Hartshorne 1.3 Exercises: Morphisms

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3.1

(a) Show that any conic in $\mathbf{A}^2$ is isomorphic either to $\mathbf{A}^1$ or $\mathbf{A}^1 - \{0\}$ *(cf. Ex.* 1.1*)*.
(b) Show that $\mathbf{A}^1$ is *not* isomorphic to any proper open subset of itself. (This result is generalized by *(Ex.* 6.7*)* below.)
(c) Any conic in $\mathbf{P}^2$ is isomorphic to $\mathbf{P}^1$.
(d) We will see later *(Ex.* 4.8*)* that any two curves are homeomorphic. But show now that $\mathbf{A}^2$ is not even homeomorphic to $\mathbf{P}^2$.
(e) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

3.2 A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

(a) For example, let $\phi : \mathbf{A}^1 \to \mathbf{A}^2$ be defined by $t \mapsto ({}^t2, t^3)$. Show that ϕ defines a bijective bicontinuous morphism of $\mathbf{A}^1$ onto the curve $y^2 = x^3$, but that ϕ is not an isomorphism.
(b) For another example, let the characteristic of the base field k be $p > 0$, and define a map $\phi : \mathbf{A}^1 \to \mathbf{A}^2$ by $t \to t^p$. Show that ϕ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism.*

3.3

(a) Let $\phi : X \to Y$ be a morphism. Then for each $P \in X$, ϕ induces a homomorphism of local rings $\phi_P^* : \mathcal{O}_{\phi(P),Y} \to \mathcal{O}_{P,X}$.
(b) Show that a morphism ϕ is an isomorphism if and only if ϕ is a homeomorphism, and the induced map ϕ_P^* on local rings is an isomorphism, for all $p \in X$.
(c) Show that if $\phi(X)$ is dense in Y, then the map ϕ_P^* is injective for all $p \in X$.

3.4 Show that the d-uple embedding of $\mathbf{P}^n$ (Ex. 2.12) is an isomorphism onto its image.

3.5 By abuse of language, we will say that a variety "is affine" if it is isomorphic to an affine variety. If $H \subseteq \mathbf{P}^n$ is any hypersurface, show that $\mathbf{P}^n \setminus H$ is affine. [*Hint*: Let H have degree d. Then consider the d-uple embedding of $\mathbf{P}^n$ in $\mathbf{P}^n$ and use the fact that $\mathbf{P}^n$ minus a hyperplane is affine.]

3.6 There are quasi-affine varieties which are not affine. For example, show that $X = \mathbf{A}^2 - \{(0,0)\}$ is not affine. [*Hint*: Show that $\mathcal{O} \cong k[x,y]$ and use (3.5). See (III, Ex. 4.3) for another proof.]

3.7

(a) Show that any two curves in $\mathbf{P}^2$ have a nonempty intersection.
(b) More generally, show that if $Y \subseteq \mathbf{P}^n$ is a projective variety of dimension $\geqslant 1$, and if H is a hypersurface, then $Y \cap H \neq \varnothing$. [*Hint*: Use (Ex. 3.5) and (Ex. 3.1e). See (7.2) for a generalization.]

3.8 Let H_i and H_j be the hyperplanes in $\mathbf{P}^n$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Show that any regular function on $\mathbf{P}^n - (H_i \cap H_j)$ is constant. (This gives an alternate proof of (3.4a) in the case $Y = \mathbf{P}^n$.)

3.9 The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbf{P}^1$, and let Y be the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$. Then $X \cong Y$ (Ex. 3.4). But show that $S(X) \ncong S(Y)$.

3.10 *Subvarieties.* A subset of a topological space is locally closed if it is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.

If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset, then Y is also a quasi-affine (respectively, quasi-projective) variety, by virtue of being a locally closed subset of the same affine or projective space. We call this the induced structure on Y, and we call Y a subvariety of X.

Now let $\phi : X \to Y$ be a morphism, let $X' \subseteq X$ and $Y' \subseteq Y$ be irreducible locally closed subsets such that $\phi(X') \subseteq Y'$. Show that $\phi|_{X'} : X' \to Y'$ is a morphism.

3.11 Let X be any variety and let $P \in X$. Show there is a 1-1 correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

3.12 If P is a point on a variety X, then $\dim \mathcal{O}_P = \dim X$. [*Hint*: Reduce to the affine case and use (3.2c).]

3.13 *The Local Ring of a Subvariety.* Let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $\langle U, f\rangle$ where $U \subseteq X$ is open, $U \cap Y \neq 0$, and f is a regular function on U. We say $\langle U, f\rangle$ is equivalent to $\langle V, g\rangle$ if $f = g$ on $U \cap V$. Show that $\mathcal{O}_{Y,X}$ is a local ring, with residue field $K(Y)$ and dimension $= \dim X - \dim Y$. It is the local ring of Y on X. Note if $Y = P$ is a point we get $\mathcal{O}_P$ and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

3.14 *Projection from a Point.* Let $\mathbf{P}^n$ be a hyperplane in $\mathbf{P}^{n+1}$ and let $P \in \mathbf{P}^{n+1} - \mathbf{P}^n$. Define a mapping ϕ : $\mathbf{P}^{n+1} - P \to \mathbf{P}^n$ by $\phi(Q) =$ the intersection of the unique line containing P and Q with $\mathbf{P}^n$.

- **(a)** Show that ϕ is a morphism.
- **(b)** Let $Y \subseteq \mathbf{P}^3$ be the twisted cubic curve which is the image of the 3-uple embedding of $\mathbf{P}^1$ (Ex. 2.12). If t,u are the homogeneous coordinates on $\mathbf{P}^1$, we say that Y is the curve given parametrically by $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$. Let $P = (0, 0, 1, 0)$, and let $\mathbf{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the plane, and find its equation.

