Hartshorne 1.3 Exercises: Morphisms Feiyang Lin and Luke Trujillo

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3.1

(a) Show that any conic in $\mathbf{A}^2$ is isomorphic either to $\mathbf{A}^1$ or $\mathbf{A}^1 - \{0\}$ *(cf. Ex.* 1.1*)*.

(b) Show that $\mathbf{A}^1$ is *not* isomorphic to any proper open subset of itself. (This result is generalized by *(Ex.* 6.7*)* below.)

(c) Any conic in $\mathbf{P}^2$ is isomorphic to $\mathbf{P}^1$.

(d) We will see later *(Ex.* 4.8*)* that any two curves are homeomorphic. But show now that $\mathbf{A}^2$ is not even homeomorphic to $\mathbf{P}^2$.

(e) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

3.2 A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

(a) For example, let $\varphi : \mathbf{A}^1 \to \mathbf{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbf{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.

(b) For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi : \mathbf{A}^1 \to \mathbf{A}^2$ by $t \to t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism.*

Solution:

(a) **Bijection:** Since k is an integral domain, this map is injective. In addition, the map is surjective: if $(x_0, y_0) \in Z(y^2 - x^3)$ are nonzero, we set $t_0 = \frac{y_0}{x_0}$. We then have that $\varphi(t_0) = (t_0^2, t_0^3) = (x_0, y_0)$.

Closed: Let $Y = Z(T) \in \mathbf{A}^1$ be a closed set, T some family of polynomials in $k[x]$. Establishing notation, write

$$\psi : k[x,y] \to k[t] \qquad x \mapsto t^2, y \mapsto t^3$$

Now observe that

$$\varphi(Y) = \left\{ (z_0^2, z_0^3) \;\middle|\; z_0 \in Y \right\} = \left\{ (z_0^2, z_0^3) \;\middle|\; f(z_0) = 0 \quad \forall f \in T \right\} = Z(y^2 - x^3, T')$$

where $T' = \{g \in k[x,y] \mid \psi(g) \in T\}$. Hence φ is a closed.

Continuous: Now we show this function is continuous. Let $W = Z(S) \cap Z(y^2 - x^3)$ where S is a family of polynomials in $k[x,y]$. Then

$$\varphi^{-1}(W) = \left\{ t_0 \in \mathbf{A}^1 \;\middle|\; f(t_0^2, t_0^3) = 0 \quad \forall f \in T \cap (y^2 - x^3) \right\}.$$

If $f(t_0^2, t_0^3) = 0$, then $\psi(f)(t_0) = 0$. Hence $\varphi^{-1}(W) \subseteq Z(\psi(T))$. The alternative direction is immediate, and we have that $\varphi^{-1}(W) = Z(\psi(T))$, so that $\varphi^{-1}(W)$ is closed.

Morphism: We show that this function is actually a morphism of varieties. Consider a regular map $f : V \subset Z(y^2 - x^3) \to k$ on some open subset $V \subset Z(y^2 - x^3)$. Let $P \in \varphi^{-1}(V)$. Since $f : V \to k$ is regular, there exits an open set $U \subseteq V$ containing $\varphi(p)$ such that

$$f(x,y) = \frac{g(x,y)}{h(x,y)} \qquad \forall (x,y) \in U.$$

Now observe that $\varphi^{-1}(U)$ is an open subset of $\varphi^{-1}(V)$ containing P, and moreover that

$$f \circ \varphi(t) = f(t^2, t^3) = \frac{g(t^2, t^3)}{h(t^2, t^3)} \qquad \forall t \in \varphi^{-1}(U).$$

Hence, $f \circ \varphi$ is regular whenever f is, making φ a morphism of varieties.

We finally show that this is not an isomorphism. This is because

$$k[x,y]/(y^2 - x^3) \not\cong k[x] \implies A(Z(y^2 - x^3)) \not\cong A(\mathbf{A}^1).$$

and this is due to the fact that ψ is not surjective (e.g., $x \notin \operatorname{im}(\psi)$, although $\ker(\psi) = I(y^2 - x^3)$). Hence, these varieties are not isomorphic.

□

3.3

(a) Let $\varphi : X \to Y$ be a morphism. Then for each $P \in X$, φ induces a homomorphism of local rings $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$.

(b) Show that a morphism φ is an isomorphism if and only if φ is a homeomorphism, and the induced map φ_P^* on local rings is an isomorphism, for all $p \in X$.

