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Graduate Texts in Mathematics

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Christian Kassel

Quantum Groups

With 88 Illustrations

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Preface

« Eh bien, Monsieur, que
pensez-vous des x et des y ? »
Je lui ai répondu :
« C'est bas de plafond. »
V. Hugo [Hug51]

The term "quantum groups" was popularized by Drinfeld in his address to the International Congress of Mathematicians in Berkeley (1986). It stands for certain special Hopf algebras which are nontrivial deformations of the enveloping Hopf algebras of semisimple Lie algebras or of the algebras of regular functions on the corresponding algebraic groups. As was soon observed, quantum groups have close connections with varied, a priori remote, areas of mathematics and physics.

The aim of this book is to provide an introduction to the algebra behind the words "quantum groups" with emphasis on the fascinating and spectacular connections with low-dimensional topology. Despite the complexity of the subject, we have tried to make this exposition accessible to a large audience. We assume a standard knowledge of linear algebra and some rudiments of topology (and of the theory of linear differential equations as far as Chapter XIX is concerned).

We divided the book into four parts we now briefly describe. In Part I we introduce the language of Hopf algebras and we illustrate it with the Hopf algebras $SL_q(2)$ and $U_q(\mathfrak{sl}(2))$ associated with the classical group SL_2. These are the simplest examples of quantum groups, and actually the only ones we treat in detail. Part II focuses on two classes of Hopf algebras that provide solutions of the Yang-Baxter equation in a systematic way. We review a method due to Faddeev, Reshetikhin, and Takhtadjian as well as Drinfeld's quantum double construction, both designed to produce quantum groups. Parts I and II may form the core of a one-year introductory course on the subject.

Parts III and IV are devoted to some of the spectacular connections alluded to before. The avowed objective of Part III is the construction of isotopy invariants of knots and links in $\mathbf{R}^3$, including the Jones polynomial, from certain solutions of the Yang-Baxter equation. To this end, we introduce various classes of tensor categories that are responsible for the close relationship between quantum groups and knot theory. Part IV presents more advanced material: it is an account of Drinfeld's elegant treatment of the monodromy of the Knizhnik-Zamolodchikov equations. Our aim is to highlight Drinfeld's deep result expressing the braided tensor category of modules over a quantum enveloping algebra in terms of the corresponding semisimple Lie algebra. We conclude the book with the construction of a "universal knot invariant". This is a nice, far-reaching application of the algebraic techniques developed in the preceding chapters.

I wish to acknowledge the inspiration I drew during the composition of this text from [Dri87] [Dri89a] [Dri89b] [Dri90] by Drinfeld, [JS93] by Joyal and Street, [Tur89] [RT90] by Reshetikhin and Turaev. After having become acquainted with quantum groups, the reader is encouraged to return to these original sources. Further references are given in the notes at the end of each chapter. Lusztig's and Turaev's monographs [Lus93] [Tur94] may complement our exposition advantageously.

This book grew out of two graduate courses I taught at the Department of Mathematics of the Université Louis Pasteur in Strasbourg during the years 1990–92. Part I is the expanded English translation of [Kas92]. It is a pleasure to express my thanks to C. Bennis, R. Berger, C. Mitschi, P. Nuss, C. Reutenauer, M. Rosso, V. Turaev, M. Wambst for valuable discussions and comments, and to Raymond Séroul who coded the figures. I owe special thanks to Patrick Ion for his marvellous job in preparing the book for printing, with his attention to mathematical, English, typographical, and computer details.

Christian Kassel
March 1994, Strasbourg

Notation. — Throughout the text, k is a field and the words "vector space", "linear map" mean respectively "k-vector space" and "k-linear map". The boldface letters $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ stand successively for the nonnegative integers, all integers, the field of rational, real, and complex numbers. The Kronecker symbol δ_{ij} is defined by $\delta_{ij} = 1$ if $i = j$ and is zero otherwise. We denote the symmetric group on n letters by S_n. The sign of a permutation σ is indicated by $\varepsilon(\sigma)$.

The symbol $\square$ indicates the end of a proof. Roman figures refer to the numbering of the chapters.

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Part One

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Chapter I
Preliminaries

The goal of this first chapter is the construction of polynomial algebras $GL(2)$ and $SL(2)$ modelling the 2×2-matrices with invertible determinant [resp. with determinant equal to 1]. The multiplication of matrices induces an additional structure on these algebras. This structure is one of the basic ingredients of what will be called a Hopf algebra in Chapter III. We complete the chapter with various concepts of ring theory to be used in the sequel. The ground field is denoted by k.

I.1 Algebras and Modules

We recall some facts on algebras and modules.

An *algebra* is a ring A together with a ring map $\eta_A : k \to A$ whose image is contained in the centre of A. The map $(\lambda, a) \mapsto \eta_A(\lambda)a$ from $k \times A$ to A equips A with a vector space structure over k and the multiplication map $\mu_A : A \times A \to A$ is bilinear.

A *morphism of algebras* or an *algebra morphism* is a ring map $f : A \to B$ such that

$$f \circ \eta_A = \eta_B. \tag{1.1}$$

As a consequence of (1.1), f preserves the units, i.e., we have $f(1) = 1$. The linear map $\eta_A : k \to A$ is a morphism of algebras. If $i : A \to B$ is an injective algebra morphism, we say that A is a *subalgebra* of the algebra B.

Let us denote by $\mathrm{Hom}_{Alg}(A, B)$ the set of algebra morphisms from A to B. In general, this set has no further structure. Nevertheless, we shall soon see how to put a group structure on $\mathrm{Hom}_{Alg}(A, B)$ when A and B satisfy some additional hypotheses.

We give a few examples of algebras that will be used frequently in this book.

1. Given an algebra A, we define the *opposite algebra* A^{op} as the algebra with the same underlying vector space as A, but with multiplication defined by

$$\mu_{A^{\mathrm{op}}} = \mu_A \circ \tau_{A,A} \tag{1.2}$$

where $\tau_{A,A}$ is the *flip* switching the two factors of $A \times A$. In other words,

$$\mu_{A^{\mathrm{op}}}(a, a') = a'a. \tag{1.3}$$

An algebra A is *commutative* if and only if

$$\mu_{A^{\mathrm{op}}} = \mu_A. \tag{1.4}$$

2. The *centre* $Z(A)$ of an algebra A is the subalgebra

$$\{a \in A \mid aa' = a'a \text{ for all } a' \in A\}.$$

We have $Z(A) = Z(A^{\mathrm{op}})$.

3. If I is a two-sided ideal of an algebra A, i.e., a subspace of A such that

$$\mu_A(I \times A) \subset I \supset \mu_A(A \times I),$$

then there exists a unique algebra structure on the quotient vector space A/I such that the canonical projection from A onto A/I is a morphism of algebras.

4. We endow the product set $A = \prod_{i \in I} A_i$ of a family $(A_i)_{i \in I}$ of algebras with the unique algebra structure such that the canonical projection from A to A_i is an algebra morphism for all $i \in I$. The algebra A is called the *product algebra* of the family $(A_i)_{i \in I}$.

5. Given an algebra A we can form the algebra $A[x]$ of all polynomials $\sum_{i=0}^{n} a_i x^i$ where n is any non-negative integer and the algebra $A[x, x^{-1}]$ of all Laurent polynomials $\sum_{i=m}^{n} a_i x^i$ where $m, n \in \mathbf{Z}$.

6. For any positive integer n we denote by $M_n(A)$ the algebra of all $n \times n$-matrices with entries in A.

7. The space $\mathrm{End}(V)$ of linear endomorphisms of a vector space V is an algebra with product given by the composition and unit by the identity map id_V of V.

Given an algebra A, a *left A-module* or, simply, an *A-module* is a vector space V together with a bilinear map $(a, v) \mapsto av$ from $A \times V$ to V such that

$$a(a'v) = (aa')v \quad \text{and} \quad 1v = v \tag{1.5}$$

for all $a, a' \in A$ and $v \in V$. One similarly defines a right A-module using a bilinear map from $V \times A$ to V. A right A-module is nothing else than a left module over the opposite algebra A^{op}. Therefore we need only consider left modules which shall for simplicity be called modules in the sequel.

If V and V' are A-modules, a linear map $f : V \to V'$ is said to be *A-linear* or a *morphism of A-modules* if

$$f(av) = af(v) \tag{1.6}$$

for all $a \in A$ and $v \in V$.

An *A-submodule* V' of an A-module V is a subspace of V with an A-module structure such that the inclusion of V' into V is A-linear.

The action of A on an A-module V defines an algebra morphism ρ from A to $\text{End}(V)$ by

$$\rho(a)(v) = av. \tag{1.7}$$

The map ρ is called a *representation* of A on V.

Given A-modules $V_1, \dots, V_n$, the direct sum $V_1 \oplus \cdots \oplus V_n$ has an A-module structure given by

$$a(v_1, \dots, v_n) = (av_1, \dots, av_n) \tag{1.8}$$

where $a \in A$, $v_1 \in V_1, \dots, v_n \in V_n$. These definitions lead us to the following ones.

Definition I.1.1. *An A-module V is simple if it has no other submodules than $\{0\}$ and V. It is semisimple if it is isomorphic to a direct sum of simple A-modules. It is indecomposable if it is not isomorphic to the direct sum of two non-zero submodules.*

In the language of representations, a simple module [resp. a semisimple module] is an *irreducible* representation [resp. a *completely reducible* representation]. The following well-known proposition will be used in Chapters V–VII.

Proposition I.1.2. *The following statements are equivalent.*

(i) *For any pair $V' \subset V$ of finite-dimensional A-modules, there exists an A-module V'' such that $V \cong V' \oplus V''$.*

(ii) *For any pair $V' \subset V$ of finite-dimensional A-modules where V' is simple, there exists an A-module V'' such that $V \cong V' \oplus V''$.*

(iii) *For any pair $V' \subset V$ of finite-dimensional A-modules, there exists an A-linear map $p : V \to V'$ with $p^2 = p$.*

(iv) *For any pair $V' \subset V$ of finite-dimensional A-modules where V' is simple, there exists an A-linear map $p : V \to V'$ with $p^2 = p$.*

(v) *Any finite-dimensional A-module is semisimple.*

PROOF. Clearly, (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv). We also have (i) $\Rightarrow$ (iii): it suffices to define p as the canonical projection from $V' \oplus V''$ onto V'. Similarly, (ii) $\Rightarrow$ (iv).

Assertion (iii) $\Rightarrow$ Assertion (i). Let $V'' = \mathrm{Ker}\,(p)$; it is a submodule of V. The relations $v = p(v) + (v - p(v))$ and $p^2 = p$ prove that V is the direct sum V' and V''. Similarly, (iv) $\Rightarrow$ (ii).

Assertion (ii) $\Rightarrow$ Assertion (v). Assuming (ii), we have to prove that any finite-dimensional A-module V is semisimple. We may also assume that $\dim(V) > 0$. Consider a non-zero submodule V_1 of V of minimal dimension; it has to be simple. By (ii) there exists a submodule V^1 such that $V \cong V_1 \oplus V^1$ and $\dim(V^1) < \dim(V)$. Iterating this procedure, we build a sequence $(V_n)_{n>0}$ of simple submodules and a sequence $(V^n)_{n>0}$ of submodules such that

$$V^n \cong V_{n+1} \oplus V^{n+1} \quad \text{and} \quad \dim(V^{n+1}) < \dim(V^n).$$

Since the dimension of V^n is strictly decreasing, there exists an integer p such that $V^p = \{0\}$. The module V is a direct sum of simple modules: $V \cong V_1 \oplus \cdots \oplus V_p$.

It remains to be shown that Assertion (v) implies Assertion (i). Let $V' \subset V$ be a pair of finite-dimensional A-modules. By (v)

$$V = \bigoplus_{i \in I} V_i$$

is a direct sum over a finite index set I of simple submodules V_i. Let J be a maximal subset of I such that

$$V' \cap (\bigoplus_{j \in J} V_j) = \{0\}. \tag{1.9}$$

If $i \notin J$, then

$$V' \cap (V_i \oplus \bigoplus_{j \in J} V_j) \neq \{0\},$$

hence

$$V_i \cap (V' + \bigoplus_{j \in J} V_j) \neq \{0\}.$$

Since V_i is simple, this implies that

$$V_i \subset V' + \bigoplus_{j \in J} V_j$$

for all $i \notin J$. This holds also for all $i \in J$. Consequently, for the sum V of all V_i we must have

$$V = V' + \bigoplus_{j \in J} V_j. \tag{1.10}$$

As a consequence of (1.9–1.10), we get $V = V' \oplus V''$ where V'' is the submodule $\oplus_{j \in J} V_j$. $\square$

I.2 Free Algebras

Let X be a set. Consider the vector space $k\{X\}$ with basis the set of all words $x_{i_1} \dots x_{i_p}$ in the alphabet X, including the empty word $\emptyset$. A word will be called a monomial. Define the degree of the monomial $x_{i_1} \dots x_{i_p}$ as its length p. *Concatenation* of words defines a multiplication on $k\{X\}$ by

$$(x_{i_1} \dots x_{i_p})(x_{i_{p+1}} \dots x_{i_n}) = x_{i_1} \dots x_{i_p} x_{i_{p+1}} \dots x_{i_n}. \tag{2.1}$$

Formula (2.1) equips $k\{X\}$ with an algebra structure, called the *free algebra* on the set X. The unit is the empty word: $1 = \emptyset$. In the sequel we shall mainly consider free algebras on finite sets. If $X = \{x_1, \dots, x_n\}$ we also denote $k\{X\}$ by $k\{x_1, \dots, x_n\}$.

Free algebras have the following universal property.

Proposition I.2.1. *Let X be a set. Given an algebra A and a set-theoretic map f from X to A, there exists a unique algebra morphism $\bar{f} : k\{X\} \to A$ such that $\bar{f}(x) = f(x)$ for all $x \in X$.*

PROOF. It is enough to define $\bar{f}$ on any word of X. For the empty word we set $\bar{f}(\emptyset) = 1$. Otherwise, if $x_{i_1}, \dots, x_{i_p}$ are elements of X, we define

$$\bar{f}(x_{i_1} \dots x_{i_p}) = f(x_{i_1}) \dots f(x_{i_p}).$$

The rest of the proof follows easily. □

An equivalent formulation of Proposition 2.1 is: There exists a natural bijection

$$\mathrm{Hom}_{Alg}(k\{X\}, A) \cong \mathrm{Hom}_{Set}(X, A) \tag{2.2}$$

where $\mathrm{Hom}_{Set}(X, A)$ is the set of all set-theoretic maps from X to A. In particular, if X is the finite set $\{x_1, \dots, x_n\}$, then $f \mapsto (f(x_1), \dots, f(x_n))$ induces a bijection

$$\mathrm{Hom}_{Alg}(k\{x_1, \dots, x_n\}, A) \cong A^n. \tag{2.3}$$

Any algebra A is the quotient of a free algebra $k\{X\}$. It suffices to take any generating set X for the algebra A (for instance $X = A$). Consequently, $A = k\{X\}/I$ where I is a two-sided ideal of $k\{X\}$. In this case, for any algebra A' we have the natural bijection

$$\mathrm{Hom}_{Alg}(k\{X\}/I, A') \cong \{f \in \mathrm{Hom}_{Set}(X, A') \mid \bar{f}(I) = 0\}. \tag{2.4}$$

Example 1. Let I be the two-sided ideal of $k\{x_1, \dots, x_n\}$ generated by all elements of the form $x_i x_j - x_j x_i$ where i, j run over all integers from 1 to n. The quotient-algebra $k\{x_1, \dots, x_n\}/I$ is isomorphic to the polynomial algebra $k[x_1, \ldots, x_n]$ in n variables with coefficients in the ground field k. As a corollary of (2.4) we have

$$\mathrm{Hom}_{Alg}(k[x_1, \ldots, x_n], A) \cong \{(a_1, \ldots, a_n) \in A^n \mid a_i a_j = a_j a_i \text{ for all } (i, j)\} \tag{2.5}$$

for any algebra A.

In the next sections we shall see more examples where families of elements subject to "universal" algebraic relations are represented by quotients of free algebras.

I.3 The Affine Line and Plane

Let us restrict to commutative algebras. As a consequence of (2.5) we have the following proposition.

Proposition I.3.1. *Let A be a commutative algebra and f be a set-theoretic map from the finite set $\{x_1, \ldots, x_n\}$ to A. There exists a unique morphism of algebras $\bar{f}$ from $k[x_1, \ldots, x_n]$ to A such that $\bar{f}(x_i) = f(x_i)$ for all i.*

In other words, giving an algebra morphism from the polynomial algebra $k[x_1, \ldots, x_n]$ to a *commutative* algebra A is equivalent to giving an n-tuple $(a_1, \ldots, a_n)$ of elements of A:

$$\mathrm{Hom}_{Alg}(k[x_1, \ldots, x_n], A) \cong A^n. \tag{3.1}$$

Let us consider the special case $n = 1$ of (3.1). For any commutative algebra A the underlying set A is in bijection with the set $\mathrm{Hom}_{Alg}(k[x], A)$:

$$\mathrm{Hom}_{Alg}(k[x], A) \cong A. \tag{3.2}$$

The algebra $k[x]$ is called the *affine line* and the set $\mathrm{Hom}_{Alg}(k[x], A)$ is called the set of A-points of the affine line. Now A has an abelian group structure. We wish to express it in a universal way using the affine line $k[x]$. The abelian group structure of A consists of three maps, namely the addition $+ : A^2 \to A$, the unit $0 : \{0\} \to A$, and the inverse $- : A \to A$, satisfying the well-known axioms which express the fact that the addition is associative and commutative, that it has 0 as a left and right unit and that

$$(-a) + a = a + (-a) = 0$$

for all $a \in A$. These laws do not depend on the particular commutative algebra A. It will therefore be possible to express them universally.

To this end, let us introduce the *affine plane* $k[x', x'']$ with the bijection

$$\mathrm{Hom}_{Alg}(k[x', x''], A) \cong A^2 \tag{3.3}$$

obtained from (3.1) for $n = 2$. An element of $\mathrm{Hom}_{Alg}(k[x', x''], A)$ is called an A-point of the affine plane. The set $\mathrm{Hom}_{Alg}(k, A)$, reduced to the single point η_A, will be denoted by $\{0\}$.

Proposition I.3.2. *Let $\Delta : k[x] \to k[x', x'']$, $\varepsilon : k[x] \to k$, $S : k[x] \to k[x]$ be the algebra morphisms defined by*

$$\Delta(x) = x' + x'', \quad \varepsilon(x) = 0, \quad S(x) = -x.$$

Under the identifications (3.2–3.3)*, the morphisms Δ, ε and S correspond to the maps $+$, 0 and $-$ respectively.*

PROOF. Left to the reader. □

The morphisms Δ, ε and S are subject to further relations which express the associativity, the commutativity, the unit and the inverse axioms of an abelian group. They equip the affine line $k[x]$ with what will be called a cocommutative Hopf algebra structure in Chapter III.

In order to illustrate better the phenomenon we have just observed, we give another example. For any algebra A denote by $A^\times$ the group of invertible elements of A. We represent the set $A^\times$ by an algebra as above. Consider the ideal I of $k[x, y]$ generated by $xy - 1$. For any commutative algebra A we have

$$\mathrm{Hom}_{Alg}(k[x, y]/I, A) \cong A^\times. \tag{3.4}$$

The set $\{x^k\}_{k \in \mathbf{Z}}$ is a basis of the vector space $k[x, y]/I$. We denote this algebra by $k[x, x^{-1}]$; it is the algebra of Laurent polynomials in one variable. One defines similarly the algebra

$$k[x', x'', x'^{-1}, x''^{-1}] = k[x', y', x'', y'']/(x'y' - 1, x''y'' - 1)$$

of Laurent polynomials in two variables. We have a bijection

$$\mathrm{Hom}_{Alg}(k[x', x'^{-1}, x'', x''^{-1}], A) \cong A^\times \times A^\times. \tag{3.5}$$

Define algebra morphisms

$$\Delta : k[x, x^{-1}] \to k[x', x'^{-1}, x'', x''^{-1}], \quad \varepsilon : k[x, x^{-1}] \to k,$$
$$S : k[x, x^{-1}] \to k[x, x^{-1}]$$

by

$$\Delta(x) = x'x'', \quad \varepsilon(x) = 1, \quad S(x) = x^{-1}. \tag{3.6}$$

Then the morphisms Δ, ε and S correspond respectively to the multiplication in $A^\times$, to the unit 1 and to the inverse under the identifications (3.4–3.5). Here again, the morphisms Δ, ε, S equip $k[x, x^{-1}]$ with a cocommutative Hopf algebra structure.

I.4 Matrix Multiplication

For any algebra A we denote by $M_2(A)$ the algebra of 2×2-matrices with entries in A. As a set, $M_2(A)$ is in bijection with the set A^4 of 4-tuples of A. By (3.1) we have a natural bijection

$$\mathrm{Hom}_{Alg}(M(2), A) \cong M_2(A) \tag{4.1}$$

for any commutative algebra A where $M(2)$ is defined as the polynomial algebra $k[a,b,c,d]$. This bijection maps an algebra morphism $f : M(2) \to A$ to the matrix

$$\begin{pmatrix} f(a) & f(b) \\ f(c) & f(d) \end{pmatrix}.$$

The multiplication of matrices is a map $M_2(A) \times M_2(A) \to M_2(A)$ we wish to represent universally on $M(2)$, in the spirit of Section 3. The set $M_2(A) \times M_2(A)$ being in bijection with A^8, we introduce the polynomial algebra

$$M(2)^{\otimes 2} = k[a', a'', b', b'', c', c'', d', d'']. \tag{4.2}$$

Proposition I.4.1. *Let $\Delta : M(2) \to M(2)^{\otimes 2}$ be the algebra morphism defined by*

$$\Delta(a) = a'a'' + b'c'', \qquad \Delta(b) = a'b'' + b'd'',$$
$$\Delta(c) = c'a'' + d'c'', \qquad \Delta(d) = c'b'' + d'd''.$$

Then for any commutative algebra A, the morphism Δ corresponds to the matrix multiplication in $M_2(A)$ under the identifications (4.1–4.2).

The proof is easy and left to the reader. It is convenient to rewrite the formulas for Δ in Proposition 4.1 in the compact matrix form

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}. \tag{4.3}$$

I.5 Determinants and Invertible Matrices

We keep the notations of the previous section. We now consider the group $GL_2(A)$ of invertible matrices of the matrix algebra $M_2(A)$. When A is commutative, we know that a matrix is invertible if and only if its determinant is invertible in A:

$$GL_2(A) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(A) \text{ such that } \alpha\delta - \beta\gamma \in A^\times \right\}.$$

Define $SL_2(A)$ as the subgroup of $GL_2(A)$ of matrices with determinant $\alpha\delta - \beta\gamma = 1$.

Proposition I.5.1. *Define the commutative algebras*

$$GL(2) = M(2)[t]/((ad - bc)t - 1)$$

and

$$SL(2) = GL(2)/(t - 1) = M(2)/(ad - bc - 1).$$

For any commutative algebra A there are bijections

$$\mathrm{Hom}_{Alg}(GL(2), A) \cong GL_2(A) \quad \text{and} \quad \mathrm{Hom}_{Alg}(SL(2), A) \cong SL_2(A) \quad (5.1)$$

sending an algebra morphism f to the matrix

$$\begin{pmatrix} f(a) & f(b) \\ f(c) & f(d) \end{pmatrix}.$$

PROOF. We give it only for $GL(2)$. Similar arguments work for $SL(2)$. Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a matrix in $GL_2(A)$. Since A is commutative, there exists a unique algebra morphism $f : M(2)[t] \to A$ such that

$$f(a) = \alpha, \quad f(b) = \beta, \quad f(c) = \gamma, \quad f(d) = \delta \quad \text{and} \quad f(t) = (\alpha\delta - \beta\gamma)^{-1}.$$

Now,

$$\begin{aligned} f\Big((ad - bc)t - 1\Big) &= \Big(f(a)f(d) - f(b)f(c)\Big)f(t) - f(1) \\ &= (\alpha\delta - \beta\gamma)(\alpha\delta - \beta\gamma)^{-1} - 1 \\ &= 0. \end{aligned}$$

This implies that the morphism f factors through the quotient algebra $GL(2)$. The rest of the proof is easy. □

The next lemma follows from a straightforward computation using the morphism Δ of Proposition 4.1.

Lemma I.5.2. *We have* $\Delta(ad - bc) = (a'd' - b'c')(a''d'' - b''c'')$.

We now lift the group structures of $GL_2(A)$ and of $SL_2(A)$ to the algebras $GL(2)$ and $SL(2)$. Consider the commutative algebras

$$GL(2)^{\otimes 2} = M(2)^{\otimes 2}[t', t'']/((a'd' - b'c')t' - 1, (a''d'' - b''c'')t'' - 1)$$

and

$$SL(2)^{\otimes 2} = GL(2)^{\otimes 2}/(t'-1, t''-1) = M(2)^{\otimes 2}/(a'd'-b'c'-1, a''d''-b''c''-1).$$

Proposition I.5.3. *The formulas of Proposition* 4.1 *define algebra morphisms*

$$\Delta : GL(2) \to GL(2)^{\otimes 2} \quad \text{and} \quad \Delta : SL(2) \to SL(2)^{\otimes 2}.$$

PROOF. The formulas of Proposition 4.1 define an algebra morphism Δ from $M(2)[t]$ to $GL(2)^{\otimes 2}$ provided we set $\Delta(t) = t't''$. In order to show that Δ factors through $GL(2)$ we have to check that $\Delta((ad - bc)t - 1)$ vanishes. Now, by Lemma 5.2 and by definition of $GL(2)^{\otimes 2}$, we have

$$\begin{aligned} \Delta((ad - bc)t - 1) &= (a'd' - b'c')(a''d'' - b''c'')t't'' - 1 \\ &= 1.1 - 1 = 0. \end{aligned}$$

The proof for $SL(2)$ is similar. □

In Section 4 we checked that the map Δ corresponded to matrix multiplication under the above identifications. Let us exhibit the algebra maps

$$\varepsilon : GL(2) \to k \quad \text{and} \quad \varepsilon : SL(2) \to k$$

corresponding to the units of the groups $GL_2(A)$ and $SL_2(A)$ and the algebra morphisms

$$S : GL(2) \to GL(2) \quad \text{and} \quad S : SL(2) \to SL(2)$$

corresponding to the inversions in the same groups. They are defined by the formulas

$$\varepsilon(a) = \varepsilon(d) = \varepsilon(t) = 1, \qquad \varepsilon(b) = \varepsilon(c) = 0,$$

$$S(a) = (ad - bc)^{-1}\, d, \qquad S(b) = -(ad - bc)^{-1}\, b,$$

$$S(c) = -(ad - bc)^{-1}\, c, \qquad S(d) = (ad - bc)^{-1}\, a,$$

and $S(t) = t^{-1} = ad - bc$. We rewrite them in the more compact and illuminating form

$$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{5.2}$$

I.6 Graded and Filtered Algebras

The remaining sections of this chapter are devoted to some concepts of ring theory.

Definition I.6.1. *An algebra A is graded if there exist subspaces $(A_i)_{i\in\mathbf{N}}$ such that*

$$A = \bigoplus_{i\in\mathbf{N}} A_i \qquad \text{and} \qquad A_i \cdot A_j \subset A_{i+j}$$

for all $i, j \in \mathbf{N}$. The elements of A_i are said to be homogeneous of degree i.

We always assume that the unit 1 of a graded algebra belongs to A_0.

Example 1. Free algebras are graded by the length of words, i.e., the subspace A_i of $A = k\{X\}$ is defined as the subspace linearly generated by all monomials of degree i. The elements of X are of degree 1.

Proposition I.6.2. *Let $A = \bigoplus_{i\geq 0} A_i$ be a graded algebra and I be a two-sided ideal generated by homogeneous elements. Then*

$$I = \bigoplus_{i\geq 0} I \cap A_i$$

and the quotient algebra A/I is graded with $(A/I)_i = A_i/(I \cap A_i)$ for all i.

PROOF. It suffices to show that $I = \bigoplus_{i\geq 0} I \cap A_i$. First observe that the sum has to be direct since the subspaces A_i form a direct sum. Therefore, it remains to be checked that $I = \sum_{i\geq 0} I \cap A_i$. The ideal I is generated by homogeneous elements x_i of degree d_i. Consequently, if $x \in I$ then

$$x = \sum_i a_i x_i b_i$$

for some $a_i, b_i \in A$. Now, $a_i = \sum_j a_i^j$ and $b_i = \sum_j b_i^j$, where a_i^j and b_i^j are homogeneous elements of degree j. It follows that

$$x = \sum_{i,j,k} a_i^j x_i b_i^k$$

is a sum of homogeneous elements of degree $d_i + j + k$ in I. This implies that I is a subspace of $\sum_{i\geq 0} I \cap A_i$. The converse inclusion is clear. □

Example 2. The polynomial algebra $k[x_1, \ldots, x_n]$ is graded as the quotient of the free algebra $A = k\{x_1, \ldots, x_n\}$ (graded as in Example 1) by the ideal I generated by the degree-2 homogeneous elements $x_i x_j - x_j x_i$ where i and j run over all integers between 1 and n. The generators $x_1, \ldots, x_n$ are of degree one.

The algebras $M(2)$ and $M(2)^{\otimes 2}$ of Section 4 are graded as polynomial algebras. On the contrary, the ideals defining the algebras $GL(2)$ and $SL(2)$ are not generated by homogeneous elements. Though not graded, $GL(2)$ and $SL(2)$ are filtered algebras in the sense of the following definition.

Definition I.6.3. *An algebra A is filtered if there exists an increasing sequence $\{0\} \subset F_0(A) \subset \ldots \subset F_i(A) \subset \ldots \subset A$ of subspaces of A such that*

$$A = \bigcup_{i\geq 0} F_i(A) \quad \textit{and} \quad F_i(A) \cdot F_j(A) \subset F_{i+j}(A).$$

The elements of $F_i(A)$ are said to be of degree $\leq i$.

For any filtered algebra A there exists a graded algebra $S = \mathrm{gr}(A)$ defined by

$$S_i = F_i(A)/F_{i-1}(A).$$

We give a few examples of filtered algebras.

Example 3. Any algebra A has a trivial filtration given by $F_i(A) = A$ for all i.

Example 4. We filter any graded algebra $A = \bigoplus_{i \geq 0} A_i$ by

$$F_i(A) = \bigoplus_{0 \leq j \leq i} A_j$$

for all $i \in \mathbf{N}$. We have $\mathrm{gr}(A) = A$.

Example 5. Let $A \supset \ldots \supset F_1(A) \supset F_0(A)$ be a filtered algebra and I be a two-sided ideal of A. The quotient-algebra A/I is filtered with

$$F_i(A/I) = F_i(A)/F_i(A) \cap I.$$

In this case we have

$$\mathrm{gr}(A/I) = \bigoplus_{i \geq 0} F_i(A)/(F_{i-1}(A) + F_i(A) \cap I).$$

As a special case, consider the algebra $SL(2)$. It is filtered as the quotient of the graded algebra $M(2)$. We have

$$\mathrm{gr}(SL(2)) \cong k[a, b, c, d,]/(ad - bc).$$

I.7 Ore Extensions

Let R be an algebra and $R[t]$ be the free (left) R-module consisting of all polynomials of the form

$$P = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0 t^0$$

with coefficients in R. If $a_n \neq 0$, we say that the degree $\deg(P)$ of P is equal to n; by convention, we set $\deg(0) = -\infty$. The aim of this section is to find all algebra structures on $R[t]$ compatible with the algebra structure on R and with the degree. We need the following definition.

Let α be an algebra endomorphism of R. An *α-derivation* of R is a linear endomorphism δ of R such that

$$\delta(ab) = \alpha(a)\delta(b) + \delta(a)b \tag{7.1}$$

for all $a, b \in R$. Observe that (7.1) implies $\delta(1) = 0$.

Theorem I.7.1. (a) *Assume that $R[t]$ has an algebra structure such that the natural inclusion of R into $R[t]$ is a morphism of algebras, and we have* $\deg(PQ) = \deg(P) + \deg(Q)$ *for any pair (P, Q) of elements of $R[t]$. Then R has no zero-divisors and there exist a unique injective algebra endomorphism α of R and a unique α-derivation δ of R such that*

$$ta = \alpha(a)t + \delta(a) \tag{7.2}$$

for all $a \in R$.

(b) *Conversely, let R be an algebra without zero-divisors. Given an injective algebra endomorphism α of R and an α-derivation δ of R, there exists a unique algebra structure on $R[t]$ such that the inclusion of R into $R[t]$ is an algebra morphism and Relation* (7.2) *holds for all a in R.*

The algebra defined by Theorem 7.1 (b), denoted $R[t, \alpha, \delta]$, is called the *Ore extension* attached to the data (R, α, δ).

PROOF. (a) Let a, b be non-zero elements of R, hence of degree 0 in $R[t]$. We have $\deg(ab) = \deg(a) + \deg(b) = 0$, which implies that $ab \neq 0$. Consequently, R has no zero-divisors.

Let us now prove the existence and the uniqueness of the endomorphisms α and δ. Take any non-zero element a of R and consider the product ta. We have $\deg(ta) = \deg(t) + \deg(a) = 1$. By definition of $R[t]$ there exist uniquely determined elements $\alpha(a) \neq 0$ and $\delta(a)$ of R such that

$$ta = \alpha(a)t + \delta(a). \tag{7.2}$$

This defines maps α and δ in a unique fashion. The left multiplication by t being linear, so are α and δ. Furthermore, α has to be injective. Let us expand both sides of the equality $(ta)b = t(ab)$ in $R[t]$ using (7.2). Here a and b are elements of R. We get

$$\alpha(a)\alpha(b)t + \alpha(a)\delta(b) + \delta(a)b = \alpha(ab)t + \delta(ab). \tag{7.3}$$

Relation (7.3) implies that

$$\alpha(ab) = \alpha(a)\alpha(b) \quad \text{and} \quad \delta(ab) = \alpha(a)\delta(b) + \delta(a)b. \tag{7.4}$$

Applying (7.2) to $t1 = t$ yields $\alpha(1) = 1$ and $\delta(1) = 0$. It follows that α is an injective algebra endomorphism and δ is an α-derivation.

(b) It clearly suffices to know the product ta for any $a \in R$ in order to determine the product on $R[t]$ completely. Thus, (7.2) defines the algebra structure on $R[t]$ uniquely.

Let us now prove the existence of the algebra structure. To this end, we shall embed $R[t]$ into the associative algebra $\mathcal{M}$ consisting of all infinite matrices $(f_{ij})_{i,j\geq 1}$ with entries in the algebra $\mathrm{End}(R)$ of linear endomorphisms of R such that each row, as well as each column, has only finitely many non-zero entries. The unit of $\mathcal{M}$ is the infinite diagonal matrix I with identities on the diagonal. Given an element a of R, we denote by $\widehat{a} \in \mathrm{End}(R)$ the left multiplication by a. The hypotheses made on α and δ translate into the relations

$$\alpha\widehat{a} = \widehat{\alpha(a)}\alpha \quad \text{and} \quad \delta\widehat{a} = \widehat{\alpha(a)}\delta + \widehat{\delta(a)} \tag{7.5}$$

in $\mathrm{End}(R)$. Now, consider the infinite matrix

$$T = \begin{pmatrix} \delta & 0 & 0 & 0 & \cdots \\ \alpha & \delta & 0 & 0 & \cdots \\ 0 & \alpha & \delta & 0 & \cdots \\ 0 & 0 & \alpha & \delta & \cdots \\ 0 & 0 & 0 & \alpha & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

in $\mathcal{M}$. It allows one to define a linear map $\Phi : R[t] \to \mathcal{M}$ by

$$\Phi\Big(\sum_{i=0}^{n} a_i t^i\Big) = \sum_{i=0}^{n} (\widehat{a_i} I)T^i. \tag{7.6}$$

Lemma I.7.2. *The map Φ is injective.*

PROOF. For any integer $i \geq 1$, let e_i be the infinite column vector whose entries are all zero, except for the i-th one which is equal to the unit 1 of R. We may apply the matrix T of endomorphisms to e_i. Since $\delta(1) = 0$ and $\alpha(1) = 1$ we get

$$T(e_i) = e_{i+1} \tag{7.7}$$

for all $i \geq 1$. Now, let $P = \sum_{i=0}^{n} a_i t^i$ be an element of $R[t]$ such that $\Phi(P) = 0$. We wish to show that all elements $a_0, \ldots, a_n$ are zero. Apply $\Phi(P)$ to the vector column e_1. By (7.7) we get

$$0 = \Phi(P)(e_1) = \sum_{i=0}^{n} (\widehat{a_i} I)T^i(e_1) = \sum_{i=0}^{n} \widehat{a_i} e_{i+1}.$$

The set $\{e_i\}_{i\geq 1}$ being free, we have $\widehat{a_i} = 0$ for all i. Since R has a unit, we get $a_i = 0$ for all i. Hence, $P = 0$. □

Relations (7.5) imply the following relation in $\mathcal{M}$ for all $a \in R$.

Lemma I.7.3. *We have $T(\widehat{a}I) = (\widehat{\alpha(a)}I)T + (\widehat{\delta(a)}I)$.*

We now complete the proof of Theorem 7.1 (b). Let S be the subalgebra of $\mathcal{M}$ generated by the elements T and $\widehat{a}I$ where a runs over R. By Lemma 7.3 it is clear that S is the image of $R[t]$ under the map Φ. Since the latter is injective, it induces a linear isomorphism from $R[t]$ to the algebra S. This allows one to lift the algebra structure of S to $R[t]$. Relation (7.2) holds in $R[t]$ in view of Lemma 7.3. $\square$

We draw a few consequences. First, we wish to give a general formula for the product in $R[t, \alpha, \delta]$. Consider $P = \sum_{i=0}^{n} a_i t^i$ and $Q = \sum_{i=0}^{m} b_i t^i$. Set $PQ = \sum_{i=0}^{n+m} c_i t^i$. Let $S_{n,k}$ be the linear endomorphism of R defined as the sum of all $\begin{pmatrix} n \\ k \end{pmatrix}$ possible compositions of k copies of δ and of $n-k$ copies of α.

Corollary I.7.4. *Under the hypotheses of Theorem* 7.1 (b), *the following holds.*

(a) *For all i with $0 \leq i \leq m+n$ we have*

$$c_i = \sum_{p=0}^{i} a_p \sum_{k=0}^{p} S_{p,k}(b_{i-p+k}) \tag{7.8}$$

and for all $a \in R$ and $n \in \mathbf{N}$ we have in $R[t, \alpha, \delta]$

$$t^n a = \sum_{k=0}^{n} S_{n,k}(a)\, t^{n-k}. \tag{7.9}$$

(b) *The algebra $R[t, \alpha, \delta]$ has no zero-divisors. As a left R-module, it is free with basis $\{t^i\}_{i \in \mathbf{N}}$.*

(c) *If α is an automorphism, then $R[t, \alpha, \delta]$ is also a right free R-module with the same basis $\{t^i\}_{i \in \mathbf{N}}$.*

PROOF. (a) Relation (7.9) follows from (7.2) by induction on n. It implies (7.8).

(b) This is a consequence of the existence of the degree and of the definition of $R[t]$.

(c) Let us first prove that the set $\{t^i\}_{i \geq 0}$ generates $R[t, \alpha, \delta]$ as a right R-module. This means that any element P of $R[t, \alpha, \delta]$ can also be written under the form $P = \sum_{i=0}^{n} t^i a^i$ where $a_0, \ldots, a_n \in R$. Let us prove this by induction on the degree n of P. For $n = 0$, it is clear. For higher n we use the relation

$$at^n = t^n \alpha^{-n}(a) + \text{lower-degree terms} \tag{7.10}$$

which makes sense once α is assumed to be invertible. It remains to be proved that the set $\{t^i\}_{i \geq 0}$ is free. Suppose it is not. Then there exists a relation of the form

$$t^n a_n + t^{n-1} a_{n-1} + \cdots + t a_1 + a_0 = 0$$

with $a_n \neq 0$. Using (7.10) once again, we get another relation of the form

$$\alpha^n(a_n) t^n + \text{lower-degree terms} = 0,$$

which, by Part (b), implies that $\alpha^n(a_n) = 0$. The map α being an isomorphism, we get $a_n = 0$, hence a contradiction. $\square$

Example 1. Consider the special case $\alpha = \mathrm{id}_R$. If $\delta = 0$, then the Ore extension $R[t, \mathrm{id}_R, 0]$ is clearly isomorphic to the polynomial algebra $R[t]$. In case of a general derivation δ, the algebra $R[t, \mathrm{id}_R, \delta]$ is an algebra of polynomial differential operators (see Exercise 8). When $R = k[x]$ and δ is the usual derivation d/dx of polynomials, then $R[t, \mathrm{id}_R, \delta]$ is the Weyl algebra which is generated by two variables x and δ subject to the well-known Heisenberg relation $\delta x - x\delta = 1$.

I.8 Noetherian Rings

Proposition I.8.1. *Let A be a ring. The following two statements are equivalent.*

(i) *Any left ideal I of A is finitely generated, i.e., there exist $a_1, \dots, a_n$ in I such that $I = Aa_1 + \cdots + Aa_n$.*

(ii) *Any ascending sequence $I_1 \subset I_2 \subset I_3 \subset \dots \subset A$ of left ideals of A is finite, i.e., there exists an integer r such that $I_{r+i} = I_r$ for all $i \geq 0$.*

PROOF. Let us first show that (i) implies (ii). Consider an ascending sequence $I_1 \subset I_2 \subset I_3 \subset \dots$ of left ideals of A. The union of these ideals is a left ideal I which, by (i), is generated by a finite number $a_1, \dots, a_n$ of elements of A. By definition of the union there exists an integer r such that $a_1, \dots, a_n$ all belong to the ideal I_r. It follows that $I \subset I_r \subset I_{r+i} \subset I$ for all $i \geq 0$.

We now establish the converse. Let I be a left ideal that is not finitely generated and a_1 be an element of I. The left ideal $I_1 = Aa_1$ is contained in I and $I_1 \neq I$. Therefore, we can find an element $a_2 \in I \setminus Aa_1$. We have $I_1 \subset I_2 = Aa_1 + Aa_2 \subset I$ and $I_1 \neq I_2 \neq I$. Proceeding inductively, we find an infinite strictly ascending sequence $I_1 \subset \cdots I_n \subset I_{n+1} \subset \cdots I$ of left ideals. $\square$

Any ring A satisfying the equivalent conditions of Proposition 8.1 is said to be *left Noetherian*. The ring A is *right Noetherian* if the opposite ring A^{op} is left Noetherian. It is *Noetherian* if it is both left and right Noetherian.

Example 1. Any (skew-)field K is Noetherian, the only ideals being $\{0\}$ and K.

The property of being Noetherian is preserved by quotients and Ore extensions, as will be seen next.

Proposition I.8.2. *Let $\varphi : A \to B$ be a surjective morphism of rings. If A is left Noetherian, then so is B.*

PROOF. Let J be a left ideal of B. The left ideal $I = \varphi_{-1}(J)$ of A is generated by elements $a_1, \ldots, a_n$. Therefore, $J = \varphi(\varphi_{-1}(J))$ is generated by $\varphi(a_1), \ldots, \varphi(a_n)$. □

The following theorem is a non-commutative version of Hilbert's basis theorem.

Theorem I.8.3. *Let R be an algebra, α be an algebra automorphism and δ be an α-derivation of R. If R is left Noetherian, then so is the Ore extension $R[t, \alpha, \delta]$.*

As a consequence of Proposition 8.2 and Theorem 8.3 applied to the case $\alpha = \mathrm{id}$ and $\delta = 0$, we have

Corollary I.8.4. *If R is left Noetherian, then so is $R[X_1, \ldots, X_n]/I$ for any ideal I.*

Proof of Theorem 8.3. Let I be a left ideal of the Ore extension $R[t, \alpha, \delta]$. We have to prove that I is finitely generated. Given an integer $d \geq 0$, define I_d as the union of $\{0\}$ and of all elements of R which appear as leading coefficients of degree d elements of I. One checks easily that I_d is a left ideal of R.

On the other hand, if a is the leading coefficient of some polynomial P, then $\alpha(a)$ is the leading coefficient of tP. Consequently, $\alpha(I_d)$ is included in I_{d+1}. We therefore have the ascending sequence

$$I_0 \subset \alpha^{-1}(I_1) \subset \alpha^{-2}(I_2) \subset \ldots \subset \alpha^{-n}(I_n) \subset \alpha^{-(n+1)}(I_{n+1}) \subset \cdots$$

of left ideals in R. Since R is left Noetherian, there exists an integer n such that $I_{n+i} = \alpha^i(I_n)$ for all $i \geq 0$.

For any d with $0 \leq d \leq n$ choose generators $a_{d,1}, \ldots, a_{d,p}$ of I_d. Let $P_{d,i}$ be a degree d element of I whose leading coefficient is $a_{d,i}$. The set $\{P_{d,i}\}_{0 \leq d \leq n, 1 \leq i \leq p}$ is finite. Let us prove by induction on the degree that any polynomial P in I belongs to the ideal $I' = \sum_{d,i} R[t, \alpha, \delta]\, P_{d,i}$. This will imply that $I = I'$ is finitely generated, hence establish the theorem.

The induction hypothesis clearly holds in degree 0. Suppose we have proved that any element of degree $< d$ in I is in I'. Let P be a degree d element of I.

(a) If $d \leq n$, the leading coefficient a of P is of the form $a = \sum_{0 \leq i \leq p} r_i a_{d,i}$ where $r_0, \ldots, r_p$ are elements of R. Consequently, $Q = P - \sum_{0 \leq i \leq p} r_i P_{d,i}$ is an element of I of degree $< d$. By induction, Q, hence P, belong to I'.

(b) If $d > n$, the leading coefficient a of P belongs to $I_d = \alpha^{d-n}(I_n)$. It can be written $a = \sum_{0 \leq i \leq p} r_i \alpha^{d-n}(a_{d,i})$ for some $r_0, \ldots, r_p$ in R. Consider the polynomial

$$Q = P - \sum_{0 \leq i \leq p} r_i t^{d-n} P_{d,i}.$$

The coefficient of t^d in Q is

$$a - \sum_{0 \le i \le p} r_i \alpha^{d-n}(a_{d,i}) = 0,$$

which shows that the degree of Q is $< d$. We can therefore apply the induction hypothesis and conclude as above. □

I.9 Exercises

1. (*Schur's lemma*) Prove that any A-linear map between simple A-modules is either zero or an isomorphism. Deduce that the A-linear endomorphisms of a simple A-module form a skew-field.

2. Let $p = p^2$ be an A-linear idempotent endomorphism of an indecomposable A-module V. Show that $p = 0$ or $p = \mathrm{id}_V$.

3. Let A_1, A_2 be algebras. Let V_1 be an A_1-module and V_2 be an A_2-module. Establish that $(a_1, a_2)(v_1, v_2) = (a_1 v_1, a_2 v_2)$ (where $a_1 \in A_1$, $a_2 \in A_2$, $v_1 \in V_1$, $v_2 \in V_2$) defines an $A_1 \times A_2$-module structure on $V_1 \times V_2$. Prove also that any $A_1 \times A_2$-module is of this form.

4. Let A be a filtered algebra and $\mathrm{gr}(A)$ the associated graded algebra. Prove that if $\mathrm{gr}(A)$ is Noetherian without zero-divisors, then so is A.

5. (*Rees algebra*) Let $A \supset \ldots \supset F_1 \supset F_0$ be a filtered algebra. Define the Rees algebra $R(A)$ as the subalgebra

$$R(A) = \sum_{n \ge 0} F_n t^n$$

of the polynomial algebra $A[t]$. Prove that

(i) there are algebra isomorphisms

$$R(A)/(t-1) \cong A, \quad R(A)/(t) \cong \mathrm{gr}(A), \quad R(A)[t^{-1}] \cong A[t, t^{-1}],$$

(ii) if the algebra $\mathrm{gr}(A)$ is generated by homogeneous elements $\bar{a}_1, \ldots,$ $\bar{a}_n$ of respective degrees $d_1, \ldots, d_n$, then $R(A)$ is generated by the elements t, $a_1 t^{d_1}, \ldots, a_n t^{d_n}$ where a_i is a lift of $\bar{a}_i$ in F_i for all i.

6. (*Poincaré series of a graded algebra*) Let $A = \bigoplus_{i \ge 0} A_i$ be a graded algebra such that the vector spaces A_i are all finite-dimensional. Define the Poincaré series of A as the formal series

$$P(A) = \sum_{i \ge 0} \dim(A_i)\, t^i.$$

Prove that

$$P(k\{x_1,\ldots,x_n\}) = \frac{1}{1-nt} \quad \text{and} \quad P(k[x_1,\ldots,x_n]) = \frac{1}{(1-t)^n}.$$

7. Compute the Poincaré series of the graded algebra associated to the filtered algebra $SL(2)$.

8. (*Leibniz formula*) Let δ be an α-derivation of an algebra R. Prove that if $a_1,\ldots,a_n$ are elements of R, then

$$\begin{aligned}\delta(a_1\ldots a_n) &= \delta(a_1)a_2\ldots a_n \\ &+ \textstyle\sum_{i=2}^{n-1} \alpha(a_1\ldots a_{i-1})\delta(a_i)a_{i+1}\ldots a_n + \alpha(a_1\ldots a_{n-1})\delta(a_n)\end{aligned}$$

and

$$\delta^n(a_1a_2) = \sum_{k=0}^{n} S_{n,k}(a_1)\,\delta^{n-k}(a_2)\,.$$

for $n \geq 1$. The endomorphisms $S_{n,k}$ were defined in Section 7.

9. Let R be an algebra with an algebra automorphism α and an α-derivation δ. Establish that $\delta\alpha^{-1}$ is an α^{-1}-derivation of the opposite algebra R^{op} and that we have an algebra isomorphism

$$R[t,\alpha,\delta]^{\mathrm{op}} \cong R^{\mathrm{op}}[t,\alpha^{-1},-\delta\alpha^{-1}].$$

Deduce that $R[t,\alpha,\delta]$ is right Noetherian if R is.

10. (*Algebra of differential operators*) Let R be an algebra over a field of characteristic zero and let δ be a derivation of R. The algebra of differential operators associated to δ is the Ore extension $R[t,\mathrm{id}_R,\delta]$, which we simply denote by $R[t,\delta]$.

(a) Prove that for any integer $n > 0$ and any element a of R we have

$$t^n a = \sum_{k=0}^{n} \binom{n}{k} \delta^k(a)\,t^{n-k}.$$

(b) Show that any trace on $R[t,\delta]$, i.e., any linear map τ on $R[t,\delta]$ such that $\tau(xy) = \tau(yx)$ for any pair (x,y) of elements of $R[t,\delta]$, is zero.

11. (*Algebra of pseudo-differential operators*) Keep the hypotheses and the notations of the previous exercise. Show that the formula

$$\Big(\sum_i a_it^i\Big)\Big(\sum_i b_it^i\Big) = \sum_i c_it^i$$

where

$$c_i = \sum_{k\in\mathbf{N},\, p\in\mathbf{Z}} \frac{p(p-1)\dots(p-k+1)}{k!}\, a_p\, \delta^k(b_{i-p+k}),$$

defines an algebra structure on the vector space $R[t,\delta][[t^{-1}]]$ of formal series of the form $\sum_{i=-\infty}^{n} a_i t^i$. Check that $R[t,\delta]$ is a subalgebra. Define the *non-commutative residue* as the linear map from $R[t,\delta][[t^{-1}]]$ to $R/([R,R]+\delta(R))$, sending the formal series $\sum_{i=-\infty}^{n} a_i t^i$ to the class of the coefficient a_{-1}. Prove that the non-commutative residue is a trace on the algebra $R[t,\delta][[t^{-1}]]$ of pseudo-differential operators.

I.10 Notes

Ore extensions were introduced by Ore in [Ore33]. They are also called *"skew polynomial rings"* in [Coh71] [MR87] (see also [Cur52]). One of Ore's motivations was to find a large class of non-commutative algebras that are embeddable into a skew-field. As is well-known, this is possible for any commutative integral domain, but not for a general non-commutative algebra. Ore proved that any algebra obtained from a skew-field by iterated Ore extensions can itself be embedded into some skew-field (see Proposition 0.8.4 in [Coh71]). For more details on Noetherian rings, we refer the reader to [Lan65] and [MR87]. The examples given in [MR87], 2.11 show that the non-commutative version of Hilbert's basis theorem is no longer true if the endomorphism α is not assumed to be bijective.

Chapter II
Tensor Products

This chapter is devoted to a few facts on tensor products of vector spaces and of algebras that will be needed in the sequel. We fix a field k once and for all.

II.1 Tensor Products of Vector Spaces

Given vector spaces U and V, we denote by $\mathrm{Hom}(U, V)$ the space of linear maps from U to V. In particular, define $\mathrm{End}(V) = \mathrm{Hom}(V, V)$, the space of linear endomorphisms of V. If W is a third vector space, we denote by $\mathrm{Hom}^{(2)}(U, V; W)$ the space of bilinear maps from $U \times V$ to W.

The tensor product $U \otimes V$ of two vector spaces can be characterized as follows.

Theorem II.1.1. *Given vector spaces U and V there exist a vector space $U \otimes V$ and a bilinear map $\varphi_0 : U \times V \to U \otimes V$ such that, for all vector spaces W, the linear map*

$$\mathrm{Hom}(U \otimes V, W) \to \mathrm{Hom}^{(2)}(U, V; W)$$

given by $f \mapsto f \circ \varphi_0$ is a linear isomorphism. The vector space $U \otimes V$ is called the tensor product of U and V. It is unique up to isomorphism.

For any $u \in U$ and $v \in V$, set $u \otimes v = \varphi_0(u, v)$. Since φ_0 is bilinear, the following relations hold in $U \otimes V$:

$$(u + u') \otimes v = u \otimes v + u' \otimes v, \tag{1.1}$$

$$u \otimes (v + v') = u \otimes v + u \otimes v', \tag{1.2}$$

$$\lambda(u \otimes v) = (\lambda u) \otimes v = u \otimes (\lambda v) \tag{1.3}$$

where $u, u' \in U$, $v, v' \in V$ and $\lambda \in k$. Moreover, as we shall see in the subsequent proof, any element of $U \otimes V$ is a finite sum of the form

$$\sum_{i=1}^{p} u_i \otimes v_i \tag{1.4}$$

where $u_1, \ldots, u_p$ belong to U and $v_1, \ldots, v_p$ belong to V.

PROOF. We indicate the proof. Consider the vector space $k[U \times V]$ whose basis is the set $U \times V$. We define $U \otimes V$ as the quotient of $k[U \times V]$ by the subspace generated by the elements

$$(u + u', v) - (u, v) - (u', v), \quad (u, v + v') - (u, v) - (u, v'),$$

$$(\lambda u, v) - \lambda(u, v), \quad (u, \lambda v) - \lambda(u, v)$$

where $u, u' \in U$, $v, v' \in V$ and $\lambda \in k$. The class of $(u, v) \in U \times V$ in $U \otimes V$ is denoted $\varphi_0(u, v) = u \otimes v$. By construction, the canonical map φ_0 from $U \times V$ to $U \otimes V$ is bilinear. The rest of the proof follows easily. For details, see [Bou70], Chap. 2 and [Lan65]. □

Corollary II.1.2. *For any triple (U, V, W) of vector spaces there is a natural isomorphism*

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W)).$$

PROOF. If φ is a bilinear map from $U \times V$ to W and u is any vector of U, then $\varphi(u, -)$ is a linear map from V to W. This sets up the desired isomorphism. □

The proof of the following easy proposition is left to the reader.

Proposition II.1.3. *Let U, V, W be vector spaces. There are isomorphisms*

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$

determined by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$,

$$k \otimes V \cong V \cong V \otimes k$$

determined by $\lambda \otimes v \mapsto \lambda v$ and $v \mapsto v \otimes 1$, and

$$V \otimes W \cong W \otimes V$$

given by the flip $\tau_{V,W}$ defined by $\tau_{V,W}(v \otimes w) = w \otimes v$.

The tensor product also commutes with the direct sum of spaces. Let $(U_i)_{i\in I}$ be a family of vector spaces indexed by a set I. Recall that there exists a vector space $\bigoplus_{i\in I} U_i$, called the *direct sum* of the family (U_i), and linear maps $q_i : U_i \to \bigoplus_{i\in I} U_i$ such that for any vector space V, the linear map

$$\mathrm{Hom}(\bigoplus_{i\in I} U_i, V) \to \prod_{i\in I} \mathrm{Hom}(U_i, V) \tag{1.5}$$

given by $f \mapsto (f \circ q_i)_i$ is an isomorphism.

Proposition II.1.4. *We have*

$$(\bigoplus_{i\in I} U_i) \otimes V \cong \bigoplus_{i\in I} (U_i \otimes V). \tag{1.6}$$

Proof. By Corollary 1.2 and (1.5) we have the chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}((\bigoplus_{i\in I} U_i) \otimes V, W) &\cong \mathrm{Hom}(\bigoplus_{i\in I} U_i, \mathrm{Hom}(V, W)) \\ &\cong \prod_{i\in I} \mathrm{Hom}(U_i, \mathrm{Hom}(V, W)) \\ &\cong \prod_{i\in I} \mathrm{Hom}(U_i \otimes V, W) \\ &\cong \mathrm{Hom}(\bigoplus_{i\in I} (U_i \otimes V), W). \end{aligned}$$

These hold for any vector space W. A classical argument given in full detail in the second proof of Proposition 5.1 (c) allows one to conclude. □

Recall also the notion of a direct product of vector spaces. Let $(V_i)_{i\in I}$ be a family of vector spaces indexed by a set I. There exists a vector space $\prod_{i\in I} V_i$, called the *direct product* of the family $(V_i)_{i\in I}$, and linear maps $p_i : \prod_{i\in I} V_i \to V_i$ such that for all vector spaces U, the map

$$\mathrm{Hom}(U, \prod_{i\in I} V_i) \to \prod_{i\in I} \mathrm{Hom}(U, V_i) \tag{1.7}$$

given by $f \mapsto (p_i \circ f)_i$ is an isomorphism. As a set, $\prod_{i\in I} V_i$ may be realized as the vector space of all families $(v_i)_{i\in I}$ such that $v_i \in V_i$ for all i. The direct sum $\bigoplus_{i\in I} V_i$ is then the subspace of $\prod_{i\in I} V_i$ consisting of the families $(v_i)_{i\in I}$ where all but finitely many v_i are zero. When the indexing set I is finite, the direct product coincides with the direct sum. Otherwise, the direct sum is a proper subspace of the direct product.

Corollary II.1.5. *Let* $\{u_i\}_{i\in I}$ *be a basis of the vector space* U *and* $\{v_j\}_{j\in J}$ *be a basis of* V. *Then the set* $\{u_i \otimes v_j\}_{(i,j)\in I\times J}$ *is a basis of the tensor product* $U \otimes V$. *Consequently, we have* $\dim(U \otimes V) = \dim(U)\dim(V)$.

PROOF. By definition of the direct sum, we have

$$U \cong \bigoplus_{i \in I} k u_i \quad \text{and} \quad V \cong \bigoplus_{j \in J} k v_j. \tag{1.8}$$

Applying Propositions 1.3–1.4 and using $k \otimes k \cong k$, we get

$$U \otimes V \cong \bigoplus_{(i,j) \in I \times J} k\,(u_i \otimes v_j).$$

□

Let us define the notion of a *free module* over an algebra A using the tensor product. It is a module of the form $A \otimes V$ where V is a vector space and A acts on $A \otimes V$ by

$$a(a' \otimes v) = aa' \otimes v$$

for $a, a' \in A$ and $v \in V$. A *basis* of an A-module M is a subset $\{x_i\}_{i \in I}$ of M such that the map

$$(a_i)_{i \in I} \mapsto \sum_{i \in I} a_i x_i$$

from the direct sum $\bigoplus_{i \in I} A$ to M is an isomorphism. By Propositions 1.3–1.4,

$$\bigoplus_{i \in I} A \cong \bigoplus_{i \in I} (A \otimes k) \cong A \otimes V$$

where $V = \bigoplus_{i \in I} k$. It follows that an A-module has a basis if and only if it is free.

II.2 Tensor Products of Linear Maps

Let $f : U \to U'$ and $g : V \to V'$ be linear maps. We define their tensor product $f \otimes g : U \otimes V \to U' \otimes V'$ by

$$(f \otimes g)(u \otimes v) = f(u) \otimes g(v) \tag{2.1}$$

for all u in U and v in V. This gives rise to a linear map

$$\lambda : \mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V') \to \mathrm{Hom}(V \otimes U, U' \otimes V') \tag{2.2}$$

defined by

$$\Big(\lambda(f \otimes g)\Big)(v \otimes u) = f(u) \otimes g(v). \tag{2.3}$$

The reasons for the switch of U and V in (2.2) will become apparent in III.5.2 and in Chapter XIV. The main result of this section is the following.

Theorem II.2.1. *The map λ is an isomorphism provided at least one of the pairs (U, U'), (V, V') or (U, V) consists of finite-dimensional vector spaces.*

PROOF. Assume that U and U' are finite-dimensional. We wish to show that the map λ of (2.2) is an isomorphism. We shall do this by reducing λ to simpler maps. We can write $U = \bigoplus_{i\in I} ku_i$ where $\{u_i\}_{i\in I}$ is a finite basis of U. As a consequence of the isomorphism (1.5–1.6), the map λ turns into a map from $(\prod_i \mathrm{Hom}(ku_i, U')) \otimes \mathrm{Hom}(V, V')$ to $\prod_i \mathrm{Hom}(V \otimes ku_i, U' \otimes V')$. The set I being finite, we may replace $\prod_i$ by $\bigoplus_i$. Applying (1.6) again, it remains to prove that the map

$$\lambda : \mathrm{Hom}(ku_i, U') \otimes \mathrm{Hom}(V, V') \to \mathrm{Hom}(V \otimes ku_i, U' \otimes V')$$

is an isomorphism in the special case $U = ku_i$.

Since ku_i is one-dimensional, this amounts to checking that the map

$$\lambda' : U' \otimes \mathrm{Hom}(V, V') \to \mathrm{Hom}(V, U' \otimes V') \tag{2.4}$$

defined by

$$\lambda'(u' \otimes f)(v) = u' \otimes f(v)$$

is an isomorphism. By assumption, we also have $U' = \bigoplus_{i\in I'} ku'_i$ for some finite basis $\{u_i\}_{i\in I'}$. We again use (1.6–1.7) and the fact that the direct product over the finite set I' is the same as the direct sum. We get

$$U' \otimes \mathrm{Hom}(V, V') \cong \bigoplus_{i\in I'} ku'_i \otimes \mathrm{Hom}(V, V')$$

and

$$\mathrm{Hom}(V, U' \otimes V') \cong \prod_{i\in I'} \mathrm{Hom}(V, ku'_i \otimes V').$$

This allows us to break λ' into the direct product of the maps

$$\lambda' : ku'_i \otimes \mathrm{Hom}(V, V') \to \mathrm{Hom}(V, ku'_i \otimes V').$$

In this special case, λ' is given by $\lambda'(u'_i \otimes f)(v) = u'_i \otimes f(v)$, which is clearly an isomorphism. Hence, so is the map λ' of (2.4), which concludes the proof.

There are similar arguments in the remaining two cases. □

We deduce two corollaries involving the dual vector space $V^* = \mathrm{Hom}(V, k)$ of a vector space V. For the first one, we specialize Theorem 2.1 by taking $U' = V' = k$.

Corollary II.2.2. *The map $\lambda : U^* \otimes V^* \to (V \otimes U)^*$ is an isomorphism provided U or V are finite-dimensional.*

For the second corollary, we take $U = V' = k$ in Theorem 2.1.

Corollary II.2.3. *The map $\lambda_{U,V} : V \otimes U^* \to \mathrm{Hom}(U,V)$ given for $u \in U$, $v \in V$ and $\alpha \in U^*$ by*

$$\lambda_{U,V}(v \otimes \alpha)(u) = \alpha(u)\, v \tag{2.5}$$

is an isomorphism if U or V are finite-dimensional. In particular, if V is a finite-dimensional vector space, the map $\lambda_{V,V}$ is an isomorphism

$$V \otimes V^* \cong \mathrm{End}(V).$$

We now wish to express the general map λ of (2.2) in terms of the special maps λ defined in Corollaries 2.2–2.3 and of the flip. This is done in the following lemma which will be useful later. Note that the map $\lambda_{U,U'} \otimes \lambda_{V,V'}$ is invertible when either U or U', and either V or V' are finite-dimensional.

Lemma II.2.4. *The following diagram commutes:*

$$\begin{tikzcd}
U' \otimes U^* \otimes V' \otimes V^* \arrow[r, "\lambda_{U,U'} \otimes \lambda_{V,V'}"] \arrow[d, "\mathrm{id} \otimes \tau_{U^*,V'} \otimes \mathrm{id}"] & \mathrm{Hom}(U,U') \otimes \mathrm{Hom}(V,V') \arrow[dd, "\lambda"] \\
U' \otimes V' \otimes U^* \otimes V^* \arrow[d, "\mathrm{id} \otimes \mathrm{id} \otimes \lambda"] & \\
U' \otimes V' \otimes (V \otimes U)^* \arrow[r, "\lambda_{V \otimes U, U' \otimes V'}"] & \mathrm{Hom}(V \otimes U, U' \otimes V')
\end{tikzcd}$$

PROOF. Easy. □

There is another important operation on linear homomorphisms that we have not yet discussed. It is the *composition* $(g, f) \mapsto g \circ f$ of two linear maps. This operation is bilinear and leads, for any triple (U, V, W) of vector spaces, to the map

$$\mathrm{Hom}(V,W) \otimes \mathrm{Hom}(U,V) \xrightarrow{\circ} \mathrm{Hom}(U,W).$$

Under some finite-dimensionality conditions, we can express the composition in simpler terms again involving the special maps λ of Corollary 2.3 as well as the *evaluation map*

$$\mathrm{ev}_V : V^* \otimes V \to k$$

which is defined as usual, namely by

$$\mathrm{ev}_V(\alpha \otimes v) = <\alpha, v> = \alpha(v) \tag{2.6}$$

for any linear form α and any vector v of V.

Lemma II.2.5. *The square*

$$\begin{tikzcd}
W \otimes V^* \otimes V \otimes U^* \arrow[r, "\mathrm{id} \otimes \mathrm{ev}_V \otimes \mathrm{id}"] \arrow[d, "\lambda_{V,W} \otimes \lambda_{U,V}"] & W \otimes U^* \arrow[d, "\lambda_{U,W}"] \\
\mathrm{Hom}(V,W) \otimes \mathrm{Hom}(U,V) \arrow[r, "\circ"] & \mathrm{Hom}(U,W)
\end{tikzcd}$$

commutes.

PROOF. Easy. □

II.3 Duality and Traces

All vector spaces considered in this section are assumed to be finite-dimensional. If V is such a vector space, we denote a basis of V by $\{v_i\}_i$ using the corresponding lower-case letter for vectors. The dual basis in the dual vector space V^* is denoted $\{v^i\}_i$. Using these bases, the evaluation map can be redefined by

$$\mathrm{ev}_V(v^i \otimes v_j) = < v^i, v_j > = \delta_{ij}. \tag{3.1}$$

Let us express the isomorphism $\lambda_{U,V} : V \otimes U^* \cong \mathrm{Hom}(U, V)$ of Corollary 2.3 in terms of bases. Let $f : U \to V$ be a linear map. Using bases for U and V, we have

$$f(u_j) = \sum_i f_j^i v_i \tag{3.2}$$

for some family $(f_j^i)_{ij}$ of scalars. It is easily checked that

$$f = \lambda_{U,V}\Big(\sum_{ij} f_j^i\, v_i \otimes u^j\Big). \tag{3.3}$$

In particular, taking for f the identity of V, we get

$$\mathrm{id}_V = \lambda_{V,V}\Big(\sum_i v_i \otimes v^i\Big). \tag{3.4}$$

This allows us to define the *coevaluation map* of any finite-dimensional vector space V as the linear map $\delta_V : k \to V \otimes V^*$ defined by

$$\delta_V(1) = \lambda_{V,V}^{-1}(\mathrm{id}_V) = \sum_i v_i \otimes v^i. \tag{3.5}$$

By its very definition, the map δ_V is independent of the choice of a basis. We now record some relations between the evaluation and coevaluation maps. These relations will turn out to be fundamental when we define duality in categories in Chapter XIV.

Proposition II.3.1. *The composition of the maps*

$$V \xrightarrow{\delta_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} V$$

is equal to the identity of V. Similarly, the composition of the maps

$$V^* \xrightarrow{\mathrm{id}_{V^*} \otimes \delta_V} V^* \otimes V \otimes V^* \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_{V^*}} V^*$$

is equal to the identity of V^*.

PROOF. Immediate. □

Let us recall the operation of *transposition*. For a linear map $f : U \to V$, define its *transpose* $f^* : V^* \to U^*$ as the linear map defined for all $\alpha \in V^*$ and all $u \in U$ by

$$< f^*(\alpha), u > = < \alpha, f(u) > . \tag{3.6}$$

In other words, f^* is the unique linear map such that the square

$$\begin{array}{ccc} V^* \otimes U & \xrightarrow{f^* \otimes \mathrm{id}_U} & U^* \otimes U \\ \Big\downarrow{\scriptstyle \mathrm{id}_{V^*} \otimes f} & & \Big\downarrow{\scriptstyle \mathrm{ev}_U} \\ V^* \otimes V & \xrightarrow{\mathrm{ev}_V} & k \end{array} \tag{3.7}$$

commutes. The transposition may be recovered from the evaluation and coevaluation maps as shown in the following result whose proof is left to the reader.

Proposition II.3.2. *Let $f : U \to V$ be a linear map. Then the transpose f^* is equal to the composition of the maps*

$$V^* \xrightarrow{\mathrm{id}_{V^*} \otimes \delta_U} V^* \otimes U \otimes U^* \xrightarrow{\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*}} V^* \otimes V \otimes U^* \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_{U^*}} U^*.$$

Observe that if (3.2) holds, then

$$f^*(v^j) = \sum_i f_i^j u^i. \tag{3.8}$$

We thus see that transposition amounts to exchanging upper and lower indices. We generalize this as follows. Let f be a linear map from $V \otimes W$ to $X \otimes Y$. Using bases on these spaces, we define the *partial transposes*

$$f^+ : X^* \otimes W \to V^* \otimes Y \quad \text{and} \quad f^\times : V \otimes Y^* \to X \otimes W^*$$

by

$$f^+(x^i \otimes w_j) = \sum_{k,\ell} f_{kj}^{i\ell}\, v^k \otimes y_\ell \tag{3.9}$$

and

$$f^\times(v_i \otimes y^j) = \sum_{k,\ell} f_{i\ell}^{kj}\, x_k \otimes w^\ell \tag{3.10}$$

if

$$f(v_i \otimes w_j) = \sum_{k,\ell} f_{ij}^{k\ell}\, x_k \otimes y_\ell. \tag{3.11}$$

Lemma II.3.3. *The definitions of f^+ and $f^\times$ are independent of the choice of bases. We also have*

$$(f^+)^\times = (f^\times)^+ = f^*.$$

PROOF. Left to the reader. □

The isomorphism $\lambda_{V,V}$ of Corollary 2.3 allows one to define the *trace of an endomorphism* in a finite-dimensional vector space V. The trace tr : $\mathrm{End}(V) \to k$ is defined as the composition

$$\mathrm{End}(V) \xrightarrow{\lambda_{V,V}^{-1}} V \otimes V^* \xrightarrow{\tau_{V,V^*}} V^* \otimes V \xrightarrow{\mathrm{ev}_V} k. \tag{3.12}$$

Proposition II.3.4. *Let f and g be endomorphisms of a finite-dimensional vector space V.*

(a) *The trace satisfies the relation*

$$\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f). \tag{3.13}$$

(b) *If $(f_j^i)_{ij}$ is the matrix of f in a basis of V, then*

$$\mathrm{tr}(f) = \sum_i f_i^i. \tag{3.14}$$

(c) *We also have*

$$\mathrm{tr}(f^*) = \mathrm{tr}(f). \tag{3.15}$$

PROOF. (a) By linearity, it suffices to prove (3.13) for

$$f = \lambda_{V,V}(v \otimes \alpha) \quad \text{and} \quad g = \lambda_{V,V}(w \otimes \beta)$$

where $v, w \in V$ and $\alpha, \beta \in V^*$. We have $f \circ g = \lambda_{V,V}(\alpha(w)\, v \otimes \beta)$ by Lemma 2.5. Consequently, $\mathrm{tr}(f \circ g) = \alpha(w)\, \beta(v)$, which clearly equals $\mathrm{tr}(g \circ f)$.

(b) From Relations (3.2–3.3) we derive

$$\mathrm{tr}(f) = \sum_{ij} f_j^i < v^j, v_i >= \sum_i f_i^i.$$

(c) Relation (3.15) follows from (3.8) and (3.14). □

The next result expresses the trace in terms of the evaluation and coevaluation maps and of the flip.

Proposition II.3.5. *The trace of $f : V \to V$ is equal to the composition of the maps*

$$k \xrightarrow{\delta_V} V \otimes V^* \xrightarrow{f \otimes \mathrm{id}_{V^*}} V \otimes V^* \xrightarrow{\tau_{V,V^*}} V^* \otimes V \xrightarrow{\mathrm{ev}_V} k.$$

We close these generalities with the *partial traces* of an endomorphism f of $U \otimes V$. By Theorem 2.1 the map $f \otimes g \mapsto \lambda(f \otimes g) \circ \tau_{U,V}$ is an isomorphism $\overline{\lambda}$ from $\mathrm{End}(U) \otimes \mathrm{End}(V)$ onto $\mathrm{End}(U \otimes V)$. We define tr_1 and tr_2 by the following commutative diagram.

$$\begin{tikzcd}
\mathrm{End}(V) & \mathrm{End}(U \otimes V) \arrow[l, "\mathrm{tr}_1"'] \arrow[r, "\mathrm{tr}_2"] & \mathrm{End}(U) \\
k \otimes \mathrm{End}(V) \arrow[u, "\cong"'] & \mathrm{End}(U) \otimes \mathrm{End}(V) \arrow[l, "\mathrm{tr} \otimes \mathrm{id}"'] \arrow[r, "\mathrm{id} \otimes \mathrm{tr}"] \arrow[u, "\overline{\lambda}"'] & \mathrm{End}(U) \otimes k \arrow[u, "\cong"']
\end{tikzcd} \tag{3.16}$$

Lemma II.3.6. *If $f(u_i \otimes v_j) = \sum_{k,\ell} f_{ij}^{k\ell}\, u_k \otimes v_\ell$ on some bases of U and V, then*

$$\operatorname{tr}_1(f)(v_j) = \sum_{i,\ell} f_{ij}^{i\ell}\, v_\ell \quad \textit{and} \quad \operatorname{tr}_2(f)(u_i) = \sum_{j,k} f_{ij}^{kj}\, u_k. \tag{3.17}$$

We also have $\operatorname{tr}_1(\operatorname{tr}_2(f)) = \operatorname{tr}_2(\operatorname{tr}_1(f)) = \operatorname{tr}(f)$.

PROOF. Left to the reader. □

II.4 Tensor Products of Algebras

Given algebras A and B, we put an algebra structure on the tensor product $A \otimes B$ by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb' \tag{4.1}$$

where $a, a' \in A$ and $b, b' \in B$. We call $A \otimes B$ the *tensor product of the algebras A and B*. Its unit is $1 \otimes 1$. Defining $i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$, we get algebra morphisms $i_A : A \to A \otimes B$ and $i_B : B \to A \otimes B$. The following relation holds in view of (4.1):

$$i_A(a) i_B(b) = i_B(b) i_A(a) = a \otimes b \tag{4.2}$$

for all $a \in A$ and $b \in B$. The tensor product of algebras enjoys the following universal property.

Proposition II.4.1. *Let $f : A \to C$ and $g : B \to C$ be algebra morphisms such that, for any pair $(a, b) \in A \times B$, the relation $f(a)g(b) = g(b)f(a)$ holds in C. Then there exists a unique morphism of algebras $f \otimes g : A \otimes B \to C$ such that $(f \otimes g) \circ i_A = f$ and $(f \otimes g) \circ i_B = g$.*

We can rephrase Proposition 4.1 by saying that $\operatorname{Hom}_{Alg}(A \otimes B, C)$ is the subset of $\operatorname{Hom}_{Alg}(A, C) \times \operatorname{Hom}_{Alg}(B, C)$ consisting of all pairs (f, g) of morphisms whose images commute in C. In particular, if C is *commutative* we have

$$\operatorname{Hom}_{Alg}(A \otimes B, C) \cong \operatorname{Hom}_{Alg}(A, C) \times \operatorname{Hom}_{Alg}(B, C). \tag{4.3}$$

PROOF. Any element of $A \otimes B$ is a finite sum of elements of the form $a \otimes b$. Therefore, by (4.2), $f \otimes g$ (if it exists) has to be of the form

$$(f \otimes g)(a \otimes b) = (f \otimes g)(i_A(a))(f \otimes g)(i_B(b)) = f(a)g(b).$$

This proves the uniqueness assertion. As for the existence of the map $f \otimes g$, one checks that the previous formula defines an algebra morphism. This uses the commutativity assumption as follows:

$$\begin{aligned}(f\otimes g)(a\otimes b)(f\otimes g)(a'\otimes b') &= f(a)g(b)f(a')g(b')\\ &= f(a)f(a')g(b)g(b')\\ &= f(aa')g(bb')\\ &= (f\otimes g)(aa'\otimes bb').\end{aligned}$$

□

We apply Proposition 4.1 to a situation encountered in Chapter I.

Proposition II.4.2. *Let $A = k\{X\}/I$ be a quotient of the free algebra on a set X. Take two copies X' and X'' of X. Let I' and I'' be the corresponding ideals in $k\{X'\}$ and $k\{X''\}$. Then the tensor product algebra $A\otimes A$ is isomorphic to the algebra*

$$A^{\otimes 2} = k\{X' \sqcup X''\}/(I', I'', X'X'' - X''X')$$

where $X' \sqcup X''$ denotes the disjoint union of the two copies and where $X'X'' - X''X'$ is the two-sided ideal generated by all elements of the form $x'x'' - x''x'$ with $x' \in X'$ and $x'' \in X''$.

PROOF. For any $x \in X$ we denote the corresponding copy in X' [resp. in X''] by x' [resp. by x'']. Setting $\varphi'(x) = x'$ and $\varphi''(x) = x''$ defines algebra morphisms $\varphi', \varphi'' : A \to A^{\otimes 2}$. Since $x'y'' = y''x'$ by definition of $A^{\otimes 2}$, we have $\varphi'(x)\varphi''(y) = \varphi''(y)\varphi'(x)$ for any pair (x, y) of elements of X. By Proposition 4.1 there exists an algebra morphism $\varphi : A\otimes A \to A^{\otimes 2}$ such that $\varphi(x\otimes y) = x'y''$.

Conversely, we get an algebra morphism ψ from $A^{\otimes 2}$ to $A\otimes A$ by setting $\psi(x') = x\otimes 1$ and $\psi(x'') = 1\otimes x$ where $x' \in X'$ and $x'' \in X''$. One easily checks that φ and ψ are inverse of each other. □

We retain from the previous statement that one passes from $A^{\otimes 2}$ to $A\otimes A$ by replacing the copy x' of x by $x\otimes 1$ and the copy x'' by $1\otimes x$ and vice versa. Let us apply this recipe to the constructions of I.4–5. Denoting $M(2)$, $GL(2)$ or $SL(2)$ by G, we see that in all three cases the algebra $G^{\otimes 2}$ defined in I.4–5 is isomorphic to the tensor product algebra $G\otimes G$. We can thus rewrite the map Δ of Proposition I.4.1 as the algebra morphism from G to $G\otimes G$ determined by

$$\Delta(a) = a\otimes a + b\otimes c, \qquad \Delta(b) = a\otimes b + b\otimes d,$$

$$\Delta(c) = c\otimes a + d\otimes c, \qquad \Delta(d) = c\otimes b + d\otimes d.$$

We rewrite these four relations in the compact matrix form

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.4}$$

Lemma I.5.2 implies that

$$\Delta(ad - bc) = (ad - bc) \otimes (ad - bc). \tag{4.5}$$

II.5 Tensor and Symmetric Algebras

Let V be a vector space. Define $T^0(V) = k$, $T^1(V) = V$ and $T^n(V) = V^{\otimes n}$ (the tensor product of n copies of V) if $n > 1$. The canonical isomorphisms

$$T^n(V) \otimes T^m(V) \cong T^{n+m}(V)$$

induce an associative product on the vector space $T(V) = \oplus_{n \geq 0} T^n(V)$. Equipped with this algebra structure, $T(V)$ is called the *tensor algebra* of V. The product in $T(V)$ is explicitly given by

$$(x_1 \otimes \ldots \otimes x_n)(x_{n+1} \otimes \ldots \otimes x_{n+m}) = x_1 \otimes \ldots \otimes x_n \otimes x_{n+1} \otimes \ldots \otimes x_{n+m} \tag{5.1}$$

where $x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}$ are elements of V. The unit for this product is the image of the unit element 1 in $k = T^0(V)$. Let i_V be the canonical embedding of $V = T^1(V)$ into $T(V)$. By (5.1) we have

$$x_1 \otimes \ldots \otimes x_n = i_V(x_1) \ldots i_V(x_n), \tag{5.2}$$

which allows us to set

$$x_1 \ldots x_n = x_1 \otimes \ldots \otimes x_n \tag{5.3}$$

whenever $x_1, \ldots, x_n$ are elements of V.

Proposition II.5.1. (a) *The algebra $T(V)$ is graded such that $T^n(V)$ is the subspace of degree n homogeneous elements.*

(b) *For any algebra A and any linear map $f : V \to A$, there exists a unique algebra morphism $\bar{f} : T(V) \to A$ such that $\bar{f} \circ i_V = f$. Consequently, the map $\bar{f} \mapsto \bar{f} \circ i_V$ is a bijection*

$$\operatorname{Hom}_{Alg}(T(V), A) \cong \operatorname{Hom}(V, A). \tag{5.4}$$

(c) *Let I be an indexing set for a basis of the vector space V. Then the tensor algebra $T(V)$ is isomorphic to the free algebra $k\{I\}$.*

PROOF. Part (a) is clear. Let us prove Part (b). If $\bar{f}$ exists, it has to be of the form

$$\bar{f}(x_1 \ldots x_n) = f(x_1) \ldots f(x_n)$$

in view of (5.3). This proves the uniqueness of $\bar{f}$. As for its existence, one checks immediately that the previous formula defines an algebra morphism from $T(V)$ into A.

(c) By Corollary 1.5, if $\{e_i\}_{i\in I}$ is a basis of V, then $\{e_{i_1}\dots e_{i_n}\}_{i_1,\dots,i_n\in I}$ is a basis of the vector space $T^n(V)$. When n runs over the set of non-negative integers we get a basis of $T(V)$ which is clearly in bijection with a basis of $k\{I\}$. This bijection induces an isomorphism between both vector spaces. The product on $T(V)$ corresponds to the concatenation in $k\{I\}$ under this isomorphism.

Let us give another, less pedestrian, proof of Part (c). By (1.5), (1.8), (5.4) and (I.2.2) we have the following chain of natural bijections:

$$\begin{aligned}
\mathrm{Hom}_{Alg}(T(V), A) &\cong \mathrm{Hom}(V, A)\\
&\cong \mathrm{Hom}(\bigoplus_{i\in I} ke_i, A)\\
&\cong \prod_{i\in I} \mathrm{Hom}(ke_i, A)\\
&\cong \mathrm{Hom}_{Set}(I, A)\\
&\cong \mathrm{Hom}_{Alg}(k\{I\}, A).
\end{aligned}$$

Let α be the composition of these bijections. First, take $A = T(V)$ and define $\varphi = \alpha(\mathrm{id}_{T(V)})$; this is an algebra morphism from $k\{I\}$ to $T(V)$. Now take $A = k\{I\}$ and define $\psi = \alpha^{-1}(\mathrm{id}_{k\{I\}})$; this is an algebra morphism from $T(V)$ to $k\{I\}$. We claim that φ and ψ are isomorphisms between $T(V)$ and $k\{I\}$. First, observe that the bijection α is natural, which means that for any algebra morphism $f : A \to A'$ we have

$$f \circ \alpha(\omega) = \alpha(f \circ \omega)$$

for any $\omega \in \mathrm{Hom}_{Alg}(T(V), A)$. Let us now compose φ and ψ. On the one hand, we get

$$\psi \circ \varphi = \psi \circ \alpha(\mathrm{id}_{T(V)}) = \alpha(\psi \circ \mathrm{id}_{T(V)}) = \alpha(\psi) = \mathrm{id}_{k\{I\}},$$

whereas on the other hand, we have

$$\alpha(\varphi \circ \psi) = \varphi \circ \alpha(\psi) = \varphi \circ \mathrm{id}_{k\{I\}} = \varphi,$$

whence $\varphi \circ \psi = \alpha^{-1}(\varphi) = \mathrm{id}_{T(V)}$. □

Let us define symmetric algebras. If V is a vector space, the *symmetric algebra* $S(V)$ is the quotient $S(V) = T(V)/I(V)$ of the tensor algebra $T(V)$ by the two-sided ideal $I(V)$ generated by all elements $xy - yx$ where x and y run over V. If $x_1, \dots, x_n$ are elements of V, we again denote by $x_1 \dots x_n$ the class of $x_1 \dots x_n$ in $S(V)$. The image of $T^n(V)$ under the projection of $T(V)$ onto $S(V)$ is denoted $S^n(V)$. Let i_V be the canonical map from $V = T^1(V)$ to $S(V)$.

Proposition II.5.2. (a) *The algebra $S(V)$ is commutative, and is graded such that $S^n(V)$ is the subspace of degree n homogeneous elements.*

(b) *For any algebra A and any linear map $f : V \to A$ such that*

$$f(x)f(y) = f(y)f(x)$$

for any pair (x, y) of elements of V, there exists a unique algebra morphism $\bar{f} : S(V) \to A$ such that $\bar{f} \circ i_V = f$.

(c) *If I is an indexing set for a basis of V, then the symmetric algebra $S(V)$ is isomorphic to the polynomial algebra $k[I]$ on the set I.*

(d) *If V' is another vector space, we have an algebra isomorphism*

$$S(V \oplus V') \cong S(V) \otimes S(V'). \tag{5.5}$$

Part (b) implies that the map $\bar{f} \mapsto \bar{f} \circ i_V$ is a bijection

$$\mathrm{Hom}_{Alg}(S(V), A) \cong \mathrm{Hom}(V, A) \tag{5.6}$$

when the algebra A is *commutative.*

Proof. We leave (a)–(c) as an exercise. Let us give a short proof of (d). Using (1.5), (4.3) and (5.6), we have the chain of natural bijections

$$\begin{aligned}
\mathrm{Hom}_{Alg}(S(V \oplus V'), A) &\cong \mathrm{Hom}(V \oplus V', A)\\
&\cong \mathrm{Hom}(V, A) \times \mathrm{Hom}(V', A)\\
&\cong \mathrm{Hom}_{Alg}(S(V), A) \times \mathrm{Hom}_{Alg}(S(V'), A)\\
&\cong \mathrm{Hom}_{Alg}(S(V) \otimes S(V'), A).
\end{aligned}$$

We then successively take A to be $S(V \oplus V')$ and $S(V) \otimes S(V')$, which produces isomorphisms between these algebras, as in the second proof of Part (c) of Proposition 5.1. □

II.6 Exercises

1. If f and f' [resp. g and g'] are composable linear maps, show that

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g).$$

2. Prove that if f is a surjective linear map, then so is $f \otimes \mathrm{id}_V$ for any vector space V. What about the kernel of $f \otimes \mathrm{id}_V$?

3. Prove that the map λ of (2.2) is injective.

4. Let U, V be finite-dimensional vector spaces, f [resp. g] be an endomorphism of U [resp. of V]. Show that $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\ \mathrm{tr}(g)$.

5. Let $A = \bigoplus_{i\geq 0} A_i$ and $A' = \bigoplus_{i\geq 0} A'_i$ be graded algebras. Show that the tensor product algebra $A \otimes A'$ is graded with

$$(A \otimes A')_n = \bigoplus_{i+j=n} A_i \otimes A'_j.$$

6. (*Exterior algebra*) For any vector space V we define the exterior algebra (or Grassmann algebra) $\Lambda(V)$ as the quotient $\Lambda(V) = T(V)/I'(V)$ of $T(V)$ by the two-sided ideal $I'(V)$ generated by the elements $x \otimes x$ where x runs over V. If $x_1, \ldots, x_n$ are elements of V, denote by $x_1 \wedge \ldots \wedge x_n$ the class of $x_1 \otimes \ldots \otimes x_n$ in $\Lambda(V)$. The subspace of $\Lambda(V)$ generated by the elements $x_1 \wedge \ldots \wedge x_n$ is denoted $\Lambda^n(V)$. Let i_V be the canonical map from $V = T^1(V)$ to $\Lambda(V)$. Prove the following statements.

   (a) The algebra $\Lambda(V)$ is graded such that $\Lambda^n(V)$ is the subspace of degree n homogeneous elements.

