Exercise 2.1 Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbf{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.

Solution: Let $\mathfrak{a}$ be a homogeneous ideal of $k[x_0, \ldots, x_n]$. Then in $\mathbf{P}^n$, we have that

$$Z(\mathfrak{a}) = \left\{ P \in \mathbf{P}^n \,\middle|\, \text{For all } f \in \mathfrak{a}, f(P) = 0 \right\}$$

while in $\mathbf{A}^{n+1}$, we have that

$$Z'(\mathfrak{a}) = \left\{ Q \in \mathbf{A}^{n+1} \,\middle|\, \text{For all } f \in \mathfrak{a}, f(Q) = 0 \right\}.$$

We can write a surjective map $\pi : Z'(\mathfrak{a}) \to Z(\mathfrak{a})$ sending an affine point to its projective equivalence class. In addition, for a projective point $P \in Z(\mathfrak{a})$, we can observe that $\pi^{-1}(P) \subset Z'(\mathfrak{a})$ (the elements of P's equivalence class).

Thus, if f is homogeneous with a nonzero degree and $f(P) = 0$ for all $P \in Z(\mathfrak{a})$, then $f(Q) = 0$ for all $Q \in Z'(\mathfrak{a})$. By the usual Nullstellensatz, this implies that $f^q \in \mathfrak{a}$ for some $q > 0$, which proves the result. □

Exercise 2.2 For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(*i.*) $Z(\mathfrak{a}) = \emptyset$ (the empty set);
(*ii.*) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
(*iii.*) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Solution:

$(i) \implies (ii)$**:** Suppose $\mathfrak{a}$ is a homogeneous ideal and $Z(\mathfrak{a}) = \varnothing$. By the homogeneous Nullstellensatz, it is vacuously true that for every $f \in \oplus_{d>0} S_d$, $f(P) = 0$ for all $P \in Z(\mathfrak{a})$, and so for every such f there exists a $q > 0$ such that $f^q \in \mathfrak{a}$. Therefore, $\oplus_{d>0} S_d \subseteq \sqrt{a}$. If $\mathfrak{a}$ contains a unit, then $\sqrt{\mathfrak{a}} = S$. If $\mathfrak{a}$ does not contain a unit, then $\sqrt{\mathfrak{a}} \subseteq \oplus_{d>0} S_d \implies \oplus_{d>0} S_d = \sqrt{\mathfrak{a}}$.

$(ii) \implies (iii)$**:** Suppose $\sqrt{a} = S$. Then this implies that $1 \in \mathfrak{a} \implies \mathfrak{a} = S$. Hence $S_d \subset \mathfrak{a}$.

Alternatively, subbPose $\sqrt{a} = \oplus_{d>0} S_d$. Then for every $f \in \oplus_{d>0} S_d$, there exists a $q > 0$ such that $f^q \in \mathfrak{a}$. In particular, there exist $r_1, \ldots, r_n$ such that

$$x_1^{r_1}, \ldots, x_n^{r_n} \in \mathfrak{a}.$$

Take $d = r_1 + \cdots + r_n$. We claim that $S_d \subset \mathfrak{a}$. To see this, note that every degree-d homogeneous polynomial is of the form

$$\sum_{k \geqslant 0} c_k x_1^{\alpha_1(k)} \cdots x_n^{\alpha_n(k)}$$

where only finitely many summands are nonzero and $\alpha_1(k) + \cdots + \alpha_n(k) = d$ is a sum of nonnegative integers. Since $d = r_1 + \cdots + r_n$, we know that for any summand $x_1^{\alpha_1(k)} \cdots x_n^{\alpha_n(k)}$ at least one $\alpha_i(k) \geqslant r_i$. Hence, the summand is in $\mathfrak{a}$, and so the whole sum is in $\mathfrak{a}$. Therefore $S_d \subseteq \mathfrak{a}$.

$(iii) \implies (i)$**:** Suppose $S_d \subseteq \mathfrak{a}$ for $d > 0$. Then $Z(\mathfrak{a})$ must at least contain points P such that $f(P) = 0$ for all $f \in S_d$. This includes the polynomials $x_1^d, \ldots, x_n^d$. However, these n polynomials cannot all be simultaneously zero. Hence $Z(\mathfrak{a}) = \varnothing$.

(i) $\Rightarrow$ (ii). Since $Z(\mathfrak{a}) = \emptyset$, the zero set of $\mathfrak{a}$ in affine space Did you mean projective space? is either $\emptyset$ or $\{\mathbf{0}\}$. In the first case, we certainly have $\sqrt{\mathfrak{a}} = S$. In the second case, we have $\sqrt{\mathfrak{a}} = \{p \in S : p(0) = 0\} = S_+$.

(ii) $\Rightarrow$ (iii). It suffices to show that there exists d such that all monomials of degree d lies in $\mathfrak{a}$. Since $S_+ \subset \sqrt{\mathfrak{a}}$, for each $0 \leq i \leq n$, there exists d_i such that $x_i^{d_i} \in \mathfrak{a}$. Let $d = n \sum_{i=0}^n d_i$. Then if a monomial has degree d, then there exists i such that the exponent of x_i in the monomial is at least $\sum_{i=0}^n d_i$, and therefore at least d_i, which implies that the monomial is in $\mathfrak{a}$.

