Hartshorne 1.1 Exercises: Affine Varieties

Feiyang Lin and Luke Trujillo

1.1

(a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

(c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ and $A(Z)$. Which one is it and when?

Solution:

(a) Consider the map $\phi : k[x, y] \to k[x]$ where $\phi(p(x, y)) = p(x, x^2)$. As this is (1) surjective and (2) has kernel $(y - x^2)$, we see that

$$k[x, y]/(y - x^2) \cong k[x].$$

Hence $(y - x^2)$ is prime. Moreover, if we denote $Y = Z(y - x^2)$, then we see that

$$A(Y) \cong k[x, y]/I(Y) = k[x, y]/I(Z(y - x^2)) = k[x, y]/(y - x^2) \cong k[x].$$

Therefore, $A(Y) \cong k[x]$.

(b) Consider the map $\phi : k[x, y] \to k[x, 1/x]$ where $\phi(p(x, y)) = p(x, 1/x)$. This is surjective with kernel $(xy - 1)$. This then gives us

$$k[x, y]/(xy - 1) \cong k[x, 1/x] \ncong k[x].$$

Denote $Y = Z(xy - 1)$. Note that $xy - 1$ is irreducible in $k[x, y]$. Hence, $(xy - 1)$ is prime. Moreover,

$$A(Y) \cong k[x, y]/I(Z(xy - 1)) = k[x, y]/(xy - 1) \ncong k[x].$$

Thus $A(Y) \ncong k[x]$.

(c) Let $f = x^2 + axy + by^2 + cx + dy + e$. Suppose b is a perfect square. Then

$$f = (x + by)^2 + cx + dy + e.$$

Write $X = x + a$. Then $f = X^2 + cx + dy + e$.

□

1.2 *The Twisted Cubic Curve.* Let $Y \subseteq \mathbf{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t$, $y = t^2$, $z = t^3$).

Solution: Construct the map $\phi : k[x, y, z] \to k[x]$ were $\phi(p(x, y, z)) = p(x, x^2, x^3)$. Then the kernel of the map is $(x^2 - y, x^3 - z)$. Therefore,

$$k[x, y, z]/(x^2 - y, x^3 - z) \cong k[x].$$

(I think my dimension calculation here can be greatly simplified by using the above isomorphism.) Hence $Y = Z(x^2 - y, x^3 - z)$ closed, irreducible, and hence an affine variety. Now observe that

$$(x^2 - y) \subset (x^2 - y, x^3 - z)$$

as prime ideals. Thus $(x^2 - y, x^3 - z)$ corresponds to an ideal J of $k[x, y, z]/(x^2 - y)$. In fact, J is generated by the coset $x^3 - z + (x^2 - y)$. As this is not a unit in $k[x, y, z]/(x^2 - y)$, we may conclude that J has height of one by Theorem 1.11A. We then have by Theorem 1.8A that

$$\operatorname{ht}(J) + \dim\big((k[x, y, z]/(x^2 - y))/J\big) = \dim\big(k[x, y, z]/(x^2 - y)\big)$$

However, we know that

$$\operatorname{ht}(x^2 - y) + \dim\big(k[x, y, z]/(x^2 - y)\big) = \dim(k[x, y, z]).$$

By Theorem 1.11A, $\operatorname{ht}(x^2 - y) = 1$ since $x^2 - y$ is not a zero divisor or unit. In addition, $\dim(k[x, y, z]) = 3$ by Proposition 1.9. Hence

$$\dim\big(k[x, y, z]/(x^2 - y)\big) = 1 \implies \operatorname{ht}(J) + \dim\big((k[x, y, z]/(x^2 - y))/(x^3 - z)\big) = 2$$

As we know $\text{ht}(J) = 1$, we see that $\dim\big((k[x,y,z]/(x^2-y))/J\big) = 1$. But

$$\dim\big((k[x,y,z]/(x^2-y))/J\big) = \dim\big((k[x,y,z]/(x^2-y,x^3-z)\big) = \dim(Z(x^2-y,x^3-z)).$$

Therefore, $\dim(Z(x^2-y,x^3-z)) = 1$.

Finally, observe that $I(Y) = I(Z(x^2-y,x^3-z)) = (x^2-y,x^3-z)$. Thus the generators are just x^2-y and x^3-z.