   (b) For any algebra A and any linear map $f : V \to A$ satisfying $f(x)^2 = 0$ for all $x \in V$, there exists a unique algebra morphism $\bar{f} : \Lambda(V) \to A$ such that $\bar{f} \circ i_V = f$.

   (c) Let I be an *ordered* set indexing a basis $\{e_i\}_{i\in I}$ of V. Then the set $\{e_{i_1} \wedge \ldots \wedge e_{i_n}\}_{i_1<\ldots<i_n\in I}$ is a basis of $\Lambda^n(V)$.

   (d) Assume V of finite dimension d. Prove that

$$\sum_{n\geq 0} \dim(\Lambda^n(V))\, t^n = (1+t)^d.$$

7. (*Symmetric and antisymmetric tensors*) The symmetric group S_n has a left action on $T^n(V)$ given by

$$\sigma(x_1 \otimes \ldots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(n)}$$

where $\sigma \in S_n$ and $x_1, \ldots, x_n \in V$. Define two endomorphisms Σ (the symmetrization operator) and A (the antisymmetrization operator) of $T^n(V)$ by

$$\Sigma(\alpha) = \sum_{\sigma\in S_n} \sigma(\alpha) \quad \text{and} \quad A(\alpha) = \sum_{\sigma\in S_n} \varepsilon(\sigma)\sigma(\alpha)$$

where $\varepsilon(\sigma)$ is the sign of the permutation σ. A tensor α of $T^n(V)$ is symmetric [resp. antisymmetric] if $\sigma(\alpha) = \alpha$ [resp. $\sigma(\alpha) = \varepsilon(\sigma)\sigma(\alpha)$] for any permutation σ. The subspace of symmetric [resp. antisymmetric] tensors of $T^n(V)$ is denoted $S'_n(V)$ [resp. $\Lambda'_n(V)$]. Prove that

   (a) $\Sigma(T^n(V)) \subset S'_n(V)$ and $A(T^n(V)) \subset \Lambda'_n(V)$,

(b) if $n!$ is invertible in the field k, the previous inclusions are equalities and the composition of the inclusion $S'_n(V) \to T^n(V)$ [resp. of the inclusion $\Lambda'_n(V) \to T^n(V)$] with the canonical projection $T^n(V) \to S^n(V)$ [resp. with the projection $T^n(V) \to \Lambda^n(V)$] is an isomorphism.

8. Let $A \otimes V$ be a free A-module. Prove that the space of A-linear maps from $A \otimes V$ to any A-module W is isomorphic to $\mathrm{Hom}(V, W)$.

II.7 Notes

For more details on the tensor, symmetric and exterior algebras as well as on the subspaces $S'_n(V)$ and $\Lambda'_n(V)$ of Exercise 7, see [Bou70], Chap. 3.

Chapter III

The Language of Hopf Algebras

In this chapter we introduce the fundamental concepts of coalgebras, bialgebras, Hopf algebras and comodules which we shall use extensively in the sequel. We shall also prove that the algebras $GL(2)$ and $SL(2)$ of Chapter I are Hopf algebras.

III.1 Coalgebras

The concept of a coalgebra is dual to the concept of an algebra in the following sense. Paraphrasing the definition of an algebra in I.1, we can say that an algebra is given by a triple (A, μ, η) where A is a vector space and $\mu : A \otimes A \to A$ and $\eta : k \to A$ are linear maps satisfying the following axioms (Ass) and (Un).

(Ass): The square

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \tag{1.1}$$

commutes.

(Un): The diagram

$$\begin{tikzcd} k \otimes A \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "\mathrm{id} \otimes \eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd} \tag{1.2}$$

commutes.

The axiom (Ass) expresses the requirement that the multiplication μ is associative whereas Axiom (Un) means that the element $\eta(1)$ of A is a left and a right unit for μ. The algebra A is *commutative* if, in addition, it satisfies the axiom

(Comm): The triangle

$$\begin{tikzcd} A\otimes A \arrow[rr, "\tau_{A,A}"] \arrow[dr, "\mu"'] & & A\otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \tag{1.3}$$

commutes, where $\tau_{A,A}$ is the flip switching the factors: $\tau_{A,A}(a\otimes a') = a'\otimes a$.

A morphism of algebras $f : (A,\mu,\eta) \to (A',\mu',\eta')$ is a linear map f from A to A' such that

$$\mu'\circ(f\otimes f) = f\circ\mu \quad \text{and} \quad f\circ\eta = \eta'. \tag{1.4}$$

We now get the definition of a coalgebra by systematically reversing all arrows in the previous diagrams.

Definition III.1.1. (a) *A coalgebra is a triple* (C,Δ,ε) *where* C *is a vector space and* $\Delta : C \to C\otimes C$ *and* $\varepsilon : C \to k$ *are linear maps satisfying the following axioms* (Coass) *and* (Coun).

(Coass)*: The square*

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C\otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C\otimes C \arrow[r, "\Delta\otimes\mathrm{id}"] & C\otimes C\otimes C \end{tikzcd} \tag{1.5}$$

commutes.

(Coun)*: The diagram*

$$\begin{tikzcd} k\otimes C & C\otimes C \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\varepsilon"] & C\otimes k \\ & C \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd} \tag{1.6}$$

commutes. The map Δ *is called the coproduct or the comultiplication while* ε *is called the counit of the coalgebra. The squares* (1.5–1.6) *express that the coproduct* Δ *is coassociative and counital.*

If, furthermore, the triangle (Cocomm)

$$\begin{tikzcd} & C \arrow[dl, "\Delta"] \arrow[dr, "\Delta"] & \\ C\otimes C \arrow[rr, "\tau_{C,C}"] & & C\otimes C \end{tikzcd} \tag{1.7}$$

commutes, where $\tau_{C,C}$ *is the flip, we say that the coalgebra* C *is cocommutative.*

(b) *Consider two coalgebras* (C,Δ,ε) *and* $(C',\Delta',\varepsilon')$*. A linear map* f *from* C *to* C' *is a morphism of coalgebras or a coalgebra morphism if*

$$(f\otimes f)\circ\Delta = \Delta'\circ f \quad \textit{and} \quad \varepsilon = \varepsilon'\circ f. \tag{1.8}$$

It is easily checked that the composition of two morphisms of coalgebras is again a morphism of coalgebras.

Let us give a few examples of coalgebras.

Example 1. (*The ground coalgebra*) The field k has a natural coalgebra structure with $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. Moreover, for any coalgebra (C, Δ, ε), the map $\varepsilon : C \to k$ is a morphism of coalgebras.

Example 2. (*Opposite coalgebra*) For any coalgebra $C = (C, \Delta, \varepsilon)$ set

$$\Delta^{\mathrm{op}} = \tau_{C,C} \circ \Delta. \tag{1.9}$$

Then $(C, \Delta^{\mathrm{op}}, \varepsilon)$ is a coalgebra which we call the opposite coalgebra and denote by C^{cop}.

The next result relates algebras and coalgebras.

Proposition III.1.2. *The dual vector space of a coalgebra is an algebra.*

PROOF. Let (C, Δ, ε) be a coalgebra. Recall the map $\lambda : C^* \otimes C^* \to (C \otimes C)^*$ of Corollary II.2.2. Set $\overline{\lambda} = \lambda \circ \tau_{C^*,C^*}$. Define $A = C^*$, $\mu = \Delta^* \circ \overline{\lambda}$ and $\eta = \varepsilon^*$ where the superscript $*$ on a linear map indicates its transpose. Then (A, μ, η) is an algebra (use the commutative diagrams (1.1–1.2) and (1.5–1.6)). $\square$

Example 3. (*Coalgebra of a set*) Let X be a set and $C = k[X] = \bigoplus_{x \in X} kx$ be the vector space with basis X. We put a coalgebra structure on C by defining

$$\Delta(x) = x \otimes x \quad \text{and} \quad \varepsilon(x) = 1 \tag{1.10}$$

where $x \in X$. The dual algebra C^* is the algebra of functions on X with values in k. Indeed, a linear form f on C is determined by its values on the basis X. Let f' be another linear form. Then

$$(ff')(x) = \mu(f \otimes f')(x) = \overline{\lambda}(f \otimes f')(\Delta(x)) = f(x)f'(x).$$

Finally, the unit of the algebra C^* is given by the constant function ε. We shall later return to this example when X has, in addition, a group structure.

In general, the dual vector space of an algebra does not carry a natural coalgebra structure. Nevertheless, we have the following result in the finite-dimensional case (see also Section 9).

Proposition III.1.3. *The dual vector space of a finite-dimensional algebra has a coalgebra structure.*

PROOF. Let (A, μ, η) be a finite-dimensional algebra. Then the map $\overline{\lambda}$ from $A^* \otimes A^*$ to $(A \otimes A)^*$ is an isomorphism, which allows us to define Δ by

$\Delta = \overline{\lambda}^{-1} \circ \mu^*$. We also set $\varepsilon = \eta^*$. Using the commutative diagrams (1.1–1.2) and (1.5–1.6), one checks that $(A^*, \Delta, \varepsilon)$ is a coalgebra. $\square$

Example 4. (*The matrix coalgebra*) Let $A = M_n(k)$ be the algebra of $n \times n$-matrices with entries in k. Denote by E_{ij} the matrix with all entries equal to 0, except for the (i,j)-entry which is equal to 1. The set of matrices E_{ij} $(1 \leq i, j \leq n)$ is a basis of $M_n(k)$. Let $\{x_{ij}\}$ be the dual basis. Then A^* is the coalgebra defined by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{ij}. \tag{1.11}$$

Indeed, we have

$$\varepsilon(x_{ij}) = x_{ij}(\eta(1)) = x_{ij}(\sum_k E_{kk}) = \sum_k \delta_{ik}\delta_{kj} = \delta_{ij}$$

and

$$\begin{aligned} \mu^*(x_{ij})(E_{k\ell} \otimes E_{mn}) &= x_{ij}(\mu(E_{k\ell} \otimes E_{mn})) \\ &= \delta_{\ell m} x_{ij}(E_{kn}) \\ &= \delta_{\ell m}\delta_{ik}\delta_{jn} \\ &= \sum_p \delta_{ik}\delta_{\ell p}\delta_{pm}\delta_{jn} \\ &= \sum_p x_{ip}(E_{k\ell})x_{pj}(E_{mn}) \\ &= \overline{\lambda}\Big(\sum_p x_{ip} \otimes x_{pj}\Big)(E_{k\ell} \otimes E_{mn}). \end{aligned}$$

Example 5. (*Tensor product of coalgebras*) The tensor product $C \otimes C'$ of two coalgebras (C, Δ, ε) and $(C', \Delta', \varepsilon')$ has a coalgebra structure with comultiplication $(\mathrm{id} \otimes \tau_{C,C'} \otimes \mathrm{id}) \circ (\Delta \otimes \Delta')$ and counit $\varepsilon \otimes \varepsilon'$.

We return to Example 3.

Proposition III.1.4. *Let X and Y be two sets and $X \times Y$ be the product set. There exists an isomorphism of coalgebras*

$$k[X] \otimes k[Y] \cong k[X \times Y].$$

PROOF. The isomorphism is given on the basis $\{x \otimes y\}_{(x,y) \in X \times Y}$ of the tensor product $k[X] \otimes k[Y]$ by

$$\psi(x \otimes y) = (x, y). \tag{1.12}$$

It is clear that

$$(\psi \otimes \psi)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta)(x \otimes y) = (x, y) \otimes (x, y) = \Delta\psi(x \otimes y)$$

and $\varepsilon\psi(x \otimes y) = 1 = \varepsilon(x)\varepsilon(y)$, which shows that ψ is a morphism of coalgebras. □

We shall also need the following concept.

Definition III.1.5. *Let (C, Δ, ε) be a coalgebra. A subspace I of C is a coideal if $\Delta(I) \subset I \otimes C + C \otimes I$ and $\varepsilon(I) = 0$.*

When I is a coideal, then Δ factors through a map $\overline{\Delta}$ from C/I to

$$C \otimes C/(I \otimes C + C \otimes I) = C/I \otimes C/I.$$

Similarly, the counit factors through a map $\overline{\varepsilon} : C/I \to k$. Then clearly, the triple $(C/I, \overline{\Delta}, \overline{\varepsilon})$ is a coalgebra. It is called the *quotient-coalgebra.* We shall give examples later.

Notation 1.6. We now present Sweedler's sigma notation which we shall use continually in the sequel. If x is an element of a coalgebra (C, Δ, ε), the element $\Delta(x)$ of $C \otimes C$ is of the form

$$\Delta(x) = \sum_i x_i' \otimes x_i''. \tag{1.13}$$

In order to get rid of the subscripts, we henceforth agree to write the sum (1.13) in the form

$$\Delta(x) = \sum_{(x)} x' \otimes x''. \tag{1.14}$$

Using (1.14) we may express the coassociativity of Δ, i.e., the commutativity of the square (1.5), by

$$\sum_{(x)} \Big(\sum_{(x')} (x')' \otimes (x')'' \Big) \otimes x'' = \sum_{(x)} x' \otimes \Big(\sum_{(x'')} (x'')' \otimes (x'')'' \Big). \tag{1.15}$$

By convention again, we identify both sides of (1.15) with

$$\sum_{(x)} x' \otimes x'' \otimes x''', \tag{1.16}$$

also written $\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$. If we apply the comultiplication to (1.16), we get the following three equal expressions

$$\sum_{(x)} \Delta(x') \otimes x'' \otimes x''', \quad \sum_{(x)} x' \otimes \Delta(x'') \otimes x''', \quad \sum_{(x)} x' \otimes x'' \otimes \Delta(x''')$$

which we agree to write

$$\sum_{(x)} x' \otimes x'' \otimes x''' \otimes x'''' \tag{1.17}$$

or $\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}$. More generally, let $\Delta^{(n)} : C \to C^{\otimes(n+1)}$ be defined inductively on $n \geq 1$ by $\Delta^{(1)} = \Delta$ and

$$\Delta^{(n)} = (\Delta \otimes \mathrm{id}_{C^{\otimes(n-1)}}) \circ \Delta^{(n-1)} = (\mathrm{id}_{C^{\otimes(n-1)}} \otimes \Delta) \circ \Delta^{(n-1)}. \tag{1.18}$$

Then by convention, we write

$$\Delta^{(n)}(x) = \sum_{(x)} x^{(1)} \otimes \cdots \otimes x^{(n+1)}. \tag{1.19}$$

These conventions and the coassociativity of Δ imply for instance that

$$\begin{aligned} (\mathrm{id}_C \otimes \Delta \otimes \mathrm{id}_{C^{\otimes 2}})\Big(\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}\Big) \\ = \sum\nolimits_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)} \otimes x^{(5)} \end{aligned} \tag{1.20}$$

Using the conventions (1.14), the condition (1.6) for counitality may be reformulated for any $x \in C$ as

$$\sum_{(x)} \varepsilon(x')x'' = x = \sum_{(x)} x'\varepsilon(x''). \tag{1.21}$$

As a consequence of (1.21) and of (1.19), we get identities such as

$$\sum_{(x)} x^{(1)} \otimes \varepsilon(x^{(2)}) \otimes x^{(3)} \otimes x^{(4)} \otimes x^{(5)} = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}. \tag{1.22}$$

Indeed, the left-hand side may be rewritten as

$$\sum_{(x)} x^{(1)} \otimes (\varepsilon \otimes \mathrm{id})(\Delta(x^{(2)})) \otimes x^{(3)} \otimes x^{(4)}.$$

Then apply (1.21).

The coalgebra C is cocommutative if

$$\sum_{(x)} x' \otimes x'' = \sum_{(x)} x'' \otimes x' \tag{1.23}$$

for all $x \in C$.

The left Relation (1.8) defining a coalgebra morphism can be reformulated as

$$\sum_{(x)} f(x') \otimes f(x'') = \sum_{(f(x))} f(x)' \otimes f(x)''. \tag{1.24}$$

The comultiplication of the tensor product $C \otimes C'$ of the coalgebras C and C' (see Example 5) is given for $x \in C$ and $y \in C'$ by

$$\Delta(x \otimes y) = \sum_{(x\otimes y)} (x \otimes y)' \otimes (x \otimes y)'' = \sum_{(x)(y)} (x' \otimes y') \otimes (x'' \otimes y''). \tag{1.25}$$

We invite the reader to play with Sweedler's sigma notation in order to acquire some familiarity with this most useful convention.

III.2 Bialgebras

Let H be a vector space equipped simultaneously with an algebra structure (H, μ, η) and a coalgebra structure (H, Δ, ε). Let us discuss two compatibility conditions between these two structures. We give $H \otimes H$ the induced structures of a tensor product of algebras (see II.4) and of a tensor product of coalgebras (see Section 1, Example 5).

Theorem III.2.1. *The following two statements are equivalent.*

(i) *The maps μ and η are morphisms of coalgebras.*

(ii) *The maps Δ and ε are morphisms of algebras.*

Proof. It consists essentially in writing down the commutative diagrams expressing both statements. The fact that μ is a morphism of coalgebras is equivalent to the commutativity of the two squares

$$\begin{tikzcd} H \otimes H \arrow[r, "\mu"] \arrow[d, "(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\Delta\otimes\Delta)"] & H \arrow[d, "\Delta"] \\ (H \otimes H) \otimes (H \otimes H) \arrow[r, "\mu\otimes\mu"] & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\mu"] & k \otimes k \arrow[d, "\mathrm{id}"] \\ H \arrow[r, "\varepsilon"] & k \end{tikzcd}$$

whereas the fact that η is a morphism of coalgebras is expressed by the commutativity of the two diagrams

$$\begin{tikzcd} k \arrow[r, "\eta"] \arrow[d, "\mathrm{id}"] & H \arrow[d, "\Delta"] \\ k \otimes k \arrow[r, "\eta\otimes\eta"] & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} k \arrow[rr, "\eta"] \arrow[dr, "\mathrm{id}"'] & & H \arrow[dl, "\varepsilon"] \\ & k & \end{tikzcd}$$

□

Observe that these four commutative diagrams are exactly the same as the following four diagrams whose commutativity express the fact that Δ and ε are morphisms of algebras:

$$\begin{tikzcd} H \otimes H \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\mu"] & (H \otimes H) \otimes (H \otimes H) \arrow[d, "(\mu\otimes\mu)(\mathrm{id}\otimes\tau\otimes\mathrm{id})"] \\ H \arrow[r, "\Delta"] & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "\eta"] \arrow[d, "\mathrm{id}"] & H \arrow[d, "\Delta"] \\ k \otimes k \arrow[r, "\eta\otimes\eta"] & H \otimes H \end{tikzcd}$$

and

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\varepsilon\otimes\varepsilon} & k \otimes k \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mathrm{id}} \\ H & \xrightarrow{\varepsilon} & k \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\eta} & H \\ & \searrow{\scriptstyle \mathrm{id}} \quad \swarrow{\scriptstyle \varepsilon} & \\ & k & \end{array}$$

This leads to the following definition.

Definition III.2.2. *A bialgebra is a quintuple $(H, \mu, \eta, \Delta, \varepsilon)$ where (H, μ, η) is an algebra and (H, Δ, ε) is a coalgebra verifying the equivalent conditions of Theorem* 2.1. *A morphism of bialgebras is a morphism for the underlying algebra and coalgebra structures.*

In the sequel, we shall mainly use Condition (ii) of Theorem 2.1 to define a bialgebra structure. Using the conventions of 1.6, we see that the condition $\Delta(xy) = \Delta(x)\Delta(y)$ is expressed for any pair (x, y) of elements in a bialgebra by

$$\sum_{(xy)} (xy)' \otimes (xy)'' = \sum_{(x)(y)} x'y' \otimes x''y''. \tag{2.1}$$

We also have

$$\Delta(1) = 1 \otimes 1, \quad \varepsilon(xy) = \varepsilon(x)\varepsilon(y), \quad \varepsilon(1) = 1. \tag{2.2}$$

The following proposition is easy to check.

Proposition III.2.3. *Let $H = (H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Then*

$$H^{\mathrm{op}} = (H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon), \quad H^{\mathrm{cop}} = (H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon),$$

and $H^{\mathrm{op\,cop}} = (H, \mu^{\mathrm{op}}, \eta, \Delta^{\mathrm{op}}, \varepsilon)$ are bialgebras.

Example 1. By Propositions 1.2–1.3 the dual vector space H^* of a finite-dimensional bialgebra H has a natural bialgebra structure.

Example 2. In Example 3 of Section 1 we associated a coalgebra $k[X]$ to a set X. Assume now that X comes with a unital *monoid* structure, i.e., with an associative map $\mu : X \times X \to X$ having a left and right unit e. The map μ induces an algebra structure on $k[X]$ with unit e. We have

$$\Delta(xy) = xy \otimes xy = (x \otimes x)(y \otimes y) = \Delta(x)\Delta(y)$$

and $\varepsilon(xy) = 1 = \varepsilon(x)\varepsilon(y)$, which implies that the maps Δ and ε are morphisms of algebras. Thus $k[X]$ becomes a bialgebra.

If, in addition, X is a finite set, then the dual of $k[X]$ also is a bialgebra. We have already observed that the algebra structure of the dual is the usual algebra structure of the space of k-valued functions on X. An easy computation shows that the comultiplication and the counit on the algebra of functions are given by

$$\Delta(f)(x \otimes y) = f(xy) \quad \text{and} \quad \varepsilon(f) = f(e). \tag{2.3}$$

Example 3. (*The bialgebra $M(n)$*) Let $M(n) = k[x_{11}, \ldots, x_{nn}]$ be the polynomial algebra in n^2 variables $\{x_{ij}\}_{1 \leq i,j \leq n}$. For all i, j, set

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{ij}. \tag{2.4}$$

These formulas define morphisms of algebras $\Delta : M(n) \to M(n) \otimes M(n)$ and $\varepsilon : M(n) \to k$ equipping $M(n)$ with a bialgebra structure. When $n = 2$, one recovers the bialgebra $M(2)$ of I.4.

We now endow the tensor algebra with a bialgebra structure.

Theorem III.2.4. *Given a vector space V, there exists a unique bialgebra structure on the tensor algebra $T(V)$ such that $\Delta(v) = 1 \otimes v + v \otimes 1$ and $\varepsilon(v) = 0$ for any element v of V. This bialgebra structure is cocommutative and for all $v_1, \ldots, v_n \in V$ we have*

$$\varepsilon(v_1 \ldots v_n) = 0 \tag{2.5}$$

and $\Delta(v_1 \ldots v_n)$

$$= 1 \otimes v_1 \ldots v_n + \sum_{p=1}^{n-1} \sum_{\sigma} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)} + v_1 \ldots v_n \otimes 1 \tag{2.6}$$

where σ runs over all permutations of the symmetric group S_n such that

$$\sigma(1) < \sigma(2) \ldots < \sigma(p) \quad \textit{and} \quad \sigma(p+1) < \sigma(p+2) \ldots < \sigma(n).$$

Such a permutation σ is called a $(p, n-p)$-*shuffle*.

PROOF. By universality of the tensor algebra, there exist unique algebra morphisms $\Delta : T(V) \to T(V) \otimes T(V)$ and $\varepsilon : T(V) \to k$ such that their restrictions to V are given by the formulas of the theorem. Now consider several elements $v_1, \ldots, v_n$ in V. Formula (2.5) is a trivial consequence of the multiplicativity of ε.

Let us now compute $\Delta(v_1 \ldots v_n)$. We shall do this by induction on n. Formula (2.6) holds for $n = 1$ by definition. Suppose it holds up to $n-1 \geq 1$. Then we have the series of equalities

$$\begin{aligned} &\Delta(v_1 \ldots v_n) \\ &\quad = \quad \Delta(v_1 \ldots v_{n-1})\Delta(v_n) \end{aligned}$$

$$
\begin{aligned}
&= \Delta(v_1 \dots v_{n-1})(1 \otimes v_n + v_n \otimes 1) \\
&= \Big(1 \otimes v_1 \dots v_{n-1} + \sum_{p=1}^{n-2} \sum_{\sigma} v_{\sigma(1)} \dots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \dots v_{\sigma(n-1)} \\
&\qquad + v_1 \dots v_{n-1} \otimes 1\Big)(1 \otimes v_n + v_n \otimes 1) \\
&= 1 \otimes v_1 \dots v_n + \sum_{p=1}^{n-2} \sum_{\sigma} v_{\sigma(1)} \dots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \dots v_{\sigma(n-1)} v_n \\
&\qquad + v_1 \dots v_{n-1} \otimes v_n + v_n \otimes v_1 \dots v_{n-1} \\
&\qquad + \sum_{p=1}^{n-2} \sum_{\sigma} v_{\sigma(1)} \dots v_{\sigma(p)} v_n \otimes v_{\sigma(p+1)} \dots v_{\sigma(n-1)} + v_1 \dots v_n \otimes 1
\end{aligned}
$$

where σ runs over all $(p, n-1-p)$-shuffles of S_{n-1}. Let us rewrite the last sum in the form

$$
\begin{aligned}
1 \otimes v_1 \dots v_n &+ \sum_{p=1}^{n-2} \sum_{\rho} v_{\rho(1)} \dots v_{\rho(p)} \otimes v_{\rho(p+1)} \dots v_{\rho(n-1)} v_n \\
&+ v_1 \dots v_{n-1} \otimes v_n + v_n \otimes v_1 \dots v_{n-1} \\
+ \sum_{p=2}^{n-1} \sum_{\tau} & v_{\tau(1)} \dots v_{\tau(p-1)} v_n \otimes v_{\tau(p)} \dots v_{\tau(n-1)} + v_1 \dots v_n \otimes 1
\end{aligned}
$$

where ρ runs over all $(p, n-1-p)$-shuffles of S_{n-1} and τ runs over all $(p-1, n-p)$-shuffles permuting the set $\{1, \dots, n\} \setminus \{p\}$. Now observe that if $\sigma \in S_n$ is a $(p, n-p)$-shuffle, then either $\sigma(n) = n$, hence the restriction ρ of σ to S_{n-1} is a $(p, n-1-p)$-shuffle, or $\sigma(p) = n$, hence $\tau = \sigma$ acting on $\{1, \dots, n\} \setminus \{p\}$ is a $(p-1, n-p)$-shuffle. This completes the proof of (2.6).

It remains to prove the coassociativity, the counitality and the cocommutativity of Δ. The counitality results from an easy computation using (2.5) and (2.6). The cocommutativity is a consequence of the fact that the permutation

$$
\begin{pmatrix} 1 & 2 & \dots & p & p+1 & p+2 & \dots & n \\ p+1 & p+2 & \dots & n & 1 & 2 & \dots & p \end{pmatrix}
$$

switches $(p, n-p)$-shuffles and $(n-p, p)$-shuffles. As for the coassociativity, one may check it directly using (2.6). But, we rather observe that $\Delta : T(V) \to T(V) \otimes T(V)$ is induced by the diagonal map $\delta(v) = (v, v)$ from V into $V \oplus V$. The coassociativity of Δ then results from the obvious relation $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta$. □

We now introduce the concept of a primitive element.

Definition III.2.5. *Let (C, Δ, ε) be a coalgebra. An element x of C is primitive if we have*

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

We denote by Prim(C) *the subspace of all primitive elements of* C.

Proposition III.2.6. *If x is a primitive element of a bialgebra, then we have $\varepsilon(x) = 0$. If y is another one, then the commutator $[x, y] = xy - yx$ is primitive too.*

Proof. By definition of the counit and of a primitive element we have

$$x = \varepsilon(1)x + \varepsilon(x)1 = x + \varepsilon(x)1.$$

The vanishing of $\varepsilon(x)$ follows immediately. As for the second assertion, we have

$$\Delta(xy) = (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) = 1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1.$$

We deduce

$$\Delta([x, y]) = 1 \otimes [x, y] + [x, y] \otimes 1,$$

which implies that $[x, y]$ is primitive. $\square$

The generators $v \in V$ of the tensor algebra $T(V)$ are primitive by Theorem 2.4. Let H be a bialgebra and $x_1, \ldots, x_n$ be *primitive* elements of H. Consider a vector space V with basis $\{v_1, \ldots, v_n\}$. There is a unique algebra morphism f from the tensor algebra $T(V)$ to H such that $f(v_i) = x_i$ for all i.

Proposition III.2.7. *The map $f : T(V) \to H$ is a morphism of bialgebras.*

Proof. We have to check that

$$\varepsilon(f(\xi)) = \varepsilon(\xi) \quad \text{and} \quad (f \otimes f)\Delta(\xi) = \Delta(f(\xi)) \tag{2.7}$$

for all $\xi \in T(V)$. Since all maps involved in (2.7) are algebra maps, it is enough to check (2.7) when $\xi = v \in V$. In this case (2.7) holds because x_i is primitive and we have Proposition 2.6. $\square$

As a consequence of Proposition 2.7, we see that for any set $\{x_1, \ldots, x_n\}$ of primitive elements in a bialgebra, $\Delta(x_1, \ldots, x_n)$ is given by Formula (2.6) of Theorem 2.4 after replacing v_i by x_i.

III.3 Hopf Algebras

Given an algebra (A, μ, η) and a coalgebra (C, Δ, ε) we define a bilinear map, the *convolution*, on the vector space $\mathrm{Hom}(C, A)$ of linear maps from C to A. By definition, if f, g are such linear maps, then the convolution $f \star g$ is the composition of the maps

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A. \tag{3.1}$$

Using Sweedler's sigma notation of 1.6, we have

$$(f \star g)(x) = \sum_{(x)} f(x')g(x'') \tag{3.2}$$

for any element $x \in C$. The convolution is clearly bilinear.

Proposition III.3.1. (a) *The triple* $(\mathrm{Hom}(C, A), \star, \eta \circ \varepsilon)$ *is an algebra.*

(b) *The map* $\lambda_{C,A} : A \otimes C^* \to \mathrm{Hom}(C, A)$ *of Corollary* II.2.3 *is a morphism of algebras where* $A \otimes C^*$ *is the tensor product algebra of* A *and of the algebra* C^* *dual to the coalgebra* C.

PROOF. (a) By (3.2), by the associativity of the product in A and by the coassociativity of the coproduct in C we have

$$\Big((f \star g) \star h\Big)(x) = \sum_{(x)} f(x')g(x'')h(x''') = \Big(f \star (g \star h)\Big)(x).$$

This proves that the convolution is associative. The map $\eta \circ \varepsilon$ is a left unit for the convolution in view of

$$((\eta\varepsilon) \star f)(x) = \sum_{(x)} \varepsilon(x')f(x'') = f\Big(\sum_{(x)} \varepsilon(x')x''\Big) = f(x),$$

which results from (1.21). One proves similarly that $\eta \circ \varepsilon$ is a right unit.

(b) Let $a, b \in A$ and $\alpha, \beta \in C^*$. Then for $x \in C$ we have

$$\begin{aligned} \Big(\lambda_{C,A}(a \otimes \alpha) \star \lambda_{C,A}(b \otimes \beta)\Big)(x) &= \sum_{(x)} \alpha(x')\beta(x'')\, ab \\ &= (\alpha\beta)(x)\, ab \\ &= \Big(\lambda_{C,A}(ab \otimes \alpha\beta)\Big)(x). \end{aligned}$$

This proves that $\lambda_{C,A}$ preserves the product. As for the unit, we have

$$\Big(\lambda_{C,A}(1 \otimes \varepsilon)\Big)(x) = \varepsilon(x)1 = (\eta \circ \varepsilon)(x).$$

□

Example 1. When $A = k$ the algebra structure $(\mathrm{Hom}(C, k), \star, \eta \circ \varepsilon)$ on the dual space C^* is the same as the one defined in Proposition 1.2.

When $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra we may consider the case $C = A = H$ and thus define the convolution on the vector space $\mathrm{End}(H)$ of endomorphisms of H.

Definition III.3.2. *Let* $(H, \mu, \eta, \Delta, \varepsilon)$ *be a bialgebra. An endomorphism* S *of* H *is called an antipode for the bialgebra* H *if*

$$S \star \mathrm{id}_H = \mathrm{id}_H \star S = \eta \circ \varepsilon.$$

A Hopf algebra is a bialgebra with an antipode. A morphism of Hopf algebras is a morphism between the underlying bialgebras commuting with the antipodes.

A bialgebra does not necessarily have an antipode. But if it does, it has only one. Indeed, if S and S' are antipodes, then

$$S = S \star (\eta\varepsilon) = S \star (\mathrm{id}_H \star S') = (S \star \mathrm{id}_H) \star S' = (\eta\varepsilon) \star S' = S'.$$

A Hopf algebra with an antipode S will be denoted by $(H, \mu, \eta, \Delta, \varepsilon, S)$.

Using Sweedler's convention 1.6, we see that an antipode satisfies the relations

$$\sum_{(x)} x' S(x'') = \varepsilon(x) 1 = \sum_{(x)} S(x') x'' \tag{3.3}$$

for all $x \in H$. In any Hopf algebra we have relations such as

$$\begin{aligned} \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes S(x^{(3)}) \otimes x^{(4)} \otimes x^{(5)} &= \sum_{(x)} x^{(1)} \otimes \varepsilon(x^{(2)}) \otimes x^{(3)} \otimes x^{(4)} \\ &= \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}. \end{aligned}$$

The first equality follows from (3.3), i.e., by definition of the antipode while the second one follows from (1.21), i.e., from the Axiom (Coun). Such computations will be performed later without further explanations.

We state the counterpart of Example 1 of Section 2.

Proposition III.3.3. *Let H be a finite-dimensional Hopf algebra with antipode S. Then the bialgebra H^* is a Hopf algebra with antipode S^*.*

PROOF. The endomorphism S^* of H^* is the transpose of S. Let us prove the first equality in (3.3). For all $\alpha \in H^*$ and $x \in H$ we have

$$\begin{aligned} \Big(\sum_{(\alpha)} \alpha' S^*(\alpha'')\Big)(x) &= \sum_{(\alpha)(x)} \alpha'(x') S^*(\alpha'')(x'') \\ &= \sum_{(\alpha)(x)} \alpha'(x')\alpha''(Sx'') \\ &= \alpha\Big(\sum_{(x)} x'(Sx'')\Big) \\ &= \alpha(\eta\varepsilon(x)) \\ &= \varepsilon^*\eta^*(\alpha)(x). \end{aligned}$$

One shows similarly that $\sum_{(\alpha)} S^*(\alpha')\alpha'' = \varepsilon^*\eta^*(\alpha)$. □

Example 2. Let G be a monoid and $k[G]$ the bialgebra of Section 2, Example 2. Then $k[G]$ has an antipode if and only if any element x of G has an inverse, i.e., if and only if G is a group. Indeed, if S exists, by definition of Δ we must have

$$xS(x) = S(x)x = \varepsilon(x)1 = 1$$

for any $x \in G$. This implies that $S(x) = x^{-1}$ for $x \in G$.

We state a few important properties of the antipode.

Theorem III.3.4. *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra.*

(a) *Then S is a bialgebra morphism from H to $H^{\mathrm{op\,cop}}$, i.e., we have*

$$S(xy) = S(y)S(x), \quad S(1) = 1 \tag{3.4}$$

for all $x, y \in H$ and

$$(S \otimes S)\Delta = \Delta^{\mathrm{op}} S, \quad \varepsilon \circ S = \varepsilon. \tag{3.5}$$

(b) *The following three statements are equivalent:*

(i) *we have $S^2 = \mathrm{id}_H$,*

(ii) *for all $x \in H$ we have $\sum_{(x)} S(x'')x' = \varepsilon(x)1$,*

(iii) *for all $x \in H$ we have $\sum_{(x)} x''S(x') = \varepsilon(x)1$.*

(c) *If H is commutative or cocommutative, then $S^2 = \mathrm{id}_H$.*

The left relation in (3.5) can be reformulated under Sweedler's convention 1.6 as

$$\sum_{(S(x))} S(x)' \otimes S(x)'' = \sum_{(x)} S(x'') \otimes S(x'). \tag{3.6}$$

Proof. (a) Let us start with (3.4). Define maps ν, ρ in $\mathrm{Hom}(H \otimes H, H)$ by

$$\nu(x \otimes y) = S(y)S(x) \quad \text{and} \quad \rho(x \otimes y) = S(xy)$$

where $x, y \in H$. We have to show that $\rho = \nu$. It is enough to prove that $\rho \star \mu = \mu \star \nu = \eta\varepsilon$. Now, by (1.21), (2.1) and (3.2)

$$\begin{aligned}
(\rho \star \mu)(x \otimes y) &= \sum_{(x \otimes y)} \rho((x \otimes y)')\mu((x \otimes y)'') \\
&= \sum_{(x)(y)} \rho(x' \otimes y')\mu(x'' \otimes y'') \\
&= \sum_{(x)(y)} S(x'y')x''y'' \\
&= \sum_{(xy)} S((xy)')(xy)'' \\
&= \eta\varepsilon(xy).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\mu \star \nu)(x \otimes y) &= \sum_{(x\otimes y)} \mu((x\otimes y)')\,\nu((x\otimes y)'') \\
&= \sum_{(x)(y)} x'y'S(y'')S(x'') \\
&= \sum_{(x)} x'\Big(\sum_{(y)} y'S(y'')\Big)S(x'') \\
&= \sum_{(x)} x'\varepsilon(y)S(x'') \\
&= \eta\varepsilon(x)\eta\varepsilon(y) \\
&= \eta\varepsilon(xy),
\end{aligned}$$

which is the same.

Applying $(\mathrm{id} \star S)(x) = \eta\varepsilon(x)$ to $x = 1$, one gets $S(1) = 1$. This proves (3.4).

Let us deal with (3.5). It is equivalent to prove $\Delta \circ S = (S \otimes S) \circ \Delta^{\mathrm{op}}$. We set $\rho = \Delta \circ S$ and $\nu = (S \otimes S) \circ \Delta^{\mathrm{op}}$. These are linear maps from H to $H \otimes H$. We wish to show that $\rho = \nu$. This will follow from $\rho \star \Delta = \Delta \star \nu = (\eta \otimes \eta)\varepsilon$, which we prove now. On the one hand, by (1.21)

$$\begin{aligned}
(\rho \star \Delta)(x) &= \sum_{(x)} \Delta(S(x'))\Delta(x'') = \Delta\Big(\sum_{(x)} S(x')x''\Big) \\
&= \Delta(\eta\varepsilon(x)) = ((\eta \otimes \eta)\varepsilon)(x)
\end{aligned}$$

for all $x \in H$. On the other hand, we have

$$\begin{aligned}
(\Delta \star \nu)(x) &= \sum_{(x)} \Delta(x')\Big((S \otimes S)(\Delta^{\mathrm{op}}(x''))\Big) \\
&= \sum_{(x)} (x' \otimes x'')\Big(S(x'''') \otimes S(x''')\Big) \\
&= \sum_{(x)} x'S(x'''') \otimes x''S(x''') \\
&= \sum_{(x)} x'S(x''') \otimes \varepsilon(x'')1 \\
&= \sum_{(x)} x'\varepsilon(x'')S(x''') \otimes 1 \\
&= \sum_{(x)} x'S(x'') \otimes 1 \\
&= \varepsilon(x)1 \otimes 1 \\
&= (\eta \otimes \eta)(\varepsilon(x)).
\end{aligned}$$

The fourth and seventh equalities follow from (3.3), the sixth one from (1.21).

We also derive

$$\varepsilon(S(x)) = \varepsilon\Big(S(\sum_{(x)} \varepsilon(x')x'')\Big) = \varepsilon\Big(\sum_{(x)} \varepsilon(x')S(x'')\Big) = \varepsilon(\eta\varepsilon(x)) = \varepsilon(x)$$

from (1.21). This completes the proof of (3.5).

(b) Let us prove that (ii) implies (i). By uniqueness of the inverse, it is enough to show that S^2 is a right inverse of S for the convolution, just as is id_H. Now, using (3.4) and Condition (ii), we get for all $x \in H$

$$\begin{aligned}(S \star S^2)(x) &= \sum_{(x)} S(x')S^2(x'') = S\Big(\sum_{(x)} S(x'')x'\Big)\\ &= S(\varepsilon(x)1) = \varepsilon(x)S(1) = \varepsilon(x)1.\end{aligned}$$

This implies that $S \star S^2 = \eta\varepsilon$, hence $S^2 = \mathrm{id}_H$. Let us prove the converse implication: if $S^2 = \mathrm{id}_H$ we have

$$\begin{aligned}\sum_{(x)} S(x'')x' &= S^2\Big(\sum_{(x)} S(x'')x'\Big)\\ &= S\Big(\sum_{(x)} S(x')S^2(x'')\Big)\\ &= S\Big(\sum_{(x)} S(x')x''\Big)\\ &= S(\varepsilon(x)1)\\ &= \varepsilon(x)1.\end{aligned}$$

One proves that (i) is equivalent to (iii) in a similar fashion.

(c) Recall Relations (3.3): we have

$$\sum_{(x)} x'S(x'') = \eta\varepsilon(x) = \sum_{(x)} S(x')x''$$

for all $x \in H$. When H is commutative, the first equality becomes

$$\sum_{(x)} S(x'')x' = \eta\varepsilon(x),$$

which implies $S^2 = \mathrm{id}_H$ by Part (b) (ii). When H is cocommutative, the second equality becomes

$$\eta\varepsilon(x) = \sum_{(x)} S(x'')x'$$

which again implies $S^2 = \mathrm{id}_H$ in view of Part (b) (iii). □

As an immediate consequence of Theorem 3.4, we have the following.

Corollary III.3.5. *Let* $H = (H, \mu, \eta, \Delta, \varepsilon, S)$ *be a Hopf algebra. Then*

$$H^{\mathrm{op\,cop}} = (H, \mu^{\mathrm{op}}, \eta, \Delta^{\mathrm{op}}, \varepsilon, S)$$

is another Hopf algebra and $S : H \to H^{\mathrm{op\,cop}}$ *is a morphism of Hopf algebras. If, moreover,* S *is an isomorphism with inverse* S^{-1}*, then*

$$H^{\mathrm{op}} = (H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon, S^{-1}) \quad \textit{and} \quad H^{\mathrm{cop}} = (H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon, S^{-1})$$

are isomorphic Hopf algebras, the isomorphism being given by S.

An endomorphism T of a bialgebra H such that

$$\sum_{(x)} T(x'')x' = \varepsilon(x)1 = \sum_{(x)} x''T(x') \tag{3.7}$$

for all $x \in H$ is sometimes called a *skew-antipode* for H. Alternatively, a skew-antipode for H is an antipode for the bialgebras H^{op} and H^{cop}. By Corollary 3.5 the inverse (if it exists) of an antipode is a skew-antipode.

It is not always easy to check the defining Relations (3.3) of an antipode for every element of a bialgebra, but it may be simpler to check only for some generators. It is convenient to have the following lemma.

Lemma III.3.6. *Let* H *be a bialgebra and* $S : H \to H^{\mathrm{op}}$ *be an algebra morphism. Assume that* H *is generated as an algebra by a subset* X *such that*

$$\sum_{(x)} x'S(x'') = \varepsilon(x)1 = \sum_{(x)} S(x')x''$$

for all $x \in X$. *Then* S *is an antipode for* H.

PROOF. It is enough to check that if (3.3) holds for x and y, then it holds for the product xy. Now, by (3.3–3.4)

$$\begin{aligned}
\sum_{(xy)} (xy)'S((xy)'') &= \sum_{(x)(y)} x'y'S(x''y'') \\
&= \sum_{(x)} x'\Big(\sum_{(y)} y'S(y'')\Big)S(x'') \\
&= \Big(\sum_{(x)} x'S(x'')\Big)\varepsilon(y) \\
&= \varepsilon(x)\varepsilon(y) \\
&= \varepsilon(xy).
\end{aligned}$$

One proves $\sum_{(xy)} S((xy)')(xy)'' = \varepsilon(xy)$ similarly. □

Use the previous lemma to show that the following provide examples of Hopf algebras.

Example 3. The tensor bialgebra $H = T(V)$ is a Hopf algebra with an antipode determined by $S(1) = 1$ and for all $v_1, v_2, \ldots, v_n \in V$ by

$$S(v_1 v_2 \ldots v_n) = (-1)^n \, v_n \ldots v_2 v_1.$$

Example 4. (*The symmetric bialgebra* $S(V)$) Let I be the kernel of the projection of $T(V)$ onto the symmetric algebra $S(V)$. Let us show that I is a coideal for the coalgebra structure put on $T(V)$ in Theorem 2.4. Any element of I is a sum of elements of the form $x[v,w]y$ where $x, y \in T(V)$ and $v, w \in V$. By Theorem 2.4 we have

$$\Delta(x[v,w]y) = \sum_{(x)(y)} \Big(x'[v,w]y' \otimes x''y'' + x'y' \otimes x''[v,w]y'' \Big)$$

which belongs to $I \otimes T(V) + T(V) \otimes I$ and

$$\varepsilon(x[v,w]y) = \varepsilon(x)[\varepsilon(v), \varepsilon(w)]\varepsilon(y) = 0,$$

which proves that I is a coideal. It follows that the bialgebra structure of $T(V)$ induces a bialgebra structure on $S(V)$ for which the elements of V are primitive. One checks that $S(V)$ has an antipode which is the multiplication by $(-1)^n$ on $S^n(V)$.

Another useful concept is the concept of a *grouplike* element of a coalgebra (H, Δ, ε), i.e., an element $x \neq 0$ such that

$$\Delta(x) = x \otimes x. \tag{3.8}$$

The set of grouplike elements of H will be denoted by $\mathcal{G}(H)$.

Proposition III.3.7. *Let H be a bialgebra. Then $\mathcal{G}(H)$ is a monoid for the multiplication of H with unit* 1. *If, furthermore, H has an invertible antipode S, then any grouplike element x has an inverse in $\mathcal{G}(H)$ which is $S(x)$. Consequently, $\mathcal{G}(H)$ is a group.*

PROOF. The first assertion is clear. As for the second, observe that (3.6) and (3.8) imply $\Delta(S(x)) = S(x) \otimes S(x)$. It follows that $S(x)$ belongs to $\mathcal{G}(H)$. To complete the proof, one checks that $\varepsilon(x) = 1$ when x is grouplike, and one uses the computation in Example 2 in order to show that $S(x)$ is the inverse of x. □

Example 5. If $k[G]$ is the Hopf algebra associated to a group G as in Example 2, then the elements of G are the only grouplike elements of $k[G]$. In other words, we have

$$\mathcal{G}(k[G]) = G. \tag{3.9}$$

III.4 Relationship with Chapter I. The Hopf Algebras $GL(2)$ and $SL(2)$

The aim of this section is to show that the algebras $M(2)$, $GL(2)$ and $SL(2)$ defined in I.4 and I.5 are bialgebras. We use Proposition II.4.2 in order to identify $M(2)^{\otimes 2}$ with $M(2) \otimes M(2)$, $GL(2)^{\otimes 2}$ with $GL(2) \otimes GL(2)$ and $SL(2)^{\otimes 2}$ with $SL(2) \otimes SL(2)$. Let us show that the morphisms Δ of I.4 and ε of I.5 equip these algebras with a cocommutative bialgebra structure. Recall (I.4.4): we have

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.1}$$

and $\Delta(t) = t \otimes t$. In order to prove that Δ is coassociative, it suffices to check this on the generators a, b, c, d, and t, which results from the fact that t is grouplike and from the matrix equality

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

Similarly, the counit axiom follows from $\varepsilon(t) = 1$ and from the matrix equalities

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.2}$$

The algebra morphism S defined in (I.5.2) is an antipode for the bialgebras $GL(2)$ and $SL(2)$ which become Hopf algebras in this way. Indeed, by Lemma 3.6, it is enough to check Relations (3.3) for the generators a, b, c, d, t. For a, b, c, d it follows from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{pmatrix}. \tag{4.3}$$

As for t, we have $tS(t) = S(t)t = \varepsilon(t) = 1$ since $S(t) = t^{-1} = ad - bc$.

The antipode is an involution due to the fact that $GL(2)$ and $SL(2)$ are both commutative. This can also be checked directly on Formula (I.5.2) defining S.

III.5 Modules over a Hopf Algebra

Let A be an algebra. The tensor product $U \otimes V$ of two A-modules is an $A \otimes A$-module by

$$(a \otimes a')(u \otimes v) = au \otimes a'v \tag{5.1}$$

where $a, a' \in A$, $u \in U$ and $v \in V$. Now, if A possesses a bialgebra structure $(A, \mu, \eta, \Delta, \varepsilon)$, then the algebra morphism $\Delta : A \to A \otimes A$ enables us to equip the $A \otimes A$-module $U \otimes V$ with an A-module structure by

$$a(u \otimes v) = \Delta(a)(u \otimes v) = \sum_{(a)} a'u \otimes a''v. \tag{5.2}$$

The counit ε equips any vector space V with a *trivial* A-module structure by

$$av = \varepsilon(a)v \tag{5.3}$$

where $a \in A$ and $v \in V$.

The following is the natural extension of Proposition II.1.3 to the framework of A-modules.

Proposition III.5.1. *If A is a bialgebra, U, V and W are A-modules and k is given the trivial A-module structure, then the canonical isomorphisms of Proposition* II.1.3

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \quad \text{and} \quad k \otimes V \cong V \cong V \otimes k$$

are A-module isomorphisms. If, furthermore, A is cocommutative, then the flip $\tau_{V,W} : V \otimes W \cong W \otimes V$ is an isomorphism of A-modules.

PROOF. The proof is easy and is left to the reader. $\square$

Let us show how an antipode allows us to give a natural A-module structure to the vector space $\mathrm{Hom}(V, V')$ of linear maps from V to V' when V and V' have A-module structures. We first observe that

$$\Big((a \otimes a')f\Big)(v) = af(a'v) \tag{5.4}$$

puts an $A \otimes A^{\mathrm{op}}$-module structure on $\mathrm{Hom}(V, V')$. Indeed, we have

$$\begin{aligned}
\Big((a \otimes a')(b \otimes b')f\Big)(v) &= \Big((ab \otimes b'a')f\Big)(v) \\
&= abf(b'a'v) \\
&= a\Big((b \otimes b')f\Big)(a'v) \\
&= \Big((a \otimes a')((b \otimes b')f)\Big)(v)
\end{aligned}$$

for $a, a', b, b' \in A$, $v \in V$ and $f \in \mathrm{Hom}(V, V')$. Now, if A is a Hopf algebra with antipode S, then the map $(\mathrm{id} \otimes S) \circ \Delta$ is a morphism of algebras from A to $A \otimes A^{\mathrm{op}}$. Pulling (5.4) back along this morphism, we get an A-module structure on $\mathrm{Hom}(V, V')$. Explicitly, if $a \in A$, $v \in V$ and $f \in \mathrm{Hom}(V, V')$, the action of A on $\mathrm{Hom}(V, V')$ is given by

$$(af)(v) = \sum_{(a)} a'f(S(a'')v). \tag{5.5}$$

In particular, if $V' = k$ is given the trivial A-module structure, then (5.5) induces an A-module structure on the dual vector space V^* which becomes

$$(af)(v) = f(S(a)v). \tag{5.6}$$

Indeed, by (5.5) and (1.21), we get

$$(af)(v) = \sum_{(a)} \varepsilon(a')f(S(a'')v) = f\Big(S(\sum_{(a)} \varepsilon(a')a'')v\Big) = f(S(a)v).$$

Proposition III.5.2. *Let $(A, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra and U, U', V and V' be A-modules such that, either U or U', and, either V or V', are finite-dimensional vector spaces. Then the linear map*

$$\lambda : \mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V') \to \mathrm{Hom}(V \otimes U, U' \otimes V')$$

of (II.2.2) *is A-linear if, in addition, the flip $\tau_{U^*,V'} : U^* \otimes V' \to V' \otimes U^*$ is A-linear. In particular, the maps*

$$\lambda : U^* \otimes V^* \to (V \otimes U)^* \quad \textit{and} \quad \lambda_{U,V} : V \otimes U^* \to \mathrm{Hom}(U, V)$$

are A-linear.

PROOF. (a) Let $f : U \to U'$, $g : V \to V'$, $u \in U$, $v \in V$ and $a \in A$. Let us first compute $\lambda(a(f \otimes g))$ using (II.2.2), (5.2) and (5.5). We have

$$\begin{aligned}
Z_1 &= \Big(\lambda(a(f \otimes g))\Big)(v \otimes u) \\
&= \sum_{(a)} \lambda(a'f \otimes a''g)(v \otimes u) \\
&= \sum_{(a)} (a'f)(u) \otimes (a''g)(v) \\
&= \sum_{(a)} (a')'f(S((a')'')u) \otimes (a'')'g(S((a'')'')v) \\
&= \sum_{(a)} a'f(S(a'')u) \otimes a'''g(S(a'''')v)
\end{aligned}$$

using Sweedler's sigma notation. On the other hand, $a\lambda(f \otimes g)$ is given by

$$\begin{aligned}
Z_2 &= \Big(a\lambda(f \otimes g)\Big)(v \otimes u) \\
&= \sum_{(a)} a'\lambda(f \otimes g)(S(a'')(v \otimes u)) \\
&= \sum_{(a)} a'\lambda(f \otimes g)(S(a'')'v \otimes S(a'')''u)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{(a)} a'\lambda(f\otimes g)(S((a'')'')v\otimes S((a'')')u)\\
&= \sum_{(a)} a'\lambda(f\otimes g)(S(a''')v\otimes S(a'')u)\\
&= \sum_{(a)} a'\Big(f(S(a'')u)\otimes g(S(a''')v)\Big)\\
&= \sum_{(a)} (a')'f(S(a'')u)\otimes (a')''g(S(a''')v)\\
&= \sum_{(a)} a'f(S(a''')u)\otimes a''g(S(a'''')v).
\end{aligned}$$

We used (3.6) for the fourth equality. Observe that $Z_1 \neq Z_2$ in general.

(b) Let $V' = k$ be given the trivial action. Replacing a''' in Z_1 [resp. a'' in Z_2] by $\varepsilon(a''')$ [resp. by $\varepsilon(a'')$] and using (1.21), we get

$$Z_1 = Z_2 = \sum_{(a)} a'f(S(a'')u)\otimes g(S(a''')v),$$

which proves that $\lambda : \mathrm{Hom}(U,U')\otimes V^* \to \mathrm{Hom}(V\otimes U, U')$ is A-linear. We get the two special cases of Proposition 5.2 with $U' = k$ and with $U = k$.

For the general case, we use Lemma II.2.4 which expresses λ in terms of the special maps λ and of the flip $\tau_{U^*,V'}$. □

As a corollary of Proposition 5.2, we see that the general map λ of Theorem II.2.1 is A-linear when A is cocommutative. This happens, for instance, when A is a group algebra or an enveloping algebra.

As for the evaluation and the coevaluation maps, we have the following result.

Proposition III.5.3. *Let V be an A-module. Then the evaluation map $\mathrm{ev}_V : V^*\otimes V \to k$ is A-linear. If, moreover, the vector space V is finite-dimensional, then the coevaluation map $\delta_V : k \to V\otimes V^*$ of* II.3 *and the composition*

$$\mathrm{Hom}(V,W)\otimes \mathrm{Hom}(U,V) \xrightarrow{\circ} \mathrm{Hom}(U,W)$$

are A-linear too.

Proof. (a) Let $a\in A$, $v \in V$ and $\alpha\in V^*$. Then

$$\begin{aligned}
\mathrm{ev}_V(a(\alpha\otimes v)) &= \sum_{(a)} \mathrm{ev}_V(a'\alpha\otimes a''v)\\
&= \sum_{(a)} (a'\alpha)(a''v)\\
&= \alpha(\sum_{(a)} S(a')a''v)
\end{aligned}$$

$$\begin{aligned} &= \alpha(\varepsilon(a)v) \\ &= \varepsilon(a)\alpha(v) \end{aligned}$$

by the rightmost relation (3.3) and by (5.6). This implies that the evaluation map is A-linear.

(b) The coevaluation map δ_V is A-linear as the composition of the unit $\eta : k \to \mathrm{End}(V)$ and of $\lambda_{V,V}^{-1}$. The latter is A-linear by Proposition 5.2. So is the map $\eta : k \to \mathrm{End}(V)$ following

$$\begin{aligned} (a\eta(1))(v) &= (a\mathrm{id}_V)(v) \\ &= \sum_{(a)} a'\mathrm{id}_V(S(a'')v) \\ &= \sum_{(a)} a'S(a'')v \\ &= \varepsilon(a)v \\ &= (\eta(a1))(v) \end{aligned}$$

for all $v \in V$ and $a \in A$. Here we used the leftmost relation (3.3).

(c) For the composition map, one uses Lemma II.2.5. □

III.6 Comodules

Algebras act on modules, coalgebras coact on comodules. This section is devoted to the definition of the latter concept. Let A be an algebra. Recall that an A-module is a pair (M, μ_M) where M is a vector space and $\mu_M : A \otimes M \to M$ is a linear map such that the following axioms (Ass) and (Un) hold.

(Ass): The square

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu_M"] & A \otimes M \arrow[d, "\mu_M"] \\ A \otimes M \arrow[r, "\mu_M"] & M \end{tikzcd} \tag{6.1}$$

commutes.

(Un): The diagram

$$\begin{tikzcd} k \otimes M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\cong"] & A \otimes M \arrow[d, "\mu_M"] \\ & M \end{tikzcd} \tag{6.2}$$

commutes.

A morphism of A-modules $f : (M, \mu_M) \to (M', \mu_{M'})$ is a linear map f from M to M' such that

$$\mu_{M'} \circ (\mathrm{id} \otimes f) = f \circ \mu_M \,. \tag{6.3}$$

The definition of a comodule over a coalgebra is obtained by reversing all arrows in the diagrams above.

Definition III.6.1. *Let (C, Δ, ε) be a coalgebra.*

(a) *A C-comodule is a pair (N, Δ_N) where N is a vector space and $\Delta_N : N \to C \otimes N$ is a linear map, called the coaction of C on N, such that the following axioms* (Coass) *and* (Coun) *are satisfied.*

(Coass)*: The square*

$$\begin{tikzcd} N \arrow[r, "\Delta_N"] \arrow[d, "\Delta_N"] & C \otimes N \arrow[d, "\mathrm{id}\otimes\Delta_N"] \\ C \otimes N \arrow[r, "\Delta\otimes\mathrm{id}"] & C \otimes C \otimes N \end{tikzcd} \tag{6.4}$$

commutes.

(Coun)*: The diagram*

$$\begin{tikzcd} k \otimes N & C \otimes N \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \\ & N \arrow[u, "\Delta_N"'] \arrow[ul, "\cong"] \end{tikzcd} \tag{6.5}$$

commutes.

(b) *Let (N, Δ_N) and $(N', \Delta_{N'})$ be C-comodules. A linear map f from N to N' is a morphism of C-comodules if*

$$(\mathrm{id} \otimes f) \circ \Delta_N = \Delta_{N'} \circ f. \tag{6.6}$$

(c) *A subspace N' of a C-comodule (N, Δ_N) is a subcomodule of N if $\Delta_N(N') \subset C \otimes N'$.*

Actually, the comodules we have just defined are left comodules. One similarly defines a right C-comodule N, using a map $N \otimes C \to N$ subject to relations parallel to (6.4–6.5). A right C-comodule is the same as a (left) comodule over the opposite coalgebra C^{cop}.

The composition of two morphisms of comodules is another morphism of comodules. Similarly, the inclusion of a subcomodule into a comodule is a morphism of comodules. Let us give a few examples of comodules.

Example 1. Let C be a coalgebra. Then (C, Δ) is a C-comodule.

Example 2. Let C be a coalgebra and C^* the dual vector space equipped with the dual algebra structure of Proposition 1.2. If (N, Δ_N) is a C-comodule, then the dual vector space N^* has the structure of a right C^*-module given by the composition of the maps

$$N^* \otimes C^* \xrightarrow{\lambda} (C \otimes N)^* \xrightarrow{\Delta_N^*} N^*. \tag{6.7}$$

Example 3. Let A be a *finite-dimensional algebra* and A^* be the dual vector space with the coalgebra structure given by Proposition 1.3. If (M, μ_M) is a right A-module, then the dual vector space M^* has a structure of A^*-comodule given by the composition of the maps

$$M^* \xrightarrow{\mu_M^*} (M \otimes A)^* \xrightarrow{\lambda^{-1}} A^* \otimes M^* . \tag{6.8}$$

In order to put a structure of comodule on the tensor product of two comodules, we need a bialgebra structure as in Section 5.

Example 4. (*Tensor product of comodules*) Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra and M and N be H-comodules. We define $\Delta_{M\otimes N}$ by

$$\Delta_{M\otimes N} = (\mu \otimes \mathrm{id}_{M\otimes N})(\mathrm{id}_H \otimes \tau_{M,H} \otimes \mathrm{id}_N)(\Delta_M \otimes \Delta_N). \tag{6.9}$$

The map $\Delta_{M\otimes N}$ endows the tensor product $M \otimes N$ with an H-comodule structure.

Example 5. (*Trivial comodule*) Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra and V be a vector space. The linear map

$$V \cong k \otimes V \xrightarrow{\eta\otimes\mathrm{id}_V} H \otimes V \tag{6.10}$$

equips V with an H-comodule structure. Such a comodule is called a trivial comodule.

Example 6. (*Free comodule*) Let (C, Δ, ε) be a coalgebra. The free C-comodule on a vector space V is the comodule $(C \otimes V, \Delta \otimes \mathrm{id}_V)$. This is a generalization of Example 1.

Proposition 5.1 has the following counterpart for comodules. The proof is left to the reader.

Proposition III.6.2. *If H is a bialgebra, M, N, P are H-comodules and k is given the trivial H-comodule structure of Example* 5, *then the canonical isomorphisms of Proposition* II.1.3

$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P) \quad \textit{and} \quad k \otimes M \cong M \cong M \otimes k$$

are isomorphisms of H-comodules. If, in addition, the bialgebra H is commutative, then the flip $\tau_{M,N} : M \otimes N \cong N \otimes M$ is an isomorphism of H-comodules too.

Notation 6.3. It is often convenient to use for comodules the same kind of notation as was introduced for coalgebras in Section 1. Let (C, Δ, ε) be a coalgebra and (N, Δ_N) be a C-comodule. By convention we shall write

$$\Delta_N(x) = \sum_{(x)} x_C \otimes x_N \tag{6.11}$$

for any $x \in N$. Relation (6.4) is equivalent to

$$\sum_{(x)} (x_C)' \otimes (x_C)'' \otimes x_N = \sum_{(x)} x_C \otimes (x_N)_C \otimes (x_N)_N \tag{6.12}$$

for all $x \in N$. Relation (6.5) is equivalent to

$$\sum_{(x)} \varepsilon(x_C) \otimes x_N = x. \tag{6.13}$$

A linear map $f : N \to N'$ is a morphism of C-comodules if

$$\sum_{(x)} x_C \otimes f(x_N) = \sum_{(x)} f(x)_C \otimes f(x)_{N'}. \tag{6.14}$$

III.7 Comodule-Algebras. Coaction of $SL(2)$ on the Affine Plane

The aim of this section is to define a coaction of the bialgebra $SL(2)$ on the affine plane of Chapter I. Before doing so, we introduce the following concept.

Definition III.7.1. *Let $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$ be a bialgebra and (A, μ_A, η_A) be an algebra. We say A is an H-comodule-algebra if*

(a) *the vector space A has an H-comodule structure given by a map $\Delta_A : A \to H \otimes A$, and*

(b) *the structure maps $\mu_A : A \otimes A \to A$ and $\eta_A : k \to A$ are morphisms of H-comodules, the tensor product $A \otimes A$ and the ground field k being given the H-comodule structures described in Section* 6.

We note the following useful characterization of comodule-algebra structures.

Proposition III.7.2. *Let H be a bialgebra and A be an algebra. Then A is an H-comodule-algebra if and only if*

(a) *the vector space A has an H-comodule structure given by a map $\Delta_A : A \to H \otimes A$, and*

(b) *the map $\Delta_A : A \to H \otimes A$ is a morphism of algebras.*

PROOF. It is similar to the proof of Theorem 2.1. We first express the fact that μ_A is a morphism of H-comodules with the commutative square

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ \Big\downarrow u & & \Big\downarrow \Delta_A \\ H \otimes (A \otimes A) & \xrightarrow{\mathrm{id} \otimes \mu_A} & H \otimes A \end{array} \tag{7.1}$$

where $u = (\mu_H \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{A,H} \otimes \mathrm{id}) \circ (\Delta_A \otimes \Delta_A)$. The fact that η_A is a morphism of H-comodules is equivalent to the commutativity of the square

$$\begin{array}{ccc} k & \xrightarrow{\eta_A} & A \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\Delta_A} \\ k \otimes k & \xrightarrow{\eta_H \otimes \eta_A} & H \otimes A \end{array} \tag{7.2}$$

Now, Diagrams (7.1–7.2) are exactly the same as Diagrams (7.3) below which express the fact that Δ_A is a morphism of algebras:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\Delta_A \otimes \Delta_A} & (H \otimes A) \otimes (H \otimes A) \\ \downarrow{\scriptstyle\mu_A} & & \downarrow{\scriptstyle v} \\ A & \xrightarrow{\Delta_A} & H \otimes A \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\cong} & k \otimes k \\ \downarrow{\scriptstyle\eta_A} & & \downarrow{\scriptstyle\eta_H \otimes \eta_A} \\ A & \xrightarrow{\Delta_A} & H \otimes A \end{array} \tag{7.3}$$

where $v = (\mu_H \otimes \mu_A) \circ (\mathrm{id} \otimes \tau_{A,H} \otimes \mathrm{id})$. Indeed, we have

$$(\mathrm{id} \otimes \mu_A) \circ u = v \circ (\Delta_A \otimes \Delta_A).$$

□

Using the conventions of Sections 1 and 6, we can rewrite Condition (b) of Proposition 7.2 as $\Delta_A(1) = 1 \otimes 1$ and

$$\sum_{(ab)} (ab)_H \otimes (ab)_A = \sum_{(a)(b)} a_H b_H \otimes a_A b_A \tag{7.4}$$

for all $a, b \in A$.

We now show that the affine plane $k[x,y]$ defined in I.3 possesses an comodule-algebra structure over the bialgebras $M(2)$ and $SL(2)$.

Theorem III.7.3. *There exists a unique $M(2)$-comodule-algebra structure and a unique $SL(2)$-comodule-algebra structure on the affine plane $A = k[x,y]$ such that*

$$\Delta_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}.$$

This matrix notation is short for the two relations

$$\Delta_A(x) = a \otimes x + b \otimes y \quad \text{and} \quad \Delta_A(y) = c \otimes x + d \otimes y. \tag{7.5}$$

PROOF. We use Proposition 7.2. First observe that Formulas (7.5) define a morphism of algebras $\Delta_A : k[x,y] \to M(2) \otimes k[x,y]$. The projection of $M(2)$ onto $SL(2)$ being an algebra morphism too, so is the resulting composition $k[x,y] \to SL(2) \otimes k[x,y]$.

It remains to be checked that Δ_A defines a comodule structure, i.e., that for all $z \in k[x,y]$ we have

$$(\mathrm{id} \otimes \Delta_A) \circ \Delta_A(z) = (\Delta \otimes \mathrm{id}) \circ \Delta_A(z) \quad \text{and} \quad (\varepsilon \otimes \mathrm{id}) \circ \Delta_A(z) = 1 \otimes z \tag{7.6}$$

where Δ and ε are as in I.4–I.5. As both sides of each equality to be proved consist solely of algebra morphisms, it suffices to check (7.6) only for $z = x$ and $z = y$. The above matrix notation allows us to do this simultaneously. We have

$$\big((\mathrm{id} \otimes \Delta_A) \circ \Delta_A\big) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \big((\Delta \otimes \mathrm{id}) \circ \Delta_A\big) \begin{pmatrix} x \\ y \end{pmatrix}$$

in view of (4.1). On the other hand, using (I.5.2) we get

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = 1 \otimes \begin{pmatrix} x \\ y \end{pmatrix}.$$

□

Let us compute $\Delta_A(x^i y^j)$ in $M(2) \otimes k[x, y]$.

Lemma III.7.4. *For all $i, j \geq 0$ we have*

$$\Delta_A(x^i y^j) = \sum_{r=0}^{i} \sum_{s=0}^{j} \begin{pmatrix} i \\ r \end{pmatrix} \begin{pmatrix} j \\ s \end{pmatrix} a^r b^{i-r} c^s d^{j-s} \otimes x^{r+s} y^{i+j-r-s}.$$

PROOF. Since Δ_A is an algebra morphism, we have

$$\Delta_A(x^i y^j) = \Delta_A(x)^i \Delta_A(y)^j = (a \otimes x + b \otimes y)^i (c \otimes x + d \otimes y)^j.$$

Next, apply the binomial formula. □

Let us denote by $k[x, y]_n$ the subspace of homogeneous polynomials of total degree n in $A = k[x, y]$. Lemma 7.4 implies that $k[x, y]_n$ is a subcomodule of the affine plane due to the fact that

$$\Delta_A(k[x, y]_n) \subset M(2) \otimes k[x, y]_n.$$

Actually, the $M(2)$-[resp. $SL(2)$-]comodule $k[x, y]$ is the direct sum of the comodules $k[x, y]_n$.

According to Section 6, Example 2, the dual vector space $k[x, y]_n^*$ of the comodule $k[x, y]_n$ has a module structure over the algebra $SL(2)^*$, the dual of the coalgebra $SL(2)$. We shall identify this module in V.7.

III.8 Exercises

1. (*Tensor product of coalgebras*) Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be coalgebras. Show that the linear maps $\pi : C \otimes C' \to C$ and $\pi' : C \otimes C' \to C'$ defined by $\pi(c \otimes c') = \varepsilon'(c')c$ and $\pi'(c \otimes c') = \varepsilon(c)c'$ are morphisms

of coalgebras and that the coalgebra $C \otimes C'$ satisfies the following universal property: for any *cocommutative* coalgebra D and any pair $f : D \to C$ and $f' : D \to C'$ of coalgebra morphisms, there exists a unique morphism of coalgebras $f \otimes f' : D \to C \otimes C'$ such that $\pi \circ (f \otimes f') = f$ and $\pi' \circ (f \otimes f') = f'$.

2. (*Divided powers*) Consider the vector space $C = k[t]$ of polynomials in one variable. Prove that there exists a unique coalgebra structure (C, Δ, ε) on C such that

$$\Delta(t^n) = \sum_{p+q=n} t^p \otimes t^q \quad \text{and} \quad \varepsilon(t^n) = \delta_{n0}$$

for all $n \geq 0$. Show that C becomes a bialgebra when given the product

$$t^p t^q = \binom{p+q}{p} t^{p+q}.$$

Find an antipode.

3. (*Tensor coalgebra*) Let V be a vector space.

   (a) Show that the canonical isomorphisms $V^{\otimes(n+m)} \cong V^{\otimes n} \otimes V^{\otimes m}$ endow $T'(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ with a coalgebra structure, called the *tensor coalgebra* of V.

   (b) Let p_V be the canonical projection of $T'(V)$ onto V. Prove that for any coalgebra C and any linear map $f : C \to V$, there exists a unique morphism of coalgebras $\bar{f} : C \to T'(V)$ such that $f = p_V \circ \bar{f}$.

   (c) Using the notation of Chapter II, Exercise 7, define the subspace $S'(V) = \bigoplus_{n \geq 0} S'_n(V)$ [resp. $\Lambda'(V) = \bigoplus_{n \geq 0} \Lambda'_n(V)$] of $T'(V)$ generated by all symmetric [resp. antisymmetric] tensors. Show that $S'(V)$ and $\Lambda'(V)$ are subcoalgebras of $T'(V)$.

   (d) Let C be a cocommutative coalgebra and f be a linear map from C to V. Prove the existence and the uniqueness of a coalgebra morphism $\bar{f} : C \to S'(V)$ such that $f = p_V \circ \bar{f}$.

4. (*Graded dual*) The graded dual vector space of a graded vector space $V = \bigoplus_{n \geq 0} V_n$ is the graded vector space $V^*_{gr} = \bigoplus_{n \geq 0} V^*_n$. Let $W = \bigoplus_{n \geq 0} W_n$ be another graded vector space. Show that there is a grading on the tensor product $V \otimes W$ such that

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j \, .$$

Prove that $V^*_{gr} \otimes W^*_{gr} \cong (V \otimes W)^*_{gr}$ if V_n is finite-dimensional for each $n \geq 0$.

5. (*Graded coalgebra*) Keep the notation of the previous exercise. A coalgebra (C, Δ, ε) is graded if there exist subspaces $(C_n)_{n\geq 0}$ of C such that $C = \bigoplus_{n\geq 0} C_n$ and $\Delta(C_n) \subset \bigoplus_{i+j=n} C_i \otimes C_j$ for all $n \geq 0$ and $\varepsilon(C_n) = \{0\}$ for all $n > 0$.

   (a) Prove that the graded dual vector space of a graded coalgebra carries a natural graded algebra structure.

   (b) Let $A = \bigoplus_{n\geq 0} A_n$ be a graded algebra whose summands A_n are all finite-dimensional. Prove that the graded dual vector space of A carries a natural graded coalgebra structure.

   (c) Check that the coalgebra C of Exercise 2 is the graded dual vector space of the polynomial algebra $k[t]$.

   (d) (*Shuffle bialgebra*) Let V be a finite-dimensional vector space. Show that the tensor coalgebra $T'(V)$ of Exercise 3 is the graded dual of the tensor algebra $T(V)$. Deduce that $T'(V)$ has a bialgebra structure whose multiplication is given by

   $$(v_1 \otimes \cdots \otimes v_p)(v_{p+1} \otimes \cdots \otimes v_{p+q}) = \sum_{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p+q)}$$

   where $v_1, \ldots, v_{p+q}$ are elements of V and where σ runs over all (p, q)-shuffles of the symmetric group S_{p+q}.

   (e) Under the same hypotheses as before, show that $S'(V)$ and $\Lambda'(V)$ are subbialgebras of $T'(V)$ whose graded duals are the bialgebras $S(V)$ and $\Lambda(V)$ respectively.

6. (*Convolution algebra*) Let G be a finite group. Equip the vector space $C(G)$ of complex-valued functions on G with the convolution product

   $$(ff')(x) = \sum_{y\in G} f(y)f'(y^{-1}x)$$

   where $x \in G$ and $f, f' \in C(G)$. Show that $C(G)$ has a Hopf algebra structure such that the linear map $f \mapsto \sum_{x\in G} f(x)x$ is a Hopf algebra isomorphism from $C(G)$ to the group Hopf algebra $\mathbf{C}[G]$. Determine the unit, the comultiplication, the counit and the antipode of $C(G)$.

7. (*An example of a non-commutative, non-cocommutative Hopf algebra*) Let H be the quotient of the free algebra $k\{t, x\}$ by the two-sided ideal generated by $t^2 - 1, x^2, xt + tx$. Prove that H is a four-dimensional vector space and that

   $$\Delta(t) = t \otimes t, \quad \Delta(x) = 1 \otimes x + x \otimes t,$$

   $$\varepsilon(t) = 1, \quad \epsilon(x) = 0, \quad S(t) = t, \quad S(x) = tx$$

   endow H with a Hopf algebra structure whose antipode is of order 4.

8. (*Convolution and composition*) Consider a morphism of algebras $f : A \to A'$ and a morphism of coalgebras $g : C' \to C$. Prove that the map $h \mapsto f \circ h \circ g$ from $\mathrm{Hom}(C, A)$ to $\mathrm{Hom}(C', A')$ is a morphism of algebras for the convolution $*$.

9. Use the previous exercise to show that a morphism of bialgebras between two Hopf algebras is necessarily a morphism of Hopf algebras (Hint: prove $S \circ f = f \circ S$ by applying left and right convolution with $f = f \circ \mathrm{id} = \mathrm{id} \circ f$).

10. Let $H = (H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra.

    (a) Set $\psi^0 = \eta\varepsilon$, and $\psi^n = \mathrm{id}_H^{\star n}$ (convolution of n morphisms all equal to the identity of H) if $n > 0$ and $\psi^n = S^{\star n}$ if $n < 0$. Prove that each map ψ^n is an endomorphism of algebras [resp. of coalgebras] when H is commutative [resp. cocommutative] and that, in both cases, we have $\psi^n \circ \psi^m = \psi^{n+m}$ for any pair (n, m) of integers.

    (b) Let $H = k[G]$ be a group. Show that ψ^n is the coalgebra endomorphism given by $\psi^n(g) = g^n$ $(g \in G)$.

    (c) Let $H = S(V)$ be a symmetric algebra. Then $\psi^n(x) = n^d x$ for any $x \in S^d(V)$.

    (d) Show that if $H = SL(2)$, then the algebra endomorphism ψ^n is determined by the matrix identity

    $$\begin{pmatrix} \psi^n(a) & \psi^n(b) \\ \psi^n(c) & \psi^n(c) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \qquad \text{if } n > 0$$

    and by

    $$\begin{pmatrix} \psi^n(a) & \psi^n(b) \\ \psi^n(c) & \psi^n(c) \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^n \qquad \text{if } n < 0.$$

11. Let H be a Hopf algebra, A be a *commutative* algebra and C be a *cocommutative* coalgebra. Prove that the set $\mathrm{Hom}_{Alg}(H, A)$ of algebra morphisms (resp. the set $\mathrm{Hom}_{\mathrm{Cog}}(C, H)$ of coalgebra morphisms) is a group for the convolution, the inverse of a morphism f being given by $f \circ S$ [resp. by $S \circ f$].

12. Let A be a commutative algebra.

    (a) Let V be a finite-dimensional vector space. Consider the symmetric algebra $S(V)$ with the Hopf algebra structure described in Section 3, Example 4. Prove that the group $\mathrm{Hom}_{Alg}(S(V), A)$ is isomorphic to the additive group underlying $A^{\dim(V)}$.

(b) Show that $\mathrm{Hom}_{Alg}(k[\mathbf{Z}], A)$ is isomorphic to the group of invertible elements of A where $k[\mathbf{Z}]$ is the Hopf algebra of the group of integers.

(c) Let C be the Hopf algebra of Exercise 2. Determine the group $\mathrm{Hom}_{Alg}(C, A)$.

13. Let (C, Δ, ε) be a coalgebra and $(C \otimes V, \Delta \otimes \mathrm{id}_V)$ be a free comodule (see Section 6, Example 6). Prove that for any comodule N the map $f \mapsto (\varepsilon \otimes \mathrm{id}_V) \circ f$ is a linear isomorphism from the space of comodule maps from N to $C \otimes V$ to the space $\mathrm{Hom}(N, V)$.

14. Let C be a coalgebra and (N, Δ_N) be a C-comodule. Prove that Δ_N is an injective morphism of comodules from N to the free comodule $(C \otimes N, \Delta \otimes \mathrm{id}_N)$.

15. Let $\{x_i\}_{i\in I}$ be a basis of a C-comodule (N, Δ_N). Define elements c_i^j of the coalgebra (C, Δ, ε) by $\Delta_N(x_i) = \sum_{j\in I} c_i^j x_j$ for all $i \in I$.

(a) Prove that $\Delta(c_i^j) = \sum_{k\in I} c_i^k \otimes c_k^j$ and $\varepsilon(c_i^j) = \delta_{ij}$ for all $i, j \in I$.

(b) Show that the subspace C_N of C linearly generated by the elements $(c_i^j)_{i,j\in I}$ is the smallest subspace C' of C such that $\Delta_N(N) \subset C' \otimes N$. Check that C_N is a coalgebra.

(c) Assume that N is finite-dimensional. Prove that the element $t_N = \sum_{i\in I} c_i^i$ of C_N is independent of the basis $\{x_i\}_{i\in I}$.

16. Prove the structure theorem for bimodules over a Hopf algebra as stated in Section 9.

III.9 Notes

The concept of a Hopf algebra was developed by algebraic topologists abstracting the work of Hopf [Hop41] on manifolds admitting a product (such as Lie groups). A basic reference is the famous article [MM65] by Milnor and Moore. Hopf algebras also came up in the representation theory of Lie groups and algebraic groups (see [Abe80] [DG70] [Hoc81] [Ser93]). For abstract Hopf algebras, we refer to Abe's and Sweedler's monographs [Abe80] [Swe69].

All examples of bialgebras given in this chapter turn out to be either commutative or cocommutative, except for the Hopf algebra of Exercise 7 which is due to Sweedler. Not many examples of non-commutative, non-cocommutative bialgebras were known before the "quantum group" era (nevertheless, see [Par81], [Rad76], [Swe69], pages 89–90, [Taf71], [TW80]). This has dramatically changed in the 1980's with the appearance of quantum groups. For details on the order of the square of the antipode of a Hopf algebra, see [Rad76][Taf71][TW80].

(*Restricted dual*) We saw in Section 1 or in Exercise 5 how to put a coalgebra structure on the dual of an algebra $A = (A, \mu, \eta)$ which is either finite-dimensional or graded. In the general case one can proceed as follows. We know that the map $\lambda : A^* \otimes A^* \to (A \otimes A)^*$ of Corollary II.2.2 allows to identify $A^* \otimes A^*$ with a subspace of $(A \otimes A)^*$. Define

$$A^o = \{\alpha \in A^* \mid \mu^*(\alpha) \in A^* \otimes A^*\}.$$

If the algebra is finite-dimensional, then λ is an isomorphism and $A^o = A^*$. One can show that A^o is the subspace of linear forms whose kernel contains an ideal of finite codimension in A. The vector space A^o enjoys the following property: the embedding λ induces an isomorphism

$$A^o \otimes A^o \cong (A \otimes A)^o.$$

Consequently, $\mu^*(\alpha)$ belongs to $A^o \otimes A^o$ whenever α is in A^o. It results that $(A^o, \mu^*_{|A^o}, \eta^*)$ defines a coalgebra structure on A^o. If, in addition, A has a bialgebra [resp. a Hopf algebra] structure, then so has A^o. For more details, read [Abe80] [Swe69] [Tak85].

(*Restricted dual of a Hopf algebra and representations*) Let H be a Hopf algebra. Its restricted dual H^o also has a Hopf algebra structure. It has the following alternative definition based on representations. Let $\rho : H \to \mathrm{End}(V)$ be a representation of H on a finite-dimensional vector space V. Consider the transpose map $\rho^* : \mathrm{End}(V)^* \to H^*$. Its image $\mathrm{Im}\,(\rho^*)$, called the *coefficient space of the representation* ρ, sits in the restricted dual H^o. Then the restricted dual may also be defined as the sum of the coefficient spaces of all finite-dimensional representations. In the case when all finite-dimensional H-modules are semisimple, H^o is the direct sum of the coalgebras $\mathrm{Im}\,(\rho_i)$ where ρ_i runs over all finite-dimensional simple H-modules up to isomorphism. (see [Abe80] [Ser93] [Swe69]).

(*Bimodules*) Let H be a bialgebra. Let M be a vector space equipped with an H-module and an H-comodule structure given by maps

$$\mu_M : H \otimes M \to M \quad \text{and} \quad \Delta_M : M \to H \otimes M.$$

Give $H \otimes M$ the induced module and comodule structures. Then μ_M is a morphism of comodules if and only if Δ_M is a morphism of modules. If these equivalent conditions are satisfied, we say that M is an H-bimodule.

Given such a bimodule M, define the subspace

$$N = \{m \in M \mid \Delta_M(m) = 1 \otimes m\}.$$

It turns out that N is a subcomodule, but not a submodule of M. Put the induced comodule structure on the free H-module $H \otimes M$. Then $H \otimes M$ becomes a bimodule. The structure theorem for bimodules can be stated as follows: if H is a Hopf algebra, then the map $x \otimes m \mapsto xm$ from $H \otimes N$ to M is an isomorphism of H-bimodules. For details, see [Abe80] [Swe69].

Chapter IV
The Quantum Plane and Its Symmetries

In Chapter I we defined the affine plane as the algebra freely generated by two variables x and y subject to the trivial commutation relation $yx = xy$. This corresponds to our classical perception of plane geometry. In this chapter, we consider a modified commutation relation depending on a parameter q, namely

$$yx = q\, xy.$$

This new relation defines the quantum plane. In Section 2 we derive a few identities well-known to combinatorialists and to the experts in the theory of linear q-difference equations. Next, investigating the self-transformations of the quantum plane, we build a bialgebra $M_q(2)$ and Hopf algebras $GL_q(2)$ and $SL_q(2)$, which are one-parameter deformations of the bialgebras $M(2)$, $GL(2)$, and $SL(2)$ defined in Chapter I. The bialgebras obtained in this way are our first examples of quantum groups. They have the peculiarity of being neither commutative nor cocommutative.

IV.1 The Quantum Plane

Let q be an invertible element of the ground field k, and let I_q be the two-sided ideal of the free algebra $k\{x, y\}$ generated by the element $yx - qxy$. We define the *quantum plane* as the quotient-algebra

$$k_q[x, y] = k\{x, y\}/I_q. \tag{1.1}$$

When $q \neq 1$, the algebra $k_q[x,y]$ is non-commutative. If we give the free algebra $k\{x,y\}$ its natural grading, then the ideal I_q is generated by a homogeneous degree-two element. It follows that the quantum plane has a grading such that the generators x and y are of degree 1. We denote by $k_q[x,y]_n$ the vector subspace of all degree-n elements of $k_q[x,y]$.

Proposition IV.1.1. (a) *If α is the automorphism of the polynomial ring $k[x]$ determined by $\alpha(x) = qx$, then the algebra $k_q[x,y]$ is isomorphic to the Ore extension $k[x][y,\alpha,0]$. Thus, $k_q[x,y]$ is Noetherian, has no zero divisors, and the set of monomials $\{x^i y^j\}_{i,j\geq 0}$ is a basis of the underlying vector space.*

(b) *For any pair (i,j) of nonnegative integers, we have*

$$y^j x^i = q^{ij}\, x^i y^j. \tag{1.2}$$

(c) *Given any k-algebra R, there is a natural bijection*

$$\mathrm{Hom}_{Alg}(k_q[x,y],R) \cong \{(X,Y) \in R\times R \mid YX = qXY\}. \tag{1.3}$$

A pair (X,Y) of elements of R subject to the relation $YX = qXY$ will be called an *R-point of the quantum plane.*

PROOF. (a) We use the theory of Ore extensions as presented in I.7–8. Define an algebra morphism $\varphi : k\{x,y\} \to k[x][y,\alpha,0]$ by $\varphi(x) = x$ and $\varphi(y) = y$. Since

$$\varphi(y)\varphi(x) - q\varphi(x)\varphi(y) = yx - qxy = \alpha(x)y - qxy = 0,$$

the morphism φ vanishes on the ideal I_q, thus defining a morphism of algebras, still denoted φ, on $k_q[x,y]$. The morphism φ is surjective because the Ore extension $k[x][y,\alpha,0]$ is generated by x and y. In order to show that φ is an isomorphism, we only have to construct a linear map ψ from $k[x][y,\alpha,0]$ to $k_q[x,y]$ such that $\psi\circ\varphi = \mathrm{id}$. We define ψ on the basis $\{x^i y^j\}_{i,j\geq 0}$ of $k[x][y,\alpha,0]$ by $\psi(x^i y^j) = x^i y^j$. The rest of the proof of (i) follows from I.7 and I.8.

Part (b) is proved by an easy induction. Part (c) is a consequence of the universal property (I.2.4) and of the definition (1.1). □

We give an example of an R-point of the quantum plane.

Example 1. Let A be the algebra of smooth complex functions on $\mathbf{C}\setminus\{0\}$ and let q be a complex number different from 0 and from 1. Consider the linear endomorphisms τ_q and δ_q in $R = \mathrm{End}(A)$ defined by

$$\tau_q(f)(x) = f(qx) \quad \text{and} \quad \delta_q(f)(x) = \frac{f(qx)-f(x)}{(qx-x)}.$$

The pair $\{\tau_q,\delta_q\}$ is an R-point of $k_q[x,y]$. The "limit" of the operator δ_q when q tends to 1 is the usual derivative d/dx.

IV.2 Gauss Polynomials and the q-Binomial Formula

We fix an invertible element q of the field k. For future applications, we need to compute the powers of $x+y$ in the quantum plane $k_q[x,y]$. To this end, we introduce the so-called Gauss polynomials which are polynomials in one variable q whose values at $q=1$ are equal to the classical binomial coefficients.

Let us start with some notation. For any integer $n>0$, set

$$(n)_q = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q-1}. \tag{2.1}$$

Define the q-factorial of n by $(0)!_q = 1$ and

$$(n)!_q = (1)_q (2)_q \dots (n)_q = \frac{(q-1)(q^2-1)\dots(q^n-1)}{(q-1)^n} \tag{2.2}$$

when $n>0$. The q-factorial of n is a polynomial in q with integral coefficients and with value at $q=1$ equal to the usual factorial $n!$. We define the *Gauss polynomials* for $0 \leq k \leq n$ by

$$\begin{pmatrix} n \\ k \end{pmatrix}_q = \frac{(n)!_q}{(k)!_q\,(n-k)!_q}. \tag{2.3}$$

Proposition IV.2.1. *Let $0 \leq k \leq n$.*

(a) $\begin{pmatrix} n \\ k \end{pmatrix}_q$ *is a polynomial in q with integral coefficients and with value at $q=1$ equal to the binomial coefficient* $\begin{pmatrix} n \\ k \end{pmatrix}$.

