
Exercise 2.1 Prove the “homogeneous Nullstellensatz,” which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in \mathbf{P}^n , then $f^q \in \mathfrak{a}$ for some $q > 0$.

Solution:

□

Exercise 2.2 For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

- (i.) $Z(\mathfrak{a}) = \emptyset$ (the empty set);
 - (ii.) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
 - (iii.) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.
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Solution: (i) \Rightarrow (ii). Since $Z(\mathfrak{a}) = \emptyset$, the zero set of \mathfrak{a} in affine space is either \emptyset or $\{0\}$. In the first case, we certainly have $\sqrt{\mathfrak{a}} = S$. In the second case, we have $\sqrt{\mathfrak{a}} = \{p \in S : p(0) = 0\} = S_+$.

(ii) \Rightarrow (iii). It suffices to show that there exists d such that all monomials of degree d lies in \mathfrak{a} . Since $S_+ \subset \sqrt{\mathfrak{a}}$, for each $0 \leq i \leq n$, there exists d_i such that $x_i^{d_i} \in \mathfrak{a}$. Let $d = n \sum_{i=0}^n d_i$. Then if a monomial has degree d , then there exists i such that the exponent of x_i in the monomial is at least $\sum_{i=0}^n d_i$, and therefore at least d_i , which implies that the monomial is in \mathfrak{a} .

(iii) \Rightarrow (i). If $S_d \subseteq \mathfrak{a}$ for some $d > 0$, then $x_i \in \sqrt{\mathfrak{a}}$ for every $0 \leq i \leq n$, which means that $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) = \emptyset$. □

Exercise 2.3

- (1) If $T_1 \subseteq T_2$ are subsets of S^h , then $Z(T_1) \supseteq Z(T_2)$.
 - (2) If $Y_1 \subseteq Y_2$ are subsets of \mathbf{P}^n , then $I(Y_1) \supseteq I(Y_2)$.
 - (3) For any two subsets Y_1, Y_2 of \mathbf{P}^n , $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
 - (4) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
 - (5) For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \overline{Y}$.
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Solution:

□

Exercise 2.4

- (1) There is a one-to-one inclusion-reversing correspondence between algebraic sets in \mathbf{P}^n and homogeneous radical ideals of S not equal to S_+ given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note:* Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S .
 - (2) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.
 - (3) Show that \mathbf{P}^n itself is irreducible.
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Solution:

□

Exercise 2.5

- (1) \mathbf{P}^n is a noetherian topological space.
 - (2) Every algebraic set in \mathbf{P}^n can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.
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Solution:

□

for 2.12(b)

We prove the harder direction that $Z(\mathfrak{a}) \subseteq \text{im}(\rho_d)$. We may index the $N+1$ coordinates of a point in \mathbf{P}^n by tuples of the form (a_0, a_1, \dots, a_n) where $a_i \in \mathbb{Z}_+$ and the sum of all a_i 's is d . Given $\mathbf{y} \in Z(\mathfrak{a})$, I claim that there exists $0 \leq i \leq n$ such that $y_{d\mathbf{e}_i} \neq 0$. Indeed, suppose towards the contrary. Then since for any index $\mathbf{v} = (a_0, a_1, \dots, a_n)$, $p(\mathbf{y}) = y_{\mathbf{v}}^d - \prod_{i=0}^n y_{d\mathbf{e}_i}^{a_i} \in \mathfrak{a}$, we have that $p(\mathbf{y}) = y_{\mathbf{v}}^d = 0$, which implies that $y_{\mathbf{v}} = 0$ for arbitrary \mathbf{v} , which is absurd.

So fix some i and some representative of \mathbf{y} such that $y_{d\mathbf{e}_i} = 1$. Let $\mathbf{x} \in \mathbf{P}^n$ be such that $x_i = y_{d\mathbf{e}_i} = 1$ and $x_j = y_{d\mathbf{e}_i - \mathbf{e}_i + \mathbf{e}_j}$. We claim that $\rho_d(\mathbf{x}) = \mathbf{y}$. By construction, we already have that $x_i^{d-1}x_j = x_j = y_{d\mathbf{e}_i - \mathbf{e}_i + \mathbf{e}_j}$. It's straightforward to check that the polynomials in \mathfrak{a} ensures that $y_{\mathbf{v}} = M_{\mathbf{v}}(\mathbf{x})$.

To give an example to make this proof clearer, consider the example when $n = 1$ and $d = 3$. Then if $\mathbf{y} = (y_{30}, y_{21}, y_{12}, y_{03}) \in Z(\mathfrak{a})$, suppose WLOG that $y_{30} = 1$. Then $\mathbf{y} = \rho_d(1, y_{21})$. We check that since $y_{30}y_{12} - y_{21}^2 = 0$, indeed $y_{12} = y_{21}^2$. Similarly, since $y_{03}y_{30}^2 - y_{21}^3$, indeed $y_{03} = y_{21}^3$.