(c) Show that if $\varphi(X)$ is dense in Y, then the map φ_P^* is injective for all $p \in X$.

Solution:

(a) Given a morphism $\varphi : X \to Y$ of varieties and a point $P \in X$, we define

$$\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X} \qquad \big(U \subset Y, f : U \to k\big) \longmapsto \big(\varphi^{-1}(U), f \circ \varphi : \varphi^{-1}(U) \to k\big).$$

Well-defined: Let $(U_1, f_1) \sim (U_2, f_2)$ be members of $\mathcal{O}_{\varphi(P),Y}$. Then $f_1 \cap f_2$ on $U_1 \cap U_2$, which implies that $f_1 \circ \varphi = f_2 \circ \varphi$ on $\varphi^{-1}(U_1 \cap U_2)$. Hence $(\varphi^{-1}(U_1), f_1 \circ \varphi) \sim (\varphi^{-1}(U_2), f_2 \circ \varphi)$.

Local: Let $\mathfrak{m}_{\varphi(P)} = \{(V, f) \in \mathcal{O}_{\varphi(P),Y} \mid f \circ \varphi(p) = 0\}$ be the maximal ideal of $\mathcal{O}_{\varphi(P),Y}$. Then observe that $\varphi_P^*(\mathfrak{m}_{\varphi(P)}) = (\varphi^{-1}(V), f \circ \varphi) \in \mathfrak{m}_P \subset \mathcal{O}_{P,X}$. Hence, we see that $\varphi_P^*(\mathfrak{m}_{\varphi(P)}) \subset \mathfrak{m}_P$, so that φ_P^* is a homomorphism of local rings.

(b) Let $\varphi : X \to Y$ be a homeomorphism such that φ_P^* is an isomorphism for each $P \in X$. Let $f : V \subseteq Y \to k$ be a regular function on V containing P. Then f corresponds via φ_P^* to the regular function $f \circ \varphi : \varphi^{-1}(V) \to k$ on $\varphi^{-1}(P)$. Hence, φ is a morphism of varieties. By a similar argument, φ^{-1} is also a morphism of varieties. As they are inverses of each other, we see that φ is an isomorphism of varieties. The other direction is not difficult to check.

(c)

□

3.4 Show that the d-uple embedding of $\mathbf{P}^n$ (Ex. 2.12) is an isomorphism onto its image.

Solution: We recall some notation.

- $M_i(x_0, \ldots, x_n)$ is the i-th d-degree monomial (from 0 to $N = \binom{n+d}{n} - 1$)
- $\rho_d : \mathbf{P}^n \to Z(\mathfrak{a}) \subset \mathbf{P}^N$ sends $(a_0, \ldots, a_n)$ to $(M_0(a_0, \ldots, a_n), \ldots, M_N(a_0, \ldots, a_n))$
- $\theta : k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ sends y_i to $M_i(x_0, \ldots, x_n)$.

From Exercise 2.12, we already know that $\rho_d : \mathbf{P}^n \to Z(\mathfrak{a}) \subset \mathbf{P}^N$ is a homeomorphism. We show that ρ_d and ρ_d^{-1} are morphisms of varieties and are inverses of each other (as morphisms of varieties).

- We show ρ_d is a morphism of varieties. Let $f : V \subseteq Z(\mathfrak{a}) \to k$ be regular. Then for each $p \in V$, there exists an open set $U \subseteq V$ containing p such that $f(y_0, \ldots, y_N) = g_1(y_0, \ldots, y_N)/g_2(y_0, \ldots, y_N)$ on U, for some homogeneous polynomials g_1, g_2 of the same degree. Consider the function $f \circ \rho_d : \rho_d^{-1}(V) \to k$. Observe that $\rho_d^{-1}(U)$ is a subset of $\rho_d^{-1}(V)$ containing $\rho_d^{-1}(p)$, and moreover that

$$f \circ \rho_d = \frac{\theta(g_1)}{\theta(g_2)} \text{ on all of } \rho_d^{-1}(U).$$

  Note that $\theta(g_1)$, $\theta(g_2)$ are homogeneous polynomials of the same degree, and moreover that $\theta(g_2) \neq 0$ on any of $\rho_d^{-1}(U)$. Hence, $f \circ \rho_d$ is regular, which implies that ρ_d is a morphism of varieties.
- We show ρ_d^{-1} is a morphism of varieties. Let $m : V \subseteq \mathbf{P}^n \to k$ be a regular function. Then for each $p \in P$, there exists some open set $U \subseteq V$ containing p and a pair of homogeneous polynomials h_1, h_2 of the same degree such that $m(x_0, \ldots, x_n) = h_1(x_0, \ldots, x_n)/h_2(x_0, \ldots, x_n)$ on all of U.