(b) *We have*

$$\begin{pmatrix} n \\ k \end{pmatrix}_q = \begin{pmatrix} n \\ n-k \end{pmatrix}_q. \tag{2.4}$$

(c) (q*-Pascal identity*) *We also have*

$$\begin{pmatrix} n \\ k \end{pmatrix}_q = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q + q^k \begin{pmatrix} n-1 \\ k \end{pmatrix}_q = \begin{pmatrix} n-1 \\ k \end{pmatrix}_q + q^{n-k} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q. \tag{2.5}$$

PROOF. Relations (2.4–2.5) follow from easy computations. For Part (a), one proceeds by induction on n using (2.5). □

We return to the quantum plane of Section 1 and prove the q-binomial formula.

Proposition IV.2.2. *Let x and y be variables subject to the quantum plane relation $yx = qxy$. Then for all $n > 0$ we have*

$$(x+y)^n = \sum_{0\le k\le n} \binom{n}{k}_q x^k y^{n-k}.$$

PROOF. Because of the universal property of the quantum plane, it suffices to prove the statement in $k_q[x,y]$. Expanding $(x+y)^n$ and using (1.2), we see that the monomials in the expansion are all scalar multiples of monomials of the form $x^k y^{n-k}$. We therefore have

$$(x+y)^n = \sum_{0\le k\le n} \binom{n}{k}' x^k y^{n-k}$$

where $\binom{n}{k}'$ is a polynomial with integral coefficients in q. Let us prove by induction on n that we have

$$\binom{n}{k}' = \binom{n}{k}_q. \tag{2.6}$$

Relation (2.6) clearly holds for $n = 1$. It thus suffices to check that the coefficients $\binom{n}{k}'$ satisfy (2.5). Using (1.2), we have

$$\begin{aligned}
\sum_{0\le k\le n} \binom{n}{k}' x^k y^{n-k} &= (x+y)\left(\sum_{0\le k\le n-1} \binom{n-1}{k}' x^k y^{n-1-k}\right)\\
&= \sum_{0\le k\le n-1} \binom{n-1}{k}' x^{k+1} y^{n-1-k}\\
&\qquad + \sum_{0\le k\le n-1} q^k \binom{n-1}{k}' x^k y^{n-k}\\
&= \sum_{0\le k\le n} \left(\binom{n-1}{k-1}' + q^k \binom{n-1}{k}'\right) x^k y^{n-k}.
\end{aligned}$$

We get (2.5) in view of the linear independence of the monomials $\{x^k y^{n-k}\}_k$. $\square$

We now derive a few q-identities from the q-binomial formula. These identities will not be needed in the sequel. The first one is the q-analogue of the *Chu-Vandermonde formula.*

Proposition IV.2.3. *For $m \ge p \le n$, we have*

$$\binom{m+n}{p}_q = \sum_{0\le k\le p} q^{(m-k)(p-k)} \binom{m}{k}_q \binom{n}{p-k}_q.$$

PROOF. Expand both sides of the identity $(x+y)^{m+n} = (x+y)^m(x+y)^n$ using Proposition 2.2 and identify the terms corresponding to $x^p y^{m+n-p}$. $\square$

We introduce a q-variant of the exponential. Let z be a variable (commuting with q). We define the *q-exponential* as the formal series

$$e_q(z) = \sum_{n\geq 0} \frac{z^n}{(n)!_q}. \tag{2.7}$$

Observe that this series is well-defined provided q is not a root of unity, which we assume until the end of this section.

Proposition IV.2.4. *Let x and y be variables such that $yx = qxy$. Then*

$$e_q(x+y) = e_q(x)e_q(y).$$

PROOF. By application of Proposition 2.2, we have

$$\begin{aligned}\Big(\sum_{k\geq 0}\frac{x^k}{(k)!_q}\Big)\Big(\sum_{\ell\geq 0}\frac{y^\ell}{(\ell)!_q}\Big) &= \sum_{n\geq 0}\frac{1}{(n)!_q}\Big(\sum_{k+\ell=n}\frac{(n)!_q}{(k)!_q(\ell)!_q}x^k y^\ell\Big)\\ &= \sum_{n\geq 0}\frac{(x+y)^n}{(n)!_q}.\end{aligned}$$

$\square$

The q-exponential is an invertible formal series, but, in contrast to the case $q = 1$, we have $e_q(z)^{-1} \neq e_q(-z)$. In order to compute the inverse of $e_q(z)$, we consider the algebra of formal series $k[[z]]$ and the algebra $\mathrm{End}(k[[z]])$ of linear endomorphisms of $k[[z]]$. Define two elements Z and τ_q of $\mathrm{End}(k[[z]])$ by $(Zf)(z) = zf(z)$ and $(\tau_q f)(z) = f(qz)$. An easy computation shows that (Z, τ_q) is an $\mathrm{End}(k[[z]])$-point of the quantum plane, which is to say we have the following lemma.

Lemma IV.2.5. *We have $\tau_q Z = q\, Z\tau_q$ in* $\mathrm{End}(k[[z]])$.

If for any scalar a of the field k we apply the endomorphism $((a-Z)\tau_q)^n$ to the constant formal series 1, we get

$$\Big((a-Z)\tau_q\Big)^n(1) = (a-z)(a-qz)\dots(a-q^{n-1}z). \tag{2.8}$$

In particular, for $a = 0$ we have

$$(-Z\tau_q)^n(1) = (-1)^n\, q^{n(n-1)/2}\, z^n. \tag{2.9}$$

Proposition IV.2.6. *The inverse of $e_q(z)$ is given by*

$$e_q(z)^{-1} = \sum_{n\geq 0}(-1)^n\, q^{n(n-1)/2}\,\frac{z^n}{(n)!_q}.$$

PROOF. Lemma 2.5 implies $(-Z\tau_q)Z = qZ(-Z\tau_q)$. Using Proposition 2.4, we get the following identity in $\mathrm{End}(k[[z]])$:

$$e_q(Z(1-\tau_q)) = e_q(Z) \circ e_q(-Z\tau_q). \tag{2.10}$$

Let us apply both sides of (2.10) to the constant formal series 1. On the one hand, we have $e_q(Z(1-\tau_q))(1) = 1$ because $(1-\tau_q)(1) = 0$. On the other hand, by (2.9) we get

$$e_q(Z)\Big(e_q(-Z\tau_q)(1)\Big) = e_q(z)\Big(\sum_{n\geq 0} (-1)^n\, q^{n(n-1)/2}\, \frac{z^n}{(n)!_q}\Big).$$

□

Here are two more general q-identities.

Proposition IV.2.7. *For any scalar a we have*

$$(a-z)(a-qz)\dots(a-q^{n-1}z) = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix}_q q^{k(k-1)/2}\, a^{n-k} z^k$$

and

$$e_q(a) = e_q(z)\Big(\sum_{n=0}^{\infty} \frac{1}{(n)!_q}\,(a-z)(a-qz)\dots(a-q^{n-1}z)\Big).$$

PROOF. One proceeds as in the proof of Proposition 2.6, but now with the operator identity $(a\tau_q)(-Z\tau_q) = q\,(-Z\tau_q)(a\tau_q)$. By Propositions 2.2 and 2.4, we get

$$\Big((a-Z)\tau_q\Big)^n = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix}_q (Z\tau_q)^k \circ (a\tau_q)^{n-k}$$

and

$$e_q((a-Z)\tau_q) = e_q(-Z\tau_q) \circ e_q(a)$$

in $\mathrm{End}(k[[z]])$. Applying again these identities to the constant formal series 1, we get the desired relations in view of $e_q(-Z\tau_q)(1) = e_q(z)^{-1}$, which was proved above. □

IV.3 The Algebra $M_q(2)$

From now on, we assume that $q^2 \neq -1$. Let us define a q-analogue of the algebra $M(2)$ of I.4. In addition to variables x, y subject to the quantum plane relation $yx = qxy$, consider four variables a, b, c, d commuting with x and y. Define x', y', x'', and y'' using the following matrix relations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.1}$$

Theorem IV.3.1. *Under the previous hypotheses, there is an equivalence between*
(i) *the two relations* $y'x' = qx'y'$ *and* $y''x'' = qx''y''$, *and*
(ii) *the six relations*

$$ba = qab, \qquad db = qbd, \tag{3.2}$$

$$ca = qac, \qquad dc = qcd, \tag{3.3}$$

$$bc = cb, \quad ad - da = (q^{-1} - q)\, bc. \tag{3.4}$$

PROOF. Let us check that (i) implies (ii). By (3.1) we have

$$(cx + dy)(ax + by) = q\,(ax + by)(cx + dy).$$

Identifying the coefficients of x^2, y^2, and of xy, we obtain

$$ca = qac, \quad db = qbd, \quad cb + qda = qad + q^2bc. \tag{3.5}$$

Dividing the latter by q yields

$$ad - da = q^{-1}cb - qbc. \tag{3.6}$$

Using x'' and y'' in a similar fashion leads to three more relations obtained from (3.5–3.6) by exchanging b and c, namely

$$ba = qab, \quad dc = qcd, \quad ad - da = q^{-1}bc - qcb. \tag{3.7}$$

From (3.6–3.7) we derive $(q^{-1} + q)(bc - cb) = 0$, which is equivalent to $bc = cb$ since $q^2 \neq -1$. We have proved that (i) implies (ii). The converse implication follows from similar straightforward computations. □

Definition IV.3.2. *The algebra* $M_q(2)$ *is the quotient of the free algebra* $k\{a, b, c, d\}$ *by the two-sided ideal* J_q *generated by the six relations* (3.2–3.4) *of Theorem* 3.1.

When $q = 1$, the algebra $M_q(2)$ is clearly isomorphic to the algebra $M(2)$ of I.4. Since the ideal J_q is generated by quadratic elements, the natural grading of the free algebra induces a grading on $M_q(2)$ such that the generators a, b, c, d are of degree 1.

Given an algebra R, we define an *R-point of $M_q(2)$* to be a quadruple $(A, B, C, D) \in R^4$ satisfying the relations

$$BA = qAB, \qquad DB = qBD, \tag{3.8}$$

$$CA = qAC, \qquad DC = qCD, \tag{3.9}$$

$$BC = CB, \quad AD - DA = (q^{-1} - q)\, BC. \tag{3.10}$$

By the very definition of $M_q(2)$, the set of R-points of $M_q(2)$ is in bijection with the set $\mathrm{Hom}_{Alg}(M_q(2), R)$ of algebra morphisms from $M_q(2)$ to R. It will sometimes be convenient and more enlightening to write an R-point (A, B, C, D) of $M_q(2)$ in the matrix form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3.11}$$

Theorem 3.1 can be paraphrased using the language of R-points as follows: a quadruple $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of elements of an algebra R is an R-point of $M_q(2)$ if and only if the following pairs (X', Y') and (X'', Y'') are R'-points of the quantum plane, where X', Y', X'', Y'' are matricially defined by

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X'' \\ Y'' \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

and where R' is the tensor product algebra

$$R' = R \otimes k_q[X, Y] = R\{X, Y\}/(YX - qXY).$$

We now introduce the quantum determinant $\det_q$ as the following element of the algebra $M_q(2)$.

Proposition IV.3.3. *The element* $\det_q = ad - q^{-1}bc = da - qbc$ *of* $M_q(2)$ *is central.*

PROOF. It suffices to show that $\det_q$ commutes with the generators a, b, c, d. Now, by (3.2–3.4) we have

$$(ad - q^{-1}bc)a = a(da - qbc), \quad (ad - q^{-1}bc)b = b(ad - q^{-1}bc),$$

$$(ad - q^{-1}bc)c = c(ad - q^{-1}bc), \quad (da - qbc)d = d(ad - q^{-1}bc). \qquad \square$$

Given an R-point $m = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $M_q(2)$, the element

$$\mathrm{Det}_q(m) = AD - q^{-1}BC = DA - qBC$$

of R is called the *quantum determinant* of m.

Proposition IV.3.4. *Let R be an algebra and*

$$m = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \textit{and} \quad m' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

be two R-points of $M_q(2)$ such that the elements A, B, C, D commute with the elements A', B', C', D'.

(a) *The element* $m'm$ *defined by the matrix product*

$$m'm = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an R*-point of* $M_q(2)$.

(b) *We have* $\mathrm{Det}_q(m'm) = \mathrm{Det}_q(m')\,\mathrm{Det}_q(m)$ *in* R.

(c) *The quadruple*

$$\begin{pmatrix} D & -qB \\ -q^{-1}C & A \end{pmatrix}$$

is an R*-point of* $M_{q^{-1}}(2)$ *and an* R^{op}*-point of* $M_q(2)$.

PROOF. (a) We use the reformulation of Theorem 3.1 stated a few lines ahead of Proposition 3.3. Let R' be the tensor product algebra

$$R' = R \otimes k_q[X,Y] = R\{X,Y\}/(YX - qXY).$$

Define X', Y', X'', Y'' by

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X'' \\ Y'' \end{pmatrix} = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

By definition, the elements X, Y of R' commute with the other variables A, A', etc. It results from Theorem 3.1 that the pairs (X', Y') and (X'', Y'') are R'-points of the quantum plane. Now, by hypothesis, the elements A', B', C', D' of R' commute with X' and Y' and the elements A, B, C, D commute with X'' and Y''. By a second application of Theorem 3.1,

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

and

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} X'' \\ Y'' \end{pmatrix} = \begin{pmatrix} A'' & C'' \\ B'' & D'' \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

are R'-points of the quantum plane. It follows that $m'm$ is an R-point of $M_q(2)$.

(b) This follows from computations we leave to the reader. A more conceptual method is suggested as an exercise at the end of this chapter.

(c) Define $A' = D$, $B' = -qB$, $C' = -q^{-1}C$, and $D' = A$. Then Relations (3.8–3.10) imply

$$A'B' = qB'A', \qquad B'D' = qD'B',$$

$$A'C' = qC'A', \qquad C'D' = qD'C',$$

$$C'B' = B'C', \quad D'A' - A'D' = (q^{-1} - q)\,B'C',$$

which means precisely that (A', B', C', D') is an R-point of $M_{q^{-1}}(2)$ or an R^{op}-point of $M_q(2)$. □

IV.4 Ring-Theoretical Properties of $M_q(2)$

The aim of this section is to show that the algebra $M_q(2)$, though non-commutative, retains certain properties of the commutative algebra $M(2)$. We freely use the notations and the results of I.7–8.

Theorem IV.4.1. *The algebra $M_q(2)$ is Noetherian and has no zero divisors. A basis for the underlying vector space is given by the set of monomials $\{a^i b^j c^k d^\ell\}_{i,j,k,\ell \geq 0}$.*

We shall prove this theorem by building a tower

$$A_0 = k \subset A_1 \subset A_2 \subset A_3 \subset A_4 = M_q(2)$$

of algebras such that each A_i is an Ore extension of A_{i-1}. As a consequence of Corollary I.7.2, we conclude that the set $\{\sigma(a)^i \sigma(b)^j \sigma(c)^k \sigma(d)^\ell\}_{i,j,k,\ell \geq 0}$ is also a basis of $M_q(2)$ for any permutation σ of the set $\{a, b, c, d\}$. Define the algebras $A_1 = k[a]$, $A_2 = k\{a, b\}/(ba - qab)$, and

$$A_3 = k\{a, b, c\}/(ba - qab, ca - qac, cb - bc).$$

The algebra A_1 is trivially an Ore extension of A_0. Let α_1 be the automorphism of A_1 determined by $\alpha_1(a) = qa$.

Lemma IV.4.2. *There is an isomorphism between A_2 and the Ore extension $A_1[b, \alpha_1, 0]$. Furthermore, the set $\{a^i b^j\}_{i,j \geq 0}$ is a basis of A_2.*

Observe that the algebra A_2 is isomorphic to the quantum plane $k_q[x, y]$ (the isomorphism sends a onto x and b onto y).

PROOF. Let us define $\varphi_1 : A_2 \to A_1[b, \alpha_1, 0]$ by $\varphi_1(a) = a$ and $\varphi_1(b) = b$. Since

$$\varphi_1(b)\varphi_1(a) - q\varphi_1(a)\varphi_1(b) = ba - qab = \alpha_1(a)b - qab = 0,$$

φ_1 defines a morphism of algebras. This morphism is surjective since the algebra $A_1[b, \alpha_1, 0]$ is generated by a and b. In order to show that it is an isomorphism, we only have to build a linear map $\psi_1 : A_1[b, \alpha_1, 0] \to A_2$ such that $\psi_1 \circ \varphi_1 = \mathrm{id}$. We define ψ_1 on the basis $\{a^i b^j\}_{i,j \geq 0}$ of $A_1[b, \alpha_1, 0]$ by $\psi_1(a^i b^j) = a^i b^j$. □

It is easy to check that $\alpha_2(a) = qa$ and $\alpha_2(b) = b$ define an automorphism α_2 of the algebra A_2. We have the following result whose proof follows the same lines as the proof of Lemma 4.2.

Lemma IV.4.3. *The algebra A_3 is isomorphic to the algebra $A_2[c, \alpha_2, 0]$; the set $\{a^i b^j c^k\}_{i,j,k \geq 0}$ is a basis of A_3.*

The last step consists in building A_4 out of A_3. This is the only step involving a non-zero derivation. First, one checks that

$$\alpha_3(a) = a, \quad \alpha_3(b) = qb, \quad \alpha_3(c) = qc$$

define an algebra automorphism of A_3. We define another endomorphism δ of A_3 on the basis $\{a^i b^j c^k\}_{i,j,k\geq 0}$ by $\delta(b^j c^k) = 0$ and by

$$\delta(a^i b^j c^k) = (q - q^{-1}) \frac{1 - q^{2i}}{1 - q^2} \, a^{i-1} b^{j+1} c^{k+1} \tag{4.1}$$

if $i > 0$. The proof of the following result is left to the reader.

Lemma IV.4.4. *The endomorphism δ is an α_3-derivation of A_3.*

We use this result to prove the next one.

Lemma IV.4.5. *The algebra $A_4 = M_q(2)$ is isomorphic to the Ore extension $A_3[d, \alpha_3, \delta]$, and $\{a^i b^j c^k d^\ell\}_{i,j,k,\ell\geq 0}$ is a basis of A_4.*

PROOF. Set $\varphi_4(a) = a$, $\varphi_4(b) = b$, $\varphi_4(c) = c$, $\varphi_4(d) = d$. This defines a surjective morphism of algebras φ_4 from A_4 onto the Ore extension $A_3[d, \alpha_3, \delta]$, provided we check that $(\varphi_4(a), \varphi_4(b), \varphi_4(c), \varphi_4(d))$ is an $A_3[d, \alpha_3, \delta]$-point of $M_q(2)$. This implies checking the six relations (3.8–3.10). Now the three relations not involving d already hold in A_3. As for the three remaining, namely

$$db = qbd, \quad dc = qcd, \quad da = ad + (q - q^{-1})bc,$$

they hold in $A_3[d, \alpha_3, \delta]$ by the very definition of α_3 and of δ. To complete the proof, one constructs a linear map ψ_4 such $\psi_4 \circ \varphi_4 = \mathrm{id}$ as in the proof of Lemma 4.2. □

Theorem 4.1 is now a consequence of Lemmas 4.2, 4.3 and 4.5, of Corollary I.7.2, and of Theorem I.8.3.

IV.5 Bialgebra Structure on $M_q(2)$

We now endow the algebra $M_q(2)$ with a bialgebra structure. The comultiplication and the counit will be the same as the comultiplication and the counit put on $M(2)$ in I.4 (see also III.4).

Theorem IV.5.1. *There exist morphisms of algebras*

$$\Delta : M_q(2) \to M_q(2) \otimes M_q(2) \quad \textit{and} \quad \varepsilon : M_q(2) \to k$$

uniquely determined by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \tag{5.1}$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d, \tag{5.2}$$

$$\varepsilon(a) = \varepsilon(d) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0. \tag{5.3}$$

Equipped with these morphisms, the algebra $M_q(2)$ becomes a bialgebra that is neither commutative nor cocommutative. Furthermore, we have

$$\Delta(\det_q) = \det_q \otimes \det_q \quad \textit{and} \quad \varepsilon(\det_q) = 1. \tag{5.4}$$

We may rewrite Relations (5.1–5.3) in the abridged matrix form

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.5}$$

PROOF. In order to show that Δ is a morphism of algebras, it suffices to check that $(\Delta(a), \Delta(b), \Delta(c), \Delta(d))$ is an $M_q(2) \otimes M_q(2)$-point of $M_q(2)$. This follows from Proposition 3.4 (a). A simple computation shows that $(\varepsilon(a), \varepsilon(b), \varepsilon(c), \varepsilon(d))$ is a k-point of $M_q(2)$, thus proving that ε also defines an algebra morphism.

We now have to check the coassociativity and counit axioms. Let us start with

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta. \tag{5.6}$$

Since both sides of (5.6) are morphisms of algebras, it is enough to verify it on the generators a, b, c, d. Using the matrix form, we have

$$\begin{aligned}
\Big((\Delta \otimes \mathrm{id})\Delta\Big)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\
&= \Big((\mathrm{id} \otimes \Delta)\Delta\Big)\begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\end{aligned}$$

A similar argument shows that the counit axiom follows from the matrix identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

As for the computation of $\Delta(\det_q)$, it results from Proposition 3.4 (b). □

IV.6 The Hopf Algebras $GL_q(2)$ and $SL_q(2)$

We proceed by analogy with I.5. Using the quantum determinant $\det_q$ of Proposition 3.3, we define the algebras

$$GL_q(2) = M_q(2)[t]/(t\det_q - 1)$$

and
$$SL_q(2) = M_q(2)/(\det_q - 1) = GL_q(2)/(t-1).$$
Given an algebra R, we define an *R-point of* $GL_q(2)$ [resp. *of* $SL_q(2)$] as an R-point $m = (A, B, C, D)$ of $M_q(2)$ whose quantum determinant

$$\mathrm{Det}_q(m) = AD - q^{-1}BC$$

is invertible in R [resp. is equal to 1]. Denoting $GL_q(2)$ and $SL_q(2)$ by G_q, we see that the set of R-points of G_q is in bijection with the set $\mathrm{Hom}_{Alg}(G_q, R)$ of algebra morphisms from G_q to R.

Theorem IV.6.1. *Relations* (5.1–5.3) *defining the comultiplication* Δ *and the counit* ε *of* $M_q(2)$ *equip the algebras* $GL_q(2)$ *and* $SL_q(2)$ *with Hopf algebra structures such that the antipode* S *is given in matrix form by*

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = {\det_q}^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}. \tag{6.1}$$

PROOF. (a) We first have to show that Δ and ε are well-defined on $GL_q(2)$ and on $SL_q(2)$. For $SL_q(2)$ this results from the following computations: by (5.4)
$$\Delta(\det_q - 1) = (\det_q - 1) \otimes \det_q + 1 \otimes (\det_q - 1)$$
and $\varepsilon(\det_q - 1) = 0$. A similar argument works for $GL_q(2)$ provided we set

$$\Delta(t) = t \otimes t \quad \text{and} \quad \varepsilon(t) = 1. \tag{6.2}$$

The coassociativity and counit axioms hold for $GL_q(2)$ and for $SL_q(2)$ since they already hold for $M_q(2)$.

(b) It remains to check that $GL_q(2)$ and $SL_q(2)$ have antipodes. Set

$$S'(a) = d, \quad S'(b) = -qb, \quad S'(c) = -q^{-1}c, \quad S'(d) = a. \tag{6.3}$$

By Proposition 3.4 (c), the quadruple $(S'(a), S'(b), S'(c), S'(d))$ is a $M_q(2)^{\mathrm{op}}$-point of $M_q(2)$. Consequently, S' defines a morphism of algebras from $M_q(2)$ to $M_q(2)^{\mathrm{op}}$. Next, we extend S' to $GL_q(2)$ and to $SL_q(2)$ by setting $S'(t) = t$. This is a well-defined algebra morphism because

$$S'(t)S'(\det_q) = \Big(S'(d)S'(a) - q^{-1}S'(c)S'(b)\Big)S'(t) = (ad - q^{-1}bc)\; t = 1.$$

Since the quantum determinant is invertible and central in $G_q = GL_q(2)$ and $SL_q(2)$, it is possible to define an algebra morphism S from G_q to G_q^{op} by $S(t) = t^{-1}$ and

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = {\det_q}^{-1}\begin{pmatrix} S'(a) & S'(b) \\ S'(c) & S'(d) \end{pmatrix} = {\det_q}^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

Finally, to check that S is an antipode, it suffices to work with the generators a, b, c, d, according to Lemma III.3.6. Relations (III.3.3) are equivalent to the matrix identities

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$= \det_q \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

In contrast to the inversion in a group and to the antipode of $GL(2)$ and of $SL(2)$, the antipode S of $GL_q(2)$ and of $SL_q(2)$ is in general not involutive. Indeed, from (6.1) we derive

$$\begin{pmatrix} S^{2n}(a) & S^{2n}(b) \\ S^{2n}(c) & S^{2n}(d) \end{pmatrix} = \begin{pmatrix} a & q^{2n}b \\ q^{-2n}c & d \end{pmatrix}$$
$$= \begin{pmatrix} q^n & 0 \\ 0 & q^{-n} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q^{-n} & 0 \\ 0 & q^n \end{pmatrix}$$

for any positive integer n. Fix such an n and let q be a root of unity of order exactly n. Then we obtain two examples of Hopf algebras for which the square of the antipode has order n. For results on the order of S^2 previous to the quantum group era, see [Rad76] [Taf71] [TW80].

IV.7 Coaction on the Quantum Plane

We saw in III.7 that the affine plane $k[x, y]$ was a comodule-algebra over either one of the bialgebras $M(2)$ and $SL(2)$. We now develop a quantum version of this.

Theorem IV.7.1. *There exists a unique $M_q(2)$-comodule-algebra structure and a unique $SL_q(2)$-comodule-algebra structure on the quantum plane $A = k_q[x, y]$ such that*

$$\Delta_A(x) = a \otimes x + b \otimes y \quad \textit{and} \quad \Delta_A(y) = c \otimes x + d \otimes y.$$

We rewrite these formulas in the matrix form

$$\Delta_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7.1}$$

PROOF. We use Proposition III.7.2. We first check that (7.1) defines an algebra morphism Δ_A from A to $M_q(2) \otimes A$. It is enough to verify that

$$\Delta_A(y)\Delta_A(x) = q\,\Delta_A(x)\Delta_A(y)$$

in $M_q(2) \otimes A$. Now, by (3.2–3.4), we have

$$\begin{aligned}
\Delta_A(y)\Delta_A(x) &= (c\otimes x + d\otimes y)(a\otimes x + b\otimes y)\\
&= qac\otimes x^2 + (bc + qda)\otimes xy + qbd\otimes y^2\\
&= q\left(ac\otimes x^2 + (q^{-1}bc + ad)\otimes xy + bd\otimes y^2\right)\\
&= q\,(a\otimes x + b\otimes y)(c\otimes x + d\otimes y)\\
&= q\,\Delta_A(x)\Delta_A(y).
\end{aligned}$$

Since the projection of $M_q(2)$ onto $SL_q(2)$ is a morphism of algebras, the resulting map $A \to SL_q(2)\otimes A$ is an algebra morphism too.

It remains to check that Δ_A defines a comodule structure on the quantum plane. This is done as in the proof of Theorem III.7.3. $\square$

We record the following quantum version of Lemma III.7.4.

Lemma IV.7.2. *For $i, j \geq 0$ we have*

$$\Delta_A(x^iy^j) = \sum_{r=0}^{i}\sum_{s=0}^{j} q^{(i-r)s}\binom{i}{r}_{q^2}\binom{j}{s}_{q^2} a^rb^{i-r}c^sd^{j-s}\otimes x^{r+s}y^{i+j-r-s}.$$

PROOF. We first observe that $\Delta_A(x^iy^j) = \Delta_A(x)^i\Delta_A(y)^j$ since Δ_A is an algebra morphism. Next, we have

$$(b\otimes y)(a\otimes x) = q^2\,(a\otimes x)(b\otimes y) \quad\text{and}\quad (d\otimes y)(c\otimes a) = q^2\,(c\otimes a)(d\otimes y)$$

in the algebra $M_q(2)\otimes A$. This allows us to apply Proposition 2.2 to both

$$\Delta_A(x)^i = (a\otimes x + b\otimes y)^i \quad\text{and}\quad \Delta_A(y)^j = (c\otimes a + d\otimes y)^j. \qquad \square$$

Denote by $k_q[x,y]_n$ the subspace of degree n elements of $A = k_q[x,y]$. As a consequence of Lemma 7.2, we see that $k_q[x,y]_n$ is a subcomodule of the quantum plane. Actually, the quantum plane is the direct sum of the comodules $k_q[x,y]_n$. By III.6, Example 2, the dual vector space $k_q[x,y]_n^*$ is a module over the algebra $SL_q(2)^*$ dual to the coalgebra $SL_q(2)$. We shall identify this module in VII.5.

IV.8 Hopf $*$-Algebras

The standing assumption in this section is that the ground field k is the field of complex numbers. Given a complex number z, we denote its complex conjugate by $\overline{z}$. Recall that an $\mathbf{R}$-linear map $u : V \to V'$ between complex vector spaces is said to be *antilinear* if $u(\lambda v) = \overline{\lambda}v$ for all $\lambda \in \mathbf{C}$ and $v \in V$.

Definition IV.8.1. *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a complex Hopf algebra. We say that H is a Hopf $*$-algebra if there exists an antilinear involution * on H satisfying the two conditions*

(i) *the map * is an antimorphism of real algebras, i.e., an algebra morphism from H into H^{op}, as well as a morphism of real coalgebras, and*

(ii) *we have $S(S(x)^*)^* = x$ for all $x \in H$.*

Two Hopf $*$-algebra structures $*_1$ and $*_2$ on H are equivalent if there exists a Hopf algebra automorphism φ of H such that $\varphi(x^{*_1}) = \varphi(x)^{*_2}$ for all x in H.

We wish to show that the Hopf algebras $GL_q(2)$ and $SL_q(2)$ have natural Hopf $*$-algebra structures given by matrix transposition. We shall need the following equivalent formulation.

Lemma IV.8.2. *A Hopf algebra H has a Hopf $*$-algebra structure if and only if there exists an antilinear automorphism γ of H such that*

(i) *the map γ is a morphism of real algebras and an antimorphism of real coalgebras, and*

(ii) *we have $\gamma^2 = (S\gamma)^2 = \mathrm{id}_H$.*

PROOF. Suppose we have an involution $*$ as in Definition 8.1. Define γ by $\gamma(x) = S^{-1}(x^*)$ for all $x \in H$. It is clear that γ is an antilinear algebra automorphism. It is an antimorphism of coalgebras because so are the antipode S and its inverse by Theorem III.3.4 (a). We have $S\gamma = *$, which shows that $S\gamma$ is an involution. Finally, γ is an involution too, as can been seen from

$$\gamma^2 = (S^{-1}*)^2 = ((*S)^2)^{-1} = \mathrm{id}_H^{-1} = \mathrm{id}_H.$$

The second equality follows from $*$ being an involution while the third one follows from Definition 8.1 (ii). Conversely, define $* = S\gamma$ from an automorphism γ as in Lemma 8.2. It is an involution by Lemma 8.2 (ii). Let us check Condition (ii) of Definition 8.1. We have

$$(*S)^2 = (S\gamma S)^2 = (S\gamma)^2\gamma^{-1}(S\gamma)^2\gamma^{-1} = \gamma^{-2} = \mathrm{id}_H. \qquad \square$$

We now present the main result of this section. We freely use the notation of the previous sections. Recall the inverse t of the element $\det_q = ad - q^{-1}bc$ of $GL_q(2)$. In $SL_q(2)$ we have $t = 1$.

Theorem IV.8.3. *There exist unique Hopf $*$-algebra structures on the Hopf algebras $GL_q(2)$ and $SL_q(2)$ such that*

$$a^* = td, \quad b^* = -q\, tc, \quad c^* = -q^{-1}tb, \quad d^* = ta, \quad t^* = t^{-1}.$$

PROOF. By Theorem 3.1, the transpose $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an $M_q(2)$-point of $M_q(2)$. Consequently, there exists a unique antilinear algebra endomorphism γ of $M_q(2)$ defined by the matrix identity

$$\begin{pmatrix} \gamma(a) & \gamma(b) \\ \gamma(c) & \gamma(d) \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \tag{8.1}$$

Since transposition is involutive, so is γ. The map γ is an antimorphism of coalgebras in view of the formula (5.5) giving the comultiplication on $M_q(2)$ and of the fact that matrix transposition reverses products.

We now extend γ to $GL_q(2)$ by $\gamma(t) = t$. Since $\gamma(t\det_q - 1) = t\det_q - 1$, it defines an antilinear algebra automorphism both on $GL_q(2)$ and on $SL_q(2)$. Let us check that $S\gamma$ is an involution. It is enough to verify this on the generators a, b, c, d, and t. For t, this is clear. For the remaining generators, we have

$$(S\gamma)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = t\begin{pmatrix} d & -qc \\ -q^{-1}b & a \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
(S\gamma)^2\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (S\gamma)\begin{pmatrix} d & -qc \\ -q^{-1}b & a \end{pmatrix}(S\gamma)(t) \\
&= t\begin{pmatrix} a & b \\ c & d \end{pmatrix}t^{-1} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\end{aligned}$$

We conclude the proof by recalling that $* = S\gamma$. □

IV.9 Exercises

1. (*Gauss*) Show that

$$\sum_{0\le k\le n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix}_q = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (1-q)(1-q^3)\ldots(1-q^{n-1}) & \text{if } n \text{ is even.} \end{cases}$$

2. (*Gauss*) Show that

$$\begin{pmatrix} n+m+1 \\ m+1 \end{pmatrix}_q = \sum_{0\le k\le n} q^k \begin{pmatrix} m+k \\ m \end{pmatrix}_q.$$

3. Let F be a finite field of order q.

   (a) Show that $\left(\begin{array}{c} n \\ k \end{array}\right)_q$ is equal to the number of k-dimensional subspaces of a n-dimensional vector space over F.

   (b) Prove Relations (2.4–2.5) using the previous assertion.

4. (*q-differentiation*) Consider the linear endomorphisms Z, τ_q, and δ_q of the polynomial algebra $k[z]$ and of the algebra $k[[z]]$ of formal series, defined by

$$(Zf)(z) = zf(z), \quad (\tau_q(f))(z) = f(qz), \quad (\delta_q(f))(z) = \frac{f(qz) - f(z)}{(qz - z)}.$$

   (a) Check that

$$\delta_q \tau_q = q\,\tau_q \delta_q, \quad [\delta_q, Z] = \tau_q, \quad \delta_q Z - q\,Z\delta_q = 1.$$

   (b) Prove that τ_q is an algebra automorphism and that δ_q is a τ_q-derivation.

   (c) Show that any τ_q-derivation δ of $k[z]$ is of the form $\delta = P\delta_q$ for some polynomial P. If, moreover, $\delta\tau_q = q\,\tau_q\delta$, then P has to be a constant.

   (d) Assume that q is not a root of unity. Check that

$$\delta_q\left(\frac{z^n}{(n)!_q}\right) = \frac{z^{n-1}}{(n-1)!_q}$$

   for all $n \geq 1$. Deduce that the q-exponential $e_q(z)$ is, up to a multiplicative constant, the only formal series solution of the equation $\delta_q(f) = f$.

5. Let $\Lambda_q[\xi, \eta]$ be the algebra $k\{\xi, \eta\}/(\xi^2, \eta^2, \xi\eta + q\,\eta\xi)$. Set

$$\left(\begin{array}{c} \xi' \\ \eta' \end{array}\right) = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\left(\begin{array}{c} \xi \\ \eta \end{array}\right)$$

   where a, b, c and d are variables commuting with ξ and η. Assume that $q^2 \neq -1$.

   (a) Prove that Assertions (i) and (ii) of Theorem 3.1 are equivalent to the relations

$$y'x' = qx'y' \quad \text{and} \quad {\xi'}^2 = {\eta'}^2 = \xi\eta + q\,\eta\xi = 0.$$

   (b) Check that $(a\xi + b\eta)(c\xi + d\eta) = \det_q \xi\eta$. Deduce Part (b) of Proposition 3.4.

   (c) Find a $M_q(2)$-comodule-algebra structure on $\Lambda_q[\xi, \eta]$.

6. Show that the centre of $M_q(2)$ is the subalgebra generated by $\det_q$ when q is not a root of unity.

7. (*Basis of* $SL_q(2)$) Show that the set $\{a^i b^j c^k\}_{i,j,k\geq 0} \cup \{b^i c^j d^k\}_{i,j\geq 0,k>0}$ is a basis of $SL_q(2)$.

8. Let q be a root of unity of order $d > 1$. Prove that $yx = qxy$ implies
$$(x+y)^d = x^d + y^d.$$

9. Let H be a complex Hopf $*$-algebra whose counit is denoted ε. Show that $\varepsilon(x^*) = \overline{\varepsilon(x)}$ for all elements x of H.

IV.10 Notes

The content of Section 2 on q-identities, as well as Exercises 1–3, is classical. We borrowed it from [And76], Chap. 3 and from [Cig79].

The q-exponential is an example of a *q-hypergeometric series* or *basic hypergeometric series*, i.e., of a formal series $\sum_{n\geq 0} a_n z^n$ such that each quotient a_{n+1}/a_n is a rational function of q^n (where q is a complex parameter different from 0 and from 1). Basic hypergeometric series first appeared in a note published by Heine [Hei46] in 1846. Since, q-analogues of most classical functions and identities have been found. F.H. Jackson [Jac10] introduced the q-differentiation operator δ_q and its inverse which is the q-integration. Nowadays, q-series appear in combinatorics, in number theory, in statistical mechanics, and in the theory of Lie algebras. There are many monographs on this vast subject, e.g., [GR90] [Sla66].

The operator τ_q introduced in Section 2 is fundamental in the theory of linear q-difference equations with polynomial coefficients. Such an equation is a functional equation of the form
$$\sum_{i=0}^{n} P_i(z) f(q^i z) = Q(z)$$
where $P_0(z), \ldots, P_n(z), Q(z)$ are polynomials and $f(z)$ is a function. Using the operator τ_q, one can rewrite the equation above as $(\sum_{i=0}^{n} P_i \tau_q^i)(f) = Q$. The articles by Adams [Ada29] and by Trjitzinsky [Trj33] are two classical references on the formalism of the q-difference equations.

Sections 3, 5 and 6 are taken from Manin's book [Man88]. With Section 3 we entered the heart of the subject of Part I of this book. The bialgebras $M_q(2)$, $GL_q(2)$, and $SL_q(2)$ of Sections 5–6 depend on one parameter. There also exist two-parameter versions such as the algebra $M_{p,q}(2)$ generated by four generators a, b, c, d and the six relations

$$ba = pab, \qquad db = qbd,$$

$$ca = qac, \qquad dc = pcd,$$

$$bc = pq^{-1}cb, \quad ad - da = (q^{-1} - p)\, cb.$$

It has the same bialgebra structure as $M(2)$. With the additional relation $ad - p^{-1}bc = 1$, one gets the Hopf algebra $SL_{p,q}(2)$ of [AST91].

In higher dimension $n > 2$, Faddeev, Reshetikhin, Takhtadjian [RTF89] defined the bialgebra $M_q(n)$ generated by the generators $(T_i^j)_{1\le i,j\le n}$ and the relations

$$T_i^m T_i^k = q\, T_i^k T_i^m, \qquad T_j^m T_i^m = q\, T_i^m T_j^m,$$

$$T_i^m T_j^k = T_j^k T_i^m, \quad T_i^k T_j^m - T_j^m T_i^k = (q^{-1} - q)\, T_i^m T_j^k$$

for $i < j$ and $k < m$. The comultiplication and the counit are given by

$$\Delta(T_i^j) = \sum_{k=1}^{n} T_i^k \otimes T_k^j \quad \text{and} \quad \varepsilon(T_i^j) = \delta_{ij}.$$

The algebra $M_q(n)$ is an iterated Ore extension and, like $M_q(2)$, it possesses a remarkable grouplike central element that is

$$\det_q = \sum_{\sigma\in S_n} (-q)^{-\ell(\sigma)}\, T_1^{\sigma(1)} \dots T_n^{\sigma(n)},$$

where $\ell(\sigma)$ is the length of a minimal decomposition of the permutation σ in product of transpositions. The quantum determinant $\det_q$ allows one to construct $GL_q(n)$ and $SL_q(n)$ as in the case $n = 2$ discussed in this chapter. The bialgebra $M_q(n)$ has two interesting comodule-algebras: the first one

$$A_q^{n|0} = k\{x_1, \dots, x_n\}/(x_j x_i - q x_i x_j \ \text{ for } i < j)$$

generalizes the quantum plane whereas the second one

$$A_q^{0|n} = k\{\xi_1, \dots, \xi_n\}/(\xi_i^2, \xi_j\xi_i + q\xi_i\xi_j \ \text{ for } i < j)$$

generalizes the algebra $\Lambda_q[\xi, \eta]$ of Exercise 5.

Both algebras $A_q^{n|0}$ and $A_q^{0|n}$ are examples of *quadratic algebras*, i.e., of quotients of free algebras by ideals generated by degree-two homogeneous elements. For authors like Manin, quadratic algebras form the starting point of the theory of quantum groups. Manin assigns to every quadratic algebra a universal Hopf algebra over which the given quadratic algebra is a comodule-algebra. When applied to the quantum plane, Manin's construction yields $GL_q(2)$. For further reading, see [Man87] [Man88].

We have just mentioned the quantum groups $SL_q(n)$. There exist quantum groups for all classical Lie groups and supergroups. For instance, Takeuchi [Tak89] constructed quantum versions of the symplectic and orthogonal groups.

Woronowicz exhibited Hopf $*$-algebra structures on quantum groups in the framework of C^*-algebras. See [Wor87b] [Wor87a] [Wor88].

The reader will find more examples of and more details on quantum groups in [AST91] [Mal90] [Mal93] [Man89] [PW91] [Res90] [Sud90] [Tak92c].

Chapter V
The Lie Algebra of $SL(2)$

In this chapter we investigate the enveloping Hopf algebra $U = U(\mathfrak{sl}(2))$ of the Lie algebra $\mathfrak{sl}(2)$ of traceless two-by-two matrices. This Hopf algebra is in duality with $SL(2)$. We also describe the finite-dimensional representations of U. Chapter V prepares for Chapters VI–VII where we shall construct a q-deformation U_q of U and study its finite-dimensional representations. The statements and proofs for U_q will essentially be copied from those of the present chapter. We start by recalling the classical concepts of Lie algebras and enveloping algebras. As usual, we denote the ground field by k.

V.1 Lie Algebras

Definition V.1.1. (a) *A Lie algebra L is a vector space with a bilinear map $[\ ,\ ] : L \times L \to L$, called the Lie bracket, satisfying the following two conditions for all $x, y, z \in L$:*

(i) (*antisymmetry*)

$$[x, y] = -[y, x],$$

(ii) (*Jacobi identity*)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

(b) *A Lie subalgebra L' of a Lie algebra L is a subspace L' of L such that for any $(x, y) \in L' \times L'$ we have $[x, y] \in L'$. An ideal I of a Lie algebra*

L is a subspace I of L such that for any element $(x, y) \in L \times I$ we have $[x, y] \in I$.

(c) *A morphism of Lie algebras f from the Lie algebra L into the Lie algebra L' is a linear map $f : L \to L'$ such that $f([x, y]) = [f(x), f(y)]$ for all $x, y \in L$.*

(d) *A Lie algebra is abelian if its Lie bracket is zero.*

Let us give a few examples of Lie algebras.

1. If L and L' are Lie algebras, we equip the direct sum $L \oplus L'$ with a Lie bracket given by

$$[(x, x'), (y, y')] = ([x, y], [x', y'])$$

for $x, y \in L$ and $x', y' \in L'$. The canonical injections of L and L' into $L \oplus L'$ and the canonical projections of $L \oplus L'$ onto L and L' are morphisms of Lie algebras.

2. Given a Lie algebra L, we define the opposite Lie algebra L^{op} as the vector space L with Lie bracket $[-, -]^{\mathrm{op}}$ given by

$$[x, y]^{\mathrm{op}} = [y, x] = -[x, y].$$

The linear map $\mathrm{op}(x) = -x$ is a Lie algebra isomorphism from L to L^{op}.

3. Let I be an ideal of a Lie algebra L. There exists a unique Lie algebra structure on the quotient vector space L/I such that the canonical projection from L onto L/I is a morphism of Lie algebras.

4. Let $f : L \to L'$ be a morphism of Lie algebras. Its kernel $\mathrm{Ker}\,(f)$ is an ideal of L, the image $f(L)$ is a subalgebra of L', and the induced map $L/\mathrm{Ker}\,(f) \to f(L)$ is an isomorphism of Lie algebras.

5. Let A be an (associative) algebra. Set $[a, b] = ab - ba$ for $a, b \in A$. It is easy to show that this bilinear map is antisymmetric and satisfies the Jacobi identity. We also have $[a, bc] = [a, b]c + b[a, c]$ for all $a, b, c \in A$. This Lie algebra will be denoted by $\mathrm{L}(A)$.

For any vector space V, we denote the Lie algebra $\mathrm{L}(\mathrm{End}(V))$ of all endomorphisms of V by $\mathfrak{gl}(V)$. When V is of finite dimension n, then $\mathfrak{gl}(V)$ is isomorphic to the Lie algebra $\mathfrak{gl}(n) = \mathrm{L}(M_n(k))$ of $n \times n$-matrices with entries in the field k. It is clear that the commutator of two matrices with zero trace is of trace zero. Consequently, the vector space $\mathfrak{sl}(n)$ of traceless n by n matrices is a Lie subalgebra of $\mathfrak{gl}(n)$.

V.2 Enveloping Algebras

To any Lie algebra L we assign an (associative) algebra $U(L)$, called the *enveloping algebra* of L, and a morphism of Lie algebras $i_L : L \to \mathrm{L}(U(L))$. We define the enveloping algebra as follows. Let $I(L)$ be the two-sided ideal of the tensor algebra $T(L)$ generated by all elements of the form $xy - yx - [x,y]$ where x, y are elements of L. We define

$$U(L) = T(L)/I(L).$$

The above generators of $I(L)$ are not homogeneous for the grading of $T(L)$ defined in II.5. Therefore there is no grading on the enveloping algebra compatible with the grading of the tensor algebra. Nevertheless, $U(L)$ is filtered as a quotient algebra of $T(L)$.

We define a map i_L as the composition of the canonical injection of L into $T(L)$ and of the canonical surjection of the tensor algebra onto the enveloping algebra. By definition of i_L, we have $i_L([x,y]) = xy - yx$, which shows that i_L is a morphism of Lie algebras.

Example 1. If L is an abelian Lie algebra, then $U(L)$ coincides with the symmetric algebra $S(L)$. In particular, if L is the zero Lie algebra $\{0\}$, then $U(\{0\}) = k$. We also have $U(L^{\mathrm{op}}) = U(L)^{\mathrm{op}}$.

We now state the universal property of $U(L)$.

Theorem V.2.1. *Let L be a Lie algebra. Given any associative algebra A and any morphism of Lie algebras f from L into $\mathrm{L}(A)$, there exists a unique morphism of algebras $\varphi : U(L) \to A$ such that $\varphi \circ i_L = f$.*

If we denote by $\mathrm{Hom}_{\mathrm{Lie}}(L, L')$ the set of morphisms of Lie algebras from L into L', we can express Theorem 2.1 by a natural bijection

$$\mathrm{Hom}_{\mathrm{Lie}}(L, \mathrm{L}(A)) \cong \mathrm{Hom}_{\mathit{Alg}}(U(L), A).$$

PROOF. By definition of the tensor algebra, f extends to a morphism of algebras $\bar{f}$ from $T(L)$ to A defined by $\bar{f}(x_1 \ldots x_n) = f(x_1) \ldots f(x_n)$ for $x_1, \ldots, x_n$ in L. The existence of φ follows from $\bar{f}(I(L)) = \{0\}$. In order to prove this fact, we only have to show that $\bar{f}(xy - yx - [x,y])$ vanishes for any pair (x,y) of elements of L. Now,

$$\bar{f}(xy - yx - [x,y]) = f(x)f(y) - f(y)f(x) - f([x,y]),$$

which is zero since f is a morphism of Lie algebras.

The uniqueness of φ is due to the fact that L generates the algebra $T(L)$, hence $U(L)$. □

We derive two corollaries from Theorem 2.1.

Corollary V.2.2. (a) *For any morphism of Lie algebras $f : L \to L'$, there exists a unique morphism of algebras $U(f) : U(L) \to U(L')$ such that $U(f) \circ i_L = i_{L'} \circ f$. We also have $U(\mathrm{id}_L) = \mathrm{id}_{U(L)}$.*

(b) *If $f' : L \to L''$ is another morphism of Lie algebras, then*

$$U(f' \circ f) = U(f') \circ U(f).$$

PROOF. (a) Apply Theorem 2.1 to $A = U(L')$ and to the morphism of Lie algebras $i_{L'} \circ f$.

(b) We have

$$U(f') \circ U(f) \circ i_L = U(f') \circ i_{L'} \circ f = i_{L''} \circ f' \circ f = U(f' \circ f) \circ i_L.$$

One concludes by appealing to the uniqueness of $U(f' \circ f)$ proved in Part (a). The uniqueness assertion also implies that $U(\mathrm{id}_L)$ is the identity of $U(L)$. □

Corollary V.2.3. *Let L and L' be Lie algebras and $L \oplus L'$ their direct sum. Then*

$$U(L \oplus L') \cong U(L) \otimes U(L').$$

PROOF. We first construct an algebra morphism φ from $U(L \oplus L')$ to the algebra $U(L) \otimes U(L')$. For any $x \in L$ and $x' \in L'$, set

$$f(x, x') = i_L(x) \otimes 1 + 1 \otimes i_{L'}(x').$$

This formula defines a linear map f from $L \oplus L'$ into $U(L) \otimes U(L')$. Let us show that f is a morphism of Lie algebras. For $x, y \in L$ and $x', y' \in L'$ we have

$$\begin{aligned}
[f(x,x'), f(y,y')] &= (i_L(x) \otimes 1 + 1 \otimes i_{L'}(x'))(i_L(y) \otimes 1 + 1 \otimes i_{L'}(y')) \\
&\quad - (i_L(y) \otimes 1 + 1 \otimes i_{L'}(y'))(i_L(x) \otimes 1 + 1 \otimes i_{L'}(x')) \\
&= [i_L(x), i_L(y)] \otimes 1 + 1 \otimes [i_{L'}(x'), i_{L'}(y')] \\
&= i_L([x,y]) \otimes 1 + 1 \otimes i_{L'}([x', y']) \\
&= f([x,y], [x',y']) = f([(x,x'), (y,y')]).
\end{aligned}$$

Applying Theorem 2.1, we get an algebra morphism φ from $U(L \oplus L')$ to $U(L) \otimes U(L')$.

We now use the universal property of the tensor product of two algebras in order to build a morphism of algebras $\psi : U(L) \otimes U(L') \to U(L \oplus L')$. The compositions of the canonical injections of L and of L' into $L \oplus L'$ and of the map $i_{L \oplus L'}$ are morphisms of Lie algebras. By Theorem 2.1 there exist morphisms of algebras $\psi_1 : U(L) \to U(L \oplus L')$ and $\psi_2 : U(L') \to U(L \oplus L')$ such that, for any $x \in L$ and $x' \in L'$, we have

$$\psi_1(x) = i_{L \oplus L'}(x, 0) \quad \text{and} \quad \psi_2(x') = i_{L \oplus L'}(0, x').$$

By Proposition II.4.1, the formula $\psi(a \otimes a') = \psi_1(a)\psi_2(a')$ defines an algebra morphism ψ from $U(L) \otimes U(L')$ into $U(L \oplus L')$ provided we show that $\psi_1(a)\psi_2(a') = \psi_2(a')\psi_1(a)$ for all $a \in U(L)$ and $a' \in U(L')$. We prove the latter by observing that it is enough to check that $\psi_1(a)$ and $\psi_2(a')$ commute when $a = x \in L$ and $a' = x' \in L'$. Now,

$$\begin{aligned}[\psi_1(x), \psi_2(x')] &= [i_{L\oplus L'}(x,0), i_{L\oplus L'}(0,x')] \\ &= i_{L\oplus L'}([(x,0),(0,x')]) \\ &= i_{L\oplus L'}([x,0],[0,x']) \\ &= 0.\end{aligned}$$

We claim that the morphisms φ and ψ are inverse of each other. Let us consider the composition $\psi \circ \varphi$. It is an endomorphism of the algebra $U(L \oplus L')$ restricting to the identity on the image of $L \oplus L'$. Indeed, for all $x \in L$ and $x' \in L'$

$$\psi(\varphi(x,x')) = \psi(x \otimes 1) + \psi(1 \otimes x') = i_{L\oplus L'}((x,0)+(0,x')) = i_{L\oplus L'}(x,x').$$

Consequently, $\psi \circ \varphi = \mathrm{id}$. A similar argument shows that $\varphi \circ \psi = \mathrm{id}$. □

Corollaries 2.2 and 2.3 allow us to put a Hopf algebra structure on the enveloping algebra $U(L)$. Indeed, a comultiplication Δ on $U(L)$ is defined by $\Delta = \varphi \circ U(\delta)$, where δ is the diagonal map $x \mapsto (x,x)$ from L into $L \oplus L$ and φ is the isomorphism $U(L \oplus L) \to U(L) \otimes U(L)$ that was built in the proof of Corollary 2.3. The counit is given by $\varepsilon = U(0)$ where 0 is the zero morphism from L into the zero Lie algebra $\{0\}$. Finally, the antipode is defined by $S = U(\mathrm{op})$ where op is the isomorphism from L onto L^{op} of Example 1.2.

Proposition V.2.4. *The enveloping algebra $U(L)$ is a cocommutative Hopf algebra for the maps Δ, ε, and S defined above. For $x_1, \ldots, x_n \in L$, we have*

$$\begin{aligned}\Delta(x_1 \ldots x_n) &= 1 \otimes x_1 \ldots x_n + \sum_{p=1}^{n-1} \sum_{\sigma} x_{\sigma(1)} \ldots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \ldots x_{\sigma(n)} \\ &\quad + x_1 \ldots x_n \otimes 1\end{aligned}$$

where σ runs over all (p,q)-shuffles of the symmetric group S_n, and

$$S(x_1 x_2 \ldots x_n) = (-1)^n \, x_n \ldots x_2 x_1.$$

PROOF. The coassociativity axiom (III.1.5) is satisfied as a consequence of the commutativity of the square

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\delta"] & L \oplus L \arrow[d, "\mathrm{id}\oplus\delta"] \\ L \oplus L \arrow[r, "\delta\oplus\mathrm{id}"] & L \oplus L \oplus L \end{tikzcd}$$

the counit axiom (III.1.6) because of the commutativity of the diagram

$$\begin{tikzcd} 0 \oplus L & L \oplus L \arrow[l, "0\oplus\mathrm{id}"'] \arrow[r, "\mathrm{id}\oplus 0"] & L \oplus 0 \\ & L \arrow[ul, "\cong"] \arrow[u, "\delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

and the cocommutativity (III.1.7) thanks to the commutativity of the triangle

$$\begin{array}{ccc} & L & \\ {}^{\delta}\swarrow & & \searrow^{\delta} \\ L \oplus L & \xrightarrow{\tau} & L \oplus L \end{array}$$

The formula for Δ results from Theorem III.2.4. The definition of S and Lemma III.3.6 imply that S is an antipode for $U(L)$. $\square$

For the sake of completeness, we give two additional important properties of enveloping algebras.

Theorem V.2.5. *Let L be a Lie algebra.*

(a) *The algebra $U(L)$ is filtered as a quotient of the tensor algebra $T(L)$ (graded as in* II.5*) and the corresponding graded algebra is isomorphic to the symmetric algebra on L:*

$$\operatorname{gr} U(L) \cong S(L).$$

Hence, if $\{v_i\}_{i\in I}$ is a totally ordered basis of L, $\{v_{i_1} \dots v_{i_n}\}_{i_1 \leq \dots \leq i_n \in I,\, n \in \mathbf{N}}$ is a basis of $U(L)$.

(b) *When the characteristic of the field k is zero, the symmetrization map $\eta : S(L) \to U(L)$ defined by*

$$\eta(v_1 \dots v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \cdots v_{\sigma(n)} \tag{2.1}$$

for $v_1, \dots, v_n \in L$, is an isomorphism of coalgebras.

Part (a) of the statement is known as the *Poincaré-Birkhoff-Witt Theorem.* For a proof of Theorem 2.5, we refer to [Bou60] [Dix74] [Hum72] [Jac79].

We end this section by a few remarks on the representations of Lie algebras. By definition, an L-module is a $U(L)$-module in the sense of I.1, which is the same as a morphism of algebras $\rho : U(L) \to \operatorname{End}(V)$. In view of the universal property of $U(L)$ stated in Theorem 2.1, it is equivalent to a morphism (still denoted ρ) of Lie algebras $\rho : L \to \mathfrak{gl}(V)$. For $x \in L$ and $v \in V$, set $xv = \rho(x)(v)$. We observe that $(x, v) \mapsto xv$ is a bilinear map from $L \times V$ to V such that

$$[x, y]v = x(yv) - y(xv) \tag{2.2}$$

for $x, y \in L$ and $v \in V$. Conversely, any bilinear map from $L \times V$ to V such that Relation (2.2) holds for all $x, y \in L$ and $v \in V$, defines an L-module.

The L-module V is trivial in the sense of III.5 if we have $xv = 0$ for all $x \in L$ and $v \in V$. By definition of the coproduct in the enveloping algebra, the structure of L-module on the tensor product of two L-modules V and V' is given by

$$x(v \otimes v') = xv \otimes v' + v \otimes xv' \tag{2.3}$$

for $x \in L$, $v \in V$, and $v' \in V'$. According to III.5, the Lie algebra acts on $\mathrm{Hom}(V, V')$ by

$$(xf)(v) = xf(v) - f(xv), \tag{2.4}$$

which can also be expressed as $\rho(x)(f) = [\rho(x), f]$ for $f \in \mathrm{Hom}(V, V')$. In particular, if V' is the trivial module k, then L acts on the dual vector space $V^* = \mathrm{Hom}(V, k)$ by

$$(xf)(v) = -f(xv). \tag{2.5}$$

Finally, L acts on itself by the so-called *adjoint representation* which is given for $x, x' \in L$ by

$$xx' = [x, x']. \tag{2.6}$$

V.3 The Lie Algebra $\mathfrak{sl}(2)$

To simplify matters, we assume for the rest of this chapter that the ground field k is the field of complex numbers. The Lie algebra $\mathfrak{gl}(2) = \mathrm{L}(M_2(k))$ of 2×2-matrices with complex entries is four-dimensional. The four matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis of $\mathfrak{gl}(2)$. Their commutators are easily computed. We get

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y,$$

and

$$[I, X] = [I, Y] = [I, H] = 0. \tag{3.1}$$

The matrices of trace zero in $\mathfrak{gl}(2)$ form the subspace $\mathfrak{sl}(2)$ spanned by the basis $\{X, Y, H\}$. Relations (3.1) show that $\mathfrak{sl}(2)$ is an ideal of $\mathfrak{gl}(2)$ and that there is an isomorphism of Lie algebras

$$\mathfrak{gl}(2) \cong \mathfrak{sl}(2) \oplus kI,$$

which reduces the investigation of the Lie algebra $\mathfrak{gl}(2)$ to that of $\mathfrak{sl}(2)$.

The enveloping algebra $U = U(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$ is isomorphic to the algebra generated by the three elements X, Y, H with the three relations

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y. \tag{3.2}$$

We prove some relations in U.

Lemma V.3.1. *The following relations hold in U for any $p, q \geq 0$:*

$$X^p H^q = (H-2p)^q X^p, \quad Y^p H^q = (H+2p)^q Y^p,$$

$$[X, Y^p] = pY^{p-1}(H-p+1) = p(H+p-1)Y^{p-1},$$
$$[X^p, Y] = pX^{p-1}(H+p-1) = p(H-p+1)X^{p-1}.$$

PROOF. One proves the first two relations by an easy double induction on p and q using the relations $XH = (H-2)X$ and $YH = (H+2)Y$, which is another way of expressing the commutation relations (3.2).

We prove the third relation by induction on p. It trivially holds for $p = 1$. When $p > 1$, we have

$$\begin{aligned}
[X, Y^p] &= [X, Y^{p-1}]Y + Y^{p-1}[X, Y] \\
&= (p-1)Y^{p-2}(H-p+2)Y + Y^{p-1}H \\
&= Y^{p-1}\Big((p-1)(H-p) + H\Big) \\
&= pY^{p-1}(H-p+1).
\end{aligned}$$

We conclude by letting Y^{p-1} jump over H according to the second relation.

As for the last relation, it can be obtained from the third one by applying the automorphism σ of $\mathfrak{sl}(2)$ defined by

$$\sigma(X) = Y, \quad \sigma(Y) = X, \quad \sigma(H) = -H. \tag{3.3}$$

□

Proposition V.3.2. *The set $\{X^i Y^j H^k\}_{i,j,k \in \mathbf{N}}$ is a basis of $U(\mathfrak{sl}(2))$.*

PROOF. It is a consequence of the Poincaré-Birkhoff-Witt Theorem 2.5. Another proof can be given, using Ore extensions, along the lines of the proof of Proposition VI.1.4. □

We close this section by a few remarks on the centre of U. Let us consider the *Casimir element* defined as the element

$$C = XY + YX + \frac{H^2}{2} \tag{3.4}$$

of the enveloping algebra U.

Lemma V.3.3. *The Casimir element C belongs to the centre of U.*

PROOF. It is enough to show that the Lie brackets of C with H, X, Y vanish. Now,

$$\begin{aligned}
[H, C] &= [H, X]Y + X[H, Y] + [H, Y]X + Y[H, X] + \frac{1}{2}[H, H^2] \\
&= 2XY - 2XY - 2YX + 2YX = 0.
\end{aligned}$$

We also have

$$\begin{aligned}[X,C] &= X[X,Y]+[X,Y]X+\frac{1}{2}[X,H]H+\frac{1}{2}H[X,H] \\ &= XH+HX-XH-HX=0.\end{aligned}$$

One shows $[Y,C]=0$ in a similar fashion. □

Harish-Chandra constructed an isomorphism of algebras from the centre of U to the polynomial algebra $k[t]$. This isomorphism sends C to the generator t (see for instance [Bou60], Chap. 8 or [Dix74], Chap. 7). As a consequence, the Casimir element generates the centre of the enveloping algebra. We shall give full details in the quantum case (see VI.4).

V.4 Representations of $\mathfrak{sl}(2)$

We now determine all finite-dimensional U-modules. We start with the concept of a highest weight vector.

Definition V.4.1. *Let V be a U-module and λ be a scalar. A vector $v \neq 0$ in V is said to be of weight $\lambda \in k$ if $Hv=\lambda v$. If, in addition, we have $Xv=0$, then we say that v is a highest weight vector of weight λ.*

Proposition V.4.2. *Any non-zero finite-dimensional U-module V has a highest weight vector.*

PROOF. Since k is algebraically closed and V is finite-dimensional, the operator H has an eigenvector $w \neq 0$ with eigenvalue α: $Hw=\alpha w$. If $Xw=0$, then w is a highest weight vector and we are done. If not, let us consider the sequence of vectors $X^n w$. By Lemma 3.1, we have

$$H(X^n w)=(\alpha+2n)(X^n w).$$

Consequently, $(X^n w)_{n\geq 0}$ is a sequence of eigenvectors for H with distinct eigenvalues. As V is finite-dimensional, H can have but a finite number of eigenvalues; consequently, there exists an integer n such that $X^n w \neq 0$ and $X^{n+1}w=0$. The vector $X^n w$ is a highest weight vector. □

Lemma V.4.3. *Let v be a highest weight vector of weight λ. For $p \in \mathbf{N}$, set $v_p=\frac{1}{p!}Y^p v$. Then*

$$Hv_p=(\lambda-2p)\,v_p,\quad Xv_p=(\lambda-p+1)\,v_{p-1},\quad Yv_p=(p+1)\,v_{p+1}.$$

PROOF. The third relation is trivial; the first two result from Lemma 3.1. □

We now state the theorem describing simple finite-dimensional U-modules.

Theorem V.4.4. (a) *Let V be a finite-dimensional U-module generated by a highest weight vector v of weight λ. Then*

(i) *The scalar λ is an integer equal to* $\dim(V) - 1$.

(ii) *Setting $v_p = 1/p!\, Y^p v$, we have $v_p = 0$ for $p > \lambda$ and, in addition, $\{v = v_0, v_1, \dots, v_\lambda\}$ is a basis for V.*

(iii) *The operator H acting on V is diagonalizable with the $(\lambda+1)$ distinct eigenvalues $\{\lambda, \lambda - 2, \dots, \lambda - 2\lambda = -\lambda\}$.*

(iv) *Any other highest weight vector in V is a scalar multiple of v and is of weight λ.*

(v) *The module V is simple.*

(b) *Any simple finite-dimensional U-module is generated by a highest weight vector. Two finite-dimensional U-modules generated by highest weight vectors of the same weight are isomorphic.*

PROOF. (a) According to Lemma 4.3, the sequence $\{v_p\}_{p\geq 0}$ is a sequence of eigenvectors for H with distinct eigenvalues. Since V is finite-dimensional, there has to exist an integer n such that $v_n \neq 0$ and $v_{n+1} = 0$. The formulas of Lemma 4.3 then show that $v_m = 0$ for all $m > n$ and $v_m \neq 0$ for all $m \leq n$. We get $n = \lambda$ since we have $0 = Xv_{n+1} = (\lambda - n)v_n$ by Lemma 4.3. The family $\{v = v_0, \dots, v_\lambda\}$ is free, for it is composed of non-zero eigenvectors for H with distinct eigenvalues. It also generates V; indeed, the formulas of Lemma 4.3 show that any element of V, which is generated by v as a module, is a linear combination of the set $\{v_i\}_i$. It results that $\dim(V) = \lambda + 1$. We have thus proved (i) and (ii). The assertion (iii) is also a consequence of Lemma 4.3.

(iv) Let v' be another highest weight vector. It is an eigenvector for the action of H; hence, it is a scalar multiple of some vector v_i. But, again by Lemma 4.3, the vector v_i is killed by X if and only $i = 0$.

(v) Let V' be a non-zero U-submodule of V and let v' be a highest weight vector of V'. Then v' also is a highest weight vector for V. By (iv), v' is a non-zero scalar multiple of v. Therefore v is in V'. Since v generates V, we must have $V \subset V'$, which proves that V is simple.

(b) Let v be a highest weight vector of V; if V is simple, then the submodule generated by v is necessarily equal to V. Consequently, V is generated by a highest weight vector.

If V and V' are generated by highest weight vectors v and v' with the same weight λ, then the linear map sending v_i to v_i' for all i is an isomorphism of U-modules. □

Up to isomorphism, the simple U-modules are classified by the nonnegative integers: given such an integer n, there exists a unique (up to isomorphism) simple U-module of dimension $n+1$, generated by a highest weight vector of weight n. We denote this module by $V(n)$ and the corresponding morphism of Lie algebras by $\rho(n) : \mathfrak{sl}(2) \to \mathfrak{gl}(n+1)$.

For instance, we have $V(0) = k$ and $\rho(0) = 0$, which means that the module $V(0)$ is trivial, as is also the case for all modules of dimension 1.

More generally, any trivial U-module is isomorphic to a direct sum of copies of $V(0)$.

Observe that the morphism $\rho(1) : \mathfrak{sl}(2) \to \mathfrak{gl}(2)$ is the natural embedding of $\mathfrak{sl}(2)$ into $\mathfrak{gl}(2)$ and that the module $V(2)$ is isomorphic to the adjoint representation of $\mathfrak{sl}(2)$ via the map sending the highest weight vector v_0 onto X, v_1 onto $-H$ and v_2 onto Y.

As for the higher-dimensional module $V(n)$, the generators X, Y, and H act by operators represented by the following matrices in the basis $\{v_0, v_1, \ldots, v_n\}$:

$$\rho(n)(X) = \begin{pmatrix} 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n-1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\rho(n)(Y) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n & 0 \end{pmatrix},$$

and

$$\rho(n)(H) = \begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ 0 & n-2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -n+2 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}.$$

Let us determine the action of the Casimir element on the simple module $V(n)$.

Lemma V.4.5. *Any central element of U acts by a scalar on the simple module $V(n)$. In particular, the Casimir element C acts on $V(n)$ by multiplication by the scalar $\frac{n(n+2)}{2}$, which is non-zero when $n > 0$.*

PROOF. Let Z be a central element in U. It commutes with H which decomposes $V(n)$ into a direct sum of one-dimensional eigenspaces. Consequently, the operator Z is diagonal with the same eigenvectors $\{v = v_0, \ldots, v_n\}$ as H. In particular, there exist scalars $\alpha_0, \ldots, \alpha_n$ such that $Zv_p = \alpha_p v_p$ for all p. Now

$$\alpha_{p+1} Y v_p = \alpha_{p+1}(p+1)v_{p+1} = (p+1)Zv_{p+1} = ZYv_p = YZv_p = \alpha_p Y v_p.$$

Consequently, all scalars α_p are equal, which shows that Z acts as a scalar.

In order to determine the action of the Casimir element on $V(n)$, we have only to compute Cv for the highest weight vector v. By (3.4) and by Lemma 4.3 we get

$$Cv = XYv + YXv + \frac{H^2}{2}v = nv + \frac{n^2}{2}v = \frac{n(n+2)}{2}v.$$

□

We finally show that any finite-dimensional U-module is a direct sum of simple U-modules.

Theorem V.4.6. *Any finite-dimensional U-module is semisimple.*

PROOF. By Proposition I.1.3, it suffices to show that for any finite-dimensional U-module V and any submodule V' of V, there exists another submodule V'' such that V is isomorphic to the direct sum $V' \oplus V''$. Set $L = \mathfrak{sl}(2)$.

1. We shall first prove the existence of such a submodule V'' in the case when V' is of codimension 1 in V. We proceed by induction on the dimension of V'.

If $\dim(V') = 0$, we may take $V'' = V$. If $\dim(V') = 1$, then necessarily V' and V/V' are trivial one-dimensional representations. Therefore there exists a basis $\{v_1 \in V', v_2\}$ of V such that $Lv_1 = 0$ and $Lv_2 \subset V' = kv_1$. Consequently, we have $[L, L]v_i = 0$ for $i = 1, 2$. Formulas (3.2) show that the action of L on V is trivial. We thus may take for V'' any supplementary subspace of V' in V.

We now assume that $\dim(V') = p > 1$ and that the assertion to be proved holds in all dimensions $< p$. We have the following alternative: either V' is simple, or it is not.

1.a. Let us first suppose that V' is not simple; then there exists a submodule V_1 of V' such that $0 < \dim(V_1) < \dim(V') = p$. Let π be the canonical projection of V onto $\overline{V} = V/V_1$. The module $\overline{V'} = \pi(V')$ is a submodule of $\overline{V}$ of codimension one and its dimension is $< p$. This allows us to apply the induction hypothesis and to find a submodule $\overline{V''}$ of $\overline{V}$ such that $\overline{V} \cong \overline{V'} \oplus \overline{V''}$. Lifting this isomorphism to V, we get

$$V = V' + \pi^{-1}(\overline{V''}).$$

Now, since $\dim(\overline{V''}) = 1$, the vector space V_1 is a submodule of codimension one of $\pi^{-1}(\overline{V''})$. We again apply the induction hypothesis in order to find a submodule V'' of $\pi^{-1}(\overline{V''})$ such that $\pi^{-1}(\overline{V''}) \cong V_1 \oplus V''$. Let us prove that the one-dimensional submodule V'' has the expected properties, namely $V \cong V' \oplus V''$. Indeed, the above argument implies that $V = V' + V_1 + V''$; now V_1 is contained in V', which shows that V is the sum of V' and of V''. The formula $\dim(V) = \dim(V') + \dim(V'')$ implies that this is a direct sum.

1.b. If the submodule V' is simple of dimension > 1, then Lemma 4.5 implies that the Casimir element C acts on V' as a scalar $\alpha \neq 0$. Consequently, the operator C/α is the identity on V'. Now V/V' is one-dimensional, hence a trivial module. Therefore C sends V into the submodule V', which means that the map C/α is a projector of V onto V'. As C/α commutes with any element of U, the map C/α is a morphism of U-modules. By Proposition I.1.3, the submodule $V'' = \operatorname{Ker}(C/\alpha)$ is a supplementary submodule to V'.

2. *General case.* We are now given two finite-dimensional modules $V' \subset V$ without any restriction on the codimension. We shall reduce the situation to the codimension-one case by considering vector spaces $W' \subset W$ defined as follows: W [resp. W'] is the subspace of all linear maps from V to V' whose restriction to V' is a homothety [resp. is zero]. It is clear that W' is of codimension one in W. In order to reduce to Part 1, we have to equip W and W' with U-module structures. We give $\operatorname{Hom}(V, V')$ the U-module structure defined by Relation (2.4). Let us check that W and W' are U-submodules. For $f \in W$, let α be the scalar such that $f(v) = \alpha v$ for all $v \in V'$; then for any $x \in L$, we have

$$(xf)(v) = xf(v) - f(xv) = x(\alpha v) - \alpha(xv) = 0.$$

A similar argument proves that W' is a submodule. Applying Part 1, we get a one-dimensional submodule W'' such that $W \cong W' \oplus W''$. Let f be a generator of W''. By definition, it acts on V' as a scalar $\alpha \neq 0$. It follows that f/α is a projector of V onto V'. To conclude, it suffices to check that f (hence f/α) is a morphism of modules. Now, since W'' is a one-dimensional submodule, it is trivial. Therefore, we have $xf = 0$ for all $x \in L$, which by (2.4) translates into $xf(v) - f(xv) = 0$ for all $v \in V$. □

V.5 The Clebsch-Gordan Formula

Given two finite-dimensional U-modules, we consider their tensor product equipped with the module structure given by Relation (2.3). By Theorem 4.6 it can be decomposed in simple modules. By the distributivity of the tensor product with respect to direct sums and by Theorems 4.4 and 4.6, it is enough to decompose $V(n) \otimes V(m)$ into simple modules. This is the object of the next assertion known as the *Clebsch-Gordan formula.*

Proposition V.5.1. *Consider two nonnegative integers $n \geq m$. Then there exists an isomorphism of U-modules*

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m+2) \oplus V(n-m).$$

PROOF. It is enough to prove that, for all p with $0 \leq p \leq m$, the module $V(n) \otimes V(m)$ contains a highest weight vector of weight $n+m-2p$. In effect, if so, there exists a non-zero morphism of modules from $V(n+m-2p)$ into

$V(n) \otimes V(m)$. The module $V(n+m-2p)$ being simple, the kernel of such a morphism has to be zero, which means that the morphism is an embedding of $V(n+m-2p)$ into $V(n) \otimes V(m)$. The submodules $V(n+m-2p)$ being simple and of distinct highest weights, their sum in $V(n) \otimes V(m)$ is direct. Thus, the right-hand side of the Clebsch-Gordan formula embeds into the left-hand side. To conclude, it suffices to check that both sides have the same dimension. Now the dimension of $V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m)$ equals

$$\begin{aligned} \sum_{p=0}^{m} (n+m-2p+1) &= (n+1)(m+1) \\ &= \dim(V(n)) \dim(V(m)) \\ &= \dim(V(n) \otimes V(m)). \end{aligned}$$

Proposition 5.1 will then be a consequence of the following lemma. □

Lemma V.5.2. *Let v be a highest weight vector of $V(n)$ and v' be a highest weight vector of $V(m)$. Define $v_p = \frac{1}{p!} Y^p v$ and $v'_p = \frac{1}{p!} Y^p v'$ for $p \geq 0$. Then*

$$\sum_{i=0}^{p} (-1)^i \frac{(m-p+i)!(n-i)!}{(m-p)!n!} v_i \otimes v'_{p-i}$$

is a highest weight vector of $V(n) \otimes V(m)$ of weight $n+m-2p$.

PROOF. Set

$$\alpha_i = (-1)^i \frac{(m-p+i)!(n-i)!}{(m-p)!n!} \quad \text{and} \quad w = \sum_{i=0}^{p} \alpha_i v_i \otimes v'_{p-i}.$$

It is enough to check that $Xw = 0$ and $Hw = (n+m-2p)w$. The latter holds because the tensors $v_i \otimes v'_{p-i}$ all are of weight $n+m-2p$. Indeed, by Lemma 4.3, we have

$$H(v_i \otimes v'_{p-i}) = H(v_i) \otimes v'_{p-i} + v_i \otimes H(v'_{p-i}) = (n+m-2p)\, v_i \otimes v'_{p-i}.$$

Let us compute Xw. By Lemma 4.3 again, we have

$$\begin{aligned} Xw &= \sum_{i=0}^{p} \alpha_i X(v_i) \otimes v'_{p-i} + \sum_{i=0}^{p} \alpha_i v_i \otimes X(v'_{p-i}) \\ &= \sum_{i=0}^{p} \alpha_i (n-i+1)\, v_{i-1} \otimes v'_{p-i} + \sum_{i=0}^{p} \alpha_i (m-p+i+1)\, v_i \otimes v'_{p-i-1} \\ &= \sum_{i=1}^{p} \Big(\alpha_i (n-i+1) + \alpha_{i-1}(m-p+i) \Big) v_{i-1} \otimes v'_{p-i}. \end{aligned}$$

Now,

$$\begin{aligned}
&\alpha_i(n-i+1)+\alpha_{i-1}(m-p+i)\\
&= (-1)^i \frac{(m-p+i)!(n-i)!}{(m-p)!n!}(n-i+1)\\
&\qquad + (-1)^{i-1}\frac{(m-p+i-1)!(n-i+1)!}{(m-p)!n!}(m-p+i)\\
&= 0.
\end{aligned}$$

□

Remark 5.3. (a) One deduces from Proposition 5.1 that the adjoint representation $V(2)$ is related to $V(0)$ and $V(1)$ by

$$V(1)^{\otimes 2} \cong V(2) \oplus V(0).$$

(b) The dual module $V(n)^*$ is isomorphic to the simple module $V(n)$ (prove it). Consequently, we have the U-linear isomorphisms

$$\mathrm{Hom}(V(n), V(m)) \cong V(m) \otimes V(n)^* \cong V(m) \otimes V(n).$$

V.6 Module-Algebra over a Bialgebra. Action of $\mathfrak{sl}(2)$ on the Affine Plane

We now introduce a concept that formalizes nicely many situations where an algebra acts on another one.

Definition V.6.1. *Let H be a bialgebra and A an algebra. We call A a module-algebra over H if*

(a) *the vector space underlying A is an H-module, and*

(b) *the multiplication $\mu : A \otimes A \to A$ and the unit $\eta : k \to A$ of A are morphisms of H-modules, the tensor product $A \otimes A$ and the ground field k being given the H-module structures described by Relations* (III.5.2–5.3).

In the literature, module-algebras over a bialgebra H are also called *H-algebras.* By making explicit Condition (b) of Definition 6.1, we see that A is a module-algebra over H if the action of H on A satisfies the following two compatibility relations with the product and the unit of A:

$$x(ab) = \sum_{(x)} (x'a)(x''b) \tag{6.1}$$

and

$$x1 = \varepsilon(x)1 \tag{6.2}$$

where x is an element of H and a, b are elements of A. Here we used Sweedler's sigma notation (see III.1.6). The map ε is the counit of the bialgebra H while 1 is the unit of A.

It is not always convenient to check Relation (6.1) for all elements x of H. The following result shows that it is enough to check it for a set of generators.

Lemma V.6.2. *Let H be a bialgebra and A be an algebra with a structure of H-module such that Relation* (6.2) *holds. Assume that H is generated as an algebra by a subset X whose elements x satisfy Relation* (6.1) *for all elements a and b in A. Then A is a module-algebra over H.*

PROOF. It suffices to check that if Relation (6.1) holds for x and y in H, then it also holds for their product xy. Now, for all $a, b \in A$, we have

$$\begin{aligned}
(xy)(ab) &= x(y(ab)) \\
&= x\Big(\sum_{(y)} (y'a)(y''b)\Big) \\
&= \sum_{(x)(y)} \Big(x'(y'(a))\Big)\Big(x''(y''(b))\Big) \\
&= \sum_{(x)(y)} \Big((x'y')a\Big)\Big((x''y'')b\Big) \\
&= \sum_{(xy)} \Big((xy)'a\Big)\Big((xy)''b\Big).
\end{aligned}$$

□

The following examples show that module-algebra structures appear in a number of situations.

Example 1. Let φ be an automorphism of an algebra A. Consider the algebra $k[\mathbf{Z}]$ of the group of integers with the bialgebra structure described in III.2, Example 2. If $k[\mathbf{Z}]$ acts on A by sending a generator of $\mathbf{Z}$ on φ, then A becomes a module-algebra over $k[\mathbf{Z}]$.

Let us describe module-algebras over enveloping algebras.

Lemma V.6.3. *Let L be a Lie algebra. An algebra A is a module-algebra over $U(L)$ if and only if A has an L-module structure such that the elements of L act on A as derivations.*

PROOF. From Section 2 we know that a $U(L)$-module is an L-module and conversely. Assume that A is a module-algebra over $U(L)$. If $x \in L$ we have $\Delta(x) = x \otimes 1 + 1 \otimes x$. For such an x, Relation (6.1) becomes

$$x(ab) = x(a)b + ax(b)$$

for all $a, b \in A$, which shows that x acts as a derivation. The converse statement results from Lemma 6.2. $\square$

We now return to the Lie algebra $\mathfrak{sl}(2)$ and show how the affine plane becomes a module-algebra over the enveloping algebra $U(\mathfrak{sl}(2))$.

Theorem V.6.4. *Define an action of the Lie algebra $\mathfrak{sl}(2)$ on the polynomial algebra $k[x,y]$ by*

$$XP = x\frac{\partial P}{\partial y}, \quad YP = y\frac{\partial P}{\partial x}, \quad HP = x\frac{\partial P}{\partial x} - y\frac{\partial P}{\partial y}$$

where P denotes any polynomial of $k[x,y]$.

(a) *Then $k[x,y]$ becomes a module-algebra over $U(\mathfrak{sl}(2))$.*

(b) *The subspace $k[x,y]_n$ of homogeneous polynomials of degree n is a submodule of $k[x,y]$ isomorphic to the simple $\mathfrak{sl}(2)$-module $V(n)$.*

We have thus succeeded in packing into a single module all simple finite-dimensional $U(\mathfrak{sl}(2))$-modules, thanks to the notion of module-algebra.

PROOF. (a) We shall first check that the above formulas define an action of $\mathfrak{sl}(2)$ on $k[x,y]$. We have

$$\begin{aligned} [X,Y]P &= x\frac{\partial}{\partial y}\Big(y\frac{\partial P}{\partial x}\Big) - y\frac{\partial}{\partial x}\Big(x\frac{\partial P}{\partial y}\Big) \\ &= x\frac{\partial P}{\partial x} + xy\frac{\partial^2 P}{\partial y \partial x} - y\frac{\partial P}{\partial y} - yx\frac{\partial^2 P}{\partial x \partial y} \\ &= HP. \end{aligned}$$

One similarly shows that $[H,X]P = 2XP$ and $[H,Y]P = -2YP$.

In order to conclude that we have a module-algebra structure, it is enough in view of Lemma 6.3 to check that the generators X, Y, H act on $k[x,y]$ as derivations, which is clearly the case.

(b) Fix a non-negative integer n and set $v = x^n \in k[x,y]_n$. Clearly, v is a highest weight vector of weight n. For all $p \geq 0$ we have

$$v_p = \frac{1}{p!}Y^p v = \binom{n}{p} x^{n-p}y^p$$

if $p \leq n$ and $v_p = 0$ if $p > n$. Since the monomials $\{v_p\}_p$ generate $k[x,y]_n$, the latter is a $\mathfrak{sl}(2)$-module generated by a highest weight vector of weight n. Hence, by Theorem 4.4, it is isomorphic to the simple module $V(n)$. $\square$

V.7 Duality between the Hopf Algebras $U(\mathfrak{sl}(2))$ and $SL(2)$

The main objective of this section is to relate this chapter to Chapter I by building a duality between $U = U(\mathfrak{sl}(2))$ and the Hopf algebra $SL(2)$ defined in I.5. We start with the following definition due to Takeuchi [Tak81].

Definition V.7.1. *Given bialgebras $(U, \mu, \eta, \Delta, \varepsilon)$ and $(H, \mu, \eta, \Delta, \varepsilon)$ and a bilinear form $< \, , \, >$ on $U \times H$, we say that the bilinear form realizes a duality between U and H, or that the bialgebras U and H are in duality, if we have*

$$< uv, x > = \sum_{(x)} < u, x' >< v, x'' >, \tag{7.1}$$

$$< u, xy > = \sum_{(u)} < u', x >< u'', y >, \tag{7.2}$$

$$< 1, x > = \varepsilon(x), \tag{7.3}$$

and

$$< u, 1 > = \varepsilon(u) \tag{7.4}$$

for all $u, v \in U$ and $x, y \in H$.

If, in addition, U and H are Hopf algebras with antipodes S, then they are said to be in duality if the underlying bialgebras are in duality and if, moreover, we have

$$< S(u), x > = < u, S(x) > \tag{7.5}$$

for all $u \in U$ and $x \in H$.

Let us motivate this definition. Let φ be the linear map from U to the dual vector space H^* defined by

$$\varphi(u)(x) = < u, x > .$$

Similarly, $\psi(x)(u) = < u, x >$ defines a linear map from H to U^*. From Proposition III.1.2 we know that the dual spaces U^* and H^* carry natural algebra structures. If, in addition, the vector space H is finite-dimensional, then the dual space H^* has a natural bialgebra structure induced by the one on H (see III.2, Example 1). We are now ready to state a characterization for duality between bialgebras.

Proposition V.7.2. *Given bialgebras U and H and a bilinear form $< \, , \, >$ on $U \times H$, the bilinear form realizes a duality between U and H if and only if the linear maps φ and ψ are morphisms of algebras.*

If, moreover, H is finite-dimensional, then the bilinear form realizes a duality if and only if φ is a morphism of bialgebras.

We shall say that the duality between U and H is *perfect* when both maps φ and ψ are injective. In case U and H are finite-dimensional, a perfect duality between them induces isomorphisms of bialgebras between U and H^* and between H and U^*.

PROOF. Let us express that φ is a morphism of algebras. Recall that the unit of H^* is equal to the counit ε of H and that the product of two linear forms α and β of H^* is given by

$$(\alpha\beta)(x) = \sum_{(x)} \alpha(x')\beta(x'')$$

for all $x \in H$. Then the relations $\varphi(1) = 1$ and $\varphi(uv) = \varphi(u)\varphi(v)$ imply $< 1, x > = \varphi(1)(x) = \varepsilon(x)$ and

$$\begin{aligned} < uv, x > &= \varphi(uv)(x) = (\varphi(u)\varphi(v))(x) \\ &= \sum_{(x)} \varphi(u)(x')\varphi(v)(x'') = \sum_{(x)} < u, x' >< v, x'' > . \end{aligned}$$

It results that Relations (7.1) and (7.3) of Definition 7.1 are equivalent to the fact that φ is a morphism of algebras. By symmetry, we see that Relations (7.2) and (7.4) are equivalent to the fact that ψ is a morphism of algebras.

Now assume that H is finite-dimensional. Then the dual space H^* is a bialgebra. We have already expressed the fact that φ is a morphism of algebras. Let us express that it is a morphism of coalgebras. On one hand, the relation $\varepsilon\varphi = \varepsilon$ expressing that φ preserves the counit reads

$$\varepsilon(u) = (\varepsilon\varphi)(u) = \varphi(u)(1) = < u, 1 > .$$

On the other hand, if φ preserves the comultiplication, we have

$$\begin{aligned} < u, xy > &= \varphi(u)(xy) = \Delta(\varphi(u))(x \otimes y) \\ &= \sum_{(u)} \varphi(u')(x)\varphi(u'')(y) \\ &= \sum_{(u)} < u', x >< u'', y > . \end{aligned}$$

Thus, the map φ is a morphism of coalgebras if and only if Relations (7.2) and (7.4) are satisfied. $\square$

We return to the enveloping algebra $U = U(\mathfrak{sl}(2))$. We wish to set it in duality with the Hopf algebra $SL(2)$. Our first task is to construct a morphism of algebras ψ from the algebra $M(2) = k[a, b, c, d]$ (introduced in I.4) to the dual algebra U^*. We shall deduce a bilinear form on $U \times M(2)$ defined by $< u, x > = \psi(x)(u)$ and satisfying Relations (7.2) and (7.4). Now, building ψ is equivalent to giving four pairwise commuting elements $A, B, C,\ D$ of U^*.

