---

1. Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution: Let x be nilpotent and n the smallest integer such that $x^n = 0$. The case for $n = 2$ is clear because $(1+x)(1-x) = 1 - x^2 = 1$. Therefore, let $n > 2$, and let k be the smallest integer such that $n \leqslant 2^k$. Then observe that

$$(1+x)(1-x)\prod_{i=2}^{k}\left(1+x^{2^i}\right) = 1.$$

Hence we see that $1+x$ is a unit. By adjusting the above argument, and noting that the set of units of a ring form a group, we can get that the sum of a nilpotent element and a unit is a unit. $\square$

---

2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n$ in $A[x]$. Prove that

i) f is a unit in $A[x] \iff a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent. [If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex. 1.]

ii) f is nilpotent $\iff a_0, a_1, \ldots a_n$ are nilpotent.

iii) f is a zero-divisor $\iff$ there exists a $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ $(0 \leqslant r \leqslant n)$.]

iv) f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff f$ and g are primitive.

Solution:

i) Suppose that f is a unit. Then there exists a polynomial g such that

$$fg = 1 \implies \sum_{k=0}^{n+m}\left(\sum_{i+j=k} a_ib_j\right)x^k = 1.$$

For this to be the case, we must have that $a_0b_0 = 1$, which implies that a_0 is a unit, and $a_nb_m = 0$, which implies that a_n is a zero divisor. We'll now argue that $a_n^{r+1}b_{m-r} = 0$ whenever $r > 0$ (here we treat $b_{m-r} = 0$ whenever $r > m$.) Since the base case is true, suppose that it is true for all integers less than $r \geqslant 0$. Then observe that the coefficient of x^{n+m-r} is given by

$$\sum_{i+j=m+n-r} a_ib_j = a_nb_{m-r} + a_{n-1}b_{m-r-1} + \cdots + a_{n-r}b_m = 0.$$

If we multiply this by a_n^r, we see that

$$a_n^r \cdot a_nb_{m-r} + a_n^r \cdot a_{n-1}b_{m-r-1} + \cdots + a_n^r \cdot a_{n-r}b_m = a_n^{r+1}b_{m-r}$$

where for every summand other than the first term we applied the induction hypothesis. This then implies that $a_n^{r+1}b_{m-r} = 0$. Hence, we have proved our claim by induction. Using our claim, note that $a_n^{m+1}b_0 = 0$. Since b_0 is a unit, we have that $a_n^{m+1} = 0$ which implies a_n is nilpotent.

ii) We can prove the forward statement by induction on degree. First note that the base case of $\deg f = 0$ is immediate. Therefore, let f have degree $n > 0$ and suppose the (forward) statement is true for polynomials of degree $n-1$. Since f is nilpotent, we have that $f^r = 0$ for some integer r. In particular, we have that $a_n^rx^{n+r} = 0 \implies a_n^r = 0$. This shows that a_n is nilpotent. Now observe that

$$f - a_nx^n = a_0 + \cdots a_{n-1}x^{n-1}.$$

The left hand side is nilpotent, and so $a_0 + \cdots + a_{n-1}x^{n-1}$ must be nilpotent. We can then apply our induction hypothesis to conclude that $a_0, \ldots, a_{n-1}$ must be nilpotent. This proves the forward direction.

To prove the reverse direction, suppose a_i are nilpotent. Then f^r is a polynomial whose coefficients are (up to a scalar multiple) of the form

$$a_0^{k_0}a_1^{k_1}\cdots a_n^{k_n}$$

where k_i are nonnegative and sum to r. Since each a_i is nilpotent, take $r = \operatorname{ord}(a_0) + \cdots + \operatorname{ord}(a_n)$. Then each coefficient $a_0^{k_0} a_1^{k_1} \cdots a_n^{k_n}$ of f^r must be zero, since at for at least one i, we have that $k_i \geqslant \operatorname{ord}(a_i)$ (or else the k_i powers cannot sum to r). In taking r to be this value, we can see that each coefficient of f^r is zero, which implies that f is nilpotent. This proves the reverse direction, and completes the if and only if proof.