  Take $U_i = \mathbf{P}^n - H_i$ such that $p \in U \cap U_i$. Let $\alpha = |d - \deg(h_1)| = |d - \deg(h_2)|$. Observe that

$$m = \frac{x_i^\alpha h_1(x_0, \ldots, x_n)}{x_i^\alpha h_2(x_0, \ldots, x_n)} = \frac{h_1'(x_0, \ldots, x_n)}{h_2'(x_0, \ldots, x_n)}.$$

  Both h_1' and h_2' are homogeneous polynomials of degree $\alpha + \deg(h_1) = \alpha + \deg(h_2)$ which is divisible by d. Hence, we see that both of these polynomials are in the image of θ. Let $k_1, k_2 \in k[y_0, \ldots, y_N]$ such that

$\theta(k_1) = h_1'$, $\theta(k_2) = h_2'$. Then we see that

$$f \circ \rho_d^{-1} = \frac{k_1}{k_2} \text{ on all of } \rho_d(U \cap U_i).$$

Thus we see that $f \circ \rho_d^{-1} : \rho_d(V) \subset Z(\mathfrak{a}) \to k$ is a regular function, so that ρ_d^{-1} is a morphism of varieties. As both ρ_d and ρ_d^{-1} are morphisms of varieties, and are homeomorphisms, we see that ρ_d establishes the isomorphism $\mathbf{P}^n \cong Z(\mathfrak{a})$, as desired. □

3.5 By abuse of language, we will say that a variety "is affine" if it is isomorphic to an affine variety. If $H \subseteq \mathbf{P}^n$ is any hypersurface, show that $\mathbf{P}^n - H$ is affine. [*Hint*: Let H have degree d. Then consider the d-uple embedding of $\mathbf{P}^n$ in $\mathbf{P}^n$ and use the fact that $\mathbf{P}^N$ minus a hyperplane is affine.]

Solution: Let H, the hypersurface, be given by the irreducible polynomial f. Since f is homogeneous, we may write it as

$$f(x_0, \ldots, x_n) = \sum_{i=0}^{N} a_i M_i(x_0, \ldots, x_n).$$

where M_i is the i-th homogeneous monomial of degree $d = \deg(f)$. If we let $y_i = M_i$, then the linear polynomial

$$p(y_0, \ldots, y_N) = \sum_{i=0}^{N} a_i y_i$$

has the property that $\theta(p(y_0, \ldots, y_N)) = f$, where θ substitutes y_i for M_i. What this tell us is that $\rho_d(H) = Z(p)$, so that $\rho_d(H)$ is actually a hyperplane. Observe now that

$$\rho_d(\mathbf{P}^n - H) \cong Z(\mathfrak{a}) - \rho_d(H) = (\mathbf{P}^N - \rho_d(H)) \cap Z(\mathfrak{a}).$$

However, $\mathbf{P}^N - \rho_d(H) \cong \mathbf{A}^{N-1}$ as we know that the $\mathbf{P}^N - Z(x_i) \cong \mathbf{A}^{N-1}$, and we can achieve a similar result for general hyperplanes by substituting variables as necessary.

□

3.6 There are quasi-affine varieties which are not affine. For example, show that $X = \mathbf{A}^2 - \{(0,0)\}$ is not affine. [*Hint*: Show that $\mathcal{O}(X) \cong k[x,y]$ and use (3.5). See (III, Ex. 4.3) for another proof.]

Solution: Consider $f \in \mathcal{O}(X)$. Let $p = (x_0, y_0) \in X$, Since f is regular on $X = \mathbf{A}^2 - \{(0,0)\}$, we know that there is an open set U of p such that

$$f(x,y) = \frac{g(x,y)}{h(x,y)} \quad \forall (x,y) \in U.$$

Now $h(x,y)$ is only allowed to have a root at $(0,0)$. Thus, observe that $h(x_0, y)$ is a single variable polynomial with no solution, as $x_0 \neq 0$. Since k is algebraically closed, $h(x_0, y)$ must be constant, so that $h(x,y)$ is just a polynomial in x. However, we can similarly observe that $h(x,y)$ would also need to be constant. This all together implies that $\mathcal{O}(X) \cong k[x,y]$.