The definitions of $A, B, C,$ and D use the simple U-module $V(1)$ with the basis $\{v_0, v_1\}$ described in Section 4. Given an element u in U, we set

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

where ρ is the representation $\rho(1)$ corresponding to $V(1)$. This defines four linear forms on U, hence four elements A, B, C, D of the dual space U^*. The comultiplication of U being cocommutative, the dual algebra U^* is commutative. Therefore, the quadruple (A, B, C, D) defines a unique morphism of algebras $\psi : M(2) \to U^*$ such that

$$\psi(a) = A, \quad \psi(b) = B, \quad \psi(c) = C, \quad \psi(d) = D. \tag{7.6}$$

Proposition V.7.3. *The bilinear form $< u, x > = \psi(x)(u)$ realizes a duality between the bialgebras U and $M(2)$.*

PROOF. It remains to check Relations (7.1) and (7.3). We start with (7.3). The identity $\rho(1) = 1$ yields

$$\begin{pmatrix} <1,a> & <1,b> \\ <1,c> & <1,d> \end{pmatrix} = \begin{pmatrix} A(1) & B(1) \\ C(1) & D(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{pmatrix} \tag{7.7}$$

by definition of the counit in $M(2)$. Now, from Relation (7.2) we get

$$<1, xy> = <1, x><1, y> .$$

Both maps $x \mapsto <1, x>$ and ε are morphisms of algebras and they coincide on the generators a, b, c, d of $M(2)$ by (7.7). Therefore, they have to be equal, which proves Relation (7.3).

We now turn to the proof of Relation (7.1). Let us denote by $C(x)$ the following condition on an element x of $M(2)$: For any pair (u, v) of elements of U, we have

$$<uv, x> = \sum_{(x)} <u, x'><v, x''> .$$

Let us first show that $C(1)$ is satisfied. Indeed, from (7.4) we get

$$<uv, 1> = \varepsilon(uv) = \varepsilon(u)\varepsilon(v) = <u, 1><v, 1> .$$

We next prove that Conditions $C(a), C(b), C(c), C(d)$ hold. By definition, we have

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = \begin{pmatrix} <u,a> & <u,b> \\ <u,c> & <u,d> \end{pmatrix}.$$

Let us express that $\rho(uv) = \rho(u)\rho(v)$. We have

$$\begin{pmatrix} <uv,a> & <uv,b> \\ <uv,c> & <uv,d> \end{pmatrix}$$
$$= \begin{pmatrix} <u,a> & <u,b> \\ <u,c> & <u,d> \end{pmatrix} \begin{pmatrix} <v,a> & <v,b> \\ <v,c> & <v,d> \end{pmatrix}. \tag{7.8}$$

Expanding this matrix product, we get exactly the four desired conditions since, as we know from Chapter I, the coproduct on $M(2)$ is defined by the matrix relation

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In order to conclude the proof of (7.1), we need to check Condition $C(x)$ for an arbitrary element x of $M(2)$. To this end, we first observe that if $C(x)$ and $C(y)$ are verified, then so is $C(\lambda x + y)$ for any scalar λ; second, we use the following lemma, which completes the proof of the proposition. □

Lemma V.7.4. *If Conditions $C(x)$ and $C(y)$ hold, then so does $C(xy)$.*

PROOF. Relation (7.2), and Conditions $C(x)$ and $C(y)$ imply that

$$\begin{aligned} < uv, xy > &= \sum_{(uv)} < (uv)', x >< (uv)'', y > \\ &= \sum_{(u)(v)} < u'v', x >< u''v'', y > \\ &= \sum_{(u)(v)(x)(y)} < u', x' >< v', x'' >< u'', y' >< v'', y'' > . \end{aligned}$$

They also yield

$$\begin{aligned} &\sum_{(xy)} < u, (xy)' >< v, (xy)'' > \\ &= \sum_{(x)(y)} < u, x'y' >< v, x''y'' > \\ &= \sum_{(u)(v)(x)(y)} < u', x' >< u'', y' >< v', x'' >< v'', y'' > \\ &= < uv, xy > . \end{aligned}$$

□

The duality between $M(2)$ and U is not perfect: the morphism ψ is not injective as the following lemma shows.

Lemma V.7.5. *We have $\psi(ad - bc) = 1$.*

Equivalently, $< u, ad - bc >= \varepsilon(u)$ for all elements u of U.

PROOF. Lemma I.5.2 as rephrased in (II.4.5) means that the element $ad-bc$ is grouplike. Consequently, by (7.1) we have

$$< uv, ad - bc >=< u, ad - bc >< v, ad - bc >$$

for any pair (u, v) of elements of U. On the other hand, by (7.3) we have

$$< 1, ad - bc >= \varepsilon(ad - bc) = 1.$$

This implies that the linear map $u \mapsto < u, ad - bc >$ is a morphism of algebras from U to k. To show that this morphism coincides with the counit ε, it suffices to check that both maps have the same values on the generators X, Y and H. Now we have

$$\begin{aligned}
&< X, ad - bc > \\
&= \quad \varepsilon(a) < X, d > + < X, a > \varepsilon(d) - \varepsilon(b) < X, c > - < X, b > \varepsilon(c) \\
&= \quad 0 = \varepsilon(X).
\end{aligned}$$

Similarly, we get $< Y, ad - bc >= 0 = \varepsilon(Y)$. Finally,

$$\begin{aligned}
&< H, ad - bc > \\
&= \quad \varepsilon(a) < H, d > + < H, a > \varepsilon(d) - \varepsilon(b) < H, c > - < H, b > \varepsilon(c) \\
&= \quad -1 + 1 = 0 = \varepsilon(H).
\end{aligned}$$

□

As a consequence of the previous lemma, the morphism of algebras $\psi : M(2) \to U^*$ factors through $SL(2) = M(2)/(ad - bc - 1)$. We still denote by ψ the induced morphism of algebras from $SL(2)$ to U^* and by $< \, , >$ the corresponding bilinear form.

Theorem V.7.6. *The bilinear form $< u, x >= \psi(x)(u)$ realizes a duality between the Hopf algebras U and $SL(2)$.*

PROOF. We already know that ψ is a morphism of algebras. By Proposition 7.2 we are left with showing that $\varphi : U \to SL(2)^*$ is a morphism of algebras too. Now, the projection from $M(2)$ onto $SL(2)$ dualizes to an injective morphism from $SL(2)^*$ into $M(2)^*$. It is clear that, when composing the latter with φ, we get the morphism of algebras $\varphi : U \to M(2)^*$ investigated earlier. Consequently, $\varphi : U \to SL(2)^*$ is a morphism of algebras. This shows that we have a duality between bialgebras.

It remains to examine the antipodes and to check Relation (7.5). Let us start with the generators. In the abridged matrix form we have

$$\begin{aligned}
< S(X), \begin{pmatrix} a & b \\ c & d \end{pmatrix} > &= \rho(S(X)) = -\rho(X) \\
&= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\
&= < X, \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} > \\
&= < X, \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} > .
\end{aligned}$$

One proceeds similarly with Y, H, and 1.

For arbitrary elements of U and $SL(2)$, we use the following result. $\square$

Lemma V.7.7. *Let u, v be elements of U. If*

$$< S(u), x > = < u, S(x) > \quad \text{and} \quad < S(v), x > = < v, S(x) >$$

for all $x \in SL(2)$, then $< S(uv), x > = < uv, S(x) >$. Similarly, let x, y be elements of $SL(2)$. If

$$< S(u), x > = < u, S(x) > \quad \text{and} \quad < S(u), y > = < u, S(y) >$$

for all $u \in U$, then $< S(u), xy > = < u, S(xy) >$.

PROOF. Theorem III.3.4 (a) and Definition 7.1 imply that

$$\begin{aligned} < S(uv), x > &= < S(v)S(u), x > \\ &= \sum_{(x)} < S(v), x' >< S(u), x'' > \\ &= \sum_{(x)} < u, S(x'') >< v, S(x') > \\ &= \sum_{(S(x))} < u, S(x)' >< v, S(x)'' > \\ &= < uv, S(x) > . \end{aligned}$$

The proof of the second assertion is similar. $\square$

To the duality between U and $SL(2)$ corresponds a duality between U-modules and $SL(2)$-comodules. We now investigate this. In III.7 we showed that the vector space $k[x, y]_n$ of homogeneous polynomials of total degree n had a natural $SL(2)$-comodule structure. By duality, the dual vector space $k[x, y]_n^*$ has a module structure over the algebra $SL(2)^*$, hence over the algebra U via the morphism $\varphi : U \to SL(2)^*$. The following result gives the structure of $k[x, y]_n^*$ as a U-module.

Theorem V.7.8. *The U-module $k[x, y]_n^*$ is simple with highest weight n.*

In other words, the $SL(2)$-comodule $k[x, y]_n$ corresponds by duality to the U-module $V(n)$.

PROOF. We shall show that the linear form on $k[x, y]_n$ defined by

$$f(x^i y^{n-i}) = \delta_{ni}$$

is a highest weight vector with weight n of the U-module $k[x, y]_n^*$, which will imply that $k[x, y]_n^*$ contains a submodule isomorphic to the simple module $V(n)$. Since

$$\dim(V(n)) = n + 1 = \dim(k[x, y]_n^*),$$

we get $k[x, y]_n^* \cong V(n)$.

In order to prove that f is a highest weight vector, we need the identity

$$(uf)(x^i y^{n-i}) = < u, a^i c^{n-i} > \tag{7.9}$$

for all $u \in U$ and for all i such that $0 \leq i \leq n$. Indeed, by definition of f, by III.6, Example 2 and by Lemma III.7.4 we have

$$\begin{aligned}
&(uf)(x^i y^{n-i}) \\
&= (u \otimes f)(\Delta_A(x^i y^{n-i})) \\
&= \sum_{r=0}^{i} \sum_{s=0}^{n-i} \binom{i}{r} \binom{n-i}{s} < u, a^r b^{i-r} c^s d^{n-i-s} > f(x^{r+s} y^{n-r-s}) \\
&= \sum_{r=0}^{i} \sum_{s=0}^{n-i} \binom{i}{r} \binom{n-i}{s} < u, a^r b^{i-r} c^s d^{n-i-s} > \delta_{n,r+s} \\
&= \sum_{r=0}^{i} \sum_{s=0}^{n-i} \binom{i}{r} \binom{n-i}{s} < u, a^r b^{i-r} c^s d^{n-i-s} > \delta_{i,r} \delta_{n-i,s} \\
&= < u, a^i c^{n-i} > .
\end{aligned}$$

Let us apply Relation (7.9) to H. A straightforward computation using (7.2–7.3) and the definition of the bilinear form yields

$$< H, a^i c^j > = i \delta_{j0}.$$

Consequently, we have $(Hf)(x^i y^{n-i}) = n\delta_{ni}$, which implies that $Hf = nf$.

It remains to prove that $Xf = 0$. This is a consequence of Relation (7.9) applied to X and of the fact that $< X, a^i c^j > = 0$ for all i and j. Let us prove the latter. First, we have $< X, 1 > = \varepsilon(X) = 0$. Next, if $i > 0$, we have by (7.2–7.3)

$$\begin{aligned}
< X, a^i > &= \varepsilon(a) < X, a^{i-1} > + < X, a > \varepsilon(a^{i-1}) \\
&= < X, a^{i-1} > = \ldots = < X, a > = 0.
\end{aligned}$$

Similarly, if $j > 0$ we get

$$< X, c^j > = \varepsilon(c) < X, c^{j-1} > + < X, c > \varepsilon(c^{j-1}) = 0.$$

Consequently,

$$< X, a^i c^j > = \varepsilon(a)^i < X, c^j > + < X, a^i > \varepsilon(c^j) = 0.$$

$\square$

V.8 Exercises

1. Let L be a Lie algebra. Show that $[L, L]$ is an ideal of L and that the quotient Lie algebra $L^{\mathrm{ab}} = L/[L, L]$ is abelian. Prove that if f is a morphism of Lie algebras from L to any abelian Lie algebra V, then there exists a unique linear map f^{ab} from L^{ab} into V such that f is the composition of f^{ab} and of the canonical projection from L onto L^{ab}.

2. For any Lie algebra L determine the group of grouplike elements of the Hopf algebra $U(L)$.

3. Let A be an algebra and $\mathrm{Der}(A)$ the vector space of all derivations of A. Show that the commutator of any two derivations is again a derivation and that $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

4. Show that any algebra A is a module-algebra over the enveloping algebra of the Lie algebra $\mathrm{Der}(A)$ and over the bialgebra $k[G]$ where G is the group of algebra automorphisms of A.

5. Let L be a Lie algebra and $\rho : L \to \mathfrak{gl}(V)$ a finite-dimensional representation of L. Define a symmetric bilinear form on L by

$$< x, y >_\rho = \mathrm{tr}(\rho(x)\rho(y))$$

where tr denotes the trace of endomorphisms.

   (a) Prove that this form is invariant, i.e., we have

$$< [x, y], z >_\rho = < x, [y, z] >_\rho$$

   for all $x, y, z \in L$.

   (b) Let $\{x_i\}_{1 \le i \le d}$ be a basis of L. Assume the form $< \, , >_\rho$ non-degenerate. Define a new basis $\{x^i\}_{1 \le i \le d}$ of L by requiring that $< x_i, x^j >_\rho = \delta_{ij}$. We get an element $C_\rho = \sum_{1 \le i \le d} x_i x^i$ of $U(L)$. Show that C_ρ belongs to the centre of the enveloping algebra and that $\mathrm{tr}(\rho(C_\rho)) = d = \dim(L)$.

   (c) (*Whitehead Lemma*) Let $f : L \to V$ be a linear map satisfying the relation

$$f([x, y]) = xf(y) - yf(x)$$

   for all $x, y \in L$. Assume that the form $< \, , >_\rho$ is non-degenerate and that C_ρ is well-defined. Show that we have

$$C_\rho f(x) = x\Big(\sum_{1 \le i \le d} x_i f(x^i)\Big).$$

   Deduce that, when $\rho(C_\rho)$ is invertible, there exists a vector v in V such that $f(x) = xv$ for all x in L.

6. Find all invariant symmetric bilinear forms of $\mathfrak{sl}(2)$ (as defined in the previous exercise; assume that the field k is of characteristic zero).

7. Show that the enveloping algebra $U(\mathfrak{sl}(2))$ is Noetherian and has no divisors of zero. Find its centre (Hint: proceed by analogy with VI.4).

8. Assume that k is a field of characteristic zero. Show that the Lie algebra $\mathfrak{sl}(2)$ has no ideals but $\{0\}$ and the algebra itself. Deduce that $\mathfrak{sl}(2) = [\mathfrak{sl}(2), \mathfrak{sl}(2)]$.

9. Show that the dual of the U-module $V(n)$ is isomorphic to $V(n)$.

10. Determine all Hopf algebra automorphisms of $U(\mathfrak{sl}(2))$.

11. Check that there is an antiautomorphism T of algebras of $U(\mathfrak{sl}(2))$ such that $T(X) = Y$, $T(Y) = X$, and $T(H) = H$. Prove that T is a morphism of coalgebras. Find all non-degenerate symmetric bilinear forms $(\ ,\ )$ on the simple module $V(n)$ such that $(xv, v') = (v, T(x)v')$ for all $x \in U(\mathfrak{sl}(2))$ and $v, v' \in V(n)$. Show that the basis of $V(n)$ consisting of the vectors $v_0, \ldots, v_n$ (defined in Section 4) is orthogonal for such a form.

12. (*Bialgebra structure on the quantum plane*) (a) Show that the formulas

$$\Delta(x) = x \otimes x, \quad \Delta(y) = x \otimes y + y \otimes 1, \quad \varepsilon(x) = 1, \quad \varepsilon(y) = 0$$

equip the free algebra $k\{x, y\}$ and the quantum plane $k_q[x, y]$ with a bialgebra structure.

(b) Prove that an algebra R is a module-algebra over the bialgebra $k\{x, y\}$ [resp. over $k_q[x, y]$] if and only if R possesses an algebra endomorphism τ and a τ-derivation δ [resp. τ and δ such that the relation $\delta\tau = q\tau\delta$ holds].

(c) Find all $k_q[x, y]$-algebra structures on the polynomial algebra $k[z]$ (consider only the ones for which τ is an automorphism). In particular, show that, when τ is the algebra automorphism τ_q of $k[z]$ considered in IV.2, then δ is necessarily a scalar multiple of δ_q (see Exercise 4 in Chapter IV).

13. Show that any antilinear involution $*$ on a complex Lie algebra L such that $[x, y]^* = [y^*, x^*]$ for all $x, y \in L$ induces a Hopf $*$-algebra structure on $U(L)$.

14. Prove that there exists a unique Hopf $*$-algebra structure on $U(\mathfrak{sl}(2))$ such that $X^* = Y$, $Y^* = X$, and $H^* = -H$.

15. Find all Hopf $*$-algebra structures on $U(\mathfrak{sl}(2))$ up to equivalence, assuming that the ground field is the field of complex numbers.

V.9 Notes

There exist numerous textbooks on the theory of Lie algebras. See, for instance, [Bou60][Dix74][Hum72][Jac79][Ser65][Var74]. The content of this chapter is essentially taken from these sources. We found the proof of Theorem 4.6 in Serre's book [Ser65]. As for Definition 7.1, we took it from [Tak81]. Let us supplement the content of this chapter with the following remarks.

(*Free Lie algebras*) Let X be a set. Consider the smallest Lie subalgebra $\mathcal{L}(X)$ of the free algebra $k\{X\}$ containing X. Denote by i_X the injection of X into $\mathcal{L}(X)$. The free Lie algebra $\mathcal{L}(X)$ enjoys the following universal property: For any set-theoretic map f from X into a Lie algebra L, there exists a unique morphism of Lie algebras $\bar{f} : \mathcal{L}(X) \to L$ such that $f = \bar{f} \circ i_X$. It follows from this universal property, from Proposition I.2.1, and from Theorem 2.1 that there is an isomorphism of algebras

$$U(\mathcal{L}(X)) \cong k\{X\}.$$

A description of bases for $\mathcal{L}(X)$ may be found in [Bou60], Chap. 2. See also [Reu93].

(*Primitive elements of the enveloping bialgebra*) Any Lie algebra L is contained in the Lie algebra of primitive elements of its enveloping algebra. In characteristic zero, this embedding is an equality:

$$L = \mathrm{Prim}(U(L)).$$

When applied to free algebras, one gets $\mathcal{L}(X) \cong \mathrm{Prim}(k\{X\})$ (see [Bou60], Chap. 2).

(*Real forms*) A real form of a complex Lie algebra L is a real Lie subalgebra $L_{\mathbf{R}}$ of L such that the embedding of the complexification $L_{\mathbf{R}} \oplus iL_{\mathbf{R}}$ into L is an isomorphism of complex Lie algebras. Here i denotes a square root of -1. To any real form of L, one associates its conjugation, which is the antilinear involutive endomorphism of Lie algebras σ given by

$$\sigma(x + iy) = x - iy$$

for all $x, y \in L_{\mathbf{R}}$. Conversely, given any such involution of L, we obtain a real form by

$$L_{\mathbf{R}} = \{x \in L \mid \sigma(x) = x\}.$$

For any real form of L with conjugation σ, we define a Hopf $*$-algebra structure on the enveloping algebra $U(L)$ by $* = S \circ U(\sigma)$. In other words, we have $1^* = 1$ and

$$(x_1 \dots x_n)^* = (-1)^n \sigma(x_n) \dots \sigma(x_1)$$

for all $x_1, \dots, x_n \in L$. Conversely, suppose we have a Hopf $*$-algebra structure on the enveloping algebra $U(L)$. Since $*$ is a coalgebra morphism, it preserves the Lie subalgebra of primitive elements, which is L (we are in characteristic zero). It is easy to check that the subspace of all elements x of L such that $x = -x^*$ is a real form of L. We thus see that the real forms on a complex Lie algebra L are in one-to-one correspondence with the Hopf $*$-algebra structures on $U(L)$.

For instance, the real Lie subalgebra $\mathfrak{su}(2)$ of 2×2-matrices M in $\mathfrak{sl}(2)$ such that $M = -{}^t\overline{M}$ is a real form of $\mathfrak{sl}(2)$. The vectors $A = \frac{1}{2}(X - Y)$, $B = \frac{i}{2}(X + Y)$, iH form a real basis of $\mathfrak{su}(2)$ such that

$$[A, B] = C, \quad [B, C] = A, \quad [C, A] = B.$$

This proves that $\mathfrak{su}(2)$ is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of real anti-symmetric 3×3-matrices.

(*Duality*) Theorem 7.6 asserts the existence of a Hopf algebra morphism from $SL(2)$ to $U(\mathfrak{sl}(2))^*$. This morphism is actually an isomorphism from $SL(2)$ to the restricted dual $U(\mathfrak{sl}(2))^o$. This holds, more generally, for any simply-connected algebraic group in characteristic zero (see [Abe80] [Hoc81] [JS91b] [Swe69]).

Chapter VI

The Quantum Enveloping Algebra of $\mathfrak{sl}(2)$

The aim of Chapters VI–VII is to construct a Hopf algebra $U_q = U_q(\mathfrak{sl}(2))$ which is a one-parameter deformation of the enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$ investigated in Chapter V, and which is in duality with the Hopf algebra $SL_q(2)$ defined in Chapter IV. It will be our second main example of a quantum group. When the parameter q is not a root of unity, the algebra U_q has properties parallel to those of the enveloping algebra of $\mathfrak{sl}(2)$. In the present chapter we classify the simple finite-dimensional modules of U_q and determine its centre. We close the chapter with a few considerations on the case when q is a root of unity.

We assume throughout this chapter that the ground field k is the field of complex numbers.

VI.1 The Algebra $U_q(\mathfrak{sl}(2))$

Let us fix an invertible element q of k different from 1 and -1 so that the fraction $\frac{1}{q-q^{-1}}$ is well-defined. We introduce some notation.

For any integer n, set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}. \tag{1.1}$$

These q-analogues are more symmetric than the ones defined in IV.2, as shown by the relations

$$[-n] = -[n] \quad \text{and} \quad [m+n] = q^n[m] + q^{-m}[n]. \tag{1.2}$$

Observe that, if q is not a root of unity, then $[n] \neq 0$ for any non-zero integer. This is not so when q is a root of unity. In that case, denote by d its order, i.e., the smallest integer > 1 such that $q^d = 1$. Since we assume $q^2 \neq 1$, we must have $d > 2$. Define also

$$e = \begin{cases} d & \text{if } d \text{ is odd} \\ d/2 & \text{when } d \text{ is even.} \end{cases} \tag{1.3}$$

Let us agree that $d = e = \infty$ when q is not a root of unity. Now it is easy to check that

$$[n] = 0 \Longleftrightarrow n \equiv 0 \text{ modulo } e. \tag{1.4}$$

We also have the following versions of factorials and binomial coefficients. For integers $0 \leq k \leq n$, set $[0]! = 1$,

$$[k]! = [1][2]\dots[k] \tag{1.5}$$

if $k > 0$, and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}. \tag{1.6}$$

These q-analogues are related to those of IV.2 by

$$[n] = q^{-(n-1)}\,(n)_{q^2}, \quad [n]! = q^{-n(n-1)/2}(n)!_{q^2}, \tag{1.7}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{-k(n-k)} \begin{pmatrix} n \\ k \end{pmatrix}_{q^2}. \tag{1.8}$$

With this new notation we can rewrite Proposition IV.2.2 as follows. If x and y are variables subject to the relation $yx = q^2xy$, then we have ($n > 0$)

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}. \tag{1.9}$$

Definition VI.1.1. *We define $U_q = U_q(\mathfrak{sl}(2))$ as the algebra generated by the four variables E, F, K, K^{-1} with the relations*

$$KK^{-1} = K^{-1}K = 1, \tag{1.10}$$

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \tag{1.11}$$

and

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{1.12}$$

The rest of the section is devoted to a few elementary properties of U_q. The following lemma has an easy proof left to the reader.

Lemma VI.1.2. *There is a unique algebra automorphism of U_q such that*

$$\omega(E) = F, \quad \omega(F) = E, \quad \omega(K) = K^{-1}.$$

The automorphism ω is sometimes called the *Cartan automorphism.* We now state a q-analogue of Lemma V.3.1.

Lemma VI.1.3. *Let $m \geq 0$ and $n \in \mathbf{Z}$. The following relations hold in U_q:*

$$E^m K^n = q^{-2mn}\, K^n E^m, \quad F^m K^n = q^{2mn}\, K^n F^m,$$

$$\begin{aligned}[E, F^m] &= [m]\, F^{m-1} \frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q-q^{-1}} \\ &= [m]\, \frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q-q^{-1}} F^{m-1},\end{aligned}$$

$$\begin{aligned}[E^m, F] &= [m]\, \frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q-q^{-1}} E^{m-1} \\ &= [m]\, E^{m-1} \frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q-q^{-1}}.\end{aligned}$$

PROOF. The first two relations result trivially from Relations (1.11). The third one is proved by induction on m using

$$[E, F^m] = [E, F^{m-1}]F + F^{m-1}[E, F] = [E, F^{m-1}]F + F^{m-1}\frac{K - K^{-1}}{q-q^{-1}}$$

as in the proof of Lemma V.3.1. Applying the automorphism ω to the third relation, one gets the fourth one. □

We now describe a basis of U_q by showing that U_q is an iterated Ore extension. We refer to I.7–8 for information concerning Ore extensions.

Proposition VI.1.4. *The algebra U_q is Noetherian and has no zero divisors. The set $\{E^i F^j K^\ell\}_{i,j \in \mathbf{N};\, \ell \in \mathbf{Z}}$ is a basis of U_q.*

PROOF. Define $A_0 = k[K, K^{-1}]$. We shall construct two Ore extensions $A_1 \subset A_2$ such that A_2 is isomorphic to U_q. First, observe that the algebra A_0 has no zero divisors and is Noetherian as a quotient of a (Noetherian) two-variable polynomial algebra. The family $\{K^\ell\}_{\ell \in \mathbf{Z}}$ is a basis of A_0.

Consider the automorphism α_1 of A_0 determined by $\alpha_1(K) = q^2 K$ and the corresponding Ore extension $A_1 = A_0[F, \alpha_1, 0]$: the latter has a basis consisting of the monomials $\{F^j K^\ell\}_{j \in \mathbf{N},\, \ell \in \mathbf{Z}}$. An argument analogous to the one used to prove Lemma IV.4.2 shows that A_1 is the algebra generated by F, K, K^{-1} and the relation $FK = q^2 KF$.

We now build an Ore extension $A_2 = A_1[E, \alpha_1, \delta]$ from an automorphism α_1 and an α_1-derivation of A_1. The automorphism α_1 is defined by

$$\alpha_1(F^j K^\ell) = q^{-2\ell} F^j K^\ell. \tag{1.13}$$

Let us take as given for a moment that there exists an α_1-derivation δ such that

$$\delta(F) = \frac{K - K^{-1}}{q - q^{-1}} \quad \text{and} \quad \delta(K) = 0.$$

Then the following relations hold in A_2:

$$EK = \alpha_1(K)E + \delta(K) = q^{-2}KE$$

and

$$EF = \alpha_1(F)E + \delta(F) = FE + \frac{K - K^{-1}}{q - q^{-1}}.$$

From these one easily concludes that A_2 is isomorphic to U_q. It then results from Corollary I.7.2 and from Theorem I.8.3 that U_q has the required properties. □

It remains to prove the following technical lemma in order to complete the proof of Proposition 1.4.

Lemma VI.1.5. *Denote by $\delta(F)(K)$ the Laurent polynomial $\frac{K-K^{-1}}{q-q^{-1}}$, and set $\delta(K^\ell) = 0$ and*

$$\delta(F^j K^\ell) = \sum_{i=0}^{j-1} F^{j-1}\delta(F)(q^{-2i}K)K^\ell \tag{1.14}$$

when $j > 0$. Then δ extends to an α_1-derivation of A_1.

PROOF. We must check that, for all $j, m \in \mathbf{N}$ and all $\ell, n \in \mathbf{Z}$, we have

$$\delta(F^j K^\ell \cdot F^m K^n) = \alpha_1(F^j K^\ell)\delta(F^m K^n) + \delta(F^j K^\ell)F^m K^n. \tag{1.15}$$

Let us compute the right-hand side of (1.15) using (1.11), (1.13), and (1.14). We have

$$\begin{aligned}
&\alpha_1(F^j K^\ell)\delta(F^m K^n) + \delta(F^j K^\ell)F^m K^n \\
&= \sum_{i=0}^{m-1} q^{-2\ell}\, F^j K^\ell F^{m-1}\delta(F)(q^{-2i}K)K^n \\
&\qquad + \sum_{i=0}^{j-1} F^{j-1}\delta(F)(q^{-2i}K)K^\ell F^m K^n
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{m-1} q^{-2\ell-2\ell(m-1)}\, F^{j+m-1}\delta(F)(q^{-2i}K)K^{\ell+n} \\
&\qquad\qquad + \sum_{i=0}^{j-1} q^{-2\ell m}\, F^{m+j-1}\delta(F)(q^{-2i-2m}K)K^{\ell+n} \\
&= \sum_{i=0}^{m-1} q^{-2\ell m}\, F^{m+j-1}\delta(F)(q^{-2i}K)K^{\ell+n} \\
&\qquad\qquad + \sum_{i=m}^{j+m-1} q^{-2\ell m}\, F^{m+j-1}\delta(F)(q^{-2i}K)K^{\ell+n} \\
&= q^{-2\ell m} \Big(\sum_{i=0}^{j+m-1} F^{j+m-1}\delta(F)(q^{-2i}K)K^{\ell+n} \Big) \\
&= q^{-2\ell m}\delta(F^{j+m}K^{\ell+n}) \\
&= \delta(F^jK^\ell \cdot F^mK^n).
\end{aligned}
$$

□

VI.2 Relationship with the Enveloping Algebra of $\mathfrak{sl}(2)$

One expects to recover $U = U(\mathfrak{sl}(2))$ from U_q by setting $q = 1$. This is impossible with Definition 1.1. So we first have to give another presentation for U_q.

Proposition VI.2.1. *The algebra U_q is isomorphic to the algebra U'_q generated by the five variables E, F, K, K^{-1}, L and the relations*

$$KK^{-1} = K^{-1}K = 1, \tag{2.1}$$

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \tag{2.2}$$

$$[E, F] = L, \quad (q - q^{-1})L = K - K^{-1}, \tag{2.3}$$

$$[L, E] = q(EK + K^{-1}E), \quad [L, F] = -q^{-1}(FK + K^{-1}F). \tag{2.4}$$

Observe that, contrary to U_q, the algebra U'_q is defined for all values of the parameter q, in particular for $q = 1$. In some sense, it would have been better to proceed through the whole theory of the quantum enveloping algebra of $\mathfrak{sl}(2)$ with U'_q rather than with U_q, but the simpler presentation given in Section 1 is sufficient for our purposes.

PROOF. Set

$$\varphi(E) = E, \quad \varphi(F) = F, \quad \varphi(K) = K$$

and

$$\psi(E) = E, \quad \psi(F) = F, \quad \psi(K) = K, \quad \psi(L) = [E, F].$$

It is clear that φ gives rise to a well-defined morphism of algebras from U_q to U'_q. Let us show that $\psi : U'_q \to U_q$ is well-defined too. It suffices to check that the images under ψ of the defining Relations (2.1) hold in the algebra U_q. This is clearly true for Relations (2.1–2.2) and for $[E, F] = L$. For the remaining relation in (2.3) we have

$$(q - q^{-1})\psi(L) = (q - q^{-1})[E, F] = K - K^{-1}.$$

For the first relation in (2.4) we get

$$\begin{aligned} [\psi(L), \psi(E)] = [[E, F], E] &= \frac{1}{q - q^{-1}}[K - K^{-1}, E] \\ &= \frac{(q^2 - 1)EK + (q^2 - 1)K^{-1}E}{q - q^{-1}} \\ &= q\,(EK + K^{-1}E). \end{aligned}$$

One derives the last relation in a similar fashion.

The reader may now verify that φ and ψ are reciprocal algebra morphisms by checking the necessary relations on the generators. □

The relationship with the enveloping algebra U is given in the following statement.

Proposition VI.2.2. *If $q = 1$, we have*

$$U'_1 \cong U[K]/(K^2 - 1) \quad \textit{and} \quad U \cong U'_1/(K - 1).$$

Proof. It suffices to prove the first isomorphism. Now U'_1 has the following presentation: it is generated by E, F, K, K^{-1}, L and Relations (2.1–2.4) in which q has been replaced by 1, namely

$$KK^{-1} = K^{-1}K = 1, \tag{2.5}$$

$$KEK^{-1} = E, \quad KFK^{-1} = F, \tag{2.6}$$

$$[E, F] = L, \quad K - K^{-1} = 0, \tag{2.7}$$

$$[L, E] = (EK + K^{-1}E), \quad [L, F] = -(FK + K^{-1}F). \tag{2.8}$$

Relations (2.5–2.6) imply that K is central. Relation (2.7) yields $K^2 = 1$, which allows one to rewrite the Relations (2.8) as

$$[L, E] = 2EK, \quad [L, F] = -2FK. \tag{2.9}$$

We then get an isomorphism from U'_1 to $U[K]/(K^2 - 1)$ by sending E to XK, F to Y, K to K, and L to HK. □

In particular, the projection of U'_1 onto U is obtained by sending E to X, F to Y, K to 1, and L to H. One may use this projection to rederive certain relations in U (for instance, Lemma V.3.1) from their q-analogues in U'_q.

VI.3 Representations of U_q

We assume in this section that the complex parameter q is not a root of unity. Our aim is to determine all finite-dimensional simple U_q-modules under this assumption by closely following the methods of Section V.4.

For any U_q-module V and any scalar $\lambda \neq 0$, we denote by V^λ the subspace of all vectors v in V such that $Kv = \lambda v$. The scalar λ is called a *weight* of V if $V^\lambda \neq \{0\}$.

Lemma VI.3.1. *We have $EV^\lambda \subset V^{q^2\lambda}$ and $FV^\lambda \subset V^{q^{-2}\lambda}$.*

PROOF. For $v \in V^\lambda$ we have

$$K(Ev) = q^2 E(Kv) = q^2\lambda\, Ev \quad \text{and} \quad K(Fv) = q^{-2}F(Kv) = q^{-2}\lambda\, Fv.$$

□

Definition VI.3.2. *Let V be a U_q-module and λ be a scalar. An element $v \neq 0$ of V is a highest weight vector of weight λ if $Ev = 0$ and if $Kv = \lambda v$. A U_q-module is a highest weight module of highest weight λ if it is generated by a highest weight vector of weight λ.*

Proposition VI.3.3. *Any non-zero finite-dimensional U_q-module V contains a highest weight vector. Moreover, the endomorphisms induced by E and F on V are nilpotent.*

PROOF. Since $k = \mathbf{C}$ is algebraically closed and V is finite-dimensional, there exists a non-zero vector w and a scalar α such that $Kw = \alpha w$. If $Ew = 0$, the vector w is a highest weight vector and we are done. If not, let us consider the sequence of vectors $E^n w$ where n runs over the non-negative integers. According to Lemma 3.1, it is a sequence of eigenvectors with distinct eigenvalues; consequently, there exists an integer n such that $E^n w \neq 0$ and $E^{n+1}w = 0$. The vector $E^n w$ is a highest weight vector.

In order to show that the action of E on V is nilpotent, it suffices to check that 0 is the only possible eigenvalue of E. Now, if v is a non-zero eigenvector for E with eigenvalue $\lambda \neq 0$, then so is $K^n v$ with eigenvalue $q^{-2n}\lambda$. The endomorphism E would then have infinitely many distinct eigenvalues, which is impossible. The same argument works for F. □

Lemma VI.3.4. *Let v be a highest weight vector of weight λ. Set $v_0 = v$ and $v_p = \frac{1}{[p]!} F^p v$ for $p > 0$. Then*

$$Kv_p = \lambda q^{-2p}\, v_p, \quad Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}}\, v_{p-1}, \quad Fv_{p-1} = [p]\, v_p.$$

PROOF. These relations result from Lemma 1.3. □

We now determine all finite-dimensional simple U_q-modules.

Theorem VI.3.5. (a) *Let V be a finite-dimensional U_q-module generated by a highest weight vector v of weight λ. Then*

(i) *The scalar λ is of the form $\lambda = \varepsilon\, q^n$ where $\varepsilon = \pm 1$ and n is the integer defined by* $\dim(V) = n + 1$.

(ii) *Setting $v_p = F^p v/[p]!$, we have $v_p = 0$ for $p > n$ and, in addition, the set $\{v = v_0, v_1, \ldots, v_n\}$ is a basis of V.*

(iii) *The operator K acting on V is diagonalizable with the $(n+1)$ distinct eigenvalues $\{\varepsilon q^n, \varepsilon q^{n-2}, \ldots, \varepsilon q^{-n+2}, \varepsilon q^{-n}\}$.*

(iv) *Any other highest weight vector in V is a scalar multiple of v and is of weight λ.*

(v) *The module V is simple.*

(b) *Any simple finite-dimensional U_q-module is generated by a highest weight vector. Two finite-dimensional U-modules generated by highest weight vectors of the same weight are isomorphic.*

PROOF. (a) According to Lemma 3.4, the sequence $\{v_p\}_{p\geq 0}$ is a sequence of eigenvectors for K with distinct eigenvalues. Since V is finite-dimensional, there has to exist an integer n such that $v_n \neq 0$ and $v_{n+1} = 0$. The formulas of Lemma 3.4 then show that $v_m = 0$ for all $m > n$ and $v_m \neq 0$ for all $m \leq n$. By Lemma 3.4, we also have

$$0 = Ev_{n+1} = \frac{q^{-n}\lambda - q^n\lambda^{-1}}{q - q^{-1}} v_n.$$

Hence, $q^{-n}\lambda = q^n\lambda^{-1}$, which is equivalent to $\lambda = \pm q^n$. The rest of the proof of (i)–(iii) is as in the classical case (see Theorem V.4.4).

(iv) Let v' be another highest weight vector. It is an eigenvector for the action of K; hence, it is a scalar multiple of some vector v_i. But, again by Lemma 3.4, the vector v_i is killed by E if and only $i = 0$.

(v) Let V' be a non-zero U_q-submodule of V and let v' be a highest weight vector of V'. Then v' also is a highest weight vector for V. By (iv), v' has to be a non-zero scalar multiple of v. Therefore v is in V'. Since v generates V, we must have $V \subset V'$, which proves that V is simple.

(b) The proof is the same as for Theorem V.4.4 (b). □

Theorem 3.5 implies that, up to isomorphism, there exists a unique simple U_q-module of dimension $n+1$ and generated by a highest weight vector of weight εq^n. We denote this module by $V_{\varepsilon,n}$ and the corresponding morphism of algebras $U_q \to \mathrm{End}(V_{\varepsilon,n})$ by $\rho_{\varepsilon,n}$. Observe that the formulas of Lemma 3.4 may be rewritten as follows for $V_{\varepsilon,n}$:

$$Kv_p = \varepsilon\, q^{n-2p}\, v_p, \tag{3.1}$$

$$Ev_p = \varepsilon\, [n - p + 1]\, v_{p-1}, \tag{3.2}$$

and

$$Fv_{p-1} = [p]\, v_p. \tag{3.3}$$

As a special case, we have $V_{\varepsilon,0} = k$. The morphism $\rho_{\varepsilon,0}$ is given by

$$\rho_{\varepsilon,0}(K) = \varepsilon, \quad \rho_{\varepsilon,0}(E) = \rho_{\varepsilon,0}(F) = 0.$$

We shall see in VII.1 that $\rho_{\varepsilon,0}$ may be identified with the counit of a Hopf algebra structure on U_q. It will imply that the module $V_{1,0}$ is trivial and that any trivial U_q-module is isomorphic to a direct sum of copies of $V_{1,0}$. On the other hand, the module $V_{-1,0}$ is not trivial.

On the $(n+1)$-dimensional module $V_{\varepsilon,n}$, the generators E, F and K act by operators that can be represented on the basis $\{v_0, v_1, \ldots, v_n\}$ by the matrices

$$\rho_{\varepsilon,n}(E) = \varepsilon \begin{pmatrix} 0 & [n] & 0 & \cdots & 0 \\ 0 & 0 & [n-1] & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\rho_{\varepsilon,n}(F) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & [2] & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & [n] & 0 \end{pmatrix},$$

and

$$\rho_{\varepsilon,n}(K) = \varepsilon \begin{pmatrix} q^n & 0 & \cdots & 0 & 0 \\ 0 & q^{n-2} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q^{-n+2} & 0 \\ 0 & 0 & \cdots & 0 & q^{-n} \end{pmatrix}.$$

So far, we have built U_q-modules generated by highest weight vectors whose weights λ had special values. Let us now show that there exist highest weight modules with arbitrary highest weights.

Let us fix a scalar $\lambda \neq 0$. Consider an infinite-dimensional vector space $V(\lambda)$ with denumerable basis $\{v_i\}_{i \in \mathbf{N}}$. For $p \geq 0$, set

$$Kv_p = \lambda q^{-2p}\, v_p, \quad K^{-1}v_p = \lambda^{-1} q^{2p}\, v_p, \tag{3.4}$$

$$Ev_{p+1} = \frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}}\, v_p, \quad Fv_p = [p+1]\, v_{p+1} \tag{3.5}$$

and $Ev_0 = 0$.

Lemma VI.3.6. *Relations* (3.4–3.5) *define a U_q-module structure on $V(\lambda)$. The element v_0 generates $V(\lambda)$ as a U_q-module and is a highest weight vector of weight λ.*

PROOF. Immediate computations yield

$$\begin{aligned} KK^{-1}v_p &= v_p, & K^{-1}Kv_p &= v_p \\ KEK^{-1}v_p &= q^2Ev_p, & KFK^{-1}v_p &= q^{-2}Fv_p. \end{aligned}$$

We also have

$$\begin{aligned} [E,F]v_p &= \left([p+1]\,\frac{q^{-p}\lambda - q^p\lambda^{-1}}{q-q^{-1}} - [p]\,\frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}\right) v_p \\ &= \frac{q^{-2p}\lambda - q^{2p}\lambda^{-1}}{q-q^{-1}}\, v_p \\ &= \frac{K-K^{-1}}{q-q^{-1}} v_p. \end{aligned}$$

This proves that Relations (3.4–3.5) define a U_q-module structure on $V(\lambda)$.

Next, we have $Kv_0 = \lambda\, v_0$ and $Ev_0 = 0$, which means that v_0 is a highest weight vector of weight λ. Finally, (3.5) implies that $v_p = F^p v_0/[p]!$ for all p, which proves that $V(\lambda)$ is generated by v_0. □

By analogy with the classical case, the highest weight U_q-module $V(\lambda)$ is called the *Verma module* of highest weight λ. It enjoys the following universal property.

Proposition VI.3.7. *Any highest weight U_q-module V of highest weight λ is a quotient of the Verma module $V(\lambda)$.*

PROOF. Let v be a highest weight vector generating V. We define a linear map f from $V(\lambda)$ to V by $f(v_p) = 1/[p]!\, F^p v$. Lemma 3.4 implies that f is U_q-linear. Since $f(v_0) = v$ generates V, the map f is surjective. □

In particular, the simple finite-dimensional module $V_{\varepsilon,n}$ described above is a quotient of the Verma module $V(\varepsilon q^n)$. As a consequence, the module $V(\lambda)$ cannot be simple when λ is of the form $\pm q^n$ where n is a nonnegative integer.

VI.4 The Harish-Chandra Homomorphism and the Centre of U_q

Our next objective is to describe the centre Z_q of U_q in case q is not a root of unity. We assume this throughout this section.

We start by introducing a special central element of U_q. It is sometimes called the *quantum Casimir element*.

Proposition VI.4.1. *The element*

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

belongs to the centre of U_q.

PROOF. It suffices to check that C_q commutes with the generators K, E, F. The commutation with K is clear from $KEFK^{-1} = EF$. As for E, we have

$$EC_q = EFE + E\frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} = EFE + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}E = C_qE.$$

Similar argument gives the result for F. □

Let U_q^K be the subalgebra of U_q of all elements commuting with K.

Lemma VI.4.2. *An element of* U_q *belongs to* U_q^K *if and only if it is of the form*

$$\sum_{i\geq 0} F^i P_i E^i \tag{4.1}$$

where $P_0, P_1, \ldots$ *are elements of* $k[K, K^{-1}]$.

PROOF. This is a consequence of the fact that $\{F^iK^\ell E^j\}_{i,j\in\mathbf{N};\,\ell\in\mathbf{Z}}$ is a basis of U_q and that $K(F^iK^\ell E^j)K^{-1} = q^{2(j-i)}\,F^iK^\ell E^j$. □

Let us consider the left ideal $I = U_qE \cap U_q^K$ of U_q^K.

Lemma VI.4.3. *We have* $I = FU_q \cap U_q^K$ *and* $U_q^K = k[K, K^{-1}] \oplus I$.

PROOF. Let $u = \sum_{i\geq 0} F^iP_iE^i$ be an element of U_q^K. If u also lies in U_qE, then $P_0 = 0$. Hence, u belongs to $FU_q \cap U_q^K$ and conversely. Since the form (4.1) is unique for any element of U_q^K, we get the desired direct sum. □

It results from $I = FU_q \cap U_q^K$ that I is a two-sided ideal and that the projection φ from U_q^K onto $k[K, K^{-1}]$ is a morphism of algebras. The map φ is called the *Harish-Chandra homomorphism*. It permits one to express the action of the centre Z_q on a highest weight module.

Proposition VI.4.4. *Let* V *be a highest weight* U_q*-module with highest weight* λ. *Then, for any central element* z *of* U_q *and any* $v \in V$, *we have*

$$zv = \varphi(z)(\lambda)v.$$

Recall that $\varphi(z)$ is a Laurent polynomial in K and that $\varphi(z)(\lambda)$ is its value at λ.

PROOF. Let v_0 be a highest weight vector generating V and z a central element of U_q. The element z can be written in the form

$$z = \varphi(z) + \sum_{i>0} F^i P_i E^i.$$

Since $Ev_0 = 0$ and $Kv_0 = \lambda v_0$, we get $zv_0 = \varphi(z)(\lambda)v_0$. If v is an arbitrary element of V, we have $v = xv_0$ for some x in U_q; hence,

$$zv = zxv_0 = xzv_0 = \varphi(z)(\lambda)xv_0 = \varphi(z)(\lambda)v.$$

□

Example 1. The definition of the central element C_q shows that

$$\varphi(C_q) = \frac{qK - q^{-1}K^{-1}}{(q-q^{-1})^2}. \tag{4.2}$$

Consequently, C_q acts on a highest weight module of highest weight λ as the multiplication by the scalar

$$\frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2}. \tag{4.3}$$

Let us now prove that the restriction of the Harish-Chandra homomorphism to the centre Z_q is injective.

Lemma VI.4.5. *Let $z \in Z_q$. If $\varphi(z) = 0$, then $z = 0$.*

PROOF. Let z be an element in the centre such that $\varphi(z) = 0$. Assume z non-zero; it can be written as $z = \sum_{i=k}^{\ell} F^i P_i E^i$ where $0 < k \leq \ell$ are integers and $P_k, \dots, P_\ell$ are non-zero Laurent polynomials in K. Consider a Verma module $V(\lambda)$ whose highest weight is not a power of q. Then Relations (3.4–3.5) show that $Ev_p = 0$ if and only if $p = 0$. Let us apply z to the vector v_k of $V(\lambda)$. On the one hand, Proposition 4.4 implies that $zv_k = \varphi(z)(\lambda)v_k = 0$; on the other, we get

$$zv_k = F^k P_k E^k v_k = c\ P_k(\lambda)v_k,$$

where c is a non-zero constant. It follows that $P_k(\lambda) = 0$. As a consequence, we have a non-zero polynomial P_k with infinitely many roots; hence a contradiction. □

Verma modules will also allow us to prove a symmetry relation for the polynomials $\varphi(z)$. Before we state this, let us introduce the following notation. For any Laurent polynomial P in $k[K, K^{-1}]$, denote by $\widetilde{P}$ the polynomial defined by the change of variable

$$\widetilde{P}(\lambda) = P(q^{-1}\lambda).$$

Lemma VI.4.6. *For any element z in the centre Z_q, we have*

$$\widetilde{\varphi(z)}(\lambda) = \widetilde{\varphi(z)}(\lambda^{-1}).$$

PROOF. For any integer $n > 0$, consider the Verma module $V(q^{n-1})$. By (3.5) we have

$$Ev_n = \frac{q^{-(n-1)}q^{n-1} - q^{n-1}q^{-(n-1)}}{q - q^{-1}} v_n = 0.$$

Thus, v_n is a highest weight vector of weight $q^{n-1-2n} = q^{-n-1}$. By Proposition 4.4, a central element z acts on the module generated by v_n as the multiplication by the scalar $\varphi(z)(q^{-n-1})$; but, since v_n is in $V(q^{n-1})$, the element z also acts as the scalar $\varphi(z)(q^{n-1})$. In other words, we have

$$\widetilde{\varphi(z)}(q^n) = \widetilde{\varphi(z)}(q^{-n}).$$

One concludes by observing that the powers of q form an infinite sequence of distinct scalars. □

We pause to record the following lemma.

Lemma VI.4.7. *Any Laurent polynomial of $k[K, K^{-1}]$ satisfying the relation $P(\lambda) = P(\lambda^{-1})$ is a polynomial in $K + K^{-1}$.*

PROOF. We proceed by induction on the degree of the polynomial. If the degree is 0, the statement holds trivially. Let us suppose that the lemma is proved for all degrees $< n$ and let P be a Laurent polynomial of degree n such that $P(\lambda) = P(\lambda^{-1})$. Then we may write P in the form

$$P(K) = c\,(K^n + K^{-n}) + \text{ (terms of degree } < n).$$

Now,

$$K^n + K^{-n} = (K + K^{-1})^n + \text{ (terms of degree } < n).$$

One concludes by applying the induction hypothesis. □

We are ready to state the main theorem.

Theorem VI.4.8. *When q is not a root of unity, the centre Z_q of U_q is a polynomial algebra generated by the element C_q. The restriction of the Harish-Chandra homomorphism to Z_q is an isomorphism onto the subalgebra of $k[K, K^{-1}]$ generated by $qK + q^{-1}K^{-1}$.*

PROOF. We already know that the restriction of φ to the centre is injective. We are left with determining its image. By Lemmas 4.6 and 4.7, the latter is contained in the subalgebra of $k[K, K^{-1}]$ generated by $qK + q^{-1}K^{-1}$. Consider the central element C_q defined above. By (4.2) we know that

$$\varphi(C_q) = \frac{1}{(q - q^{-1})^2}(qK + q^{-1}K^{-1}),$$

which proves that the image of Z_q is the whole subalgebra and that C_q generates the centre. The latter is a polynomial algebra because the powers of $qK + q^{-1}K^{-1}$ are linearly independent for obvious reasons of degree.
□

VI.5 Case when q is a Root of Unity

Our next aim is to find all finite-dimensional simple U_q-modules in the case when the complex parameter q is a root of unity $\neq \pm 1$. As we shall quickly see, the situation is much more complicated than in the generic case when q is not a root of unity. Define the order d of q and the integer e as in (1.3). Recall that $[e] = 0$.

The following theorem asserts that the simple U_q-modules of sufficiently low dimensions are the same as in the generic case.

Proposition VI.5.1. *Any simple non-zero U_q-module of dimension $< e$ is isomorphic to a module of the form $V_{\varepsilon,n}$ where $\varepsilon = \pm 1$ and $0 \leq n < e-1$.*

The modules $V_{\varepsilon,n}$ have been described in Section 3.

PROOF. The proof is exactly the same as the proof of Theorem 3.5. One uses the fact that $1, q^2, \ldots, q^{2n}$ are distinct scalars when $n < e$. □

The first big difference with the generic case appears in the following statement.

Proposition VI.5.2. *There is no simple finite-dimensional U_q-module of dimension $> e$.*

Before we prove this proposition, we state two lemmas. The first one implies that the centre of U_q is much bigger when q is a root of unity than when it is not. The second one is a special case of a general statement on finite-dimensional modules.

Lemma VI.5.3. *The elements E^e, F^e, and K^e belong to the centre of U_q.*

PROOF. This is a consequence of Relation (1.1) and of Lemma 1.3. Indeed, E^e commutes with K because $q^{2e} = 1$ and with F because $[e] = 0$. Similar arguments can be applied to F^e and to K^e. □

Lemma VI.5.4. *Let z be a central element of U_q. Then z acts on any finite-dimensional simple U_q-module V by multiplication by a scalar.*

PROOF. Let u be the endomorphism induced by the action of z on V: it is U_q-linear because z is central. Since V is finite-dimensional, the endomorphism u has an eigenvalue λ. Consider the U_q-linear endomorphism

$u - \lambda \mathrm{id}_V$. Its kernel K is a submodule of the simple module V. Since $K \neq \{0\}$, we must have $K = V$. □

Proof of Proposition 5.2. Let us assume that there exists a simple finite-dimensional module V of dimension $> e$. We shall prove that V has a non-zero submodule of dimension $\leq e$. Hence, a contradiction.

(a) Suppose there exists a non-zero eigenvector $v \in V$ for the action of K such that $Fv = 0$. We claim that the subspace V' generated by $v, Ev, \ldots, E^{e-1}v$ is a submodule of dimension $\leq e$. It is enough to check that V' is stable under the action of the generators E, F, K. This is clear for K. Let us check that V' is stable under E. The vector $E(E^p v) = E^{p+1}v$ belongs to V' if $p < e-1$. If $p = e-1$, we have

$$E(E^{e-1}v) = E^e v = c_1 v$$

where c_1 is a scalar in view of Lemmas 5.3 and 5.4. Finally, V' is stable under F thanks to $Fv = 0$ and Lemma 1.3.

(b) Now, suppose there is no non-zero eigenvector $v \in V$ for the action of K such that $Fv = 0$. Let v be a non-zero eigenvector for the action of K. We have $Fv \neq 0$. We claim that the subspace V'' generated by $v, Fv, \ldots, F^{e-1}v$ is also a submodule of dimension $\leq e$. Again, V'' is clearly stable under K. It is also stable under F since the vector $F(F^p v) = F^{p+1}v$ belongs to V'' if $p < e-1$. If $p = e-1$, we have

$$F(F^{e-1}v) = F^e v = c_2 v$$

where c_2 is another scalar, again in view of Lemmas 5.3 and 5.4. The scalar c_2 is not zero; otherwise, there would exist an integer $p < e$ such that $F^p v$ would be an eigenvector for K killed by F, which would contradict our assumption.

In order to check that V'' is stable under E, we use the central element C_q defined in Section 4. By Lemma 5.4, it acts on V by multiplication by a scalar c_3. By definition of C_q we get for $p > 0$

$$\begin{aligned} E(F^p v) &= EF(F^{p-1}v) \\ &= \Big(C_q - \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}\Big)(F^{p-1}v) \\ &= c_3\, F^{p-1}v - \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}(F^{p-1}v), \end{aligned}$$

which shows that $E(F^p v)$ sits in V''. When $p = 0$, we use the same argument after observing that $v = c_2^{-1}F^e v$. □

It remains now to find the simple U_q-modules in dimension e. We shall content ourselves with their descriptions, omitting proofs. First, we give two families of e-dimensional modules.

The first one depends on three complex numbers λ, a, and b. We assume that $\lambda \neq 0$. Consider an e-dimensional vector space equipped with a basis $\{v_0, \ldots, v_{e-1}\}$. For $0 \leq p < e-1$, set

$$Kv_p = \lambda q^{-2p} v_p, \tag{5.1}$$

$$Ev_{p+1} = \Big(\frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}}\,[p+1] + ab\Big)\, v_p, \tag{5.2}$$

$$Fv_p = v_{p+1}, \tag{5.3}$$

and $Ev_0 = av_{e-1}$, $Fv_{e-1} = bv_0$, and $Kv_{e-1} = \lambda q^{-2(e-1)} v_{e-1}$. These formulas endow this vector space with a U_q-module structure, denoted $V(\lambda, a, b)$.

The second family depends on two scalars $\mu \neq 0$ and c. We let E, F, K act on a vector space with basis $\{v_0, \ldots, v_{e-1}\}$ by

$$Kv_p = \mu q^{2p} v_p, \tag{5.4}$$

$$Fv_{p+1} = \frac{q^{-p}\mu^{-1} - q^p\mu}{q - q^{-1}}\,[p+1]\, v_p, \tag{5.5}$$

$$Ev_p = v_{p+1} \tag{5.6}$$

if $0 \leq p < e-1$ and by $Fv_0 = 0$, $Ev_{e-1} = cv_0$, and $Kv_{e-1} = \mu q^{-2} v_{e-1}$ otherwise. These formulas determine another U_q-module, denoted $\widetilde{V}(\mu, c)$.

The following theorem which we admit without proof closes the list of all simple finite-dimensional U_q-modules when q is a root of unity.

Theorem VI.5.5. *Any simple U_q-module of dimension e is isomorphic to a module of the following list:*

(i) $V(\lambda, a, b)$ *with* $b \neq 0$,

(ii) $V(\lambda, a, 0)$ *where* λ *is not of the form* $\pm q^{j-1}$ *for any* $1 \leq j \leq e-1$,

(iii) $\widetilde{V}(\pm q^{1-j}, c)$ *with* $c \neq 0$ *and* $1 \leq j \leq e-1$.

It should be added that all modules $V(\lambda, a, b)$ and $\widetilde{V}(\mu, c)$, including the ones that are not in the list of Theorem 5.5, are indecomposable.

In the situation under investigation, the algebra U_q possesses an interesting finite-dimensional quotient-algebra.

Definition VI.5.6. *The algebra $\overline{U}_q$ is the quotient of the algebra U_q by the two-sided ideal generated by the central elements E^e, F^e, and $K^e - 1$.*

It is not difficult to convince oneself that a finite-dimensional $\overline{U}_q$-module is simple [resp. indecomposable] if and only if it is simple [resp. indecomposable] as a U_q-module. Therefore, in order to have a complete list of all simple finite-dimensional $\overline{U}_q$-modules, it is enough to determine the simple finite-dimensional U_q-modules on which E^e, F^e and $K^e - 1$ act by 0. This is done without any difficulty using Theorem 5.5 and Relations (5.1–5.6). We get the following:

Theorem VI.5.7. *Any non-zero simple finite-dimensional $\overline{U}_q$-module is isomorphic to a module of the form*

(i) $V_{1,n}$ *with* $0 \leq n < e-1$, *or* $V(q^{-1},0,0)$ *if* $d = e$ *is odd,*

(ii) $V_{\pm 1,n}$ *with* n *even* $< e-1$ *if* d *and* e *are even,*

(iii) $V_{1,n}$ *with* n *even* $< e-1$, *or* $V_{-1,n}$ *with* n *odd* $< e-1$, *or* $V(-q^{-1},0,0)$ *if* d *is even and* e *is odd.*

We shall need the following proposition in IX.6.

Proposition VI.5.8. *The finite set* $\{E^iF^jK^\ell\}_{0\leq i,j,\ell\leq e-1}$ *is a basis of* $\overline{U}_q$.

PROOF. Thanks to the commutation relations between the generators, we are reduced to showing that $\{F^jK^\ell E^i\}_{0\leq i,j,\ell\leq e-1}$ is a basis of $\overline{U}_q$. By Proposition 1.4 it is clear that this set generates $\overline{U}_q$. It remains to check that it is free. To this end, we introduce an intermediate quotient-algebra $\widetilde{U}_q$ defined by $\widetilde{U}_q = U_q/(E^e, F^e)$ and we show first that the set $\{F^jK^\ell E^i\}_{0\leq i,j\leq e-1;\,\ell\in\mathbf{Z}}$ is a basis of $\widetilde{U}_q$. Let us prove this claim. Again, it is enough to prove that the set is free.

Let us consider a linear relation of the form

$$Z = \sum_{0\leq i,j\leq e-1;\, r\leq \ell\leq s} \alpha_{ij\ell}\, F^jK^\ell E^i = 0. \tag{5.7}$$

We let it act on the vectors v_p of the canonical basis of the module $V(\lambda,0,0)$ (check that this module is killed by E^e and F^e, but in general not by K^e-1). We assume that λ is neither zero, nor a root of unity. Since $Ev_0 = 0$, we have

$$Zv_0 = \sum_{0\leq i,j\leq e-1;\, r\leq \ell\leq s} \alpha_{0j\ell}\, F^j\lambda^\ell v_0 = \sum_{0\leq i,j\leq e-1;\, r\leq \ell\leq s} \alpha_{0j\ell}\, \lambda^\ell v_j = 0. \tag{5.8}$$

Since $v_0, \ldots, v_{e-1}$ are linearly independent, Relation (5.8) implies that

$$\sum_{\ell=0}^{s-r} \alpha_{0,j,\ell+r}\, \lambda^\ell = 0 \tag{5.9}$$

for all j. Writing (5.9) for $s-r+1$ distinct complex numbers λ, we get a linear system whose determinant is a non-zero Vandermonde determinant. Consequently, $\alpha_{0j\ell} = 0$ for all j and ℓ. Next, we apply Z to the vector v_1. The hypothesis made on λ implies that Ev_1 is a non-zero multiple of v_0; hence we get $\alpha_{1j\ell} = 0$ for all j and ℓ by the same argument as above. Applying Z successively to the vectors v_2 up to v_{e-1}, one shows that all coefficients $\alpha_{ij\ell}$ vanish.

Now that we have secured a basis for $\widetilde{U}_q$, we prove Proposition 5.8. We consider a linear relation of the form

$$\sum_{0\leq i,j,\ell\leq e-1} \alpha_{ij\ell}\, F^jK^\ell E^i = 0. \tag{5.10}$$

in $\overline{U}_q$. Denoting by Z the element of $\widetilde{U}_q$ represented by the left-hand side of (5.10), we see Z belongs to the two-sided ideal of $\widetilde{U}_q$ generated by $K^e - 1$. Hence, we have $Z = (K^e - 1)Y$ where $Y = \sum_{0\leq i,j\leq e-1;\, \ell\in\mathbf{Z}} \beta_{ij\ell} F^j K^\ell E^i$. Since K^e is central, we get

$$Z = \sum_{0\leq i,j\leq e-1;\, \ell\in\mathbf{Z}} \beta_{ij\ell} F^j K^{\ell+e} E^i - \sum_{0\leq i,j\leq e-1;\, \ell\in\mathbf{Z}} \beta_{ij\ell} F^j K^\ell E^i. \tag{5.11}$$

Assume $Z \neq 0$, hence $Y \neq 0$. Denote by $d(Z)$ [resp. by $\delta(Z)$] the degree in K [resp. the degree in K^{-1}] of the non-zero element Z of $\widetilde{U}_q$ written in the above-mentioned basis. Relation (5.11) implies that

$$d(Z) = d(Y) + e \quad \text{and} \quad \delta(Z) = \delta(Y). \tag{5.12}$$

Now, by definition of Z, we have

$$0 \leq \delta(Z) \leq d(Z) < e. \tag{5.13}$$

Combining (5.12–5.13), we get $d(Y) < 0 \leq \delta(Z) = \delta(Y)$. This is impossible; hence, $Z = 0$. $\square$

VI.6 Exercises

1. Compute $[E^i, F^j]$ in U_q.

2. (*Simple Verma modules*) Assume that q is not a root of unity. Show that the Verma module $V(\lambda)$ is simple if and only if λ is not of the form $\lambda = \pm q^n$ with $n \in \mathbf{N}$.

3. Prove Theorem 5.5.

4. Prove Theorem 5.7.

5. Assume that q is of finite order $d > 2$. Let λ be a non-zero scalar. Consider the Verma module $V(\lambda)$. Show that $F^e v_0$ generates a highest weight submodule of weight λ and that the quotient $\overline{V(\lambda)}$ of $V(\lambda)$ by this submodule is a simple U_q-module of dimension e.

6. Under what conditions on λ, a, and b is the module $V(\lambda, a, b)$ of Section 5 a highest weight module?

VI.7 Notes

The algebra $U_q = U_q(\mathfrak{sl}(2))$ is due to Kulish and Reshetikhin [KR81]. Drinfeld [Dri85][Dri87] and Jimbo [Jim85] independently generalized this construction by defining an algebra $U_q(\mathfrak{g})$ for any complex semisimple Lie algebra (more generally, for any symmetrizable Kac-Moody Lie algebra) $\mathfrak{g}$.

A complex semisimple Lie algebra is determined by its so-called Cartan matrix $(a_{ij})_{1\leq i,j\leq \ell}$ (see [Bou60], chap. 8, [Hum72], [Ser65]). In case $\mathfrak{g}$ is of type A, D or E, the Cartan matrix $(a_{ij})_{1\leq i,j\leq \ell}$ is symmetric, positive definite with integral coefficients such that $a_{ii} = 2$ and $a_{ij} = 0, -1$ if $i \neq j$. Then Drinfeld-Jimbo's algebra $U_q(\mathfrak{g})$ can be presented as follows: it is the algebra generated by $(E_i, F_i, K_i, K_i^{-1})_{1\leq i\leq \ell}$ and the relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \qquad K_iK_j = K_jK_i,$$

$$K_iE_jK_i^{-1} = q^{a_{ij}}E_j, \qquad K_iF_jK_i^{-1} = q^{-a_{ij}}F_j,$$

$$[E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$E_iE_j = E_jE_i \quad \text{and} \quad F_iF_j = F_jF_i \qquad \text{if } a_{ij} = 0,$$

$$E_i^2E_j - [2]\ E_iE_jE_i + E_jE_i^2 = 0 \quad \text{and} \quad F_i^2F_j - [2]\ F_iF_jF_i + F_jF_i^2 = 0,$$

if $a_{ij} = -1$. When $a_{ij} = 0$ if $|i-j| > 1$ and $a_{ij} = -1$ if $|i-j| = 1$, we obtain $U_q(\mathfrak{sl}(\ell+1))$. A presentation of $U_q(\mathfrak{g})$ corresponding to the algebra U_q' was given by Lusztig [Lus89].

The algebra $U_q(\mathfrak{g})$ possesses a Poincaré-Birkhoff-Witt-type basis ([Lus90a] [Lus90b] [Ros89] [Yam89]) and a quantum Casimir element (see [Jim85]). Lusztig [Lus88] and Rosso [Ros88] proved that, when q is not a root of unity, any finite-dimensional simple $\mathfrak{g}$-module could be deformed into a finite-dimensional simple $U_q(\mathfrak{g})$-module. A quantum Harish-Chandra homomorphism was constructed by [CK90] [JL92] [Ros90] [Tan90].

Numerous authors have investigated the algebras $U_q(\mathfrak{g})$ and their representations when q is a root of unity, for instance, [CK90] [CKP92] [DJMM91] [Lus89] [Lus90b] [RA89] [Sal90] (see also [Ros92]). We refer to [CK90] [CKP92] for a description of the centre of U_q: it is a finite extension of the polynomial subalgebra generated by E^e, F^e and K^e. Contrary to the generic case, there is a bound for the dimension of the finite-dimensional simple U_q-modules. For $\mathfrak{g} = \mathfrak{sl}(2)$, this bound is e (see Proposition 5.2).

We owe the treatment of Section 5 (including statements and proofs) to R. Berger.

Chapter VII

A Hopf Algebra Structure on $U_q(\mathfrak{sl}(2))$

We assume in this chapter that the field k is the field of complex numbers and that q is *not* a root of unity. We now equip the algebra $U_q = U_q(\mathfrak{sl}(2))$ defined in Chapter VI with a Hopf algebra structure. Then we prove that any finite-dimensional U_q-module is a direct sum of the simple modules described in VI.3. We show later that U_q acts naturally on the quantum plane of IV.1 and that it is in duality with the Hopf algebra $SL_q(2)$ of Chapter IV. We shall also build scalar products on the simple finite-dimensional U_q-modules. We describe the quantum Clebsch-Gordan formula and give the main properties of the quantum Clebsch-Gordan coefficients.

VII.1 Comultiplication

We resume the notation of the previous chapter. Set

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \tag{1.1}$$

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \tag{1.2}$$

$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = \varepsilon(K^{-1}) = 1, \tag{1.3}$$

and

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K. \tag{1.4}$$

Proposition VII.1.1. *Relations* (1.1–1.4) *endow U_q with a Hopf algebra structure.*

PROOF. (a) We first show that Δ defines a morphism of algebras from U_q into $U_q \otimes U_q$. It is enough to check that

$$\Delta(K)\Delta(K^{-1}) = \Delta(K^{-1})\Delta(K) = 1, \tag{1.5}$$

$$\Delta(K)\Delta(E)\Delta(K^{-1}) = q^2\Delta(E), \tag{1.6}$$

$$\Delta(K)\Delta(F)\Delta(K^{-1}) = q^{-2}\Delta(F), \tag{1.7}$$

$$[\Delta(E), \Delta(F)] = \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}. \tag{1.8}$$

Relations (1.5) are clear. As for (1.6), we have

$$\begin{aligned}
\Delta(K)\Delta(E)\Delta(K^{-1}) &= (K \otimes K)(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1}) \\
&= 1 \otimes KEK^{-1} + KEK^{-1} \otimes K \\
&= q^2\,(1 \otimes E + E \otimes K) \\
&= q^2\Delta(E).
\end{aligned}$$

Relation (1.7) is proved in a similar way. Finally, for (1.8) we have

$$\begin{aligned}
&[\Delta(E), \Delta(F)] \\
&= (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) \\
&\qquad - (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K) \\
&= K^{-1} \otimes EF + F \otimes E + EK^{-1} \otimes KF + EF \otimes K \\
&\qquad - K^{-1} \otimes FE - K^{-1}E \otimes FK - F \otimes E - FE \otimes K \\
&= K^{-1} \otimes [E, F] + [E, F] \otimes K \\
&= \frac{K^{-1} \otimes (K - K^{-1}) + (K - K^{-1}) \otimes K}{q - q^{-1}} \\
&= \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}.
\end{aligned}$$

(b) Next, we check that Δ is coassociative. It suffices to do it on the four generators. We give a sample calculation for E. On the one hand, we have

$$(\Delta \otimes \mathrm{id})\Delta(E) = (\Delta \otimes \mathrm{id})(1 \otimes E + E \otimes K) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K.$$

On the other hand, we have

$$(\mathrm{id} \otimes \Delta)\Delta(E) = (\mathrm{id} \otimes \Delta)(1 \otimes E + E \otimes K) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K,$$

which is the same.

(c) It is easy to check that ε defines a morphism of algebras from U_q onto k and satisfies the counit axiom.

(d) It remains to see that S defines an antipode for U_q. We have first to check that S is a morphism of algebras from U_q into U_q^{op}, namely that the following four relations hold:

$$S(K^{-1})S(K) = S(K)S(K^{-1}) = 1, \tag{1.9}$$

$$S(K^{-1})S(E)S(K) = q^2 S(E), \tag{1.10}$$

$$S(K^{-1})S(F)S(K) = q^{-2} S(F), \tag{1.11}$$

$$[S(F), S(E)] = \frac{S(K) - S(K^{-1})}{q - q^{-1}}. \tag{1.12}$$

We give the computations for (1.10) and (1.12). We have

$$S(K^{-1})S(E)S(K) = -K(EK^{-1})K^{-1} = -q^2 EK^{-1} = q^2 S(E)$$

and

$$\begin{aligned} [S(F), S(E)] &= KFEK^{-1} - EK^{-1}KF = [F, E] \\ &= \frac{K^{-1} - K}{q - q^{-1}} = \frac{S(K) - S(K^{-1})}{q - q^{-1}}. \end{aligned}$$

To conclude that S is an antipode, we appeal to Lemma III.3.6. It suffices to check that the relations

$$\sum_{(x)} x' S(x'') = \sum_{(x)} S(x')x'' = \varepsilon(x)1$$

hold when x is any of the generators E, F, K, K^{-1}. This verification is left to the reader. □

We have thus defined a Hopf algebra that is *neither commutative nor cocommutative.* Observe also that the square of the antipode is not the identity (when $q^2 \neq 1$). Nevertheless, it is an inner automorphism, as expressed by the following statement.

Proposition VII.1.2. *We have $S^2(u) = KuK^{-1}$ for any $u \in U_q$.*

PROOF. In effect, we have

$$S^2(E) = q^2 E = KEK^{-1}, \quad S^2(F) = q^{-2} F = KFK^{-1},$$

and $S^2(K) = K$. □

We thus get, just as in Chapter IV, examples of Hopf algebras whose antipodes have a finite order $2N$ for any integer $N > 1$; it suffices to take any primitive $2N$-th root of unity as the parameter q.

The algebra U'_q of VI.2 can be endowed with a Hopf algebra structure such that the isomorphism $\varphi : U_q \to U'_q$ of Proposition VI.2.1 preserves the Hopf algebra structures. In addition to Relations (1.1–1.4), it suffices to set

$$\Delta(L) = K^{-1} \otimes L + L \otimes K, \quad \varepsilon(L) = 0, \quad S(L) = -L. \tag{1.13}$$

It follows easily that the isomorphism $U(\mathfrak{sl}(2)) \cong U'_1/(K-1)$ is an isomorphism of Hopf algebras. In other words, the Hopf algebra structure of U_q extends the Hopf algebra structure of the enveloping algebra $U(\mathfrak{sl}(2))$.

We end this section by expressing the comultiplication of U_q in the basis described in Proposition VI.1.4.

Proposition VII.1.3. *For all $i, j \in \mathbf{N}$ and $\ell \in \mathbf{Z}$ we have*

$$\begin{aligned} \Delta(E^i F^j K^\ell) &= \sum_{r=0}^{i} \sum_{s=0}^{j} q^{r(i-r)+s(j-s)-2(i-r)(j-s)} \begin{bmatrix} i \\ r \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} \\ &\qquad \times E^{i-r} F^s K^{\ell-(j-s)} \otimes E^r F^{j-s} K^{\ell+(i-r)}. \end{aligned}$$

PROOF. First observe that

$$\begin{aligned} \Delta(E^i F^j K^\ell) &= \Delta(E)^i \Delta(F)^j \Delta(K)^\ell \\ &= (1 \otimes E + E \otimes K)^i (K^{-1} \otimes F + F \otimes 1)^j (K^\ell \otimes K^\ell). \end{aligned}$$

Now,

$$(E \otimes K)(1 \otimes E) = q^2\,(1 \otimes E)(E \otimes K)$$

and

$$(K^{-1} \otimes F)(F \otimes 1) = q^2\,(F \otimes 1)(K^{-1} \otimes F).$$

Applying Relation (VI.1.9), we get

$$\Delta(E)^i = \sum_{r=0}^{i} q^{r(i-r)} \begin{bmatrix} i \\ r \end{bmatrix} E^{i-r} \otimes E^r K^{i-r}$$

and

$$\Delta(F)^j = \sum_{s=0}^{j} q^{s(j-s)} \begin{bmatrix} j \\ s \end{bmatrix} F^s K^{-(j-s)} \otimes F^{j-s}.$$

One concludes with (VI.1.11). □

VII.2 Semisimplicity

In this section we shall prove that any finite-dimensional U_q-module is the direct sum of simple U_q-modules when q is not a root of unity, which we assume in this chapter. Let us start with a technical lemma on the simple modules $V_{\varepsilon,n}$ of VI.3.

Lemma VII.2.1. *There exists an element C of the centre of U_q acting by 0 on $V_{\varepsilon,0}$ and by a non-zero scalar on $V_{\varepsilon',n}$ when n is an integer > 0 and $\varepsilon, \varepsilon' = \pm 1$.*

PROOF. Define

$$C = C_q - \varepsilon \frac{q+q^{-1}}{(q-q^{-1})^2}$$

where C_q is the central element introduced in VI.4. By (VI.4.3), C acts on $V_{\varepsilon,0}$ by

$$\varepsilon \frac{q+q^{-1}}{(q-q^{-1})^2} - \varepsilon \frac{q+q^{-1}}{(q-q^{-1})^2} = 0,$$

and on $V_{\varepsilon',n}$ by

$$\varepsilon' \frac{q^{n+1}+q^{-(n+1)}}{(q-q^{-1})^2} - \varepsilon \frac{q+q^{-1}}{(q-q^{-1})^2}.$$

We have to show that the latter is not 0. If it were, we would have

$$q^{2n+2} - \varepsilon\varepsilon' q^{n+2} - \varepsilon\varepsilon' q^n + 1 = 0,$$

or, equivalently,

$$(q^{n+2} - \varepsilon\varepsilon')(q^n - \varepsilon\varepsilon') = 0,$$

which would be contrary to the assumptions. □

We now state a quantum version of Theorem V.4.6.

Theorem VII.2.2. *When q is not a root of unity, any finite-dimensional U_q-module is semisimple.*

PROOF. We follow the proof of Theorem V.4.6 step by step. Recall that it is enough to prove that if V is any finite-dimensional U_q-module and V' is any submodule of V, then there exists another submodule V'' such that V is isomorphic to the direct sum $V' \oplus V''$ as a module.

1. We shall first prove the existence of such a submodule V'' in the case when V' is of codimension one in V. We proceed by induction on the dimension of V'.

If $\dim(V') = 0$, we may take $V'' = V$. If $\dim(V') = 1$, then necessarily V' and V/V' are simple one-dimensional modules of respective weights ε_1 and ε_2. If the weights ε_1 and ε_2 differ, there exists a basis $\{v_1, v_2\}$ of V in which K acts diagonally. Since Ev_i is an eigenvector for K with eigenvalue $\varepsilon_i q^2 \neq \varepsilon_j$, we must have $Ev_i = 0$ for $i = 1, 2$. Similarly, F acts trivially on V. Hence, the module V is the direct sum of the submodules $V' = kv_1$ and $V'' = kv_2$.

Otherwise, there exists a basis $\{v_1, v_2\}$ with $V' = kv_1$ such that we have $Kv_1 = \varepsilon v_1$ and $Kv_2 = \varepsilon v_2 + \alpha v_1$. Again, Ev_1 is an eigenvector for K with eigenvalue $\varepsilon q^2 \neq \varepsilon$, hence it is zero. Let us prove that Ev_2 is zero too. Indeed, writing $Ev_2 = \lambda v_1 + \mu v_2$, we have

$$\varepsilon\lambda v_1 + \mu(\varepsilon v_2 + \alpha v_1) = KEv_2 = q^2 EKv_2 = q^2 E(\varepsilon v_2 + \alpha v_1) = \varepsilon q^2(\lambda v_1 + \mu v_2),$$

which implies $\mu\varepsilon(q^2 - 1) = 0$ and $\lambda\varepsilon(q^2 - 1) = \mu\alpha$. Thus, $\lambda = \mu = 0$. One proves in a similar way that F acts as 0 on V. Since $[E, F]$ acts as 0, we have $K = K^{-1}$ on V. In particular, since $K^{-1}v_2 = \varepsilon v_2 - \alpha v_1$, we have $\alpha = -\alpha$, hence $\alpha = 0$. In this situation K is also diagonalizable and we reach the same conclusion as before.

We now assume that $\dim(V') = p > 1$ and that the assertion to be proved holds in dimension $< p$. There is the following alternative: either V' is simple, or it is not.

1.a. If V' is not simple, one uses the same argument as in Part 1.a of the proof of Theorem V.4.6.

1.b. Suppose now that the submodule V' is simple of dimension > 1. The one-dimensional quotient module V/V' has weight $\varepsilon = \pm 1$. Let us consider the operator C of Lemma 2.1; it acts by 0 on V/V'. Consequently, we have $CV \subset V'$. On the other hand, C acts on V' as multiplication by a scalar $\alpha \neq 0$. It follows that C/α is the identity on V'. Therefore the map C/α is a projector of V onto V'. This projector is U_q-linear since C is central. By Proposition I.1.3, the submodule $V'' = \operatorname{Ker}(C/\alpha)$ meets the requirements.

2. *General case.* We are now given finite-dimensional modules $V' \subset V$ without any restriction on the codimension. We shall reduce to the codimension-one case by considering vector spaces $W' \subset W$ defined as follows: W [resp. W'] is the subspace of all linear maps from V to V' whose restriction to V' is a homothety [resp. is zero]. It is clear that W' is of codimension one in W. In order to reduce to Part 1, we have to equip W and W' with U_q-module structures. We give $\operatorname{Hom}(V, V')$ the U_q-module structure defined in III.5. Let us check that W and W' are submodules of $\operatorname{Hom}(V, V')$. For $f \in W$, let α be the scalar such that $f(v) = \alpha v$ for all $v \in V'$; then for all $x \in U_q$ and $v \in V'$, we have

$$(xf)(v) = \sum_{(x)} x' f(S(x'')v) = \alpha\Big(\sum_{(x)} x' S(x'')\Big)v = \alpha\varepsilon(x)v.$$

A similar argument proves that W' is a submodule too. Applying Part 1, we get a one-dimensional submodule W'' such that $W \cong W' \oplus W''$. Let f be a generator of W''. By definition, it acts on V' as a scalar $\alpha \neq 0$. It follows that f/α is a projector of V onto V' and that $V'' = \operatorname{Ker}(f)$ is a supplementary subspace of V'. To conclude, it suffices to check that V'' is a U_q-submodule of V. Now, since W'' is a one-dimensional submodule, it is simple of weight ± 1. Therefore, for all $x \in U_q$ we have $xf = \pm\,\varepsilon(x) f$. In particular, if v belongs to V'', we have

$$K^{-1} f(Kv) = (K^{-1}f)(v) = \pm\varepsilon(K^{-1}) f(v) = 0,$$

which implies $f(Kv) = 0$. This proves that $KV'' \subset V''$. Similarly, V'' is stable under K^{-1}. On the other hand, we have for v, hence for Kv in V'',

$$\begin{aligned} 0 &= \pm\varepsilon(E)f(Kv) = (Ef)(Kv \\ &= f(S(E)Kv) + Ef(K^{-1}Kv) = -f(Ev) + Ef(v). \end{aligned}$$

Consequently, $f(Ev) = 0$, which implies that V'' is stable under the action of E. A similar computation shows that $FV'' \subset V''$. The subspace V'' is therefore a submodule. □

VII.3 Action of $U_q(\mathfrak{sl}(2))$ on the Quantum Plane

This section is the quantum version of V.6. We start with a few generalities on skew-derivations of an algebra A. For $a \in A$, denote by a_ℓ [resp. a_r] the left [resp. right] multiplication by the element a. If σ is an automorphism of the algebra A, we have

$$\sigma a_\ell = \sigma(a)_\ell \sigma \quad \text{and} \quad \sigma a_r = \sigma(a)_r \sigma. \tag{3.1}$$

Given two automorphisms σ and τ of an algebra A, a linear endomorphism δ of A is called a (σ, τ)-*derivation* if

$$\delta(aa') = \sigma(a)\delta(a') + \delta(a)\tau(a') \tag{3.2}$$

for all a, a' in A. Relation (3.2) is equivalent to

$$\delta a_\ell = \sigma(a)_\ell \delta + \delta(a)_\ell \tau \tag{3.3}$$

or to

$$\delta a_r = \tau(a)_r \delta + \delta(a)_r \sigma. \tag{3.4}$$

It is well-known that, if δ is a derivation of a commutative algebra, then $a_\ell\delta$ is a derivation too. In a non-commutative situation, this is no longer the case. Nevertheless, the following assertion holds.

Lemma VII.3.1. *Let δ be a (σ, τ)-derivation of A and a be an element of A. If there exist algebra automorphisms σ' and τ' of A such that*

$$a_r\sigma' = a_\ell\sigma \quad \textit{and} \quad a_\ell\tau' = a_r\tau,$$

then the linear endomorphism $a_\ell\delta$ is a (σ', τ)-derivation and $a_r\delta$ is a (σ, τ')-derivation.

PROOF. This follows from straightforward computations. □

We now return to the quantum plane $A = k_q[x, y]$ of IV.1. Let us consider its algebra automorphisms σ_x and σ_y defined by

$$\sigma_x(x) = qx, \quad \sigma_x(y) = y, \quad \sigma_y(x) = x, \quad \sigma_y(y) = qy. \tag{3.5}$$

When $q = 1$, we have $\sigma_x = \sigma_y = \mathrm{id}$. We define q-analogues $\partial_q/\partial x$ and $\partial_q/\partial y$ of the classical partial derivatives by

$$\frac{\partial_q(x^m y^n)}{\partial x} = [m]\, x^{m-1}y^n \quad \text{and} \quad \frac{\partial_q(x^m y^n)}{\partial y} = [n]\, x^m y^{n-1} \tag{3.6}$$

for all $m, n \geq 0$. Let us describe all commutation relations between the endomorphisms x_ℓ, x_r, y_ℓ, y_r, σ_x, σ_y, $\partial_q/\partial x$, $\partial_q/\partial y$. We say that a commutation relation between two endomorphisms u and v is trivial if $uv = vu$.

Proposition VII.3.2. (a) *Within the algebra of linear endomorphisms of $k_q[x, y]$, all commutation relations between the above six endomorphisms are trivial, except the following ones:*

$$y_\ell x_\ell = q x_\ell y_\ell, \qquad x_r y_r = q y_r x_r,$$

$$\sigma_x x_{\ell,r} = q x_{\ell,r}\sigma_x, \qquad \sigma_y y_{\ell,r} = q y_{\ell,r}\sigma_y,$$

$$\frac{\partial_q}{\partial x}\sigma_x = q\sigma_x\frac{\partial_q}{\partial x}, \qquad \frac{\partial_q}{\partial y}\sigma_y = q\sigma_y\frac{\partial_q}{\partial y},$$

$$\frac{\partial_q}{\partial x}y_\ell = q y_\ell\frac{\partial_q}{\partial x}, \qquad \frac{\partial_q}{\partial y}x_r = q x_r\frac{\partial_q}{\partial y},$$

$$\frac{\partial_q}{\partial x}x_\ell = q^{-1}x_\ell\frac{\partial_q}{\partial x} + \sigma_x = q x_\ell\frac{\partial_q}{\partial x} + \sigma_x^{-1},$$

$$\frac{\partial_q}{\partial y}y_r = q^{-1}y_r\frac{\partial_q}{\partial y} + \sigma_y = q y_r\frac{\partial_q}{\partial y} + \sigma_y^{-1}.$$

We also have

$$x_\ell\frac{\partial_q}{\partial x} = \frac{\sigma_x - \sigma_x^{-1}}{q - q^{-1}} \quad \text{and} \quad y_r\frac{\partial_q}{\partial y} = \frac{\sigma_y - \sigma_y^{-1}}{q - q^{-1}}.$$

(b) *The endomorphism $\frac{\partial_q}{\partial x}$ is a $(\sigma_x^{-1}\sigma_y, \sigma_x)$-derivation and, similarly, $\frac{\partial_q}{\partial y}$ is a $(\sigma_y, \sigma_x\sigma_y^{-1})$-derivation.*

PROOF. (a) This part results from easy, but fastidious computations.

(b) First observe that, if Relation (3.3) holds for two elements a, a' of A, then it holds for their product aa'. Indeed, we have

$$\begin{aligned} \delta(aa')_\ell &= \delta a_\ell a'_\ell \\ &= \sigma(a)_\ell\delta a'_\ell + \delta(a)_\ell\tau a'_\ell \\ &= \sigma(a)_\ell\sigma(a')_\ell\delta + \sigma(a)_\ell\delta(a')_\ell\tau + \delta(a)_\ell\tau(a')_\ell\tau \\ &= \sigma(aa')_\ell\delta + \delta(aa')_\ell\tau. \end{aligned}$$

We are reduced to checking Relation (3.3) for $\partial_q/\partial x$ and $\partial_q/\partial y$ in the case when $a = x$ and $a = y$. For $\partial_q/\partial x$ we have

$$(\sigma_x^{-1}\sigma_y)(x)_\ell \frac{\partial_q}{\partial x} + \left(\frac{\partial_q x}{\partial x}\right)_\ell \sigma_x = q^{-1} x_\ell \frac{\partial_q}{\partial x} + \sigma_x = \frac{\partial_q}{\partial x} x_\ell$$

by Proposition 3.2(a). We also have

$$(\sigma_x^{-1}\sigma_y)(y)_\ell \frac{\partial_q}{\partial x} + \left(\frac{\partial_q y}{\partial x}\right)_\ell \sigma_x = q y_\ell \frac{\partial_q}{\partial x} = \frac{\partial_q}{\partial x} y_\ell .$$

Similar computations can be carried out for $\partial_q/\partial y$. $\square$

We now show how the "quantum partial derivatives" $\frac{\partial_q}{\partial x}$ and $\frac{\partial_q}{\partial y}$ endow the quantum plane with the structure of a module-algebra (as defined in V.6.1) over the Hopf algebra U_q.

Theorem VII.3.3. *For any $P \in k_q[x, y]$, set*

$$EP = x\frac{\partial_q P}{\partial y}, \quad FP = \frac{\partial_q P}{\partial x} y,$$

$$KP = (\sigma_x \sigma_y^{-1})(P), \quad K^{-1}P = (\sigma_y \sigma_x^{-1})(P). \tag{3.7}$$

(a) *Formulas (3.7) define the structure of a U_q-module-algebra on $k_q[x, y]$.*

(b) *The subspace $k_q[x, y]_n$ of homogeneous elements of degree n is a U_q-submodule of the quantum plane. It is generated by the highest weight vector x^n and is isomorphic to the simple module $V_{1,n}$.*

Theorem 3.3 is the quantum version of Theorem V.6.4. It shows that the quantum plane contains all finite-dimensional simple U_q-modules.

PROOF. (a) We first show that the formulas (3.7) equip $k_q[x, y]$ with a U_q-module structure. In other words, we have to check Relations (VI.1.10–1.12). We use Proposition 3.2.

