

Exercise 2.1 Prove the “homogeneous Nullstellensatz,” which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in \mathbf{P}^n , then $f^q \in \mathfrak{a}$ for some $q > 0$.

Solution: Let \mathfrak{a} be a homogeneous ideal of $k[x_0, \dots, x_n]$. Then in \mathbf{P}^n , we have that

$$Z(\mathfrak{a}) = \left\{ P \in \mathbf{P}^n \mid \text{For all } f \in \mathfrak{a}, f(P) = 0 \right\}$$

while in \mathbf{A}^{n+1} , we have that

$$Z'(\mathfrak{a}) = \left\{ Q \in \mathbf{A}^{n+1} \mid \text{For all } f \in \mathfrak{a}, f(Q) = 0 \right\}.$$

We can write a surjective map $\pi : Z'(\mathfrak{a}) \rightarrow Z(\mathfrak{a})$ sending an affine point to its projective equivalence class. In addition, for a projective point $P \in Z(\mathfrak{a})$, we can observe that $\pi^{-1}(P) \subset Z'(\mathfrak{a})$ (the elements of P 's equivalence class).

Thus, if f is homogeneous with a nonzero degree and $f(P) = 0$ for all $P \in Z(\mathfrak{a})$, then $f(Q) = 0$ for all $Q \in Z'(\mathfrak{a})$. By the usual Nullstellensatz, this implies that $f^q \in \mathfrak{a}$ for some $q > 0$, which proves the result. \square

Exercise 2.2 For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

- (i.) $Z(\mathfrak{a}) = \emptyset$ (the empty set);
- (ii.) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
- (iii.) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Solution:

- (i) \implies (ii): Suppose \mathfrak{a} is a homogeneous ideal and $Z(\mathfrak{a}) = \emptyset$. By the homogeneous Nullstellensatz, it is vacuously true that for every $f \in \bigoplus_{d>0} S_d$, $f(P) = 0$ for all $P \in Z(\mathfrak{a})$, and so for every such f there exists a $q > 0$ such that $f^q \in \mathfrak{a}$. Therefore, $\bigoplus_{d>0} S_d \subseteq \sqrt{\mathfrak{a}}$. If \mathfrak{a} contains a unit, then $\sqrt{\mathfrak{a}} = S$. If \mathfrak{a} does not contain a unit, then $\sqrt{\mathfrak{a}} \subseteq \bigoplus_{d>0} S_d \implies \bigoplus_{d>0} S_d = \sqrt{\mathfrak{a}}$.

- (ii) \implies (iii): Suppose $\sqrt{\mathfrak{a}} = S$. Then this implies that $1 \in \mathfrak{a} \implies \mathfrak{a} = S$. Hence $S_d \subset \mathfrak{a}$.

Alternatively, suppose $\sqrt{\mathfrak{a}} = \bigoplus_{d>0} S_d$. Then for every $f \in \bigoplus_{d>0} S_d$, there exists a $q > 0$ such that $f^q \in \mathfrak{a}$. In particular, there exist r_1, \dots, r_n such that

$$x_1^{r_1}, \dots, x_n^{r_n} \in \mathfrak{a}.$$

Take $d = r_1 + \dots + r_n$. We claim that $S_d \subset \mathfrak{a}$. To see this, note that every degree- d homogeneous polynomial is of the form

$$\sum_{k \geq 0} c_k x_1^{\alpha_1(k)} \dots x_n^{\alpha_n(k)}$$

where only finitely many summands are nonzero and $\alpha_1(k) + \dots + \alpha_n(k) = d$ is a sum of nonnegative integers. Since $d = r_1 + \dots + r_n$, we know that for any summand $x_1^{\alpha_1(k)} \dots x_n^{\alpha_n(k)}$ at least one $\alpha_i(k) \geq r_i$. Hence, the summand is in \mathfrak{a} , and so the whole sum is in \mathfrak{a} . Therefore $S_d \subseteq \mathfrak{a}$.

- (iii) \implies (i): Suppose $S_d \subseteq \mathfrak{a}$ for $d > 0$. Then $Z(\mathfrak{a})$ must at least contain points P such that $f(P) = 0$ for all $f \in S_d$. This includes the polynomials x_1^d, \dots, x_n^d . However, these n polynomials cannot all be simultaneously zero. Hence $Z(\mathfrak{a}) = \emptyset$.

(i) \implies (ii). Since $Z(\mathfrak{a}) = \emptyset$, the zero set of \mathfrak{a} in affine space Did you mean projective space? is either \emptyset or $\{0\}$. In the first case, we certainly have $\sqrt{\mathfrak{a}} = S$. In the second case, we have $\sqrt{\mathfrak{a}} = \{p \in S : p(0) = 0\} = S_+$.

