Atiyah-Macdonald Chapter 1 Exercises Feiyang Lin and Luke Trujillo

---

1. Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution: Let x be nilpotent and n the smallest integer such that $x^n = 0$. The case for $n = 2$ is clear because $(1+x)(1-x) = 1 - x^2 = 1$. Therefore, let $n > 2$, and let k be the smallest integer such that $n \leqslant 2^k$. Then observe that

$$(1+x)(1-x)\prod_{i=2}^{k}\left(1+x^{2^i}\right) = 1.$$

Hence we see that $1+x$ is a unit. By adjusting the above argument, and noting that the set of units of a ring form a group, we can get that the sum of a nilpotent element and a unit is a unit. □

---

2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n$ in $A[x]$. Prove that

i) f is a unit in $A[x] \iff a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent. [If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex. 1.]

ii) f is nilpotent $\iff a_0, a_1, \ldots a_n$ are nilpotent.

iii) f is a zero-divisor $\iff$ there exists a $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ $(0 \leqslant r \leqslant n)$.]

iv) f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff$ f and g are primitive.

Solution:

i) Suppose that f is a unit. Then there exists a polynomial g such that

$$fg = 1 \implies \sum_{k=0}^{n+m}\left(\sum_{i+j=k} a_ib_j\right)x^k = 1.$$

For this to be the case, we must have that $a_0b_0 = 1$, which implies that a_0 is a unit, and $a_nb_m = 0$, which implies that a_n is a zero divisor. We'll now argue that $a_n^{r+1}b_{m-r} = 0$ whenever $r > 0$ (here we treat $b_{m-r} = 0$ whenever $r > m$.) Since the base case is true, suppose that it is true for all integers less than $r \geqslant 0$. Then observe that the coefficient of x^{n+m-r} is given by

$$\sum_{i+j=m+n-r} a_ib_j = a_nb_{m-r} + a_{n-1}b_{m-r-1} + \cdots + a_{n-r}b_m = 0.$$

If we multiply this by a_n^r, we see that

$$a_n^r \cdot a_nb_{m-r} + a_n^r \cdot a_{n-1}b_{m-r-1} + \cdots + a_n^r \cdot a_{n-r}b_m = a_n^{r+1}b_{m-r}$$

where for every summand other than the first term we applied the induction hypothesis. This then implies that $a_n^{r+1}b_{m-r} = 0$. Hence, we have proved our claim by induction. Using our claim, note that $a_n^{m+1}b_0 = 0$. Since b_0 is a unit, we have that $a_n^{m+1} = 0$ which implies a_n is nilpotent.

ii) We can prove the forward statement by induction on degree. First note that the base case of $\deg f = 0$ is immediate. Therefore, let f have degree $n > 0$ and suppose the (forward) statement is true for polynomials of degree $n-1$. Since f is nilpotent, we have that $f^r = 0$ for some integer r. In particular, we have that $a_n^rx^{n+r} = 0 \implies a_n^r = 0$. This shows that a_n is nilpotent. Now observe that

$$f - a_nx^n = a_0 + \cdots a_{n-1}x^{n-1}.$$

The left hand side is nilpotent, and so $a_0 + \cdots + a_{n-1}x^{n-1}$ must be nilpotent. We can then apply our induction hypothesis to conclude that $a_0, \ldots, a_{n-1}$ must be nilpotent. This proves the forward direction.

To prove the reverse direction, suppose a_i are nilpotent. Then f^r is a polynomial whose coefficients are (up to a scalar multiple) of the form

$$a_0^{k_0}a_1^{k_1}\cdots a_n^{k_n}$$

where k_i are nonnegative and sum to r. Since each a_i is nilpotent, take $r = \operatorname{ord}(a_0) + \cdots + \operatorname{ord}(a_n)$. Then each coefficient $a_0^{k_0} a_1^{k_1} \cdots a_n^{k_n}$ of f^r must be zero, since at for at least one i, we have that $k_i \geqslant \operatorname{ord}(a_i)$ (or else the k_i powers cannot sum to r). In taking r to be this value, we can see that each coefficient of f^r is zero, which implies that f is nilpotent. This proves the reverse direction, and completes the if and only if proof.