Relation (1.10) is trivially verified. For Relation (1.11) we have

$$\begin{aligned} KEK^{-1} &= \sigma_x \sigma_y^{-1} x_\ell \frac{\partial_q}{\partial y} \sigma_y \sigma_x^{-1} \\ &= q x_\ell \sigma_y^{-1} \frac{\partial_q}{\partial y} \sigma_y \\ &= q^2 x_\ell \frac{\partial_q}{\partial y} = q^2 E. \end{aligned}$$

One proves $KFK^{-1} = q^{-2}F$ in a similar fashion. As for (1.12), we have

$$\begin{aligned}
[E,F] &= x_\ell \frac{\partial_q}{\partial y} y_r \frac{\partial_q}{\partial x} - y_r \frac{\partial_q}{\partial x} x_\ell \frac{\partial_q}{\partial y} \\
&= q^{-1} x_\ell y_r \frac{\partial_q}{\partial y}\frac{\partial_q}{\partial x} + x_\ell \sigma_y \frac{\partial_q}{\partial x} - q^{-1} y_r x_\ell \frac{\partial_q}{\partial x}\frac{\partial_q}{\partial y} - y_r \sigma_x \frac{\partial_q}{\partial y} \\
&= x_\ell \sigma_y \frac{\partial_q}{\partial x} - y_r \sigma_x \frac{\partial_q}{\partial y} \\
&= \frac{\sigma_y(\sigma_x - \sigma_x^{-1}) - \sigma_x(\sigma_y - \sigma_y^{-1})}{q - q^{-1}} \\
&= \frac{\sigma_x \sigma_y^{-1} - \sigma_y \sigma_x^{-1}}{q - q^{-1}} \\
&= \frac{K - K^{-1}}{q - q^{-1}}.
\end{aligned}$$

We now prove that the quantum plane is a U_q-algebra. By Lemma V.6.2, it is enough to check that for any $u \in U_q$, we have

$$u1 = \varepsilon(u)1, \tag{3.8}$$

and

$$K(PQ) = K(P)K(Q), \tag{3.9}$$

$$E(PQ) = PE(Q) + E(P)K(Q), \tag{3.10}$$

$$F(PQ) = K^{-1}(P)F(Q) + F(P)Q, \tag{3.11}$$

for any pair (P, Q) of elements of the quantum plane. Relation (3.8) follows easily from (3.5–3.7) and Relation (3.9) from the fact that K acts as an algebra automorphism. By Lemma 3.1 and by Proposition 3.2(b), the endomorphism $x_\ell \frac{\partial_q}{\partial y}$ is a $(\mathrm{id}, \sigma_x \sigma_y^{-1})$-derivation and $y_r \frac{\partial_q}{\partial x}$ is a $(\sigma_x^{-1}\sigma_y, \mathrm{id})$-derivation, which implies Relations (3.10–3.11).

(b) We have $Ex^n = 0$, $Kx^n = q^n x^n$, and

$$\frac{1}{[p]!} F^p(x^n) = q^{-p} \frac{[n]!}{[p]![n-p]!} x^{n-p} y^p.$$

Consequently, x^n is a highest weight vector of weight q^n and generates the submodule $k_q[x,y]_n$. □

Observe that $[E,F]$ acts on the quantum plane as the operator

$$x_\ell \sigma_y \frac{\partial_q}{\partial x} - y_r \sigma_x \frac{\partial_q}{\partial y}.$$

Its "limit when q tends to 1" is the operator $x\partial/\partial x - y\partial/\partial y$ by which the element H of $\mathfrak{sl}(2)$ acts on the affine plane (see Theorem V.6.4).

VII.4 Duality between the Hopf Algebras $U_q(\mathfrak{sl}(2))$ and $SL_q(2)$

We now relate this chapter to Chapter IV by showing that U_q is in duality with the Hopf algebra $SL_q(2)$ defined in IV.6. We use the concept of duality introduced in V.7.

As in V.7, our first task is to construct an algebra morphism ψ from the algebra $M_q(2)$ (defined in IV.3) into the dual algebra U_q^*. We shall deduce a bilinear form on $U_q \times M_q(2)$ defined by $< u, x > = \psi(x)(u)$ and satisfying Relations (V.7.2) and (V.7.4). Giving the morphism ψ is equivalent to giving four elements A, B, C, D in U_q^* satisfying the six defining relations of $M_q(2)$ (see IV.3).

The definitions of A, B, C, D use the simple U_q-module $V_{1,1}$ of highest weight q and with basis $\{v_0, v_1\}$. The matrix representations of the generators E, F and K in this basis have been given in VI.3. Setting $\rho = \rho_{1,1}$, we have

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}. \tag{4.1}$$

More generally, for any element u of U_q, define

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{4.2}$$

We thus get four linear forms on U_q, hence four elements A, B, C, D of U_q^*.

Lemma VII.4.1. *The quadruple (A, B, C, D) is a U_q^*-point of $M_q(2)$.*

PROOF. This is done by a direct, but laborious checking. First, one has to compute in U_q^* the twelve products AB, BA, AC, $CA, \ldots$ formed by all pairs of distinct elements of the set $\{A, B, C, D\}$. Recall that the product of any two elements x, y of U_q^* is given by

$$(xy)(u) = \sum_{(u)} x(u')y(u''). \tag{4.3}$$

It suffices to evaluate $(xy)(u)$ on the basis $\{E^i F^j K^\ell\}$ of U_q. Let us set $u = E^i F^j K^\ell$. When $i > 2$ or when $j > 2$, we see from Proposition 1.3 that in the sum $\sum_{(u)} u' \otimes u''$, either u' or u'' contains powers of E or of F with exponents > 1. Now, by (4.1), $\rho(E^i F^j K^\ell) = \rho(E)^i \rho(F)^j \rho(K)^\ell$ vanishes when $i > 1$ or $j > 1$. Consequently, if $x, y \in \{A, B, C, D\}$, we have

$$(xy)(E^i F^j K^\ell) = 0$$

whenever $i > 2$ or $j > 2$. It therefore remains to evaluate the products on the elements $E^i F^j K^\ell$ where $0 \leq i \leq 2$ and $0 \leq j \leq 2$.

(i) If $u = K^\ell$, we have $\Delta(K^\ell) = K^\ell \otimes K^\ell$ and all products evaluated on u vanish, except that

$$(AD)(u) = (DA)(u) = 1. \tag{4.4}$$

(ii) If $u = FK^\ell$, we have $\Delta(FK^\ell) = K^{\ell-1} \otimes FK^\ell + FK^\ell \otimes K^\ell$ and all products evaluated on u vanish, except that

$$(CA)(u) = q\,(AC)(u) = q^{2\ell} \quad \text{and} \quad (DC)(u) = q\,(CD)(u) = q. \tag{4.5}$$

(iii) If $u = F^2K^\ell$, we have

$$\Delta(F^2K^\ell) = \alpha_1\, FK^{\ell-1} \otimes FK^\ell + (\text{terms of degree } > 2 \text{ in } F)$$

and all products evaluated on u vanish.

(iv) If $u = EK^\ell$, we have $\Delta(EK^\ell) = EK^\ell \otimes K^{\ell+1} + K^\ell \otimes EK^\ell$ and all products evaluated on u vanish, except that

$$(BA)(u) = q\,(AB)(u) = q \quad \text{and} \quad (DB)(u) = q\,(BD)(u) = q^{-2\ell}. \tag{4.6}$$

(v) If $u = EFK^\ell$, we have

$$\begin{aligned}\Delta(EFK^\ell) \;&=\; K^{\ell-1} \otimes EFK^\ell + EFK^\ell \otimes K^{\ell+1} \\ &\qquad + FK^\ell \otimes EK^\ell + q^{-2}\, EK^{\ell-1} \otimes FK^{\ell+1}\end{aligned}$$

and all products evaluated on u vanish, except that

$$(BC)(u) = (CB)(u) = 1, \quad (DA)(u) = q, \quad \text{and} \quad (AD)(u) = q^{-1}. \tag{4.7}$$

(vi) If $u = EF^2K^\ell$, we have

$$\begin{aligned}\Delta(EF^2K^\ell) \;&=\; \alpha_2\,(FK^{\ell-1} \otimes EFK^\ell + q^{-2}\, EFK^{\ell-1} \otimes FK^{\ell+1}) \\ &\qquad + (\text{terms of degree } > 2 \text{ in } F)\end{aligned}$$

and all products evaluated on u vanish, except

$$(CA)(u) = q\,(AC)(u) = \alpha_2\, q^{2\ell-1}. \tag{4.8}$$

(vii) If $u = E^2K^\ell$, we have

$$\Delta(E^2K^\ell) = \alpha_3\, EK^\ell \otimes EK^{\ell+1} + (\text{terms of degree } > 2 \text{ in } E)$$

and all products evaluated on u vanish.

(viii) If $u = E^2FK^\ell$, we have

$$\begin{aligned}\Delta(E^2FK^\ell) \;&=\; \alpha_4\,(EFK^\ell \otimes EK^{\ell+1} + q^{-2}\, EK^{\ell-1} \otimes EFK^{\ell+1}) \\ &\qquad + (\text{terms of degree } > 2 \text{ in } E)\end{aligned}$$

and all products evaluated on u vanish, except

$$(BA)(u) = q\,(AB)(u) = \alpha_4. \tag{4.9}$$

(ix) If $u = E^2F^2K^\ell$, we have

$$\Delta(E^2F^2K^\ell) = \alpha_5\, EFK^{\ell-1} \otimes EFK^{\ell+1} + (\text{terms of degree } > 2 \text{ in } E \text{ and } F)$$

and all products evaluated on u vanish.

In Cases (iii) and (vi–ix) we denoted by α_1, α_2, α_3, α_4, and α_5 scalars that are well-defined, but about which we need not be explicit. From this case-by-case analysis, it is easy to check that A, B, C, D satisfy the six defining relations of $M_q(2)$. As a sample calculation, we check the most involved relation, namely

$$DA - AD = (q - q^{-1})\, BC.$$

From the above observations, we see that it is enough to perform the checking for $u = K^\ell$, which is trivial, and for $u = EFK^\ell$. In the latter case, (4.7) implies

$$(DA - AD)(u) = q - q^{-1} = (q - q^{-1})\,(BC)(u).$$

□

As a consequence of Lemma 4.1 and of IV.3, there exists a unique morphism of algebras ψ from $M_q(2)$ into U_q^* such that

$$\psi(a) = A, \quad \psi(b) = B, \quad \psi(c) = C, \quad \psi(d) = D.$$

Proposition VII.4.2. *The bilinear form $< u, x > = \psi(x)(u)$ realizes a duality between the bialgebras U_q and $M_q(2)$.*

PROOF. The comultiplication and the counit of $M_q(2)$ being the same as those of $M(2)$, the proof follows along the same lines as in the proof of Proposition V.7.3. □

The duality between $M_q(2)$ and U_q is not perfect, just as in the classical case.

Lemma VII.4.3. *For the quantum determinant* $\det_q = da - qbc$ *of* $M_q(2)$, *we have* $\psi(\det_q) = 1$.

Equivalently, $< u, \det_q > = \varepsilon(u)$ for all elements u of U_q.

PROOF. By Theorem IV.5.1, the element $\det_q$ is grouplike, i.e., we have $\Delta(\det_q) = \det_q \otimes \det_q$. It results that the map $u \mapsto < u, \det_q >$ is a morphism of algebras from U_q to k. To show that this morphism coincides with the counit ε, it suffices to check that both maps take the same values on the generators E, F, K and K^{-1}. Using (V.7.2–7.3) and (1.1), we get for E:

$$\begin{aligned}
&< E, \det_q > \\
&\quad = \ < E, da > -q < E, bc > \\
&\quad = \ \varepsilon(d) < E, a > + < E, d >< K, a > \\
&\qquad\qquad\qquad -q\varepsilon(b) < E, c > -q < E, b >< K, c > \\
&\quad = \ 0 = \varepsilon(E).
\end{aligned}$$

For K we have

$$\begin{aligned}
< K, \det_q > &= < K, da > -q < K, bc > \\
&= < K, d >< K, a > -q < K, b >< K, c > \\
&= q^{-1}q = 1 = \varepsilon(K).
\end{aligned}$$

Similar computations can be carried out for F and K^{-1}. □

As a consequence of Lemma 4.3, the algebra morphism ψ from $M_q(2)$ to U_q^* factors through $SL_q(2) = M_q(2)/(\det_q - 1)$. We still denote by ψ the induced morphism of algebras from $SL_q(2)$ into U_q^* and by $< \, , \, >$ the corresponding bilinear form.

Theorem VII.4.4. *The bilinear form $< u, x > = \psi(x)(u)$ realizes a duality between the Hopf algebras U_q and $SL_q(2)$.*

PROOF. We use the same argument as in the proof of Theorem V.7.6. The only difference lies with the antipodes. We first check Relation (V.7.5) for the generators. Using the condensed matrix form, we have

$$\begin{aligned}
< S(E), \begin{pmatrix} a & b \\ c & d \end{pmatrix} > &= \rho(S(E)) = -\rho(E)\rho(K^{-1}) = \begin{pmatrix} 0 & -q \\ 0 & 0 \end{pmatrix} \\
&= < E, \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} > = < E, \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} > .
\end{aligned}$$

For F we have

$$\begin{aligned}
< S(F), \begin{pmatrix} a & b \\ c & d \end{pmatrix} > &= \rho(S(F)) = -\rho(K)\rho(F) = \begin{pmatrix} 0 & 0 \\ -q^{-1} & 0 \end{pmatrix} \\
&= < F, \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} > = < F, \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} > .
\end{aligned}$$

One proceeds with K and K^{-1} similarly. To conclude, one appeals to Lemma V.7.7. □

VII.5 Duality between $U_q(\mathfrak{sl}(2))$-Modules and $SL_q(2)$-Comodules

Exactly as in the classical case considered in V.7, there is a duality between U_q-modules and $SL_q(2)$-comodules. We have seen in IV.7 that the vector space $k_q[x,y]_n$ of homogeneous elements of degree n of the quantum plane has a natural structure as an $SL_q(2)$-comodule. By duality, the dual vector space $k_q[x,y]_n^*$ has a module structure over the algebra $SL_q(2)^*$, hence over the algebra U_q via the morphism $\varphi : U_q \to SL_q(2)^*$. The following statement gives the structure of $k_q[x,y]_n^*$ as a U_q-module.

Theorem VII.5.1. *The U_q-module $k_q[x,y]_n^*$ is isomorphic to the simple module $V_{1,n}$ of highest weight q^n.*

Thus, the $SL_q(2)$-comodule $k_q[x,y]_n$ corresponds by duality to the U_q-module $V_{1,n}$.

PROOF. We shall show that the linear form on $k_q[x,y]_n$ defined by

$$f(x^i y^{n-i}) = \delta_{ni}$$

is a highest weight vector, with weight q^n, of the U_q-module $k_q[x,y]_n^*$, which implies that $k_q[x,y]_n^*$ contains a submodule isomorphic to the simple module $V_{1,n}$. Since

$$\dim(V_{1,n}) = n+1 = \dim(k_q[x,y]_n^*),$$

we get $k_q[x,y]_n^* \cong V_{1,n}$.

In order to prove that f is a highest weight vector, we need the relation

$$(uf)(x^i y^{n-i}) = < u, a^i c^{n-i} > \tag{5.1}$$

for all $u \in U_q$ and for all i such that $0 \le i \le n$. But this is so since, by definition of f, by III.6, Example 2, by Lemma IV.7.2, and using the abbreviation

$$C_{r,s} = q^{(i-r)s} \begin{pmatrix} i \\ r \end{pmatrix}_{q^2} \begin{pmatrix} n-i \\ s \end{pmatrix}_{q^2}$$

to shorten the formulas, we have

$$\begin{aligned}
(uf)(x^i y^{n-i}) &= (u \otimes f)(\Delta_A(x^i y^{n-i})) \\
&= \sum_{r=0}^{i} \sum_{s=0}^{n-i} C_{r,s} < u, a^r b^{i-r} c^s d^{n-i-s} > f(x^{r+s} y^{n-r-s}) \\
&= \sum_{r=0}^{i} \sum_{s=0}^{n-i} C_{r,s} < u, a^r b^{i-r} c^s d^{n-i-s} > \delta_{n,r+s} \\
&= \sum_{r=0}^{i} \sum_{s=0}^{n-i} C_{r,s} < u, a^r b^{i-r} c^s d^{n-i-s} > \delta_{i,r} \delta_{n-i,s} \\
&= < u, a^i c^{n-i} > .
\end{aligned}$$

Let us apply Relation (5.1) to K. A straightforward computation yields

$$< K, a^i c^j > = < K, a >^i < K, c >^j = \delta_{j0}\, q^i.$$

Consequently, we have $(Kf)(x^i y^{n-i}) = \delta_{ni}\, q^i = \delta_{ni}\, q^n$, which implies that $Kf = q^n f$.

It remains to prove that $Ef = 0$. This is a consequence of Relation (5.1) applied to E and of the fact that $< E, a^i c^j > = 0$ for all i and j. Let us prove the latter. First, we have $< E, 1 > = \varepsilon(E) = 0$. Next, if $i > 0$ we have by (V.7.2–7.3)

$$\begin{aligned} < E, a^i > &= \varepsilon(a) < E, a^{i-1} > + < E, a >< K, a^{i-1} > \\ &= < E, a^{i-1} > = \ldots = < E, a > = 0. \end{aligned}$$

Similarly, if $j > 0$ we get

$$< E, c^j > = \varepsilon(c) < E, c^{j-1} > + < E, c >< K, c^{j-1} > = 0.$$

Consequently,

$$< E, a^i c^j > = \varepsilon(a)^i < E, c^j > + < E, a^i >< K, c^j > = 0.$$

□

VII.6 Scalar Products on $U_q(\mathfrak{sl}(2))$-Modules

In this section, given any finite-dimensional U_q-module V, we construct a *scalar product*, i.e., a non-degenerate symmetric bilinear form $(\ ,\ )$ on V such that

$$(xv, v') = (v, T(x)v') \tag{6.1}$$

for all $x \in U_q$ and $v, v' \in V$. The linear map T is the algebra antiautomorphism of U_q defined as follows.

Proposition VII.6.1. *There exists a unique algebra antiautomorphism T of U_q such that $T(E) = KF$, $T(F) = EK^{-1}$, and $T(K) = K$. The automorphism T is also a morphism of coalgebras.*

PROOF. Left to the reader. □

By Theorem 2.2, it is enough to construct a scalar product on any simple U_q-module of the form $V_{\varepsilon,n}$. This is done in the following theorem.

Theorem VII.6.2. *On the simple U_q-module $V_{\varepsilon,n}$ generated by the highest weight vector v, there exists a unique scalar product such that $(v, v) = 1$. If we define the vectors v_i for all $i \geq 0$ by $v_i = F^i v/[i]!$, then they are pairwise orthogonal and we have*

$$(v_i, v_i) = q^{-(n-i-1)i} \begin{bmatrix} n \\ i \end{bmatrix}.$$

PROOF. Let us first assume that there exists a scalar product on $V_{\varepsilon,n}$ such that $(v,v) = 1$. Let us show that (v_i, v_j) is necessarily of the prescribed form. By definition and by (6.1) we have

$$(v_i, v_j) = \frac{1}{[i]!}(F^i v, v_j) = \frac{1}{[i]!}(v, T(F)^i v_j) = \frac{1}{[i]!}(v, (EK^{-1})^i v_j).$$

An easy induction on i shows that $(EK^{-1})^i = q^{i(i+1)} K^{-i} E^i$ for any $i > 0$. Consequently, the vector $T(F)^i v_j$ is a scalar multiple of $E^i v_j$ which vanishes as soon as $i > j$. Therefore $(v_i, v_j) = 0$ if $i > j$. By symmetry, we also have $(v_i, v_j) = 0$ if $i < j$.

We need the formula

$$E^i v_j = \varepsilon^i \frac{[n-j+i]!}{[n-j]!} v_{j-i}$$

to compute (v_i, v_i). We have

$$\begin{aligned} (v_i, v_i) &= \frac{1}{[i]!} q^{i(i+1)} (v, K^{-i} E^i v_i) \\ &= \varepsilon^i q^{i(i+1)} \frac{[n]!}{[i]![n-i]!} (v, K^{-i} v) \\ &= q^{i(i+1)-ni} \begin{bmatrix} n \\ i \end{bmatrix} (v, v). \end{aligned}$$

This proves the uniqueness of the scalar product. Let us now prove its existence.

Clearly, there exists a non-degenerate symmetric bilinear form such that

$$(v_i, v_j) = q^{-(n-i-1)i} \begin{bmatrix} n \\ i \end{bmatrix} \delta_{ij}. \tag{6.2}$$

We have to check that it satisfies Relation (6.1). It is enough to check this for $x = E$, F, K and K^{-1}. We shall do this for $x = E$ leaving all other computations to the reader. On the one hand, we have

$$(Ev_i, v_j) = \varepsilon[n-i+1](v_{i-1}, v_j) = \varepsilon\delta_{i-1,j}\, q^{-(n-i)(i-1)} \frac{[n]!}{[i-1]![n-i]!}.$$

On the other hand, by (VI.3.1–3.3) and by (6.2), we have

$$\begin{aligned} (v_i, T(E)v_j) &= (v_i, KFv_j) \\ &= \varepsilon q^{n-2(j+1)} [j+1]\, (v_i, v_{j+1}) \\ &= \varepsilon\delta_{i,j+1}\, q^{-(n-i-1)i+n-2(j+1)}\, [j+1] \frac{[n]!}{[i]![n-i]!} \\ &= \varepsilon\delta_{i,j+1}\, q^{-(n-i)(i-1)} \frac{[n]!}{[i-1]![n-i]!} = (Ev_i, v_j). \end{aligned}$$

□

VII.7 Quantum Clebsch-Gordan

We now prove a quantum Clebsch-Gordan formula for the finite-dimensional simple U_q-modules. Since

$$V_{\varepsilon,n} \cong V_{\varepsilon,0} \otimes V_{1,n} \cong V_{1,n} \otimes V_{\varepsilon,0}, \tag{7.1}$$

we need give this formula only for the modules $V_{1,n}$, henceforth denoted for simplicity by V_n.

Theorem VII.7.1. *Let $n \geq m$ be two nonnegative integers. There exists an isomorphism of U_q-modules*

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m}.$$

One proves Theorem 7.1 in the same way as Proposition V.5.1. It suffices to check that the module $V_n \otimes V_m$ contains a highest weight vector of weight q^{n+m-2p} for any integer p such that $0 \leq p \leq m$.

Lemma VII.7.2. *Let $v^{(n)}$ be a highest weight vector of weight q^n in V_n and $v^{(m)}$ be a highest weight vector of weight q^m in V_m. Let us define $v_p^{(n)} = \frac{1}{[p]!}F^p v^{(n)}$ and $v_p^{(m)} = \frac{1}{[p]!}F^p v^{(m)}$ for all $p \geq 0$. Then,*

$$v^{(n+m-2p)} = \sum_{i=0}^{p} (-1)^i \, \frac{[m-p+i]![n-i]!}{[m-p]![n]!} \, q^{-i(m-2p+i+1)} \, v_i^{(n)} \otimes v_{p-i}^{(m)}$$

is a highest weight vector of weight q^{n+m-2p} in $V_n \otimes V_m$.

PROOF. It is clear that $v_i^{(n)} \otimes v_{p-i}^{(m)}$ has weight $q^{n-2i+m-2(p-i)} = q^{n+m-2p}$. Let us prove that $Ev^{(n+m-2p)} = 0$. Recall that $\Delta(E) = 1 \otimes E + E \otimes K$. It follows that

$$\begin{aligned}
&Ev^{(n+m-2p)} \\
&= \sum_{i=0}^{p} (-1)^i \, \frac{[m-p+i]![n-i]!}{[m-p]![n]!} \, q^{-i(m-2p+i+1)} \, v_i^{(n)} \otimes Ev_{p-i}^{(m)} \\
&\quad + \sum_{i=0}^{p} (-1)^i \, \frac{[m-p+i]![n-i]!}{[m-p]![n]!} \, q^{-i(m-2p+i+1)} \, Ev_i^{(n)} \otimes Kv_{p-i}^{(m)} \\
&= \sum_{i=0}^{p} (-1)^i \, [m-p+i+1] \frac{[m-p+i]![n-i]!}{[m-p]![n]!} \, q^{-i(m-2p+i+1)} \\
&\qquad \times v_i^{(n)} \otimes v_{p-i-1}^{(m)} \\
&\quad + \sum_{i=0}^{p} (-1)^i \, [n-i+1] \frac{[m-p+i]![n-i]!}{[m-p]![n]!} \, q^{-i(m-2p+i+1)+(m-2p+2i)}
\end{aligned}$$

$$
\begin{aligned}
& \times v_{i-1}^{(n)} \otimes v_{p-i}^{(m)} \\
= & \sum_{i=0}^{p} (-1)^i \Big(\frac{[m-p+i]![n-i+1]!}{[m-p]![n]!} q^{-(i-1)(m-2p+i)} \\
& \quad - \frac{[m-p+i]![n-i+1]!}{[m-p]![n]!} q^{-(i-1)(m-2p+i)} \Big) v_{i-1}^{(n)} \otimes v_{p-i}^{(m)} \\
= & \; 0.
\end{aligned}
$$

□

This concludes the proof of Theorem 7.1. We wish to go one step further and address the following problem. We now have two bases of $V_n \otimes V_m$ at our disposal. They are of different natures: the first one, adapted to the tensor product, is the set

$$\{v_i^{(n)} \otimes v_j^{(m)}\}_{0 \leq i \leq n,\, 0 \leq j \leq m} \;;$$

the second one, formed by the vectors

$$v_k^{(n+m-2p)} = \frac{1}{[k]!} F^k v^{(n+m-2p)}$$

with $0 \leq p \leq m$ and $0 \leq k \leq n+m-2p$, is better adapted to the U_q-module structure. Comparing both bases leads us to the so-called *quantum Clebsch-Gordan coefficients*

$$\begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix}$$

defined for $0 \leq p \leq m$ and $0 \leq k \leq n+m-2p$ by

$$v_k^{(n+m-2p)} = \sum_{0 \leq i \leq n,\, 0 \leq j \leq m} \begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix} v_i^{(n)} \otimes v_j^{(m)}. \qquad (7.2)$$

The remainder of this section is devoted to a few properties of these coefficients, also called *quantum 3j-symbols* in the physics literature.

Lemma VII.7.3. *Fix p and k. The vector $v_k^{(n+m-2p)}$ is a linear combination of vectors of the form $v_i^{(n)} \otimes v_{p-i+k}^{(m)}$. Therefore, we have*

$$\begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix} = 0 \qquad (7.3)$$

when $i + j \neq p + k$. We also have the induction relation

$$\begin{bmatrix} n & m & n+m-2p \\ i & j+1 & k+1 \end{bmatrix} = \frac{[j+1]q^{-(n-2i)} + [i]}{[k+1]} \begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix}. \qquad (7.4)$$

PROOF. This goes by induction on k. The assertion holds for $k = 0$ thanks to Lemma 7.2. Supposing

$$v_k^{(n+m-2p)} = \sum_i \alpha_i \, v_i^{(n)} \otimes v_{p-i+k}^{(m)},$$

we get

$$\begin{aligned}
[k+1]v_{k+1}^{(n+m-2p)} &= Fv_k^{(n+m-2p)} \\
&= \sum_i \alpha_i \left(K^{-1}v_i^{(n)} \otimes Fv_{p-i+k}^{(m)} + Fv_i^{(n)} \otimes v_{p-i+k}^{(m)}\right) \\
&= \sum_i \alpha_i \Big([p-i+k+1]q^{-(n-2i)}v_i^{(n)} \otimes v_{p-i+k+1}^{(m)} \\
&\qquad\qquad + [i+1]v_{i+1}^{(n)} \otimes v_{p-i+k}^{(m)}\Big) \\
&= \sum_i \alpha_i \left([p-i+k+1]q^{-(n-2i)} + [i]\right)v_i^{(n)} \otimes v_{p-i+k+1}^{(m)}.
\end{aligned}$$

The rest follows easily. □

We now prove some orthogonality relations for the quantum Clebsch-Gordan coefficients, which will allow us to express the basis $\{v_i^{(n)} \otimes v_j^{(m)}\}_{i,j}$ in terms of the basis $\{v_k^{(n+m-2p)}\}_{p,k}$. Let us equip V_n and V_m with the scalar product $(\ ,\ )$ defined in Section 6. Consider the symmetric bilinear form on $V_n \otimes V_m$ given by

$$(v_1 \otimes v_1', v_2 \otimes v_2') = (v_1, v_2)(v_1', v_2') \tag{7.5}$$

where $v_1, v_2 \in V_n$ and $v_1', v_2' \in V_m$.

Lemma VII.7.4. *The symmetric bilinear form* (7.5) *is non-degenerate and the basis* $\{v_i^{(n)} \otimes v_j^{(m)}\}_{i,j}$ *is orthogonal. Furthermore, for all* $x \in U_q$ *and all* $w_1, w_2 \in V_n \otimes V_m$*, we have*

$$(xw_1, w_2) = (w_1, T(x)w_2).$$

PROOF. The first two assertions are clear. Let us prove the last one. If $w_1 = v_1 \otimes v_1'$ and $w_2 = v_2 \otimes v_2'$, we have

$$\begin{aligned}
(xw_1, w_2) &= (\Delta(x)(v_1 \otimes v_1'), v_2 \otimes v_2') \\
&= \sum_{(x)} (x'v_1, v_2)(x''v_1', v_2') \\
&= \sum_{(x)} (v_1, T(x')v_2)(v_1', T(x'')v_2') \\
&= \sum_{(T(x))} (v_1, T(x)'v_2)(v_1', T(x)''v_2') \\
&= (w_1, T(x)w_2),
\end{aligned}$$

using the fact that T is an automorphism of coalgebras (see Proposition 6.1). □

The second basis of $V_n \otimes V_m$ is orthogonal too.

Proposition VII.7.5. (a) *The basis* $\{v_k^{(n+m-2p)}\}_{0\le p\le m,\, 0\le k\le n+m-2p}$ *is orthogonal.*

(b) *Fix integers p, q, k, ℓ. We have the following orthogonality relations:*

$$0 = \sum_{i,j} q^{-i(n-i-1)-j(m-j-1)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \times \begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix} \begin{bmatrix} n & m & n+m-2q \\ i & j & \ell \end{bmatrix}$$

when $p \neq q$ or $k \neq \ell$, and

$$\sum_{i,j} q^{-i(n-i-1)-j(m-j-1)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix}^2 = q^{-k(n+m-2p-k-1)} \begin{bmatrix} n+m-2p \\ k \end{bmatrix}.$$

(c) *Given i and j, we have*

$$v_i^{(n)} \otimes v_j^{(m)} = q^{-i(n-i-1)-j(m-j-1)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \times \sum_{p=0}^{m} \sum_{k=0}^{n+m-2p} q^{k(n+m-2p-k-1)} \frac{\begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix}}{\begin{bmatrix} n+m-2p \\ k \end{bmatrix}} v_k^{(n+m-2p)}.$$

PROOF. (a) Arguing as in the proof of Theorem 6.2, one shows that

$$(v_k^{(n+m-2p)}, v_\ell^{(n+m-2p)}) = 0$$

whenever $k \neq \ell$. Let us examine the case when $p \neq q$. Let us first show that the highest weight vectors $v^{(n+m-2p)}$ and $v^{(n+m-2q)}$ are orthogonal. In fact, Lemma 7.2 implies that $(v^{(n+m-2p)}, v^{(n+m-2q)})$ can be written

$$\begin{aligned} (v^{(n+m-2p)}, v^{(n+m-2q)}) &= \sum_{i,j} \alpha_i \beta_j (v_i^{(n)}, v_j^{(n)})(v_{p-i}^{(m)}, v_{q-j}^{(m)}) \\ &= \sum_{i} \alpha_i \beta_i (v_i^{(n)}, v_i^{(n)})(v_{p-i}^{(m)}, v_{q-i}^{(m)}), \end{aligned}$$

which is zero because $p - i \neq q - i$. It remains to show that

$$(v_k^{(n+m-2p)}, v_\ell^{(n+m-2q)}) = 0$$

when $k, \ell > 0$.

By symmetry, it is enough to consider the case $k \geq \ell$. We have

$$\begin{aligned}(v_k^{(n+m-2p)}, v_\ell^{(n+m-2q)}) &= \gamma\,(F^k v^{(n+m-2p)}, v_\ell^{(n+m-2q)}) \\ &= \gamma'\,(v^{(n+m-2p)}, E^k v_\ell^{(n+m-2q)})\end{aligned}$$

for some scalars γ and γ'. Now, if $k \geq \ell$, the vector $E^k v_\ell^{(n+m-2q)}$ is zero or is a scalar multiple of the highest weight vector $v^{(n+m-2q)}$, which brings us back to a previous case.

(b) Let us compute $(v_k^{(n+m-2p)}, v_\ell^{(n+m-2q)})$. It is equal to

$$\begin{aligned}&\sum_{i+j=p+k}\ \sum_{r+s=q+\ell} \begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix} \begin{bmatrix} n & m & n+m-2q \\ r & s & \ell \end{bmatrix} \\ &\qquad\qquad \times (v_i^{(n)}, v_r^{(n)})(v_j^{(m)}, v_s^{(m)}) \\ =\ &\sum_{i+j=p+k} \begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix} \begin{bmatrix} n & m & n+m-2q \\ i & j & \ell \end{bmatrix} \\ &\qquad\qquad \times (v_i^{(n)}, v_i^{(n)})(v_j^{(m)}, v_j^{(m)}) \\ =\ &\sum_{i+j=p+k} q^{-i(n-i-1)-j(m-j-1)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix} \\ &\qquad\qquad \times \begin{bmatrix} n & m & n+m-2q \\ i & j & \ell \end{bmatrix}.\end{aligned}$$

On the other hand, we have

$$(v_k^{(n+m-2p)}, v_\ell^{(n+m-2q)}) = \delta_{pq}\delta_{k\ell}\, q^{-k(n+m-2p-k-1)} \begin{bmatrix} n+m-2p \\ k \end{bmatrix}.$$

(c) We have $v_i^{(n)} \otimes v_j^{(m)} = \sum_{p=0}^{m} \sum_{k=0}^{n+m-2p} \gamma_{pk}\, v_k^{(n+m-2p)}$ for some coefficients γ_{pk}. Therefore,

$$\begin{aligned}&\gamma_{pk}(v_k^{(n+m-2p)}, v_k^{(n+m-2p)}) \\ &\qquad = (v_i^{(n)} \otimes v_j^{(m)}, v_k^{(n+m-2p)}) \\ &\qquad = \begin{bmatrix} n & m & n+m-2p \\ i & j & k \end{bmatrix} (v_i^{(n)}, v_i^{(n)})(v_j^{(m)}, v_j^{(m)}).\end{aligned}$$

Applying (6.2), one gets the desired explicit expression for γ_{pk}. □

For more details on the quantum Clebsch-Gordan coefficients, see [KR89] [KK89] [Vak89] where they are expressed in terms of *q-Hahn polynomials*, i.e., of certain orthogonal q-hypergeometric series (see also [GR90], Chap. 7). Koelink-Koornwinder and Vaksman showed that the orthogonality relations of the q-Hahn polynomials were equivalent to the orthogonality relations of the quantum Clebsch-Gordan coefficients. The corresponding property for the classical Clebsch-Gordan coefficients was known already (see [Koo90]).

VII.8 Exercises

1. Compute $S(E^i F^j K^\ell)$ in U_q.

2. Let x be an element of U_q. Prove successively that

   (a) x is grouplike if and only if x is of the form $x = K^n$;

   (b) if $\Delta(x) = 1 \otimes x + x \otimes K$ and $\varepsilon(x) = 0$, then x is a linear combination of E and of KF;

   (c) if $\Delta(x) = K^{-1} \otimes x + x \otimes 1$ and $\varepsilon(x) = 0$, then x is a linear combination of F and of EK^{-1};

   (d) if $\Delta(x) = 1 \otimes x + x \otimes K^{-1}$, then $x = 0$.

3. Use Exercise 2 to show that there exists an isomorphism of Hopf algebras from U_q onto $U_{q'}$ if and only if $q' = \pm q^{\pm 1}$, and that any Hopf algebra automorphism φ of U_q is of the form

$$\varphi(E) = \alpha E, \quad \varphi(F) = \alpha^{-1} F, \quad \varphi(K) = K$$

   where α is a non-zero scalar.

4. (*Hopf $*$-algebra structures on U_q*) We use the concepts introduced in IV.8.

   (a) Prove that U_q is a Hopf $*$-algebra if and only if q^2 is a real number or q is a complex number of modulus 1.

   (b) Check that the following formulas determine five Hopf $*$-algebra structures on U_q:

      (i) $E^* = E$, $F^* = F$, and $K^* = K$ if $|q| = 1$;

      (ii) $E^* = KF$, $F^* = EK^{-1}$, and $K^* = K$ if q is real > 0;

      (iii) $E^* = -KF$, $F^* = -EK^{-1}$, and $K^* = K$ if q is real < 0;

      (iv) $E^* = iKF$, $F^* = iEK^{-1}$, and $K^* = K$ if $q = \lambda i$ with λ real > 0;

      (v) $E^* = -iKF$, $F^* = -iEK^{-1}$, and $K^* = K$ if $q = \lambda i$ with λ real < 0.

   (c) Show that any Hopf $*$-algebra structure on U_q is equivalent to one of the previous five ones (Hint: use Exercise 2).

5. Given a Hopf $*$-algebra structure on U_q and a U_q-module V, define a Hermitian scalar product as a definite positive Hermitian form $(\ ,\ )$ such that $(xv, v') = (v, x^* v')$ for all $x \in U_q$ and $v, v' \in V$. Determine all Hermitian scalar products on the simple module $V_{\varepsilon,n}$.

6. Prove that there exists a U_q-linear isomorphism between the simple module $V_{\varepsilon,n}$ and its dual module.

VII.9 Notes

The Hopf algebra structure of $U_q(\mathfrak{sl}(2))$ is due to Sklyanin [Skl85]. The Drinfeld-Jimbo algebras $U_q(\mathfrak{g})$ also have a non-commutative, non-cocommutative Hopf algebra structure. In the cases A, D, E considered in VI.7, it is given on the generators $(E_i, F_i, K_i)_{1\leq i\leq \ell}$ by

$$\Delta(E_i) = 1\otimes E_i + E\otimes K_i, \quad \Delta(F_i) = K_i^{-1}\otimes F_i + F_i\otimes 1, \quad \Delta(K_i) = K_i\otimes K_i,$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1,$$

and

$$S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i, \quad S(K_i) = K_i^{-1}.$$

In this chapter we adopted the conventions of Takeuchi [Tak92c] rather than those of Drinfeld and Jimbo. In the special case $\mathfrak{g} = \mathfrak{sl}(2)$, Takeuchi's conventions allow U_q to act on the quantum plane of Chapter IV. Following Drinfeld [Dri87], Takeuchi [Tak92c] [Tak92b] also showed the existence of a duality between $U_q(\mathfrak{sl}(n))$ and the Hopf algebra $SL_q(n)$ of IV.9, embedding the latter into the restricted dual of $U_q(\mathfrak{sl}(n))$.

The semisimplicity of the finite-dimensional U_q-modules is due to Rosso [Ros88]. We followed his proof closely.

For more details on quantum Clebsch-Gordan coefficients, read [KR89] [KK89] [Koo90] [Vak89]. For the Hopf $*$-algebra structures on U_q (determined in Exercise 4), see [MMN$^+$90].

Part Two

Universal R-Matrices

Chapter VIII
The Yang-Baxter Equation and (Co)Braided Bialgebras

Part II is centered around the now famous Yang-Baxter equation whose solutions are the so-called R-matrices. We introduce the concept of braided bialgebras due to Drinfeld. These are bialgebras with a universal R-matrix inducing a solution of the Yang-Baxter equation on any of their modules. This provides a systematic method to produce solutions of the Yang-Baxter equation. There is a dual notion of cobraided bialgebras. We show how to construct a cobraided bialgebra out of any solution of the Yang-Baxter equation by a method due to Faddeev, Reshetikhin and Takhtadjian [RTF89]. We conclude this chapter by proving that the quantum groups $GL_q(2)$ and $SL_q(2)$ of Chapter IV can be obtained by this method and that they are cobraided.

VIII.1 The Yang-Baxter Equation

Definition VIII.1.1. *Let V be a vector space over a field k. A linear automorphism c of $V \otimes V$ is said to be an R-matrix if it is a solution of the Yang-Baxter equation*

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)$$

that holds in the automorphism group of $V \otimes V \otimes V$.

Finding all solutions of the Yang-Baxter equation is a difficult task, as will appear from the examples given below. Let $\{v_i\}_i$ be a basis of the vector space V. An automorphism c of $V \otimes V$ is defined by the family

$(c_{ij}^{k\ell})_{i,j,k,\ell}$ of scalars determined by

$$c(v_i \otimes v_j) = \sum_{k,\ell} c_{ij}^{k\ell}\, v_k \otimes v_\ell.$$

Then c is a solution of the Yang-Baxter equation if and only if for all i, j, k, ℓ, m, n, we have

$$\sum_{p,q,r,x,y,z} (c_{ij}^{pq}\delta_{kr})(\delta_{px}c_{qr}^{yz})(c_{xy}^{\ell m}\delta_{zn}) = \sum_{p,q,r,x,y,z} (\delta_{ip}c_{jk}^{qr})(c_{pq}^{xy}\delta_{rz})(\delta_{x\ell}c_{yz}^{mn}),$$

which is equivalent to

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{\ell m} = \sum_{y,q,r} c_{jk}^{qr} c_{iq}^{\ell y} c_{yr}^{mn}. \tag{1.1}$$

for all i, j, k, ℓ, m, n. Solving the non-linear equations (1.1) is a highly non-trivial problem. Nevertheless, numerous solutions of the Yang-Baxter equation have been discovered in the 1980's. Let us list a few examples.

Example 1. For any vector space V we denote by $\tau_{V,V} \in \mathrm{Aut}(V \otimes V)$ the *flip* switching the two copies of V. It is defined by

$$\tau_{V,V}(v_1 \otimes v_2) = v_2 \otimes v_1,$$

for any $v_1, v_2 \in V$. The flip satisfies the Yang-Baxter equation because of the Coxeter relation $(12)(23)(12) = (23)(12)(23)$ in the symmetry group S_3.

Here is a way to generate new R-matrices from old ones.

Lemma VIII.1.2. *If $c \in \mathrm{Aut}(V \otimes V)$ is an R-matrix, then so are λc, c^{-1} and $\tau_{V,V} \circ c \circ \tau_{V,V}$ where λ is any non-zero scalar.*

PROOF. This follows from the identities

$$(\lambda c \otimes \mathrm{id}_V) = \lambda(c \otimes \mathrm{id}_V), \quad (\mathrm{id}_V \otimes \lambda c) = \lambda(\mathrm{id}_V \otimes c),$$

$$(c^{-1} \otimes \mathrm{id}_V) = (c \otimes \mathrm{id}_V)^{-1}, \quad (\mathrm{id}_V \otimes c^{-1}) = (\mathrm{id}_V \otimes c)^{-1},$$

$$(c' \otimes \mathrm{id}_V) = \sigma(\mathrm{id}_V \otimes c)\sigma^{-1}, \quad (\mathrm{id}_V \otimes c') = \sigma(c \otimes \mathrm{id}_V)\sigma^{-1},$$

where $c' = \tau_{V,V} \circ c \circ \tau_{V,V}$ and σ is the automorphism of $V \otimes V \otimes V$ defined by $\sigma(v_1 \otimes v_2 \otimes v_3) = v_3 \otimes v_2 \otimes v_1$ for $v_1, v_2, v_3 \in V$. $\square$

Example 2. Let us solve the Yang-Baxter equation when $V = V_1 = V_{1,1}$ is the 2-dimensional simple module over the Hopf algebra $U_q = U_q(\mathfrak{sl}(2))$ considered in Chapters VI–VII. More precisely, let us find all U_q-automorphisms of $V_1 \otimes V_1$ that are R-matrices. We freely use the notation of the above-mentioned chapters. Recall that if v_0 is a highest weight vector of V_1, then the set $\{v_0, v_1 = Fv\}$ is a basis of V_1. By the Clebsch-Gordan Theorem VII.7.1 we have $V_1 \otimes V_1 \cong V_2 \oplus V_0$. Lemma VII.7.2 implies that the vectors

$$w_0 = v_0 \otimes v_0 \quad \text{and} \quad t = v_0 \otimes v_1 - q^{-1} v_1 \otimes v_0$$

are highest weight vectors of respective weights q^2 and 1. We complete the set of linearly independent vectors $\{w_0, t\}$ into a basis for $V \otimes V$ by setting

$$w_1 = Fw_0 = q^{-1} v_0 \otimes v_1 + v_1 \otimes v_0 \quad \text{and} \quad w_2 = \frac{1}{[2]} F^2 w_0 = v_1 \otimes v_1$$

where $[2] = q + q^{-1}$.

Proposition VIII.1.3. *Any U_q-linear automorphism φ of $V_1 \otimes V_1$ is diagonalizable and of the form $\varphi(w_i) = \lambda w_i$ $(i = 0, 1, 2)$ and $\varphi(t) = \mu t$ where λ and μ are non-zero scalars. The automorphism φ is an R-matrix if and only if*

$$(\lambda - \mu)(q\lambda + q^{-1}\mu)(q^{-1}\lambda + q\mu) = 0.$$

PROOF. Since φ is U_q-linear, the image under φ of a highest weight vector is a highest weight vector of the same weight. Now, w_0 and t have different weights (we still assume that $q^2 \neq 1$); therefore, there exist λ and μ such that $\varphi(w_0) = \lambda w_0$ and $\varphi(t) = \mu t$.

As for the remaining basis vectors, we have

$$\varphi(w_i) = \frac{1}{[i]} \varphi(F^i w_0) = \frac{1}{[i]} F^i \varphi(w_0) = \lambda w_i$$

for $i = 1, 2$. This completes the proof of the first assertion in Proposition 1.3.

The second assertion results from tedious computation. Let us give some details. We first observe that the matrix Φ of φ with respect to the basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$ is given by

$$\Phi = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \alpha & \gamma & 0 \\ 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

where

$$\alpha = \frac{q^{-1}\lambda + q\mu}{[2]}, \quad \beta = \frac{q\lambda + q^{-1}\mu}{[2]}, \quad \gamma = \frac{\lambda - \mu}{[2]}.$$

The automorphisms $\varphi \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varphi$ can be expressed, respectively, by the 8×8-matrices Φ_{12} and Φ_{23} in the basis consisting of the elements $v_0 \otimes v_0 \otimes v_0$, $v_0 \otimes v_0 \otimes v_1$, $v_0 \otimes v_1 \otimes v_0$, $v_0 \otimes v_1 \otimes v_1$, $v_1 \otimes v_0 \otimes v_0$, $v_1 \otimes v_0 \otimes v_1$, $v_1 \otimes v_1 \otimes v_0$, and $v_1 \otimes v_1 \otimes v_1$ of $V \otimes V \otimes V$ where

$$\Phi_{12} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

and

$$\Phi_{23} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Now, $\Phi_{12}\Phi_{23}\Phi_{12} - \Phi_{23}\Phi_{12}\Phi_{23}$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K & -\alpha\beta\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha\beta\gamma & L & 0 & \alpha\beta\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & -K & 0 & \alpha\beta\gamma & 0 & 0 \\ 0 & 0 & \alpha\beta\gamma & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\beta\gamma & 0 & -L & \alpha\beta\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\beta\gamma & -M & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $K = \alpha((\lambda - \alpha)\lambda - \gamma^2)$, $L = \alpha\beta(\alpha - \beta)$ and $M = \beta(\gamma^2 + \lambda(\beta - \lambda))$. Suppose that we have proved that K, L and M are multiples of $\alpha\beta\gamma$. Then

$$\Phi_{12}\Phi_{23}\Phi_{12} - \Phi_{23}\Phi_{12}\Phi_{23} = \alpha\beta\gamma \times \Psi$$

where Ψ is a non-zero matrix. It follows that Φ is an R-matrix if and only if $\alpha\beta\gamma = 0$, which would complete the proof of Proposition 1.3.

It remains to show that K, L and M are multiples of $\alpha\beta\gamma$. An easy computation proves that

$$\lambda - \alpha = q\gamma, \quad \lambda - \beta = q^{-1}\gamma, \quad q^{-1}\lambda - \gamma = q^{-1}\alpha, \quad q\lambda - \gamma = q\beta$$

and $\beta - \alpha = (q - q^{-1})\gamma$. Therefore,

$$K = \alpha\gamma(q\lambda - \gamma) = q\,\alpha\beta\gamma, \quad L = -(q - q^{-1})\,\alpha\beta\gamma$$

and $M = \beta\gamma(\gamma - q^{-1}\lambda) = -q^{-1}\alpha\beta\gamma$. □

To sum up, the R-matrices of the U_q-module $V_1 \otimes V_1$ belong to the following three types depending on a parameter $\lambda \neq 0$:

1. If $\lambda = \mu$, φ is a homothety.
2. If $q\lambda + q^{-1}\mu = 0$, then

$$\Phi = q\lambda \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

3. If $q^{-1}\lambda + q\mu = 0$, then

$$\Phi = q^{-1}\lambda \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

It is clear that Cases 2 and 3 are equivalent within a change of basis after exchanging q and q^{-1}. As we shall see in the next example, the minimal polynomial of Φ is of degree ≤ 2.

Example 3. We now give an important class of R-matrices with *quadratic minimal polynomial.* Such R-matrices will be used in Chapter XII to construct isotopy invariants of links in $\mathbf{R}^3$.

Let V be a finite-dimensional vector space with a basis $\{e_1, \ldots, e_N\}$. For two invertible scalars p, q and for any family $\{r_{ij}\}_{1 \leq i,j \leq N}$ of scalars in k such that $r_{ii} = q$ and $r_{ij}r_{ji} = p$ when $i \neq j$, we define an automorphism c of $V \otimes V$ by

$$\begin{aligned} c(e_i \otimes e_i) &= q\, e_i \otimes e_i \\ c(e_i \otimes e_j) &= \begin{cases} r_{ji}\, e_j \otimes e_i & \text{if } i < j \\ r_{ji}\, e_j \otimes e_i + (q - pq^{-1})\, e_i \otimes e_j & \text{if } i > j. \end{cases} \end{aligned}$$

Proposition VIII.1.4. *The automorphism c is a solution of the Yang-Baxter equation. Moreover, we have*

$$(c - q\,\mathrm{id}_{V \otimes V})(c + pq^{-1}\,\mathrm{id}_{V \otimes V}) = 0,$$

or, equivalently, $c^2 - (q - pq^{-1})c - p\,\mathrm{id}_{V \otimes V} = 0.$

Proof. (a) We first show that c is an R-matrix. In order to simplify the proof, let us introduce the following notation. The symbol (ijk) will stand for the vector $e_i \otimes e_j \otimes e_k$, and $[i > j]$ for the integer 1 if $i > j$ and for 0 otherwise. Then c can be redefined as

$$c(e_i \otimes e_j) = r_{ji} e_j \otimes e_i + [i > j]\beta e_i \otimes e_j$$

where $\beta = q - pq^{-1}$.

An immediate computation yields

$$\begin{aligned}
&(c\otimes \mathrm{id})(\mathrm{id}\otimes c)(c\otimes \mathrm{id})((ijk))\\
&\quad= r_{ji}r_{ki}r_{kj}(kji) + r_{ji}r_{ki}[j>k]\beta(jki)\\
&\qquad +r_{kj}r_{ki}[i>j]\beta(kij) + r_{kj}[i>j][j>k]\beta^2(ikj)\\
&\qquad +r_{ji}\Big([j>i][i>k] + [i>j][j>k]\Big)\beta^2(jik)\\
&\qquad +\Big(r_{ji}r_{ij}[i>k]\beta + [i>j][j>k]\beta^3\Big)(ijk)
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{id}\otimes c)(c\otimes \mathrm{id})(\mathrm{id}\otimes c)((ijk))\\
&\quad= r_{ji}r_{ki}r_{kj}(kji) + r_{ji}r_{ki}[j>k]\beta(jki)\\
&\qquad +r_{kj}r_{ki}[i>j]\beta(kij) + r_{ji}[i>k][j>k]\beta^2(jik)\\
&\qquad +r_{kj}\Big([i>k][k>j] + [i>j][j>k]\Big)\beta^2(ikj)\\
&\qquad +\Big(r_{jk}r_{kj}[i>k]\beta + [i>j][j>k]\beta^3\Big)(ijk).
\end{aligned}$$

We have to prove that these expressions are equal for all i, j, k. This is clearly the case if $i = j = k$. If i, j, k are distinct indices, they are equal in view of relations of the type

$$[i>j][i>k] = [i>j][j>k] + [i>k][k>j]$$

which express the fact that for distinct indices, we have $i > j$ and $i > k$ if and only if $i > j > k$ or $i > k > j$. If exactly two indices are equal, say $i = j \neq k$, then the desired equality is equivalent to $r_{ii}^2 = \beta r_{ii} + p$, which holds since $r_{ii} = q$ and $\beta = q - pq^{-1}$.

(b) One computes $c^2 - \beta c - p\,\mathrm{id}_{V\otimes V}$ on any vector of the form $e_i \otimes e_j$. If $i \neq j$, one immediately obtains 0. If $i = j$, one gets $(q^2 - \beta q - p)(e_i \otimes e_i)$, which is zero because of the value given to β. □

Consider the following two special cases:

(i) If $p = q^2$ and $r_{ij} = q$ for all i, j, then c is a homothety.

(ii) Take $p = 1$ and $r_{ij} = 1$ for $i \neq j$. Then c takes the form shown in Case 3 of Example 2 when V is two-dimensional. Thus, Example 2 turns out to be a special case of Example 3.

VIII.2 Braided Bialgebras

The aim of this section is to define the concept of a braided bialgebra. The importance of this concept comes from the fact proved in Section 3 that braided bialgebras generate solutions of the Yang-Baxter equation.

Definition VIII.2.1. *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. We call it quasi-cocommutative if there exists an invertible element R of the algebra $H \otimes H$ such that for all $x \in H$ we have*

$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1}. \tag{2.1}$$

Here $\Delta^{\mathrm{op}} = \tau_{H,H} \circ \Delta$ denotes the opposite coproduct on H. An element R satisfying this condition is called a *universal R-matrix*. It is part of the data of a quasi-cocommutative bialgebra. Any cocommutative bialgebra is quasi-cocommutative with universal R-matrix equal to $R = 1 \otimes 1$. Thus we can look upon a quasi-cocommutative bialgebra as a bialgebra whose non-cocommutativity is controlled by its universal R-matrix.

If we set $R = \sum_i s_i \otimes t_i$, then Relation (2.1) can be expressed, for all $x \in H$, by

$$\sum_{(x),i} x''s_i \otimes x't_i = \sum_{(x),i} s_i x' \otimes t_i x'' \tag{2.2}$$

using Sweedler's sigma notation introduced in III.1. We also define a *quasi-cocommutative Hopf algebra* as a Hopf algebra whose underlying bialgebra has a universal R-matrix.

Convention. The following notation will be used extensively in the sequel. Let H be an algebra and $X = \sum_i x_i^{(1)} \otimes \ldots \otimes x_i^{(p)} \in H^{\otimes p}$ $(p > 1)$. For any p-tuple $(k_1, \ldots, k_p)$ of distinct elements of $\{1, \ldots, n\}$ $(n \geq p)$, we denote by $X_{k_1 \ldots k_p}$ the element of $H^{\otimes n}$ given by

$$X_{k_1 \ldots k_p} = \sum_i y_i^{(1)} \otimes \ldots \otimes y_i^{(n)}$$

where $y_i^{(k_j)} = x_i^{(j)}$ for any $j \leq p$ and $y_i^{(k)} = 1$ otherwise. For instance, if $R = \sum_i s_i \otimes t_i$, then R_{31} will be the element of $H^{\otimes 3}$ given by

$$R_{31} = \sum_i t_i \otimes 1 \otimes s_i.$$

We now introduce the main concept of this section.

Definition VIII.2.2. *A quasi-cocommutative bialgebra $(H, \mu, \eta, \Delta, \varepsilon, R)$ or a quasi-cocommutative Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ is braided if the universal R-matrix R satisfies the two relations*

$$(\Delta \otimes \mathrm{id}_H)(R) = R_{13}R_{23} \tag{2.3}$$

and

$$(\mathrm{id}_H \otimes \Delta)(R) = R_{13}R_{12}. \tag{2.4}$$

Braided bialgebras are central in the theory of quantum groups and R-matrices. In the literature, notably in Drinfeld's papers [Dri87] [Dri89a] where this concept was defined for the first time, braided bialgebras are called *quasi-triangular* bialgebras. We call them braided because their categories of modules are braided in a sense that will be explained in Chapter XIII.

If $R = \sum_i s_i \otimes t_i$, Relations (2.3) and (2.4) can be expressed respectively as

$$\sum_{i,(s_i)} (s_i)' \otimes (s_i)'' \otimes t_i = \sum_{i,j} s_i \otimes s_j \otimes t_i t_j \tag{2.5}$$

and

$$\sum_{i,(t_i)} s_i \otimes (t_i)' \otimes (t_i)'' = \sum_{i,j} s_i s_j \otimes t_j \otimes t_i. \tag{2.6}$$

Example 1. Cocommutative bialgebras are braided with universal R-matrix $R = 1 \otimes 1$.

Here is a non-trivial example.

Example 2. (*Sweedler's four-dimensional Hopf algebra*) Let H be the algebra generated by two elements x, y and relations

$$x^2 = 1, \quad y^2 = 0, \quad yx + xy = 0.$$

The set $\{1, x, y, xy\}$ forms a basis of the underlying vector space. There is a unique Hopf algebra structure on H such that

$$\begin{aligned} \Delta(x) &= x \otimes x, & \varepsilon(x) &= 1, & S(x) &= x, \\ \Delta(y) &= 1 \otimes y + y \otimes x, & \varepsilon(y) &= 0, & S(y) &= xy. \end{aligned}$$

Observe that the antipode S is of order 4 and that, for any $a \in H$, we have $S^2(a) = xax^{-1}$. Set

$$R_\lambda = \frac{1}{2}\Big(1\otimes 1 + 1\otimes x + x\otimes 1 - x\otimes x\Big) + \frac{\lambda}{2}\Big(y\otimes y + y\otimes xy + xy\otimes xy - xy\otimes y\Big)$$

where λ is any scalar. It is easy to show that R_λ satisfies the conditions of Definition 2.2, thus endowing H with the structure of a braided Hopf algebra for any scalar λ. Observe that $R_\lambda^{-1} = \tau_{H,H}(R_\lambda)$.

We now investigate a few properties of universal R-matrices. The following lemma will be useful later. It shows how to form new quasi-cocommutative Hopf algebras from a given one.

Lemma VIII.2.3. (a) *If $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ is a quasi-cocommutative Hopf algebra whose antipode S is bijective, then so are*

$$(H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon, S^{-1}, S, R^{-1}), \quad (H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon, S^{-1}, S, R^{-1})$$

and $(H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon, S^{-1}, S, \tau_{H,H}(R))$.

(b) *If, furthermore,* $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ *is braided, then so is*

$$(H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon, S^{-1}, S, \tau_{H,H}(R)).$$

PROOF. (a) As a result of Corollary III.3.5, we see that $(H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon, S^{-1})$ and $(H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon, S^{-1})$ are Hopf algebras. In $(H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon, S^{-1})$, Relation (2.1) reads $\Delta^{\mathrm{op}}(x) = R^{-1}\Delta(x)R$, whereas it becomes

$$\Delta(x) = R^{-1}\Delta^{\mathrm{op}}(x)R \quad \text{and} \quad \Delta(x) = \tau_{H,H}(R)\Delta^{\mathrm{op}}(x)\tau_{H,H}(R)^{-1}$$

in $(H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon, S^{-1})$, which proves Part (a).

(b) According to (a), the Hopf algebra $(H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon, S^{-1}, S, \tau(R))$ is quasi-commutative. We now have to check Relations (2.3) and (2.4).

Let us start with $(\Delta \otimes \mathrm{id}_H)(R) = R_{13}R_{23}$ and let us apply the transposition (12) to it. We get

$$(\Delta^{\mathrm{op}} \otimes \mathrm{id}_H)(R) = R_{23}R_{13}.$$

We now use the circular permutation (123) to obtain

$$(\mathrm{id}_H \otimes \Delta^{\mathrm{op}})(R) = \Big(\tau_{H,H}(R)\Big)_{13}\Big(\tau_{H,H}(R)\Big)_{12}.$$

Similarly, one shows that Relation (2.4) for R implies Relation (2.3) for $\tau_{H,H}(R)$. $\square$

Theorem VIII.2.4. *Let* $(H, \mu, \eta, \Delta, \varepsilon, R)$ *be a braided bialgebra.*

(a) *Then the universal R-matrix R satisfies the equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.7}$$

and we have

$$(\varepsilon \otimes \mathrm{id}_H)(R) = 1 = (\mathrm{id}_H \otimes \varepsilon)(R). \tag{2.8}$$

(b) *If, moreover, H has an invertible antipode, then*

$$(S \otimes \mathrm{id}_H)(R) = R^{-1} = (\mathrm{id}_H \otimes S^{-1})(R) \tag{2.9}$$

and

$$(S \otimes S)(R) = R. \tag{2.10}$$

Using the above conventions, in any braided Hopf algebra H whose universal R-matrix is of the form $R = \sum_i s_i \otimes t_i$, Relations (2.7–2.9) are equivalent to

$$\sum_{i,j,k} s_k s_j \otimes t_k s_i \otimes t_j t_i = \sum_{i,j,k} s_j s_i \otimes s_k t_i \otimes t_k t_j, \tag{2.11}$$

$$\sum_i \varepsilon(s_i)t_i = \sum_i s_i \varepsilon(t_i) = 1, \tag{2.12}$$

and

$$R^{-1} = \sum_i S(s_i) \otimes t_i = \sum_i s_i \otimes S^{-1}(t_i). \tag{2.13}$$

PROOF. (a) Relation (2.3) and the definition of R imply

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes \mathrm{id})(R) \\ &= (\Delta^{\mathrm{op}} \otimes \mathrm{id})(R)R_{12} \\ &= (\tau_{H,H} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R)R_{12} \\ &= (\tau_{H,H} \otimes \mathrm{id})(R_{13}R_{23})R_{12} \\ &= R_{23}R_{13}R_{12}. \end{aligned}$$

From $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$ and from (2.3), we get

$$R = (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(R_{13}R_{23}) = (\varepsilon \otimes \mathrm{id})(R)\varepsilon(1)R.$$

Since $\varepsilon(1) = 1$ and R is invertible, we obtain $(\varepsilon \otimes \mathrm{id})(R) = 1$. Similarly, we use the relation $(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$ and (2.4) to derive $(\mathrm{id} \otimes \varepsilon)(R) = 1$.

(b) Now suppose that H has an invertible antipode S. We know that the antipode verifies $\mu(S \otimes \mathrm{id})\Delta(x) = \varepsilon(x)1$ for all $x \in H$. This implies

$$(\mu \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (\varepsilon \otimes \mathrm{id})(R) = 1$$

from (2.8). Consequently,

$$1 = (\mu \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(R_{13}R_{23}) = (S \otimes \mathrm{id})(R)\, S(1)R.$$

Since $S(1) = 1$, we get

$$(S \otimes \mathrm{id})(R) = R^{-1}. \tag{2.14}$$

Replace $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ by the braided Hopf algebra

$$(H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon, S^{-1}, S, \tau_{H,H}(R))$$

of Lemma 2.3 (b). Then Relation (2.14) becomes

$$(S^{-1} \otimes \mathrm{id})(\tau_{H,H}(R)) = \tau_{H,H}(R)^{-1},$$

which is clearly equivalent to $(\mathrm{id} \otimes S^{-1})(R) = R^{-1}$. Finally, we have

$$\begin{aligned} (S \otimes S)(R) &= (\mathrm{id} \otimes S)(S \otimes \mathrm{id})(R) \\ &= (\mathrm{id} \otimes S)(R^{-1}) \\ &= (\mathrm{id} \otimes S)(\mathrm{id} \otimes S^{-1})(R) \\ &= (\mathrm{id} \otimes \mathrm{id})(R) \\ &= R. \end{aligned}$$

□

In Chapter XIII we shall give a categorical interpretation of Relations (2.3) and (2.4). Here we give another one in terms of algebra and coalgebra maps. Indeed, with the universal R-matrix R we can build two linear maps ${}_R\lambda$ and λ_R from the dual vector space H^* into H. They are defined by

$$ {}_R\lambda(\alpha) = \sum_i \alpha(s_i)\, t_i \quad \text{and} \quad \lambda_R(\alpha) = \sum_i s_i \alpha(t_i) \tag{2.15} $$

where $R = \sum_i s_i \otimes t_i$ and α is any linear form on H. We endow the dual space H^* with its canonical algebra structure and, if H is finite-dimensional, with its canonical coalgebra structure.

Proposition VIII.2.5. *Let $(H, \mu, \eta, \Delta, \varepsilon, R)$ be a braided bialgebra. Then ${}_R\lambda$ is an algebra map and λ_R is an algebra antimorphism. Moreover, if H is finite-dimensional, then λ_R is a coalgebra map and ${}_R\lambda$ is a coalgebra antimorphism.*

PROOF. We first prove that ${}_R\lambda$ is an algebra map. Let us compute ${}_R\lambda(\varepsilon)$. From (2.12) we get ${}_R\lambda(\varepsilon) = \sum_i \varepsilon(s_i)t_i = 1$, which shows that ${}_R\lambda$ sends the unit of H^* to the unit of H. Now, let α and β be linear forms on H. Then by (2.3), or its equivalent form (2.5), we have

$$ \begin{aligned} {}_R\lambda(\alpha\beta) &= \sum_i (\alpha\beta)(s_i)\, t_i = \sum_i (\alpha \otimes \beta)(\Delta(s_i))\, t_i \\ &= \sum_{i,j} \alpha(s_i)\beta(s_j)\, t_i t_j = {}_R\lambda(\alpha)\, {}_R\lambda(\beta), \end{aligned} $$

which proves that ${}_R\lambda$ preserves the multiplications. One may show in an analogous way that λ_R is an algebra antimorphism using (2.4).

Now assume that H is finite-dimensional. Then the dual space H^* has a coalgebra structure. Its comultiplication Δ satisfies

$$ \alpha(xy) = \Delta(\alpha)(x \otimes y) = \sum_{(\alpha)} \alpha'(x)\alpha''(y). $$

In order to prove that λ_R is a coalgebra map, we first have to check that

$$ \Delta\, \lambda_R = (\lambda_R \otimes \lambda_R)\Delta. $$

Now, we have

$$ \Delta(\lambda_R(\alpha)) = \sum_i \Delta(s_i)\alpha(t_i) = (\mathrm{id} \otimes \alpha)(\Delta \otimes \mathrm{id})(R); $$

so using (2.5) we get $\Delta(\lambda_R(\alpha)) = \sum_{i,j} \alpha(t_i t_j)\, s_i \otimes s_j$. On the other hand,

$$\begin{aligned}(\lambda_R \otimes \lambda_R)\Delta(\alpha) &= \sum_{(\alpha)} \lambda_R(\alpha') \otimes \lambda_R(\alpha'') \\ &= \sum_{i,j,(\alpha)} s_i \alpha'(t_i) \otimes s_j \alpha''(t_j) \\ &= \sum_{i,j} \alpha(t_i t_j)\, s_i \otimes s_j \\ &= \Delta(\lambda_R(\alpha)).\end{aligned}$$

We next prove that λ_R preserves counits. Using (2.12), we get

$$\varepsilon\, \lambda_R(\alpha) = \varepsilon\Big(\sum_{(\alpha),i} s_i \alpha(t_i)\Big) = \alpha\Big(\sum_{(\alpha),i} \varepsilon(s_i) t_i\Big) = \alpha(1) = \varepsilon(\alpha).$$

One similarly proves that ${}_R\lambda$ is a coalgebra antimorphism using (2.4). □

VIII.3 How a Braided Bialgebra Generates R-Matrices

We now prove the existence of a solution of the Yang-Baxter equation on every module over a braided bialgebra $(H, \mu, \eta, \Delta, \varepsilon, R)$.

Let V and W be two H-modules. The universal R-matrix R in $H \otimes H$ allows us to build a natural isomorphism $c^R_{V,W}$ of H-modules between $V \otimes W$ and $W \otimes V$. This isomorphism generalizes the flip $\tau_{V,W}$ between the factors V and W and is defined for all $v \in V$ and $w \in W$ by

$$c^R_{V,W}(v \otimes w) = \tau_{V,W}\Big(R(v \otimes w)\Big) = \sum_i t_i w \otimes s_i v \tag{3.1}$$

where $R = \sum_i s_i \otimes t_i$. By (2.13) $c^R_{V,W}$ is an isomorphism with inverse given by

$$(c^R_{V,W})^{-1}(w \otimes v) = R^{-1}(v \otimes w) = \sum_i S(s_i)v \otimes t_i w = \sum_i s_i v \otimes S^{-1}(t_i) w. \tag{3.2}$$

The latter two equalities hold only when H has an invertible antipode.

Proposition VIII.3.1. *Under the previous hypotheses,*
(a) *the map $c^R_{V,W}$ is an isomorphism of H-modules, and*
(b) *for any triple (U, V, W) of H-modules, we have*

$$c^R_{U\otimes V,W} = (c^R_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c^R_{V,W}), \quad c^R_{U,V\otimes W} = (\mathrm{id}_V \otimes c^R_{U,W})(c^R_{U,V} \otimes \mathrm{id}_W)$$

and

$$\begin{aligned}&(c^R_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c^R_{U,W})(c^R_{U,V} \otimes \mathrm{id}_W) \\ &\qquad = (\mathrm{id}_W \otimes c^R_{U,V})(c^R_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c^R_{V,W}).\end{aligned}$$

PROOF. (a) We have to prove that $c^R_{V,W}$ is H-linear. Now, by (2.1), for any $x \in H$ we have

$$\begin{aligned} c^R_{V,W}(x(v\otimes w)) &= \tau_{V,W}\Big(R\Delta(x)(v\otimes w)\Big) \\ &= \tau_{V,W}\Big(\Delta^{\mathrm{op}}(x)R(v\otimes w)\Big) \\ &= \Delta(x)\tau_{V,W}\Big(R(v\otimes w)\Big) \\ &= x\Big(c^R_{V,W}(v\otimes w)\Big). \end{aligned}$$

(b) We prove the second and the last relations, leaving the first one to the reader. For $u \in U$, $v \in V$ and $w \in W$ we get using (2.6)

$$\begin{aligned} (\mathrm{id}_V \otimes c^R_{U,W})(c^R_{U,V}\otimes \mathrm{id}_W)(u\otimes v\otimes w) &= \sum_{i,j} t_iv\otimes t_jw\otimes s_js_iu \\ &= \sum_i (t_i)'v\otimes (t_i)''w\otimes s_iu \\ &= \sum_i \Delta(t_i)(v\otimes w)\otimes s_iu \\ &= c^R_{U,V\otimes W}(u\otimes v\otimes w). \end{aligned}$$

As for the last relation in Part (b) of Proposition 3.1, we have

$$(c^R_{V,W}\otimes \mathrm{id}_U)(\mathrm{id}_V\otimes c^R_{U,W})(c^R_{U,V}\otimes\mathrm{id}_W)(u\otimes v\otimes w) = \sum_{i,j,k} t_kt_jw\otimes s_kt_iv\otimes s_js_iu$$

and

$$(\mathrm{id}_W\otimes c^R_{U,V})(c^R_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes c^R_{V,W})(u\otimes v\otimes w) = \sum_{i,j,k} t_jt_iw\otimes t_ks_iv\otimes s_ks_ju.$$

Both right-hand sides are equal in view of (2.11). An alternative proof will be given in XIII.1. □

Setting $U = V = W$ in Part (b) of Proposition 3.1, we conclude that $c^R_{V,V}$ is a solution of the Yang-Baxter equation for any H-module V. This efficient way of producing R-matrices explains why the element R is called the universal R-matrix of H. Observe that if $R = 1\otimes 1$, then $c^R_{V,W} = \tau_{V,W}$ is the flip. We have already remarked in Proposition III.5.1 that the flip was an isomorphism of H-modules for cocommutative H.

VIII.4 The Square of the Antipode in a Braided Hopf Algebra

As we observed in Theorem III.3.4, the antipode S of a cocommutative Hopf algebra is an involution: $S^2 = \mathrm{id}_H$. In the quasi-cocommutative case,

S^2 is in general not equal to the identity. Nevertheless, as we shall see in this section, it is an inner automorphism.

Let $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ be a quasi-cocommutative Hopf algebra with an invertible antipode. Consider the element u of H given by

$$u = \sum_i S(t_i)s_i \tag{4.1}$$

where $R = \sum_i s_i \otimes t_i$. Set $R^{-1} = \sum_i \bar{s}_i \otimes \bar{t}_i$.

Proposition VIII.4.1. *Under the previous hypothesis, the element u is invertible in H with inverse given by*

$$u^{-1} = \sum_i S^{-1}(\bar{t}_i)\,\bar{s}_i \tag{4.2}$$

and for all $x \in H$ we have

$$S^2(x) = uxu^{-1}. \tag{4.3}$$

PROOF. Let us first show that $S^2(x)u = ux$ for all x. If y belongs to $H \otimes H$, Relation (2.1) implies the equality

$$(\Delta^{\mathrm{op}} \otimes \mathrm{id})(y)(R \otimes 1) = (R \otimes 1)(\Delta \otimes \mathrm{id})(y)$$

in $H \otimes H \otimes H$. When $y = \Delta(x)$ for some $x \in H$, we get

$$\sum_{i,(x)} x''s_i \otimes x't_i \otimes x''' = \sum_{i,(x)} s_ix' \otimes t_ix'' \otimes x'''.$$

To the latter relation we apply the linear map from $H \otimes H \otimes H$ to H which is $\mathrm{id}_H \otimes S \otimes S^2$ composed with the multiplication from right to left. This yields

$$\sum_{i,(x)} S^2(x''')S(x't_i)x''s_i = \sum_{i,(x)} S^2(x''')S(t_ix'')s_ix',$$

or, equivalently,

$$\sum_{i,(x)} S^2(x''')S(t_i)S(x')x''s_i = \sum_{i,(x)} S^2(x''')S(x'')S(t_i)s_ix', \tag{4.4}$$

since the antipode is an antiautomorphism of algebra. Let us first evaluate the left-hand side of (4.4). By definition of the antipode and of the counit, we have

$$\sum_{(x)} S(x')x'' \otimes x''' = \sum_{(x)} \varepsilon(x')1 \otimes x'' = 1 \otimes x.$$

Hence, $\sum_{(x)} S(x')x'' \otimes S^2(x''') = 1 \otimes S^2(x)$. Multiplying both sides on the right by $\sum_i s_i \otimes S(t_i)$, we get

$$\sum_{(x),i} S(x')x''s_i \otimes S^2(x''')S(t_i) = \sum_i s_i \otimes S^2(x)S(t_i).$$

Consequently, for the left-hand side of (4.4), we have

$$\sum_{i,(x)} S^2(x''')S(t_i)S(x')x''s_i = \sum_i S^2(x)S(t_i)s_i = S^2(x)u. \tag{4.5}$$

The relation $S^2(x)u = ux$ will then be a consequence of (4.4–4.5) and of

$$\sum_{i,(x)} S^2(x''')S(x'')S(t_i)s_i x' = ux. \tag{4.6}$$

Let us prove (4.6).

$$\sum_{(x)} x' \otimes S(x''S(x''')) = \sum_{(x)} x' \otimes S(\varepsilon(x'')1) = \sum_{(x)} x'\varepsilon(x'') \otimes S(1) = x \otimes 1.$$

Multiplying by $u \otimes 1$ on the left, we get

$$\sum_{i,(x)} S(t_i)s_i x' \otimes S^2(x''')S(x'') = ux \otimes 1,$$

which implies (4.6) after applying the multiplication in H.

It remains to show that u is invertible. Set

$$v = \sum_i S^{-1}(\bar{t}_i)\,\bar{s}_i \tag{4.7}$$

where $R^{-1} = \sum_i \bar{s}_i \otimes \bar{t}_i$. Then

$$uv = \sum_i uS^{-1}(\bar{t}_i)\,\bar{s}_i = \sum_i S(\bar{t}_i)u\bar{s}_i$$

from the first part of the proof. Consequently,

$$uv = \sum_{i,j} S(t_j\bar{t}_i)\,s_j\bar{s}_i = S(1)1 = 1$$

since $\sum_{i,j} s_j\bar{s}_i \otimes t_j\bar{t}_i = RR^{-1} = 1 \otimes 1$. It follows that $1 = uv = S^2(v)u$, which implies that u is left and right invertible with inverse v. □

Observe that $S^2(u) = u$ and $S^2(u^{-1}) = u^{-1}$.

Corollary VIII.4.2. *Under the hypotheses of Proposition* 4.1, *we have* $uS(u) = S(u)u$. *This element is central in* H.

Proof. Let x be any element in H. Applying S to $ux = S^2(x)u$ implies $S(x)S(u) = S(u)S^3(x)$. Replacing x by $S^{-1}(x)$, we get

$$xS(u) = S(u)S^2(x) = S(u)uxu^{-1},$$

hence $xS(u)u = S(u)ux$. This proves that $S(u)u$ is central in H. For $x = u$, this formula leads to $uS(u) = S(u)u$. □

As we already know, any module V over a Hopf algebra H with invertible antipode has two duals V^* and ${}^*V$. As vector spaces, both coincide with the vector space of linear forms on V. However, the H-actions are different: On V^* an element a of H acts on a linear form α by

$$< a\alpha, - > = < \alpha, S(a)- >$$

whereas on ${}^*V$ it acts by

$$< a\alpha, - > = < \alpha, S^{-1}(a) - > .$$

Using the defining property of the antipode we observe that the evaluation maps $V^* \otimes V \to k$ and $V \otimes {}^*V \to k$ are H-linear (notice the precise order of the tensorands). The element u induces an isomorphism between both duals as recorded in the next proposition.

Proposition VIII.4.3. *If H is a quasi-cocommutative Hopf algebra, then the map $\alpha \mapsto \alpha(u?)$ from V^* to ${}^*V$ is an isomorphism of H-modules.*

PROOF. By $\alpha(u?)$ we mean the linear form $v \mapsto \alpha(uv)$. Set $j(\alpha) = \alpha(u?)$. The map j is bijective because u is invertible. Let us show that j is H-linear. For any $v \in V$, Relation (4.3) implies

$$\begin{aligned} < j(a\alpha), v > &= < \alpha(S(a)u?), v > \\ &= < \alpha, S(a)uv > \\ &= < \alpha, S^2(S^{-1}(a))uv > \\ &= < \alpha, uS^{-1}(a)v > \\ &= < j(\alpha), S^{-1}(a)v > \\ &= < aj(\alpha), v > . \end{aligned}$$

□

Define the biduals V^{**} and ${}^{**}V$ by $V^{**} = (V^*)^*$ and ${}^{**}V = {}^*({}^*V)$. The reader is invited to prove the following proposition.

Proposition VIII.4.4. *Under the hypotheses of Proposition* 4.3, *the map $v \mapsto < - , uv >$ [resp. the map $v \mapsto < - , u^{-1}v >$] from V to V^{**} [resp. to ${}^{**}V$] is an H-linear injective map.*

We now assume that H is *braided*. Then by (2.13) and by Proposition 4.1, the inverse of u is given by

$$u^{-1} = \sum_i S^{-1}(t_i)S(s_i) = \sum_i S^{-2}(t_i)s_i. \tag{4.8}$$

In the braided case, we have the following additional relations for u.

Proposition VIII.4.5. *If H is a braided Hopf algebra, then the element u satisfies the relations*

$$\varepsilon(u) = 1, \quad \Delta(u) = (R_{21}R)^{-1}(u \otimes u) = (u \otimes u)(R_{21}R)^{-1},$$

$$\Delta(S(u)) = (R_{21}R)^{-1}\Big(S(u) \otimes S(u)\Big) = \Big(S(u) \otimes S(u)\Big)(R_{21}R)^{-1}$$

and for the central element $uS(u)$ we have

$$\Delta(uS(u)) = (R_{21}R)^{-2}\Big(uS(u) \otimes uS(u)\Big) = \Big(uS(u) \otimes uS(u)\Big)(R_{21}R)^{-2}.$$

PROOF. (a) The relation $\varepsilon(u) = 1$ follows from (2.12).

(b) Let us compute $\Delta(u)$. Applying the flip $\tau_{H,H}$ to (2.1), we get

$$\Delta(a) = R_{21}\Delta^{\mathrm{op}}(a)R_{21}^{-1} \tag{4.9}$$

for all $a \in H$. Relations (2.1) and (4.9) imply

$$\Delta(a)R_{21}R = R_{21}R\,\Delta(a) \tag{4.10}$$

for all $a \in H$. In view of (4.10) it is enough to show that $\Delta(u)R_{21}R = u \otimes u$. By (4.10) again and by Theorem III.3.4 we have

$$\begin{aligned} \Delta(u)R_{21}R &= \sum_i \Delta(S(t_i))\Delta(s_i)R_{21}R \\ &= \sum_i (S \otimes S)(\Delta^{\mathrm{op}}(t_i))\Delta(s_i)R_{21}R \\ &= \sum_i (S \otimes S)(\Delta^{\mathrm{op}}(t_i))R_{21}R\Delta(s_i). \end{aligned}$$

We now let the algebra $H^{\otimes 4}$ act on $H \otimes H$ on the *right* by

$$(a \otimes b) \cdot (A \otimes B) = (S \otimes S)(B)(a \otimes b)A$$

where $a, b \in H$ and $A, B \in H \otimes H$. We can rewrite the previous equalities as

$$\Delta(u)R_{21}R = R_{21} \cdot (R_{12}R_{13}R_{23}R_{14}R_{24}).$$

By (2.7) this equals $R_{21} \cdot (R_{23}R_{13}R_{12}R_{14}R_{24})$, which we now evaluate. Using Relation (2.13), which gives the inverse of R, we get

$$\begin{aligned} R_{21} \cdot R_{23} &= \sum_{i,j} S(t_j)\,t_i \otimes s_i s_j \\ &= (S \otimes \mathrm{id})\Big(\sum_{i,j} S^{-1}(t_i)t_j \otimes s_i s_j\Big) \\ &= (S \otimes \mathrm{id})(R_{21}^{-1}R_{21}) \\ &= (S \otimes \mathrm{id})(1 \otimes 1) \\ &= 1 \otimes 1. \end{aligned}$$

Hence,

$$R_{21} \cdot (R_{23}R_{13}) = (1 \otimes 1) \cdot R_{13} = \sum_i S(t_i)s_i \otimes 1 = u \otimes 1.$$

Next,

$$R_{21} \cdot (R_{23}R_{13}R_{12}) = (u \otimes 1) \cdot R_{12} = (u \otimes 1)R$$

and

$$\begin{aligned}
R_{21} \cdot (R_{23}R_{13}R_{12}R_{14}) &= (u \otimes 1)\Big(\sum_{i,j} s_i s_j \otimes S(t_j)t_i\Big) \\
&= (u \otimes 1)(\mathrm{id} \otimes S)\Big(\sum_{i,j} s_i s_j \otimes S^{-1}(t_i)t_j\Big) \\
&= (u \otimes 1)(\mathrm{id} \otimes S)(R^{-1}R) \\
&= (u \otimes 1)(\mathrm{id} \otimes S)(1 \otimes 1) \\
&= (u \otimes 1).
\end{aligned}$$

Finally, we have

$$R_{21} \cdot (R_{23}R_{13}R_{12}R_{14}R_{24}) = (u \otimes 1) \cdot R_{24} = (u \otimes 1)(1 \otimes u) = u \otimes u,$$

which is what we wished to prove.

(c) The formula for $\Delta(S(u))$ is an easy consequence of the formula for $\Delta(u)$ and of $(S \otimes S) \circ \Delta = \Delta^{\mathrm{op}} \circ S$, which was proved in Theorem III.3.4.

(d) The last relation follows from (b), (c) and the centrality of $uS(u)$.

□

VIII.5 A Dual Concept: Cobraided Bialgebras

Just as braided bialgebras induce R-matrices on their modules, there are bialgebras inducing R-matrices on their comodules. These are the cobraided bialgebras which we now define.

Definition VIII.5.1. *A cobraided bialgebra $(H, \mu, \eta, \Delta, \varepsilon, r)$ is a bialgebra H together with a linear form r on $H \otimes H$ satisfying the conditions*

(i) *there exists a linear form $\bar{r}$ on $H \otimes H$ such that*

$$r * \bar{r} = \bar{r} * r = \varepsilon, \tag{5.1}$$

(ii) *we have*

$$\mu^{\mathrm{op}} = r * \mu * \bar{r}, \tag{5.2}$$

(iii) *and*

$$r(\mu \otimes \mathrm{id}_H) = r_{13} * r_{23} \quad \textit{and} \quad r(\mathrm{id}_H \otimes \mu) = r_{13} * r_{12} \tag{5.3}$$

where $*$ is the convolution operation on linear forms, and the linear forms r_{12}, r_{23} and r_{13} are defined by

$$r_{12} = r \otimes \varepsilon, \quad r_{23} = \varepsilon \otimes r, \quad r_{13} = (\varepsilon \otimes r)(\tau_{H,H} \otimes \mathrm{id}_H).$$

The linear form r is called the universal R-form of H. A Hopf algebra is cobraided if the underlying bialgebra is.

This definition is dual to Definition 2.2. More precisely, Relation (5.2) is dual to Relation (2.1), whereas Relations (5.3) correspond to Relations (2.3–2.4). Conditions (5.1–5.3) can be reexpressed in the following way. For any triple (x, y, z) of elements of H we have

(i)

$$\sum_{(x)(y)} r(x' \otimes y')\bar{r}(x'' \otimes y'') = \sum_{(x)(y)} \bar{r}(x' \otimes y')r(x'' \otimes y'') = \varepsilon(x)\varepsilon(y), \quad (5.4)$$

(ii)

$$yx = \sum_{(x)(y)} r(x' \otimes y')x''y''\bar{r}(x''' \otimes y'''), \quad (5.5)$$

(iii)

$$r(xy \otimes z) = \sum_{(x)(y)(z)} r(x' \otimes z')\varepsilon(y')\varepsilon(x'')r(y'' \otimes z'') = \sum_{(z)} r(x \otimes z')r(y \otimes z'') \quad (5.6)$$

and

$$r(x \otimes yz) = \sum_{(x)(y)(z)} r(x' \otimes z')\varepsilon(y')\varepsilon(z'')r(x'' \otimes y'') = \sum_{(x)} r(x' \otimes z)r(x'' \otimes y). \quad (5.7)$$

A bialgebra satisfying only Conditions (i) and (ii) of Definition 5.1 may be called *quasi-commutative* by analogy with the quasi-cocommutative case of Section 2.

We now show how the universal R-form r of a cobraided bialgebra H induces a solution of the Yang-Baxter equation on any H-comodule. The map $c^R_{V,W}$ defined in (3.1) for a braided bialgebra H with a universal R-matrix R and H-modules V, W is the composition of the maps

$$V \otimes W \xrightarrow{R \otimes \mathrm{id}_{V \otimes W}} H \otimes H \otimes V \otimes W \longrightarrow$$
$$\xrightarrow{\mathrm{id}_H \otimes \tau_{H,V} \otimes \mathrm{id}_W} H \otimes V \otimes H \otimes W \quad \xrightarrow{\mu_V \otimes \mu_W} V \otimes W \xrightarrow{\tau_{V,W}} W \otimes V$$

where μ_V and μ_W are the actions of H on V and W respectively and where we have identified R with the linear map from k to $H \otimes H$, sending 1 to R.

Let H be a cobraided bialgebra with universal R-form r. Given the H-comodules V and W with respective coactions $\Delta_V : V \to H \otimes V$ and $\Delta_W : W \to H \otimes W$, we define the linear map

$$c^r_{V,W} : V \otimes W \to W \otimes V$$

by analogy with the above definition of $c^R_{V,W}$ as the composition of the maps

$$V \otimes W \xrightarrow{\tau_{V,W}} W \otimes V \xrightarrow{\Delta_W \otimes \Delta_V} H \otimes W \otimes H \otimes V \longrightarrow \\ \xrightarrow{\mathrm{id}_H \otimes \tau_{W,H} \otimes \mathrm{id}_V} H \otimes H \otimes W \otimes V \xrightarrow{r \otimes \mathrm{id}_{W \otimes V}} W \otimes V \quad (5.8)$$

(this was obtained by reversing the arrows and interchanging V and W). Using the conventions of III.6 we can rewrite this definition for any $v \in V$ and $w \in W$ as

$$c^r_{V,W}(v \otimes w) = \sum_{(v)(w)} r(w_H \otimes v_H)\, w_W \otimes v_V. \quad (5.9)$$

Proposition VIII.5.2. (a) *Under the previous hypotheses, the map $c^r_{V,W}$ is an isomorphism of H-comodules.*

(b) *If U is a third H-comodule, we have*

$$c^r_{U \otimes V,W} = (c^r_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c^r_{V,W})$$

and

$$c^r_{U,V \otimes W} = (\mathrm{id}_V \otimes c^r_{U,W})(c^r_{U,V} \otimes \mathrm{id}_W).$$

Moreover, we have

$$(c^r_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c^r_{U,W})(c^r_{U,V} \otimes \mathrm{id}_W) \\ = (\mathrm{id}_W \otimes c^r_{U,V})(c^r_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c^r_{V,W}).$$

Setting $U = V = W$ in the last relation, we see that $c^r_{V,V}$ is a solution of the Yang-Baxter equation.