□

1.3 Let Y be the algebraic set in $\mathbf{A}^3$ defined by the two polynomials x^2-yz and $xz-x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

Solution: Since $Y = Z(x^2-y, xz-z)$, we see that it consists of all $\mathbf{A}^3$ that satisfy:

$$\begin{cases} x^2 - yz = 0 \\ xz - x = 0 \end{cases}$$

There are three main ways we can satisfy the above equations.

- We could set $z = 1 \implies x = y^2$. This consists of $Z(z-1, x = y^2)$.
- We could set $z = x = 0$. This consists of the points of $Z(x,z)$.
- Finally, we could set $x = y = 0$. This consists of $Z(x,y)$.

Thus $Z(z-1,x-y^2)\cup Z(x,z)\cup Z(x,y) \subset Y$. It is not hard to see that conversely any $(x_0,y_0,z_0)\in Y$ must be in one of the three sets. Therefore, $Y = Z(z-1,x-y^2)\cup Z(x,z)\cup Z(x,y)$. Moreover, each of these are affine varities, and as none are contained in any other, we see that these are the unique irreducible components of Y.

□

1.4 If we identify $\mathbf{A}^2$ with $\mathbf{A}^1\times\mathbf{A}^1$ in the natural way, show that the Zariski topology on $\mathbf{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbf{A}^1$.

Solution:

□

1.5 Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbf{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

Solution: Thanks for pointing out the mistakes Feiyang! I cleared up a lot of my confusion.
If B is a k-algebra isomorphic to an affine coordinate ring, then

$$B \cong k[x_1,\ldots,x_n]/I(Y)$$

with Y an algebraic set. By definition, this is a finitely genereated k-algebra. Additionally, $I(Y)$ is radical, and so B cannot have any nilpotents.

Converseley, suppose B is finitely generated and has no nilpotents. By definition, there exists elements $b_1,\ldots,b_n\in B$ and a surjection $\phi : k[x_1,\ldots,x_n]\to B$ where $p(x_1,\ldots,x_n)\mapsto p(b_1,\ldots,b_n)$. This establishes the isomorphism

$$B \cong k[x_1,\ldots,x_n]/\ker(\phi).$$

Since B has no nilpotents, we know that for any $x\in k[x_1,\ldots,x_n]-\ker(\phi)$, $x^r\notin\ker(\phi)$ for any $r>0$. Hence, $\sqrt{\ker(\phi)} = \ker(\phi)$, so that $\ker(\phi)$ is radical. Therefore,

$$B \cong k[x_1,\ldots,x_n]/\ker(\phi) \cong k[x_1,\ldots,x_n]/I(Z(\ker(\phi))).$$

We now see that B is isomorphic to the affine coordinate ring of $Z(\ker(\phi))$, which is an algebraic set. □

1.6 Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Solution: We first prove the first sentence. Let U be a nonempty open subset of X, an irreducible space. Observe that $\overline{U}\cup U^c = X$. Since X is irreducible and U is nonempty, we see that $\overline{U} = X$. Therefore, U is dense.

Now suppose U was reducible (in its subspace topology). Then this implies that $U = Y_1 \cup Y_2$ with Y_1, Y_2 closed and proper (in U's subspace topology). Now we may express $Y_1 = Z_1 \cap U$ with Z_1 closed in X; similarly, there is a closed Z_2 corresponding to Y_2. Therefore,

$$U \subset Z_1 \cup Z_2 \implies \overline{U} \subset \overline{Z_1 \cup Z_2} \implies X = Z_1 \cup Z_2.$$

Hence either $X = Z_1$ or Z_2, so Y_1 or Y_2 is either U, contradicting our assumption that Y_1 and Y_2 are proper. Therefore, U is irreducible.

Now we prove the second sentence. Let Y be irreducible in its subspace topology, and suppose $\overline{Y}$ is reducible in X. Then there exists proper, closed subsets Z_1, Z_2 of $\overline{Y}$ such that $\overline{Y} = Z_1 \cup Z_2$. Hence, $Y = (Y \cap Z_1) \cup (Y \cap Z_2)$, which implies that $Y = Z_1$ or $Y = Z_2$. However, this implies that $\overline{Y} = Z_1$ or Z_2, a contradiction. Therefore $\overline{Y}$ is irreducible.

