# Hartshorne 1.1 Exercises: Affine Varieties    Feiyang Lin and Luke Trujillo

## 1.1

(a) Let $Y$ be the plane curve $y = x^2$ (i.e., $Y$ is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$

(b) Let $Z$ be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over $k$.

(c) Let $f$ be any irreducible quadratic polynomial in $k[x, y]$, and let $W$ be the conic defined by $f$. Show that $A(W)$ is isomorphic to $A(Y)$ and $A(Z)$. Which one is it and when?

*Solution:*

(a) Consider the map $\varphi : k[x, y] \to k[x]$ where $\varphi(p(x, y)) = p(x, x^2)$. As this is (1) surjective and (2) has kernel $(y - x^2)$, we see that
$$k[x, y]/(y - x^2) \cong k[x].$$
Hence $(y - x^2)$ is prime. Moreover, if we denote $Y = Z(y - x^2)$, then we see that
$$A(Y) \cong k[x, y]/I(Y) = k[x, y]/I(Z(y - x^2)) = k[x, y]/(y - x^2) \cong k[x].$$
Therefore, $A(Y) \cong k[x]$.

(b) Consider the map $\varphi : k[x, y] \to k[x, 1/x]$ where $\varphi(p(x, y)) = p(x, 1/x)$. This is surjective with kernel $(xy - 1)$. This then gives us
$$k[x, y]/(xy - 1) \cong k[x, 1/x] \ncong k[x].$$
Denote $Y = Z(xy - 1)$. Note that $xy - 1$ is irreducible in $k[x, y]$. Hence, $(xy - 1)$ is prime. Moreover,
$$A(Y) \cong k[x, y]/I(Z(xy - 1)) = k[x, y]/(xy - 1) \ncong k[x].$$
Thus $A(Y) \ncong k[x]$.

(c) Let $f = x^2 + axy + by^2 + cx + dy + e$. Suppose $b$ is a perfect square. Then
$$f = (x + by)^2 + cx + dy + e.$$
Write $X = x + a$. Then $f = X^2 + cx + dy + e$.

$\square$

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**1.2** *The Twisted Cubic Curve.* Let $Y \subseteq \mathbf{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that $Y$ is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$. We say that $Y$ is given by the *parametric representation* $x = t$, $y = t^2$, $z = t^3$).

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*Solution:*   Construct the map $\varphi : k[x, y, z] \to k[x]$ were $\varphi(p(x, y, z)) = p(x, x^2, x^3)$. Then the kernel of the map is $(x^2 - y, x^3 - z)$. Therefore,
$$k[x, y, z]/(x^2 - y, x^3 - z) \cong k[x].$$
(I think my dimension calculation here can be greatly simplified by using the above isomorphism.)  Hence $Y = Z(x^2 - y, x^3 - z)$ closed, irreducible, and hence an affine variety. Now observe that
$$(x^2 - y) \subset (x^2 - y, x^3 - z)$$
as prime ideals. Thus $(x^2 - y, x^3 - z)$ corresponds to an ideal $J$ of $k[x, y, z]/(x^2 - y)$. In fact, $J$ is generated by the coset $x^3 - z + (x^2 - y)$. As this is not a unit in $k[x, y, z]/(x^2 - y)$, we may conclude that $J$ has height of one by Theorem 1.11A. We then have by Theorem 1.8A that
$$\mathrm{ht}(J) + \dim \left( (k[x, y, z]/(x^2 - y))/J \right) = \dim \left( k[x, y, z]/(x^2 - y) \right)$$

However, we know that
$$\mathrm{ht}(x^2 - y) + \dim\left(k[x,y,z]/(x^2-y)\right) = \dim(k[x,y,z]).$$

By Theorem 1.11A, $\mathrm{ht}(x^2 - y) = 1$ since $x^2 - y$ is not a zero divisor or unit. In addition, $\dim(k[x,y,z]) = 3$ by Proposition 1.9. Hence

$$\dim\left(k[x,y,z]/(x^2-y)\right) = 1 \implies \mathrm{ht}(J) + \dim\left((k[x,y,z]/(x^2-y))/(x^3-z)\right) = 2$$

As we know $\mathrm{ht}(J) = 1$, we see that $\dim\left((k[x,y,z]/(x^2-y))/J\right) = 1$. But

$$\dim\left((k[x,y,z]/(x^2-y))/J\right) = \dim\left((k[x,y,z]/(x^2-y, x^3-z)\right) = \dim(Z(x^2-y, x^3-z)).$$

Therefore, $\dim(Z(x^2 - y, x^3 - z)) = 1$.