PROOF. (a) We use Condition (5.1) to prove that $c^r_{V,W}$ is invertible. Define a linear map $\bar{c}^r_{V,W}$ from $W \otimes V$ to $V \otimes W$ by

$$\bar{c}^r_{V,W}(w \otimes v) = \sum_{(v)(w)} \bar{r}(v_H \otimes w_H)\, v_V \otimes w_W.$$

We claim that $\bar{c}^r_{V,W}$ is an inverse to $c^r_{V,W}$. Let us show that it is a left inverse. We have

$$\begin{aligned}
&(\bar{c}^r_{V,W} \circ c^r_{V,W})(v \otimes w) \\
&= \sum_{(v)(w)} r(w_H \otimes v_H)\, \bar{r}\Big((v_V)_H \otimes (w_W)_H\Big)\, (v_V)_V \otimes (w_W)_W \\
&= \sum_{(v)(w)} r\Big((w_H)' \otimes (v_H)'\Big)\, \bar{r}\Big((w_H)'' \otimes (v_H)''\Big)\, v_V \otimes w_W \\
&= \sum_{(v)(w)} \varepsilon(w_H)\varepsilon(v_H) v_V \otimes w_W \\
&= v \otimes w.
\end{aligned}$$

The second equality follows from the coassociativity of the coactions while the third one is a consequence of Relation (5.1) and the last one follows from the counitarity of the coactions. One proves that $\bar{c}^r_{V,W}$ is a right inverse to $c^r_{V,W}$ in a similar way.

We now prove that Relation (5.2) implies that $c^r_{V,W}$ is a map of comodules, namely we have

$$\Delta_{W\otimes V} \circ c^r_{V,W} = (\mathrm{id}_H \otimes c^r_{V,W}) \circ \Delta_{V\otimes W}.$$

This is equivalent to

$$\begin{aligned} & r(w_H \otimes v_H)\,(w_W)_H(v_V)_H \otimes (w_W)_W \otimes (v_V)_V \\ &= \sum_{(v)(w)} r\Big((w_W)_H \otimes (v_V)_H\Big)\, v_H w_H \otimes (w_W)_W \otimes (v_V)_V \end{aligned}$$

for any $v \in V$ and $w \in W$. Now by the coassociativity of the coactions, the previous relation can be rewritten as

$$\begin{aligned} & \sum_{(v)(w)} r\Big((w_H)' \otimes (v_H)'\Big)\,(w_H)''(v_H)'' \otimes w_W \otimes v_V \\ &= \textstyle\sum_{(v)(w)} (v_H)'(w_H)' r\Big((w_H)'' \otimes (v_H)''\Big) \otimes w_W \otimes v_V. \end{aligned}$$

The latter is a consequence of $r * \mu = \mu^{\mathrm{op}} * r$, which is equivalent to Relation (5.2) after convolution with r.

(b) Let us prove that $c^r_{U\otimes V,W} = (c^r_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c^r_{V,W})$. We have

$$\begin{aligned} & (c^r_{U,W} \otimes \mathrm{id}_V)\Big((\mathrm{id}_U \otimes c^r_{V,W})(u \otimes v \otimes w)\Big) \\ &= \sum_{(u)(v)(w)} r(w_H \otimes v_H)\, r\Big((w_W)_H \otimes u_H\Big)\,(w_W)_W \otimes u_U \otimes v_V \\ &= \sum_{(u)(v)(w)} r\Big((w_H)' \otimes v_H\Big)\, r\Big((w_H)'' \otimes u_H\Big)\, w_W \otimes u_U \otimes v_V \\ &= \sum_{(u)(v)(w)} r(w_H \otimes u_H v_H)\, w_W \otimes u_U \otimes v_V \\ &= c^r_{U\otimes V,W}(u \otimes v \otimes w). \end{aligned}$$

The second equality follows from the coassociativity, and the third one from Relation (5.7). One proves that $c^r_{U,V\otimes W} = (\mathrm{id}_V \otimes c^r_{U,W})(c^r_{U,V} \otimes \mathrm{id}_W)$ in a similar way.

The last relation of Proposition 5.2 is a consequence of the previous relations and of the naturality of the maps c^r. We leave the proof to the reader. A proof in a more general context will be given in XIII.1. $\square$

VIII.6 The FRT Construction

We have just seen that a cobraided bialgebra gives rise to an R-matrix on each comodule. Conversely, Faddeev, Reshetikhin and Takhtadjian showed in [RTF89] that any R-matrix c in $\mathrm{Aut}(V \otimes V)$ on a finite-dimensional vector space V can be obtained as in Section 5 from a cobraided bialgebra $A(c)$ coacting on V. The Faddeev-Reshetikhin-Takhtadjian construction (FRT construction for short) is based on the following theorem.

Theorem VIII.6.1. *Let V be a finite-dimensional vector space and c an endomorphism of $V \otimes V$. There exists a bialgebra $A(c)$ together with a linear map $\Delta_V : V \to A(c) \otimes V$ such that*

(i) *the map Δ_V equips V with the structure of a comodule over $A(c)$,*

(ii) *the map c becomes a comodule map with respect to this structure,*

(iii) *if A' is another bialgebra coacting on V via a linear map Δ'_V such that Condition* (ii) *is satisfied, then there exists a unique bialgebra morphism $f : A(c) \to A'$ such that*

$$\Delta'_V = (f \otimes \mathrm{id}_V) \circ \Delta_V.$$

The bialgebra $A(c)$ is unique up to isomorphism.

The proof will be given in several steps.

1. In the first one, we define $A(c)$ as an algebra. Let $\{v_i\}_{1 \leq i \leq N}$ be a basis of V and let the coefficients c_{ij}^{mn} be defined by

$$c(v_i \otimes v_j) = \sum_{1 \leq m,n \leq N} c_{ij}^{mn} \, v_m \otimes v_n.$$

Pick a family of indeterminates T_i^j, where i and j both run over the set $\{1, \ldots, N\}$.

Definition VIII.6.2. *The algebra $A(c)$is the quotient of the free algebra F generated by the family $(T_i^j)_{1 \leq i,j \leq N}$ by the two-sided ideal $I(c)$ generated by all elements C_{ij}^{mn} where*

$$C_{ij}^{mn} = \sum_{1 \leq k,\ell \leq N} c_{ij}^{k\ell} T_k^m T_\ell^n - \sum_{1 \leq k,\ell \leq N} T_i^k T_j^\ell c_{k\ell}^{mn} \tag{6.1}$$

and i, j, m and n run over the indexing set.

2. We put a bialgebra structure on $A(c)$.

Lemma VIII.6.3. *There is a unique bialgebra structure on $A(c)$ such that*

$$\Delta(T_i^j) = \sum_{1 \leq k \leq N} T_i^k \otimes T_k^j \quad \textit{and} \quad \varepsilon(T_i^j) = \delta_{ij}. \tag{6.2}$$

PROOF. It is clear that the above formulas define unique algebra maps

$$\Delta : F \to F \otimes F \quad \text{and} \quad \varepsilon : F \to k.$$

To check the coassociativity and the counitarity, it is enough to check these on the generators T_i^j, which is easy and done as in III.4 and in IV.5. We also have to prove that $I(c)$ is a coideal, i.e., that

$$\Delta(I(c)) \subset I(c) \otimes F + F \otimes I(c) \quad \text{and} \quad \varepsilon(I(c)) = 0.$$

We have

$$\begin{aligned}
\Delta(C_{ij}^{mn}) &= \sum_{k,\ell,p,q} c_{ij}^{k\ell} T_k^p T_\ell^q \otimes T_p^m T_q^n - \sum_{k,\ell,p,q} T_i^p T_j^q \otimes T_p^k T_q^\ell c_{k\ell}^{mn} \\
&= \sum_{p,q} C_{ij}^{pq} \otimes T_p^m T_q^n + \sum_{k,\ell,p,q} T_i^k T_j^\ell c_{k\ell}^{pq} \otimes T_p^m T_q^n \\
&\qquad + \sum_{p,q} T_i^p T_j^q \otimes C_{pq}^{mn} - \sum_{k,\ell,p,q} T_i^p T_j^q \otimes c_{pq}^{k\ell} T_k^m T_\ell^n \\
&= \sum_{p,q} C_{ij}^{pq} \otimes T_p^m T_q^n + \sum_{p,q} T_i^p T_j^q \otimes C_{pq}^{mn}
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon(C_{ij}^{mn}) &= \sum_{k,\ell} c_{ij}^{k\ell}\, \varepsilon(T_k^m T_\ell^n) - \sum_{k,\ell} \varepsilon(T_i^k T_j^\ell)\, c_{k\ell}^{mn} \\
&= \sum_{k,\ell} c_{ij}^{k\ell} \delta_{km} \delta_{\ell n} - \sum_{k,\ell} \delta_{ik} \delta_{j\ell} c_{k\ell}^{mn} \\
&= c_{ij}^{mn} - c_{ij}^{mn} = 0.
\end{aligned}$$

□

3. We now let $A(c)$ coact on V. Define a linear map Δ_V from V to $A(c) \otimes V$ on the basis $\{v_i\}_{1 \le i \le N}$ by

$$\Delta_V(v_i) = \sum_{1 \le j \le N} T_i^j \otimes v_j. \tag{6.3}$$

It is an easy exercise to check that this endows V with a left comodule structure over the bialgebra $A(c)$.

4. We prove that the endomorphism c of $V \otimes V$ is a comodule map for the coaction we have just defined. The coaction Δ_V induces on $V \otimes V$ a coaction $\Delta_{V \otimes V}$ defined by

$$\Delta_{V \otimes V}(v_i \otimes v_j) = \sum_{1 \le k,\ell \le N} T_i^k T_j^\ell \otimes v_k \otimes v_\ell.$$

Then c is a comodule map if and only if

$$\begin{aligned} &\Delta_{V\otimes V}\Big(c(v_i\otimes v_j)\Big) - (\mathrm{id}_{A(c)}\otimes c)\Big(\Delta_{V\otimes V}(v_i\otimes v_j)\Big) \\ &= \sum_{k,\ell,m,n} T_k^m T_\ell^n \otimes c_{ij}^{k\ell} v_m \otimes v_n - \sum_{k,\ell,m,n} T_i^k T_j^\ell \otimes c_{k\ell}^{mn} v_m \otimes v_n \end{aligned}$$

vanishes in $A(c)\otimes V$. Now, it is clear that the last expression is equal to

$$\sum_{m,n} C_{ij}^{mn} \otimes v_m \otimes v_n,$$

which is zero by definition of $A(c)$.

5. We now establish the universality of $A(c)$. Let (A', Δ'_V) be a pair satisfying the conditions of Theorem 6.1. Then there exists a family $(u_i^j)_{1\leq i,j\leq N}$ of elements of A' uniquely determined by

$$\Delta'_V(v_i) = \sum_{1\leq j\leq N} u_i^j \otimes v_j.$$

The coassociativity and the counitarity of Δ'_V imply that

$$\Delta(u_i^j) = \sum_{1\leq k\leq N} u_i^k \otimes u_k^j \quad \text{and} \quad \varepsilon(u_i^j) = \delta_{ij}.$$

Condition (ii) of Theorem 6.1 is equivalent to the vanishing of

$$\Delta'_{V\otimes V}\Big(c(v_i\otimes v_j)\Big) - (\mathrm{id}_{A(c)}\otimes c)\Big(\Delta'_{V\otimes V}(v_i\otimes v_j)\Big)$$

for all i and j, in other words to the vanishing of

$$\sum_{1\leq k,\ell\leq N} c_{ij}^{k\ell} u_k^m u_\ell^n - \sum_{1\leq k,\ell\leq N} u_i^k u_j^\ell c_{k\ell}^{mn}$$

for all i, j, m and n. From this it is clear that the map f from F to A' defined by $f(T_i^j) = u_i^j$ for all i and j extends to a bialgebra map factoring through $A(c)$. Let us check the relation $\Delta'_V = (f\otimes \mathrm{id}_V)\Delta_V$. For any i we have

$$(f\otimes \mathrm{id}_V)(\Delta_V(v_i)) = \sum_{1\leq j\leq N} f(T_i^j)\otimes v_j = \sum_{1\leq j\leq N} u_i^j \otimes v_j = \Delta'_V(v_i).$$

Conversely, the relation $\Delta'_V = (f\otimes \mathrm{id}_V)\Delta_V$ necessarily implies $f(T_i^j) = u_i^j$, which proves the uniqueness of f along with the fact that the family (T_i^j) generates the algebra $A(c)$. This completes the proof of Theorem 6.1.

Theorem VIII.6.4. *Assume in addition to the hypotheses of Theorem* 6.1 *that the endomorphism c of $V\otimes V$ is a solution of the Yang-Baxter equation. Then there exists a unique linear form r on $A(c)\otimes A(c)$ turning $A(c)$ into a cobraided bialgebra such that $c^r_{V,V} = c$. We have*

$$r(T_i^m \otimes T_j^n) = c_{ji}^{mn} \tag{6.4}$$

for all i, j, m and n.

The rest of this section is devoted to the proof of Theorem 6.4.

(a) Suppose $A(c)$ is cobraided with a universal R-form r such that the automorphism $c^r_{V,V}$ coincides with the given R-matrix c. By (5.9) and (6.3) we have

$$c(v_j \otimes v_i) = \sum_{m,n} r(T_i^m \otimes T_j^n) v_m \otimes v_n.$$

On the other hand, we have $c(v_j \otimes v_i) = \sum_{m,n} c_{ji}^{mn} v_m \otimes v_n$. It follows that $r(T_i^m \otimes T_j^n) = c_{ji}^{mn}$ for all i, j, m, n. Relations (5.6–5.7) imply the uniqueness assertion in Theorem 6.4.

(b) We now prove the existence of r. We first have to define r on the whole space $A(c)\otimes A(c)$. Let W be the vector subspace of F spanned by the set $\{T_i^j\}_{1\leq i,j\leq N}$. We define $r : W \otimes W \to k$ by (6.4). Conditions (5.3) and

$$r(1\otimes T_i^j) = r(T_i^j \otimes 1) = \varepsilon(T_i^j) = \delta_{ij}$$

allow one to extend r into a linear form, still denoted r, on $F\otimes F$.

In order to prove that r defines a form on $A(c)\otimes A(c)$, we have to prove the following lemma.

Lemma VIII.6.5. *We have $r(F\otimes I(c)) = r(I(c)\otimes F) = 0$.*

PROOF. First, we observe that

$$r(1\otimes I(c)) = r(I(c)\otimes 1) = \varepsilon(I(c)) = 0.$$

Using Conditions (5.3), we see that it is now enough to show that the images $r(T_p^q \otimes C_{ij}^{mn})$ and $r(C_{ij}^{mn}\otimes T_p^q)$ vanish for all i, j, m, n, p, q. We have

$$\begin{aligned} r(T_p^q \otimes C_{ij}^{mn}) &= \sum_{k,\ell} c_{ij}^{k\ell}\, r(T_p^q \otimes T_k^m T_\ell^n) - \sum_{k,\ell} r(T_p^q \otimes T_i^k T_j^\ell)\, c_{k\ell}^{mn} \\ &= \sum_{k,\ell,r} c_{ij}^{k\ell}\, r(T_p^r \otimes T_\ell^n)\, r(T_r^q \otimes T_k^m) \\ &\qquad - \sum_{k,\ell,r} r(T_p^r \otimes T_j^\ell)\, r(T_r^q \otimes T_i^k)\, c_{k\ell}^{mn} \\ &= \sum_{k,\ell,r} c_{ij}^{k\ell} c_{\ell p}^{rn} c_{kr}^{qm} - \sum_{k,\ell,r} c_{jp}^{r\ell} c_{ir}^{qk} c_{k\ell}^{mn}, \end{aligned}$$

which is zero in view of (1.1), i.e., of the Yang-Baxter equation. $\square$

(c) Now that r is defined, we have to check the conditions of Definition 5.1. This will be done in several steps.

1. Conditions (iii) are satisfied by definition of r.

2. Condition (i): we have to prove that r is invertible with respect to the convolution, namely that there exists a linear form $\bar{r}$ on $A(c) \otimes A(c)$ such that $r * \bar{r} = \bar{r} * r = \varepsilon$. We define $\bar{r}$ on the generators T_i^j by

$$\bar{r}(T_i^m \otimes T_j^n) = (c^{-1})_{ij}^{nm} \quad \text{and} \quad \bar{r}(1 \otimes T_i^m) = \bar{r}(T_i^m \otimes 1) = \varepsilon(T_i^m) = \delta_{im}$$

where the coefficients $(c^{-1})_{ij}^{nm}$ are defined in terms of the inverse c^{-1} of c by

$$c^{-1}(v_i \otimes v_j) = \sum_{m,n} (c^{-1})_{ij}^{mn} v_m \otimes v_n.$$

Lemma VIII.6.6. *The above formulas define a unique linear form $\bar{r}$ on $A(c) \otimes A(c)$ such that for all x, y in $A(c)$ we have*

$$\bar{r}(xy \otimes z) = \sum_{(z)} \bar{r}(y \otimes z')\,\bar{r}(x \otimes z'') \tag{6.5}$$

and

$$\bar{r}(x \otimes yz) = \sum_{(x)} \bar{r}(x' \otimes y)\,\bar{r}(x'' \otimes z). \tag{6.6}$$

PROOF. The proof is similar to the proof of Lemma 6.5. Use the fact that c^{-1} is also a solution of the Yang-Baxter equation. □

We now check Relation (5.4). Let us prove that

$$\sum_{(x)(y)} r(x' \otimes y')\,\bar{r}(x'' \otimes y'') = \varepsilon(x)\varepsilon(y) \tag{6.7}$$

by induction on the degrees of x and y. If x or y is of degree zero, this is immediate. If both x and y are of degree 1, this follows from the subsequent computation. For $x = T_i^m$ and $y = T_j^n$ we have

$$\sum_{p,q} r(T_i^p \otimes T_j^q)\,\bar{r}(T_p^m \otimes T_q^n) = \sum_{p,q} c_{ji}^{pq}(c^{-1})_{pq}^{nm} = \delta_{im}\delta_{jn} = \varepsilon(T_i^m)\varepsilon(T_j^n).$$

The second equality results from the fact that c^{-1} is the inverse of c. The general case follows from the next lemma.

Lemma VIII.6.7. *If Relation* (6.7) *is verified by the couples (x, y), (x, z) and (y, z), then it also holds for the couples (x, yz) and (xy, z).*

PROOF. We give the proof for the couple (x, yz). The proof for (xy, z) is similar. In view of Relation (5.7) and Relations (6.6–6.7) we have

$$\begin{aligned}
&\sum_{(x)(y)(z)} r(x' \otimes y'z')\bar{r}(x'' \otimes y''z'') \\
&\quad = \sum_{(x)(y)(z)} r(x' \otimes z')r(x'' \otimes y')\bar{r}(x''' \otimes y'')\bar{r}(x'''' \otimes z'') \\
&\quad = \sum_{(x)(z)} r(x' \otimes z')\varepsilon(x'')\varepsilon(y)\bar{r}(x''' \otimes z'') \\
&\quad = \sum_{(x)(z)} \varepsilon(y)r(x' \otimes z')\bar{r}(x'' \otimes z'') \\
&\quad = \varepsilon(y)\varepsilon(x)\varepsilon(z) \\
&\quad = \varepsilon(x)\varepsilon(yz).
\end{aligned}$$

□

The relation $\sum_{(x),(y)} \bar{r}(x' \otimes y')\, r(x'' \otimes y'') = \varepsilon(x)\varepsilon(y)$ is proved similarly.

3. Condition (ii): We have to check that for any x and y in $A(c)$ we have

$$\sum_{(x),(y)} r(x' \otimes y')x''y'' = \sum_{(x),(y)} y'x'r(x'' \otimes y''). \tag{6.8}$$

We proceed as for Condition (i), namely we first check (6.8) in case $x = 1$ or $y = 1$ when it is trivial and in case $x = T_i^m$ and $y = T_j^n$, then show that if (6.8) is true for (x, y), (x, z) and (y, z), then it is for (x, yz) and (xy, z). Firstly, we have

$$\begin{aligned}
\sum_{p,q} r(T_i^p \otimes T_j^q)T_p^m T_q^n &= \sum_{p,q} c_{ji}^{pq} T_p^m T_q^n \\
&= \sum_{p,q} T_j^q T_i^p c_{qp}^{mn} \\
&= \sum_{p,q} T_j^q T_i^p r(T_p^m \otimes T_q^n)
\end{aligned}$$

because of the defining relations of $A(c)$.

We continue with the following analogue of Lemma 6.7.

Lemma VIII.6.8. *If (6.8) is verified by the couples (x, y), (x, z) and (y, z), then it is by the couples (x, yz) and (xy, z).*

PROOF. Suppose (6.8) is true for (x, y) and for (x, z). Then for (x, yz) we have

$$\sum_{(x)(y)(z)} r(x' \otimes y'z')x''y''z'' = \sum_{(x)(y)(z)} r(x' \otimes z')r(x'' \otimes y')x'''y''z''$$

$$\begin{aligned}
&= \sum_{(x)(y)(z)} r(x' \otimes z')y'x''r(x''' \otimes y'')z'' \\
&= \sum_{(x)(y)(z)} y'r(x' \otimes z')x''z''r(x''' \otimes y'') \\
&= \sum_{(x)(y)(z)} y'z'x'r(x'' \otimes z'')r(x''' \otimes y'') \\
&= \sum_{(x)(y)(z)} y'z'x'r(x'' \otimes y''z'').
\end{aligned}$$

The other cases are proved similarly. □

This completes the proof of Theorem 6.4.

VIII.7 Application to $GL_q(2)$ and $SL_q(2)$

In this section we show that the bialgebra $M_q(2)$ and the Hopf algebras $GL_q(2)$ and $SL_q(2)$ defined in Chapter IV are cobraided.

Let V be a two-dimensional vector space with basis $\{v_1, v_2\}$ and let c be the automorphism of $V \otimes V$ whose matrix with respect to the basis $\{v_1 \otimes v_1, v_2 \otimes v_2, v_1 \otimes v_2, v_2 \otimes v_1\}$ is

$$q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & q - q^{-1} \end{pmatrix} \tag{7.1}$$

where $q^{1/2}$ is an invertible scalar. This matrix has been displayed in Section 1 where we proved it was an R-matrix. The FRT construction associates to c a cobraided bialgebra $A(c)$ which we now describe.

Proposition VIII.7.1. *The bialgebra $A(c)$ associated to the R-matrix* (7.1) *is isomorphic to the bialgebra $M_q(2)$ of Definition* IV.3.2.

PROOF. Let $T_1^1 = a$, $T_1^2 = b$, $T_2^1 = c$ and $T_2^2 = d$. By the FRT construction, $A(c)$ is the algebra generated by a, b, c, d and the sixteen relations which can be written in the following compact matrix form

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & q - q^{-1} \end{pmatrix} \begin{pmatrix} a^2 & b^2 & ab & ba \\ c^2 & d^2 & cd & dc \\ ac & bd & ad & bc \\ ca & db & cb & da \end{pmatrix}$$
$$= \begin{pmatrix} a^2 & b^2 & ab & ba \\ c^2 & d^2 & cd & dc \\ ac & bd & ad & bc \\ ca & db & cb & da \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & q - q^{-1} \end{pmatrix}.$$

An easy computation shows that these relations are equivalent to the six relations

$$\begin{array}{rclrcl} ba & = & qab, & db & = & qbd, \\ ca & = & qac, & dc & = & qcd, \\ cb & = & bc, & da - ad & = & (q - q^{-1})bc, \end{array}$$

defining the algebra $M_q(2)$. This identifies $A(c)$ and $M_q(2)$ as algebras. The corresponding comultiplications are clearly the same (compare (6.2) and Theorem IV.5.1). □

From this and from Theorem 6.4, we deduce the following important result on $M_q(2)$.

Corollary VIII.7.2. *The bialgebra $M_q(2)$ has a unique structure as a cobraided bialgebra with universal R-form r determined by*

$$\begin{pmatrix} r(a \otimes a) & r(b \otimes b) & r(a \otimes b) & r(b \otimes a) \\ r(c \otimes c) & r(d \otimes d) & r(c \otimes d) & r(d \otimes c) \\ r(a \otimes c) & r(b \otimes d) & r(a \otimes d) & r(b \otimes c) \\ r(c \otimes a) & r(d \otimes b) & r(c \otimes b) & r(d \otimes a) \end{pmatrix} = \lambda \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 1 & q - q^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\lambda = q^{-1/2}$.

It is easy to check that the coaction of $A(c)$ on the two-dimensional vector space V coincides with the coaction of $M_q(2)$ on the elements of degree 1 of the quantum plane $k_q[x, y]$ (see IV.7).

We now show that $GL_q(2)$ and $SL_q(2)$ are cobraided with the same universal R-form. Since $SL_q(2)$ is a quotient of $GL_q(2)$, it is enough to prove this for $SL_q(2)$. We start with the following lemma.

Lemma VIII.7.3. *For all $x \in M_q(2)$ we have*

$$r(x \otimes \det_q) = r(\det_q \otimes x) = \varepsilon(x).$$

Recall that $\det_q = da - qbc$ is the quantum determinant introduced in Chapter IV.

PROOF. Suppose we have proved that the relations in Lemma 7.3 hold for two elements x and y. Since $\det_q$ is grouplike by Theorem IV.5.1, we deduce from (5.6) that

$$r(xy \otimes \det_q) = r(x \otimes \det_q)\, r(y \otimes \det_q) = \varepsilon(x)\varepsilon(y) = \varepsilon(xy),$$

which reduces a proof of Lemma 7.3 to checking it for $x = a, b, c, d$.

For $x = a$ we have

$$\begin{aligned} r(a \otimes \det_q) = r(a \otimes a)r(a \otimes d) + r(b \otimes a)r(d \otimes d) \\ -qr(a \otimes c)r(a \otimes b) - qr(b \otimes c)r(d \otimes b) \end{aligned}$$

since $\Delta(a) = a \otimes a + b \otimes d$. Using Corollary 7.2, we get

$$r(a \otimes \det_q) = q^{-1/2} q q^{-1/2} + 0 - 0 - 0 = 1 = \varepsilon(a).$$

We leave the other verifications to the reader. □

Corollary VIII.7.4. *The Hopf algebras $GL_q(2)$ and $SL_q(2)$ are cobraided with universal R-form r.*

PROOF. Recall that $SL_q(2)$ is the quotient of $M_q(2)$ by the ideal I generated by the element $\det_q - 1$. Now, Lemma 7.3 is equivalent to the statement that

$$r\left((\det_q - 1) \otimes x\right) = r\left(x \otimes (\det_q - 1)\right) = 0$$

for all $x \in M_q(2)$. Therefore r vanishes on $I \otimes M_q(2)$ and on $M_q(2) \otimes I$, which proves that r defines a bilinear form on $SL_q(2)$. □

Remark 7.5. The normalization constant $q^{-1/2}$ in front of the R-matrix in (7.1) has been introduced precisely so as to have r vanish on the ideal I defining $SL_q(2)$.

VIII.8 Exercises

1. Consider a matrix of the form

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

   Show that it is a solution of the Yang-Baxter equation if and only if the following conditions are satisfied:

$$adb = adc = ad(a-d) = 0,$$

$$p^2 a = pa^2 + abc, \quad q^2 a = qa^2 + abc,$$

$$p^2 d = pd^2 + dbc, \quad q^2 d = qd^2 + dbc.$$

2. Consider the Hopf algebra H of Section 2, Example 2. Show that there exists an automorphism φ of the Hopf algebra H such that $(\varphi \otimes \varphi)(R_\lambda) = R_{\lambda'}$ if and only if there exists a non-zero scalar μ such that $\lambda' = \mu^2 \lambda$.

3. Find all (co)braided bialgebra structures on the group bialgebra of a finite group.

4. Let H be a finite-dimensional bialgebra and H^* be the dual bialgebra. Show that H^* is cobraided if and only if H is braided.

5. Let H be a cobraided bialgebra with universal R-form r. Show that

$$r_{23} * r_{13} * r_{12} = r_{12} * r_{13} * r_{23}.$$

6. Let $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, r)$ be a cobraided Hopf algebra with invertible antipode S.

   (a) Show that $r \circ (\eta \otimes \mathrm{id}_H) = r \circ (\mathrm{id}_H \otimes \eta) = \varepsilon$ and

$$r \circ (S \otimes \mathrm{id}_H) = \bar{r}, \quad \bar{r} \circ (\mathrm{id}_H \otimes S) = r, \quad r \circ (S \otimes S) = r.$$

   (b) Define a linear form u on H by $u = r \circ (\mathrm{id}_H \otimes S) \circ \Delta^{\mathrm{op}}$. Show that u is invertible as an element of H^* and that $S^2 = u * \mathrm{id}_H * \bar{u}$ where $\bar{u}$ denotes the inverse of u for the convolution.

7. Let $(H, \mu, \eta, \Delta, \varepsilon, r)$ be a cobraided bialgebra. Define linear maps ${}_r\lambda$ and λ_r from H to H^* by ${}_r\lambda(x) = r(- \otimes x)$ and $\lambda_r(x) = r(x \otimes -)$. Show that ${}_r\lambda$ is an algebra antimorphism and λ_r is an algebra map and, in case H is finite-dimensional, ${}_r\lambda$ is a coalgebra map and λ_r a coalgebra antimorphism.

8. Let A be the algebra $k\{s, t, t^{-1}\}/(s^2, st + ts)$. Show that the following formulas define a unique cobraided Hopf algebra structure on A:

$$\Delta(t) = t \otimes t, \quad \Delta(s) = s \otimes 1 + t^{-1} \otimes s,$$

$$\varepsilon(t) = 1, \quad \varepsilon(s) = 0, \quad S(t) = t^{-1}, \quad S(s) = st,$$

$$r(t \otimes t) = -1, \quad r(s \otimes t) = r(t \otimes s) = r(s \otimes s) = 0.$$

   Check that the antipode S is of order 4.

9. Let $c \in \mathrm{Aut}(V \otimes V)$ be a solution of the Yang-Baxter equation and let $c' = \tau_{V,V} \circ c \circ \tau_{V,V}$. Show that we have the following isomorphisms

$$A(c^{-1}) \cong A(c) \quad \text{and} \quad A(c') \cong A(c)^{\mathrm{op}}.$$

10. Let c be the R-matrix of Proposition 1.4. Prove that $A(c)$ is the algebra generated by $(T_i^j)_{1 \le i,j \le N}$ and the relations

$$r_{nm}\, T_i^m T_i^n = q\, T_i^n T_i^m, \quad r_{ji}\, T_j^m T_i^m = q\, T_i^m T_j^m,$$

$$r_{ji}\, T_j^n T_i^m = r_{nm}\, T_i^m T_j^n, \quad r_{ji}\, T_j^m T_i^n - r_{mn}\, T_i^n T_j^m = (q - pq^{-1})\, T_i^m T_j^n$$

   where i, j, m, n run over all positive integers $\le N$ such that $i < j$ and $m > n$.

11. Use the description of the universal R-form on $SL_q(2)$ to find an R-matrix on the $SL_q(2)$-comodule $k_q[x, y]$ described in IV.7.

VIII.9 Notes

The Yang-Baxter equation first came up in a paper by Yang [Yan67] as a factorization condition of the scattering S-matrix in the many-body problem in one dimension and in work of Baxter [Bax72] [Bax82] on exactly solvable models in statistical mechanics. It also played an important rôle in the quantum inverse scattering method created around 1978–79 by Faddeev, Sklyanin, Takhtadjian [Fad84] for the construction of quantum integrable systems. Attempts to find solutions of the Yang-Baxter equation in a systematic way have led to the theory of quantum groups (see [Dri87]). Many papers in the literature are devoted to the construction of R-matrices, e.g., [Dri85] [Dri87] [Jim86a] [Jim86b] [KS80], to quote but a few.

The concept of a quasi-cocommutative and of a braided (or quasi-triangular) Hopf algebra is due to Drinfeld [Dri87] [Dri89a]. For a review, see [Maj90b]. The four-dimensional Hopf algebra of Example 2 of Section 2 is due to Sweedler. The universal R-matrices R_λ were found by Radford [Rad93a].

The dual concept of cobraided bialgebras appears in [Hay92] [LT91] [Maj91b] [Sch92]. Cobraided bialgebras have properties dual to braided bialgebras. We gave some of them in Exercises 5–7.

The FRT construction is due to Faddeev, Reshetikhin and Takhtadjian [RTF89]. The bialgebras $M_{p,q}(2)$ and $M_q(n)$ of IV.10 can be obtained by this method (see Exercise 10). In Sections 5–6 we followed the treatment proposed by [LT91].

Exercise 1 is taken from [Kau91] and Exercise 2 from [Rad93a]. The cobraided Hopf algebra of Exercise 8 was found by Pareigis [Par81] before the advent of quantum groups.

Chapter IX
Drinfeld's Quantum Double

In the previous chapter we showed that braided Hopf algebras provided solutions of the Yang-Baxter equation. The problem is now to find enough such Hopf algebras. Drinfeld [Dri87] devised an ingenious method, the "quantum double construction", which builds a braided Hopf algebra out of any finite-dimensional Hopf algebra with invertible antipode. It is the goal of this chapter to describe this construction in full detail, and to show how to apply it to the finite-dimensional quotient of the Hopf algebra $U_q(\mathfrak{sl}(2))$ considered in VI.5. We also give a characterization of the modules over the quantum double in Section 5.

IX.1 Bicrossed Products of Groups

The quantum double construction is a special case of the bicrossed product construction. Since the latter is rather involved, we start with an analogous construction for groups, namely the bicrossed product of groups due to Takeuchi [Tak81]; it generalizes the notion of a semidirect product of groups.

Let G be a group with subgroups H and K. We assume that for any element x in G there exists a unique pair $(y, z) \in H \times K$ satisfying

$$x = yz. \tag{1.1}$$

This allows us to attach to any pair $(y, z) \in H \times K$ a unique element $z \cdot y$

in H and a unique element z^y in K such that

$$zy = (z \cdot y)\, z^y. \tag{1.2}$$

Let y, y' be elements of H and z, z' be elements of K. Expanding the associativity relations

$$(zz')y = z(z'y) \quad \text{and} \quad z(yy') = (zy)y'$$

gives us the relations

$$
\begin{aligned}
(zz') \cdot y &= z \cdot (z' \cdot y), && (1.3)\\
(zz')^y &= z^{z' \cdot y} z'^y, && (1.4)\\
z \cdot (yy') &= (z \cdot y)(z^y \cdot y'), && (1.5)\\
z^{yy'} &= (z^y)^{y'}. && (1.6)
\end{aligned}
$$

Similarly, we expand $z = z1$ and $y = 1y$, which implies

$$
\begin{aligned}
z \cdot 1 &= 1, && (1.7)\\
z^1 &= z, && (1.8)\\
1 \cdot y &= y, && (1.9)\\
1^y &= 1. && (1.10)
\end{aligned}
$$

Relations (1.3) and (1.9) mean precisely that the map $\alpha : K \times H \to H$ defined by

$$\alpha(z, y) = z \cdot y$$

is a left action of the group K on the set H. Similarly (1.6) and (1.8) mean that the map $\beta : K \times H \to K$ defined by

$$\beta(z, y) = z^y$$

is a right action of the group H on the set K. We make the following definition.

Definition IX.1.1. *A pair (H, K) of groups is said to be matched if there exist a left action α of the group K on the set H and a right action β of the group H on the set K, such that for all $y, y' \in H$ and $z, z' \in K$ we have*

$$(zz')^y = z^{z' \cdot y} z'^y, \tag{1.4}$$

$$z \cdot (yy') = (z \cdot y)(z^y \cdot y'), \tag{1.5}$$

$$z \cdot 1 = 1, \tag{1.7}$$

$$1^y = 1, \tag{1.10}$$

where $\alpha(z, y) = z \cdot y$ and $\beta(z, y) = z^y$.

Proposition IX.1.2. (a) *Let (H, K) be a matched pair of groups. There exists a unique group structure, denoted $H \bowtie K$, on the set-theoretic product $H \times K$ with unit $(1, 1)$ such that*

$$(y, z)(y', z') = (y(z \cdot y'), z^{y'} z')$$

for all $y, y' \in H$ and $z, z' \in K$. This group structure is called the bicrossed product of H and K. Furthermore, the groups H and K can be identified respectively with the subgroups $H \times \{1\}$ and $\{1\} \times K$ of $H \bowtie K$, and every element (y, z) in $H \bowtie K$ can be written uniquely as the product of an element of $H \times \{1\}$ and an element of $\{1\} \times K$:

$$(y, z) = (y, 1)(1, z)$$

where $y \in H$ and $z \in K$.

(b) *Conversely, let G be a group and H, K be subgroups of G such that the multiplication on G induces a set-theoretic bijection from $H \times K$ onto G. Then the pair (H, K) is necessarily matched and the previous bijection induces a group isomorphism from the bicrossed product $H \bowtie K$ onto G.*

PROOF. (a) It is easy to check that the above-defined product on $H \bowtie K$ is associative with $(1, 1)$ as unit. Details are left to the reader.

To prove that (y, z) is invertible in the bicrossed product, let us first look for elements $y' \in H$ and $z' \in K$ such that

$$(y, z)(y', z') = (1, 1).$$

By definition of the product, this is equivalent to the following two relations:

$$y(z \cdot y') = 1 \quad \text{and} \quad z^{y'} z' = 1.$$

From the first one we derive

$$y' = z^{-1} \cdot (z \cdot y') = z^{-1} \cdot y^{-1},$$

and then from the second one we get

$$z' = (z^{z^{-1} \cdot y^{-1}})^{-1}.$$

Set $(y', z')(y, z) = (Y, Z)$ where y' and z' are given the above values. We have to show that $(Y, Z) = (1, 1)$. Multiplying the last identity by (y, z) on the left, we get

$$(y, z) = (y, z)(Y, Z) = (y(z \cdot Y), z^Y Z).$$

This implies that

$$Y = z^{-1} \cdot (z \cdot Y) = z^{-1} \cdot 1 = 1 \quad \text{and} \quad Z = z^Y z^{-1} = z^1 z^{-1} = 1.$$

Thus, the element (y, z) is invertible with inverse equal to

$$(y, z)^{-1} = (z^{-1} \cdot y^{-1}, (z^{z^{-1} \cdot y^{-1}})^{-1}).$$

It is easy to check that

$$(y, 1)(y', 1) = (yy', 1), \quad (1, z)(1, z') = (1, zz') \quad \text{and} \quad (y, 1)(1, z) = (y, z),$$

which proves the remaining assertions of Part (a).

For the proof of Part (b), it suffices to review the arguments that led us to Definition 1.1. □

Example 1. (*Product of groups*) Let H and K be groups. We let each one act trivially on the other, which means, using the above notation, that

$$z \cdot y = y \quad \text{and} \quad z^y = z.$$

Then (H, K) is a matched pair, and the bicrossed product $H \bowtie K$ is isomorphic to the usual product of groups $H \times K$.

Example 2. (*Semidirect product of groups*) Let H and K be groups. We suppose that H acts trivially on K, which means that $z^y = z$, and that K acts on H by group automorphisms, which means that

$$z \cdot (yy') = (z \cdot y)(z \cdot y') \quad \text{and} \quad z \cdot 1 = 1$$

for all $y, y' \in H$ and $z \in K$. Then (H, K) is a matched pair and the bicrossed product $H \bowtie K$ is isomorphic to the semidirect product of K by H. In this case, the identity $(1, z)(y, 1)(1, z)^{-1} = ((z \cdot y), 1)$ proves that $H \times \{1\}$ is a normal subgroup of $H \bowtie K$ and that the action of K on H corresponds to the conjugation in the bicrossed product.

IX.2 Bicrossed Products of Bialgebras

We observed in Chapter III that the algebra of a group has a natural Hopf algebra structure. The question we raise now is this: Given a matched pair (H, K) of groups, can we build the algebra of the bicrossed product $H \bowtie K$ out of the group algebras $k[H]$ and $k[K]$? In order to answer this question, we first give a group algebra version of the action of a group on a set. Let us consider the case of a group G acting on a set X via a map

$$\alpha : G \times X \to X.$$

Linearizing, we get a morphism of coalgebras

$$\alpha : k[G \times X] \to k[X]$$

for the coalgebra structures introduced in III.1, Example 3. Composing with the natural isomorphism

$$k[G] \otimes k[X] \cong k[G \times X],$$

which is a coalgebra isomorphism by Proposition III.1.4, we see that the group action of G on the set X gives rise to an action of the Hopf algebra $k[G]$ on the coalgebra $k[X]$ such that the structural map

$$k[G] \otimes k[X] \to k[X]$$

is a morphism of coalgebras. The coalgebra $k[X]$ is thus a module-coalgebra on the Hopf algebra $k[G]$ in the sense of the following definition.

Definition IX.2.1. *Let H be a bialgebra and C be a coalgebra. We say that C is a module-coalgebra over H if there exists a morphism of coalgebras $H \otimes C \to C$ inducing an H-module structure on C.*

We are ready to give the definition of a matched pair of bialgebras.

Definition IX.2.2. *A pair (X, A) of bialgebras is matched if there exist linear maps $\alpha : A \otimes X \to X$ and $\beta : A \otimes X \to A$ turning X into a module-coalgebra over A, and turning A into a right module-coalgebra over X, such that, if we set*

$$\alpha(a \otimes x) = a \cdot x \quad \text{and} \quad \beta(a \otimes x) = a^x,$$

the following conditions are satisfied:

$$a \cdot (xy) = \sum_{(a)(x)} (a' \cdot x')(a''^{x''} \cdot y), \tag{2.1}$$

$$a \cdot 1 = \varepsilon(a)1, \tag{2.2}$$

$$(ab)^x = \sum_{(b)(x)} a^{b' \cdot x'}\, b''^{x''}, \tag{2.3}$$

$$1^x = \varepsilon(x)1, \tag{2.4}$$

$$\sum_{(a)(x)} a'^{x'} \otimes a'' \cdot x'' = \sum_{(a)(x)} a''^{x''} \otimes a' \cdot x' \tag{2.5}$$

for all $a, b \in A$ and $x, y \in X$.

Observe that Condition (2.5) is automatically satisfied when both bialgebras A and X are cocommutative. We also remark that Definition 2.2 is an immediate generalization of Definition 1.1. As a basic example of a matched pair of bialgebras, we may take the pair $(k[H], k[K])$ of group bialgebras where (H, K) is a matched pair of groups.

The maps α and β being morphisms of coalgebras, we have

$$\Delta(a \cdot x) = \sum_{(a)(x)} a' \cdot x' \otimes a'' \cdot x'' \quad \text{and} \quad \varepsilon(a \cdot x) = \varepsilon(a)\varepsilon(x)1 \tag{2.6}$$

in X, and

$$\Delta(a^x) = \sum_{(a)(x)} a'^{x'} \otimes a''^{x''} \quad \text{and} \quad \varepsilon(a^x) = \varepsilon(a)\varepsilon(x)1 \tag{2.7}$$

in A. We state the main result of this section; it is a natural extension of Proposition 1.2.

Theorem IX.2.3. *Let (X, A) be a matched pair of bialgebras. There exists a unique bialgebra structure on the vector space $X \otimes A$, with unit equal to $1 \otimes 1$, such that its product is given by*

$$(x \otimes a)(y \otimes b) = \sum_{(a)(y)} x(a' \cdot y') \otimes a''^{y''} b,$$

its coproduct by

$$\Delta(x \otimes a) = \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a''),$$

and its counit by

$$\varepsilon(x \otimes a) = \varepsilon(x)\varepsilon(a)$$

for all $x, y \in X$ and $a, b \in A$. Equipped with this bialgebra structure, $X \otimes A$ is called the bicrossed product of X and A and denoted $X \bowtie A$. Furthermore, the injective maps $i_X(x) = x \otimes 1$ and $i_A(a) = 1 \otimes a$ from X and from A into $X \bowtie A$ are bialgebra morphisms. We also have

$$x \otimes a = (x \otimes 1)(1 \otimes a)$$

for $a \in A$ and $x \in X$.

If the bialgebras X and A have antipodes, respectively denoted S_X and S_A, then the bicrossed product is a Hopf algebra with antipode S given by

$$S(x \otimes a) = \sum_{(x)(a)} S_A(a'') \cdot S_X(x'') \otimes S_A(a')^{S_X(x')}.$$

PROOF. The above formulas show that we equipped the bicrossed product with the coalgebra structure of the tensor product of coalgebras X and A. It is then clear that i_X and i_A are coalgebra morphisms. It remains to be proved that $X \bowtie A$ has an algebra structure and that the coproduct and the counit, as well as the embeddings i_X and i_A, are algebra morphisms.

Let us start with the associativity of the product. An easy but tedious computation using Relations (2.1) and (2.3) and the fact that both α and

β are coalgebra morphisms, shows that if $x, y, z \in X$ and $a, b, c \in A$, then both

$$\Big((x \otimes a)(y \otimes b)\Big)(z \otimes c) \quad \text{and} \quad (x \otimes a)\Big((y \otimes b)(z \otimes c)\Big)$$

are equal to

$$\sum_{(a)(b)(y)(z)} x(a' \cdot y')\,((a''^{y''} b') \cdot z') \otimes a'''^{y'''(b'' \cdot z'')}\, b'''^{z'''}\, c.$$

For the unit we get, using (2.2) and (2.4),

$$(1 \otimes 1)(x \otimes a) = \sum_{(x)} (1 \cdot x') \otimes 1^{x''} a = \sum_{(x)} x' \otimes \varepsilon(x'') a = \sum_{(x)} x' \varepsilon(x'') \otimes a = x \otimes a$$

and

$$(x \otimes a)(1 \otimes 1) = \sum_{(a)} x(a' \cdot 1) \otimes a''^{1} = \sum_{(a)} x\varepsilon(a') \otimes a'' = \sum_{(a)} x \otimes \varepsilon(a') a'' = x \otimes a.$$

Let us prove that the counit is a morphism of algebras. We have to check that

$$\varepsilon((x \otimes a)(y \otimes b)) = \varepsilon(x \otimes a)\varepsilon(y \otimes b) = \varepsilon(x)\varepsilon(a)\varepsilon(y)\varepsilon(b).$$

Now the left-hand side is equal to

$$\begin{aligned}
\varepsilon\Big(\sum_{(a)(y)} x(a' \cdot y') \otimes a''^{y''} b\Big) &= \sum_{(a)(y)} \varepsilon(x)\varepsilon(a' \cdot y')\varepsilon(a''^{y''})\varepsilon(b) \\
&= \varepsilon(x)\varepsilon(b)\Big(\sum_{(a)(y)} \varepsilon(a')\varepsilon(y')\varepsilon(a'')\varepsilon(y'')\Big) \\
&= \varepsilon(x)\varepsilon(b)\varepsilon(a)\varepsilon(y)
\end{aligned}$$

in view of (2.6) and (2.7). To conclude, we show that Relation (2.5) implies that the coproduct is a morphism of algebras. We have

$$\Delta(x \otimes a)\Delta(y \otimes b) = \sum_{(x)(a)(y)(b)} x'(a' \cdot y') \otimes a''^{y''} b' \otimes x''(a''' \cdot y''') \otimes a''''^{y''''} b'',$$

and, on the other hand,

$$\Delta\Big((x \otimes a)(y \otimes b)\Big) = \sum_{(x)(a)(y)(b)} x'(a' \cdot y') \otimes a'''^{y'''} b' \otimes x''(a'' \cdot y'') \otimes a''''^{y''''} b''.$$

Both expressions are equal in view of Relation (2.5).

Now suppose that A and X have antipodes. We have to check that the formula

$$S(x \otimes a) = \sum_{(x)(a)} S_A(a'') \cdot S_X(x'') \otimes S_A(a')^{S_X(x')}$$

defines an antipode on the bialgebra $X \bowtie A$. Using the fact that S_A and S_X are antipodes, we get

$$\begin{aligned}
&\sum_{(x)(a)} (x' \otimes a')S(x'' \otimes a'') \\
&\quad = \sum_{(x)(a)} (x' \otimes a')(S_A(a''') \cdot S_X(x''') \otimes S_A(a'')^{S_X(x'')}) \\
&\quad = \sum_{(x)(a)} x'\Big((a'S_A(a'''')) \cdot S_X(x'''')\Big) \otimes (a''S_A(a'''))^{S_X(x'')} \\
&\quad = \sum_{(x)(a)} x'\Big((a'S_A(a''')) \cdot S_X(x''')\Big) \otimes \varepsilon(a'')\varepsilon(S_X(x''))1 \\
&\quad = \sum_{(x)(a)} x'\Big((a'S_A(a'')) \cdot S_X(x'')\Big) \otimes 1 \\
&\quad = \varepsilon(a)\Big(\sum_{(x)} x'S_X(x'') \otimes 1\Big) \\
&\quad = \varepsilon(a)\varepsilon(x)1 \otimes 1 \\
&\quad = \varepsilon(x \otimes a)1 \otimes 1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\sum_{(x)(a)} S(x' \otimes a')(x'' \otimes a'') \\
&\quad = \sum_{(x)(a)} (S_A(a'') \cdot S_X(x'') \otimes S_A(a')^{S_X(x')})(x''' \otimes a''') \\
&\quad = \sum_{(x)(a)} (S_A(a''') \cdot S_X(x'''))(S_A(a'')^{S_X(x'')} \cdot x'''') \\
&\qquad\qquad \otimes S_A(a')^{S_X(x')x'''''} a'''' \\
&\quad = \sum_{(x)(a)} S_A(a'') \cdot (S_X(x'')x''') \otimes S_A(a')^{S_X(x')x''''} a''' \\
&\quad = \sum_{(x)(a)} \varepsilon(x'')\,(S_A(a'') \cdot 1) \otimes S_A(a')^{S_X(x')x'''} a''' \\
&\quad = \sum_{(x)(a)} \varepsilon(S_A(a''))\,1 \otimes S_A(a')^{S_X(x')x''} a''' \\
&\quad = \varepsilon(x)\sum_{(a)} 1 \otimes S_A(a')^1 a'' \\
&\quad = \varepsilon(x)\varepsilon(a)1 \otimes 1 \\
&\quad = \varepsilon(x \otimes a)1 \otimes 1.
\end{aligned}$$

□

We conclude this section with three examples.

Example 1. As already observed, if (H, K) is a matched pair of groups, then the pair $(k[H], k[K])$ is a matched pair of bialgebras. Furthermore, the group algebra of the bicrossed product is isomorphic to the bicrossed product of the group algebras

$$k[H \bowtie K] \cong k[H] \bowtie k[K].$$

Example 2. (*Tensor product of bialgebras*) Let X and A be bialgebras. We let each one act trivially on the other one by

$$a \cdot x = \varepsilon(a)x \quad \text{and} \quad a^x = \varepsilon(x)a$$

for all $a \in A$ and $x \in X$. It is easy to see that these trivial actions satisfy the conditions of Definition 2.2. In particular, both sides of Relation (2.5) are equal to $x \otimes a$. The formulas given in Theorem 2.3 show that in this case the bicrossed product $X \bowtie A$ is isomorphic to the tensor product bialgebra $X \otimes A$.

Example 3. (*Crossed product of bialgebras*) This notion is parallel to the semidirect product of groups. Let X and A be bialgebras. Suppose firstly that X acts trivially on A as in Example 2, namely that $a^x = \varepsilon(x)a$ for all $a \in A$ and $x \in X$, secondly that A acts on X via a map α which turns X not only into a module-coalgebra, but also into a module-algebra, and thirdly that we have the compatibility relation

$$\sum_{(a)} a' \otimes a'' \cdot x = \sum_{(a)} a'' \otimes a' \cdot x, \tag{2.8}$$

which is satisfied, for instance, when A is cocommutative. Then X and A are matched bialgebras, and the corresponding bicrossed product is called the *crossed product of A by X*. The multiplication in the crossed product is given by

$$(x \otimes a)(y \otimes b) = \sum_{(a)} x(a' \cdot y) \otimes a''b. \tag{2.9}$$

IX.3 Variations on the Adjoint Representation

Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. If a and x are elements of H, we set

$$a \cdot x = \sum_{(a)} a'xS(a'') \quad \text{and} \quad x^a = \sum_{(a)} S(a')xa''. \tag{3.1}$$

Proposition IX.3.1. *The map $(a, x) \mapsto a \cdot x$ endows H with the structure of a left module-algebra on the bialgebra H. We denote by ${}_{ad}H$ the thus* *defined H-module, and we call this action the left adjoint representation of H. Similarly, the map $(x, a) \mapsto x^a$ endows H with the structure of a right module-algebra on the bialgebra H. We denote by H_{ad} the H-module defined this way, and we call this action the right adjoint representation of H.*

PROOF. We give the proof for the left adjoint representation. We first check that $(a, x) \mapsto a \cdot x$ puts an H-module structure on H. Indeed, we have $1 \cdot x = x$ and

$$b \cdot (a \cdot x) = \sum_{(a)(b)} b'a'xS(a'')S(b'') = \sum_{(ba)} (ba)'xS((ba)'') = (ba) \cdot x$$

for all $a, b, x \in H$. Let us show it is a module-algebra over H. We have

$$a \cdot 1 = \sum_{(a)} a'S(a'') = \varepsilon(a)1$$

and

$$\begin{aligned}
\sum_{(a)} (a' \cdot x)(a'' \cdot y) &= \sum_{(a)} a'xS(a'')a'''yS(a'''') \\
&= \sum_{(a)} a'x\varepsilon(a'')yS(a''') \\
&= \sum_{(a)} a'xyS(a'') \\
&= a \cdot (xy).
\end{aligned}$$

□

Example 1. (*Conjugacy in a group*) Let G be a group and $k[G]$ be the corresponding Hopf algebra. The left adjoint representation of $k[G]$ is given by the formula

$$a \cdot x = axa^{-1}$$

for $a, x \in G$.

Example 2. (*Adjoint representation of a Lie algebra*) Let L be a Lie algebra and $U(L)$ be its enveloping algebra equipped with its canonical Hopf algebra structure (see V.2). The left adjoint representation of $U(L)$ is given by the formula

$$a \cdot x = ax - xa$$

for $a, x \in L$. The corresponding representation of L is called the adjoint representation of the Lie algebra L.

We now wish to deduce the so-called coadjoint representations of H on the dual vector space H^* from the above-defined adjoint representations. We use the following lemma.

Lemma IX.3.2. *Consider a Hopf algebra H with invertible antipode S and an algebra A that is a left [resp. right] module-algebra over H. Let us put on the dual vector space A^* the left [resp. right] H-module structure given by*

$$< a, xf > = < S^{-1}(x)a, f > \quad [\textit{resp.} \quad < a, fx > = < aS^{-1}(x), f >]$$

for all $a \in A$, $x \in H$, and $f \in A^*$. If A is finite-dimensional, then the coalgebra $(A^{\mathrm{op}})^*$ is a module-coalgebra over H.

The comultiplication on the finite-dimensional coalgebra $(A^{\mathrm{op}})^*$ is the opposite comultiplication of the dual coalgebra A^*; in other words,

$$< ab, f > = \sum_{(f)} < b, f' >< a, f'' > \tag{3.2}$$

whenever $a, b \in A$ and $f \in (A^{\mathrm{op}})^*$.

PROOF. Checking that A^* is a left H-module is easy. Let us show that the left action of H on A^* defines an H-module-coalgebra structure on A^*. It suffices to check that the map from $H \otimes A^*$ to A^* which defines the action of H on A^* is a coalgebra morphism. More precisely, we should have

$$\varepsilon(xf) = \varepsilon(x)\varepsilon(f)$$

and

$$\sum_{(xf)} (xf)' \otimes (xf)'' = \sum_{(x)(f)} x'f' \otimes x''f''. \tag{3.3}$$

Now,

$$\varepsilon(xf) = (xf)(1) = f(S^{-1}(x)1) = \varepsilon(S^{-1}(x))f(1) = \varepsilon(x)\varepsilon(f)$$

since $x1 = \varepsilon(x)1$. Let us check (3.3) by evaluating both sides on an element $a \otimes b$ in $A \otimes A$. We have

$$\begin{aligned}
< a \otimes b, \sum_{(xf)} (xf)' \otimes (xf)'' > &= \sum_{(xf)} < a, (xf)' >< b, (xf)'' > \\
&= < ba, xf > \\
&= < S^{-1}(x)(ba), f > \\
&= \sum_{(x)} < (S^{-1}(x)'b)(S^{-1}(x)''a), f > \\
&= \sum_{(x)(f)} < S^{-1}(x'')b, f'' >< S^{-1}(x')a, f' > \\
&= \sum_{(x)(f)} < a, x'f' >< b, x''f'' > \\
&= < a \otimes b, \sum_{(x)(f)} x'f' \otimes x''f'' > .
\end{aligned}$$

One proceeds in a similar fashion for the right action. □

As an immediate consequence of Proposition 3.1 and of Lemma 3.2, we deduce

Corollary IX.3.3. *Let* $H = (H, \mu, \eta, \Delta, \varepsilon, S, S^{-1})$ *be a finite-dimensional Hopf algebra with invertible antipode* S. *There is a unique left [resp. right]* H*-module-coalgebra structure on the dual of the opposite Hopf algebra, that is, on* $(H^{\mathrm{op}})^* = (H^*, \Delta^*, \varepsilon^*, (\mu^{\mathrm{op}})^*, \eta^*, (S^{-1})^*, S^*)$, *given for* $a, x \in H$ *and* $f \in H^*$ *by*

$$\begin{aligned} < a, x \cdot f > &= \sum_{(x)} < S^{-1}(x'')ax', f > \\ [\textit{resp.} \quad < a, f^x > &= \sum_{(x)} < x''aS^{-1}(x'), f >]. \end{aligned}$$

These actions will be called the left and right *coadjoint representations* of H. Applying Corollary 3.3 to the Hopf algebra

$$(H^{\mathrm{cop}})^* = (H^*, (\Delta^{\mathrm{op}})^*, \varepsilon^*, \mu^*, \eta^*, (S^{-1})^*, S^*)$$

and using the natural identification between the bidual H^{**} and H, we get a right $(H^{\mathrm{cop}})^*$-module-coalgebra structure on the Hopf algebra

$$H = (H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}).$$

By Corollary III.3.5, the Hopf algebra $(H^{\mathrm{cop}})^*$ is isomorphic via the map S^* to the Hopf algebra $(H^{\mathrm{op}})^* = (H^*, \Delta^*, \varepsilon^*, (\mu^{\mathrm{op}})^*, \eta^*, (S^{-1})^*, S^*)$. This isomorphism induces a right action of $(H^{\mathrm{op}})^*$ on H. We summarize this with the following statement.

Proposition IX.3.4. *Under the hypotheses of Corollary* 3.3, *there exists a unique right* $(H^{\mathrm{op}})^*$*-module-coalgebra structure on* H *given for* $a \in H$ *and* $f \in H^*$ *by*

$$a^f = \sum_{(a)} f(S^{-1}(a''')a')a''.$$

PROOF. Let $f, g \in H^*$ and $a \in H$. By Corollary 3.3, the action of $(H^{\mathrm{cop}})^*$ on H is given by

$$< a^f, g > = \sum_{(f)} < a, f''gS^*(f') > .$$

Computing in $(H^{\mathrm{cop}})^*$, we get

$$\begin{aligned} < a^f, g > &= \sum_{(f)(a)} < a''', f'' >< a'', g >< a', S^*(f') > \\ &= \sum_{(f)(a)} < S(a'), f' >< a''', f'' >< a'', g > \end{aligned}$$

$$= \sum_{(a)} << S(a')a''', f > a'', g >$$
$$= \sum_{(a)} < f(S(a')a''')a'', g > .$$

Therefore, the right coadjoint action of $(H^{\mathrm{cop}})^*$ on H is given by

$$a^f = \sum_{(a)} f(S(a')a''')a''.$$

Composing with $(S^{-1})^*$, we get a right action of $(H^{\mathrm{op}})^*$ on H given by

$$a^{(S^{-1})^*(f)} = a^{f \circ S^{-1}} = \sum_{(a)} f(S^{-1}(S(a')a''))a'' = \sum_{(a)} f(S^{-1}(a''')a')a''.$$

□

We now state the main result of this section. It will allow us to construct Drinfeld's quantum double in the next section.

Theorem IX.3.5. *Let $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1})$ be a finite-dimensional Hopf algebra with invertible antipode. Consider the Hopf algebra*

$$X = (H^{\mathrm{op}})^* = (H^*, \Delta^*, \varepsilon, (\mu^{\mathrm{op}})^*, \eta, (S^{-1})^*, S^*).$$

Let $\alpha : H \otimes X \to X$ and $\beta : H \otimes X \to H$ be the linear maps given by

$$\alpha(a \otimes f) = a \cdot f = \sum_{(a)} f(S^{-1}(a'')?a')$$

and

$$\beta(a \otimes f) = a^f = \sum_{(a)} f(S^{-1}(a''')a')a''$$

where $a \in H$ and $f \in X$. Then the pair (H, X) of Hopf algebras is matched in the sense of Definition 2.2.

PROOF. In this proof we systematically use Sweedler's sigma notation (defined in III.1) as well as the definitions of α and β, the counitality of ε and relations of the form $\sum_{(a)} a''S^{-1}(a') = \varepsilon(a)$. The question mark ? serves as a mute variable. Corollary 3.3 and Proposition 3.4 show that α and β endow each Hopf algebra with the structure of a module-coalgebra over the other one. We yet have to check Relations (2.1–2.5) of Definition 2.2.

Relation (2.1): For $x \in H$, we have

$$\begin{aligned}
&< x, \sum_{(a)(f)} (a' \cdot f')(a''^{f''} \cdot g) > \\
&= \sum_{(a)(f)(x)} (a' \cdot f')(x')(a''^{f''} \cdot g)(x'') \\
&= \sum_{(a)(f)(x)} f'(S^{-1}(a^{(2)})x'a^{(1)})f''(S^{-1}(a^{(5)})a^{(3)})(a^{(4)} \cdot g)(x'') \\
&= \sum_{(a)(f)(x)} f'(S^{-1}(a^{(2)})x'a^{(1)})f''(S^{-1}(a^{(6)})a^{(3)})g(S^{-1}(a^{(5)})x''a^{(4)}) \\
&= \sum_{(a)(x)} f(S^{-1}(a^{(6)})a^{(3)}S^{-1}(a^{(2)})x'a^{(1)})g(S^{-1}(a^{(5)})x''a^{(4)}) \\
&= \sum_{(a)(x)} \varepsilon(a^{(2)})f(S^{-1}(a^{(5)})x'a^{(1)})g(S^{-1}(a^{(4)})x''a^{(3)}) \\
&= \sum_{(a)(x)} f(S^{-1}(a'''')x'a')g(S^{-1}(a''')x''a''),
\end{aligned}$$

which proves Relation (2.1).

Relation (2.2): We have

$$a \cdot \varepsilon = \sum_{(a)} \varepsilon(S^{-1}(a'')?a')\varepsilon(a)\varepsilon = \sum_{(a)} \varepsilon(a')\varepsilon(a'')\varepsilon = \varepsilon(a)\varepsilon.$$

Relation (2.3): We have to show that

$$(ab)^f = \sum_{(b)(f)} a^{b'\cdot f'}\, b''^{f''}.$$

Now,

$$\begin{aligned}
\sum_{(b)(f)} a^{b'\cdot f'}\, b''^{f''} &= \sum_{(a)(b)(f)} f'(S^{-1}(b'')S^{-1}(a''')a'b')f''(S^{-1}(b'''''')b''')a''b'''' \\
&= \sum_{(a)(b)} f(S^{-1}(b'''''')b'''S^{-1}(b'')S^{-1}(a''')a'b')a''b'''' \\
&= \sum_{(a)(b)} \varepsilon(b'')f(S^{-1}(b'''')S^{-1}(a''')a'b')a''b''' \\
&= \sum_{(a)(b)} f\Big(S^{-1}(b''')S^{-1}(a''')a'b'\Big)a''b'' \\
&= \sum_{(ab)} f\Big(S^{-1}((ab)''')(ab)'\Big)(ab)'' \\
&= (ab)^f.
\end{aligned}$$

Relation (2.4): We have $1^f = f(1) = \varepsilon(f)1$.
Relation (2.5): We have to check that

$$\sum_{(a)(f)} a'^{f'} \otimes a'' \cdot f'' = \sum_{(a)(f)} a''^{f''} \otimes a' \cdot f'. \tag{3.4}$$

For the left-hand side of (3.4) we get

$$\begin{aligned}
\sum_{(a)(f)} a'^{f'} \otimes a'' \cdot f'' &= \sum_{(a)(f)} f'(S^{-1}(a''')a')a'' \otimes f''(S^{-1}(a''''')?a'''') \\
&= \sum_{(a)} a'' \otimes f(S^{-1}(a''''')?a''''S^{-1}(a''')a') \\
&= \sum_{(a)} \varepsilon(a''')a'' \otimes f(S^{-1}(a'''')?a') \\
&= \sum_{(a)(f)} a'' \otimes f(S^{-1}(a''')?a')
\end{aligned}$$

whereas for the right-hand side we have

$$\begin{aligned}
\sum_{(a)(f)} a''^{f''} \otimes a' \cdot f' &= \sum_{(a)(f)} f''(S^{-1}(a''''')a''')a'''' \otimes f'(S^{-1}(a'')?a') \\
&= \sum_{(a)} a'''' \otimes f(S^{-1}(a''''')a'''S^{-1}(a'')?a') \\
&= \sum_{(a)} \varepsilon(a'')a''' \otimes f(S^{-1}(a'''')?a') \\
&= \sum_{(a)} a'' \otimes f(S^{-1}(a''')?a'),
\end{aligned}$$

which proves (2.5). □

IX.4 Drinfeld's Quantum Double

Let $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1})$ be a finite-dimensional Hopf algebra with invertible antipode S. Let $X = (H^{\mathrm{op}})^* = (H^*, \Delta^*, \varepsilon, (\mu^{\mathrm{op}})^*, \eta, (S^{-1})^*, S^*)$ be the dual Hopf algebra. We have just proved that (H, X) is a matched pair of Hopf algebras.

IX.4.1 The quantum double as a Hopf algebra

Definition IX.4.1. *The quantum double $D(H)$ of the Hopf algebra H is the bicrossed product of H and of $X = (H^{\mathrm{op}})^*$:*

$$D(H) = X \bowtie H = (H^{\mathrm{op}})^* \bowtie H.$$

We first give a more explicit description of $D(H)$, then in the next subsection we prove that the quantum double is a braided Hopf algebra in the sense of VIII.2.

As a vector space, we have $D(H) = X \otimes H$. The unit of $D(H)$ is $1 \otimes 1$. Its counit and its comultiplication are given by

$$\varepsilon(f \otimes a) = \varepsilon(a) f(1) \tag{4.1}$$

and

$$\Delta(f \otimes a) = \sum_{(a)(f)} (f' \otimes a') \otimes (f'' \otimes a'') \tag{4.2}$$

where $f \in X$ and $a, b \in H$.

Lemma IX.4.2. *The multiplication in $D(H)$ is given by*

$$(f \otimes a)(g \otimes b) = \sum_{(a)} f\, g(S^{-1}(a''')?a') \otimes a''b \tag{4.3}$$

where $f, g \in X$ and $a, b \in H$.

Here $g(S^{-1}(a''')?a')$ means the map $x \mapsto g(S^{-1}(a''')xa')$.

PROOF. By definition of the bicrossed product, the product of $D(H)$ is given by

$$(f \otimes a)(g \otimes b) = \sum_{(a)(g)} f\,(a' \cdot g') \otimes a''^{g''} b.$$

Computing the right-hand side using the formulas of Theorem 3.5, we get

$$\begin{aligned}
&\sum_{(a)(g)} f\, g'(S^{-1}(a'')?a') \otimes g''(S^{-1}(a''''')a''')a''''b \\
&\quad = \sum_{(a)} f\, g(S^{-1}(a''''')a'''S^{-1}(a'')?a') \otimes a''''b \\
&\quad = \sum_{(a)} \varepsilon(a'')\, f\, g(S^{-1}(a'''')?a') \otimes a'''b \\
&\quad = \sum_{(a)} f\, g(S^{-1}(a''')?a') \otimes a''b.
\end{aligned}$$

□

The quantum double $D(H)$ contains H and X as Hopf subalgebras via the embeddings i_H and i_X given by

$$i_H(a) = 1 \otimes a \quad \text{and} \quad i_X(f) = f \otimes 1.$$

Formula (4.3) implies that

$$f \otimes a = i_X(f) i_H(a) \tag{4.4}$$

for all $f \in X$ and $a \in H$.

We shall use Relation (4.4) in order to simplify our notations and write fa instead of $f \otimes a = i_X(f)i_H(a)$ whenever a is an element of H and f is a linear form on H. Under this convention, the multiplication in $D(H)$ is determined by the straightening formula

$$af = \sum_{(a)} f(S^{-1}(a''')?a')a'' \tag{4.5}$$

where $f \in X$ and $a \in H$.

When H is cocommutative, the bicrossed product construction of the double of H can be reduced to the crossed product construction of Section 2, Example 3, as shown in the following statement.

Proposition IX.4.3. *Let H be a cocommutative finite-dimensional Hopf algebra with invertible antipode. Then the quantum double $D(H)$ is isomorphic, as a Hopf algebra, to the crossed product of H with $(H^{\mathrm{op}})^*$, the first algebra acting on the second one by the left coadjoint representation of Corollary* 3.3.

PROOF. We first have to prove that we are in the situation of Example 3 of Section 2, namely that $(H^{\mathrm{op}})^*$ acts trivially on H and that $(H^{\mathrm{op}})^*$ is a module-algebra over H for the left coadjoint representation. The compatibility condition (2.8) is trivially satisfied since H is cocommutative.

Resuming the notations of Proposition 3.4 and using the cocommutativity of H, we have

$$\begin{aligned} a^f &= \sum_{(a)} f(S^{-1}(a''')a')a'' \\ &= \sum_{(a)} f(S^{-1}(a''')a'')a' \\ &= \sum_{(a)} f(1)\varepsilon(a'')a' \\ &= \varepsilon(f)a, \end{aligned}$$

which proves that $(H^{\mathrm{op}})^*$ acts trivially on H.

In order to prove that $(H^{\mathrm{op}})^*$ is a module-algebra over H, we have to check that

$$a \cdot 1 = \varepsilon(a)1 \quad \text{and} \quad a \cdot (fg) = \sum_{(a)} (a \cdot f)(a \cdot g)$$

for $a \in H$ and $f, g \in (H^{\mathrm{op}})^*$. This is left to the reader.

Let us now prove that the multiplication in $D(H)$ coincides with the multiplication of the crossed product given in (2.9). For $a \in H$ and $f \in H^*$, we have the following equalities in the quantum double:

$$\begin{aligned} af &= \sum_{(a)} f(S^{-1}(a''')?a')a'' \\ &= \sum_{(a)} f(S^{-1}(a''')?a'')a' \\ &= \sum_{(a)} (a'' \cdot f)a' \\ &= \sum_{(a)} (a' \cdot f)a''. \end{aligned}$$

Here we used the cocommutativity of H in the second and fourth equalities as well as the definition of the coadjoint representation (see Corollary 3.3). The last term of this series of equalities is the multiplication formula given in (2.9) for the crossed product algebra. The coalgebra structures coincide for both constructions. □

IX.4.2 Description of the universal R-matrix

Let us consider the map $\lambda_{H,H} : H \otimes X \to \mathrm{End}(H)$ defined in II.2 for $a, b \in H$ and $f \in X$ by $\lambda_{H,H}(a \otimes f)(b) = f(b)a$. Since H is finite-dimensional, the map $\lambda_{H,H}$ is an isomorphism, which allows us to set

$$\rho = \lambda_{H,H}^{-1}(\mathrm{id}_H) \in H \otimes X.$$

We define the universal R-matrix of the quantum double as the element

$$R = (i_H \otimes i_X)(\rho) \in D(H) \otimes D(H).$$

We get a more explicit formula for R by choosing a basis $\{e_i\}_{i \in I}$ of the vector space H together with its dual basis $\{e^i\}_{i \in I}$ in X. Then

$$\rho = \sum_{i \in I} e_i \otimes e^i \quad \text{and} \quad R = \sum_{i \in I} (1 \otimes e_i) \otimes (e^i \otimes 1). \tag{4.6}$$

We state the main theorem of this section.

Theorem IX.4.4. *Under the previous hypotheses, the Hopf algebra $D(H)$ equipped with the element $R = \sum_{i \in I} (1 \otimes e_i) \otimes (e^i \otimes 1) \in D(H) \otimes D(H)$ is braided.*

PROOF. We have to prove that R satisfies the conditions of Definitions VIII.2.1–2.2. More precisely, we must prove

(1) that R is invertible in $D(H) \otimes D(H)$,

(2) that $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$ and $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$, and

(3) that for all $f \in X$ and $a \in H$, we have

$$\Delta^{\mathrm{op}}(f \otimes a)R = R\Delta(f \otimes a).$$

(1) We claim that R is invertible with inverse $\overline{R}$ equal to

$$\overline{R} = \sum_{i\in I} (1 \otimes e_i) \otimes ((e^i \circ S) \otimes 1).$$

Consider an element $\xi = b \otimes u \otimes c \otimes v$ in $H \otimes X \otimes H \otimes X$. Let us pair it with $R\overline{R}$ using the duality between H and X. We get

$$\begin{aligned} < R\overline{R}, \xi > &= \sum_{i,j\in I} < (1 \otimes e_i e_j) \otimes (e^i(e^j \circ S) \otimes 1), b \otimes u \otimes c \otimes v > \\ &= \varepsilon(b)v(1) \sum_{(c)} u\Big(\Big(\sum_{i\in I} e^i(c')\Big)\Big(\sum_{j\in I} e^j(S(c''))\Big)e_i e_j\Big) \\ &= \varepsilon(b)v(1)\, u\Big(\sum_{(c)} c'S(c'')\Big) \\ &= \varepsilon(b)v(1)\varepsilon(c)u(1) \\ &= < 1 \otimes 1 \otimes 1 \otimes 1, \xi > . \end{aligned}$$

Consequently, $R\overline{R} = 1 \otimes 1 \otimes 1 \otimes 1$. One proves that $\overline{R}$ is a left inverse of R in a similar way.

(2) We now check that $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$ or, equivalently, that

$$\sum_{i\in I,(e_i)} 1 \otimes e_i' \otimes 1 \otimes e_i'' \otimes e^i \otimes 1 = \sum_{i,j\in I} 1 \otimes e_i \otimes 1 \otimes e_j \otimes e^i e^j \otimes 1.$$

Let us evaluate the left-hand side on an element $\theta = a \otimes t \otimes b \otimes u \otimes c \otimes v$ of the tensor product $D(H) \otimes D(H) \otimes D(H)$. We have

$$\begin{aligned} < (\Delta \otimes \mathrm{id})(R), \theta > &= < \sum_{i\in I,(e_i)} 1 \otimes e_i' \otimes 1 \otimes e_i'' \otimes e^i \otimes 1, \theta > \\ &= \varepsilon(a)\varepsilon(b)v(1)\Big(\sum_{i\in I,(e_i)} e^i(c)t(e_i')u(e_i'')\Big). \end{aligned}$$

We now remark that

$$\sum_{(a)} a' \otimes a'' = \sum_{(a),\, i\in I,(e_i)} e^i(a)\, e_i' \otimes e_i'' \tag{4.7}$$

by application of the coproduct of H to $a = \sum_{i\in I} e^i(a)\, e_i$. Using (4.7), we obtain

$$\sum_{i\in I} e^i(c)t(e_i')u(e_i'') = \sum_{(c)} t(c')u(c'').$$

Therefore,

$$< (\Delta \otimes \mathrm{id})(R), \theta >= \sum_{(c)} \varepsilon(a)\varepsilon(b)v(1)t(c')u(c'').$$

On the other hand, we have

$$\begin{aligned}
< R_{13}R_{23}, \theta > &= \varepsilon(a)\varepsilon(b)v(1) \sum_{i,j\in I} t(e_i)u(e_j)(e^i e^j)(c) \\
&= \varepsilon(a)\varepsilon(b)v(1) \sum_{(c)} t\Big(\sum_{i\in I} e^i(c')e_i\Big)\, u\Big(\sum_{j\in I} e^j(c'')e_j\Big) \\
&= \varepsilon(a)\varepsilon(b)v(1) \sum_{(c)} t(c')u(c'') \\
&= < (\Delta \otimes \mathrm{id})(R), \theta > .
\end{aligned}$$

One checks similarly that $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$.

(3) Let us evaluate $\Delta^{\mathrm{op}}(f \otimes a)R$ on $\xi = b \otimes u \otimes c \otimes v$. We have

$$\begin{aligned}
&< \Delta^{\mathrm{op}}(f \otimes a)R, \xi > \\
&= \sum_{(a)(f),i\in I} < (f'' \otimes a'')(1 \otimes e_i) \otimes (f' \otimes a')(e^i \otimes 1), \xi > \\
&= \sum_{(a)(f),\, i\in I} < f'' \otimes a''''e_i \otimes f'e^i(S^{-1}(a''')?a') \otimes a'', \xi > \\
&= \sum_{(a)(c)(f),\, i\in I} f''(b)u(a''''e_i)f'(c')e^i(S^{-1}(a''')c''a')v(a'') \\
&= \sum_{(a)(c),\, i\in I} f(bc')u(a''''e^i(S^{-1}(a''')c''a')e_i)v(a'') \\
&= \sum_{(a)(c)} f(bc')u(a''''S^{-1}(a''')c''a')v(a'') \\
&= \sum_{(a)(c)} \varepsilon(a''')f(bc')u(c''a')v(a'') \\
&= \sum_{(a)(c)} f(bc')u(c''a')v(a'').
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&< R\Delta(f \otimes a), \xi > \\
&= \sum_{(a)(f),i\in I} < (1 \otimes e_i)(f' \otimes a') \otimes (e^i \otimes 1)(f'' \otimes a''), \xi > \\
&= \sum_{(a)(f),i\in I,(e_i)} < f'(S^{-1}(e_i''')?e_i') \otimes e_i''a' \otimes e^i f'' \otimes a'', \xi >
\end{aligned}$$

$$= \sum_{(a)(c)(f),\, i\in I,(e_i)} f'(S^{-1}(e_i''')be_i')u(e_i''a')e^i(c')f''(c'')v(a'')$$
$$= \sum_{(a)(c),\, i\in I,(e_i)} f(c''S^{-1}(e_i''')be_i')u(e_i''a')e^i(c')v(a'').$$

Applying $(\Delta \otimes \mathrm{id}_H)\Delta$ to $c = \sum_{i\in I} e^i(c)e_i$, we get

$$\sum_{(c)} c' \otimes c'' \otimes c''' = \sum_{i\in I,(e_i)} e^i(c)\, e_i' \otimes e_i'' \otimes e_i'''. \tag{4.8}$$

Using (4.8), we obtain

$$\begin{aligned} < R\Delta(f\otimes a), \xi > &= \sum_{(a)(c)} f\Big(c''''S^{-1}(c''')bc'\Big)u(c''a')v(a'') \\ &= \sum_{(a)(c)} \varepsilon(c''')f(bc')u(c''a')v(a'') \\ &= \sum_{(a)(c)} f(bc')u(c''a')v(a'') \\ &= < \Delta^{\mathrm{op}}(f\otimes a)R, \xi > . \end{aligned}$$

□

IX.4.3 Quantum double of a group algebra

We end this section by applying the quantum double construction to the finite-dimensional cocommutative Hopf algebra $k[G]$ where G is a finite group. By Proposition 4.3 we know that $D(k[G])$ is a crossed product.

Let $\{e_g\}_{g\in G}$ be the dual basis of the basis $\{g\}_{g\in G}$ of $k[G]$. It is easy to check that the dual algebra $(k[G]^{\mathrm{op}})^*$ is the algebra k^G with multiplication given by

$$e_g e_h = \delta_{gh}\, e_g \tag{4.9}$$

for all $g, h \in G$ and with unit $\sum_{g\in G} e_g = 1$. The comultiplication Δ, the counit ε, and the antipode S of $(k[G]^{\mathrm{op}})^*$ are defined by

$$\Delta(e_g) = \sum_{uv=g} e_v \otimes e_u, \quad \varepsilon(e_g) = \delta_{g1}, \quad S(e_g) = e_{g^{-1}} \tag{4.10}$$

for each element g of the group.

The above description of a quantum double shows the set $\{e_g h\}_{(g,h)\in G\times G}$ is a basis of $D(k[G])$. The product of the quantum double is determined by

$$he_g = e_{hgh^{-1}}h, \tag{4.11}$$

which proves again that $D(k[G])$ is the crossed product of $k[G]$ by itself, where the algebra acts on itself by conjugation. Its universal R-matrix is

$$R = \sum_{g \in G} g \otimes e_g. \tag{4.12}$$

Despite the fact that the quantum double is not cocommutative when G is not abelian, its antipode is involutive, which implies that the element

$$u = \sum_{g \in G} e_{g^{-1}} g \tag{4.13}$$

introduced in VIII.4 is central in $D(k[G])$. We also have

$$S(u) = u. \tag{4.14}$$

IX.5 Representation-Theoretic Interpretation of the Quantum Double

Let $H = (H, \mu, \eta, \Delta, \varepsilon, S)$ be a finite-dimensional Hopf algebra with invertible antipode. Again we choose a finite basis $\{a_i\}_{i \in I}$ of H along with its dual basis $\{a^i\}_{i \in I}$. The purpose of this section is to characterize modules over the quantum double $D(H)$. In view of Relation (4.5), a $D(H)$-module is nothing but a vector space V with a left H-module structure as well as a left H^*-module structure such that for all $a \in H$, $f \in H^*$, and $v \in V$ we have

$$a(fv) = \sum_{(a)} f(S^{-1}(a''')?a')\Big(a''v\Big). \tag{5.1}$$

We wish to rephrase such data purely in terms of H without any reference to the dual algebra H^*. We first introduce the following concept.

Definition IX.5.1. *A crossed H-bimodule is a vector space V together with linear maps $\mu_V : H \otimes V \to V$ and $\Delta_V : V \to V \otimes H$ such that*

(i) *the map μ_V [resp. Δ_V] turns V into a left H-module [resp. into a right H-comodule], and*

(ii) *the diagram*

$$\begin{tikzcd}
H \otimes V \arrow[rr, "\Delta \otimes \mathrm{id}_V"] \arrow[d, "\Delta \otimes \Delta_V"] & & H \otimes H \otimes V \arrow[d, "\mathrm{id}_H \otimes \mu_V"] \\
H \otimes H \otimes V \otimes H \arrow[d, "\mathrm{id}_H \otimes \tau_{H,V} \otimes \mathrm{id}_H"] & & H \otimes V \arrow[d, "\tau_{H,V}"] \\
H \otimes V \otimes H \otimes H \arrow[d, "\mu_V \otimes \mu"] & & V \otimes H \arrow[d, "\Delta_V \otimes \mathrm{id}_H"] \\
V \otimes H & & V \otimes H \otimes H \arrow[ll, "\mathrm{id}_V \otimes \mu"']
\end{tikzcd}$$

commutes.

Set $\mu_V(a \otimes v) = av$ and $\Delta_V(v) = \sum_{(v)} v_V \otimes v_H$ for $a \in H$ and $v \in V$. Then, according to the conventions set up in III.1 and III.6, the commutativity of the above diagram is equivalent to

$$\sum_{(a)(v)} a' v_V \otimes a'' v_H = \sum_{(a)(v)} (a''v)_V \otimes (a''v)_H \, a' \tag{5.2}$$

where a runs over all elements in H and v over all elements in V.

We state the main result of this section.

Theorem IX.5.2. *Let H be a finite-dimensional Hopf algebra with invertible antipode. Any left $D(H)$-module has a natural structure as a crossed H-bimodule. Conversely, any crossed H-bimodule has a natural structure as a left module over the quantum double $D(H)$.*

Proof. (a) Let V be a left $D(H)$-module. As we mentioned before, the space V is a left H-module as well as a left H^*-module satisfying Relation (5.1). We wish to show that V can be endowed with a crossed bimodule structure.

Given a basis $\{a_i\}_i$ of H and its dual basis $\{a^i\}_i$, we use the left action of H^* on V in order to define a map $\Delta_V : V \to V \otimes H$ by

$$\Delta_V(v) = \sum_i a^i v \otimes a_i \tag{5.3}$$

for any $v \in V$. Let us show that Δ_V defines a right coaction of H on V. We have to check that Δ_V is coassociative and counitary. Rather than verify this directly, we observe that Δ_V is the transpose of the associative, unitary right action $V^* \otimes H^* \to V^*$ of H^* on V^* given by

$$< \alpha f, v > = < \alpha, fv >$$

for $\alpha \in V^*$, $v \in V$ and $f \in H^*$. Indeed, we have

$$\begin{aligned} < \alpha \otimes f, \Delta_V(v) > &= \sum_i \alpha(a^i v) f(a_i) \\ &= < \alpha, \Big(\sum_i f(a_i) a^i\Big) v > \\ &= < \alpha, fv > \\ &= < \alpha f, v > \end{aligned}$$

since $f = \sum_i f(a_i) a^i$. Incidentally, this observation implies that Δ_V is independent of the choice of the basis.

In order to complete the proof that V is a crossed H-bimodule, we have to check Relation (5.2) using (5.1). If $a \in H$, $v \in V$ and $f \in H^*$, then

$$
\begin{aligned}
(\mathrm{id}\otimes f)\Big(\sum_{(a)(v)} a'v_V \otimes a''v_H\Big) &= (\mathrm{id}\otimes f)\Big(\sum_{(a),i} a'(a^i v)\otimes a''a_i\Big)\\
&= \sum_{(a),i} a'(a^i v) f(a''a_i)\\
&= \sum_{(a)(f),i} f'(a_i)f''(a'')a'(a^i v)\\
&= \sum_{(a)(f)} f''(a'')a'\Big((\sum_i f'(a_i)a^i)v\Big)\\
&= \sum_{(a)(f)} f''(a'')a'(f'v)\\
&= \sum_{(a)(f)} f''(a'''')f'(S^{-1}(a''')?a')(a''v)\\
&= \sum_{(a)} f(a''''S^{-1}(a''')?a')(a''v)\\
&= \sum_{(a)} \varepsilon(a''')f(?a')(a''v)\\
&= \sum_{(a)} f(?a')(a''v)\\
&= \sum_{(a)(f)} f'(a')\, f''a''v\\
&= \sum_{(a)(f),i} a^i(a''v)f''(a_i)f'(a')\\
&= \sum_{(a),i} a^i(a''v)f(a_i a')\\
&= (\mathrm{id}\otimes f)\Big(\sum_{(a),i} a^i(a''v)\otimes a_i a'\Big)\\
&= (\mathrm{id}\otimes f)\Big(\sum_{(a)(v)} (a''v)_V\otimes (a''v)_H a'\Big).
\end{aligned}
$$

This, being true for any linear form f, implies (5.2). In the previous series of equalities, we used the comultiplication on H^*, Relation (5.1), the fact that S^{-1} is a skew-antipode, that ε is a counit, and that $f = \sum_i f(a_i)a^i$.

(b) Conversely, let V be a crossed H-bimodule. We show that V can be given a $D(H)$-module structure. Observe that if $(V, \Delta_V : V \to V\otimes H)$ is a right H-comodule, then V becomes a left module over the dual algebra $X = H^*$ by

$$H^* \otimes V \xrightarrow{\mathrm{id}_{H^*} \otimes \Delta_V} H^* \otimes V \otimes H \xrightarrow{\mathrm{id}_{H^*} \otimes \tau_{V,H}} H^* \otimes H \otimes V \xrightarrow{\mathrm{ev}_H \otimes \mathrm{id}_V} V$$

where ev_H is the evaluation map. In other words, a linear form $f \in H^*$ acts on an element $v \in V$ by

$$f \cdot v = \sum_{(v)} < f, v_H > v_V. \tag{5.4}$$

In view of this observation, we see that a crossed bimodule has a left H-action as well as a left H^*-action. In order to prove V is a $D(H)$-module, it is enough to check Relation (5.1). We have

$$\begin{aligned}
\sum_{(a)} f(S^{-1}(a''')?a') \cdot (a''v) &= \sum_{(a)(v)} < f, S^{-1}(a''')(a''v)_H\, a' > (a''v)_V \\
&= \sum_{(a)(v)} < f, S^{-1}(a''')a''\ v_H > a'v_V \\
&= \sum_{(a)(v)} \varepsilon(a'') < f, v_H > a'v_V \\
&= \sum_{(v)} < f, v_H > av_V \\
&= a(f \cdot v)
\end{aligned}$$

for any $a \in H$, $f \in H^*$ and $v \in V$. The second equality is a consequence of Relation (5.2). The third one follows from the fact that S^{-1} is a skew-antipode. □

Remark 5.3. Formula (5.3) defining the coaction Δ_V may be rewritten as

$$\Delta_V(v) = R_{21}(v \otimes 1) \tag{5.5}$$

where R_{21} is obtained from the universal R-matrix of $D(H)$ by applying the flip. We shall use Relation (5.5) in order to determine the universal R-matrix of $U_q(\mathfrak{sl}(2))$ in XVII.4.