□

1.7

- **(a)** Show that the following conditions are equivalent for a topological space X: (i) X is noetherian; (ii) every nonempty family of closed subsets has a minimal element; (iii) X satisfies the ascending chain condition for open subsets; (iv) every nonempty family of open subsets has a maximal element.
- **(b)** A noetherian topological space is *quasi-compact,* i.e., every open cover has a finite subcover.
- **(c)** Any subset of a noetherian topological space is noetherian in its induced topology.
- **(d)** A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Solution:

- **(a)** First note that $(ii) \implies (i)$ and $(iv) \implies (iii)$ are immediate by definition of a Noetherian space.

  We show $(iii) \implies (iv)$. Since the ascending chain condition is satisfied, we may use Zorn's Lemma to deduce that any nonempty family of open subsets has a maximal element (we order it by inclusion, then apply the lemma). We can prove $(ii) \implies (i)$ similarly.
- **(b)** Let X be a Noetherian space and suppose $\mathcal{U} = \{U_i\}_{i \in \lambda}$ is an open cover of X. By (**a**), there exists a maximal element V_1 of $\mathcal{U}$. Using V_1 as our base case, inductively build the sets

  $$V_{i+1} = \max\left(\left\{U_i \in \mathcal{U} \;\middle|\; U_i \not\subset V_1 \cup \cdots \cup V_i\right\}\right) \qquad i = 1, 2, \ldots$$

  The maximum will exist by repeatedly applying (**a**). Now the chain

  $$V_1 \subset V_1 \cup V_2 \subset \cdots V_1 \cup \cdots \cup V_j \subset \cdots$$

  must have stabilize for some finite number of unions. This then implies that $X = V_1 \cup \cdots \cup V_r$ for some r. Hence, $V_1, \ldots, V_r$ is our finite subcover of $\mathcal{U}$, so that X is compact.

□

1.8 Let Y be an affine variety of dimension r in $\mathbf{A}^n$. Let H be a hypersurface in $\mathbf{A}^n$, and assume $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$. (See (7.1) for a generalization.)

Solution: First denote $Y = Z(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal in $k[x_1, \ldots, x_n]$. By Corollary 1.6, we can express the algebraic set $Y \cap H$ uniquely as

$$Y \cap H = V_1 \cup \cdots \cup V_\ell$$

where each V_i is an affine variety and $V_i \not\subset V_j$ for $i \neq j$. For each affine variety V_i denote $V_i = Z(\mathfrak{p}_i)$ with $\mathfrak{p}_i$ prime. We make some observations.

- Each prime ideal $\mathfrak{p}_i$ contains $\mathfrak{p}$, and hence corresponds to a prime ideal $\mathfrak{p}_i'$ in $k[x_1, \ldots, x_n]/\mathfrak{p}$.
- Since $Y \not\subset H$, we know that $(f) \not\subset \mathfrak{p}$ which implies $f \notin \mathfrak{p}$. Hence, we see that $f + \mathfrak{p} \in k[x_1, \ldots, x_n]/\mathfrak{p}$ is not a zero divisor (as it is an integral domain). It is also not a unit as f is irreducible. Write $\bar{f}$ as the representative of f in $k[x_1, \ldots, x_n]/\mathfrak{p}$. If $\bar{f}\bar{g} = \bar{0}$, then $fg \in \mathfrak{p}$. Since $f \in \mathfrak{p}$ and $\mathfrak{p}$ is prime, this means that $g \in \mathfrak{p}$, i.e. $\bar{g} = \bar{0}$. So $\bar{f}$ is not a zero-divisor.
- The ring $k[x_1, \ldots, x_n]/\mathfrak{p}$ is Noetherian. Thus, by Theorem 1.11A, every minimal prime ideal in $k[x_1, \ldots, x_n]/\mathfrak{p}$ containing $f + \mathfrak{p}$ must have height one.