Finally, observe that $I(Y) = I(Z(x^2-y, x^3-z)) = (x^2-y, x^3-z)$. Thus the generators are just $x^2 - y$ and $x^3 - z$.

$\square$

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**1.3** Let $Y$ be the algebraic set in $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that $Y$ is a union of three irreducible components. Describe them and find their prime ideals.

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*Solution:* Since $Y = Z(x^2 - y, xz - z)$, we see that it consists of all $\mathbf{A}^3$ that satisfy:

$$\begin{cases} x^2 - yz = 0 \\ xz - x = 0 \end{cases}$$

There are three main ways we can satisfy the above equations.

- We could set $z = 1 \implies x = y^2$. This consists of $Z(z - 1, x = y^2)$.

- We could set $z = x = 0$. This consists of the points of $Z(x, z)$.

- Finally, we could set $x = y = 0$. This consists of $Z(x, y)$.

Thus $Z(z - 1, x - y^2) \cup Z(x, z) \cup Z(x, y) \subset Y$. It is not hard to see that conversely any $(x_0, y_0, z_0) \in Y$ must be in one of the three sets. Therefore, $Y = Z(z - 1, x - y^2) \cup Z(x, z) \cup Z(x, y)$. Moreover, each of these are affine varities, and as none are contained in any other, we see that these are the unique irreducible components of $Y$.

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**1.4** If we identify $\mathbf{A}^2$ with $\mathbf{A}^1 \times \mathbf{A}^1$ in the natural way, show that the Zariski topology on $\mathbf{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbf{A}^1$.

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*Solution:*

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**1.5** Show that a $k$-algebra $B$ is isomorphic to the affine coordinate ring of some algebraic set in $\mathbf{A}^n$, for some $n$, if and only if $B$ is a finitely generated $k$-algebra with no nilpotent elements.

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*Solution:* If $B$ is a $k$-algebra isomorphic to an affine coordinate ring, then

$$B \cong k[x_1, \ldots, x_n]/I(Y)$$

with $Y$ an affine variety. By definition, this is a finitely genereated $k$-algebra. As it is also an integral domain, $B$ cannot have any nilpotents.

Converseley, suppose $B$ is finitely generated and has no nilpotents. By definition, there exists elements $b_1, \ldots, b_n \in B$ and a map $\varphi : k[x_1, \ldots, x_n] \to B$ where $p(x_1, \ldots, x_n) \mapsto p(b_1, \ldots, b_n)$. This establishes the isomorphism

$$B \cong k[x_1, \ldots, x_n]/\ker(\varphi).$$

Since $B$ has no nilpotents, every ideal is radical. Therefore,

$$B \cong k[x_1, \ldots, x_n]/\ker(\varphi) \cong k[x_1, \ldots, x_n]/I(Z(\ker(\varphi))).$$

Hence, $B$ is isomorphic to the affine coordinate ring of $Z(\ker(\varphi))$, which is an algebraic set. $\qquad\square$

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**1.6** Any nonempty open subset of an irreducible topological space is dense and irreducible. If $Y$ is a subset of a topological space $X$, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

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*Solution:* We first prove the first sentence. Let $U$ be a nonempty open subset of $X$, an irreducible space. Observe that $\overline{U} \cup U^c = X$. Since $X$ is irreducible and $U$ is nonempty, we see that $\overline{U} = X$. Therefore, $U$ is dense.