(iii) $\Rightarrow$ (i). If $S_d \subseteq \mathfrak{a}$ for some $d > 0$, then $x_i \in \sqrt{\mathfrak{a}}$ for every $0 \leq i \leq n$, which means that $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) = \emptyset$. □

Exercise 2.3

(a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.
(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.

(c) For any two subsets Y_1, Y_2 of $\mathbf{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
(d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
(e) For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \overline{Y}$.

Solution: I did **(a)** – **(c)**, but I don't feel like TEX-ing them.

(d) Suppose f is homogeneous and $f \in I(Z(\mathfrak{a}))$. Then $f(P) = 0$ for all $P \in Z(\mathfrak{a})$. By the homogeneous Nullstellensatz, we see that $f \in \sqrt{\mathfrak{a}}$.

Suppose $f \in \mathfrak{a}$. Then $f^q \in \mathfrak{a}$ for some $q > 0$. As $f^q(P) = 0$ for all $P \in Z(\mathfrak{a})$, $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ (as k is an integral domain) and so it follows that $f \in I(Z(\mathfrak{a}))$. Our total work then shows that $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, as desired.

(e) We first know that $Y \subset Z(I(Y))$; we show that it is the smallest closed set containing Y. Thus let $Z(J)$ be a closed set where $Y \subset Z(J) \subset I(Z(Y))$. Then $I(Z(J)) \subset I(Y)$. Since $I(Z(J)) = \sqrt{J}$, we see that $J \subset I(Y)$. Therefore, $Z(I(Y)) \subset Z(J)$. Hence, $Z(I(Y)) = Z(J)$, and so $Z(I(Y)) = \overline{Y}$.

□

Exercise 2.4

(a) There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbf{P}^n$ and homogeneous radical ideals of S not equal to S_+ given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note:* Since S_+ does not occur in this correspondence, it is sometimes called te *irrelevant* maximal ideal of S.
(b) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.
(c) Show that $\mathbf{P}^n$ itself is irreducible.

Solution:

□

Exercise 2.5

(a) $\mathbf{P}^n$ is a noetherian topological space.
(b) Every algebraic set in $\mathbf{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

Solution:

□

2.6 If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint*: Let $\phi_i : U_i \to \mathbf{A}^n$ be the homeomorphism of (2.2), let Y_i be the affine variety $\phi_i(Y \cap U_i)$ and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)x_i$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at the transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is empty.]

Solution:

□

2.7

(a) $\dim \mathbf{P}^n = n$.
(b) If $Y \subseteq \mathbf{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.

Solution:

□

2.8 A projective variety $Y \subseteq \mathbf{P}^n$ has dimension $n-1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbf{P}^n$.

Solution:

□

2.9 If $Y \subseteq \mathbf{A}^n$ is an affine variety, we identify $\mathbf{A}^n$ with an open set $U_0 \subset \mathbf{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbf{P}^n$, which is called the *projective closure* of Y.

Solution:

□

2.10 Let $Y \subset \mathbf{P}^n$ be a nonempty algebraic set, and let $\theta\colon\ \mathbf{A}^{n+1} \setminus \{(0, \cdots, 0\} \to \mathbf{P}^n$ be the map which sends the point with affine coordinates $(a_0, \cdots, a_n)$ to the point with homogeneous coordinates $[a_0 : \cdots : a_n]$. We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \cdots, 0)\}.$$

- **(a)** Show that $C(Y)$ is an algebraic set in $\mathbf{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \cdots, x_n]$.
- **(b)** $C(Y)$ is irreducible if and only if Y is irreducible.
- **(c)** $\dim C(Y) = \dim Y + 1$.

Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbf{P}^{n+1}$. This is called the *projective cone* over Y.

Solution:

□

2.11 A hypersurface defined by a linear polynomial is called a *hyperplane*.

- **(a)** Show that the following two conditions are equivalent for a variety $Y \subset \mathbf{P}^n$:
  - ($i.$) $I(Y)$ can be generated by linear polynomials
  - ($ii.$) Y can be written as an intersection of hyperplanes.

  In this case we say that Y is a *linear variety* in $\mathbf{P}^n$.
- **(b)** If Y is a linear variety of dimension r in $\mathbf{P}^n$, show that $I(Y)$ is minimally generated by $n-r$ linear polynomials
- **(c)** Let Y, Z be linear varieties in $\mathbf{P}^n$, with $\dim Y = r$, $\dim Z = s$. If $r + s - n \geq 0$, then $Y \cap Z \neq \emptyset$. Furthermore, if $Y \cap Z \neq \emptyset$, then $Y \cap Z$ is a linear variety of dimension $\geq r + s - n$ (Think of $\mathbf{A}^{n+1}$ as a vector space over k, and work with its subspaces.)