(ii) \implies (iii). It suffices to show that there exists d such that all monomials of degree d lies in \mathfrak{a} . Since $S_+ \subset \sqrt{\mathfrak{a}}$, for each $0 \leq i \leq n$, there exists d_i such that $x_i^{d_i} \in \mathfrak{a}$. Let $d = n \sum_{i=0}^n d_i$. Then if a monomial has degree d , then there exists i such that the exponent of x_i in the monomial is at least $\sum_{i=0}^n d_i$, and therefore at least d_i , which implies that the monomial is in \mathfrak{a} .

(iii) \implies (i). If $S_d \subseteq \mathfrak{a}$ for some $d > 0$, then $x_i \in \sqrt{\mathfrak{a}}$ for every $0 \leq i \leq n$, which means that $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) = \emptyset$. \square

Exercise 2.3

- (a) If $T_1 \subseteq T_2$ are subsets of S^h , then $Z(T_1) \supseteq Z(T_2)$.
- (b) If $Y_1 \subseteq Y_2$ are subsets of \mathbf{P}^n , then $I(Y_1) \supseteq I(Y_2)$.

- (c) For any two subsets Y_1, Y_2 of \mathbf{P}^n , $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
- (d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
- (e) For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \overline{Y}$.

Solution: I did (a) – (c), but I don't feel like \TeX -ing them.

- (d) Suppose f is homogeneous and $f \in I(Z(\mathfrak{a}))$. Then $f(P) = 0$ for all $P \in Z(\mathfrak{a})$. By the homogeneous Nullstellensatz, we see that $f \in \sqrt{\mathfrak{a}}$.
Suppose $f \in \mathfrak{a}$. Then $f^q \in \mathfrak{a}$ for some $q > 0$. As $f^q(P) = 0$ for all $P \in Z(\mathfrak{a})$, $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ (as k is an integral domain) and so it follows that $f \in I(Z(\mathfrak{a}))$. Our total work then shows that $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, as desired.
- (e) We first know that $Y \subset Z(I(Y))$; we show that it is the smallest closed set containing Y . Thus let $Z(J)$ be a closed set where $Y \subset Z(J) \subset I(Z(Y))$. Then $I(Z(J)) \subset I(Y)$. Since $I(Z(J)) = \sqrt{J}$, we see that $J \subset I(Y)$. Therefore, $Z(I(Y)) \subset Z(J)$. Hence, $Z(I(Y)) = Z(J)$, and so $Z(I(Y)) = \overline{Y}$.

□

Exercise 2.4

- (a) There is a one-to-one inclusion-reversing correspondence between algebraic sets in \mathbf{P}^n and homogeneous radical ideals of S not equal to S_+ given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note:* Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S .
- (b) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.
- (c) Show that \mathbf{P}^n itself is irreducible.

Solution:

□

Exercise 2.5

- (a) \mathbf{P}^n is a noetherian topological space.
- (b) Every algebraic set in \mathbf{P}^n can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

Solution:

□

2.6 If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint:* Let $\phi_i : U_i \rightarrow \mathbf{A}^n$ be the homeomorphism of (2.2), let Y_i be the affine variety $\phi_i(Y \cap U_i)$ and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at the transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is empty.]

Solution:

□

2.7

- (a) $\dim \mathbf{P}^n = n$.
- (b) If $Y \subseteq \mathbf{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.

Solution:

□

2.8 A projective variety $Y \subseteq \mathbf{P}^n$ has dimension $n - 1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in \mathbf{P}^n .

Solution:

□

2.9 If $Y \subseteq \mathbf{A}^n$ is an affine variety, we identify \mathbf{A}^n with an open set $U_0 \subset \mathbf{P}^n$ by the homeomorphism φ_0 . Then we can speak of \overline{Y} , the closure of Y in \mathbf{P}^n , which is called the *projective closure* of Y .

Solution:

- (a) Since $\overline{Y} = Z(I(\phi_o Y))$, $I(\overline{Y}) = I(\phi_o Y)$. So it suffices to show that $I(\phi_o Y) = \beta(I(Y))$. We have $I(\phi_o Y) \supseteq \beta(I(Y))$ because $\phi_o(Y) \subseteq Z(\beta(I(Y)))$. Conversely, we have $I(\phi_o Y) \subseteq \beta(I(Y))$ because any point of Y vanishes on $p(1, x_1, \dots, x_n)$ for $p \in I(\phi_o(Y))$. In other words, if $p \in I(\phi_o(Y))$, then $p(1, x_1, \dots, x_n) \in I(Y)$, which implies that $p \in \beta(I(Y))$.
- (b) Recall that the twisted cubic is the variety $Y = \{(t, t^2, t^3) : t \in k\}$. We claim that $I(Y) = J = (y - x^2, z - x^3)$. Since $Z(J) = Y$, it suffices to check that J is radical. Notice that $k[x, y, z]/J \cong k[x]$ has no nilpotents, which shows that J is radical. Let $f_1 = y - x^2$ and $f_2 = z - x^3$. We claim also that $\beta f_1 = wy - x^2$ and $\beta f_2 = w^2 z - x^3$ don't generate $I(\overline{Y}) = I(\phi_o(Y))$. Since $\phi_o(Y) = \{[1 : t : t^2 : t^3] : t \in k\}$, we have $wz - xy \in I(\overline{Y})$. But this cannot be generated by βf_1 and βf_2 since the only term involving z in βf_1 and βf_2 is $w^2 z$.

□

2.10 Let $Y \subset \mathbf{P}^n$ be a nonempty algebraic set, and let $\theta: \mathbf{A}^{n+1} \setminus \{(0, \dots, 0)\} \rightarrow \mathbf{P}^n$ be the map which sends the point with affine coordinates (a_0, \dots, a_n) to the point with homogeneous coordinates $[a_0 : \dots : a_n]$. We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \dots, 0)\}.$$

- (a) Show that $C(Y)$ is an algebraic set in \mathbf{A}^{n+1} , whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \dots, x_n]$.
- (b) $C(Y)$ is irreducible if and only if Y is irreducible.
- (c) $\dim C(Y) = \dim Y + 1$.

Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in \mathbf{P}^{n+1} . This is called the *projective cone* over Y .

Solution:

□

2.11 A hypersurface defined by a linear polynomial is called a *hyperplane*.

- (a) Show that the following two conditions are equivalent for a variety $Y \subset \mathbf{P}^n$:
 - (i.) $I(Y)$ can be generated by linear polynomials
 - (ii.) Y can be written as an intersection of hyperplanes.
 In this case we say that Y is a *linear variety* in \mathbf{P}^n .
- (b) If Y is a linear variety of dimension r in \mathbf{P}^n , show that $I(Y)$ is minimally generated by $n - r$ linear polynomials
- (c) Let Y, Z be linear varieties in \mathbf{P}^n , with $\dim Y = r$, $\dim Z = s$. If $r + s - n \geq 0$, then $Y \cap Z \neq \emptyset$. Furthermore, if $Y \cap Z \neq \emptyset$, then $Y \cap Z$ is a linear variety of dimension $\geq r + s - n$ (Think of \mathbf{A}^{n+1} as a vector space over k , and work with its subspaces.)

Solution:

□

2.12 For given $n, d > 0$, let M_0, \dots, M_N be all the monomials of degree d in the $n + 1$ variables x_0, \dots, x_n , where $N = \binom{n+d}{d} - 1$. We define a map $\rho_d: \mathbf{P}^n \rightarrow \mathbf{P}^N$ by sending the point $P = (a_0, \dots, a_n)$ to the point $\rho_d(P) = (M_0(a), \dots, M_N(a))$ obtained by substituting the a_i in the monomials M_j . This is called the *d-uple embedding* of \mathbf{P}^n in \mathbf{P}^N . For example, if $n = 1, d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of \mathbf{P}^1 in \mathbf{P}^2 is a conic.

- (a) Let $\theta: k[y_0, \dots, y_N] \rightarrow k[x_0, \dots, x_n]$ be the homomorphism defined by sending y_i to M_i , and let \mathfrak{a} be the kernel of θ . Then \mathfrak{a} is homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in \mathbf{P}^N .
- (b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$.

- (c) Now show that ρ_d is a homeomorphism of \mathbf{P}^n onto the projective variety $Z(\mathfrak{a})$.
 (d) Show that the twisted cubic curve in \mathbf{P}^3 (Ex. 2.9) is equal to the 3-uple embedding of \mathbf{P}^1 in \mathbf{P}^3 , for suitable choice of coordinates.

Solution:

- (a)
 (b) To give an example to make this proof clearer, consider the example when $n = 1$ and $d = 3$. Then if $\mathbf{y} = (y_{30}, y_{21}, y_{12}, y_{03}) \in Z(\mathfrak{a})$, suppose WLOG that $y_{30} = 1$. Then $\mathbf{y} = \rho_d(1, y_{21})$. We check that since $y_{30}y_{12} - y_{21}^2 = 0$, indeed $y_{12} = y_{21}^2$. Similarly, since $y_{03}y_{30} - y_{21}^3$, indeed $y_{03} = y_{21}^3$.
 (c) The map ρ_d is clearly a bijection between \mathbf{P}^n and $\text{im } \rho_d = Z(\mathfrak{a})$. So it suffices to show that ρ_d is bicontinuous, or equivalently, that it identifies the closed sets in \mathbf{P}^n and $Z(\mathfrak{a})$.
 (ρ_d continuous.) We claim that for any ideal $I \subset k[y_0, \dots, y_N]$,

