Hartshorne 1.2 Exercises: Projective Varieties Feiyang Lin and Luke Trujillo

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Exercise 2.1 Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbf{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.

Solution:

□

Exercise 2.2 For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(*i.*) $Z(\mathfrak{a}) = \emptyset$ (the empty set);
(*ii.*) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
(*iii.*) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Solution: (i) $\Rightarrow$ (ii). Since $Z(\mathfrak{a}) = \emptyset$, the zero set of $\mathfrak{a}$ in affine space is either $\emptyset$ or $\{\mathbf{0}\}$. In the first case, we certainly have $\sqrt{\mathfrak{a}} = S$. In the second case, we have $\sqrt{\mathfrak{a}} = \{p \in S : p(0) = 0\} = S_+$.

(ii) $\Rightarrow$ (iii). It suffices to show that there exists d such that all monomials of degree d lies in $\mathfrak{a}$. Since $S_+ \subset \sqrt{\mathfrak{a}}$, for each $0 \le i \le n$, there exists d_i such that $x_i^{d_i} \in \mathfrak{a}$. Let $d = n\sum_{i=0}^n d_i$. Then if a monomial has degree d, then there exists i such that the exponent of x_i in the monomial is at least $\sum_{i=0}^n d_i$, and therefore at least d_i, which implies that the monomial is in $\mathfrak{a}$.

(iii) $\Rightarrow$ (i). If $S_d \subseteq \mathfrak{a}$ for some $d > 0$, then $x_i \in \sqrt{\mathfrak{a}}$ for every $0 \le i \le n$, which means that $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) = \emptyset$. □

Exercise 2.3

(1) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.
(2) If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.
(3) For any two subsets Y_1, Y_2 of $\mathbf{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
(4) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
(5) For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \overline{Y}$.

Solution:

□

Exercise 2.4

(1) There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbf{P}^n$ and homogeneous radical ideals of S not equal to S_+ given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note:* Since S_+ does not occur in this correspondence, it is sometimes called te *irrelevant* maximal ideal of S.
(2) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.
(3) Show that $\mathbf{P}^n$ itself is irreducible.

Solution:

□

Exercise 2.5

(1) $\mathbf{P}^n$ is a noetherian topological space.
(2) Every algebraic set in $\mathbf{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

Solution:

□

for 2.12(b)

We prove the harder direction that $Z(\mathfrak{a}) \subseteq \operatorname{im}(\rho_d)$. We may index the $N+1$ coordinates of a point in $\mathbf{P}^n$ by tuples of the form $(a_0, a_1, \ldots, a_n)$ where $a_i \in \mathbb{Z}_+$ and the sum of all a_i's is d. Given $\mathbf{y} \in Z(\mathfrak{a})$, I claim that there exists $0 \le i \le n$ such that $y_{d\mathbf{e}_i} \neq 0$. Indeed, suppose towards the contrary. Then since for any index $\mathbf{v} = (a_0, a_1, \ldots, a_n)$, $p(\mathbf{y}) = y_{\mathbf{v}}^d - \prod_{i=0}^n y_{d\mathbf{e}_i}^{a_i} \in \mathfrak{a}$, we have that $p(\mathbf{y}) = y_{\mathbf{v}}^d = 0$, which implies that $y_{\mathbf{v}} = 0$ for arbitrary $\mathbf{v}$, which is absurd.

So fix some i and some representative of $\mathbf{y}$ such that $y_{d\mathbf{e}_i} = 1$. Let $\mathbf{x} \in \mathbf{P}^n$ be such that $x_i = y_{d\mathbf{e}_i} = 1$ and $x_j = y_{d\mathbf{e}_i - \mathbf{e}_i + \mathbf{e}_j}$. We claim that $\rho_d(\mathbf{x}) = \mathbf{y}$. By construction, we already have that $x_i^{d-1} x_j = x_j = y_{d\mathbf{e}_i - \mathbf{e}_i + \mathbf{e}_j}$. It's straightforward to check that the polynomials in $\mathfrak{a}$ ensures that $y_{\mathbf{v}} = M_{\mathbf{v}}(\mathbf{x})$.

To give an example to make this proof clearer, consider the example when $n = 1$ and $d = 3$. Then if $\mathbf{y} = (y_{30}, y_{21}, y_{12}, y_{03}) \in Z(\mathfrak{a})$, suppose WLOG that $y_{30} = 1$. Then $\mathbf{y} = \rho_d(1, y_{21})$. We check that since $y_{30} y_{12} - y_{21}^2 = 0$, indeed $y_{12} = y_{21}^2$. Similarly, since $y_{03} y_{30}^2 - y_{21}^3$, indeed $y_{03} = y_{21}^3$.

2.12(c)

The map ρ_d is clearly a bijection between $\mathbf{P}^n$ and $\operatorname{im} \rho_d = Z(\mathfrak{a})$. So it suffices to show that ρ_d is bicontinuous, or equivalently, that it identifies the closed sets in $\mathbf{P}^n$ and $Z(\mathfrak{a})$.

(ρ_d continuous.) We claim that for any ideal $I \subset k[y_0, \ldots, y_N]$,

$$\rho_d^{-1}(Z(I)) = Z(\theta(I)).$$

Notice that if $(x_0, \ldots, x_n) \in \rho_d^{-1}(Z(I))$, then $p(M_0(\mathbf{x}), \ldots, M_N(\mathbf{x})) = 0$ for all $p(y_0, \ldots, y_N) \in I$. If $(x_0, \ldots, x_n) \in Z(\theta(I))$, then for all $p(y_0, \ldots, y_N) \in I$, $\theta(p)(x_0, \ldots, x_n) = 0$. But $\theta(p)(x_0, \ldots, x_n) = p(M_0(\mathbf{x}), \ldots, M_N(\mathbf{x}))$. So these two conditions are equivalent.

(ρ_d^{-1} continuous.) We claim that for any ideal $J \subset k[x_0, \ldots, x_n]$,

$$\rho_d(Z(J)) = Z(\theta^{-1} J) \cap Z(\mathfrak{a}).$$

Indeed, if $\mathbf{y} = \rho_d(\mathbf{x})$ where $\mathbf{x} \in Z(J)$, then for any $p \in \theta^{-1} J$, $p(\mathbf{y}) = p(\rho_d(\mathbf{x})) = 0$ because $p \circ \rho_d = \theta(p) \in J$. Conversely, if $\mathbf{y} \in Z(\theta^{-1} J) \cap Z(\mathfrak{a})$, then by part (ii), since $Z(\mathfrak{a}) = \operatorname{im} \rho_d$, there exists $\mathbf{x}$ such that $\mathbf{y} = \rho_d(\mathbf{x})$. We check that $\mathbf{x} \in Z(J)$: for all q such that $q \circ \rho_d \in J$, we have that $q(\mathbf{y}) = q(\rho_d(\mathbf{x})) = 0$. In other words, for all $p \in J$, $p(\mathbf{x}) = 0$.