iii) We follow the hint which proves the base case of our induction: as g is supposed to have minimal degree for which $fg = 0$, we can only have that $a_n g = 0$. To prove further that $a_{n-r} g = 0$, we suppose that the statement is true for all nonnegative integers less than r. Observe that the coefficient of x^{m+n-r} is given by

$$\sum_{i+j=m+n-r} a_i b_j = a_n b_{m-r} + a_{n-1} b_{m-r-1} + \cdots + a_{n-(r+1)} b_{m+1} + a_{n-r} b_m = 0.$$

By our induction hypothesis, each term $a_n b_{m-r}, \ldots, a_{n-(r+1)} b_{m+1}$ must be zero since $a_n g, \ldots, a_{n-(r+1)} g$ are all zero. This leaves just $a_{n-r} b_m = 0$, from which we can again deduce that $a_{n-r} g = 0$ as it has degree less than m and annihilates f. Hence, we have that the statement is true for all nonnegative integers.

Using this claim, we can then prove the main result by picking a nonzero coefficient of g. Take for instance b_m: Then $b_m f = b_m a_0 + \cdots + b_m a_n x^n = 0$. This proves the main result.

iv) If fg is primitive, then this implies that the ideal generated by the coefficients of fg is the entire ring. Explicitly, there exist coefficients $c_0, \ldots, c_{n+m}$ such that

$$c_0 \left(\sum_{i+j=0} a_i b_j \right) + c_1 \left(\sum_{i+j=1} a_i b_j \right) + \cdots + c_{n+m} \left(\sum_{i+j=n+m} a_i b_j \right) = 1$$

Since the above is a linear relation, we can rearrange the coefficients to obtain a summation of coefficients in a_i to 1. This can similarly be done for b_i. Hence $(a_0, \ldots, a_n) = (1)$ and $(b_0, \ldots, b_m) = (1)$.

□

---

3. Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Solution:

i) We claim that a multivariate polynomial $f(x_1, \ldots, x_n) = \sum a(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n}$ is a unit if and only if $a(0, \ldots, 0)$ is a unit and the rest of $a(i_1, \ldots, i_n)$ are nilpotent. We can prove this by induction on the number of variables: Suppose the statement is true for polynomials in $n-1$ variables. Then any polynomial $f(x_1, \ldots, x_n)$ in n-variables is technically a polynomial in $A[x_1, \ldots, x_{n-1}][x_n]$. By Exercise 2.1, this holds if and only if each $a(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_{n-1}}$ is nilpotent with $a(0, \ldots, 0)$ a unit. However, this occurs if and only if the $a(i_1, \ldots, i_n)$ (of course excluding $a(0, \ldots, 0)$) are all nilpotent, which proves the result.

ii) We claim that a multivariate polynomial is nilpotent if and only if each of its coefficients are nilpotent in A. This is achieved by induction on the number of indeterminates as in the previous example exercise

iii) We can prove this similarly to Exercise 1.2.3. First, we define the degree of a multivariate monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ to be the sum of the degrees of each α_i. We denote this degree by $(\alpha_1, \ldots, \alpha_n)$. We then impose a total ordering on monomial degree via lexicographical ordering. In other words, we say $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ has less degree than $x_1^{\beta_1} \cdots x_n^{\beta_n}$ if and only if $\alpha_i \leqslant \beta_i$ for all i.

With that said, let $f(x_1, \ldots, x_n)$ be a zero divisor with degree $(k_1, \ldots, k_n)$, and let $g(x_1, \ldots, x_n)$ be the polynomial with least degree $(m_1, \ldots, m_n)$. If we express

$$f = \sum a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \quad g = \sum b_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

Then we see that $a(k_1, \ldots, k_n) \cdot b_{m_1, \ldots, m_n} = 0$. Hence $a(k_1, \ldots, k_n) \cdot g$ annihilates f, but has less degree than g, and so it must be zero.