IX.6 Application to $U_q(\mathfrak{sl}(2))$

We now return to the Hopf algebra $U_q = U_q(\mathfrak{sl}(2))$ studied in detail in Chapters VI–VII. We wish to show that it has a universal R-matrix using the quantum double construction of Section 4. However, we only gave this construction for finite-dimensional Hopf algebras, which is not the case of U_q. Therefore we postpone the construction of the universal R-matrix of U_q to Chapter XVII. Instead, we now work with the finite-dimensional quotient $\overline{U}_q$ introduced in VI.5.

We assume until the end of this chapter that q is a root of unity of order d in the field k where d is an *odd* integer > 1. Let us resume the notation of VI.1. Recall

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$

which is defined for any integer n, and the corresponding q-factorials $[n]!$. We have $[n] \neq 0$ if $0 < n < d$ and $[d] = 0$.

In VI.5 we defined the algebra $\overline{U}_q$ as the quotient of U_q by the two-sided ideal generated by the three elements E^d, F^d, $K^d - 1$. We proved in Proposition VI.5.7 that the finite set $\{E^i F^j K^\ell\}_{0 \leq i,j,\ell \leq d-1}$ was a basis of the underlying vector space of $\overline{U}_q$. We endow the algebra $\overline{U}_q$ with a Hopf algebra structure.

Proposition IX.6.1. *The algebra $\overline{U}_q$ has a unique Hopf algebra structure such that the canonical projection from U_q to $\overline{U}_q$ is a morphism of Hopf algebras.*

In other words, the comultiplication, the counit and the antipode of $\overline{U}_q$ are determined by Formulas (VII.1.1–1.4) defining the Hopf algebra structure of U_q.

PROOF. The proof proceeds as for Proposition VII.1.1. We still have to check that

$$\Delta(E)^d = \Delta(F)^d = \Delta(K)^d - 1 = 0,$$
$$\varepsilon(E)^d = \varepsilon(F)^d = \varepsilon(K)^d - 1 = 0,$$
$$S(E)^d = S(F)^d = S(K)^d - 1 = 0.$$

The only non-trivial computations concern the vanishing of $\Delta(E)^d$ and of $\Delta(F)^d$. Following Proposition VII.1.3, we get

$$\Delta(E)^d = E^d \otimes K^d + \sum_{r=1}^{r=d-1} q^{r(d-r)} \begin{bmatrix} d \\ r \end{bmatrix} E^{d-r} \otimes E^r K^{d-r} + 1 \otimes E^d = 0$$

because $E^d = 0$ on one hand and

$$\begin{bmatrix} d \\ r \end{bmatrix} = \frac{[d]!}{[r]![d-r]!} = 0$$

on the other. One proves that $\Delta(F)^d = 0$ in a similar way. □

The goal of this section is to establish that $\overline{U}_q$ is a braided Hopf algebra. To this end, we shall present $\overline{U}_q$ as a quotient of the quantum double of a Hopf subalgebra B_q of $\overline{U}_q$. We define B_q as the subspace of $\overline{U}_q$ linearly generated by the set $\{E^m K^n\}_{0 \leq m,n \leq d-1}$. Formulas (VII.1.1–1.4) show that

B_q is a Hopf subalgebra of $\overline{U}_q$. The reader may check that, as an algebra, B_q is generated by E and K and the relations

$$KEK^{-1} = q^2E, \quad E^d = 0, \quad \text{and} \quad K^d = 1. \tag{6.1}$$

We now apply the quantum double construction of Section 4 to the Hopf algebra $H = B_q$. We first determine $X = (B_q^{\mathrm{op}})^*$ as a Hopf algebra. Consider the linear forms α and η on B_q defined on the basis $\{E^mK^n\}_{0\leq m,n\leq d-1}$ by

$$< \alpha, E^mK^n >= \delta_{m0}\, q^{2n} \quad \text{and} \quad < \eta, E^mK^n >= \delta_{m1}. \tag{6.2}$$

Proposition IX.6.2. *The following relations hold in the Hopf algebra X:*

$$\alpha^d = 1, \quad \eta^d = 0, \quad \alpha\eta\alpha^{-1} = q^{-2}\eta,$$
$$\Delta(\alpha) = \alpha \otimes \alpha, \quad \Delta(\eta) = 1 \otimes \eta + \eta \otimes \alpha,$$
$$\varepsilon(\alpha) = 1, \quad \varepsilon(\eta) = 0,$$
$$S(\alpha) = \alpha^{d-1}, \quad S(\eta) = -\eta\alpha^{d-1}.$$

Moreover, the set $\{\eta^i\alpha^j\}_{0\leq i,j\leq d-1}$ forms a basis of X.

PROOF. We start with the following lemma.

Lemma IX.6.3. *For all integers i, j, m, n, we have*

$$< \eta^i\alpha^j, E^mK^n >= \delta_{mi}\, (i)!_{q^2}\, q^{2j(i+n)}.$$

PROOF. By Proposition VII.1.3, if α and β are linear forms on H, then the product $\alpha\beta$ in X is given by

$$< \alpha\beta, E^mK^n >= \sum_{r=0}^{r=m} \begin{pmatrix} m \\ r \end{pmatrix}_{q^2} < \alpha, E^{m-r}K^n >< \beta, E^rK^{m+n-r} > . \tag{6.3}$$

One uses (6.3) to show that

$$< \eta^i, E^mK^n >= \delta_{mi}\, (i)!_{q^2}$$

by induction on i, and that

$$< \alpha^j, E^mK^n >= \delta_{m0}\, q^{2jn}$$

by induction on j. Then

$$\begin{aligned} < \eta^i\alpha^j, E^mK^n > &= \sum_{r=0}^{r=m} \begin{pmatrix} m \\ r \end{pmatrix}_{q^2} < \eta^i, E^{m-r}K^n >< \alpha^j, E^rK^{m+n-r} > \\ &= \sum_{r=0}^{r=m} \begin{pmatrix} m \\ r \end{pmatrix}_{q^2} \delta_{m-r,i}\, \delta_{r0}\, (i)!_{q^2}\, q^{2j(m+n-r)} \\ &= \delta_{mi}\, (i)!_{q^2}\, q^{2j(i+n)}. \end{aligned}$$

□

Let us prove Proposition 6.2.

(1) Using the previous lemma and $q^d = 1$, we get

$$< \alpha^d, E^m K^n > = \delta_{m0} = < \varepsilon, E^m K^n > .$$

Therefore, $\alpha^d = \varepsilon$ is the unit of X. Analogously,

$$< \eta^d, E^m K^n > = \delta_{md}\,(d)!_{q^2} = 0$$

since $(d)_{q^2} = (q^{2d} - 1)/(q^2 - 1) = 0$. As for $\alpha\eta$, we have

$$< \alpha\eta, E^m K^n > = \sum_{r=0}^{r=m} \left(\begin{array}{c} m \\ r \end{array} \right)_{q^2} \delta_{m-r,0}\,\delta_{r1}\,q^{2n} = \delta_{m1}\,q^{2n}.$$

In view of Lemma 6.3 we can write

$$< \eta\alpha, E^m K^n > = \delta_{m1}\,q^{2(n+1)} = q^2 \;< \alpha\eta, E^m K^n >,$$

whence $\eta\alpha = q^2\alpha\eta$.

(2) Let us deal with the comultiplication of X. By definition, if α is a linear form on H, then $\Delta(\alpha)$ is given by $\Delta(\alpha)(x \otimes y) = \alpha(yx)$ for $x, y \in H$. Therefore

$$\begin{aligned}
\Delta(\alpha)(E^i K^j \otimes E^m K^n) &= q^{2ni} < \alpha, E^{i+m} K^{j+n} > \\
&= \delta_{i+m,0}\,q^{2ni} q^{2(j+n)} \\
&= \delta_{i0}\,\delta_{m0}\,q^{2j} q^{2n} \\
&= < \alpha \otimes \alpha, E^i K^j \otimes E^m K^n >,
\end{aligned}$$

which implies that $\Delta(\alpha) = \alpha \otimes \alpha$.

Similarly, we have

$$\begin{aligned}
\Delta(\eta)(E^i K^j \otimes E^m K^n) &= q^{2ni} < \eta, E^{i+m} K^{j+n} > \\
&= \delta_{i+m,1}\,q^{2ni} \\
&= \delta_{i0}\,\delta_{m1} + \delta_{i1}\,\delta_{m0}\,q^{2n} \\
&= < 1 \otimes \eta + \eta \otimes \alpha, E^i K^j \otimes E^m K^n > .
\end{aligned}$$

Consequently, $\Delta(\eta) = 1 \otimes \eta + \eta \otimes \alpha$.

(3) Concerning the counit, we have

$$\varepsilon(\alpha) = < \alpha, 1 > = 1 \quad \text{and} \quad \varepsilon(\eta) = < \eta, 1 > = 0.$$

The computation of $S(\alpha)$ and of $S(\eta)$ is left to the reader.

(4) Let us prove the last assertion of Proposition 6.2. As the dimension of X is d^2, it is enough to show that the set $\{\eta^i\alpha^j\}_{0\le i,j\le d-1}$ is linearly independent. Suppose there exists a relation of the form

$$\sum_{0\le i,j\le d-1} \lambda_{ij}\eta^i\alpha^j = 0.$$

Applying it to the vector E^mK^n, we get

$$\sum_{0\le i,j\le d-1} \lambda_{ij}\delta_{mi}(i)!_{q^2}q^{2j(i+n)} = (m)!_{q^2}\left(\sum_{0\le j\le d-1} \lambda_{mj}q^{2j(m+n)}\right) = 0.$$

Letting m fixed and running n over the d integers between 0 and $d-1$, we obtain a system of d linear equations with unknowns $\lambda_{m0}, \lambda_{m1}, \dots, \lambda_{m,d-1}$. The determinant of this system is the determinant of the matrix $(A_{k\ell})_{k\ell}$ defined by $A_{k\ell} = (q^{2(m+\ell)})^k$. It is a Vandermonde determinant which does not vanish because $q^{2(m+\ell)} \neq q^{2(m+\ell')}$ whenever $0 \le \ell \neq \ell' \le d-1$. Therefore, the system has 0 as its unique solution; in other words, we have $\lambda_{mj} = 0$ for all j. □

We now construct the quantum double $D = D(B_q)$. By definition, the set $\{\eta^i\alpha^j \otimes E^kK^\ell\}_{0\le i,j,k,\ell\le d-1}$ is a basis of D. To simplify notation, we identify an element x of $H = B_q$ with its image $1\otimes x$ in D and an element α of the dual X with its image $\alpha\otimes 1$. Under the convention already set up in Section 4, the elements of the previous basis can be rewritten in the form

$$\eta^i\alpha^j \otimes E^kK^\ell = \eta^i\alpha^jE^kK^\ell.$$

To determine the multiplication of the double D, it is enough to know how the generators α, η, E, K in D multiply together.

Proposition IX.6.4. *The following relations hold in $D = D(B_q)$:*

$$K\alpha = \alpha K, \quad K\eta = q^{-2}\,\eta K,$$

$$E\alpha = q^{-2}\,\alpha E, \quad E\eta = -q^{-2}(1 - \eta E - \alpha K).$$

PROOF. By (4.5) the product $x\alpha$ in D of $x \in H$ of $\alpha \in X$ is given by

$$x\alpha = \sum_{(x)} \alpha(S^{-1}(x''')?x')\,x''.$$

Let us apply this formula to the generators. First, we have $S^{-1}(K) = K^{-1}$ and $(\Delta\otimes\mathrm{id})(\Delta(K)) = K\otimes K\otimes K$. Consequently, for any linear form $\beta \in X$ we have

$$K\beta = \beta(K^{-1}?K)\,K. \tag{6.4}$$

Next, $S^{-1}(E) = -K^{-1}E$ and

$$(\Delta \otimes \mathrm{id})(\Delta(E)) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K.$$

Hence,

$$E\beta = -\beta(K^{-1}E?) + \beta(K^{-1}?)\,E + \beta(K^{-1}?E)\,K. \qquad (6.5)$$

Proposition 6.4 is then a consequence of (6.4–6.5) and of the following lemma. □

Lemma IX.6.5. *We have*

$$\begin{array}{rclcrcl}
\alpha(K^{-1}?K) & = & \alpha, & \quad & \alpha(K^{-1}E?) & = & 0, \\
\alpha(K^{-1}?) & = & q^{-2}\,\alpha, & & \alpha(K^{-1}?E) & = & 0, \\
\eta(K^{-1}?K) & = & q^{-2}\,\eta, & & \eta(K^{-1}E?) & = & q^{-2}\,\varepsilon, \\
\eta(K^{-1}?) & = & q^{-2}\,\eta, & & \eta(K^{-1}?E) & = & q^{-2}\,\alpha.
\end{array}$$

PROOF. Left to the reader. □

We now relate the quantum double $D(B_q)$ and the Hopf algebra $\overline{U}_q$.

Theorem IX.6.6. *Let $\chi : D(B_q) \to \overline{U}_q$ be the linear map determined by*

$$\chi(\eta^i\alpha^j E^k K^\ell) = \Big(\frac{q-q^{-1}}{q^2}\Big)^i q^{2(i+j)k-i(i-1)}\,F^i E^k K^{i+j+\ell} \qquad (6.6)$$

where $0 \leq i,j,k,\ell \leq d-1$. Then χ is a surjective Hopf algebra morphism.

PROOF. The surjectivity of χ follows from the fact that the image of the basis $\{\eta^i\alpha^j E^k K^\ell\}$ generates $\overline{U}_q$.

In order to show that χ is a map of algebras, it is enough to check that the images under χ of the generators E, K, α, η satisfy the relations of Proposition 6.4. Observe that (6.6) implies

$$\chi(E) = E, \qquad \chi(K) = K,$$

$$\chi(\alpha) = K, \qquad \chi(\eta) = \frac{q-q^{-1}}{q^2}\,FK.$$

Now, by definition of $\overline{U}_q$ we have

$$\begin{array}{rcl}
\chi(K)\chi(\alpha) & = & \chi(\alpha)\chi(K), \\
\chi(K)\chi(\eta) & = & \dfrac{q-q^{-1}}{q^2}\,KFK = q^{-2}\,\chi(\eta)\chi(K), \\
\chi(E)\chi(\alpha) & = & q^{-2}\,\chi(\alpha)\chi(E).
\end{array}$$

Finally, we get

$$\begin{aligned}
\chi(E)\chi(\eta) &= \frac{q-q^{-1}}{q^2}\, EFK \\
&= \frac{q-q^{-1}}{q^2}\, FEK + \frac{1}{q^2}\,(K-K^{-1})K \\
&= -q^{-2}\left(1 - \frac{q-q^{-1}}{q^2}\, FKE - K^2\right) \\
&= -q^{-2}\Big(1 - \chi(\eta)\chi(E) - \chi(\alpha)\chi(K)\Big).
\end{aligned}$$

This proves that χ is a morphism of algebras.

Again, to show that χ respects the comultiplication and the antipode, it is enough to check on the generators. For E, K, and α, this is clear. We still have to examine the case of η for which we have

$$\begin{aligned}
\Delta(\chi(\eta)) &= \frac{q-q^{-1}}{q^2}\,\Delta(FK) \\
&= \frac{q-q^{-1}}{q^2}\,(1\otimes FK + FK\otimes K) \\
&= \chi(1)\otimes\chi(\eta) + \chi(\eta)\otimes\chi(\alpha) \\
&= (\chi\otimes\chi)\Big(\Delta(\eta)\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\chi(S(\eta)) &= -\chi(\eta\alpha^{-1}) \\
&= -\frac{q-q^{-1}}{q^2}\, F \\
&= \frac{q-q^{-1}}{q^2}\, S(K)S(F) \\
&= \frac{q-q^{-1}}{q^2}\, S(FK) \\
&= S(\chi(\eta)).
\end{aligned}$$

□

We draw the following consequence which was our main goal.

Corollary IX.6.7. *The Hopf algebra $\overline{U}_q$ is braided.*

PROOF. The Hopf algebra $D = D(B_q)$ is braided by Theorem 4.3. Let $R_D \in D\otimes D$ be its universal R-matrix. Define the invertible element $\overline{R}$ of $\overline{U}_q\otimes\overline{U}_q$ by

$$\overline{R} = (\chi\otimes\chi)(R_D). \tag{6.7}$$

Since χ is a surjective morphism of Hopf algebras, it is clear that $\overline{R}$ satisfies Conditions (VIII.2.1–2.3). $\square$

We shall compute the universal R-matrix $\overline{R}$ of $\overline{U}_q$ in the next section.

IX.7 R-Matrices for $\overline{U}_q$

We keep the hypotheses and the notations of Section 6.

Theorem IX.7.1. *The universal R-matrix of $\overline{U}_q$ is given by*

$$\overline{R} = \frac{1}{d} \sum_{0\leq i,j,k\leq d-1} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2+2k(i-j)-2ij} E^k K^i \otimes F^k K^j.$$

PROOF. According to Section 4, we have $R_D = \sum_{i\in I} e_i \otimes e^i$ where $\{e_i\}_{i\in I}$ is any basis of the vector space B_q and $\{e^i\}_{i\in I}$ is its dual basis. Consequently, by (6.7)

$$\overline{R} = \sum_{i\in I} \chi(e_i) \otimes \chi(e^i). \tag{7.1}$$

As above, we take the set $\{E^i K^j\}_{0\leq i,j\leq d-1}$ as a basis of B_q. Denote by $\{\beta^{ij}\}_{0\leq i,j\leq d-1}$ the dual basis. By Proposition 6.2 we know there exist scalars $\{\mu_{k\ell}^{ij}\}_{0\leq i,j,k,\ell\leq d-1}$ such that

$$\beta^{ij} = \sum_{0\leq k,\ell\leq d-1} \mu_{k\ell}^{ij}\, \eta^k \alpha^\ell. \tag{7.2}$$

Apply Relation (7.2) to the vector $E^m K^n$: using Lemma 6.3, we obtain the linear system of equations

$$\begin{aligned}\delta_{im}\delta_{jn} &= \sum_{0\leq k,\ell\leq d-1} \mu_{k\ell}^{ij}\, \delta_{km} (m)!_{q^2} q^{2\ell(k+n)} \\ &= (m)!_{q^2} \left(\sum_{0\leq \ell\leq d-1} \mu_{m\ell}^{ij}\, q^{2\ell(m+n)} \right).\end{aligned}$$

An argument similar to the one that proved the linear independence of the family $\{\eta^i \alpha^j\}_{0\leq i,j\leq d-1}$ in Proposition 6.2 shows that $\mu_{m\ell}^{ij} = 0$ for $m \neq i$. Computing the coefficients $\mu_{i\ell}^{ij}$ that are solutions of the linear system

$$\sum_{0\leq \ell\leq d-1} \mu_{i\ell}^{ij}\, q^{2\ell(i+n)} = \frac{1}{(m)!_{q^2}}\, \delta_{jn}$$

requires inverting a Vandermonde matrix. We shall not do this since we are interested in $\overline{R}$, not in R_D. Instead, we shall use a simpler and more direct method.

Indeed, from the above arguments we know that $\overline{R}$ is a tensor of the form

$$\overline{R} = \sum_{0 \le i,j,\ell \le d-1} \mu_{i\ell}^{ij}\, \chi(E^i K^j) \otimes \chi(\eta^i \alpha^\ell).$$

Now $\chi(\eta^i \alpha^\ell)$ is a scalar multiple of $F^i K^{i+\ell}$. Therefore, $\overline{R}$ has the more precise form

$$\overline{R} = \sum_{0 \le i,j,k \le d-1} c_{i,j,k}\, E^k K^i \otimes F^k K^j.$$

We now determine the coefficients $c_{i,j,k}$. Theorem 7.1 will follow from Lemma 7.2. □

Lemma IX.7.2. *For all i, j, k we have*

$$c_{i,j,k} = \frac{1}{d} \frac{(q - q^{-1})^k}{[k]!}\, q^{k(k-1)/2 + 2k(i-j) - 2ij}.$$

Proof. (a) We first express $c_{i,j,k}$ in terms of $c = c_{0,0,0}$ using the relations

$$\Delta^{\mathrm{op}}(x) R = R \Delta(x)$$

for $x = E$ and $x = F$. We have

$$\begin{aligned}
\Delta^{\mathrm{op}}(E) R &= (E \otimes 1 + K \otimes E)\Bigg(\sum_{0 \le i,j,k \le d-1} c_{i,j,k}\, E^k K^i \otimes F^k K^j\Bigg) \\
&= \sum_{0 \le i,j,k \le d-1} c_{i,j,k}\, E^{k+1} K^i \otimes F^k K^j \\
&\qquad + \sum_{0 \le i,j,k \le d-1} q^{2k} c_{i,j,k}\, E^k K^{i+1} \otimes E F^k K^j,
\end{aligned}$$

and

$$\begin{aligned}
R\Delta(E) &= \Bigg(\sum_{0 \le i,j,k \le d-1} c_{i,j,k}\, E^k K^i \otimes F^k K^j\Bigg)(1 \otimes E + E \otimes K) \\
&= \sum_{0 \le i,j,k \le d-1} q^{2j} c_{i,j,k}\, E^k K^i \otimes F^k E K^j \\
&\qquad + \sum_{0 \le i,j,k \le d-1} q^{2i} c_{i,j,k}\, E^{k+1} K^i \otimes F^k K^{j+1} \\
&= \sum_{0 \le i,j,k \le d-1} q^{2j} c_{i,j,k}\, E^k K^i \otimes E F^k K^j \\
&\qquad - \sum_{0 \le i,j,k \le d-1} [k] \frac{q^{2j-(k-1)}}{q - q^{-1}} c_{i,j,k}\, E^k K^i \otimes F^{k-1} K^{j+1} \;+
\end{aligned}$$

$$+ \sum_{0\le i,j,k\le d-1} [k]\frac{q^{2j+(k-1)}}{q-q^{-1}}\, c_{i,j,k}\, E^kK^i \otimes F^{k-1}K^{j-1}$$
$$+ \sum_{0\le i,j,k\le d-1} q^{2i}\, c_{i,j,k}\, E^{k+1}K^i \otimes F^kK^{j+1}.$$

Identifying the coefficients of $E^kK^i \otimes EF^kK^j$, we get

$$c_{i,j,k} = q^{2(k-j)}\, c_{i-1,j,k}. \tag{7.3}$$

Starting all over again with F, we get

$$\begin{aligned}
\Delta^{\mathrm{op}}(F)R &= (F\otimes K^{-1} + 1\otimes F)\left(\sum_{0\le i,j,k\le d-1} c_{i,j,k}\, E^kK^i\otimes F^kK^j\right)\\
&= \sum_{0\le i,j,k\le d-1} q^{2k}\, c_{i,j,k}\, FE^kK^i \otimes F^kK^{j-1}\\
&\qquad + \sum_{0\le i,j,k\le d-1} c_{i,j,k}\, E^kK^i\otimes F^{k+1}K^j\\
&= \sum_{0\le i,j,k\le d-1} q^{2k}\, c_{i,j,k}\, E^kFK^i \otimes F^kK^{j-1}\\
&\qquad - \sum_{0\le i,j,k\le d-1} [k]\frac{q^{2k+(k-1)}}{q-q^{-1}}\, c_{i,j,k}\, E^{k-1}K^{i+1}\otimes F^kK^{j-1}\\
&\qquad + \sum_{0\le i,j,k\le d-1} [k]\frac{q^{2k-(k-1)}}{q-q^{-1}}\, c_{i,j,k}\, E^{k-1}K^{i-1}\otimes F^kK^{j-1}\\
&\qquad + \sum_{0\le i,j,k\le d-1} c_{i,j,k}\, E^kK^i\otimes F^{k+1}K^j.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
R\Delta(F) &= \left(\sum_{0\le i,j,k\le d-1} c_{i,j,k}\, E^kK^i\otimes F^kK^j\right)(K^{-1}\otimes F + F\otimes 1)\\
&= \sum_{0\le i,j,k\le d-1} q^{-2j}\, c_{i,j,k}\, E^kK^{i-1}\otimes F^{k+1}K^j\\
&\qquad + \sum_{0\le i,j,k\le d-1} q^{-2i}\, c_{i,j,k}\, E^kFK^i\otimes F^kK^j.
\end{aligned}$$

We identify the coefficients of $E^kFK^i \otimes F^kK^{j-1}$, which yields

$$c_{i,j,k} = q^{-2(k+i)}\, c_{i,j-1,k}, \tag{7.4}$$

and the coefficients of $E^kK^i \otimes F^{k+1}K^j$, which leads to

$$q^{-2j}\, c_{i+1,j,k} = c_{i,j,k} - [k+1]\frac{q^{2(k+1)+k}}{q-q^{-1}}\, c_{i-1,j+1,k+1} + [k+1]\frac{q^{2(k+1)-k}}{q-q^{-1}}\, c_{i+1,j+1,k+1}. \tag{7.5}$$

From (7.3) and (7.4) we get

$$c_{i,j,0} = q^{-2ij}\, c_{0,0,0} = q^{-2ij}\, c. \tag{7.6}$$

Combining (7.3) and (7.5), we obtain

$$q^{-2j+2(k-j)}\, c_{i,j,k} = c_{i,j,k} - [k+1]\frac{q^{4j-k+2}}{q-q^{-1}}\, c_{i+1,j+1,k+1} + [k+1]\frac{q^{k+2}}{q-q^{-1}}\, c_{i+1,j+1,k+1},$$

hence

$$c_{i+1,j+1,k+1} = \frac{q-q^{-1}}{[k+1]}\, q^{k-4j-2}\, c_{i,j,k},$$

or, equivalently,

$$c_{i,j,k} = \frac{q-q^{-1}}{[k]}\, q^{k-4j+1}\, c_{i-1,j-1,k-1}.$$

Therefore, we get

$$\begin{aligned} c_{i,j,k} &= \frac{(q-q^{-1})^k}{[k]!}\, q^{k(k+1)/2-4kj+2k(k-1)+k}\, c_{i-k,j-k,0} \\ &= \frac{(q-q^{-1})^k}{[k]!}\, q^{k(k-1)/2+2k(k-2j)}\, q^{-2(i-k)(j-k)}\, c \end{aligned}$$

by (7.6). In other words, we have

$$c_{i,j,k} = c\,\frac{(q-q^{-1})^k}{[k]!}\, q^{k(k-1)/2+2k(i-j)-2ij}.$$

(b) It is now enough to prove that $c = 1/d$. From Part (a) we know that $\overline{R}$ is of the form

$$\overline{R} = c \sum_{0\le i,j<d} q^{-2ij}\, K^i \otimes K^j + \dots$$

where $+\dots$ stands for a sum of monomials containing only positive powers of E or of F. We now use $(\Delta\otimes \mathrm{id})(\overline{R}) = \overline{R}_{13}\overline{R}_{23}$, which is Relation (VIII.2.3). We have

$$(\Delta\otimes \mathrm{id})(\overline{R}) = c \sum_{0\le i,j<d} q^{-2ij}\, K^i\otimes K^i \otimes K^j + \dots \tag{7.7}$$

whereas

$$\begin{aligned}\overline{R}_{13}\overline{R}_{23} &= c^2 \sum_{0\leq i,\ell,m,j<d} q^{-2i\ell-2mj}\, K^i \otimes K^m \otimes K^{\ell+j} + \dots \\ &= c^2 \sum_{0\leq i,\ell,m,j<d} q^{-2i\ell-2m(j-\ell)}\, K^i \otimes K^m \otimes K^j + \dots \\ &= c^2 \sum_{0\leq i,m,j<d} q^{-2mj}\Big(\sum_{0\leq \ell<d} q^{(2m-2i)\ell}\Big)\, K^i \otimes K^m \otimes K^j + \dots .\end{aligned}$$

Now, $\sum_{0\leq j<d} q^{Nj}$ vanishes except when N is a multiple of d, in which case the sum equals d. Therefore,

$$\overline{R}_{13}\overline{R}_{23} = dc^2 \sum_{0\leq i,j<d} q^{-2ij}\, K^i \otimes K^i \otimes K^j + \dots . \tag{7.8}$$

We deduce from (7.7–7.8) that $c = dc^2$. Since $\overline{R}$ is invertible, c does not vanish. This implies the announced value for c. □

We conclude this section by deriving a few R-matrices from Theorem 7.1. Let $0 < n < d$. Consider the simple $\overline{U}_q$-module $V_n = V_{1,n}$ defined in Chapter VI. As a module, it is generated by a highest weight vector $v_0^{(n)}$ of weight q^n. Recall that the action of $\overline{U}_q$ on the canonical basis $\{v_0^{(n)}, v_1^{(n)}, \dots, v_n^{(n)}\}$ of V_n is given by the relations

$$Kv_p^{(n)} = q^{n-2p}\, v_p^{(n)}, \quad Ev_p^{(n)} = [n-p+1]\, v_{p-1}^{(n)}, \quad Fv_p^{(n)} = [p+1]\, v_{p+1}^{(n)}.$$

We use them to deduce the form of the R-matrix

$$c^{\overline{R}}_{V_n,V_m} : V_n \otimes V_m \to V_m \otimes V_n$$

obtained from $\overline{R}$ via the construction of (VIII.3.1). Recall that $c^{\overline{R}}_{V_n,V_m}$ is a solution of the Yang-Baxter equation.

Corollary IX.7.3. *The isomorphism $c^{\overline{R}}_{V_n,V_m} : V_n \otimes V_m \to V_m \otimes V_n$ is the $\overline{U}_q$-linear map determined by*

$$\begin{aligned}&c^{\overline{R}}_{V_n,V_m}(v_p^{(n)} \otimes v_r^{(m)}) \\ &= \sum_{0\leq k\leq d-1} \frac{(q-q^{-1})^k}{[k]!}\, \frac{[n-p+k]![r+k]!}{[n-p]![r]!}\, q_{pr}^{nm}(k,\alpha)\, v_{r+k}^{(m)} \otimes v_{p-k}^{(n)}\end{aligned}$$

where α is any integer such that $m + \alpha d$ is even and

$$q_{pr}^{nm}(k,\alpha) = q^{k(k-1)/2+k(m-n)-pm-rn-2(k-p)(k+r)+(m+\alpha d)n/2}.$$

PROOF. By definition of $c^{\overline{R}}_{V_n,V_m}$ and in view of Theorem 7.1, we have

$$\begin{aligned} & c^{\overline{R}}_{V_n,V_m}(v_p^{(n)} \otimes v_r^{(m)}) \\ & = \frac{1}{d} \sum_{0\le i,j,k\le d-1} \frac{(q-q^{-1})^k}{[k]!} \frac{[n-p+k]![r+k]!}{[n-p]![r]!} q^{k(k-1)/2} \\ & \qquad \times Q_{pr}^{nm}(k)\, v_{r+k}^{(m)} \otimes v_{p-k}^{(n)} \end{aligned}$$

where

$$Q_{pr}^{nm}(k) = \sum_{0\le i,j<d} q^{2(i-j)k-2ij+i(n-2p)+j(m-2r)},$$

which we can rewrite as

$$Q_{pr}^{nm}(k) = \sum_{0\le i<d} q^{2ik+i(n-2p)} \Big(\sum_{0\le j<d} q^{(m-2r-2i-2k)j} \Big).$$

Again, $\sum_{0\le j<d} q^{Nj}$ vanishes except when N is a multiple of d. Thus,

$$Q_{pr}^{nm}(k) = d \sum_i q^{2ik+i(n-2p)}$$

where i runs over the set of all integers in $[0, d-1]$ such that

$$2i = m - 2r - 2k + \alpha d.$$

As 2 is invertible modulo d, there exists only one integer i satisfying these conditions. Therefore,

$$Q_{pr}^{nm}(k) = d\, q^{2ik+i(n-2p)} = d\, q^{k(m-n)-pm-rn-2(k-p)(k+r)+(m+\alpha d)n/2}. \qquad \square$$

Application 7.4. Consider the case $n = m = 1$. We may take $\alpha = 1$. Corollary 7.3 implies that

$$\begin{aligned} c^{\overline{R}}_{V_1,V_1}(v_0 \otimes v_0) &= \lambda\, q\, v_0 \otimes v_0, \\ c^{\overline{R}}_{V_1,V_1}(v_0 \otimes v_1) &= \lambda\, v_1 \otimes v_0, \\ c^{\overline{R}}_{V_1,V_1}(v_1 \otimes v_0) &= \lambda\, (v_0 \otimes v_1 + (q - q^{-1}) v_1 \otimes v_0), \\ c^{\overline{R}}_{V_1,V_1}(v_1 \otimes v_1) &= \lambda\, q\, v_1 \otimes v_1 \end{aligned}$$

where $\lambda = q^{(d-1)/2}$, $v_0 = v_0^{(1)}$ and $v_1 = v_1^{(1)}$. The reader is invited to compare these formulas with the R-matrices of VIII.1, Example 2.

IX.8 Exercises

1. Let H be a bialgebra and C a coalgebra. Prove that C is a module-coalgebra over H if and only if there exists an H-module structure on C such that the comultiplication $\Delta : C \to C \otimes C$ and the counit $\varepsilon : C \to k$ of C are H-module maps for the tensor product H-module structure on $C \otimes C$ and for the trivial H-module structure on k.

2. Let H be a bialgebra and C a coalgebra. Then C is a *comodule-coalgebra* over H if there exists an H-comodule structure on C such that the coproduct $\Delta : C \to C \otimes C$ and the counit $\varepsilon : C \to k$ of C are morphisms of H-comodules for the tensor product H-comodule structure on $C \otimes C$ and for the trivial H-comodule structure on k. Draw the commutative diagrams expressing an H-coalgebra-comodule structure on C. Deduce that C is a comodule-coalgebra over H if and only if there exists a linear map $\Delta_C : C \to H \otimes C$ inducing an H-comodule structure on C and satisfying for all $x \in H$ and $c \in C$ the relations
$$\sum_{(c)} c_H \otimes (c_C)' \otimes (c_C)'' = \sum_{(c)} c'_H c''_H \otimes (c')_C \otimes (c'')_C$$
and $\sum_{(c)} c_H \varepsilon(c_C) = \varepsilon(c)1$ where $\Delta_C(c) = \sum_{(c)} c_H \otimes c_C$.

3. Let H be a bialgebra and C a coalgebra equipped with a comodule-coalgebra structure on H. Show that the dual algebra C^* can be given a comodule-algebra structure on H.

4. Let H be a bialgebra and C a coalgebra equipped with a module-coalgebra structure on H. Show that if C is finite-dimensional, then the dual algebra C^* can be given a module-algebra structure on H.

5. (*Adjoint corepresentation*) Let H be a Hopf algebra. Define a linear map Δ_{ad} from H to $H \otimes H$ by
$$\Delta_{ad}(a) = \sum_{(a)} a' S(a''') \otimes a''.$$
Prove that Δ_{ad} endows H with a comodule-coalgebra structure over itself.

6. (*Coadjoint corepresentation*) Let H be a finite-dimensional Hopf algebra with invertible antipode. Prove that the adjoint corepresentation induces an H-comodule-algebra structure on the dual vector space H^*.

7. Let G be a finite group. (a) Show that a left module over the quantum double $D(k[G])$ is a left G-module V with a decomposition of the form $V = \bigoplus_{g \in G} V_g$ such that $hV_g \subset V_{hgh^{-1}}$ for all $g, h \in G$.

(b) Let $W = \bigoplus_{g\in G} W_g$ be another left $D(k[G])$-module. Show that the automorphism c^R defined in VIII.3 sends $V_g \otimes W_h$ to $W_h \otimes V_{hgh^{-1}}$ as the map $v \otimes w \mapsto w \otimes hv$.

8. (*Tensor product of crossed bimodules*) Use Theorem 5.2 to define the tensor product of two crossed bimodules.

9. Compute the central element $uS(u)$ (defined in VIII.4) for $\overline{U}_q$.

10. Determine $c^{\overline{R}}_{V_2,V_2}$ associated to the simple $\overline{U}_q$-module V_2 under the form of a 9×9 matrix.

11. (*A cobraided Hopf algebra structure on* $\mathrm{End}(H)$) Suppose given a finite-dimensional Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1})$ with bijective antipode. Let $E = \mathrm{End}(H)$, and identify $E \otimes E$ with $\mathrm{End}(H \otimes H)$.

    (a) Prove that there exists a Hopf algebra structure on E for which the product is the convolution of III.3, the unit is $\eta \circ \varepsilon$, the coproduct Δ', the counit ε', and the antipode S' are given by

$$\Delta'(f)(x \otimes y) = \sum_{(x)} (1 \otimes x')\Delta(f(yx''))(1 \otimes S(x''')),$$

$$\varepsilon'(f) = \varepsilon(f(1)), \quad S'(f)(x) = \sum_{(x)} S(x')(SfS^{-1})(x'')x'''$$

    for all $x, y \in H$ and $f \in E$.

    (b) Identifying E with $H \otimes H^*$ via the map $\lambda_{H,H}$ of Corollary II.2.3, define maps $p_H : E \to H$ and $p_{H^*} : E \to H^{*cop}$ by

$$p_H(x \otimes \alpha) = \alpha(1)x \quad \text{and} \quad p_{H^*}(x \otimes \alpha) = \varepsilon(x)\alpha$$

    where $x \in H$ and $\alpha \in H^*$. Prove that p_H and p_{H^*} are morphisms of Hopf algebras such that the composition of the maps

$$E \xrightarrow{\Delta'} E \otimes E \xrightarrow{p_H \otimes p_{H^*}} H \otimes H^*$$

    is $\lambda^{-1}_{H,H}$.

    (c) Check that the linear form r on $E \otimes E$ defined by

$$r(f \otimes g) = < p_{H^*}(f), p_H(g) >$$

    for $f, g \in E$ equips E with a cobraided Hopf algebra structure.

    (d) Show that the dual braided Hopf algebra E^* is isomorphic to Drinfeld's quantum double $D(H)$.

IX.9 Notes

The quantum double construction is due to Drinfeld [Dri87]. Our presentation is inspired from [Maj90a] [Tak81] (see also [RSTS88]). Radford [Rad93a] proved that the quantum double is a minimal braided Hopf algebra, i.e., it has no proper braided Hopf subalgebras. Conversely, any minimal braided Hopf algebra is finite-dimensional and is a quotient of the quantum double of some Hopf algebra. More generally, if H is braided with universal R-matrix R, consider the subspace A of H generated by all elements of the form $(\mathrm{id}_H \otimes \alpha)(R)$ where α is any linear form on H. Radford showed that the subspace A can be given the structure of a Hopf subalgebra, and that there exists a map of braided Hopf algebras from $D(A)$ to H whose image is a minimal braided Hopf subalgebra of H.

In Section 4 we proved that the quantum double of H was isomorphic to a crossed product when H is cocommutative. This is true more generally when H is braided. For more details, see [Maj91a].

Exercise 11 presents a construction dual to Drinfeld's quantum double, yielding cobraided Hopf algebras. We took it from Takeuchi [Tak92a] where a dual version of Theorem 5.2 is also given (see also [PW90] [RSTS88]).

The term "crossed bimodule" is due to [Yet90]. It was called a "quantum Yang-Baxter module" in [Rad93b].

The Hopf algebra $\overline{U}_q$ has been considered in [Lus90a][Lus90b]. A computation of its universal R-matrix was performed in [KM91] using a different method. It also appears in work by Reshetikhin-Turaev [RT91] constructing quantum invariants for 3-dimensional manifolds.

Part Three

Low-Dimensional Topology and Tensor Categories

Chapter X
Knots, Links, Tangles, and Braids

We now embark into a topological digression which will lead us into the world of knots. The reason for the presence of this chapter in a book devoted to quantum groups is the close relationship between the newly discovered invariants of links (such as the celebrated Jones polynomial) and R-matrices. This relationship will become more precise in Chapter XII. In this one we proceed to describe several classes of one-dimensional submanifolds of the three-dimensional space $\mathbf{R}^3$, such as knots, links, tangles, and braids. Since there are excellent textbooks on knot theory, we shall not prove all assertions that can be found elsewhere. Nevertheless, all results pertaining to the matter of this book, namely those connecting topological problems with the algebra of quantum groups, will be proved in detail.

After defining knots and links in $\mathbf{R}^3$, we recall the classical problem of their classification up to isotopy. Traditionally, one approaches this problem by constructing algebraic isotopy invariants. One major step in this direction was undertaken in the 1920's by Alexander, who associated a polynomial to each isotopy class of oriented links. The Alexander polynomial was used to distinguish many links and has been a powerful tool in knot theory since.

In the summer of 1984 Vaughan Jones found a different one-variable polynomial which distinguished knots that the Alexander polynomial could not distinguish [Jon85]. Shortly after, a new invariant appeared, the so-called Jones-Conway polynomial, which is a two-variable generalization of both the Alexander and the Jones polynomials. Another aim of this chapter is to establish the existence and the main properties of the Jones-Conway polynomial.

X.1 Knots and Links

Let us start with some vocabulary from general topology. The only topological spaces considered here are the real Euclidean vector spaces $\mathbf{R}^n$ with their standard topology as well as their subsets and quotients with the induced topologies.

A continuous map f from a subset U of $\mathbf{R}^m$ to a subset X of $\mathbf{R}^n$ is *piecewise-linear* if there exists a finite partition $(U_i)_i$ of U such that the n components of the restriction of f to any U_i are maps of the form $(z_1, \ldots, z_m) \mapsto a_0 + a_1 z_1 + \cdots a_m z_m$ where $a_0, a_1, \ldots, a_m$ are real numbers.

Let X be a convex topological subspace of the Euclidean space $\mathbf{R}^3$. In the sequel, X will be either $\mathbf{R}^3$, $\mathbf{R}^2$, $\mathbf{R}^2 \times [0,1]$, or $\mathbf{R} \times [0,1]$. Given a finite sequence $(M_1, \ldots, M_n)$ of points in X, we denote by $[M_1, \ldots, M_n]$ [resp. $]M_1, \ldots, M_n[$] their closed [resp. open] convex envelope, i.e., the set of all points of the form $\lambda_1 M_1 + \cdots + \lambda_n M_n$ where $(\lambda_1, \ldots, \lambda_n)$ is a sequence of real numbers ≥ 0 [resp. > 0] such that $\lambda_1 + \cdots + \lambda_n = 1$.

Definition X.1.1. *A polygonal arc L in X is the union*

$$L = \bigcup_{i=1}^{n-1} [M_i, M_{i+1}]$$

of a finite number of segments such that $]M_i, M_{i+1}[\cap]M_j, M_{j+1}[= \emptyset$ if $i \neq j$. The points $M_1, \ldots, M_n$ are called the vertices of the polygonal arc and the segments $[M_i, M_{i+1}]$ are its edges. We say that the polygonal arc is simple if the points $M_1, \ldots, M_{n-1}$ are all distinct. The polygonal arc L is closed if $M_1 = M_n$; in this case, we say that the boundary ∂L is empty. If $M_1 \neq M_n$, we set $\partial L = \{M_1, M_n\}$; the point M_1 is the origin of the simple polygonal arc L and M_n is its endpoint.

By ordering the vertices of L we define an orientation on L. It will be materialized in the figures by arrows on the edges such that on the edge $[M_i, M_{i+1}]$ the arrow points to M_{i+1}.

Definition X.1.2. *A link L in X is the union of a finite number m of pairwise disjoint simple closed polygonal arcs in X. The closed arcs are called the connected components of L. The integer m is called the order of the link. A knot is a link of order* 1.

A link is oriented by giving an orientation to each of its connected components. In the sequel we consider only oriented links. Following Reidemeister [Rei32], we define a combinatorial operation Δ on links. We assume X to be $\mathbf{R}^3$ until the end of this section.

Definition X.1.3. (a) *Let L be a link in X and M_i, M_{i+1} two consecutive vertices in a connected component of L. Given a point N in X such that*

$N \notin L$, $M_i \notin [N, M_{i+1}]$, $M_{i+1} \notin [M_i, N]$, and

$$[M_i, N, M_{i+1}] \cap L = [M_i, M_{i+1}],$$

we denote by L' the link

$$L' = (L \setminus [M_i, M_{i+1}]) \cup [M_i, N] \cup [N, M_{i+1}].$$

We say that L' is obtained from L by a Δ-operation.

(b) *Two links L and L' are combinatorially equivalent — we write this $L \sim_c L'$ — if there exist links $L = L_0, L_1, \dots, L_k = L'$ such that for all i, one of the two links L_i, L_{i+1} is obtained from the other one by a Δ-operation.*

The relation $\sim_c$ is an equivalence relation: it is the equivalence relation generated by the Δ-operations.

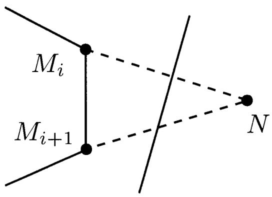


Figure 1.1. A Δ-operation

It is also possible to deform links continuously. This leads us to the concept of isotopy.

Definition X.1.4. (a) *An isotopy of X is a piecewise-linear map h from $[0,1] \times X$ to X such that, for any $t \in I$, the mapping $h(t,-)$ is a homeomorphism of X, and $h(0,-)$ is the identity of X.*

(b) *Two links L and L' are isotopic — we write $L \sim_i L'$ — if there exists an orientation-preserving isotopy h of X such that $h(1, L) = L'$.*

Lemma X.1.5. *Isotopy is an equivalence relation for links.*

Proof. Let L, L', and L'' be links. (a) Set $h(t,-) = \mathrm{id}_X$ for all $t \in I$. It is clear that h is an isotopy between L and itself: $L \sim_i L$.

(b) Let us suppose that $L \sim_i L'$ via an isotopy h. Let $h'(t,-) = h(t,-)^{-1}$ be the inverse homeomorphism. It is an isotopy between L' and L. Hence $L' \sim_i L$.

(c) If, moreover, $L' \sim_i L''$ via an isotopy h', then

$$h''(t,-) = \begin{cases} h(2t,-) & \text{if} \quad 0 \le t \le 1/2 \\ h'(2t-1,-) \circ h(1,-) & \text{if} \quad 1/2 \le t \le 1 \end{cases}$$

defines an isotopy between L and L''. In other words, the relation $\sim_i$ is transitive. □

We now have two equivalence relations on links. The following proposition identifies them with each other.

Proposition X.1.6. *Let L and L' be links in $\mathbf{R}^3$. Then*

$$L \sim_c L' \iff L \sim_i L'.$$

The reader will find a proof of this result in [BZ85], Prop. 1.10. As a consequence, we shall suppress the subscripts i and c from the symbols $\sim_i$ and $\sim_c$ and henceforth speak of isotopic or equivalent links.

We end this section with a definition of a trivial link.

Definition X.1.7. *A link of order m in $\mathbf{R}^3$ is trivial if it is isotopic to the union of m disjoint triangles in a plane. A trivial knot is a trivial link of order* 1.

We denote a trivial link of order m by

$$O^{\otimes m} = OO \cdots O \qquad (m \text{ times}).$$

Trivial links of the same order, but with different orientations, are always isotopic. Therefore we need not specify the orientation of a trivial link.

X.2 Classification of Links up to Isotopy

The fundamental problem in knot theory is to classify all links in $\mathbf{R}^3$ up to isotopy. In particular, one would like to have convenient criteria for two links to be isotopic or for a link to be trivial. This is a difficult problem.

A classical way of approching this problem is to assign to each link L an algebraic object I_L such that $I_L = I_{L'}$ whenever L and L' are equivalent. Such a function I is called an *isotopy invariant* for links. Let us give some examples.

(a) (*The order*) It is clear that the number of connected components of a link is preserved by an isotopy or a Δ-operation. Therefore the order of a link, i.e., the number of its connected components, is an isotopy invariant. However, this invariant is weak since it is clearly insensitive to how much a link is "knotted". Indeed all knots have the same order and nevertheless there exists non-trivial knots such as the right-handed trefoil knot drawn in Figure 2.1.

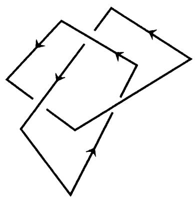


Figure 2.1. A right-handed trefoil knot

(b) (*The linking number*) This is a more refined invariant which dates back to Gauss. Let us consider two connected components L_1 and L_2 of a link L. Consider a diagram of L (to be defined in Section 3). It shows crossings of L_1 and of L_2. We associate to each crossing P an integer $\varepsilon(P) = \pm 1$ defined as in Figure 2.2.

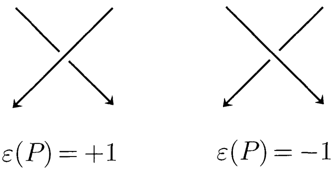


Figure 2.2. The linking number

Then the linking number of the components L_1 and L_2 is the integer

$$\mathrm{lk}(L_1, L_2) = \frac{1}{2} \sum_P \varepsilon(P)$$

where P runs over all crossings of L_1 and L_2. This number does not depend on the projection and is an isotopy invariant for links of order 2. For instance, we have $\mathrm{lk}(OO) = 0$ for the trivial link with two components, and $\mathrm{lk}(H) = 1$ for the Hopf link H drawn in Figure 2.3. It follows that the Hopf link is not trivial.

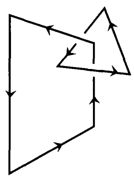


Figure 2.3. The Hopf link

(c) (*The fundamental group of a link*) Define $\pi(L) = \pi_1(\mathbf{R}^3 \setminus L)$ as the fundamental group of the complement of the link in $\mathbf{R}^3$ (the definition of the fundamental group is given in the Appendix to this chapter). For the trivial knot, the group $\pi(O)$ is isomorphic to $\mathbf{Z}$. More generally, the group of the trivial link of order m is isomorphic to the free group F_m on m generators. By the very definition of isotopy, the fundamental group of a link is an isotopy invariant. It is a very powerful invariant as one can see from a theorem of Dehn's which asserts that a link L of order m is trivial if and only if $\pi(L) = F_m$. In general, the group of a link is non-abelian. Though it is possible to give a presentation of $\pi(L)$ by generators and relations from a plane projection of L, it is very difficult to use this presentation, for instance, to decide whether L is isotopic to another given link. For more details, see [Bir74][BZ85].

(d) (*Alexander and Conway polynomials*) In 1928 Alexander [Ale28] constructed for each link L a polynomial $\Delta_L \in \mathbf{Z}[t, t^{-1}]$ defined only up to $\pm$ a power of t, which he proved to be an isotopy invariant. This invariant was a very efficient tool for distinguishing links that were not equivalent. In 1970 Conway [Con70] showed that a suitable normalization of the Alexander polynomial was of the form $\Delta_L(t) = \nabla_L(t - t^{-1})$ where $\nabla_L(z)$ is a polynomial, now called the *Conway polynomial*, in $\mathbf{Z}[z]$. Moreover, the Conway polynomial has a simple characterization in terms of the skein relations that will be described in Section 4.

X.3 Link Diagrams

The simplest way to describe a link in $\mathbf{R}^3$ is to represent it by a planar diagram. We have already used this technique for the figures of Sections 1–2. We now give a definition of what we mean by a link diagram. We first need the notion of a regular projection.

Definition X.3.1. (a) *A link projection* Π *is the union of a finite number of closed polygonal arcs in* $\mathbf{R}^2$ *such that no vertex lies in the interior of any edge. A crossing point of* Π *is a point of the link projection lying in the interior of at least two edges. The order of a crossing point* P *is the number of distinct edges in the interior of which* P *lies.*

(b) *A link projection is regular if each crossing point is of order exactly* 2.

It is not difficult to see that a crossing point cannot be a vertex, and that a link projection has only finitely many crossing points. The ordering of each component will be represented by arrows on the edges of the projection of the link following the rule given in Section 1.

Let Π be a *regular* link projection in the plane. Given a crossing point P we may consider the set E_P consisting of the two edges on which P lies. A priori, the set E_P is unordered. This brings us to the following definition.

Definition X.3.2. *A link diagram is a regular link projection in* $\mathbf{R}^2$ *for which all sets* E_P *(indexed by the crossing points* P*) are ordered. Given a crossing point* P*, the first edge of the set* E_P *with respect to the ordering is called the overcrossing edge whereas the other edge is called the undercrossing edge.*

Observe that an overcrossing edge for a crossing point may be undercrossing for another crossing point. Changing the ordering in some sets E_P will be called a change of crossings. If a regular link projection has m crossing points, then clearly there are 2^m link diagrams with the same underlying link projection.

We represent a link diagram by a drawing of the regular link projection in which the undercrossing edges are interrupted in the neighbourhood of the crossings (as in Figures 2.1 and 2.3). From such a picture we observe that any link diagram defines a link in $\mathbf{R}^3$ by letting any undercrossing edge pass under the corresponding overcrossing edge in the neighbourhood of the crossing point. This link is defined only up to isotopy. There is no reason — and in general it is false — why two link diagrams differing by a change of crossings should define equivalent links. Nevertheless, the following should be noted.

Lemma X.3.3. *Any link diagram may be turned after appropriate changes of crossings into a link diagram representing a trivial link in* $\mathbf{R}^3$.

PROOF. Consider a link diagram. Pick a vertex and start moving along the link, leaving a trail of red paint on the edges. At each crossing point, make the red edge into an overcrossing edge unless the other edge is already red, in which case the first edge is made into an undercrossing one. Apply this procedure to each connected component. The resulting link diagram (obtained from the original ones by a series of changes of crossings) represents a trivial link. □

The obvious question is now: Can any link in $\mathbf{R}^3$ be represented by a link diagram, at least up to isotopy? The answer is yes and provided by the following proposition where we fix a linear projection π_0 of the space $\mathbf{R}^3$ on the plane $\mathbf{R}^2$.

Proposition X.3.4. *Any link in* $\mathbf{R}^3$ *is equivalent to a link* L *whose image* $\pi_0(L)$ *is a regular link projection.*

PROOF. We sketch the proof. For details, see [BZ85]. Let L be a link in $\mathbf{R}^3$. Consider the set S of all possible linear projections of $\mathbf{R}^3$ onto a fixed plane. Given a projection π of S, there exists a homeomorphism h of $\mathbf{R}^3$ such that $\pi_0(h(L)) = \pi(L)$. It is therefore enough to show that the subset S_{reg} of those projections π of S such that $\pi(L)$ is a regular link projection is not empty. Now S is in bijection with $\mathbf{R}^2$. Therefore we can transport the topology of $\mathbf{R}^2$ onto S. What we shall actually prove, is that S_{reg} is dense in S for this topology.

Let π be an element of $S \setminus S_{reg}$. Then in the projection $\pi(L)$ we may have the following singularities: some crossing point may be of order ≥ 3 or some vertex may sit in the interior of some edge. This happens when the direction of the projection π passes through three edges or when it passes through a vertex and an edge. In the first case, the direction sweeps over a portion of a quadric; this projects to a part of a conic. In the second case, it determines by projection a segment of the plane. Identifying S with $\mathbf{R}^2$, we see that $S \setminus S_{reg}$ is contained in a finite number of straight lines and conics of the plane. Therefore S_{reg} is dense in S. □

Having expressed the problem of classification of links in $\mathbf{R}^3$ in purely two-dimensional terms, we now ask: When do two link diagrams represent isotopic links? Before we answer this important question, let us again follow Reidemeister by introducing the four transformations on link diagrams shown in Figures 3.1–3.4. These transformations are also called *Reidemeister moves.*

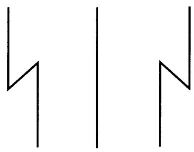


Figure 3.1. Reidemeister move (0)

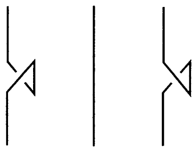


Figure 3.2. Reidemeister move (I)

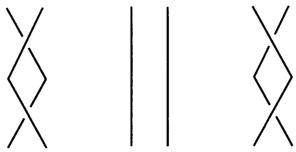


Figure 3.3. Reidemeister move (II)

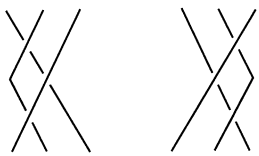


Figure 3.4. Reidemeister move (III)

Applying Transformation (0) to a link diagram means that one modifies it locally by substituting one of the drawings of the figure of Transformation (0) by another one without touching the rest of the link diagram. Similarly for the other Transformations. Figures 3.5–3.8 show that Transformations (0), (I), (II), and (III) are obtained by projection of Δ-operations. Consequently, applying these transformations to link diagrams does not change the isotopy class of the link in $\mathbf{R}^3$.

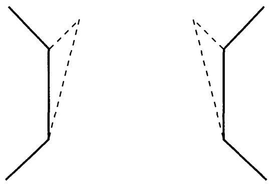


Figure 3.5. (0 projected)

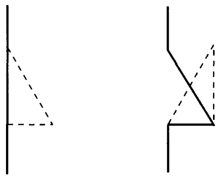


Figure 3.6. (I projected)

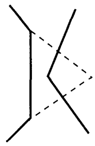


Figure 3.7. (II projected)

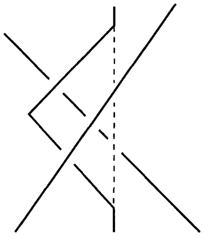


Figure 3.8. (III projected)

Reidemeister Transformations are sufficient in a sense we shall make precise below after we defined the following additional concepts. Two link diagrams Π, Π' are *isotopic* if there exists an isotopy h of $\mathbf{R}^2$ (see Definition 1.4) such that $h(1, \Pi) = \Pi'$. By this we mean that the underlying projections are isotopic in the plane and that the orders of the sets E_P are preserved in the course of the isotopy. Two isotopic link diagrams represent isotopic links in $\mathbf{R}^3$.

Introducing the *height* as the second projection of $\mathbf{R}^2$ onto $\mathbf{R}$, we define a *generic link diagram* to be a link diagram for which any two distinct vertices have different heights. In particular, a generic link diagram cannot have a horizontal edge, i.e., an edge parallel to $\mathbf{R} \times \{0\}$. A *generic isotopy* between two generic link diagrams Π, Π' is an isotopy of $\mathbf{R}^2$ such that $h(1, \Pi) = \Pi'$ and such that $h(t, \Pi)$ is a generic diagram for all $t \in [0, 1]$. The following statement is a criterion for generic diagrams to be isotopic as diagrams.

Lemma X.3.5. *Two generic diagrams are isotopic if and only if they are obtained from one another by a finite number of operations belonging to the following set:*

(A) *a generic isotopy,*

(B) *an isotopy interchanging the order of the vertices with respect to the height,*

(C) *a Reidemeister Transformation* (0)*, and*

(D) *an isotopy in the neighbourhood of a local maximum as depicted in Figure* 3.9 *and its images under reflection in the plane of the page, in a horizontal line and in a vertical line.*

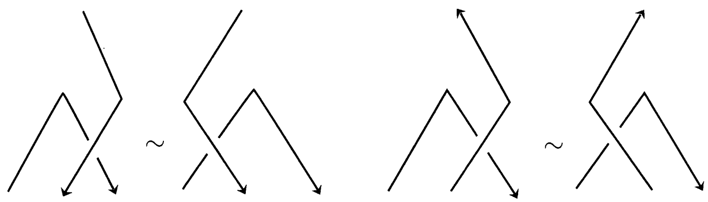


Figure 3.9. An isotopy in the neighbourhood of a local maximum

We replace Transformation (D) of the previous lemma by another set of operations that will turn out to be more adequate in Chapter XII.

Lemma X.3.6. *Two generic diagrams are isotopic if and only if they are obtained from one another by a finite number of the following operations:*

(A) *a generic isotopy,*

(B) *an isotopy interchanging the order of the vertices with respect to the height,*

(C) *a Reidemeister Transformation* (0)*, and*

(E) *an isotopy in the neighbourhood of a crossing point as shown in Figure* 3.10 *and its images under reflection in the plane of the page.*

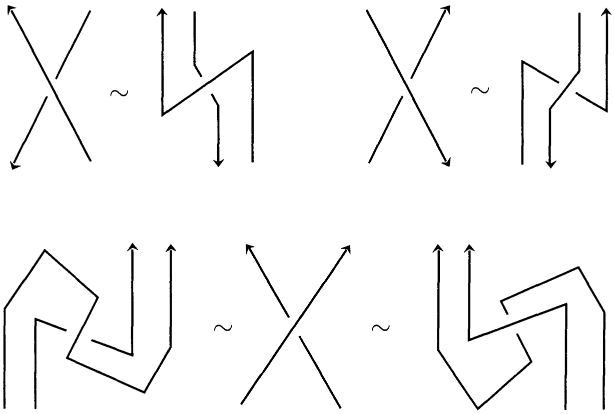


Figure 3.10. An isotopy in the neighbourhood of a crossing point

PROOF. It is clear that the operations in (E) are obtained by isotopies of diagrams. By Lemma 3.5 they follow from (A), (B), (C), and (D).

It remains to prove that Transformation (D) is a consequence of Transformations (A), (B), (C), and (E). Figures 3.11–3.12 below give a proof of this fact for the two operations represented in Figure 3.9.

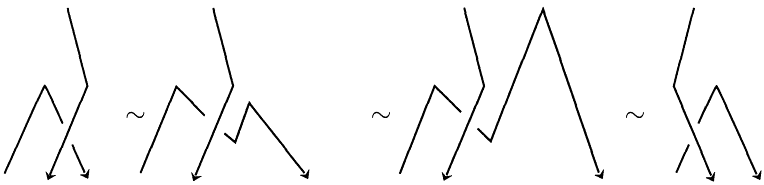


Figure 3.11.

In Figure 3.11 the first operation is of type (C), the second one of type (A) and (B) while the third one is of type (E).

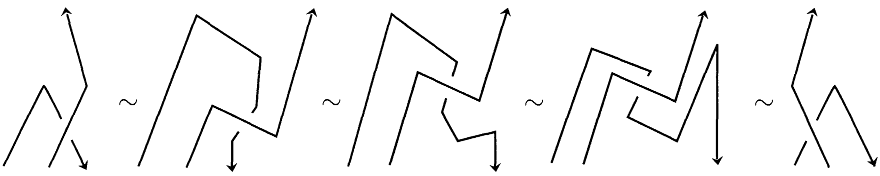


Figure 3.12.

In Figure 3.12 the first and fourth operations are of type (E), the second one of type (C), and the third one of type (A) and (B). Reflecting the previous transformations with respect to the plane of the paper or with respect to a vertical line takes care of their images under these reflections. As for the reflections with respect to a horizontal line (they involve local minima), our assertion follows from the set of operations of Figure 3.13 where the first and last ones are Transformations (C), the middle one is authorized by Figure 3.11, and the remaining ones are of type (A) and (B). □

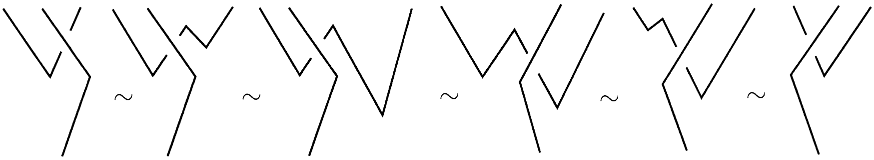


Figure 3.13.

We return to the problem of representing links in $\mathbf{R}^3$ by (generic) link diagrams. Since moving vertices up and down locally allows us to turn any link diagram into a generic one, we see that Proposition 3.4 implies that any link in $\mathbf{R}^3$ is equivalent to a link L whose projection $\pi_0(L)$ is a generic diagram.

Reidemeister [Rei32] proved the following important theorem which expresses isotopy classes of links in $\mathbf{R}^3$ in terms of purely two-dimensional link diagrams.

Theorem X.3.7. *Two generic link diagrams represent equivalent links in* $\mathbf{R}^3$ *if and only if one is obtained from the other by a finite sequence of Reidemeister Transformations* (I), (II), (III), *and of isotopies of diagrams.*

X.4 The Jones-Conway Polynomial

We now construct the Jones-Conway polynomial. This is an isotopy invariant of oriented links satisfying what knot theorists call "skein relations". In order to formulate these relations, we introduce the concept of a Conway triple. This concept already appears in [Ale28] p. 301, but Conway [Con70] was the first one to observe that it could characterize knot invariants such as the Alexander or the Conway polynomials.

Definition X.4.1. *A triple* (L_+, L_-, L_0) *of oriented links in* $\mathbf{R}^3$ *is a Conway triple if they can be represented by link diagrams* D_+, D_-, D_0 *which coincide outside a disk in* $\mathbf{R}^2$ *and which are respectively isotopic to* X_+, X_- *and* $\downarrow\downarrow$ *inside the disk.*

The diagrams X_+ and X_- are represented in Figure 4.1.

Figure 4.1.

We now state the main theorem of this chapter.

Theorem X.4.2. *There exists a unique map $L \mapsto P_L$ from the set of all oriented links in $\mathbf{R}^3$ to the ring $\mathbf{Z}[x, x^{-1}, y, y^{-1}]$ of two-variable Laurent polynomials such that*

(i) *if $L \sim L'$, then $P_L = P_{L'}$,*

(ii) *the value of P on the trivial knot is 1, and*

(iii) *whenever (L_+, L_-, L_0) is a Conway triple, we have*

$$xP_{L_+} - x^{-1}P_{L_-} = yP_{L_0}. \tag{4.1}$$

The invariant P_L is called the *Jones-Conway polynomial*, or the *two-variable Jones polynomial*, or the *Homfly polynomial* (after the initials of the six authors of [FYH+85]) of the link L. Relations (4.1) are called *skein relations*. The polynomial $\nabla_L \in \mathbf{Z}[z]$ that Conway [Con70] devised as a variant of the Alexander polynomial is characterized by Properties (i)–(ii) of Theorem 4.2 along with the skein relation

$$\nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0}. \tag{4.2}$$

Similarly, the polynomial $V_L \in \mathbf{Z}[t^{1/2}, t^{-1/2}]$ which was discovered in 1984 by Vaughan Jones [Jon85][Jon87] is characterized by Properties (i)–(ii) and by the skein relation

$$t^{-1}V_{L_+} - tV_{L_-} = \Big(\sqrt{t} - \frac{1}{\sqrt{t}}\Big)V_{L_0}. \tag{4.3}$$

As a consequence of Theorem 4.2, the Conway polynomial and the Jones polynomial exist, and they are related to the two-variable Jones-Conway polynomial by

$$\nabla_L(z) = P_L(1, z) \quad \text{and} \quad V_L(t) = P_L(t^{-1}, t^{1/2} - t^{-1/2}).$$

The Jones-Conway polynomial can help distinguish a link L from its mirror image $\widetilde{L}$, i.e., its image under a reflection with respect to a plane in $\mathbf{R}^3$. Theorem 4.2 has the following corollary.

Corollary X.4.3. *We have $P_{\widetilde{L}}(x, y) = P_L(x^{-1}, -y)$.*

PROOF. The mirror image of the Conway triple (L_+, L_-, L_0) is the triple $(\widetilde{L}_-, \widetilde{L}_+, \widetilde{L}_0)$. Consequently, we have

$$x^{-1}P_{\widetilde{L}_-} - xP_{\widetilde{L}_+} = -yP_{\widetilde{L}_0}.$$

One concludes by appealing to the characteristic properties of P. □

Let us prove Theorem 4.2. Consider the ring $\Lambda = \mathbf{Z}[x, x^{-1}, y, y^{-1}]$, the set $\mathcal{K}$ of equivalence classes of all oriented links in $\mathbf{R}^3$, and $\Lambda[\mathcal{K}]$ the free Λ-module generated by $\mathcal{K}$. We denote by Υ the quotient of $\Lambda[\mathcal{K}]$ by the Λ-submodule generated by

$$x[L_+] - x^{-1}[L_-] - y[L_0] \tag{4.4}$$

where (L_+, L_-, L_0) runs over all Conway triples of $\mathcal{K}$. The Λ-module Υ is called the *skein module* of $\mathbf{R}^3$.

Proposition X.4.4. *Let $Q : \Lambda \to \Upsilon$ be the Λ-linear map sending 1 on the class $[O]$ of the trivial knot. Then Q is an isomorphism.*

Consequently, the skein module Υ is a free Λ-module of rank one generated by $[O]$. Proposition 4.4 implies Theorem 4.2. Indeed, let L be an oriented link and $[L]$ its class in Υ. Set

$$P_L = Q^{-1}([L]) \in \mathbf{Z}[x, x^{-1}, y, y^{-1}].$$

It is clear that P satisfies all three conditions of Theorem 4.2. It remains to establish Proposition 4.4. The proof of the latter divides into two parts consisting in proving successively that the map Q is surjective, then injective.

Surjectivity of Q. This is purely topological and is essentially independent of the nature of the ring Λ. It is enough to check that the Λ-module Υ is generated by the class $[O]$ of the trivial knot. This will be shown in two steps.

Lemma X.4.5. *The Λ-module Υ is generated by the family $\{[O^{\otimes n}]\}_{n>0}$ of isotopy classes of all trivial links.*

PROOF. Let Υ_m be the Λ-submodule generated by the isotopy classes of links representable by link diagrams with $\leq m$ crossing points. Clearly, Υ_m maps to Υ_{m+1} and Υ is the union of all Υ_m. It is therefore enough to prove Lemma 4.5 for each Υ_m. This is done by induction on m. The case $m = 0$ holds by definition of trivial links. Suppose the assertion is proved for all integers $< m$. Let $[L]$ be the class of a link L in Υ_m. It may be represented by a link diagram with m crossing points. Consider one of them. Then there exists a Conway triple (L_+, L_-, L_0) such that $L = L_+$ or $L = L_-$ and the diagram L_0 has less than m crossing points. It follows from (4.4) that $[L_+] \equiv x^{-2}[L_-]$ modulo Υ_{m-1}. In other words, a change of crossings changes the class of L modulo Υ_{m-1} by multiplication by an invertible element of Λ. By Lemma 3.3 this implies that the class of L belongs to the submodule generated by the trivial links and Υ_{m-1}. The latter is also generated by trivial links in view of the induction hypothesis. □

The second step in the proof of the surjectivity of Q is the following lemma which, incidentally, shows the necessity for y to be invertible (unless $x = \pm 1$ as in the case of the Conway polynomial).

Lemma X.4.6. *For any integer $n > 1$, we have*

$$[O^{\otimes n}] = \left(\frac{x - x^{-1}}{y}\right)^{n-1} [O].$$

PROOF. Figure 4.2 implies that $(O^{\otimes n}, O^{\otimes n}, O^{\otimes n+1})$ is a Conway triple for all $n \geq 1$. By definition of Υ we get

$$[O^{\otimes n+1}] = \frac{x - x^{-1}}{y} [O^{\otimes n}].$$

One concludes by induction on n. □

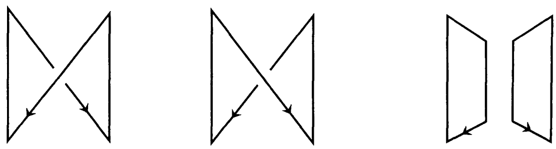


Figure 4.2. A Conway triple

Injectivity of Q. This part of the proof is algebraic in contrast with the surjectivity part. We use the following proposition whose proof will be given in Chapter XII.

Proposition X.4.7. *Let $q \neq 0$ be a complex number that is not a root of unity and let m be an integer > 1. There exists a unique map $\Phi_{m,q}$ from the set of all oriented links in $\mathbf{R}^3$ to the field $\mathbf{C}$ of complex numbers such that*

(i) *if $L \sim L'$, then $\Phi_{m,q}(L) = \Phi_{m,q}(L')$,*

(ii) *the value of $\Phi_{m,q}$ on the trivial knot is*

$$\Phi_{m,q}(O) = \frac{q^m - q^{-m}}{q - q^{-1}} \neq 0,$$

(iii) *and, whenever (L_+, L_-, L_0) is a Conway triple, we have*

$$q^m \Phi_{m,q}(L_+) - q^{-m}\Phi_{m,q}(L_-) = (q - q^{-1})\Phi_{m,q}(L_0).$$

Assuming Proposition 4.7 and using the ring map $\zeta_{q,m} : \Lambda \to \mathbf{C}$ defined by $\zeta_{q,m}(x) = q^m$ and $\zeta_{q,m}(y) = q - q^{-1}$, which furnishes $\mathbf{C}$ with a Λ-module structure, we see that $\Phi_{m,q}$ extends by linearity to a unique Λ-linear map $\Phi'_{m,q}$ from $\Lambda[\mathcal{K}]$ to $\mathbf{C}$. Now for any Conway triple (L_+, L_-, L_0), we have

$$\begin{aligned}
&\Phi'_{m,q}\Big(x[L_+] - x^{-1}[L_-] - y[L_0]\Big) \\
&= \zeta_{q,m}(x)\Phi_{m,q}(L_+) - \zeta_{q,m}(x^{-1})\Phi_{m,q}(L_-) - \zeta_{q,m}(y)\Phi_{m,q}(L_0) \\
&= q^m\Phi_{m,q}(L_+) - q^{-m}\Phi_{m,q}(L_-) - (q - q^{-1})\Phi_{m,q}(L_0) \\
&= 0
\end{aligned}$$

by Proposition 4.7. Consequently, $\Phi'_{m,q}$ factors through a unique Λ-linear map, denoted $\Phi''_{m,q}$, from Υ into $\mathbf{C}$ such that $\Phi''_{m,q}([L]) = \Phi_{m,q}(L)$ for any oriented link L.

We are now ready to prove the injectivity of the map Q from Λ into Υ, which will complete the proof of Proposition 4.4, hence of Theorem 4.2.

Let $f(x, y) \in \Lambda$ be a two-variable Laurent polynomial chosen such that $Q(f(x, y)) = f(x, y)[O]$ vanishes in Υ. Then, using the above-defined map $\Phi''_{m,q} : \Upsilon \to \mathbf{C}$, we have

$$0 = \Phi''_{m,q}\Big(f(x, y)[O]\Big) = f(q^m, q - q^{-1})\Phi_{m,q}(O)$$

for any integer $m > 1$ and any complex number q that is not a root of unity. Since $\Phi_{m,q}(O) \neq 0$, we have $f(q^m, q - q^{-1}) = 0$. Since this is true for an infinite number of distinct powers of q, the polynomial f is divisible by the polynomial $y - (q - q^{-1})$. The latter assertion holds for infinitely many complex numbers q, which is possible only if the polynomial f is zero. This proves the injectivity of Q. □

Application 4.8. We end this section with the computation of the Jones-Conway polynomial of the right-handed trefoil knot K and of the Hopf link H. Figures 4.3–4.4 show that (H, OO, O) and (K, O, H) are Conway triples.

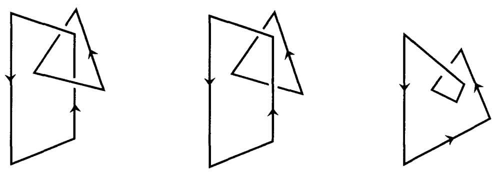


Figure 4.3.

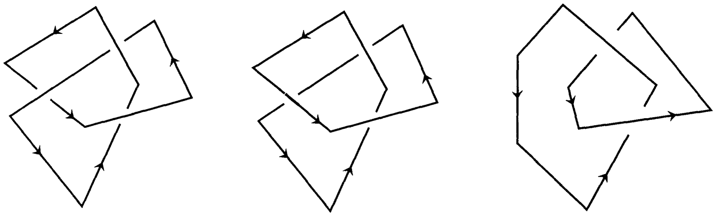


Figure 4.4.

By (4.1) we have

$$xP_H = x^{-1}P_{OO} + yP_O = x^{-1}\frac{x - x^{-1}}{y} + y.$$

Hence,

$$P_H = (x^{-1} - x^{-3})y^{-1} + x^{-1}y.$$

A similar computation yields

$$P_K = 2x^{-2} - x^{-4} + x^{-2}y^2$$

for the right-handed trefoil knot. By Corollary 4.3, we see that the Jones-Conway polynomial of the mirror image $\widetilde{K}$ is given by

$$P_{\widetilde{K}} = 2x^2 - x^4 + x^2y^2 \neq P_K.$$

This proves that the trefoil knot is not isotopic to its mirror image, a fact already observed by Dehn [Deh14] in 1914.

X.5 Tangles

This section is devoted to the concept of tangles which generalizes the notion of links. Tangles will be used extensively in Chapter XII, in particular for the proof of Proposition 4.7.

For any integer $n > 0$, we set $[n] = \{1, 2, \ldots, n\}$. When $n = 0$, we agree that $[0]$ is the empty set. We denote by I the closed interval $[0, 1]$ and by $\mathbf{R}^2$ the real plane.

Definition X.5.1. *Let k and ℓ be nonnegative integers. A tangle L of type (k, ℓ) is the union of a finite number of pairwise disjoint simple oriented polygonal arcs in $X = \mathbf{R}^2 \times I$ such that the boundary ∂L of L satisfies the condition*

$$\partial L = L \cap (\mathbf{R}^2 \times \{0, 1\}) = ([k] \times \{0\} \times \{0\}) \cup ([\ell] \times \{0\} \times \{1\}).$$

The boundary condition in Definition 5.1 means that the tangle intersects the two boundary planes of $\mathbf{R}^2 \times I$ transversally. Observe that a link in $\mathbf{R}^2 \times I$ is a tangle of type $(0,0)$. Figure 5.1 shows an example of a tangle that is not a link.

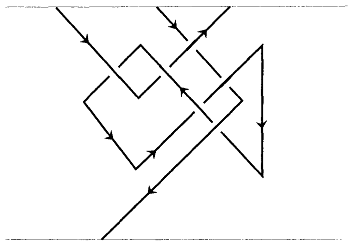


Figure 5.1. A tangle

Given a tangle L of type (k,ℓ), we define two finite sequences $s(L)$ and $b(L)$ consisting of $+$ and $-$ signs. If $k=0$, then $s(L)=\emptyset$ is the empty set by convention. Similarly if $\ell=0$, we set $b(L)=\emptyset$. In the general case, we define

$$s(L)=(\varepsilon_1,\dots,\varepsilon_k) \quad \text{and} \quad b(L)=(\eta_1,\dots,\eta_\ell)$$

where $\varepsilon_i=+$ [resp. $\eta_i=+$] if the point $(i,0,0)$ [resp. the point $(i,0,1)$] is an endpoint [resp. an origin] of L. We have $\varepsilon_i=-$ and $\eta_i=-$ in the remaining cases.

Let us give a few examples of tangles that shall be used in the sequel.

1. We denote the polygonal arcs $[(1,0,1),(1,0,0)]$ and $[(1,0,0),(1,0,1)]$ by $\downarrow$ and $\uparrow$ respectively. We have $s(\downarrow)=(+)$, $b(\downarrow)=(+)$, $s(\uparrow)=(-)$, and $b(\uparrow)=(-)$.

2. The tangles X_+ and X_- of Figure 4.1 can be defined by

$$X_\pm = [M_1, M_2^\pm] \cup [M_2^\pm, M_3] \cup [N_1, N_2^\mp] \cup [N_2^\mp, N_3] \tag{5.1}$$

where $M_1, M_2^\pm, M_3, N_1, N_2^\pm, N_3$ are the points whose coordinates in $\mathbf{R}^2\times I$ are

$$\begin{array}{rclcrcl} M_1 &=& (2,0,1), & \qquad & N_1 &=& (1,0,1),\\ M_2^\pm &=& (3/2,\mp 1,1/2), && N_2^\pm &=& (3/2,\pm 1,1/2),\\ M_3 &=& (1,0,0), && N_3 &=& (2,0,0). \end{array}$$

We have $s(X_\pm)=b(X_\pm)=(+,+)$.

3. The tangles $\cap$ and $\overleftarrow{\cap}$ of type $(2,0)$ are defined by

$$\cap = [(1,0,0),(3/2,0,1/2)] \cup [(3/2,0,1/2),(2,0,0)] \tag{5.2}$$

and

$$\overleftarrow{\cap} = [(2,0,0),(3/2,0,1/2)] \cup [(3/2,0,1/2),(1,0,0)]. \tag{5.3}$$

We have $s(\cap) = (-,+)$, $b(\cap) = \emptyset$, $s(\overleftarrow{\cap}) = (+,-)$, and $b(\overleftarrow{\cap}) = \emptyset$ (see Figure 5.2).

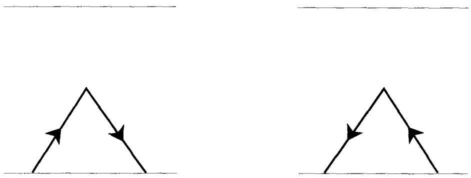


Figure 5.2. The tangles $\cap$ and $\overleftarrow{\cap}$

4. Similarly, we define tangles $\cup$ and $\overleftarrow{\cup}$ of type $(0,2)$ by

$$\cup = [(1,0,1),(3/2,0,1/2)] \cup [(3/2,0,1/2),(2,0,1)] \tag{5.4}$$

and

$$\overleftarrow{\cup} = [(2,0,1),(3/2,0,1/2)] \cup [(3/2,0,1/2),(1,0,1)]. \tag{5.5}$$

We have $s(\cup) = \emptyset$, $b(\cup) = (+,-)$, $s(\overleftarrow{\cup}) = \emptyset$, and $b(\overleftarrow{\cup}) = (-,+)$ (see Figure 5.3).

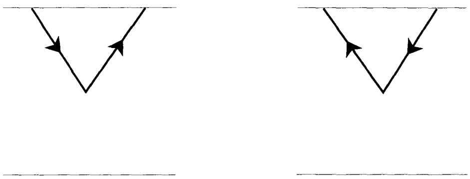


Figure 5.3. The tangles $\cup$ and $\overleftarrow{\cup}$

We have the same equivalence relations for tangles as for links. Let us adapt their definition to the case of tangles. We start with the combinatorial relation $\sim_c$.

Definition X.5.2. (a) *Let L be a tangle in X and M_i, M_{i+1} be two consecutive vertices of L. We are also given a point N in $\mathbf{R}^2\times]0,1[$ such that $N \notin L$, $M_i \notin [N, M_{i+1}]$, $M_{i+1} \notin [M_i, N]$, and*

$$[M_i, N, M_{i+1}] \cap L = [M_i, M_{i+1}].$$

We define L' as the tangle

$$L' = (L \setminus [M_i, M_{i+1}]) \cup [M_i, N] \cup [N, M_{i+1}].$$

We say that L' is obtained from L by a Δ-operation.

(b) *Two tangles L and L' are combinatorially equivalent – we write this $L \sim_c L'$ – if there exist tangles $L = L_0, L_1, \ldots, L_k = L'$ such that for all i, one of the tangles L_i, L_{i+1} is obtained from the other one by a Δ-operation.*

Clearly, if L and L' are combinatorially equivalent, they have the same boundaries and are of the same type. Isotopies are defined as follows.

Definition X.5.3. (a) *An isotopy of* $X = \mathbf{R}^2 \times I$ *is a piecewise-linear map* $h : I \times X \to X$ *such that for all* $t \in I$, *the mapping* $h(t, -)$ *is a homeomorphism of* X *restricting to the identity map on the boundary* $\partial X = \mathbf{R}^2 \times \{0, 1\}$ *and such that* $h(0, -)$ *is the identity of* X.

(b) *Two tangles* L *and* L' *are isotopic – we write* $L \sim_i L'$ *– if there exists an isotopy* h *of* X *such that* $h(1, L) = L'$.

Again if L and L' are isotopic, they have the same boundaries and are of the same type. The isotopy is shown to be an equivalence relation for tangles in the same way as it was for links (see Lemma 1.5). We have the following counterpart of Proposition 1.6.

Proposition X.5.4. *Let* L *and* L' *be two tangles. Then*

$$L \sim_c L' \iff L \sim_i L'.$$

As for links, we shall suppress the subscripts i and c from the symbols $\sim_i$ and $\sim_c$ and we shall henceforth speak of isotopic or equivalent tangles.

Tangles can also be represented by planar diagrams. We adapt the following concepts and results from Section 3 without bothering to give unnecessary details.

Definition X.5.5. (a) *A tangle projection* Π *is the union of a finite number of (not necessarily closed) polygonal arcs in* $\mathbf{R}^2$ *such that no vertex sits in the interior of any edge and such that the boundary* $\partial\Pi$ *of* Π *satisfies the condition*

$$\partial\Pi = \Pi \cap (\mathbf{R} \times \{0, 1\}) = ([k] \times \{0\}) \cup ([\ell] \times \{1\}).$$

A crossing point of Π *is a point of the tangle projection sitting in the interior of at least two edges. The order of a crossing point* P *is the number of distinct edges in the interior of which* P *sits.*

(b) *A tangle projection is regular if each crossing point is of order exactly* 2.

Let Π be a regular tangle projection in the plane. Given a crossing point P we may again consider the unordered set E_P consisting of the two edges on which P sits.

Definition X.5.6. *A tangle diagram is a regular tangle projection in* $\mathbf{R} \times I$ *for which all the sets* E_P *(indexed by the crossing points* P*) are ordered. Given a crossing point* P, *the first edge of the set* E_P *with respect to the ordering is called the overcrossing edge whereas the other edge is called the undercrossing edge.*

Replacing $\mathbf{R}^2$ by $\mathbf{R} \times [0,1]$, one defines the concepts of isotopy of tangle diagrams, of generic tangle diagrams, and of generic isotopy as for links. Similar results hold. We record here the counterpart of Lemma 3.6.

Lemma X.5.7. *Two generic tangle diagrams are isotopic if and only if they are obtained from one another by a finite number among the following operations:*

(A) *a generic isotopy,*

(B) *an isotopy interchanging the order of the vertices with respect to the height,*

(C) *a Reidemeister Transformation (0), and*

(E) *an isotopy in the neighbourhood of a crossing point as shown in Figure* 3.10 *and their images under reflection in the plane of the page.*

As in the case of links, any (generic) tangle diagram defines a tangle in $\mathbf{R}^2 \times I$ which is unique up to isotopy. Fix a linear projection π_0 of the space $\mathbf{R}^2 \times I$ on the strip $\mathbf{R} \times I$.

Proposition X.5.8. *Any tangle in $\mathbf{R}^2 \times I$ is equivalent to a tangle L whose projection $\pi_0(L)$ is a generic tangle diagram.*

When do two tangle diagrams represent isotopic tangles? The answer to this question is the same as in the case of links. It uses the Reidemeister moves already defined in Section 3.

Theorem X.5.9. *Two generic tangle diagrams represent equivalent tangles in $\mathbf{R}^2 \times I$ if and only if one is obtained from the other by a finite sequence of Reidemeister Transformations* (I), (II), (III), *and of isotopies of diagrams.*

We close this section by defining a partial binary operation on tangles. Consider the piecewise-linear mappings a_1, a_2 from the topological space $\mathbf{R}^2 \times I$ into itself defined by

$$a_1(p,z) = (p, z/2) \quad \text{and} \quad a_2(p,z) = (p,(z+1)/2)$$

where $p \in \mathbf{R}^2$ and $z \in I$. When L and L' are oriented tangles such that $b(L) = s(L')$, then

$$L' \circ L = a_1(L) \cup a_2(L')$$

is an oriented tangle with

$$s(L' \circ L) = s(L) \quad \text{and} \quad b(L' \circ L) = b(L').$$

The tangle $L' \circ L$ is called the *composition* of L and L'. It is obtained by placing L' on top of L, by glueing their middle ends together and by squeezing the whole into $\mathbf{R}^2 \times [0,1]$. Let us prove that composition is compatible with the equivalence of tangles.

Lemma X.5.10. *Let L_1, L_2, L_3, L_4 be oriented tangles with $b(L_1) = s(L_2)$ and $b(L_3) = s(L_4)$.*

(a) *If $L_1 \sim L_3$ and $L_2 \sim L_4$, then $L_2 \circ L_1 \sim L_4 \circ L_3$.*

(b) *If, furthermore, $b(L_2) = s(L_3)$, then $(L_3 \circ L_2) \circ L_1 \sim L_3 \circ (L_2 \circ L_1)$.*

PROOF. (a) Use Reidemeister Transformations.

(b) The tangles $(L_3 \circ L_2) \circ L_1$ and $L_3 \circ (L_2 \circ L_1)$ are isotopic through the isotopy $h(t, p, z) = (p, \varphi_t(z))$ where $p \in \mathbf{R}^2$, $t, z \in [0, 1]$ and φ is the continuous mapping from $I \times I$ into I defined by

$$\varphi_t(z) = \begin{cases} z(1 - \frac{t}{2}) & \text{if} \quad 0 \leq z \leq 1/2, \\ z - \frac{t}{4} & \text{if} \quad 1/2 \leq z \leq 3/4, \\ (1 + t)z - t & \text{if} \quad 3/4 \leq z \leq 1. \end{cases}$$

□

The composition has partial left and right units up to isotopy. Indeed, for any finite sequence ε of $\pm$ signs of length n, define the tangle id_ε as the union of the n intervals $\{1, \ldots, n\} \times \{0\} \times [0, 1]$, their origins and endpoints being uniquely determined by the requirement $s(\mathrm{id}_\varepsilon) = b(\mathrm{id}_\varepsilon) = \varepsilon$. If the sequence ε is empty, take $\mathrm{id}_\emptyset$ to be the empty tangle. An immediate application of Δ-operations proves the following lemma.

Lemma X.5.11. *For any tangle L, we have*

$$\mathrm{id}_{b(L)} \circ L \sim L \sim L \circ \mathrm{id}_{s(L)}.$$

X.6 Braids

We now consider a special class of tangles, called braids. Fix an integer $n \geq 1$.