Now suppose X is affine. Then by (3.5), we know that since $\mathcal{O}(X) \cong k[x,y] \cong A(\mathbf{A}^2)$, which implies that $\mathbf{A}^2 \cong X$. As this is a contradiction, we see that X is not affine.

□

3.7

(a) Show that any two curves in $\mathbf{P}^2$ have a nonempty intersection.

(b) More generally, show that if $Y \subseteq \mathbf{P}^n$ is a projective variety of dimension $\geqslant 1$, and if H is a hypersurface, then $Y \cap H \neq \varnothing$. [*Hint*: Use (Ex. 3.5) and (Ex. 3.1e). See (7.2) for a generalization.]

3.8 Let H_i and H_j be the hyperplanes in $\mathbf{P}^n$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Show that any regular function on $\mathbf{P}^n - (H_i \cap H_j)$ is constant. (This gives an alternate proof of (3.4a) in the case $Y = \mathbf{P}^n$.)

3.9 The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbf{P}^1$, and let Y be the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$. Then $X \cong Y$ (Ex. 3.4). But show that $S(X) \ncong S(Y)$.

3.10 *Subvarieties.* A subset of a topological space is locally closed if it is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.

If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset, then Y is also a quasi-affine (respectively, quasi-projective) variety, by virtue of being a locally closed subset of the same affine or projective space. We call this the induced structure on Y, and we call Y a subvariety of X.

Now let $\varphi : X \to Y$ be a morphism, let $X' \subseteq X$ and $Y' \subseteq Y$ be irreducible locally closed subsets such that $\varphi(X') \subseteq Y'$. Show that $\varphi|_{X'} : X' \to Y'$ is a morphism.

3.11 Let X be any variety and let $P \in X$. Show there is a 1-1 correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

3.12 If P is a point on a variety X, then $\dim \mathcal{O}_P = \dim X$. [*Hint*: Reduce to the affine case and use (3.2c).]

3.13 *The Local Ring of a Subvariety.* Let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $\langle U, f\rangle$ where $U \subseteq X$ is open, $U \cap Y \neq 0$, and f is a regular function on U. We say $\langle U, f\rangle$ is equivalent to $\langle V, g\rangle$ if $f = g$ on $U \cap V$. Show that $\mathcal{O}_{Y,X}$ is a local ring, with residue field $K(Y)$ and dimension $= \dim X - \dim Y$. It is the local ring of Y on X. Note if $Y = P$ is a point we get $\mathcal{O}_P$ and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

3.14 *Projection from a Point.* Let $\mathbf{P}^n$ be a hyperplane in $\mathbf{P}^{n+1}$ and let $P \in \mathbf{P}^{n+1} - \mathbf{P}^n$. Define a mapping $\varphi : \mathbf{P}^{n+1} - P \to \mathbf{P}^n$ by $\varphi(Q) =$ the intersection of the unique line containing P and Q with $\mathbf{P}^n$.

- **(a)** Show that φ is a morphism.
- **(b)** Let $Y \subseteq \mathbf{P}^3$ be the twisted cubic curve which is the image of the 3-uple embedding of $\mathbf{P}^1$ (Ex. 2.12). If t,u are the homogeneous coordinates on $\mathbf{P}^1$, we say that Y is the curve given parametrically by $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$. Let $P = (0, 0, 1, 0)$, and let $\mathbf{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the plane, and find its equation.

3.15 *Products of Affine Varieties.* Let $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ be affine varieties.

- **(a)** Show that $X \times Y \subseteq A^{n+m}$ with its induced topology is irreducible. [*Hint*: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$, $i = 1, 2$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.] The affine variety $X \times Y$ is called the product of X and Y. Note that its topology is in general not equal to the product topology (Ex. 1.4).
- **(b)** Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.
- **(c)** Show that $X \times Y$ is a product in the category of varieties, i.e., show (i) the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms, and (ii) given a variety Z, and the morphisms $Z \to X$, $Z \to Y$, there is a unique morphism $Z \to X \times Y$ making a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, dashed] \arrow[dr] \arrow[drrr] & & X \times Y \arrow[dl] \arrow[dr] & \\ & X & & Y \end{tikzcd}$$

(d) Show that $\dim X \times Y = \dim X + \dim Y$.