We show by induction that $a_{i_1, \ldots, i_n} g = 0$ for each coefficient $a_{i_1, \ldots, i_n}$ of f. To see this, suppose f has j-many monomial terms; then order them and denote them as $a_0, \ldots, a_j$; similarly suppose g has ℓ-many monomial terms and order them as $b_0, \ldots, b_\ell$. To perform induction, suppose that $a_{j-r} g = 0$ for all $0 \leqslant r < t$ where $0 \leqslant t < j$ (note the base case is true). Then observe that

$$\begin{aligned} fh = 0 &\implies (a_1 x_1^{i_1^{(1)}} \cdots x_n^{i_n^{(1)}} + \cdots + a_j x_1^{i_1^{(j)}} \cdots x_n^{i_n^{(j)}}) h = 0 \\ &\implies (a_1 x_1^{i_1^{(1)}} \cdots x_n^{i_n^{(1)}} + \cdots + a_k x_1^{i_1^{(k)}} \cdots x_n^{i_n^{(k)}}) h = 0 \\ &\implies a_k b_\ell = 0 \\ &\implies a_k h = 0. \end{aligned}$$

This completes our inductive step. We then take any nonzero coefficient of h and can now use our above result to conclude that f may be annihilated by a single element of A, as desired.

$\square$

---

4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution: It is clear that the nilradical is contained in the Jacobson radical. Therefore, let f be in the Jacobson radical. Then $1 - fg$ is a unit for all $g \in A[x]$. In particular, $1 - f \cdot x$ is a unit. By Ex 1.2.1, this implies that the coefficients of f are nilpotent. By Ex 1.2.2, this implies that f is nilradical, so the Jacobson is contained in the nilradical. This then implies that they are equal. $\square$

---

5. Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that

i) f is a unit in $A[[x]] \iff a_0$ is a unit in A.
ii) If f is nilpotent, then a_n is nilpotent for all $n \geqslant 0$. Is the converse true? (See Chapter 7, Exercise 2.)
iii) f belongs to the Jacobson radical of $A[[x]] \iff a_0$ belongs to the Jacobson radical of A.
iv) The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.
v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution:

i) Suppose $f = \sum_{n=0}^{\infty} a_n x^n$ is a power series and a_0 is a unit. We prove that there exists an inverse of $g = \sum_{n=0}^{\infty} b_n x^n$ by inductively constructing b_n.

For $n = 0$, we let b_0 be the inverse of a_0. Thus suppose that the coefficients have been defined for all nonnegative integers less than $n > 0$. Then observe that we require

$$\sum_{i=0}^{n} a_i b_{n-i} = 0.$$

The above is a linear equation which is in terms of $b_0, \ldots, b_n$. We can use our induction hypothesis to solve for b_n and construct it in this way. We see by induction that we may construct an inverse g, demonstrating that f is a unit.

ii) Suppose f is nilpotent. Then clearly a_0 is nilpotent. Suppose now that a_r is nilpotent for all $0 \leqslant r < k$ where $0 < k < n$. To show that a_k is nilpotent, note that

$$f - (a_0 + a_1 x + \cdots + a_r x^r) = \sum_{i=n}^{\infty} a_i x^i$$

is nilpotent. Hence, $\sum_{i=n}^{\infty} a_i x^i = x^k \sum_{i=n}^{\infty} a_i x^{i-k}$ is nilpotent, which implies that $\sum_{i=n}^{\infty} a_i x^{i-k}$ is nilpotent. In this case, we clearly have that a_k is nilpotent, which proves our inductive step. We thus have the general claim by induction.

iii) We know that f is in the Jacobson $\iff 1 - fg$ is a unit for all $g \in A[[x]]$. However, this is the case if and only if $1 - a_0 y$ is a unit for all $y \in A$, so that a_0 is in the Jacobson if and ony if f is in the Jacobson of $A[[x]]$.

