .
- (c) Show that if  $\varphi(X)$  is dense in  $Y$ , then the map  $\varphi_P^*$  is injective for all  $p \in X$ .

*Solution:*

- (a) Given a morphism  $\varphi : X \rightarrow Y$  of varieties and a point  $P \in X$ , we define

$$\varphi_P^* : \mathcal{O}_{\varphi(P), Y} \rightarrow \mathcal{O}_{P, X} \quad (U \subset Y, f : U \rightarrow k) \mapsto (\varphi^{-1}(U), f \circ \varphi : \varphi^{-1}(U) \rightarrow k).$$

**Well-defined:** Let  $(U_1, f_1) \sim (U_2, f_2)$  be members of  $\mathcal{O}_{\varphi(P), Y}$ . Then  $f_1 \cap f_2$  on  $U_1 \cap U_2$ , which implies that  $f_1 \circ \varphi = f_2 \circ \varphi$  on  $\varphi^{-1}(U_1 \cap U_2)$ . Hence  $(\varphi^{-1}(U_1), f_1 \circ \varphi) \sim (\varphi^{-1}(U_2), f_2 \circ \varphi)$ .

**Local:** Let  $\mathfrak{m}_{\varphi(P)} = \{(V, f) \in \mathcal{O}_{\varphi(P), Y} \mid f \circ \varphi(p) = 0\}$  be the maximal ideal of  $\mathcal{O}_{\varphi(P), Y}$ . Then observe that  $\varphi_P^*(\mathfrak{m}_{\varphi(P)}) = (\varphi^{-1}(V), f \circ \varphi) \in \mathfrak{m}_P \subset \mathcal{O}_{P, X}$ . Hence, we see that  $\varphi_P^*(\mathfrak{m}_{\varphi(P)}) \subset \mathfrak{m}_P$ , so that  $\varphi_P^*$  is a homomorphism of local rings.

- (b) Let  $\varphi : X \rightarrow Y$  be a homeomorphism such that  $\varphi_P^*$  is an isomorphism for each  $P \in X$ . Let  $f : V \subseteq Y \rightarrow k$  be a regular function on  $V$  containing  $P$ . Then  $f$  corresponds via  $\varphi_P^*$  to the regular function  $f \circ \varphi : \varphi^{-1}(V) \rightarrow k$  on  $\varphi^{-1}(P)$ . Hence,  $\varphi$  is a morphism of varieties. By a similar argument,  $\varphi^{-1}$  is also a morphism of varieties. As they are inverses of each other, we see that  $\varphi$  is an isomorphism of varieties. The other direction is not difficult to check.
- (c)

□

### 3.4 Show that the $d$ -uple embedding of $\mathbf{P}^n$ (Ex. 2.12) is an isomorphism onto its image.

*Solution:* We recall some notation.

- $M_i(x_0, \dots, x_n)$  is the  $i$ -th  $d$ -degree monomial (from 0 to  $N = \binom{n+d}{n} - 1$ )
- $\rho_d : \mathbf{P}^n \rightarrow Z(\mathfrak{a}) \subset \mathbf{P}^N$  sends  $(a_0, \dots, a_n)$  to  $(M_0(a_0, \dots, a_n), \dots, M_N(a_0, \dots, a_n))$
- $\theta : k[y_0, \dots, y_N] \rightarrow k[x_0, \dots, x_n]$  sends  $y_i$  to  $M_i(x_0, \dots, x_n)$ .

From Exercise 2.12, we already know that  $\rho_d : \mathbf{P}^n \rightarrow Z(\mathfrak{a}) \subset \mathbf{P}^N$  is a homeomorphism. We show that  $\rho_d$  and  $\rho_d^{-1}$  are morphisms of varieties and are inverses of each other (as morphisms of varieties).