Now suppose $U$ was reducible (in its subspace topology). Then this implies that $U = Y_1 \cup Y_2$ with $Y_1, Y_2$ closed and proper (in $U$'s subspace topology). Now we may express $Y_1 = Z_1 \cap U$ with $Z_1$ closed in $X$; similarly, there is a closed $Z_2$ corresponding to $Y_2$. Therefore,

$$U \subset Z_1 \cup Z_2 \implies \overline{U} \subset \overline{Z_1 \cup Z_2} \implies X = Z_1 \cup Z_2.$$

Hence either $X = Z_1$ or $Z_2$, so $Y_1$ or $Y_2$ is either $U$, contradicting our assumption that $Y_1$ and $Y_2$ are proper. Therefore, $U$ is irreducible.

Now we prove the second sentence. Let $Y$ be irreducible in its subspace topology, and suppose $\overline{Y}$ is reducible in $X$. Then there exists proper, closed subsets $Z_1$, $Z_2$ of $\overline{Y}$ such that $\overline{Y} = Z_1 \cup Z_2$. Hence, $Y = (Y \cap Z_1) \cup (Y \cap Z_2)$, which implies that $Y = Z_1$ or $Y = Z_2$. However, this implies that $\overline{Y} = Z_1$ or $Z_2$, a contradiction. Therefore $\overline{Y}$ is irreducible.
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**1.7**

**(a)** Show that the following conditions are equivalent for a topological space $X$: $(i)$ $X$ is noetherian; $(ii)$ every nonempty family of closed subsets has a minimal element; $(iii)$ $X$ satisfies the ascending chain condition for open subsets; $(iv)$ every nonempty family of open subsets has a maximal element.

**(b)** A noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.

**(c)** Any subset of a noetherian topological space is noetherian in its induced topology.

**(d)** A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

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*Solution:*

**(a)** First note that $(ii) \implies (i)$ and $(iv) \implies (iii)$ are immediate by definition of a Noetherian space.

We show $(iii) \implies (iv)$. Since the ascending chain condition is satisfied, we may use Zorn's Lemma to deduce that any nonempty family of open subsets has a maximal element (we order it by inclusion, then apply the lemma). We can prove $(ii) \implies (i)$ similarly.

**(b)** Let $X$ be a Noetherian space and suppose $\mathcal{U} = \{U_i\}_{i \in \lambda}$ is an open cover of $X$. By **(a)**, there exists a maximal element $V_1$ of $\mathcal{U}$. Using $V_1$ as our base case, inductively build the sets

$$V_{i+1} = \max\left(\left\{ U_i \in \mathcal{U} \;\middle|\; U_i \not\subset V_1 \cup \cdots \cup V_i \right\}\right) \qquad i = 1, 2, \ldots$$

The maximum will exist by repeatedly applying **(a)**. Now the chain

$$V_1 \subset V_1 \cup V_2 \subset \cdots V_1 \cup \cdots \cup V_j \subset \cdots$$

must have stabilize for some finite number of unions. This then implies that $X = V_1 \cup \cdots \cup V_r$ for some $r$. Hence, $V_1, \ldots, V_r$ is our finite subcover of $\mathcal{U}$, so that $X$ is compact.

$\qquad\square$

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**1.8** Let $Y$ be an affine variety of dimension $r$ in $\mathbf{A}^n$. Let $H$ be a hypersurface in $\mathbf{A}^n$, and assume $Y \nsubseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$. (See (7.1) for a generalization.)

*Solution:* First denote $Y = Z(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal in $k[x_1, \ldots, x_n]$. By Corollary 1.6, we can express the algebraic set $Y \cap H$ uniquely as

$$Y \cap H = V_1 \cup \cdots \cup V_\ell$$

where each $V_i$ is an affine variety and $V_i \not\subset V_j$ for $i \neq j$. For each affine variety $V_i$ denote $V_i = Z(\mathfrak{p}_i)$ with $\mathfrak{p}_i$ prime. We make some observations.

- Each prime ideal $\mathfrak{p}_i$ contains $\mathfrak{p}$, and hence corresponds to a prime ideal $\mathfrak{p}'_i$ in $k[x_1, \ldots, x_n]/\mathfrak{p}$.