$\square$

---

6. A ring A is such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Solution: We already know that the nilradical is contained in the Jacobson radical. To show the opposite inclusion, suppose the contrary. Then since the Jacobson is not contained in the nilradical, there exists an element e in the Jacobson such that $e^2 = e \neq 0$. As an element of the Jacobson we know that $1 - e$ is a unit. However, note that $(1-e)(1+e) = 1 - e^2 = 1 - e$. Thus $(1-e)(1+e-1) = 0 \implies e(1-e) = 0$. However, $e \neq 0$ and $1 - e$ is not a zero divisor, so this is a contradiction. Hence the Jacobson is contained in the nilradical, so that they are equal. $\square$

---

7. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution: Let P be a prime ideal, and consider the ring A/P. Consider an element $a+P$ in A/P. Then $(a+P)^n = a+P$ for some n. In particular, we have that $(a+P)((a+P)^{n-1} - (1+P)) = 0$. Since A/P is an integral domain, this implies that $(a+P)^{n-1} = 1+P$. Hence, $(a+P)$ is invertible. Since this was an arbitray element this implies that A/P is a field, so that P is maximal. $\square$

---

8. Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Solution: Zorn's Lemma can be applied in this case if we interpert ordering in an opposite manner. Therefore, we first want to show that each *descending* chain of prime ideals $(\mathfrak{p}_\mathrm{i})_i$ has a minimal element in Σ. Our claim is that $\mathfrak{p} = \bigcap_i \mathfrak{p}_i$ is such a minimal element. We already know it's an ideal. To show it is prime, let $xy \in \mathfrak{p}$. Then either $xy \in \mathfrak{p}_i$ for all i. In particular, this product is in $\mathfrak{p}_1$. Since $\mathfrak{p}_1$ is prime, suppose without loss of generality that $x \in \mathfrak{p}_1$. Then $x \in \mathfrak{p}_i$ for all i and so $x \in \mathfrak{p}$. Therefore, $\mathfrak{p}$ is prime.

We conclude our "backwards" (but logically valid) Zorn Lemma: Since each chain has of prime ideals has a minimal element, the set Σ of all prime ideals must have minimal elements with respect to conclusion.

$\square$

---

9. Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \iff \mathfrak{a}$ is an intersection of prime ideals.

Solution: Suppose $\mathfrak{a} = r(\mathfrak{a})$. In the ring $A/\mathfrak{a}$, an element $x + \mathfrak{a}$ is nilpotent if and only if $x \in r(\mathfrak{a})$. Hence, the nilpotents of $A/\mathfrak{a}$ correspond to the elements of $r(\mathfrak{a})$. By Proposition 1.8, we know that the nilradical of $A/\mathfrak{a}$ is the intersection of all prime ideals in this ring. By the Fourth Isomorphism theorem this intersection corresponds to the intersection of all prime ideals in A that contain $\mathfrak{a}$. Hence $r(\mathfrak{a})$ is the intersection of all prime ideals containing $\mathfrak{a}$, which completes this direction.

Conversely, suppose $\mathfrak{a}$ is the intersection of some set of prime ideals Σ (each which obviously must contain $\mathfrak{a}$). If $x \in r(\mathfrak{a})$, then $x^n \in \mathfrak{a}$ for some n. This implies that $x^n \in \mathfrak{p}$ for each $\mathfrak{p} \in \Sigma$, so that $x \in \mathfrak{p}$ for each Σ, and hence $x \in \mathfrak{a}$. Since we already know that $\mathfrak{a} \subset r(\mathfrak{a})$, this then shows that $\mathfrak{a} = r(\mathfrak{a})$, as desired. $\square$

---

10. Let A be a ring, $\mathfrak{R}$ its nilradical. Show that the following are equivalent:

i) A has exactly one prime ideal;
ii) every element of A is either a unit or nilpotent;
iii $A/\mathfrak{R}$ is a field.

Solution:

i $\iff$ ***ii*:** Suppose A has exactly on prime ideal. Then it must be a maximal ideal. Additionally, since $\mathfrak{R}$ is the intersection of all the prime ideals of A, this means that $\mathfrak{R}$ is maximal. Therefore, every element of A is either a unit or a nilpotent element.