3.15 *Products of Affine Varieties.* Let $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ be affine varieties.

- **(a)** Show that $X \times Y \subseteq A^{n+m}$ with its induced topology is irreducible. [Hint: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$, $i = 1, 2$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.] The affine variety $X \times Y$ is called the product of X and Y. Note that its topology is in general not equal to the product topology (Ex. 1.4).
- **(b)** Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties, i.e., show (i) the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms, and (ii) given a variety Z, and the morphisms $Z \to X$, $Z \to Y$, there is a unique morphism $Z \to X \times Y$ making a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, dashed] \arrow[dr] \arrow[drrr] & & X \times Y \arrow[dl] \arrow[dr] & \\ & X & & Y \end{tikzcd}$$

(d) Show that $\dim X \times Y = \dim X + \dim Y$.

3.16 *Products of Quasi-Projective Varieties.* Use the Segre embedding (Ex. 2.14) to identify $\mathbf{P}^n \times \mathbf{P}^m$ with its image and hence give it a structure of projective variety. Now for any two quasi-projective varieties $X \subseteq \mathbf{P}^n$ and $Y \subseteq \mathbf{P}^m$, consider $X \times Y \subseteq \mathbf{P}^n \times \mathbf{P}^m$

(a) Show that $X \times Y$ is a quasi-projective variety.
(b) If X, Y are both projective, show that $X \times Y$ is projective.
***(c)** Show that $X \times Y$ is a product in the category of varieties.

3.17 *Normal Varieties.* A variety Y is normal at a point $P \in Y$ if $\mathcal{O}_p$ is an integrally closed ring. Y is normal if it is normal at every point.

(a) Show that every conic in $\mathbf{P}^2$ is normal.
(b) Show that the quadric surfaces Q_1,Q_2 in $\mathbf{P}^3$ given by equations $Q_1 : xy = zw$; $Q2 : xy = z^2$ are normal (cf. (II. Ex. 6.4) for the latter.)
(c) Show that the cuspidal cubic $y^2 = x^3$ in $\mathbf{A}^2$ is not normal.
(d) If Y is affine, then Y is normal $\iff$ $A(Y)$ is integrally closed.
(e) Let Y be an affine variety. Show that there is a normal affine variety Y, and a morphism $\pi : \tilde{Y} \to Y$, with the property that whenever Z is a normal variety, and $\phi : Z \to Y$ is a dominant morphism (i.e., $\phi(Z)$ is dense in Y), then there is a unique morphism $e : Z \to \tilde{Y}$ such that $\phi = \pi \circ \theta$. $\tilde{Y}$ is called the normalization of Y. You will need (3.9A) above.

3.18 *Projectively Normal Varieties.* A projective variety $Y \subseteq \mathbf{P}^n$ is projectively normal (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

(a) If Y is projectively normal, then Y is normal.
(b) There are normal varieties in projective space which are not projectively normal. For example, let Y be the twisted quartic curve in $\mathbf{P}^3$ given parametrically by $(x, y, z, w) = (t^4, t^3u, tu^3, u^4)$. Then Y is normal but not projectively normal. See (III, Ex. 5.6) for more examples.
(c) Show that the twisted quartic curve Y above is isomorphic to $\mathbf{P}^1$, which is projectively normal. Thus projective normality depends on the embedding.

3.19 *Automorphisms of* $\mathbf{A}^m$. Let $\phi : \mathbf{A}^n \to \mathbf{A}^n$ be a morphism of $\mathbf{A}^n$ to $\mathbf{A}^n$ given by n polynomials $f_1, ..., f_n$ of n variables $x_1, ..., x_n$ Let $J = \det[\partial f_i / \partial x_j]$ be the *Jacobian* polynomial of ϕ.

(a) If ϕ is an isomorphism (in which case we call ϕ an automorphism of $\mathbf{A}^n$) show that J is a nonzero constant polynomial.
(b) The converse of **(a)** is an unsolved problem, even for $n = 2$. See, for example. Vitushkin [1].

3.20 Let Y be a variety of dimension $\geqslant 2$, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

(a) Show that f extends to a regular function on Y.

(b) Show this would be false for $\dim Y = 1$. See (III, Ex. 3.5) for generalization.

3.21 *Group Varieties.* A group variety consists ofa variety Y together with a morphism $\mu : Y \times Y \to Y$, such that the set of points of Y with the operation given by μ is a group, and such that the inverse map $y \to y^{-1}$ is also a morphism of $Y \to Y$.

(a) The *additive group* $\mathbf{G_a}$ is given by the variety $\mathbf{A}^1$ and the morphism $\mu : \mathbf{A}^2 \to \mathbf{A}^1$. defined by $\mu(a, b) = a + b$. Show it is a group variety.

(b) The multiplicative qroup $\mathbf{G_m}$ is given by the variety $\mathbf{A}^1 - \{(0)\}$ and the morphism $\mu(a, b) = ab$. Show it is a group variety.

(c) If G is a group variety, and X is any variety, show that the set $\operatorname{Hom}(X, \mathbf{G_a})$ has a natural group structure.

(d) For any variety X, show that $\operatorname{Hom}(X, \mathbf{G_a})$ is isomorphic to $\mathcal{O}(X)$ as a group under addition.

(e) For any variety X, show that $\operatorname{Hom}(X, \mathbf{G_m})$ is isomorphic to the group of units in $\mathcal{O}(X)$, under multiplication.