- We show  $\rho_d$  is a morphism of varieties. Let  $f : V \subseteq Z(\mathfrak{a}) \rightarrow k$  be regular. Then for each  $p \in V$ , there exists an open set  $U \subseteq V$  containing  $p$  such that  $f(y_0, \dots, y_N) = g_1(y_0, \dots, y_N)/g_2(y_0, \dots, y_N)$  on  $U$ , for some homogeneous polynomials  $g_1, g_2$  of the same degree. Consider the function  $f \circ \rho_d : \rho_d^{-1}(U) \rightarrow k$ . Observe that  $\rho_d^{-1}(U)$  is a subset of  $\rho_d^{-1}(V)$  containing  $\rho_d^{-1}(p)$ , and moreover that

$$f \circ \rho_d = \frac{\theta(g_1)}{\theta(g_2)} \text{ on all of } \rho_d^{-1}(U).$$

Note that  $\theta(g_1), \theta(g_2)$  are homogeneous polynomials of the same degree, and moreover that  $\theta(g_2) \neq 0$  on any of  $\rho_d^{-1}(U)$ . Hence,  $f \circ \rho_d$  is regular, which implies that  $\rho_d$  is a morphism of varieties.

- We show  $\rho_d^{-1}$  is a morphism of varieties. Let  $m : V \subseteq \mathbf{P}^n \rightarrow k$  be a regular function. Then for each  $p \in P$ , there exists some open set  $U \subseteq V$  containing  $p$  and a pair of homogeneous polynomials  $h_1, h_2$  of the same degree such that  $m(x_0, \dots, x_n) = h_1(x_0, \dots, x_n)/h_2(x_0, \dots, x_n)$  on all of  $U$ .

Take  $U_i = \mathbf{P}^n - H_i$  such that  $p \in U \cap U_i$ . Let  $\alpha = |d - \deg(h_1)| = |d - \deg(h_2)|$ . Observe that

$$m = \frac{x_i^\alpha h_1(x_0, \dots, x_n)}{x_i^\alpha h_2(x_0, \dots, x_n)} = \frac{h_1'(x_0, \dots, x_n)}{h_2'(x_0, \dots, x_n)}.$$

Both  $h_1'$  and  $h_2'$  are homogeneous polynomials of degree  $\alpha + \deg(h_1) = \alpha + \deg(h_2)$  which is divisible by  $d$ . Hence, we see that both of these polynomials are in the image of  $\theta$ . Let  $k_1, k_2 \in k[y_0, \dots, y_N]$  such that

$\theta(k_1) = h'_1$ ,  $\theta(k_2) = h'_2$ . Then we see that

$$f \circ \rho_d^{-1} = \frac{k_1}{k_2} \text{ on all of } \rho_d(U \cap U_i).$$

Thus we see that  $f \circ \rho_d^{-1} : \rho_d(V) \subset Z(\mathfrak{a}) \rightarrow k$  is a regular function, so that  $\rho_d^{-1}$  is a morphism of varieties.

As both  $\rho_d$  and  $\rho_d^{-1}$  are morphisms of varieties, and are homeomorphisms, we see that  $\rho_d$  establishes the isomorphism  $\mathbf{P}^n \cong Z(\mathfrak{a})$ , as desired.  $\square$

**3.5** By abuse of language, we will say that a variety “is affine” if it is isomorphic to an affine variety. If  $H \subseteq \mathbf{P}^n$  is any hypersurface, show that  $\mathbf{P}^n - H$  is affine. [Hint: Let  $H$  have degree  $d$ . Then consider the  $d$ -uple embedding of  $\mathbf{P}^n$  in  $\mathbf{P}^N$  and use the fact that  $\mathbf{P}^N$  minus a hyperplane is affine.]

*Solution:* Let  $H$ , the hypersurface, be given by the irreducible polynomial  $f$ . Since  $f$  is homogeneous, we may write it as

$$f(x_0, \dots, x_n) = \sum_{i=0}^N a_i M_i(x_0, \dots, x_n).$$

where  $M_i$  is the  $i$ -th homogeneous monomial of degree  $d = \deg(f)$ . If we let  $y_i = M_i$ , then the linear polynomial

$$p(y_0, \dots, y_N) = \sum_{i=0}^N a_i y_i$$

has the property that  $\theta(p(y_0, \dots, y_N)) = f$ , where  $\theta$  substitutes  $y_i$  for  $M_i$ . What this tells us is that  $\rho_d(H) = Z(p)$ , so that  $\rho_d(H)$  is actually a hyperplane. Observe now that

$$\rho_d(\mathbf{P}^n - H) \cong Z(\mathfrak{a}) - \rho_d(H) = (\mathbf{P}^N - \rho_d(H)) \cap Z(\mathfrak{a}).$$

However,  $\mathbf{P}^N - \rho_d(H) \cong \mathbf{A}^{N-1}$  as we know that the  $\mathbf{P}^N - Z(x_i) \cong \mathbf{A}^{N-1}$ , and we can achieve a similar result for general hyperplanes by substituting variables as necessary.  $\square$

**3.6** There are quasi-affine varieties which are not affine. For example, show that  $X = \mathbf{A}^2 - \{(0,0)\}$  is not affine. [Hint: Show that  $\mathcal{O}(X) \cong k[x, y]$  and use (3.5). See (III, Ex. 4.3) for another proof.]