Conversely, if every nonunit is nilpotent, this means that every non unit must be in $\mathfrak{R}$. Additionally, $\mathfrak{R}$ must be maximal because there cannot be any maximal ideal containing $\mathfrak{R}$ as every nonunit is nilpotent. Hence A has exactly one prime ideal.

i $\iff$ ***iii*:** If A has exactly one prime ideal, then again, $\mathfrak{R}$ must be prime, in fact it must be maximal. Therefore, $A/\mathfrak{R}$ is a field. If conversely we are given the fact that $A/\mathfrak{R}$ is a field, this implies that $\mathfrak{R}$ is maximal. Hence, the intersection of the prime ideals of A must consist of one element, so that A has exactly one prime ideal.

$\square$

---

11. A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that

i) $2x = 0$ for all $x \in A$
ii) Every prime ideal $\mathfrak{p}$ is maximal and $A/\mathfrak{p}$ is a field with two elements;
iii) Every finitely generated ideal in A is principal.

Solution:

i) Note that $(x+1)^2 = x^2 + 2x + 1 = x + 1 \implies 2x = 0$.

ii) Let $\mathfrak{p}$ be prime and consider the integral domain $A/\mathfrak{p}$. Then for all $x \in A/\mathfrak{p}$, we have that $x^2 = x \implies x(x-1) = 0$. Thus we see that $x = 0$ or 1 in $A/\mathfrak{p}$, which implies that this ring is actually a field with two elements.

iii If $\mathfrak{a}$ is finitely generated, then it must be generated by one element. This is because if a_1, a_2 are two distinct generators, then we must be able to construct the chain $(a_1) \subset (a_1, a_2)$. However, (a_1) is prime and therefore is maximal, so $(a_1, a_2) = (1)$. Hence, a finitely generated ideal must be principal.

□

---

12. A local ring contains no idempotent $\neq 0, 1$.

Solution: Let e be a nonzero idempotent $\neq 0, 1$ in A. Since e is idempotent we have that $e^2 - e = 0 \implies e(e-1) = 0$. Since $e \neq 0, 1$, this implies that e is a zero divisor. Therefore, e cannot be a unit, and so e and $e-1$ must be in the unique maximal ideal $\mathfrak{m}$ of A. However, this would imply that $e + (e-1) = 1 \in \mathfrak{m}$, a contradiction. Hence a local ring cannot have a nontrivial idempotent. □

---

13. *Construction of an algebraic closure of a field* (E. Artin).
Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{a}$, and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2 , and so on. Let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\overline{K}$ be the set of all elements of L which are algebraic over K. Then $\overline{K}$ is an algebraic closure of K.

Solution: Observe that each expression in $\mathfrak{a}$ is one of the form $c_1 f_1(x_{f_1}) + \cdots + c_n f_n(x_{f_n})$ where $c_i \in A$. As each f_i is monic and irreducible, none are constant. Since none are constant, and each f_i is in a different variable, the above expression always has degree of at least one and so it can never trivially evaluate to 1. Hence, $\mathfrak{a} \neq (1)$.

If $\mathfrak{m}$ is the maximal ideal of $\mathfrak{a}$, then $K_1 = A/\mathfrak{m}$ is a field, and there is a clear injection $K \to K_1$. Hence it is a field extension of K, and each $f \in \Sigma$ has a root in $A/\mathfrak{m}$; namely, the root of $f \in \Sigma$ is x_f. □

---

14. In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Solution: Note that $(0) \in \Sigma$ so that Σ is nonempty. Observe also that if $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ is a chain of elements in Σ, then the upper bound is given by the ideal $\mathfrak{a} = \bigcup_i \mathfrak{a}_i$, which is in Σ since it consists entirely of zero divisors. Hence Σ has maximal elements with respect to inclusion.

Let $\mathfrak{p}$ be a maximal element. To show that it is prime, suppose $xy \in \mathfrak{p}$. Then this implies that $p + xy$ is a zero divisor for all $p \in \mathfrak{p}$. Now suppose for a contradiction that $\mathfrak{p} \subset (\mathfrak{p}, x)$ and $(\mathfrak{p}, y)$. Then it must be the case that for some $p_1, p_2 \in \mathfrak{p}$ and $a, b \in A$, the two elements $p_1 + ax$ $p_2 + by$ are not zero divisors. However, this leads to a contradiction as this would imply that $(p_1 + ax)(p_2 + by) = (p_1 p_2 + p_1 by + p_2 ax) + abxy$ is not a zero divisor. Hence we see that either $(\mathfrak{p}, x)$ or $(\mathfrak{p}, y)$ are contained in $\mathfrak{p}$, which implies that either x or $y \in \mathfrak{p}$. So $\mathfrak{p}$ is a prime ideal.