Our claim is that each $\mathfrak{p}_i'$ is a minimal prime ideal containing $f+\mathfrak{p}$. Assuming this is true, we can observe by Theorem 1.8A that

$$\begin{aligned} \operatorname{ht}(\mathfrak{p}_i') + \dim\left((k[x_1,\dots,x_n]/\mathfrak{p})/\mathfrak{p}_i'\right) = \dim\left(k[x_1,\dots,x_n]/\mathfrak{p}\right) &\implies 1+\dim\left(A(V_i)\right) = r \\ &\implies \dim\left(A(V_i)\right) = r-1 \end{aligned}$$

Thus we show the claim. Suppose $\mathfrak{q}$ is a prime ideal in $k[x_1,\dots,x_n]/\mathfrak{p}$ containing $f+\mathfrak{p}$, and that $\mathfrak{q}\subseteq\mathfrak{p}_i'$ for some i. Then $\mathfrak{q}$ corresponds to a prime ideal $\mathfrak{q}'$ of $k[x_1,\dots,x_n]$ (1) containing $\mathfrak{p}$ and (2) containing f. However,

$$\sqrt{\langle \mathfrak{p}, f\rangle} = \mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_\ell$$

and as the radical of $\langle\mathfrak{p},f\rangle$ (the smallest ideal containing $\mathfrak{p}$ and f) is the intersection of all prime ideals containing this ideal, we see that $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_\ell\subseteq\mathfrak{q}'$. By Proposition 1.1.11(b) in Atiyah-MacDonald, this implies that $\mathfrak{p}_j\subseteq\mathfrak{q}'$. However, it must be that $j=i$, since none of these prime ideals are contained in each other. This then implies that $\mathfrak{p}_i'\subseteq\mathfrak{q}$ in $k[x_1,\dots,x_n]/\mathfrak{p}$, which gives us that $\mathfrak{q}=\mathfrak{p}_i'$. Hence, $\mathfrak{p}_i$ is a minimal prime ideal in $k[x_1,\dots,x_n]/\mathfrak{p}$ containing $f+\mathfrak{p}$, and so we may apply our above calculation. This then shows that

$$\dim(A(V_i)) = r-1$$

as desired. □

1.9 Let $\mathfrak{a}\subseteq A = k[x_1,\dots,x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geqslant n-r$.

Solution: We prove this by induction.

Base Case.: Consider an ideal $\mathfrak{a}=(a)$ which is generated by a single element. We assume $\mathfrak{a}$ is not all of $k[x_1,\dots,x_n]$ (i.e., a is not a unit); otherwise, $Z(\mathfrak{a})$ is empty, which is not irreducible, and further it does not make sense to talk about the irreducible components for the empty set.

- Suppose $a=0$. Then $\mathfrak{a}=0 \implies Z(\mathfrak{a})=\mathbf{A}^n$ which is irreducible and has dimension n.
- Suppose a is not a unit and is nonzero. Since $k[x_1,\dots,x_n]$ is a UFD, then we may uniquely express a as $a=u\cdot f_1\cdots f_m$ with u a unit, eac f_i irreducible. We then have that

$$Z(\mathfrak{a}) = Z(f_1)\cup Z(f_2)\cdots\cup Z(f_m).$$

  Now by Theorem 1.11A, we can conclude that for each $i=1,2,\dots,m$, $\operatorname{ht}(f_i)=1$. Hence

$$\operatorname{ht}(f_i)+\dim(Z(f_i)) = n \implies \dim(Z(f_i)) = n-1.$$

  Hence, every irreducible component of $Z(\mathfrak{a})$ has dimension $n-1$.

In each case we see that the irreducible components of (a) have dimension dim $\geqslant n-1$, which proves the base case.

Inductive Step.: Let $\mathfrak{a}=(a_1,\dots,a_r)$ be our ideal, and suppose the statement is true for all ideals generated by $(r-1)$-many elements. Let a_i be nonzero. Denote the decomposition of $Z(a_1,\dots,a_r)$ into its irreducible components as below

$$Z(a_1,\dots,a_r) = V_1\cup\cdots\cup V_\ell$$

with $V_j = Z(\mathfrak{p}_j)$. Similarly for $(a_1,\dots,a_{i-1},a_{i+1},\dots,a_r)$ write

$$Z(a_1,\dots,a_{i-1},a_{i+1},\dots,a_r) = Y_1\cup\cdots\cup Y_m.$$

with $Y_j = Z(\mathfrak{q}_j)$.