iii) We follow the hint which proves the base case of our induction: as g is supposed to have minimal degree for which $fg = 0$, we can only have that $a_n g = 0$. To prove further that $a_{n-r} g = 0$, we suppose that the statement is true for all nonnegative integers less than r. Observe that the coefficient of x^{m+n-r} is given by

$$\sum_{i+j=m+n-r} a_i b_j = a_n b_{m-r} + a_{n-1} b_{m-r-1} + \cdots + a_{n-(r+1)} b_{m+1} + a_{n-r} b_m = 0.$$

By our induction hypothesis, each term $a_n b_{m-r}, \ldots, a_{n-(r+1)} b_{m+1}$ must be zero since $a_n g, \ldots, a_{n-(r+1)} g$ are all zero. This leaves just $a_{n-r} b_m = 0$, from which we can again deduce that $a_{n-r} g = 0$ as it has degree less than m and annihilates f. Hence, we have that the statement is true for all nonnegative integers.

Using this claim, we can then prove the main result by picking a nonzero coefficient of g. Take for instance b_m: Then $b_m f = b_m a_0 + \cdots + b_m a_n x^n = 0$. This proves the main result.

iv) If fg is primitive, then this implies that the ideal generated by the coefficients of fg is the entire ring. Explicitly, there exist coefficients $c_0, \ldots, c_{n+m}$ such that

$$c_0 \left(\sum_{i+j=0} a_i b_j \right) + c_1 \left(\sum_{i+j=1} a_i b_j \right) + \cdots + c_{n+m} \left(\sum_{i+j=n+m} a_i b_j \right) = 1$$

Since the above is a linear relation, we can rearrange the coefficients to obtain a summation of coefficients in a_i to 1. This can similarly be done for b_i. Hence $(a_0, \ldots, a_n) = (1)$ and $(b_0, \ldots, b_m) = (1)$.

$\square$

---

3. Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Solution:

i) We claim that a multivariate polynomial $f(x_1, \ldots, x_n) = \sum a(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n}$ is a unit if and only if $a(0, \ldots, 0)$ is a unit and the rest of $a(i_1, \ldots, i_n)$ are nilpotent. We can prove this by induction on the number of variables: Suppose the statement is true for polynomials in $n-1$ variables. Then any polynomial $f(x_1, \ldots, x_n)$ in n-variables is technically a polynomial in $A[x_1, \ldots, x_{n-1}][x_n]$. By Exercise 2.1, this holds if and only if each $a(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_{n-1}}$ is nilpotent with $a(0, \ldots, 0)$ a unit. However, this occurs if and only if the $a(i_1, \ldots, i_n)$ (of course excluding $a(0, \ldots, 0)$) are all nilpotent, which proves the result.

ii) We claim that a multivariate polynomial is nilpotent if and only if each of its coefficients are nilpotent in A. This is achieved by induction on the number of indeterminates as in the previous example exercise

iii) We can prove this similarly to Exercise 1.2.3. First, we define the degree of a multivariate monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ to be the sum of the degrees of each α_i. We denote this degree by $(\alpha_1, \ldots, \alpha_n)$. We then impose a total ordering on monomial degree via lexicographical ordering. In other words, we say $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ has less degree than $x_1^{\beta_1} \cdots x_n^{\beta_n}$ if and only if $\alpha_i \leqslant \beta_i$ for all i.

With that said, let $f(x_1, \ldots, x_n)$ be a zero divisor with degree $(k_1, \ldots, k_n)$, and let $g(x_1, \ldots, x_n)$ be the polynomial with least degree $(m_1, \ldots, m_n)$. If we express

$$f = \sum a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \quad g = \sum b_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

Then we see that $a(k_1, \ldots, k_n) \cdot b_{m_1, \ldots, m_n} = 0$. Hence $a(k_1, \ldots, k_n) \cdot g$ annihilates f, but has less degree than g, and so it must be zero.