- Since $Y \not\subset H$, we know that $(f) \not\subset \mathfrak{p}$ which implies $f \notin \mathfrak{p}$. Hence, we see that $f + \mathfrak{p} \in k[x_1, \ldots, x_n]/\mathfrak{p}$ is not a zero divisor (as it is an integral domain). It is also not a unit as $f$ is irreducible.

- The ring $k[x_1, \ldots, x_n]/\mathfrak{p}$ is Noetherian. Thus, by Theorem 1.11A, every minimal prime ideal in $k[x_1, \ldots, x_n]/\mathfrak{p}$ containing $f + \mathfrak{p}$ must have height one.

Our claim is that each $\mathfrak{p}'_i$ is a minimal prime ideal containing $f + \mathfrak{p}$. Assuming this is true, we can observe by Theorem 1.8A that

$$\operatorname{ht}(\mathfrak{p}'_i) + \dim\left((k[x_1, \ldots, x_n]/\mathfrak{p})/\mathfrak{p}'_i\right) = \dim\left(k[x_1, \ldots, x_n]/\mathfrak{p}\right) \implies 1 + \dim\left(A(V_i)\right) = r$$
$$\implies \dim\left(A(V_i)\right) = r - 1$$

Thus we show the claim. Suppose $\mathfrak{q}$ is a prime ideal in $k[x_1, \ldots, x_n]/\mathfrak{p}$ containing $f + \mathfrak{p}$, and that $\mathfrak{q} \subseteq \mathfrak{p}'_i$ for some $i$. Then $\mathfrak{q}$ corresponds to a prime ideal $\mathfrak{q}'$ of $k[x_1, \ldots, x_n]$ (1) containing $\mathfrak{p}$ and (2) containing $f$. However,

$$\sqrt{\langle \mathfrak{p}, f \rangle} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_\ell$$

and as the radical of $\langle \mathfrak{p}, f \rangle$ (the smallest ideal containing $\mathfrak{p}$ and $f$) is the intersection of all prime ideals containing this ideal, we see that $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_\ell \subseteq \mathfrak{q}'$. By Proposition 1.1.11(b) in Atiyah-MacDonald, this implies that $\mathfrak{p}_j \subseteq \mathfrak{q}'$. However, it must be that $j = i$, since none of these prime ideals are contained in each other. This then implies that $\mathfrak{p}'_i \subseteq \mathfrak{q}$ in $k[x_1, \ldots, x_n]/\mathfrak{p}$, which gives us that $\mathfrak{q} = \mathfrak{p}'_i$. Hence, $\mathfrak{p}_i$ is a minimal prime ideal in $k[x_1, \ldots, x_n]/\mathfrak{p}$ containing $f + \mathfrak{p}$, and so we may apply our above calculation. This then shows that

$$\dim(A(V_i)) = r - 1$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

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**1.9** Let $\mathfrak{a} \subseteq A = k[x_1, \ldots, x_n]$ be an ideal which can be generated by $r$ elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n - r$.

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*Solution:* We prove this by induction.

**Base Case.** Consider an ideal $\mathfrak{a} = (a)$ which is generated by a single element. We assume $\mathfrak{a}$ is not all of $k[x_1, \ldots, x_n]$ (i.e., $a$ is not a unit); otherwise, $Z(\mathfrak{a})$ is empty, which is not irreducible, and further it does not make sense to talk about the irreducible components for the empty set.

- Suppose $a = 0$. Then $\mathfrak{a} = 0 \implies Z(\mathfrak{a}) = \mathbf{A}^n$ which is irreducible and has dimension $n$.
- Suppose $a$ is not a unit and is nonzero. Since $k[x_1, \ldots, x_n]$ is a UFD, then we may uniquely express $a$ as $a = u \cdot f_1 \cdots f_m$ with $u$ a unit, eac $f_i$ irreducible. We then have that

$$Z(\mathfrak{a}) = Z(f_1) \cup Z(f_2) \cdots \cup Z(f_m).$$

Now by Theorem 1.11A, we can conclude that for each $i = 1, 2, \ldots, m$, $\operatorname{ht}(f_i) = 1$. Hence

$$\operatorname{ht}(f_i) + \dim(Z(f_i)) = n \implies \dim(Z(f_i)) = n - 1.$$

Hence, every irreducible component of $Z(\mathfrak{a})$ has dimension $n - 1$.