$$\rho_d^{-1}(Z(I)) = Z(\theta(I)).$$

Notice that if $(x_0, \dots, x_n) \in \rho_d^{-1}(Z(I))$, then $p(M_0(\mathbf{x}), \dots, M_N(\mathbf{x})) = 0$ for all $p(y_0, \dots, y_N) \in I$. If $(x_0, \dots, x_n) \in Z(\theta(I))$, then for all $p(y_0, \dots, y_N) \in I$, $\theta(p)(x_0, \dots, x_n) = 0$. But $\theta(p)(x_0, \dots, x_n) = p(M_0(\mathbf{x}), \dots, M_N(\mathbf{x}))$. So these two conditions are equivalent.

(ρ_d^{-1} continuous.) We claim that for any ideal $J \subset k[x_0, \dots, x_n]$,

$$\rho_d(Z(J)) = Z(\theta^{-1}J) \cap Z(\mathfrak{a}).$$

Indeed, if $\mathbf{y} = \rho_d(\mathbf{x})$ where $\mathbf{x} \in Z(J)$, then for any $p \in \theta^{-1}J$, $p(\mathbf{y}) = p(\rho_d(\mathbf{x})) = 0$ because $p \circ \rho_d = \theta(p) \in J$. Conversely, if $\mathbf{y} \in Z(\theta^{-1}J) \cap Z(\mathfrak{a})$, then by part (ii), since $Z(\mathfrak{a}) = \text{im } \rho_d$, there exists \mathbf{x} such that $\mathbf{y} = \rho_d(\mathbf{x})$. We check that $\mathbf{x} \in Z(J)$: for all q such that $q \circ \rho_d \in J$, we have that $q(\mathbf{y}) = q(\rho_d(\mathbf{x})) = 0$. In other words, for all $p \in J$, $p(\mathbf{x}) = 0$.

□

2.13 Let Y be the image of the 2-uple embedding of \mathbf{P}^2 in \mathbf{P}^5 . This is the *Veronese surface*. If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbf{P}^5$ such that $V \cap Y = Z$.

Solution:

□

2.14 Let $\psi: \mathbf{P}^r \times \mathbf{P}^s \rightarrow \mathbf{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_i b_j, \dots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a subvariety of \mathbf{P}^N .

Solution:

□

2.15 Consider the surface Q (a *surface* is variety of dimension 2) in \mathbf{P}^3 defined by the equation $xy - zw = 0$.

- (a) Show that Q is equal to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in \mathbf{P}^3 , for suitable choice of coordinates.
 (b) Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbf{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, $M_t \cap M_u = \emptyset$, and for all t, u , $L_t \cap M_u = \text{one point}$.
 (c) Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbf{P}^1 \times \mathbf{P}^1$ (where each \mathbf{P}^1 has its Zariski topology).

Solution:

□

- (a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in \mathbf{P}^3 given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.
- (b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in \mathbf{P}^2 given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P , but that $I(C) + I(L) \neq I(P)$.

Solution:

□

2.17 A variety Y of dimension r in \mathbf{P}^n is a (*strict*) *complete intersection* if $I(Y)$ can be generated by $n - r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.

- (a) Let Y be a variety in \mathbf{P}^n , let $Y = Z(\mathfrak{a})$; and suppose that \mathfrak{a} can be generated by q elements. Then show that $\dim Y \geq n - q$.
- (b) Show that a strict complete intersection is a set-theoretic complete intersection.
- (c) The converse of (b) is false. For example let Y be the twisted cubic curve in \mathbf{P}^3 (Ex. 2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees 2, 3 respectively, such that $Y = H_1 \cap H_2$.
- (d) It is an unsolved problem whether every closed irreducible curve in \mathbf{P}^3 is a set-theoretic intersection of two surfaces.

Solution:

- (a) If $\mathfrak{a} =$

□

3.15

Solution:

- (a) As hinted, let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$ for $i = 1, 2$. We first prove that $X = X_1 \cup X_2$. If this doesn't happen, then there exists $y_1, y_2 \in Y$ such that $(x, y_1) \in Z_1 \setminus Z_2$, $(x, y_2) \in Z_2 \setminus Z_1$. However, this implies that $Z_1 \cap x \times Y$ and $Z_2 \cap x \times Y$ are both (non-empty) proper subsets of $x \times Y$. But they are both closed, and $\{x\} \times Y$ is irreducible, which is a contradiction.

□