Observe that for each $j=1,2,\dots,\ell$,

$$\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_m\subset\mathfrak{p}_j.$$

By Proposition 1.11(b) in Atiyah MacDonald, this implies that $\mathfrak{q}_s\subset\mathfrak{p}_j$ for some $s=1,2,\dots,m$. Thus, denote $\mathfrak{p}_j'$ as the prime ideal in $k[x_1,\dots,x_n]/\mathfrak{q}_s$ corresponding to $\mathfrak{p}_j$. By Theorem 1.8A, we have that

$$\operatorname{ht}(\mathfrak{p}_j') + \dim\left((k[x_1,\dots,x_n]/\mathfrak{q}_s)/\mathfrak{p}_j'\right) = \dim(k[x_1,\dots,x_n]/\mathfrak{q}_s).$$

Now $\mathfrak{p}_i'$ is a minimal prime ideal containing $a_i+\mathfrak{q}_s$, which is not a unit or zero divisor. Hence, its height is one. Therefore,

$$1+\dim(V_i) = \dim(k[x_1,\dots,x_n]/\mathfrak{q}_s)\geqslant n-(r-1) \implies \dim(V_i)\geqslant n-r.$$

This completes the inductive step and the proof is complete.

□

1.10

- **(a)** If Y is any subset of a topological space X then $\dim Y \leq \dim X$.
- **(b)** If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.
- **(c)** Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$.
- **(d)** If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.
- **(e)** Give an example of a noetherian topological space of infinite dimension.

Solution:

- **(a)**
- **(b)** It suffices to show that there exists i such that $\dim X \leq \dim U_i$. Given a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X, there exists U_i such that $U_i \cap Z_{n-1} \neq \emptyset$. Then we claim that $Z_0 \cap U_i \subset Z_1 \cap U_i \subset \cdots \subset Z_n \cap U_i$ is a chain of distinct irreducible closed subsets of U_i. It suffices to show distinctness. Suppose that $Z_j \cap U_j = Z_{j+1} \cap U_i$. Notice that

$$Z_{j+1} = Z_j \cup Z_{j+1} = Z_j \cup (Z_{j+1} \cap U_i) \cup (Z_{j+1} \cap U_i^c) = Z_j \cup (Z_{j+1} \cap U_i^c).$$

Since Z_{j+1} is irreducible and since $Z_j \neq \emptyset$ (the empty set is not considered irreducible), we have that $Z_{j+1} \cap U_i^c = \emptyset$. However, this implies that $Z_{j+1} = Z_{j+1} \cap U_i = Z_j \cap U_i$, which is a contradiction to $Z_j \subsetneq Z_{j+1}$.

□

1.11 Let $Y \subseteq \mathbf{A}^3$ be the curve given parametrically by $x = t^3$, $y = t^4$, $z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say Y is *not a local complete intersection*—cf. *(Ex.* 2.17*)*.

Solution: Construct the map $\phi : k[x, y, z] \to k[t]$ where $p(x, y, z) \mapsto p(t^3, t^4, t^5)$. The kernel of this map is given by $I(Y) = \{f \in k[x, y, z] \mid f(t^3, t^4, t^5) = 0 \text{ for all } t \in k\}$. However, the map is not surjective. Thus we have that

$$k[x, y, z]/I(Y) \cong \operatorname{im}(\phi).$$

Since $k[t]$ is an integral domain, and $\operatorname{im}(\phi)$ is a subring, this nevertheless implies that $I(Y)$ is a prime ideal. Alternatively we have an isomorphism of varieties between Y and $X = \{(t, t^3, t^4, t^5)\}$ via $(x, y, z) \mapsto (x^{-1}y, x, y, z)$ and $(t, t^3, t^4, t^5) \mapsto (t^3, t^4, t^5)$. □

1.12 Give an example of an irreducible polynomial $f \in \mathbf{R}[x, y]$, whose zero set $Z(f)$ in $\mathbf{A}^2_{\mathbf{R}}$ is not irreducible *(cf.* 1.4.2*)*.

Solution: Consider $f = (x^2 - 1)^2 + y^2$. $Z(f) = Z((x - 1, y)) \cup Z((x + 1, y))$, and so is reducible. Now we prove that f is irreducible. Suppose f is reducible. Then $f = g(x, y)h(x, y)$. One can check that $f(x, 1) = (x^2 - 1)^2 + 1$ has no factorization two non-units over $\mathbb{R}$. Therefore $f(x, y) = g(y)h(x, y)$. However f has a term x^4 and no other x^4y^i terms, which implies that $g(y)$ is a unit. □