Now since the union of all the zero divisors reduces to a union of its maximal elements, which are all prime, we see taht the union of all zero divisors is a union of prime ideals.

□

---

15. *The Prime Spectrum of a Ring*
Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

ii) $V(0) = X$, $V(1) = \varnothing$

iii) if $(E_i)_{i\in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i)$$

iv) $V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a})\cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology.* The topological space X is called the prime spectrum of A, and is written $\mathrm{Spec}(A)$.

---

16. Draw pictures of $\mathrm{Spec}(\mathbb{Z})$, $\mathrm{Spec}(\mathbb{R})$, $\mathrm{Spec}(\mathbb{C}[x])$, $\mathrm{Spec}(\mathbb{R}[x])$, $\mathrm{Spec}(\mathbb{Z}[x])$.

---

17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \mathrm{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

i) $X_f \cap X_g = X_{fg}$;
ii) $X_f = \varnothing \iff f$ is nilpotent;
iii) $X_f = X \iff f$ is a unit;
iv) $X_f = X_g \iff r((f)) = r((g))$;
v) X is quasi-compact (that is, every open covering of X has a finite subcovering)
vi) More generally, each X_f is quasi-compact.
vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.

[To prove (v), remark that it is enough to consider a covering of X by basic open sets $X_{f_i}(i \in I)$. Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i\in J} g_i f_i \qquad (g_i \in A)$$

where J is some *finite* subset of I. Then $X_{f_i}(i \in J)$ cover X.

---

18. For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \mathrm{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that

i) the set $\{x\}$ is closed (we say that x is a "closed point") in $\mathrm{Spec}(A) \iff \mathfrak{p}_x$ is maximal;
ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;
iii) $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subset \mathfrak{p}_y$;
iv) X is a T_0-space (this means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x).

---

19. A topological space X is said to be irreducible if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that $\mathrm{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

---

20. Let X be a topological space.

i) If Y is an irreducible (Exercise 19) subspace of X, then the closure $\overline{Y}$ of Y in X is irreducible.
ii) Every irreducible subspace of X is contained in a maximal irreducible subspace.
iii) The maximal irreducible subspaces of X are closed and cover X. They are called the *irreducible components* of X. What are the irreducible components of a Hausdorff space?

iv) If A is a ring and $X = \mathrm{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$ where $\mathfrak{p}$ is a minimal prime ideal of A (Exercise 8).

---

21. Let $\phi : A \to B$ be a ring homomorphism. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e., a point of X. Hence ϕ induces a mapping $\phi^* : Y \to X$. Show that

i) If $f \in A$ then ${\phi^*}^{-1}(X) = Y_{\phi(f)}$, and hence ϕ^* is continuous.
ii) If $\mathfrak{a}$ is an ideal of A, then $\overline{{\phi^*}^{-1}(V(\mathfrak{a}))} = V(\mathfrak{a}^e)$.
iii) If $\mathfrak{b}$ is an ideal of A, then $\overline{{\phi^*}^{-1}(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.
iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker(\phi))$ of X. (In particular, $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A/\mathfrak{R})$ (where $\mathfrak{R}$ is the nilradical of A) are naturally homeomorphic.)
v) If ϕ is injective, then $\phi^*(Y)$ is dense in X. More precisely, $\phi^*(Y)$ is dense in $X \iff \ker(\phi) \subset \mathfrak{R}$.
vi) Let $\psi : B \to C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
vii) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$ and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (\overline{x}, x)$, where $\overline{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.

---

22. Let $A = \prod_{i=1}^n A_i$ be the direct product of rings A_i. Show that $\mathrm{Spec}(A)$ is the disjoint union of open (and closed) subspaces X_i, where X_i is canonically homeomorphic with $\mathrm{Spec}(A_i)$.