Solution:

□

2.12 For given $n, d > 0$, let $M_0, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \ldots x_n$, where $N = \binom{n+d}{n} - 1$. We define a mabbPing $\rho_d\colon \mathbf{P}^n \to \mathbf{P}^N$ by sending the point $P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbf{P}^n$ in $\mathbf{P}^N$. For example, if $n = 1, d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ is a conic.

- **(a)** Let $\theta\colon k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ be the homorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbf{P}^N$.
- **(b)** Show that the image of ρ_d is exactly $Z(\mathfrak{a})$.
- **(c)** Now show that ρ_d is a homeomorphism of $\mathbf{P}^n$ onto the projective variety $Z(\mathfrak{a})$.
- **(d)** Show that the twisted cubic curve in $\mathbf{P}^3$ *(Ex.* 2.9*)* is equal to the 3-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

Solution: for 2.12(b) We prove the harder direction that $Z(\mathfrak{a}) \subseteq \operatorname{im}(\rho_d)$. We may index the $N+1$ coordinates of a point in $\mathbf{P}^n$ by tuples of the form $(a_0, a_1, \ldots, a_n)$ where $a_i \in \mathbb{Z}_+$ and the sum of all a_i's is d. Given $\mathbf{y} \in Z(\mathfrak{a})$, I claim that there exists $0 \leq i \leq n$ such that $y_{d\mathbf{e}_i} \neq 0$. Indeed, subbPose towards the contrary. Then since for any index

$\mathbf{v} = (a_0, a_1, \ldots, a_n)$, $p(\mathbf{y}) = y_{\mathbf{v}}^d - \prod_{i=0}^n y_{d\mathbf{e}_i}^{a_i} \in \mathfrak{a}$, we have that $p(\mathbf{y}) = y_{\mathbf{v}}^d = 0$, which implies that $y_{\mathbf{v}} = 0$ for arbitrary $\mathbf{v}$, which is absurd.

So fix some i and some representative of $\mathbf{y}$ such that $y_{d\mathbf{e}_i} = 1$. Let $x_1, \ldots, x_n \in \mathbf{P}^n$ be such that $x_i = y_{d\mathbf{e}_i} = 1$ and $x_j = y_{d\mathbf{e}_i - \mathbf{e}_i + \mathbf{e}_j}$. We claim that $\rho_d(x_1, \ldots, x_n) = \mathbf{y}$. By construction, we already have that $x_i^{d-1} x_j = x_j = y_{d\mathbf{e}_i - \mathbf{e}_i + \mathbf{e}_j}$. It's straightforward to check that the polynomials in $\mathfrak{a}$ ensures that $y_{\mathbf{v}} = M_{\mathbf{v}}(x_1, \ldots, x_n)$.

To give an example to make this proof clearer, consider the example when $n = 1$ and $d = 3$. Then if $\mathbf{y} = (y_{30}, y_{21}, y_{12}, y_{03}) \in Z(\mathfrak{a})$, subbPose WLOG that $y_{30} = 1$. Then $\mathbf{y} = \rho_d(1, y_{21})$. We check that since $y_{30}y_{12} - y_{21}^2 = 0$, indeed $y_{12} = y_{21}^2$. Similarly, since $y_{03}y_{30}^2 - y_{21}^3$, indeed $y_{03} = y_{21}^3$.

□

2.13 Let Y be the image of the 2-uple embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$. This is the *Veronese surface*. If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbf{P}^5$ such that $V \cap Y = Z$.

Solution:

□

2.14 Let $\psi \colon \mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a subvariety of $\mathbf{P}^N$.

Solution:

□

2.15 Consider the surface Q (a *surface* is variety of dimension 2) in $\mathbf{P}^3$ defined by the equation $xy - zw = 0$.

- **(a)** Show that Q is equal to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.
- **(b)** Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbf{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, $M_t \cap M_u = \emptyset$, and for all $t, u, L_t \cap M_u =$ one point.
- **(c)** Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbf{P}^1 \times \mathbf{P}^1$ (where each $\mathbf{P}^1$ has its Zariski topology).

Solution:

□

2.16

- **(a)** The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbf{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.
- **(b)** Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbf{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

Solution:

□

2.17 A variety Y of dimension r in $\mathbf{P}^n$ is a *(strict) complete intersection* if $I(Y)$ can be generated by $n - r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.

- **(a)** Let Y be a variety in $\mathbf{P}^n$, let $Y = Z(\mathfrak{a})$; and subbPose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geqslant n - q$.
- **(b)** Show that a strict complete intersection is a set-theoretic complete intersection.
- **(c)** The converse of (*b*) is false. For example let Y be the twisted cubic curve in $\mathbf{P}^3$ *(Ex. 2.9)*. Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees $2, 3$ respectively, such that $Y = H_1 \cap H_2$.

(d) It is an unsolved problem whether every closed irreducible curve in $\mathbf{P}^3$ is a set-theoretic intersection of two surfaces.

Solution:

$\square$