*Solution:* Consider  $f \in \mathcal{O}(X)$ . Let  $p = (x_0, y_0) \in X$ . Since  $f$  is regular on  $X = \mathbf{A}^2 - \{(0,0)\}$ , we know that there is an open set  $U$  of  $p$  such that

$$f(x, y) = \frac{g(x, y)}{h(x, y)} \quad \forall (x, y) \in U.$$

Now  $h(x, y)$  is only allowed to have a root at  $(0,0)$ . Thus, observe that  $h(x_0, y)$  is a single variable polynomial with no solution, as  $x_0 \neq 0$ . Since  $k$  is algebraically closed,  $h(x_0, y)$  must be constant, so that  $h(x, y)$  is just a polynomial in  $x$ . However, we can similarly observe that  $h(x, y)$  would also need to be constant. This all together implies that  $\mathcal{O}(X) \cong k[x, y]$ .

Now suppose  $X$  is affine. Then by (3.5), we know that since  $\mathcal{O}(X) \cong k[x, y] \cong A(\mathbf{A}^2)$ , which implies that  $\mathbf{A}^2 \cong X$ . As this is a contradiction, we see that  $X$  is not affine.  $\square$

### 3.7

- (a) Show that any two curves in  $\mathbf{P}^2$  have a nonempty intersection.
- (b) More generally, show that if  $Y \subseteq \mathbf{P}^n$  is a projective variety of dimension  $\geq 1$ , and if  $H$  is a hypersurface, then  $Y \cap H \neq \emptyset$ . [Hint: Use (Ex. 3.5) and (Ex. 3.1e). See (7.2) for a generalization.]

**3.8** Let  $H_i$  and  $H_j$  be the hyperplanes in  $\mathbf{P}^n$  defined by  $x_i = 0$  and  $x_j = 0$ , with  $i \neq j$ . Show that any regular function on  $\mathbf{P}^n - (H_i \cap H_j)$  is constant. (This gives an alternate proof of (3.4a) in the case  $Y = \mathbf{P}^n$ .)

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**3.9** The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let  $X = \mathbf{P}^1$ , and let  $Y$  be the 2-uple embedding of  $\mathbf{P}^1$  in  $\mathbf{P}^2$ . Then  $X \cong Y$  (Ex. 3.4). But show that  $S(X) \not\cong S(Y)$ .

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**3.10 Subvarieties.** A subset of a topological space is locally closed if it is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.

If  $X$  is a quasi-affine or quasi-projective variety and  $Y$  is an irreducible locally closed subset, then  $Y$  is also a quasi-affine (respectively, quasi-projective) variety, by virtue of being a locally closed subset of the same affine or projective space. We call this the induced structure on  $Y$ , and we call  $Y$  a subvariety of  $X$ .

Now let  $\varphi : X \rightarrow Y$  be a morphism, let  $X' \subseteq X$  and  $Y' \subseteq Y$  be irreducible locally closed subsets such that  $\varphi(X') \subseteq Y'$ . Show that  $\varphi|_{X'} : X' \rightarrow Y'$  is a morphism.

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**3.11** Let  $X$  be any variety and let  $P \in X$ . Show there is a 1-1 correspondence between the prime ideals of the local ring  $\mathcal{O}_P$  and the closed subvarieties of  $X$  containing  $P$ .

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**3.12** If  $P$  is a point on a variety  $X$ , then  $\dim \mathcal{O}_P = \dim X$ . [*Hint:* Reduce to the affine case and use (3.2c).]