We show by induction that $a_{i_1, \ldots, i_n} g = 0$ for each coefficient $a_{i_1, \ldots, i_n}$ of f. To see this, suppose f has j-many monomial terms; then order them and denote them as $a_0, \ldots, a_j$; similarly suppose g has ℓ-many monomial terms and order them as $b_0, \ldots, b_\ell$. To perform induction, suppose that $a_{j-r} g = 0$ for all $0 \leqslant r < t$ where $0 \leqslant t < j$ (note the base case is true). Then observe that

$$\begin{aligned}
fh = 0 &\implies (a_1 x_1^{i_1^{(1)}} \cdots x_n^{i_n^{(1)}} + \cdots + a_j x_1^{i_1^{(j)}} \cdots x_n^{i_n^{(j)}}) h = 0 \\
&\implies (a_1 x_1^{i_1^{(1)}} \cdots x_n^{i_n^{(1)}} + \cdots + a_k x_1^{i_1^{(k)}} \cdots x_n^{i_n^{(k)}}) h = 0 \\
&\implies a_k b_\ell = 0 \\
&\implies a_k h = 0.
\end{aligned}$$

This completes our inductive step. We then take any nonzero coefficient of h and can now use our above result to conclude that f may be annihilated by a single element of A, as desired.

□

---

4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution: It is clear that the nilradical is contained in the Jacobson radical. Therefore, let f be in the Jacobson radical. Then $1 - fg$ is a unit for all $g \in A[x]$. In particular, $1 - f \cdot x$ is a unit. By Ex 1.2.1, this implies that the coefficients of f are nilpotent. By Ex 1.2.2, this implies that f is nilradical, so the Jacobson is contained in the nilradical. This then implies that they are equal. □

---

5. Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that

i) f is a unit in $A[[x]] \iff a_0$ is a unit in A.

ii) If f is nilpotent, then a_n is nilpotent for all $n \geqslant 0$. Is the converse true? (See Chapter 7, Exercise 2.)

iii) f belongs to the Jacobson radical of $A[[x]] \iff a_0$ belongs to the Jacobson radical of A.

iv) The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution:

i) Suppose $f = \sum_{n=0}^{\infty} a_n x^n$ is a power series and a_0 is a unit. We prove that there exists an inverse of $g = \sum_{n=0}^{\infty} b_n x^n$ by inductively constructing b_n.

For $n = 0$, we let b_0 be the inverse of a_0. Thus suppose that the coefficients have been defined for all nonnegative integers less than $n > 0$. Then observe that we require

$$\sum_{i=0}^{n} a_i b_{n-i} = 0.$$

The above is a linear equation which is in terms of $b_0, \ldots, b_n$. We can use our induction hypothesis to solve for b_n and construct it in this way. We see by induction that we may construct an inverse g, demonstrating that f is a unit.

ii) Suppose f is nilpotent. Then clearly a_0 is nilpotent. Suppose now that a_r is nilpotent for all $0 \leqslant r < k$ where $0 < k < n$. To show that a_k is nilpotent, note that

$$f - (a_0 + a_1 x + \cdots + a_r x^r) = \sum_{i=n}^{\infty} a_i x^i$$

is nilpotent. Hence, $\sum_{i=n}^{\infty} a_i x^i = x^k \sum_{i=n}^{\infty} a_i x^{i-k}$ is nilpotent, which implies that $\sum_{i=n}^{\infty} a_i x^{i-k}$ is nilpotent. In this case, we clearly have that a_k is nilpotent, which proves our inductive step. We thus have the general claim by induction.

iii) We know that f is in the Jacobson $\iff$ $1 - fg$ is a unit for all $g \in A[[x]]$. However, this is the case if and only if $1 - a_0 y$ is a unit for all $y \in A$, so that a_0 is in the Jacobson if and ony if f is in the Jacobson of $A[[x]]$.

□

---

6. A ring A is such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Solution: We already know that the nilradical is contained in the Jacobson radical. To show the opposite inclusion, suppose the contrary. Then since the Jacobson is not contained in the nilradical, there exists an element e in the Jacobson such that $e^2 = e \neq 0$. As an element of the Jacobson we know that $1 - e$ is a unit. However, note that $(1 - e)(1 + e) = 1 - e^2 = 1 - e$. Thus $(1 - e)(1 + e - 1) = 0 \implies e(1 - e) = 0$. However, $e \neq 0$ and $1 - e$ is not a zero divisor, so this is a contradiction. Hence the Jacobson is contained in the nilradical, so that they are equal. □

---

7. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution: Let P be a prime ideal, and consider the ring A/P. Consider an element $a+P$ in A/P. Then $(a+P)^n = a+P$ for some n. In particular, we have that $(a+P)((a+P)^{n-1} - (1+P)) = 0$. Since A/P is an integral domain, this implies that $(a+P)^{n-1} = 1+P$. Hence, $(a+P)$ is invertible. Since this was an arbitray element this implies that A/P is a field, so that P is maximal. □

8. Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

9. Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \iff \mathfrak{a}$ is an intersection of prime ideals.