Conversely, let A be any ring. Show that the following statements are equivalent:

i) $X = \mathrm{Spec}(A)$ is disconnected.
ii) $A \cong A_1 \times A_2$ where neither of the rings A_1, A_2 is the zero ring.
iii) A contains an idempotent $\neq 0, 1$.

In particular, the spectrum of a local ring is always connected (Exercise 12).

---

23. Let A be a Boolean ring (Exercise 11), and let $X = \mathrm{Spec}(A)$.

i) For each $f \in A$, the set X_f (Exercise 17) is both open and closed in X.
ii) Let $f_1 \ldots, f_n \in A$ Show that $X_{f_1} \cup \cdots \cup X_{f_n} = X_f$ for some $f \in A$.
iii) The sets X_f are the only subsets of X which are both open and closed. [Let $Y \subset X$ be both open and closed. Since Y is open, it is a union of basic open sets X_f. Since Y is closed and X is quasi-compact (Exercise 17), Y is quasi-compact. Hence Y is a finite union of basic open sets; now use (ii) above.]
iv) X is a compact Hausdorff space.

---

24. Let L be a lattice, in which the sup and inf of two elements a, b are denoted by $a \vee b$ and $a \wedge b$ respectively. L is a Boolean lattice (or Boolean algebra) if

i) L has a least element and a greatest element (denoted by 0, 1 respectively).
ii) Each of $\vee$, $\wedge$ is distributive over the other.
iii) Each $a \in L$ has a unique "complement" $a' \in L$ such that $a \vee a' = 1$ and $a \wedge a' = 0$.

(For example, the set of all subsets of a set, ordered by inclusion, is a Boolean lattice.)

Let L be a Boolean lattice. Define addition and multiplication in L by the rules

$$a + b = (a \wedge b') \vee (a' \wedge b), \qquad ab = a \wedge b.$$

Verify that in this way L becomes a Boolean ring, say $A(L)$. Conversely, starting from a Boolean ring A, define an ordering on A as follows: $a \leqslant b$ means that $a = ab$. Show that, with respect to this ordering, A is a Boolean lattice. [The sup and inf are given by $a \vee b = a + b + ab$ and $a \wedge b = ab$,and the complement by $a' = 1 - a$.] In this way we obtain a one-to-one correspondence between (isomorphism classes of) Boolean rings and (isomorphism classes of) Boolean lattices.

---

25. From the last two exercises deduce Stone's theorem, that every Boolean lattice is isomorphic to the lattice of open-and-closed subsets of some compact Hausdorff topological space.

---

26. Let A be a ring. The subspace of $\mathrm{Spec}(A)$ consisting of the maximal ideals of A, with the induced topology, is called the *maximal spectrum* of A and is denoted by $\mathrm{Max}(A)$. For arbitrary commutative rings it does not have the nice functorial properties of $\mathrm{Spec}(A)$ (see Exercise 21), because the inverse image of a maximal ideal under a ring homomorphism need not be maximal. Let X be a compact Hausdorff space and let $C(X)$ denote the ring of all real-valued continuous functions on X (add and multiply functions by adding and multiplying their values). For each $x \in X$, let $\mathfrak{m}_x$ be the set of all $f \in C(X)$ such that $f(x) = 0$. The ideal $\mathfrak{m}_x$ is maximal, because it is the kernel of the (surjective) homomorphism $C(X) \to \mathbb{R}$ which takes f to $f(x)$. If X denotes $\mathrm{Max}(C(X))$, we have therefore defined a mapping $\mu : X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

We shall show that μ is a homeomorphism of X onto $\tilde{X}$.

i) Let $\mathfrak{m}$ be any maximal ideal of $C(X)$, and let $V = V(\mathfrak{m})$ be the set of common zeros of the functions in $\mathfrak{m}$: that is,

$$V = \{x \in X : f(x) = 0 \text{ for all } f\ in \mathfrak{m}\}.$$

Suppose V is empty. Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighborhood U_x of x in X on which f_x does not vanish. By compactness of finite number of neighborhoods, say $U_{x_1}, \ldots, U_{x_n}$, cover X. Let

$$f = f_{x_1}^2 + \cdots + f_{x_n}^2.$$

Then f does nto vanish at any point of X, hence is a unit in $C(X)$. But this contradicts $f \in \mathfrak{m}$, hence V is not empty.