In each case we see that the irreducible components of $(a)$ have dimension $\dim \geq n - 1$, which proves the base case.

**Inductive Step.** Let $\mathfrak{a} = (a_1, \ldots, a_r)$ be our ideal, and suppose the statement is true for all ideals generated by $(r-1)$-many elements. Let $a_i$ be nonzero. Denote the decomposition of $Z(a_1, \ldots, a_r)$ into its irreducible components as below

$$Z(a_1, \ldots, a_r) = V_1 \cup \cdots \cup V_\ell$$

with $V_j = Z(\mathfrak{p}_j)$. Similarly for $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r)$ write

$$Z(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r) = Y_1 \cup \cdots \cup Y_m.$$

with $Y_j = Z(\mathfrak{q}_j)$.

Observe that for each $j = 1, 2, \ldots, \ell$,

$$\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m \subset \mathfrak{p}_j.$$

By Proposition 1.11(b) in Atiyah MacDonald, this implies that $\mathfrak{q}_s \subset \mathfrak{p}_j$ for some $s = 1, 2, \ldots, m$. Thus, denote $\mathfrak{p}_j'$ as the prime ideal in $k[x_1, \ldots, x_n]/\mathfrak{q}_s$ corresponding to $\mathfrak{p}_j$. By Theorem 1.8A, we have that

$$\mathrm{ht}(\mathfrak{p}_j') + \dim \left( (k[x_1, \ldots, x_n]/\mathfrak{q}_s) / \mathfrak{p}_j' \right) = \dim(k[x_1, \ldots, x_n]/\mathfrak{q}_s).$$

Now $\mathfrak{p}_i'$ is a minimal prime ideal containing $a_i + \mathfrak{q}_s$, which is not a unit or zero divisor. Hence, its height is one. Therefore,

$$1 + \dim(V_i) = \dim(k[x_1, \ldots, x_n]/\mathfrak{q}_s) \geq n - (r-1) \implies \dim(V_i) \geq n - r.$$

This completes the inductive step and the proof is complete.

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**1.10**

(a) If $Y$ is any subset of a topological space $X$ then $\dim Y \leq \dim X$.

(b) If $X$ is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.

(c) Give an example of a topological space $X$ and a dense open subset $U$ with $\dim U < \dim X$.

(d) If $Y$ is a closed subset of an irreducible finite-dimensional topological space $X$, and if $\dim Y = \dim X$, then $Y = X$.

(e) Give an example of a noetherian topological space of infinite dimension.

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*Solution:*

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**1.11** Let $Y \subseteq \mathbf{A}^3$ be the curve given parametrically by $x = t^3$, $y = t^4$, $z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say $Y$ is *not a local complete intersection*—cf. *(Ex. 2.17)*.

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*Solution:* Construct the map $\varphi : k[x, y, z] \to k[t]$ where $p(x, y, z) = f(t^3, t^4, t^5)$. The kernel of this map is given by $I(Y) = \{f \in k[x, y, z] \mid f(t^3, t^4, t^5) = 0 \text{ for all } t \in k\}$. However, the map is not surjective. Thus we have that

$$k[x, y, z]/I(Y) \cong \mathrm{Im}(\varphi).$$

Since $k[t]$ is an integral domain, and $\mathrm{Im}(\varphi)$ is a subring, this nevertheless implies that $I(Y)$ is a prime ideal. $\square$

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**1.12** Give an example of an irreducible polynomial $f \in \mathbf{R}[x, y]$, whose zero set $Z(f)$ in $\mathbf{A}_{\mathbf{R}}^2$ is not irreducible *(cf. 1.4.2)*.

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*Solution:*

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