Exercise 17.

Solution: Recall from Exercise 14 that
$$\lim_{\to} M_i \cong C/D = \bigoplus_{i\in I} M_i/D,$$
where D is the submodule of $C = \bigoplus_{i\in I} M_i$ generated by all elements of the form $x_i - \mu_{ij}(x_i)$. Also recall that $\sum M_i$ is the set of all finite sums. Consider the map
$$\phi : \bigoplus_{i\in I} M_i \to \sum M_i.$$
For the first isomorphism $\lim_{\to} M_i \cong \sum M_i$, it suffices to show that $\ker\phi = D$.

(a) Clearly $\phi(x_i - \mu_{ij}(x_i)) = 0$. So $D \subseteq \ker\phi$.

(b) Conversely, let $x = (x_i)_{i\in I} \in C$. Then $x_i \neq 0$ for finitely many $i \in I$. Therefore, there exists $k \in I$ such that if $x_i \neq 0$, then $M_i \subseteq M_k$. Since $\phi(x) = 0$, we must have
$$x_k = \sum_{i\in I, i\neq k, x_i\neq 0} \mu_{ik}(-x_i).$$
Therefore, we may write
$$x = (x_i)_{i\in I} = x_k + \sum_{i\in I, i\neq k, x_i\neq 0} x_i = \sum_{i\in I, i\neq k, x_i\neq 0} (\mu_{ik}(-x_i) + x_i) \in D.$$

The second isomorphism $\sum M_i \cong \bigcup M_i$ follows from the fact that a finite sum of elements in various M_i's may be rewritten as a sum of elements in some M_k.

$\square$