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**3.13 The Local Ring of a Subvariety.** Let  $Y \subseteq X$  be a subvariety. Let  $\mathcal{O}_{Y,X}$  be the set of equivalence classes  $\langle U, f \rangle$  where  $U \subseteq X$  is open,  $U \cap Y \neq \emptyset$ , and  $f$  is a regular function on  $U$ . We say  $\langle U, f \rangle$  is equivalent to  $\langle V, g \rangle$  if  $f = g$  on  $U \cap V$ . Show that  $\mathcal{O}_{Y,X}$  is a local ring, with residue field  $K(Y)$  and dimension  $= \dim X - \dim Y$ . It is the local ring of  $Y$  on  $X$ . Note if  $Y = P$  is a point we get  $\mathcal{O}_P$  and if  $Y = X$  we get  $K(X)$ . Note also that if  $Y$  is not a point, then  $K(Y)$  is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

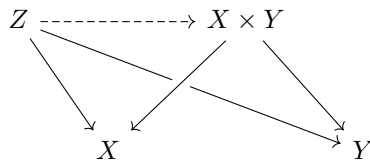
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**3.14 Projection from a Point.** Let  $\mathbf{P}^n$  be a hyperplane in  $\mathbf{P}^{n+1}$  and let  $P \in \mathbf{P}^{n+1} - \mathbf{P}^n$ . Define a mapping  $\varphi : \mathbf{P}^{n+1} - P \rightarrow \mathbf{P}^n$  by  $\varphi(Q) =$  the intersection of the unique line containing  $P$  and  $Q$  with  $\mathbf{P}^n$ .

- (a) Show that  $\varphi$  is a morphism.
  - (b) Let  $Y \subseteq \mathbf{P}^3$  be the twisted cubic curve which is the image of the 3-uple embedding of  $\mathbf{P}^1$  (Ex. 2.12). If  $t, u$  are the homogeneous coordinates on  $\mathbf{P}^1$ , we say that  $Y$  is the curve given parametrically by  $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$ . Let  $P = (0, 0, 1, 0)$ , and let  $\mathbf{P}^2$  be the hyperplane  $z = 0$ . Show that the projection of  $Y$  from  $P$  is a cuspidal cubic curve in the plane, and find its equation.
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**3.15 Products of Affine Varieties.** Let  $X \subseteq \mathbf{A}^n$  and  $Y \subseteq \mathbf{A}^m$  be affine varieties.

- (a) Show that  $X \times Y \subseteq \mathbf{A}^{n+m}$  with its induced topology is irreducible. [*Hint:* Suppose that  $X \times Y$  is a union of two closed subsets  $Z_1 \cup Z_2$ . Let  $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$ ,  $i = 1, 2$ . Show that  $X = X_1 \cup X_2$  and  $X_1, X_2$  are closed. Then  $X = X_1$  or  $X_2$  so  $X \times Y = Z_1$  or  $Z_2$ .] The affine variety  $X \times Y$  is called the product of  $X$  and  $Y$ . Note that its topology is in general not equal to the product topology (Ex. 1.4).
- (b) Show that  $A(X \times Y) \cong A(X) \otimes_k A(Y)$ .
- (c) Show that  $X \times Y$  is a product in the category of varieties, i.e., show (i) the projections  $X \times Y \rightarrow X$  and  $X \times Y \rightarrow Y$  are morphisms, and (ii) given a variety  $Z$ , and the morphisms  $Z \rightarrow X$ ,  $Z \rightarrow Y$ , there is a unique morphism  $Z \rightarrow X \times Y$  making a commutative diagram



- (d) Show that  $\dim X \times Y = \dim X + \dim Y$ .

**3.16 Products of Quasi-Projective Varieties.** Use the Segre embedding (Ex. 2.14) to identify  $\mathbf{P}^n \times \mathbf{P}^m$  with its image and hence give it a structure of projective variety. Now for any two quasi-projective varieties  $X \subseteq \mathbf{P}^n$  and  $Y \subseteq \mathbf{P}^m$ , consider  $X \times Y \subseteq \mathbf{P}^n \times \mathbf{P}^m$

- (a) Show that  $X \times Y$  is a quasi-projective variety.
- (b) If  $X, Y$  are both projective, show that  $X \times Y$  is projective.
- \*(c) Show that  $X \times Y$  is a product in the category of varieties.

**3.17 Normal Varieties.** A variety  $Y$  is normal at a point  $P \in Y$  if  $\mathcal{O}_P$  is an integrally closed ring.  $Y$  is normal if it is normal at every point.