Let x be a point of V. Then $\mathfrak{m} \subset \mathfrak{m}_x$, hence $\mathfrak{m} = \mathfrak{m}_x$ because $\mathfrak{m}$ is maximal. Hence μ is sujrective.

ii) By Urysohn's lemma (this is the only non-trivial fact required in the argument) the continuous functions separate the points of X. Hence $x \neq y \implies \mathfrak{m}_x \neq \mathfrak{m}_y$ and therefore μ is injective.

iii) Let $f \in C(X)$; let

$$U_f = \{x \in X : f(x) \neq 0\}$$

and let

$$\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} : f \notin \mathfrak{m}\}.$$

Show that $\mu(U_f) = \tilde{U}_f$. The open sets U_f (resp. $\tilde{U}_f$) form a basis of the topology of X (resp. $\tilde{X}$) and therefore μ is a homeomorphism.

Thus X can be reconstructed from the ring of functions $C(X)$.

---

27. *Affine Algebraic Varieties.* Let k be an algebraically closed field and let

$$f_\alpha(t_1, \ldots, t_n) = 0$$

be a set of polynomial equations in n variables with coefficients in k. The set X of all points $x = (x_1, \ldots, x_n) \in k^n$ which satisfy these equations is an *affine algebraic variety.*

Consider the set of all polynomials $g \in k[t_1, \ldots, t_n]$ with the property that $g(x) = 0$ for all $x \in X$. This set is an ideal $I(X)$ in the polynomial ring, and is called the ideal of the variety X. The quotient ring

$$P(X) = k[t_1, \ldots, t_n]/I(X)$$

is the ring of polynomial functions on X, because two polynomials g, h define the same polynomial function on X if and only if $g - h$ vanishes at every point of X, that is, if and only if $g - h \in I(X)$.

Let η_1 be the image of t_i in $P(X)$. The $\eta_i (1 \leqslant i \leqslant n)$ are the *coordinate functions* on X: if $x \in X$, then $\eta_i(x)$ is the ith coordinate of x. $P(X)$ is generated as a k-algebra by the coordinate functions, and is called the *coordinate ring* (or affine algebra) of X.

As in Exercise 26, for each $x \in X$ let $\mathfrak{m}_x$, be the ideal of all $f \in P(X)$ such that $f(x) = 0$; it is a maximal ideal of $P(X)$. Hence, if $X = \mathrm{Max}(P(X))$, we have defined a mapping $\mu : X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

It is easy to show that μ is injective: if $x \neq y$, we must have $x_i \neq y_i$ for for some $i(1 \leqslant i \leqslant n)$, and hence $\eta_i - x_i$ is in $\mathfrak{m}_x$ but not in $\mathfrak{m}_y$, so that $\mathfrak{m}_x \neq \mathfrak{m}_y$. What is less obvious (but still true) is that μ is *surjective*. This is one form of Hilbert's Nullstellensatz (see Chapter 7).

---

28. Let $f_1, \ldots, f_m$ be elements of $k[t_1, \ldots, t_n]$. They determine a *polynomial mapping* $\phi : k^n \to k^m$ if $x \in k^n$, the coordinates of $\phi(x)$ are $f_1(x), \ldots, f_m(x)$.

Let X, Y be affine algebraic varieties in k^n, k^m respectively. A mapping $\phi : X \to Y$ is said to be regular if ϕ is the restriction to X of a polynomial mapping from k^n to k^m. If η is a polynomial function on Y, then $\eta \circ \phi$ is a polynomial function on X. Hence ϕ induces a k-algebra homomorphism $P(Y) \to P(X)$, namely $\eta \mapsto \eta \circ \phi$. Show that in this way we obtain a one-to-one correspondence between the regular mappings $X \to Y$ and the k-algebra homomorphisms $P(Y) \to P(X)$.