3.16 *Products of Quasi-Projective Varieties.* Use the Segre embedding (Ex. 2.14) to identify $\mathbf{P}^n \times \mathbf{P}^m$ with its image and hence give it a structure of projective variety. Now for any two quasi-projective varieties $X \subseteq \mathbf{P}^n$ and $Y \subseteq \mathbf{P}^m$, consider $X \times Y \subseteq \mathbf{P}^n \times \mathbf{P}^m$

(a) Show that $X \times Y$ is a quasi-projective variety.

(b) If X, Y are both projective, show that $X \times Y$ is projective.

***(c)** Show that $X \times Y$ is a product in the category of varieties.

3.17 *Normal Varieties.* A variety Y is normal at a point $P \in Y$ if $\mathcal{O}_p$ is an integrally closed ring. Y is normal if it is normal at every point.

(a) Show that every conic in $\mathbf{P}^2$ is normal.

(b) Show that the quadric surfaces Q_1,Q_2 in $\mathbf{P}^3$ given by equations $Q_1 : xy = zw$; $Q2 : xy = z^2$ are normal (cf. (II. Ex. 6.4) for the latter.)

(c) Show that the cuspidal cubic $y^2 = x^3$ in $\mathbf{A}^2$ is not normal.

(d) If Y is affine, then Y is normal $\iff$ $A(Y)$ is integrally closed.

(e) Let Y be an affine variety. Show that there is a normal affine variety Y, and a morphism $\pi : \tilde{Y} \to Y$, with the property that whenever Z is a normal variety, and $\varphi : Z \to Y$ is a dominant morphism (i.e., $\varphi(Z)$ is dense in Y), then there is a unique morphism $e : Z \to \tilde{Y}$ such that $\varphi = \pi \circ \theta$. $\tilde{Y}$ is called the normalization of Y. You will need (3.9A) above.

3.18 *Projectively Normal Varieties.* A projective variety $Y \subseteq \mathbf{P}^n$ is projectively normal (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

(a) If Y is projectively normal, then Y is normal.

(b) There are normal varieties in projective space which are not projectively normal. For example, let Y be the twisted quartic curve in $\mathbf{P}^3$ given parametrically by $(x, y, z, w) = (t^4, t^3u, tu^3, u^4)$. Then Y is normal but not projectively normal. See (III, Ex. 5.6) for more examples.

(c) Show that the twisted quartic curve Y above is isomorphic to $\mathbf{P}^1$, which is projectively normal. Thus projective normality depends on the embedding.

3.19 *Automorphisms of* $\mathbf{A}^n$. Let $\varphi : \mathbf{A}^n \to \mathbf{A}^n$ be a morphism of $\mathbf{A}^n$ to $\mathbf{A}^n$ given by n polynomials $f_1, ..., f_n$ of n variables $x_1, ..., x_n$ Let $J = \det[\partial f_i / \partial x_j]$ be the *Jacobian* polynomial of φ.

(a) If φ is an isomorphism (in which case we call φ an automorphism of $\mathbf{A}^n$) show that J is a nonzero constant polynomial.

****(b)** The converse of **(a)** is an unsolved problem, even for $n = 2$. See, for example. Vitushkin [1].

3.20 Let Y be a variety of dimension $\geqslant 2$, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

(a) Show that f extends to a regular function on Y.

(b) Show this would be false for $\dim Y = 1$. See (III, Ex. 3.5) for generalization.

3.21 *Group Varieties.* A group variety consists of a variety Y together with a morphism $\mu : Y \times Y \to Y$, such that the set of points of Y with the operation given by μ is a group, and such that the inverse map $y \to y^{-1}$ is also a morphism of $Y \to Y$.

- **(a)** The *additive group* $\mathbf{G_a}$ is given by the variety $\mathbf{A}^1$ and the morphism $\mu : \mathbf{A}^2 \to \mathbf{A}^1$. defined by $\mu(a, b) = a + b$. Show it is a group variety.
- **(b)** The multiplicative qroup $\mathbf{G_m}$ is given by the variety $\mathbf{A}^1 - \{(0)\}$ and the morphism $\mu(a, b) = ab$. Show it is a group variety.
- **(c)** If G is a group variety, and X is any variety, show that the set $\operatorname{Hom}(X, \mathbf{G_a})$ has a natural group structure.
- **(d)** For any variety X, show that $\operatorname{Hom}(X, \mathbf{G_a})$ is isomorphic to $\mathcal{O}(X)$ as a group under addition.
- **(e)** For any variety X, show that $\operatorname{Hom}(X, \mathbf{G_m})$ is isomorphic to the group of units in $\mathcal{O}(X)$, under multiplication.