- (a) Show that every conic in  $\mathbf{P}^2$  is normal.
- (b) Show that the quadric surfaces  $Q_1, Q_2$  in  $\mathbf{P}^3$  given by equations  $Q_1 : xy = zw$ ;  $Q_2 : xy = z^2$  are normal (cf. (II, Ex. 6.4) for the latter.)
- (c) Show that the cuspidal cubic  $y^2 = x^3$  in  $\mathbf{A}^2$  is not normal.
- (d) If  $Y$  is affine, then  $Y$  is normal  $\iff A(Y)$  is integrally closed.
- (e) Let  $Y$  be an affine variety. Show that there is a normal affine variety  $\tilde{Y}$ , and a morphism  $\pi : \tilde{Y} \rightarrow Y$ , with the property that whenever  $Z$  is a normal variety, and  $\varphi : Z \rightarrow Y$  is a dominant morphism (i.e.,  $\varphi(Z)$  is dense in  $Y$ ), then there is a unique morphism  $e : Z \rightarrow \tilde{Y}$  such that  $\varphi = \pi \circ e$ .  $\tilde{Y}$  is called the normalization of  $Y$ . You will need (3.9A) above.

**3.18 Projectively Normal Varieties.** A projective variety  $Y \subseteq \mathbf{P}^n$  is projectively normal (with respect to the given embedding) if its homogeneous coordinate ring  $S(Y)$  is integrally closed.

- (a) If  $Y$  is projectively normal, then  $Y$  is normal.
- (b) There are normal varieties in projective space which are not projectively normal. For example, let  $Y$  be the twisted quartic curve in  $\mathbf{P}^3$  given parametrically by  $(x, y, z, w) = (t^4, t^3u, tu^3, u^4)$ . Then  $Y$  is normal but not projectively normal. See (III, Ex. 5.6) for more examples.
- (c) Show that the twisted quartic curve  $Y$  above is isomorphic to  $\mathbf{P}^1$ , which is projectively normal. Thus projective normality depends on the embedding.

**3.19 Automorphisms of  $\mathbf{A}^n$ .** Let  $\varphi : \mathbf{A}^n \rightarrow \mathbf{A}^n$  be a morphism of  $\mathbf{A}^n$  to  $\mathbf{A}^n$  given by  $n$  polynomials  $f_1, \dots, f_n$  of  $n$  variables  $x_1, \dots, x_n$ . Let  $J = \det[\partial f_i / \partial x_j]$  be the *Jacobian* polynomial of  $\varphi$ .

- (a) If  $\varphi$  is an isomorphism (in which case we call  $\varphi$  an automorphism of  $\mathbf{A}^n$ ) show that  $J$  is a nonzero constant polynomial.
- \*\* (b) The converse of (a) is an unsolved problem, even for  $n = 2$ . See, for example. Vitushkin [1].

**3.20** Let  $Y$  be a variety of dimension  $\geq 2$ , and let  $P \in Y$  be a normal point. Let  $f$  be a regular function on  $Y - P$ .

- (a) Show that  $f$  extends to a regular function on  $Y$ .
- (b) Show this would be false for  $\dim Y = 1$ . See (III, Ex. 3.5) for generalization.

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**3.21 Group Varieties.** A group variety consists of a variety  $Y$  together with a morphism  $\mu : Y \times Y \rightarrow Y$ , such that the set of points of  $Y$  with the operation given by  $\mu$  is a group, and such that the inverse map  $y \rightarrow y^{-1}$  is also a morphism of  $Y \rightarrow Y$ .

- (a) The *additive group*  $\mathbf{G}_a$  is given by the variety  $\mathbf{A}^1$  and the morphism  $\mu : \mathbf{A}^2 \rightarrow \mathbf{A}^1$  defined by  $\mu(a, b) = a + b$ . Show it is a group variety.
  - (b) The multiplicative group  $\mathbf{G}_m$  is given by the variety  $\mathbf{A}^1 - \{(0)\}$  and the morphism  $\mu(a, b) = ab$ . Show it is a group variety.
  - (c) If  $G$  is a group variety, and  $X$  is any variety, show that the set  $\text{Hom}(X, \mathbf{G}_a)$  has a natural group structure.
  - (d) For any variety  $X$ , show that  $\text{Hom}(X, \mathbf{G}_a)$  is isomorphic to  $\mathcal{O}(X)$  as a group under addition.
  - (e) For any variety  $X$ , show that  $\text{Hom}(X, \mathbf{G}_m)$  is isomorphic to the group of units in  $\mathcal{O}(X)$ , under multiplication.
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