10. Let A be a ring, $\mathfrak{N}$ its nilradical. Show that the following are equivalent:

i) A has exactly one prime ideal;
ii) every element of A is either a unit or nilpotent;
iii $A/\mathfrak{N}$ is a field.

Solution:

i $\iff$ ***ii*:** Suppose A has exactly on prime ideal. Then it must be a maximal ideal. Additionally, since $\mathfrak{N}$ is the intersection of all the prime ideals of A, this means that $\mathfrak{N}$ is maximal. Therefore, every element of A is either a unit or a nilpotent element.

Conversely, if every nonunit is nilpotent, this means that every non unit must be in $\mathfrak{N}$. Additionally, $\mathfrak{N}$ must be maximal because there cannot be any maximal ideal containing $\mathfrak{N}$ as every nonunit is nilpotent. Hence A has exactly one prime ideal.

i $\iff$ ***iii*:** If A has exactly one prime ideal, then again, $\mathfrak{N}$ must be prime, in fact it must be maximal. Therefore, $A/\mathfrak{N}$ is a field. If conversely we are given the fact that $A/\mathfrak{N}$ is a field, this implies that $\mathfrak{N}$ is maximal. Hence, the intersection of the prime ideals of A must consist of one element, so that A has exactly one prime ideal.

□

11. A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that

i) $2x = 0$ for all $x \in A$
ii) Every prime ideal $\mathfrak{p}$ is maximal and $A/\mathfrak{p}$ is a field with two elements;
iii) Every finitely generated ideal in A is principal.

12. A local ring contains no idempotent $\neq 0, 1$.

13. *Construction of an algebraic closure of a field* (E. Artin).
Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{a}$, and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2 , and so on. Let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\overline{K}$ be the set of all elements of L which are algebraic over K. Then $\overline{K}$ is an algebraic closure of K.

14. In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

---

15. *The Prime Spectrum of a Ring*
Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.
ii) $V(0) = X$, $V(1) = \varnothing$
iii) if $(E_i)_{i\in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i)$$

iv) $V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.
These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology.* The topological space X is called the prime spectrum of A, and is written $\operatorname{Spec}(A)$.

---

16. Draw pictures of $\operatorname{Spec}(\mathbb{Z})$, $\operatorname{Spec}(\mathbb{R})$, $\operatorname{Spec}(\mathbb{C}[x])$, $\operatorname{Spec}(\mathbb{R}[x])$, $\operatorname{Spec}(\mathbb{Z}[x])$.

---

17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

i) $X_f \cap X_g = X_{fg}$;
ii) $X_f = \varnothing \iff f$ is nilpotent;
iii) $X_f = X \iff f$ is a unit;
iv) $X_f = X_g \iff r((f)) = r((g))$;
v) X is quasi-compact (that is, every open covering of X has a finite subcovering)
vi) More generally, each X_f is quasi-compact.
vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.

[To prove (v), remark that it is enough to consider a covering of X by basic open sets $X_{f_i} (i \in I)$. Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i\in J} g_i f_i \qquad (g_i \in A)$$

where J is some *finite* subset of I. Then $X_{f_i} (i \in J)$ cover X.

---

18. For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \operatorname{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that

i) the set $\{x\}$ is closed (we say that x is a "closed point") in $\operatorname{Spec}(A) \iff \mathfrak{p}_x$ is maximal;
ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;
iii) $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subset \mathfrak{p}_y$;
iv) X is a T_0-space (this means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x).

---

19. A topological space X is said to be irreducible if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that $\mathrm{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

---

20. Let X be a topological space.

i) If Y is an irreducible (Exercise 19) subspace of X, then the closure $\overline{Y}$ of Y in X is irreducible.

ii) Every irreducible subspace of X is contained in a maximal irreducible subspace.

iii) The maximal irreducible subspaces of X are closed and cover X. They are called the *irreducible components* of X. What are the irreducible components of a Hausdorff space?

iv) If A is a ring and $X = \mathrm{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$ where $\mathfrak{p}$ is a minimal prime ideal of A (Exercise 8).