Definition X.6.1. *A braid L with n strands is a tangle of type (n, n) such that*

(i) *$s(L) = b(L) = (+, +, \cdots, +)$,*

(ii) *L contains no closed arc, and*

(iii) *for all $z \in I$, the intersection of L with theplane $\mathbf{R}^2 \times \{z\}$ consists of exactly n distinct points.*

In other words, a braid with n strands is the union of n pairwise disjoint simple polygonal arcs, relating the set $[n] \times \{0\} \times \{1\}$ to the set $[n] \times \{0\} \times \{0\}$ and having no local maximum or minimum with respect to the "height" projection $\mathbf{R}^2 \times I \to I$. Figure 6.1 shows a braid with 5 strands.

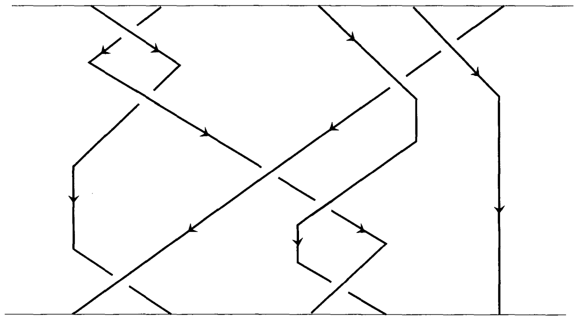


Figure 6.1. A braid

By definition, two braids are equivalent if they can be obtained from each other by a finite sequence of Δ-operations performed within the set of braids. Up to equivalence, a braid with n strands can be represented by what may be called a *braid diagram*, i.e., a tangle diagram such that for all $z \in I$ the intersection of the diagram with $\mathbf{R} \times \{z\}$ consists of exactly n distinct points. There is also a notion of isotopy of braid diagrams which is the restriction to braid diagrams of the corresponding notion for tangle diagrams. Braid diagrams are isotopic if and only if they are obtained from each other by moving vertices up and down. Concerning Reidemeister moves, Transformations (0) and (I) are clearly forbidden for braid diagrams. Only Transformations (II) and (III) may occur. They are sufficient to generate the braid equivalence, as witnessed by the following proposition whose proof is similar to (and simpler than) the proof of the corresponding Theorem 5.9 for tangles.

Proposition X.6.2. *Two braid diagrams represent equivalent braids if and only if they are obtained from each other by a finite sequence of Reidemeister Transformations* (II), (III), *and of isotopies of braid diagrams.*

X.6.1 The braid group B_n

In Section 5 we defined the composition $L' \circ L$ for tangles L, L' such that $b(L) = s(L')$. We see from the definitions that the composition of two braids with n strands is still a braid with n strands. A special braid with n strands is id_ε (as defined at the end of Section 5) where ε is the sequence consisting of n signs $+$. We denote its equivalence class by 1_n. Given a braid L we define the *inverse braid* L^{-1} as the image of L under the reflection through the plane $\mathbf{R}^2 \times \{1/2\}$.

Denote the set of equivalence classes of braids with n strands by B_n. The set B_0 has one single element, namely the empty braid. Restricted to braids, Lemma 5.10 implies that the composition of braids is compatible with braid equivalence. Therefore the composition induces a product on B_n. As a matter of fact, we have the following result.

Proposition X.6.3. *The composition of braids induces a group structure on B_n with 1_n as a unit.*

PROOF. Associativity of the product follows from Lemma 5.10 (b) whereas Lemma 5.11 implies that 1_n is a left and right unit for the product on B_n. Repeated use of Reidemeister Transformation (II) shows that

$$L^{-1} \circ L \sim 1_n \sim L \circ L^{-1},$$

which implies that the equivalence class of L^{-1} is an inverse for the equivalence class of L. □

The group B_n was introduced by E. Artin [Art25]. It is called the *braid group* (on n strands). The groups $B_0 = B_1$ are isomorphic to the trivial group $\{1\}$.

We now give a presentation by generators and relations of B_n. First, we define special elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ in B_n. A braid diagram of the braid σ_i is shown in Figure 6.2.

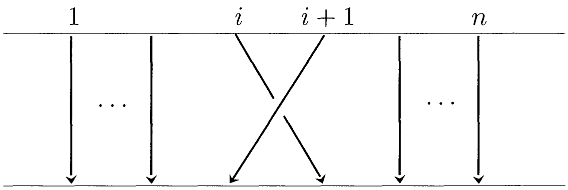


Figure 6.2. The braid σ_i

Using the tangle X_+ of Section 5, we see that σ_i is equivalent to the braid

$$\downarrow \cdots \downarrow X_+ \downarrow \cdots \downarrow$$

exchanging the i-th and the $(i+1)$-st strands and leaving the other ones untouched. Its inverse σ_i^{-1} is equivalent to the braid

$$\downarrow \cdots \downarrow X_- \downarrow \cdots \downarrow$$

with the opposite crossing (see Figure 6.3).

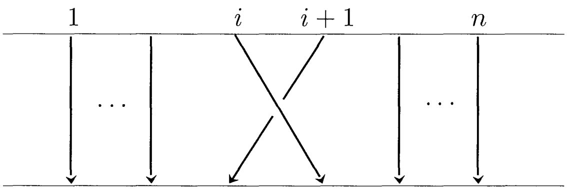


Figure 6.3. The braid σ_i^{-1}

Lemma X.6.4. (a) *The group B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$.*

(b) *When $n \geq 3$ and $1 \leq i, j \leq n-1$, we have the following relations in the braid group B_n:*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \tag{6.1}$$

if $|i-j| > 1$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{6.2}$$

Relations (6.1–6.2) are called the *braid group relations* .

PROOF. (a) Represent a braid by a braid diagram. Move crossing points up or down so that one can find $0 < t_1 < \ldots < t_r < 1$ such that for all i there is only one crossing point in $\mathbf{R} \times [t_i, t_{i+1}]$. This means that the braid is equivalent to a braid whose restriction to $\mathbf{R} \times [t_i, t_{i+1}]$ is equivalent to some σ_k or its inverse. Using the definition of the product in the braid group, we see that the given braid can be expressed in B_n as the product of the elements σ_k and their inverses.

(b) If $|i-j| > 1$, then clearly $\sigma_i \sigma_j$ and $\sigma_j \sigma_i$ are equivalent (draw a picture). Both sides of Relation (6.2) are represented in Figure 3.4. One passes from one diagram to the other by Reidemeister Transformation (III). □

We now state an important theorem due to E. Artin [Art25] [Art47].

Theorem X.6.5. *Given a group G and elements $c_1, \ldots, c_{n-1}$ ($n > 2$) such that for all i, j we have $c_i c_j = c_j c_i$ if $|i-j| > 1$ and*

$$c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1},$$

then there exists a unique group morphism from B_n to G mapping σ_i to c_i.

Corollary X.6.6. *The group B_n is isomorphic to the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the braid group relations (6.1–6.2).*

PROOF. Let G be the latter group. By Theorem 6.5, there exists a unique group morphism $\rho : B_n \to G$ such that $\rho(\sigma_i) = \sigma_i$ for all i. Now by definition of a group given by generators and relations, there exists a unique group morphism $\rho' : G \to B_n$ sending σ_i onto σ_i for all i. Then ρ' is inverse to ρ. □

Proof of Theorem 6.5. The uniqueness of the group morphism follows from the fact proven in Lemma 6.4 (a) that $\sigma_1, \ldots, \sigma_{n-1}$ generate B_n.

Now, let us establish the existence of a group morphism $\rho : B_n \to G$ such that $\rho(\sigma_i) = c_i$ for $i = 1, \ldots, n-1$. We sketch a geometric proof. Consider a braid L represented by a generic diagram as in the proof of Lemma 6.4 (a), namely for which two different crossing points have different heights. To such a diagram we can assign a unique braid word w in the generators σ_i and their inverses as in the proof of Lemma 6.4 (a). Define $\rho(w)$ to be the element of G obtained by replacing σ_i by c_i and σ_i^{-1} by c_i^{-1} in the word w. Let us show that $\rho(w)$ depends only on the initial braid.

According to Proposition 6.2, we have to check that $\rho(w)$ does not change when we perform an isotopy of diagram or Reidemeister Transformations of type (II) and (III). In the first case, moving crossing points up and down amounts to changing the braid word by products of commutators of the form $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$ for $|i-j|>1$. This leaves $\rho(w)$ unchanged because $c_ic_jc_i^{-1}c_j^{-1}=1$ for $|i-j|>1$. Under a Reidemeister Transformation (II), braid words differ by $\sigma_i\sigma_i^{-1}$ or by $\sigma_i^{-1}\sigma_i$ whose images under ρ is 1. Under a Reidemeister Transformation (III), braid words differ by $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ or its inverse. Their images under ρ again is 1 because of the relation $c_ic_{i+1}c_i=c_{i+1}c_ic_{i+1}$. □

We still have to investigate the case of braids with two strands. An adaptation of the proofs of Lemma 6.4 and Theorem 6.5 proves the following.

Proposition X.6.7. *The group B_2 is generated by σ_1 and is isomorphic to the group $\mathbf{Z}$ of integers.*

X.6.2 Braid group representations from R-matrices

We show how Theorem 6.5 allows us to construct braid group representations from any solution of the Yang-Baxter equation.

Let V be a vector space, c a linear automorphism of $V\otimes V$, and $n>1$ an integer. Then for $1\leq i\leq n-1$, define a linear automorphism c_i of $V^{\otimes n}$ by

$$c_i=\begin{cases} c\otimes \mathrm{id}_{V^{\otimes(n-2)}} & \text{if} \quad i=1,\\ \mathrm{id}_{V^{\otimes(i-1)}}\otimes c\otimes \mathrm{id}_{V^{\otimes(n-i-1)}} & \text{if} \quad 1<i<n-1,\\ \mathrm{id}_{V^{\otimes(n-2)}}\otimes c & \text{if} \quad i=n-1.\end{cases} \tag{6.3}$$

Clearly $c_ic_j=c_jc_i$ if $|i-j|>1$. It is easy to check the following lemma.

Lemma X.6.8. *Under the previous hypothesis, we have*

$$c_ic_{i+1}c_i=c_{i+1}c_ic_{i+1}$$

for all i if and only c is a solution of the Yang-Baxter equation.

The Yang-Baxter equation of VIII.1 can be expressed with the present notation as the equation $c_1c_2c_1=c_2c_1c_2$ holding in $\mathrm{Aut}(V\otimes V\otimes V)$. The following is a consequence of Theorem 6.5 and Lemma 6.8.

Corollary X.6.9. *Let $c\in \mathrm{Aut}(V\otimes V)$ be a solution of the Yang-Baxter equation. Then, for any $n>0$, there exists a unique group morphism $\rho_n^c: B_n\to \mathrm{Aut}(V^{\otimes n})$ such that $\rho_n^c(\sigma_i)=c_i$ for $i=1,\ldots,n-1$.*

Consequently, by this procedure, any linear automorphism c of $V \otimes V$ that is a solution of the Yang-Baxter gives rise to a representation of the braid group B_n on the tensor power $V^{\otimes n}$ where n is any integer ≥ 2.

X.6.3 Relation with the symmetric group S_n

Given a braid L, there exists a unique permutation $\sigma(L)$ of the set $\{1, \ldots, n\}$ such that for all $k \in \{1, \ldots, n\}$ the endpoint $(k, 0, 0)$ lies in the same connected component as the origin $(\sigma(k), 0, 1)$. The permutation $\sigma(L)$ is called the *permutation of the braid L*.

Lemma X.6.10. *The map $f \mapsto \sigma(f)$ induces a surjective morphism of groups from the braid group B_n onto the symmetric group S_n.*

PROOF. First it is clear that equivalent braids have the same permutation. Thus the map factors through B_n. It is a morphism of groups because we have $\sigma(L' \circ L) = \sigma(L') \circ \sigma(L)$, $\sigma(1_n) = \mathrm{id}$, and $\sigma(L^{-1}) = \sigma(L)^{-1}$. The permutation of the braid σ_i is the transposition $(i, i+1)$. The surjectivity of the map follows from the fact that such transpositions generate the symmetric group S_n. □

This lemma is not surprising in view of Moore's theorem which gives the following presentation in terms of generators and relations for the symmetric group S_n: it is generated by the $n-1$ transpositions $s_i = (i, i+1)$ and by Relations (6.1–6.2) where σ_i has been replaced by s_i, as well as by the additional relations $s_i^2 = 1$ for $i = 1, \ldots, n-1$.

One big difference between symmetric groups and braid groups is that the former are finite groups while the latter are infinite groups when $n > 1$. Moreover, the group B_n has no torsion, that is to say, all elements $\neq 1_n$ have infinite order.

X.6.4 Representing braids as loops

We end this section by giving a function-theoretic definition of braids. Let n be an integer ≥ 1. Consider the set

$$Y_n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid i \neq j \Rightarrow z_i \neq z_j\}$$

endowed with the subset topology of $\mathbf{C}^n$. The symmetric group S_n acts on Y_n by permutation of the coordinates. Let $X_n = Y_n/S_n$ be the quotient space with the quotient topology. The space X_n is the *configuration space* of n distinct points in $\mathbf{C}$. Consider the following set $p = \{1, 2, \ldots, n\}$ of n distinct points in X_n.

Definition X.6.11. *A loop in X_n is a piecewise linear map*

$$f = (f_1, f_2, \ldots, f_n) : I = [0, 1] \to \mathbf{C}^n$$

such that for all $t \in I$ we have $f_i(t) \neq f_j(t)$ whenever $i \neq j$ and

$$f(0) = (1, 2, \ldots, n) \quad \textit{and} \quad \{f_1(1), f_2(1), \ldots, f_n(1)\} = \{1, 2, \ldots, n\}.$$

Loops in X_n and braids with n strands in $\mathbf{R}^2 \times [0, 1]$ are equivalent notions after we have identified $\mathbf{C}$ with $\mathbf{R}^2$. Indeed, given a loop $f = (f_1, f_2, \ldots, f_n)$ in X_n, the union of the graphs of the maps f_i is a braid with n strands. Conversely, for any braid L with n strands, we define $f_i(z)$ to be the projection onto $\mathbf{R}^2 = \mathbf{C}$ of the intersection of the plane $\mathbf{R}^2 \times \{z\}$ with the connected component of L ending at the point $(i, 0, 0)$. This defines a loop $f = (f_1, f_2, \ldots, f_n)$ in X_n.

For any loop $f = (f_1, \ldots, f_n)$ in X_n, define the permutation $\sigma(f)$ of the set $\{1, \ldots, n\}$ by $\sigma(f)(k) = f_k(1)$ for all k. Check that, if f is the loop corresponding to a braid L, then $\sigma(f) = \sigma(L)$.

The equivalence of braids can be expressed on loops in X_n as follows. Two loops $f = (f_1, f_2, \ldots, f_n)$ and $g = (g_1, g_2, \ldots, g_n)$ in X_n are homotopic in X_n — we write $f \sim g$ — if there exists a piecewise linear map, called an *isotopy*,

$$H = (H_1, H_2, \ldots, H_n) : I \times I \to \mathbf{C}^n$$

such that for all $(s, t) \in I \times I$ and $i \neq j$ we have $H_i(s, t) \neq H_j(s, t)$, for all $s \in I$ and k with $1 \leq k \leq n$ we have

$$H_k(s, 0) = f_k(0) = g_k(0) \quad \text{and} \quad H_k(s, 1) = f_k(1) = g_k(1),$$

for all $t \in I$ and $1 \leq k \leq n$ we have

$$H_k(0, t) = f_k(t) \quad \text{and} \quad H_k(1, t) = g_k(t).$$

Proposition X.6.12. *Two braids with n strands are equivalent if and only if the corresponding loops are isotopic in X_n.*

We can transpose the composition of tangles on the level of loops. Let f [resp. f'] be the loop in X_n corresponding to a braid L [resp. L']. It is easy to see that the loop $ff' = \big((ff')_1, \ldots, (ff')_n\big)$ corresponding to the braid $L' \circ L$x, composed in the sense of tangles, (see Section 5) is given for all i by

$$(ff')_i(t) = \begin{cases} f_i(2t) & \text{if} \quad 0 \leq t \leq 1/2, \\ f'_{\sigma(i)}(2t - 1) & \text{if} \quad 1/2 \leq t \leq 1 \end{cases}$$

where $\sigma = \sigma(f)$ is the permutation of f. The loop ff' is called the *product* of the loops f, and f'. We have $\sigma(ff') = \sigma(f') \circ \sigma(f)$.

Given a loop $f = (f_1, \ldots, f_n)$ corresponding to a braid L, the loop f^{-1} defined by

$$f_i^{-1}(z) = f_{\sigma^{-1}(i)}(1 - z),$$

where again $\sigma = \sigma(f)$, corresponds to the inverse braid L^{-1}. The loop corresponding to the braid 1_n is the constant map $z \mapsto (1, 2, \ldots, n)$.

We have the following lemma which is the counterpart for loops of Lemmas 5.10–5.11.

Lemma X.6.13. *Given loops f, f', f'', g, g' in X_n,*
(a) *if $f \sim g$ and $f' \sim g'$, then $ff' \sim gg'$,*
(b) $(ff')f'' \sim f(f'f'')$,
(c) $1_n f \sim f \sim f 1_n$, *and*
(d) $ff^{-1} \sim 1_n \sim f^{-1}f$.

PROOF. See the Appendix to this chapter. □

We have the following important result as a consequence of the presentation of braids by loops and of the definitions of the Appendix.

Proposition X.6.14. *The braid group B_n is isomorphic to the fundamental group of the configuration space X_n of n distinct points in* $\mathbf{C}$*:*

$$B_n \cong \pi_1(X_n, p)$$

where p is the set $\{1, \dots, n\}$.

X.7 Exercises

1. (*Centre of the braid group*) Let n be an integer > 2. Show that the centre of the braid group B_n is generated by the element $(\sigma_1 \dots \sigma_{n-1})^n$.

2. For an integer $n > 1$ let F_n be the free group generated by $x_1, \dots, x_n$. Define automorphisms $X_1, \dots, X_{n-1}$ of F_n by

$$X_i(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } \; j = i, \\ x_i & \text{if } \; j = i+1, \\ x_j & \text{if } \; j \neq i, i+1. \end{cases}$$

Prove that there exists a morphism X of the braid group B_n into the group of automorphisms of F_n such that $X(\sigma_i) = X_i$ for all i.

3. (*Burau representations*) (a) Let $n > 1$ be an integer and $\{v_1, \dots, v_n\}$ be a basis of a free $\mathbf{Z}[t, t^{-1}]$-module V_n of rank n. For any i such that $1 \leq i \leq n-1$ define an automorphism β_i of V_n by $\beta_i(v_k) = v_k$ if $k \neq i, i+1$, and

$$\beta_i(v_i) = (1-t)v_i + v_{i+1} \quad \text{and} \quad \beta_i(v_{i+1}) = t v_i.$$

Show that there exists a unique morphism β of the braid group B_n into the group of automorphisms of V_n such that $\beta(\sigma_i) = \beta_i$ for all i.

(b) Let $\{e_1, \dots, e_{n-1}\}$ be a basis of a free $\mathbf{Z}[t, t^{-1}]$-module V_{n-1} of rank $n-1$. For any i such that $1 < i < n-1$ define an automorphism

$\widetilde{\beta}_i$ of V_{n-1} by $\widetilde{\beta}_i(e_k) = e_k$ if $k \neq i$ and $\widetilde{\beta}_i(e_i) = te_{i-1} - te_i + e_{i+1}$ with the convention $e_0 = e_n = 0$. Show that there exists a unique morphism $\widetilde{\beta}$ of the braid group B_n into the group of automorphisms of V_{n-1} such that $\widetilde{\beta}(\sigma_i) = \widetilde{\beta}_i$ for all i. Prove that

$$\widetilde{\beta}\Big((\sigma_1 \ldots \sigma_{n-1})^n\Big) = t^n \,\mathrm{id}_{V_{n-1}}.$$

(c) When $n = 3$ and $t = -1$, prove that $\widetilde{\beta}$ induces a surjection of groups from B_3 onto the group $SL_2(\mathbf{Z})$ of integral 2×2-matrices with determinant one.

4. Define the pure braid group P_n as the kernel of the map $f \mapsto \sigma(f)$ from the braid group B_n to the symmetric group S_n. Show that P_n is isomorphic to the fundamental group of the space Y_n defined in 6.4.

5. (*Kauffman's bracket*) Show that the Kauffman bracket as defined in Section 8 is invariant under Reidemeister Transformations (0), (I'), (II) and (III) (Transformation (I') is the variant of Transformation (I) defined in Section 8).

X.8 Notes

Classical references on knot theory are [Bir74][BZ85][Kau87a][Rei32][Rol76]. The Jones polynomial V_L was defined in [Jon85] [Jon87]. Its two-variable extension P_L appeared in a number of papers written almost simultaneously [FYH$^+$85] [Hos86] [LM87] [PT87] (see also [HKW86] [Kau91]). For Theorem 4.2 we followed the proof given by Turaev in [Tur89].

(*Smooth tangles*) There is a version of tangles and isotopies where piecewise-linear maps are replaced by C^∞ maps and the boundary condition of Definition 5.1 is replaced by a transversality condition. Such smooth tangles project to smooth tangle diagrams. It may be shown that smooth isotopy classes of smooth tangles are in bijection with isotopy classes of tangles as defined in Section 5 (see [BZ85]).

(*Framed tangles*) Let us define a *normal vector field* on a smooth tangle L as a C^∞ vector field on L that is nowhere tangent to L and that is given by the vector $(0, -1, 0)$ at all points of the boundary ∂L. One may suggestively think of a tangle with a normal vector field as a tangled ribbon defined as follows: one edge of the ribbon is the tangle itself whereas the other one is obtained from the first one by a small translation along the vector field. A *framing* of the tangle L is a homotopy class of normal vector fields on L where two normal vector fields are said to be homotopic if they can be deformed into one another within the class of normal vector fields. One can extend the concept of isotopy from tangles to tangles with framings. Isotopy classes of tangles with framings are called *framed tangles* or *ribbons*. We shall see in Chapter XIV that ribbons give rise to an interesting categorical structure, the so-called ribbon categories.

Framed tangles can be represented by tangle diagrams in the sense of Section 5 just as ordinary tangles are. Take a tangle diagram. By definition, it represents the following framed tangle: the underlying tangle is the tangle represented by the diagram and the framing is determined by the constant normal vector field $(0, -1, 0)$ that is perpendicular to the plane of the diagram and points to the reader. Any framed tangled may be represented by a planar diagram in such a way. We already know this for the underlying unframed tangle. To represent a general framed tangle with a vector field whirling around it, it is enough to know how to represent a vertical tangle around which the vector field turns by an angle of 2π or of -2π. The corresponding ribbons appear in Figure 8.1 and may be represented by the diagrams of Figure 8.2.

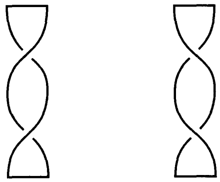


Figure 8.1.

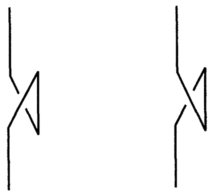


Figure 8.2.

There is an analogue of Reidemeister theorem for framed tangles. For this we need a variant (I') of Reidemeister Transformation (I). It is depicted in Figure 8.3. Two tangle diagrams represent isotopic *framed* tangles if and only if they can be obtained from one another by a finite sequence of Reidemeister Transformations (I'), (II), (III), and of isotopies of diagrams.

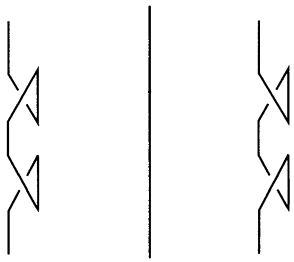


Figure 8.3. Reidemeister move (I')

(*Kauffman's bracket*) Shortly after Vaughan Jones's discovery of the link polynomial V_L, Louis Kauffman [Kau87b] found an isotopy invariant, now called the *Kauffman bracket*, for framed links. It is with values in the ring $\mathbf{Z}[x, x^{-1}]$ of Laurent polynomials. The Kauffman bracket $< L >$ can be characterized as follows. Take any diagram representing the framed link L. Single out a crossing. Define L_0 [resp. L_∞] to be the diagram where the crossing has been replaced by $||$ [resp. by $\overset{\cup}{\cap}$]. Then the bracket is determined by the rules

$$< L >= x < L_0 > +x^{-1} < L_\infty >$$

and $< O^{\otimes n} >= (-1)^{n-1}(x^2 + x^{-2})^{n-1}$. The Jones polynomial can be recovered from the Kauffman bracket (see [Kau87b]).

(*Braid groups*) The braid groups were defined by E. Artin in [Art25]. Their presentation, as in Corollary 6.6, is also due to him [Art25][Art47]. The representations described in Exercise 3 were found by Burau in 1936 [Bur36]. The Burau representation is faithful for the braid group B_3. It has long been conjectured that the general Burau representation was faithful too. This was disproven recently by J.A. Moody [Moo91]. It is still an open question whether the braid groups B_n (for n large) have faithful finite-dimensional representations at all.

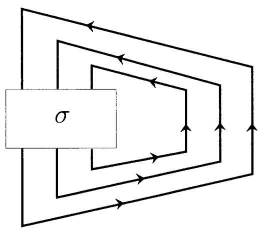


Figure 8.4. Closure of a braid

(*Closure of a braid*) For any braid $\sigma \in B_n$ define a link $\widetilde{\sigma}$ by

$$\widetilde{\sigma} = \sigma \bigcup \bigcup_{k=1}^{n} \Big([(k,0,0),(k,1,1/2)] \cup [(k,1,1/2)],(k,0,1)] \Big).$$

The link $\widetilde{\sigma}$, called the closure of σ, is isotopic to the one represented in Figure 8.4. Alexander [Ale23] showed that any link in $\mathbf{R}^3$ was equivalent to the closure of some braid. Non-equivalent braids may have equivalent closures. Define an equivalence relation $\approx$ on the set of all braids by: $\sigma \approx \sigma'$ if σ and σ' have equivalent closures. Then Markov's theorem ([Mar36]; for a proof, see [Bir74]) states that $\approx$ is the equivalence relation generated by conjugation in the braid groups and by relations of the form $\sigma \approx i_n(\sigma)\sigma_n^{\pm 1}$ where $\sigma \in B_n$ and i_n is the morphism of B_n to B_{n+1} defined by $i_n(\sigma_i) = \sigma_i$ for $i = 1, \ldots, n-1$. As a consequence, any family $(f_n : B_n \to C)_{n>0}$ of set-theoretic maps with values in a set C such that for all n and all $\sigma, \tau \in B_n$

$$f_n(\tau\sigma\tau^{-1}) = f_n(\sigma) \quad \text{and} \quad f_n(\sigma) = f_{n+1}(i_n(\sigma)\sigma_n^{\pm 1}),$$

gives rise to a unique C-valued isotopy invariant f of links in $\mathbf{R}^3$ defined by $f(L) = f_n(\sigma)$ when L is equivalent to the closure of the braid $\sigma \in B_n$. This approach was used by V. Jones to construct the polynomial V_L in [Jon85] [Jon87]. It is to be observed that the approach using Markov's theorem is less elementary than the one by Reidemeister moves.

X.9 Appendix. The Fundamental Group

We briefly recall the definition of the fundamental group of a topological space. Set $I = [0, 1]$.

Let X be a topological space with a distinguished point $\star$. A *loop* in X at the point $\star$ is a continuous map $f : I \to X$ such that $f(0) = f(1) = \star$. Denote the set of such maps by $\mathcal{L}_\star X$. Given elements f, g in $\mathcal{L}_\star X$ we define their *product* fg by

$$(fg)(t) = \begin{cases} f(2t) & \text{if} \quad 0 \leq t \leq 1/2, \\ g(2t-1) & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

The *constant loop* e is given by $e(t) = \star$. The *inverse* f^{-1} of f is defined by $f^{-1}(t) = f(1-t)$ for $t \in I$.

A *homotopy* from f to g is a continuous map $h : I \times I \to X$ such that

$$h(0, -) = f, \quad h(1, -) = g, \quad h(s, 0) = h(s, 1) = \star$$

for all $s \in I$. If such a homotopy exists, we write $f \sim g$. Homotopy is an equivalence relation. Indeed, it is

(a) reflexive because $(s, t) \mapsto f(t)$ is a homotopy from f to itself;

(b) symmetric: if h is a homotopy from f to g, then $h(1-s, -)$ is a homotopy from g to f;

(c) transitive: if h_1 and h_2 are homotopies from f_1 to f_2 and from f_2 to f_3 respectively, then

$$h(s,-) = \begin{cases} h_1(2s,-) & \text{if} \quad 0 \leq s \leq 1/2, \\ h_2(2s-1,-) & \text{if} \quad 1/2 \leq s \leq 1 \end{cases}$$

is a homotopy from f_1 to f_2.

We define $\pi_1(X,\star)$ as the set of homotopy classes in $\mathcal{L}_\star X$. We have the following lemma.

Lemma X.9.1. *Let f, f', f'', g, g' be elements of $\mathcal{L}_\star X$. Then*
(a) *$f \sim g$ and $f' \sim g'$ imply $ff' \sim gg'$,*
(b) *$(ff')f'' \sim f(f'f'')$,*
(c) *$fe \sim f \sim ef$, and*
(d) *$ff^{-1} \sim e \sim f^{-1}f$.*

PROOF. (a) If h [resp. h'] is a homotopy from f to g [resp. from f' to g'], then $(s,t) \mapsto (h(s,-)h'(s,-))(t)$ is a homotopy from ff' to gg'.

(b) A homotopy from $(ff')f''$ to $f(f'f'')$ is given by

$$h(s,t) = \begin{cases} f(\frac{4t}{s+1}) & \text{if} \quad 0 \leq t \leq \frac{s+1}{4}, \\ f'(4t-s-1) & \text{if} \quad \frac{s+1}{4} \leq t \leq \frac{s+2}{4}, \\ f''(\frac{4t-s-2}{2-s}) & \text{if} \quad \frac{s+2}{4} \leq t \leq 1. \end{cases}$$

(c) The map

$$h(s,t) = \begin{cases} f(\frac{2t}{s+1}) & \text{if} \quad 0 \leq t \leq \frac{s+1}{2}, \\ \star & \text{if} \quad \frac{s+1}{2} \leq t \leq 1 \end{cases}$$

is a homotopy from fe to f. One can also exhibit a homotopy from ef to f.

(d) A homotopy from e to ff^{-1} is given by

$$h(s,t) = \begin{cases} f(2t) & \text{if} \quad 0 \leq 2t \leq s, \\ f(s) & \text{if} \quad s \leq 2t \leq 2-s, \\ f^{-1}(2t-1) & \text{if} \quad 2-s \leq 2t \leq 2. \end{cases}$$

Exchange f and f^{-1} to get an homotopy from e to $f^{-1}f$. □

As a consequence, we see that the product of loops equips $\pi_1(X,\star)$ with the structure of a group in which the unit is the homotopy class of the constant loop e. This group is called the *fundamental group* of the topological space X at the point $\star$.

In the above definitions one may replace continuous maps by piecewise-linear or by C^∞ maps when X is an open subset of $\mathbf{R}^3$ or a quotient space of it. One gets a piecewise-linear or a smooth version of the fundamental group. These variants are isomorphic to the fundamental group defined above.

Chapter XI
Tensor Categories

This is our first chapter on tensor categories. As will become apparent in the sequel, tensor categories form the right framework for representations of Hopf algebras as well as for the topological objects of Chapter X. They provide a bridge between quantum groups and knot theory.

XI.1 The Language of Categories and Functors

We start with a few elementary definitions from category theory.

XI.1.1 Categories

Definition XI.1.1. *A category $\mathcal{C}$ consists*

(1) *of a class* $\mathrm{Ob}(\mathcal{C})$ *whose elements are called the objects of the category,*

(2) *of a class* $\mathrm{Hom}(\mathcal{C})$ *whose elements are called the morphisms of the category, and*

(3) *of maps*

$$\begin{array}{rcll} \textit{identity} & \mathrm{id} & : & \mathrm{Ob}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}), \\ \textit{source} & s & : & \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}), \\ \textit{target} & b & : & \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}), \\ \textit{composition} & \circ & : & \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}), \end{array}$$

such that

(a) *for any object* $V \in \mathrm{Ob}(\mathcal{C})$, *we have*

$$s(\mathrm{id}_V) = b(\mathrm{id}_V) = V,$$

(b) *for any morphism* $f \in \mathrm{Hom}(\mathcal{C})$, *we have*

$$\mathrm{id}_{b(f)} \circ f = f \circ \mathrm{id}_{s(f)} = f,$$

(c) *for any morphisms* f, g, h *satisfying* $b(f) = s(g)$ *and* $b(g) = s(h)$, *we have*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Here $\mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(\mathcal{C})$ denotes the class of couples (f, g) of *composable* morphisms in the category, i.e., such that $b(f) = s(g)$. The conventional notation for the composition of f and g is $g \circ f$ or gf. The object $s(f)$ is called the *source* of the morphism f and $b(f)$ is its *target*. For the *identity morphism* of an object V we write id_V. We denote by $\mathrm{Hom}_{\mathcal{C}}(V, W)$ the class of morphisms of the category $\mathcal{C}$ whose source is the object V and whose target is the object W. If $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$, we write

$$f : V \to W \quad \text{or} \quad V \xrightarrow{f} W.$$

A morphism from an object V to itself is called an *endomorphism* of V. The class of all endomorphisms of V is denoted $\mathrm{End}(V)$. A morphism f from V to W in the category is an *isomorphism* if there exists a morphism $g : W \to V$ such that $g \circ f = \mathrm{id}_V$ and $f \circ g = \mathrm{id}_W$.

Everybody knows (or at least uses) the category *Set* of sets and the category *Gr* of groups. We have already made use of the category *Vect*(k) [resp. of *Vect*$_f(k)$] consisting of vector spaces [resp. of finite-dimensional vector spaces] and of linear maps over a field k. In Chapter I we used the category *Alg* of algebras and the category A-*Mod* of left A-modules where A is an algebra. We have also considered the category Cog of coalgebras.

We define the *product of two categories* $\mathcal{C}$ and $\mathcal{D}$ as the category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs of objects $(V, W) \in \mathcal{C} \times \mathcal{D}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((V, W), (V', W')) = \mathrm{Hom}_{\mathcal{C}}(V, V') \times \mathrm{Hom}_{\mathcal{D}}(W, W').$$

A *subcategory* $\mathcal{C}$ of a category $\mathcal{D}$ consists of a subclass $\mathrm{Ob}(\mathcal{C})$ of $\mathrm{Ob}(\mathcal{D})$ and of a subclass $\mathrm{Hom}(\mathcal{C})$ of $\mathrm{Hom}(\mathcal{D})$ that are stable under the identity, source, target, and composition maps in $\mathcal{D}$.

Let us present two examples of categories that are *groupoids*, i.e., categories in which all morphisms are isomorphisms.

Example 1. (*Category associated to a family of groups*) Let $(G_i)_{i\in I}$ be a family of groups indexed by a set I. We consider the category $\mathcal{G}$ defined by $\mathrm{Ob}(\mathcal{G}) = I$ and

$$\mathrm{Hom}_{\mathcal{G}}(i, j) = \begin{cases} \emptyset & \text{if } i \neq j, \\ G_i & \text{if } i = j, \end{cases}$$

the composition on $\mathrm{Hom}_{\mathcal{G}}(i,i)$ being given by the multiplication of the group G_i:

$$h \circ g = gh$$

where $g, h \in G_i$. Note that any morphism g in $\mathcal{G}$ is an isomorphism whose inverse morphism is g^{-1}.

As a special case, consider a set I reduced to a single element 0. We get a groupoid $\mathcal{G}$ with one object and with G_0 as the set of morphisms.

Example 2. (*Category of isomorphisms in a category*) Let $\mathcal{C}$ be a category. If we set $\mathrm{Ob}(\mathcal{C}_{is}) = \mathrm{Ob}(\mathcal{C})$ and define $\mathrm{Hom}(\mathcal{C}_{is})$ as the subclass of isomorphisms of $\mathcal{C}$, then $\mathcal{C}_{is}$ is a category called the groupoid of isomorphisms of $\mathcal{C}$.

XI.1.2 Functors and natural transformations

Definition XI.1.2. *A functor $F : \mathcal{C} \to \mathcal{C}'$ from the category $\mathcal{C}$ to the category $\mathcal{C}'$ consists of a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ and of a map $F : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}')$ such that*

(a) *for any object $V \in \mathrm{Ob}(\mathcal{C})$, we have $F(\mathrm{id}_V) = \mathrm{id}_{F(V)}$,*

(b) *for any morphism $f \in \mathrm{Hom}(\mathcal{C})$, we have*

$$s(F(f)) = F(s(f)) \quad \textit{and} \quad b(F(f)) = F(b(f)),$$

(c) *if f, g are composable morphisms in the category $\mathcal{C}$, we have*

$$F(g \circ f) = F(g) \circ F(f).$$

It is clear that if $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ are functors, then the composition GF is a functor from $\mathcal{C}$ to $\mathcal{C}''$. For any category $\mathcal{C}$, there exists a functor $\mathrm{id}_{\mathcal{C}}$, called the identity functor of $\mathcal{C}$, which is the identity on the objects and on the morphisms in $\mathcal{C}$. The inclusion of a subcategory in a category is a functor.

Definition XI.1.3. *Let F, G be functors from the category $\mathcal{C}$ to the category $\mathcal{C}'$. A natural transformation η from F to G — we write $\eta : F \to G$ — is a family of morphisms $\eta(V) : F(V) \to G(V)$ in $\mathcal{C}'$ indexed by the objects V of $\mathcal{C}$ such that, for any morphism $f : V \to W$ in $\mathcal{C}$, the square*

$$\begin{tikzcd} F(V) \arrow[r, "\eta(V)"] \arrow[d, "F(f)"] & G(V) \arrow[d, "G(f)"] \\ F(W) \arrow[r, "\eta(W)"] & G(W) \end{tikzcd}$$

commutes.

If, furthermore, $\eta(V)$ is an isomorphism of $\mathcal{C}'$ for any object V in $\mathcal{C}$, we say that $\eta : F \to G$ is a natural isomorphism.

If $\eta : F \to G$ is a natural isomorphism, then the collection of all morphisms $\eta(V)^{-1}$ defines a natural isomorphism η^{-1} from G to F. We next define the important concept of an equivalence of categories.

Definition XI.1.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an equivalence of categories if there exist a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms*

$$\eta : \mathrm{id}_{\mathcal{D}} \to FG \quad \textit{and} \quad \theta : GF \to \mathrm{id}_{\mathcal{C}}.$$

We now give a useful criterion for a functor $F : \mathcal{C} \to \mathcal{D}$ to be an equivalence of categories. Let us first say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if, for any object W of $\mathcal{D}$, there exists an object V of $\mathcal{C}$ such that $F(V) \cong W$ in $\mathcal{D}$. It is said to be *faithful* [resp. *fully faithful*] if, for any couple (V, V') of objects of $\mathcal{C}$, the map

$$F : \mathrm{Hom}_{\mathcal{C}}(V, V') \to \mathrm{Hom}_{\mathcal{D}}(F(V), F(V'))$$

on morphisms is injective [resp. bijective].

Proposition XI.1.5. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is essentially surjective and fully faithful.*

PROOF. (a) Suppose that F is an equivalence. Then there exist a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{id}_{\mathcal{D}} \to FG$ and $\theta : GF \to \mathrm{id}_{\mathcal{C}}$. The first isomorphism shows that $W \cong F(G(W))$ for any object W of $\mathcal{D}$. In other words, F is essentially surjective. Now consider a morphism $f : V \to V'$ in $\mathcal{C}$. The square

$$\begin{array}{ccc} GF(V) & \xrightarrow{\theta(V)} & V \\ \Big\downarrow{\scriptstyle GF(f)} & & \Big\downarrow{\scriptstyle f} \\ GF(V') & \xrightarrow{\theta(V')} & V' \end{array}$$

commutes. It results that if $F(f) = F(f')$, hence $GF(f) = GF(f')$, then we have $f = f'$. Therefore, the functor F is faithful. Using the natural isomorphism η in a similar way, we prove that G is faithful too. Now consider a morphism $g : F(V) \to F(V')$. Let us show that $g = F(f)$ where $f = \theta(V') \circ G(g) \circ \theta(V)^{-1}$. Indeed,

$$\theta(V') \circ GF(f) \circ \theta(V)^{-1} = f = \theta(V') \circ G(g) \circ \theta(V)^{-1}.$$

Therefore $GF(f) = G(g)$. As G is faithful, we get $g = F(f)$. This proves that F is fully faithful.

(b) Let F be an essentially surjective and fully faithful functor. For any object W in $\mathcal{D}$, we choose an object $G(W)$ of $\mathcal{C}$ and an isomorphism $\eta(W) : W \to FG(W)$ in $\mathcal{D}$. If $g : W \to W'$ is a morphism of $\mathcal{D}$, we may consider

$$\eta(W') \circ g \circ \eta(W)^{-1} : FG(W) \to FG(W').$$

Since F is fully faithful, there exists a unique morphism $G(g)$ from $G(W)$ to $G(W')$ such that

$$FG(g) = \eta(W') \circ g \circ \eta(W)^{-1} : FG(W) \to FG(W').$$

One checks easily that this defines a functor G from $\mathcal{D}$ into $\mathcal{C}$ and that $\eta : \mathrm{id}_{\mathcal{D}} \to FG$ is a natural isomorphism. In order to show that F and G are equivalences of categories, we have only to find a natural isomorphism $\theta : GF \to \mathrm{id}_{\mathcal{C}}$. We define $\theta(V) : GF(V) \to V$ for any object $V \in \mathrm{Ob}(\mathcal{C})$ as the unique morphism such that $F(\theta(V)) = \eta(F(V))^{-1}$. It is easily checked that this formula defines a natural isomorphism. □

Corollary XI.1.6. *Let $\mathcal{C}$ be a category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$ such that any object of $\mathcal{C}$ is isomorphic to an object of $\mathcal{C}'$ and such that we have $\mathrm{Hom}_{\mathcal{C}'}(V, V') = \mathrm{Hom}_{\mathcal{C}}(V, V')$ for all $V, V' \in \mathrm{Ob}(\mathcal{C}')$. Then the inclusion of $\mathcal{C}'$ into $\mathcal{C}$ is an equivalence of categories.*

We deduce the following examples of equivalent categories.

Example 3. (*The groupoid $GL(k)$*) Let $\mathrm{GL}_n(k)$ be the group of invertible matrices of order n with entries in a field k. Set $\mathrm{GL}_0(k) = \{1\}$. By Example 1 we can associate to the family $(\mathrm{GL}_n(k))_{n \geq 0}$ a groupoid denoted $GL(k)$. By the previous corollary, the category $GL(k)$ is equivalent to the groupoid $(\mathit{Vect}_f(k))_{is}$ of all finite dimensional k-vector spaces whose morphisms are the linear isomorphisms.

Example 4. (*The groupoid $\mathcal{S}$*) Let S_n be the symmetric group of all permutations of the finite set $\{1, 2, \ldots, n\}$. Set $S_0 = \{1\}$. Again by the construction of Example 1, we get a groupoid $\mathcal{S}$. The category $\mathcal{S}$ is equivalent to the groupoid $(\mathit{Set}_f)_{is}$ of finite sets whose morphisms are bijective.

XI.1.3 Adjoint functors

We end these preliminaries on categories with the concept of adjoint functors. As we may observe from Proposition 1.8, as well from the examples of this section and the exercises of this chapter, the concept of adjoint functors is nothing but the categorical translation of the idea of a universal property.

Definition XI.1.7. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. Then F is right adjoint to G or G is left adjoint to F if there exist natural transformations $\eta : \mathrm{id}_{\mathcal{D}} \to FG$ and $\theta : GF \to \mathrm{id}_{\mathcal{C}}$ such that the composite maps*

$$F(V) \xrightarrow{\eta(F(V))} (FGF)(V) \xrightarrow{F(\theta(V))} F(V)$$

and

$$G(W) \xrightarrow{G(\eta(W))} (GFG)(W) \xrightarrow{\theta(G(W))} G(W)$$

are identity morphisms for all objects V of $\mathcal{C}$ and W in $\mathcal{D}$.

The following characterizes adjoint functors in terms of natural bijections.

Proposition XI.1.8. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. Then F is right adjoint to G if and only if for all objects V in $\mathcal{C}$ and all objects W in $\mathcal{D}$ there exists a natural bijection*

$$\Phi(V, W) : \mathrm{Hom}_{\mathcal{C}}(G(W), V) \to \mathrm{Hom}_{\mathcal{D}}(W, F(V)),$$

i.e., such that, for all morphisms f in $\mathcal{C}$ and all morphisms g in $\mathcal{D}$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(G(W'), V) \arrow[r, "\Phi(V,W')"] \arrow[d, "\mathrm{Hom}(G(g),V)"] & \mathrm{Hom}_{\mathcal{D}}(W', F(V)) \arrow[d, "\mathrm{Hom}(g,F(V))"] \\
\mathrm{Hom}_{\mathcal{C}}(G(W), V) \arrow[r, "\Phi(V,W)"] \arrow[d, "\mathrm{Hom}(G(W),f)"] & \mathrm{Hom}_{\mathcal{D}}(W, F(V)) \arrow[d, "\mathrm{Hom}(W,F(f))"] \\
\mathrm{Hom}_{\mathcal{C}}(G(W), V') \arrow[r, "\Phi(V',W)"] & \mathrm{Hom}_{\mathcal{D}}(W, F(V'))
\end{tikzcd}$$

commutes where $V = s(f)$, $V' = b(f)$, $W = s(g)$ and $W' = b(g)$.

PROOF. The vertical maps of the above diagram are the obvious maps obtained by composition with f, $F(f)$, g, and $G(g)$. We sketch the proof of this proposition. For details, see [Mac71], Chapter IV.

(a) Let F be a right adjoint to G. Set $\Phi(V, W)(f) = F(f) \circ \eta(W)$ for any morphism $f : G(W) \to V$, and $\Psi(V, W)(g) = \theta(V) \circ G(g)$ for any morphism $g : W \to F(V)$. Using the definition of adjoint functors, one checks that the map $\Phi(V, W)$ is bijective with inverse $\Psi(V, W)$.

(b) Suppose given the bijections $\Phi(V, W)$. We have to construct natural transformations $\eta : \mathrm{id}_{\mathcal{D}} \to FG$ and $\theta : GF \to \mathrm{id}_{\mathcal{C}}$. They are defined by

$$\eta(W) = \Phi(G(W), W)(\mathrm{id}_{G(W)}) \quad \text{and} \quad \theta(V) = \Phi^{-1}(V, F(V))(\mathrm{id}_{F(V)}).$$

The reader will easily check that η and θ are natural transformations. □

An equivalence of categories is always left and right adjoint to another equivalence. We give two examples of adjoint functors already encountered in this book.

Example 5. (*Free algebra on a set*) Let X be a set and $k\{X\}$ be the free k-algebra associated to X as in I.2. Then $X \mapsto k\{X\}$ is a left adjoint functor to the forgetful functor which assigns to any algebra its underlying set.

Example 6. (*Tensor products*) Any vector space V determines two functors F, G from the category of vector spaces into itself: $F(U) = \mathrm{Hom}(V, U)$ and $G(U) = U \otimes V$. The natural isomorphism

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$

of Corollary II.1.2 shows that G is left adjoint to F.

XI.2 Tensor Categories

XI.2.1 Definitions

Let $\mathcal{C}$ be a category and $\otimes$ be a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$. This means that

(a) we have an object $V \otimes W$ associated to any pair (V, W) of objects of the category,

(b) we have a morphism $f \otimes g$ associated to any pair (f, g) of morphisms of $\mathcal{C}$ such that

$$s(f \otimes g) = s(f) \otimes s(g) \quad \text{and} \quad b(f \otimes g) = b(f) \otimes b(g),$$

(c) if f' and g' are morphisms such that $s(f') = b(f)$ and $s(g') = b(g)$, then

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g), \tag{2.1}$$

(d) and

$$\mathrm{id}_{V \otimes W} = \mathrm{id}_V \otimes \mathrm{id}_W. \tag{2.2}$$

Relation (2.1) implies that

$$f \otimes g = (f \otimes \mathrm{id}_{b(g)}) \circ (\mathrm{id}_{s(f)} \otimes g) = (\mathrm{id}_{b(f)} \otimes g) \circ (f \otimes \mathrm{id}_{s(g)}). \tag{2.3}$$

Example 1. Let $\mathcal{C} = \mathit{Vect}(k)$ be the category of vector spaces over a field k. Then the tensor product of vector spaces (see II.1–2) defines a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$.

Any functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ will be called a *tensor product* by analogy with Example 1. Let $\mathcal{C}$ be a category with a tensor product $\otimes$. An *associativity constraint* for $\otimes$ is a natural isomorphism

$$a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes).$$

This means that, for any triple (U, V, W) of objects of $\mathcal{C}$, there exists an isomorphism

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W) \tag{2.4}$$

such that the square

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, "a_{U,V,W}"] \arrow[d, "(f\otimes g)\otimes h"] & U \otimes (V \otimes W) \arrow[d, "f\otimes(g\otimes h)"] \\
(U' \otimes V') \otimes W' \arrow[r, "a_{U',V',W'}"] & U' \otimes (V' \otimes W')
\end{tikzcd} \tag{2.5}$$

commutes whenever f, g, h are morphisms in the category.

The associativity constraint a satisfies the *Pentagon Axiom* if the pentagonal diagram

$$\begin{tikzcd}
\big(U \otimes (V \otimes W)\big) \otimes X \arrow[dd, "a_{U,V\otimes W,X}"] & \big((U \otimes V) \otimes W\big) \otimes X \arrow[l, "a_{U,V,W}\otimes \mathrm{id}_X"'] \arrow[d, "a_{U\otimes V,W,X}"] \\
 & (U \otimes V) \otimes (W \otimes X) \arrow[d, "a_{U,V,W\otimes X}"] \\
U \otimes \big((V \otimes W) \otimes X\big) \arrow[r, "\mathrm{id}_U \otimes a_{V,W,X}"] & U \otimes \big(V \otimes (W \otimes X)\big)
\end{tikzcd} \tag{2.6}$$

commutes for all objects U, V, W, X of $\mathcal{C}$.

Fix an object I in the category. A *left unit constraint* [resp. a *right unit constraint*] with respect to I is a natural isomorphism

$$l : \otimes(I \times \mathrm{id}) \to \mathrm{id} \quad [\text{resp. } r : \otimes(\mathrm{id} \times I) \to \mathrm{id}].$$

This means that for any object V of $\mathcal{C}$ there exists an isomorphism

$$l_V : I \otimes V \to V \quad [\text{resp. } r_V : V \otimes I \to V] \tag{2.7}$$

such that

$$\begin{tikzcd}
I \otimes V \arrow[r, "l_V"] \arrow[d, "\mathrm{id}_I \otimes f"] & V \arrow[d, "f"] \\
I \otimes V' \arrow[r, "l_{V'}"] & V'
\end{tikzcd}
\quad \left[\text{resp.} \quad
\begin{tikzcd}
V \otimes I \arrow[r, "r_V"] \arrow[d, "f \otimes \mathrm{id}_I"] & V \arrow[d, "f"] \\
V' \otimes I \arrow[r, "r_{V'}"] & V'
\end{tikzcd}\right] \tag{2.8}$$

commutes for any morphism f.

Given an associativity constraint a, and left and right unit constraints l, r with respect to an object I, we say that they satisfy the *Triangle Axiom* if the triangle

$$\begin{tikzcd}
(V \otimes I) \otimes W \arrow[rr, "a_{V,I,W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (I \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
 & V \otimes W &
\end{tikzcd} \tag{2.9}$$

commutes for all pairs (V, W) of objects.

Definition XI.2.1. *A tensor category $(\mathcal{C}, \otimes, I, a, l, r)$ is a category $\mathcal{C}$ which is equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, with an object I, called the unit of the tensor category, with an associativity constraint a, a left unit constraint l and a right unit constraint r with respect to I such that the Pentagon Axiom* (2.6) *and the Triangle Axiom* (2.9) *are satisfied.*

The tensor category is said to be strict if the associativity and unit constraints a, l, r are all identities of the category.

Examples of tensor categories will be given in Section 3.

XI.2.2 Properties of the unit

Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category. We state a few properties of the unit I.

Lemma XI.2.2. *The triangles*

$$\begin{tikzcd} (I\otimes V)\otimes W \arrow[rr, "a_{I,V,W}"] \arrow[dr, "l_V\otimes \mathrm{id}_W"'] & & I\otimes(V\otimes W) \arrow[dl, "l_{V\otimes W}"] \\ & V\otimes W & \end{tikzcd}$$

and

$$\begin{tikzcd} (V\otimes W)\otimes I \arrow[rr, "a_{V,W,I}"] \arrow[dr, "r_{V\otimes W}"'] & & V\otimes(W\otimes I) \arrow[dl, "\mathrm{id}_V\otimes r_W"] \\ & V\otimes W & \end{tikzcd}$$

commute for any pair (V, W) of objects of $\mathcal{C}$.

PROOF. Consider the diagram

$$\begin{tikzcd}[column sep=small]
 & & \Big(U\otimes(I\otimes V)\Big)\otimes W \arrow[drr, "\mathrm{id}"] & & \\
\Big((U\otimes I)\otimes V\Big)\otimes W \arrow[urr, "a\otimes\mathrm{id}_W"] \arrow[drr, "(r_U\otimes\mathrm{id}_V)\otimes\mathrm{id}_W"] \arrow[ddd, "a"'] & & & & \Big(U\otimes(I\otimes V)\Big)\otimes W \arrow[dll, "(\mathrm{id}_U\otimes l_V)\otimes\mathrm{id}_W"'] \arrow[ddd, "a"] \\
 & & (U\otimes V)\otimes W \arrow[d, "a"] & & \\
 & & U\otimes(V\otimes W) & & \\
(U\otimes I)\otimes(V\otimes W) \arrow[urr, "r_U\otimes\mathrm{id}_{V\otimes W}"] \arrow[drr, "a"] & & & & U\otimes\Big((I\otimes V)\otimes W\Big) \arrow[ull, "\mathrm{id}_U\otimes(l_V\otimes\mathrm{id}_W)"'] \arrow[dll, "\mathrm{id}_U\otimes a"] \\
 & & U\otimes\Big(I\otimes(V\otimes W)\Big) \arrow[uu, "\mathrm{id}_U\otimes l_{V\otimes W}"'] & &
\end{tikzcd}$$

Here we dropped the subscripts of the associativity constraint a for simplicity. The outside hexagon commutes by the Pentagon Axiom (2.6). The naturality (2.5) of a implies the commutativity of the two middle squares whereas (2.9) implies the commutativity of the top square and of the lower left triangle. Consequently, the lower right triangle commutes as well. Setting $U = I$, we get

$$\mathrm{id}_I \otimes (l_{V\otimes W} \circ a) = \mathrm{id}_I \otimes (l_V \otimes \mathrm{id}_W).$$

This relation, together with the naturality of the left unit constraint (2.8) and the fact that l is an isomorphism, implies $l_{V\otimes W} \circ a = l_V \otimes \mathrm{id}_W$, which expresses the commutativity of the upper triangle in the statement of Lemma XI.2.2. A similar proof works for the other triangle. □

Lemma XI.2.3. *Let I be the unit of a tensor category. For any object V we have*

$$l_{I\otimes V} = \mathrm{id}_I \otimes l_V, \quad r_{V\otimes I} = r_V \otimes \mathrm{id}_I, \quad \textit{and} \quad l_I = r_I.$$

PROOF. By naturality (2.8) of l, we have $l_V \circ l_{I\otimes V} = l_V \circ (\mathrm{id}_I \otimes l_V)$. Since l_V is an isomorphism, we get the first equality of the lemma. The second one is similarly a consequence of the naturality of r.

Let us prove that $l_I = r_I$. By Lemma 2.2 and the first equality of Lemma 2.3, we have

$$l_I \otimes \mathrm{id}_I = l_{I\otimes I} \circ a = (\mathrm{id}_I \otimes l_I) \circ a.$$

From (2.9) we get $r_I \otimes \mathrm{id}_I = (\mathrm{id}_I \otimes l_I) \circ a$. Combining both relations yields $l_I \otimes \mathrm{id}_I = r_I \otimes \mathrm{id}_I$. This implies $l_I = r_I$ in view of the fact that r is a natural isomorphism. □

We are now ready to prove the main result of this subsection.

Proposition XI.2.4. *The set* $\mathrm{End}(I)$ *of endomorphisms of the unit object I is a commutative monoid for the composition. Moreover, for any pair (f, g) of endomorphisms of I, we have*

$$f \otimes g = g \otimes f = r_I^{-1} \circ (f \circ g) \circ r_I = r_I^{-1} \circ (g \circ f) \circ r_I.$$

In other words, if we identify $I \otimes I$ with I via $r_I = l_I$, then the tensor product of morphisms coincides in $\mathrm{End}(I)$ with their composition.

PROOF. The composition equips $\mathrm{End}(I)$ with the structure of a monoid whose unit is id_I. Let us prove that it is commutative. By (2.8) we have

$$f \otimes \mathrm{id}_I = r_I^{-1} \circ f \circ r_I \quad \text{and} \quad \mathrm{id}_I \otimes g = l_I^{-1} \circ g \circ l_I.$$

Combining $r_I = l_I$ of Lemma 2.3 with Relation (2.3) implies that

$$f \otimes g = r_I^{-1} \circ (f \circ g) \circ r_I = r_I^{-1} \circ (g \circ f) \circ r_I = g \otimes f.$$

It follows that $f \circ g = g \circ f$. □

XI.3 Examples of Tensor Categories

In this book we shall be concerned with two main types of tensor categories. The first type is built on vector spaces and their tensor products as introduced in Chapter II. The second one uses the 1-dimensional objects of Chapter X such as links, tangles and braids. We shall connect both types in Chapter XII.

XI.3.1 Tensor categories of vector spaces

The most fundamental example of a tensor category is given by the category $\mathcal{C} = \mathit{Vect}(k)$ of vector spaces over a field k. The latter is equipped with a tensor structure for which $\otimes$ is the tensor product (defined in II.1) of the vector spaces over k, the unit object I is the ground field k itself, and the associativity and unit constraints are the natural isomorphisms

$$a\Big((u \otimes v) \otimes w\Big) = u \otimes (v \otimes w) \quad \text{and} \quad l(1 \otimes v) = v = r(v \otimes 1) \tag{3.1}$$

of Proposition II.1.3. The pentagon and triangle axioms are clearly satisfied.

There are some important examples of subcategories of $\mathit{Vect}(k)$ preserving the tensor structure. For instance, if G is a group, then the category $k[G]$-*Mod* of representations of G over k, or, equivalently, of $k[G]$-modules, is a subtensor category of $\mathit{Vect}(k)$ where the tensor product $U \otimes V$ of two G-modules and the field k are given the following G-structures:

$$g(u \otimes v) = gu \otimes gv \quad \text{and} \quad g\lambda = \lambda$$

for $g \in G$, $u \in U$, $v \in V$ and $\lambda \in k$.

We know from Chapter III that the group algebra $k[G]$ is an associative algebra over k with a comultiplication and a counit. We now investigate such types of algebras. Let A be an associative unital k-algebra with a morphism of algebras $\Delta : A \to A \otimes A$, called the *comultiplication*, and a morphism of algebras $\varepsilon : A \to k$, called the *counit*. Let us denote by A-*Mod* the category of left A-modules (alias, representations of A). If U, V are left A-modules, the tensor product $U \otimes V$ is a left $A \otimes A$-module. The comultiplication allows to pull back this $A \otimes A$-module structure into an A-module structure. It is given by

$$a(u \otimes v) = \Delta(a)(u \otimes v) \tag{3.2}$$

for $a \in A$, $u \in U$ and $v \in V$. We endow k with an A-module structure given by

$$a\lambda = \varepsilon(a)\lambda. \tag{3.3}$$

It is now clear that the tensor product in $\mathit{Vect}(k)$ restricts to a functor

$$\otimes : A\text{-}\mathit{Mod} \times A\text{-}\mathit{Mod} \to A\text{-}\mathit{Mod}$$

for which $I = k$ is a unit. The following characterizes bialgebras in terms of their categories of modules.

Proposition XI.3.1. *Let $A = (A, \Delta, \varepsilon)$ be an algebra with comultiplication and counit as above. It is a bialgebra if and only if the category A-Mod equipped with the tensor product described above and the constraints a, l, r of $\mathit{Vect}(k)$ is a tensor category.*

PROOF. Let $(A, \varphi, \eta, \Delta, \varepsilon)$ be a bialgebra. It follows from Proposition III.5.1 that $(A\text{-}Mod, \otimes, I = k, a, l, r)$ is a tensor category.

Conversely, let $(A, \varphi, \eta, \Delta, \varepsilon)$ be an algebra with comultiplication and counit. Suppose that $(A\text{-}Mod, \otimes, I = k, a, l, r)$ is a tensor category. Let us prove that Δ is coassociative and that ε is a counit in the sense of Definition III.1.1.

Let us start with the coassociativity of Δ. Consider the associativity constraint $a_{A,A,A}$. By hypothesis, it is A-linear, which means that for $a, u, v, w \in A$ we have

$$a_{A,A,A}\Big(a\big((u \otimes v) \otimes w\big)\Big) = a\, a_{A,A,A}\big((u \otimes v) \otimes w\big).$$

By definition of the associativity constraint, this can be reexpressed as

$$(\Delta \otimes \mathrm{id})(\Delta(a))(u \otimes (v \otimes w)) = (\mathrm{id} \otimes \Delta)(\Delta(a))(u \otimes (v \otimes w)).$$

Setting $u = v = w = 1 \in A$, we get

$$(\Delta \otimes \mathrm{id})(\Delta(a)) = (\mathrm{id} \otimes \Delta)(\Delta(a)).$$

Similarly, l_A is A-linear if and only if $(\varepsilon \otimes \mathrm{id})(\Delta(a)) = a$, and r_A is A-linear if and only if $(\mathrm{id} \otimes \varepsilon)(\Delta(a)) = a$ for all $a \in A$. □

XI.3.2 Tensor categories built on groups

We now give examples of *strict* tensor categories. Let $(G_i)_{i \in \mathbf{N}}$ be a family of groups indexed by the monoid $\mathbf{N}$ of nonnegative integers. We may form a category $\mathcal{G}$ as in Section 1, Example 1. Suppose that $G_0 = \{1\}$, and for any pair (n, m) of integers we have a group morphism $\rho_{n,m} : G_n \times G_m \to G_{n+m}$. We now define a tensor product on the category $\mathcal{G}$ by $n \otimes m = n + m$ and, if $g \in G_n$ and $h \in G_m$, we set

$$g \otimes h = \rho_{n,m}(g, h) \in G_{n+m}.$$

Check that $(\mathcal{G}, \otimes, I = 0, a = \mathrm{id}, l = \mathrm{id}, r = \mathrm{id})$ is a strict tensor category provided the morphisms $\rho_{n,m}$ are subject to the relations

$$\rho_{n+m,p} \circ (\rho_{n,m} \otimes \mathrm{id}_{G_p}) = \rho_{n,m+p} \circ (\mathrm{id}_{G_n} \otimes \rho_{m,p}) \qquad (3.4)$$

and $\rho_{0,n} = \rho_{n,0} = \mathrm{id}_{G_n}$ after natural identification. This construction can be applied to the following families of groups.

(a) Consider the groupoid $GL(k)$ of Section 1, Example 3 built from the family of groups $(\mathrm{GL}_n(k))$. Define maps $\rho_{n,m}$ by

$$\rho_{n,m}(g, h) = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \in \mathrm{GL}_{n+m}(k).$$

Relations (3.4) are satisfied. The category $GL(k)$ becomes a tensor category in this way.

(b) A family of subgroups $\{G_n\}_n$ of $\mathrm{GL}_n(k)$ preserved by the maps $\rho_{n,m}$ gives also rise to a strict tensor category. For instance, take the family of symmetric groups S_n, the latter being realized as a subgroup of $\mathrm{GL}_n(k)$ via the permutation matrices. The resulting category $\mathcal{S}$ is a tensor category.

XI.4 Tensor Functors

Definition XI.4.1. (a) *Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes, I, a, l, r)$ be tensor categories. A tensor functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor, φ_0 is an isomorphism from I to $F(I)$, and*

$$\varphi_2(U,V) : F(U) \otimes F(V) \to F(U \otimes V)$$

is a family of natural isomorphisms indexed by all couples (U,V) of objects of $\mathcal{C}$ such that the diagrams

$$\begin{tikzcd}
\big(F(U) \otimes F(V)\big) \otimes F(W) \arrow[r, "a_{F(U),F(V),F(W)}"] \arrow[d, "\varphi_2(U,V)\otimes \mathrm{id}_{F(W)}"] & F(U) \otimes \big(F(V) \otimes F(W)\big) \arrow[d, "\mathrm{id}_{F(U)}\otimes\varphi_2(V,W)"] \\
F(U \otimes V) \otimes F(W) \arrow[d, "\varphi_2(U\otimes V,W)"] & F(U) \otimes F(V \otimes W) \arrow[d, "\varphi_2(U,V\otimes W)"] \\
F((U \otimes V) \otimes W) \arrow[r, "F(a_{U,V,W})"] & F(U \otimes (V \otimes W))
\end{tikzcd} \tag{4.1}$$

$$\begin{tikzcd}
I \otimes F(U) \arrow[r, "l_{F(U)}"] \arrow[d, "\varphi_0\otimes \mathrm{id}_{F(U)}"] & F(U) \\
F(I) \otimes F(U) \arrow[r, "\varphi_2(I,U)"] & F(I \otimes U) \arrow[u, "F(l_U)"]
\end{tikzcd} \tag{4.2}$$

and

$$\begin{tikzcd}
F(U) \otimes I \arrow[r, "r_{F(U)}"] \arrow[d, "\mathrm{id}_{F(U)}\otimes\varphi_0"] & F(U) \\
F(U) \otimes F(I) \arrow[r, "\varphi_2(U,I)"] & F(U \otimes I) \arrow[u, "F(r_U)"]
\end{tikzcd} \tag{4.3}$$

commute for all objects (U,V,W) in $\mathcal{C}$. The tensor functor $(F, \varphi_0, \varphi_2)$ is said to be strict if the isomorphisms φ_0 and φ_2 are identities of $\mathcal{D}$.

(b) *A natural tensor transformation $\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi_0', \varphi_2')$ between tensor functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \to F'$*

such that the following diagrams commute for each couple (U,V) of objects in $\mathcal{C}$:

$$\begin{tikzcd} & F(I) \arrow[dd, "\eta(I)"] \\ I \arrow[ur, "\varphi_0"] \arrow[dr, "\varphi_0'"'] & \\ & F'(I) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} F(U)\otimes F(V) \arrow[r, "\varphi_2(U,V)"] \arrow[d, "\eta(U)\otimes\eta(V)"] & F(U\otimes V) \arrow[d, "\eta(U\otimes V)"] \\ F'(U)\otimes F'(V) \arrow[r, "\varphi_2'(U,V)"] & F'(U\otimes V) \end{tikzcd} \tag{4.4}$$

A natural tensor isomorphism is a natural tensor transformation that is also a natural isomorphism.

(c) *A tensor equivalence between tensor categories is a tensor functor $F:\mathcal{C}\to\mathcal{D}$ such that there exist a tensor functor $F':\mathcal{D}\to\mathcal{C}$ and natural tensor isomorphisms $\eta:\mathrm{id}_{\mathcal{D}}\xrightarrow{\cong}FF'$ and $\theta:F'F\xrightarrow{\cong}\mathrm{id}_{\mathcal{C}}$.*

In case there exists a tensor equivalence between $\mathcal{C}$ and $\mathcal{D}$, we say that $\mathcal{C}$ and $\mathcal{D}$ are *tensor equivalent.* Observe that if (F,φ_0,φ_2) and $(F',\varphi_0',\varphi_2')$ are tensor functors, then so is the composition $(F'F, F'(\varphi_0)\varphi_0', F'(\varphi_2)\varphi_2')$. The identity functor is a strict tensor functor.

We denote by $\mathit{Tens}(\mathcal{C},\mathcal{D})$ [resp. $\mathit{Tens}_{str}(\mathcal{C},\mathcal{D})$] the category whose objects are the tensor functors [resp. the strict tensor functors] from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are the natural tensor transformations.

Example 1. Let A be a bialgebra. The forgetful functor associating to an A-module its underlying vector space is a strict tensor functor.

Example 2. Let $f:A_1\to A_2$ be a map of bialgebras. Given an A_2-module V we can equip V with a A_1-module structure given by $a\cdot v=f(a)v$ for $a\in A_1$ and $v\in V$. This construction yields a strict tensor functor f^* from A_2-*Mod* to A_1-*Mod*.

We shall encounter our first examples of non-strict tensor functors in Chapter XV devoted to quasi-bialgebras.

XI.5 Turning Tensor Categories into Strict Ones

Since the tensor product in a tensor category is associative only up to isomorphism, one has to keep track of parentheses very carefully. This is rather fastidious and should be avoided as much as possible. We now indicate a way out of this problem: given a tensor category $\mathcal{C}=(\mathcal{C},\otimes,I,a,l,r)$, we construct a strict tensor category $\mathcal{C}^{str}$ which is tensor equivalent to $\mathcal{C}$. This is done as follows.

Let $\mathcal{S}$ be the class of all finite sequences $S=(V_1,\dots,V_k)$ of objects of $\mathcal{C}$, including the empty sequence $\emptyset$. The integer k is by definition the length of the sequence $S=(V_1,\dots,V_k)$. The length of the empty sequence is 0 by convention. If $S = (V_1, \ldots, V_k)$ and $S' = (V_{k+1}, \ldots, V_{k+n})$ are nonempty sequences of $\mathcal{S}$, we denote by $S * S'$ the sequence

$$S * S' = (V_1, \ldots, V_k, V_{k+1}, \ldots, V_{k+n}) \tag{5.1}$$

obtained by placing S' after S. We also agree that $S * \emptyset = S = \emptyset * S$. To any sequence S of $\mathcal{S}$, we assign an object $F(V)$ of $\mathcal{C}$ defined inductively by

$$F(\emptyset) = I, \quad F((V)) = V, \quad F(S * (V)) = F(S) \otimes V.$$

In other words,

$$F((V_1, V_2, \ldots, V_{k-1}, V_k)) = ((\cdots (V_1 \otimes V_2) \otimes \cdots) \otimes V_{k-1}) \otimes V_k \tag{5.2}$$

where all opening parentheses are placed on the left-hand side of V_1.

We are now ready to define the category $\mathcal{C}^{str}$: its objects are the elements of $\mathcal{S}$, i.e., the finite sequences of objects of $\mathcal{C}$, and its morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}^{str}}(S, S') = \mathrm{Hom}_{\mathcal{C}}(F(S), F(S')).$$

This defines a category whose identities and composition are taken from $\mathcal{C}$.

The rest of the section is devoted to the proof that $\mathcal{C}^{str}$ is a strict tensor category equivalent to $\mathcal{C}$.

Proposition XI.5.1. *The categories $\mathcal{C}^{str}$ and $\mathcal{C}$ are equivalent.*

PROOF. The map F defined above on the objects of $\mathcal{C}^{str}$ extends to a functor $F : \mathcal{C}^{str} \to \mathcal{C}$ which is the identity on morphisms, hence fully faithful. As any object in $\mathcal{C}$ is clearly isomorphic to the image under F of a sequence of length one, we see that F is essentially surjective. This proves the proposition in view of Proposition 1.5. Observe that $G(V) = (V)$ defines a functor $G : \mathcal{C} \to \mathcal{C}^{str}$ which is the inverse equivalence to F. Indeed, we have $FG = \mathrm{id}_{\mathcal{C}}$ and $\theta : GF \to \mathrm{id}_{\mathcal{C}^{str}}$ via the natural isomorphism

$$\theta(S) = \mathrm{id}_{F(S)} : GF(S) \to S.$$

□

We now equip $\mathcal{C}^{str}$ with the structure of a strict tensor category. Defining the tensor product on objects of $\mathcal{C}^{str}$ is easy: we set $S \otimes S' = S * S'$. It is clearly associative on objects.

In order to define the tensor product of two morphisms of $\mathcal{C}^{str}$, we first construct a natural isomorphism

$$\varphi(S, S') : F(S) \otimes F(S') \to F(S * S')$$

for any pair (S, S') of objects in $\mathcal{C}^{str}$. This isomorphism is defined by induction on the length of the sequence S'. First, we set $\varphi(\emptyset, S) = l_S$ and $\varphi(S, \emptyset) = r_S$. Next,

$$\varphi(S, (V)) = \mathrm{id}_{F(S) \otimes V} : F(S) \otimes V \to F(S \otimes (V))$$

and

$$\varphi(S, S' * (V)) = (\varphi(S, S') \otimes \mathrm{id}_V) \circ a^{-1}_{F(S),F(S'),V}. \tag{5.3}$$

The following lemma will be used in the proof of Theorem 5.3.

Lemma XI.5.2. *If S, S', S'' are objects on $\mathcal{C}^{str}$, we have*

$$\varphi(S, S' * S'') \circ (\mathrm{id}_S \otimes \varphi(S', S'')) \circ a_{F(S),F(S'),F(S'')}$$
$$= \varphi(S * S', S'') \circ (\varphi(S, S') \otimes \mathrm{id}_{S''}).$$

PROOF. We proceed by induction on the length of S''. If $S'' = \emptyset$, we have

$$\begin{aligned}
&\varphi(S, S')(\mathrm{id}_S \otimes \varphi(S', \emptyset))\, a_{F(S),F(S'),I} \\
&= \varphi(S, S')(\mathrm{id}_{F(S)} \otimes r_{F(S')})\, a_{F(S),F(S'),I} \\
&= \varphi(S, S')\, r_{F(S)\otimes F(S')} \\
&= r_{F(S*S')}\, (\varphi(S, S') \otimes \mathrm{id}_I) \\
&= \varphi(S * S', \emptyset)(\varphi(S, S') \otimes \mathrm{id}_I).
\end{aligned}$$

The first and last equalities are by definition, the second one by Lemma 2.2, and the third one by naturality of r.

Let V be an object of the category. Let us prove that the equality of Lemma 5.2 for the triple (S, S', S'') implies the equality for $(S, S', S'' * (V))$. We have

$$\begin{aligned}
&\varphi(S, S' * S'' * (V))(\mathrm{id}_S \otimes \varphi(S', S'' * (V)))\, a_{F(S),F(S'),F(S''*(V))} \\
&= \Big(\varphi(S, S' * S'') \otimes \mathrm{id}_V\Big)\, a^{-1}_{F(S),F(S'*S''),V} \Big(\mathrm{id}_S \otimes \varphi(S', S'') \otimes \mathrm{id}_V\Big) \\
&\qquad (\mathrm{id}_S \otimes a^{-1}_{F(S'),F(S''),V})\, a_{F(S),F(S'),F(S'')\otimes V} \\
&= \Big(\varphi(S, S' * S'') \otimes \mathrm{id}_V\Big) \Big(\mathrm{id}_S \otimes \varphi(S', S'') \otimes \mathrm{id}_V\Big)\, a^{-1}_{F(S),F(S')\otimes F(S''),V} \\
&\qquad (\mathrm{id}_S \otimes a^{-1}_{F(S'),F(S''),V})\, a_{F(S),F(S'),F(S'')\otimes V} \\
&= \Big(\varphi(S, S' * S'') \otimes \mathrm{id}_V\Big) \Big(\mathrm{id}_S \otimes \varphi(S', S'') \otimes \mathrm{id}_V\Big) \\
&\qquad (a_{F(S),F(S'),F(S'')} \otimes \mathrm{id}_V)\, a^{-1}_{F(S)\otimes F(S'),F(S''),V} \\
&= \Big(\varphi(S * S', S'') \otimes \mathrm{id}_V\Big) \Big(\varphi(S, S') \otimes \mathrm{id}_{S''} \otimes \mathrm{id}_V\Big)\, a^{-1}_{F(S)\otimes F(S'),F(S''),V} \\
&= \Big(\varphi(S * S', S'') \otimes \mathrm{id}_V\Big)\, a^{-1}_{F(S*S'),F(S''),V} \Big(\varphi(S, S') \otimes \mathrm{id}_{S''} \otimes \mathrm{id}_V\Big) \\
&= \varphi(S * S', S'' * (V))(\varphi(S, S') \otimes \mathrm{id}_{S''*(V)}).
\end{aligned}$$

The first and last equalities follow from (5.3), the second and fifth ones from the naturality of the associativity constraint, i.e., from Relation (2.5), the third from the Pentagon Axiom (2.6), and the fourth one from the induction hypothesis. □

We can now define the tensor product $f * f'$ of two morphisms $f : S \to T$ and $f' : S' \to T'$ of $\mathcal{C}^{str}$. By definition, f is a morphism from $F(S)$ to $F(T)$ and f' is another one from $F(S')$ to $F(T')$ in $\mathcal{C}$. We define the tensor product $f * g$ in $\mathcal{C}^{str}$ by the commutative square

$$\begin{array}{ccc} F(S) \otimes F(S') & \xrightarrow{\varphi(S,S')} & F(S * S') \\ \downarrow{\scriptstyle f \otimes f'} & & \downarrow{\scriptstyle f * f'} \\ F(T) \otimes F(T') & \xrightarrow{\varphi(T,T')} & F(T * T') \,. \end{array} \tag{5.4}$$

Theorem XI.5.3. *Equipped with this tensor product $\mathcal{C}^{str}$ is a strict tensor category. The categories $\mathcal{C}$ and $\mathcal{C}^{str}$ are tensor equivalent.*

PROOF. It is easy to check that the above-defined $*$ is a functor. This functor is strictly associative by construction. Therefore $\mathcal{C}^{str}$ is a strict tensor category.

In order to prove that it is tensor equivalent to $\mathcal{C}$, we have to exhibit tensor functors and natural tensor isomorphisms. We first claim that the triple $(F, \mathrm{id}_I, \varphi)$ is a tensor functor from $\mathcal{C}^{str}$ to $\mathcal{C}$ where φ is the natural isomorphism defined above. Indeed, Lemma 5.2 is a reformulation of Relation (4.1) while Relations (4.2–4.3) follow from the definition of $\varphi(S, \emptyset)$ and of $\varphi(\emptyset, S)$. The functor G of the proof of Proposition 5.1 is a strict tensor functor. Finally, the natural isomorphism θ is a natural tensor isomorphism. $\square$

Theorem 5.3 implies Mac Lane's coherence theorem which states that in a tensor category any diagram built from the constraints a, l, r, and the identities by composing and tensoring, commutes. In other words, the commutation of all such diagrams is equivalent to the commutation of the pentagon (2.6) and of the triangle (2.9).

XI.6 Exercises

1. Let I be a pre-ordered set, i.e., a set with a binary relation $\leq$ such that $x \leq x$, and $(x \leq y$ and $y \leq z) \Rightarrow x \leq z$. Set $\mathrm{Ob}(\mathcal{J}) = I$, $\mathrm{Hom}(\mathcal{J}) = \{(x, y) \in I \times I | x \leq y\}$, $s(x, y) = x$, $b(x, y) = y$, and $(y, z) \circ (x, y) = (x, z)$. Show that these data define a category $\mathcal{J}$.

2. Prove that the class of all categories form a category *Cat* whose objects are the categories and whose morphisms are the functors.

3. Prove that the class of functors form a category *Funct* whose objects are the functors and whose morphisms are the natural transformations between functors.

4. Express in terms of adjoint functors the following natural bijections:

(a)
$$\mathrm{Hom}_{Alg}(k[G], A) \cong \mathrm{Hom}_{Gr}(G, A^{\times})$$

where G is a group, k a commutative ring, A an algebra, and $A^{\times}$ the group of invertible elements in A.

(b)
$$\mathrm{Hom}_{Alg}(U(L), A) \cong \mathrm{Hom}_{Lie}(L, \mathrm{L}(A))$$

where *Lie* is the category of Lie algebras, L an object of *Lie*, A an algebra, $\mathrm{L}(A)$ the underlying Lie algebra, and $U(L)$ the enveloping algebra of L (see V.2).

(c)
$$\mathrm{Hom}_{\mathrm{Cog}}(k[X], C) \cong \mathrm{Hom}_{Set}(X, G(C))$$

where X is a set, $k[X]$ the corresponding coalgebra (see III.1, Example 3), and $G(C)$ the set of grouplike elements of C.

5. Let I be a set and $Vect^I$ the category whose objects are families $(U_i)_{i \in I}$ of vector spaces indexed by I. The set of morphisms in $Vect^I$ from $(U_i)_i$ to $(V_i)_i$ is the product set $\prod_{i \in I} \mathrm{Hom}(U_i, V_i)$. For any vector space U, we set $\Delta(U) = (U_i)_i$ where $U_i = U$ for all $i \in I$. Show that Δ defines a functor from *Vect* to $Vect^I$ and that the direct sum $\bigoplus$ and the direct product $\prod$ of vector spaces define functors $\bigoplus, \prod : Vect^I \to Vect$. Prove that the diagonal functor Δ is right adjoint to the functor $\bigoplus$ and left adjoint to the functor $\prod$.

6. Let R be the category of commutative rings without zero divisors and F the category of fields. Show that the correspondence assigning to any ring in R its field of fractions is a functor from R into F which is left adjoint to the "forgetful" functor.

7. Let G be a group and $\mathcal{G}$ be the corresponding category (as in Section 1, Example 1). For any $x \in G$ define a functor Ad_x from $\mathcal{G}$ to itself by $\mathrm{Ad}_x(g) = xgx^{-1}$. Show that there exists a natural isomorphism from the functor Ad_x to the identity functor.

8. Let $Vect_{gr}(k)$ be the category of nonnegatively graded vector spaces over a field k with linear maps of degree zero. Equip it with the graded tensor product (see Chapter III, Exercise 3). Define constraints a, l, r by
$$a((u \otimes v) \otimes w) = \alpha(m, n, p) u \otimes (v \otimes w),$$
$$l(1 \otimes v) = \lambda(n) v, \quad r(v \otimes 1) = \rho(n) v$$
where u, v, w are homogeneous vectors of respective degrees m, n, p and where α, λ, ρ are functions on $\mathbf{N}$ with values in $k \setminus \{0\}$. Show

that these constraints satisfy the Pentagon and the Triangle Axioms if and only if α, λ, ρ satisfy the functional equations

$$\alpha(n,p,q)\alpha(m+n,p,q)^{-1}\alpha(m,n+p,q)\alpha(m,n,p+q)^{-1}\alpha(m,n,p) = 1$$

and $\alpha(m,0,p) = \rho(m)\lambda(p)^{-1}$ for all integers m, n, p, q.

9. Show that the subcategory $\mathit{Vect}_f(k)_{is}$ of all finite-dimensional vector spaces of $\mathit{Vect}(k)$ with their isomorphisms is tensor equivalent to the tensor category $GL(k)$ of 3.2.

XI.7 Notes

Tensor categories were introduced in 1963 by Bénabou [Bén63]. See also [Mac63] where constraints as well as the Pentagon and the Triangle Axioms were defined. Tensor categories are also called *monoidal categories* in the literature. Our terminology is taken from Joyal and Street [JS91a] [JS93]. Lemma 2.2 is due to Kelly [Kel64]. For a proof of Mac Lane's coherence theorem, see [Mac63] [Mac71]. Exercise 8 was taken from [Kel64].

Chapter XII
The Tangle Category

The aim of this chapter is to set up a categorical construction of isotopy invariants of links. To this end, we build a strict tensor category $\mathcal{T}$ out of the tangles defined in X.5. Any strict tensor functor from $\mathcal{T}$ to a category of finite-dimensional vector spaces gives rise to an isotopy invariant. Using a presentation of $\mathcal{T}$ by generators and relations, we shall reduce in Section 4 the construction of such a functor to an algebraic data, called an enhanced R-matrix, consisting of a finite-dimensional vector space, an R-matrix, and a compatible automorphism. We shall apply this method in Section 5 to exhibit explicit isotopy invariants that will allow us to complete the proof of Theorem X.4.2 asserting the existence of the Jones-Conway polynomial.

We start with the notion of a presentation of a strict tensor category by generators and relations.

XII.1 Presentation of a Strict Tensor Category by Generators and Relations

One of the most efficient way of comprehending a group G is to present it by generators and relations. Recall the following facts: Let F be a subset of G and R be a set of pairs of words in the alphabet F. Then (F, R) is a presentation of the group G if the two following conditions are satisfied:

(i) the subset F generates G, and

(ii) two words a and b in the alphabet F represent the same element in G

if and only if one may pass from a to b by operations replacing any subword of the form c by a subword of the form d where (c,d) belongs to R.

Example 1. The abelian group $\mathbf{Z}^2$ has a presentation (F,R) where

$$F = \{x, y\} \quad \text{and} \quad R = \{(xy, yx)\}.$$

As an application of group presentations, we see we can define a group morphism by its restriction to a generating subset. In fact, let (F,R) be a presentation of a group G. Suppose we have a map $f_0 : F \to H$ with values in another group H. It extends to a multiplicative map, still denoted f_0, from the set of words in the alphabet F to H. Then there exists a unique morphism of groups $f : G \to H$ restricting to f_0 on F if and only if $f_0(c) = f_0(d)$ for any element (c,d) in R.

A similar formalism works for tensor categories. Its description is the main objective of this section.

Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category and $\mathcal{F}$ be a collection of morphisms of $\mathcal{C}$. We wish to define certain symbols which we shall call words in $\mathcal{F}$. Any word a will possess subwords and will be assigned a morphism $\overline{a}$ of $\mathcal{C}$. We say that the word a represents the morphism $\overline{a}$ of $\mathcal{C}$.

By definition, a word of rank 1 is a symbol of the form $[f]$ where f is an element of $\mathcal{F}$ or of the form $[\mathrm{id}_V]$ where V is an object of $\mathcal{C}$. We define the morphism of $\mathcal{C}$ represented by such symbols by $\overline{[f]} = f$ and $\overline{[\mathrm{id}_V]} = \mathrm{id}_V$. By definition, a subword of a word of rank 1 is the word itself.

Suppose defined all words of rank $\leq n$ where $n \geq 1$, the morphisms they represent and their subwords. Define a word of rank $n+1$ as a symbol of the form $a \circ b$ or of the form $a \otimes b$ where a and b are words of rank $\leq n$. We define the corresponding morphism by setting

$$\overline{a \circ b} = \overline{a} \circ \overline{b} \quad \text{and} \quad \overline{a \otimes b} = \overline{a} \otimes \overline{b} \tag{1.1}$$

where the symbols $\circ$ and $\otimes$ in the right-hand sides denote the composition and the tensor product in the tensor category $\mathcal{C}$ respectively. The subwords of $a \circ b$ and those of $a \otimes b$ consist of the word itself, the subwords of a and those of b.

The class of words in $\mathcal{F}$ is the union of all words of positive rank. We introduce an equivalence relation on words.

Definition XII.1.1. *Two words a and a' in $\mathcal{F}$ are equivalent if there exist words $a_0 = a, a_1, \ldots, a_k = a'$ such that for all i, the word a_{i+1} is obtained from a_i by replacing a subword x of one of them by a subword y of the other where x and y are the two sides of any of the following relations:*

$$([f] \circ [g]) \circ [h] \sim [f] \circ ([g] \circ [h]), \tag{1.2}$$

$$[\mathrm{id}_{b(f)}] \circ [f] \sim [f], \quad [f] \circ [\mathrm{id}_{s(f)}] \sim [f], \tag{1.3}$$

$$[\mathrm{id}_V] \circ [\mathrm{id}_V] \sim [\mathrm{id}_V], \tag{1.4}$$

$$([f]\otimes[g])\otimes[h]\sim[f]\otimes([g]\otimes[h]), \tag{1.5}$$

$$[\mathrm{id}_I]\otimes[f]\sim[f],\quad [f]\otimes[\mathrm{id}_I]\sim[f], \tag{1.6}$$

$$[\mathrm{id}_V]\otimes[\mathrm{id}_W]\sim[\mathrm{id}_{V\otimes W}], \tag{1.7}$$

$$([f]\otimes[g])\circ([f']\otimes[g'])\sim([f]\circ[f'])\otimes([g]\circ[g']) \tag{1.8}$$

where V,W are objects of $\mathcal{C}$ and f,f',g,g',h are elements of $\mathcal{F}$.

We write $a\sim b$ if a and b are equivalent words. Observe that if $a\sim b$, then $\overline{a}=\overline{b}$ holds for the corresponding morphisms in $\mathcal{C}$. The following lemma gives examples of equivalent words.

Lemma XII.1.2. (a) *If $f,g\in\mathcal{F}$, then*

$$([f]\otimes[\mathrm{id}_{b(g)}])\circ([\mathrm{id}_{s(f)}]\otimes[g])\sim([\mathrm{id}_{b(f)}\otimes[g])\circ([f]\otimes[\mathrm{id}_{s(g)}]).$$

(b) *If $f_1,f_2,\dots,f_k\in\mathcal{F}$ are morphisms such that $b(f_i)=s(f_{i+1})$ for all i, then*

$$([\mathrm{id}_V]\otimes[f_1]\otimes[\mathrm{id}_W])\circ([\mathrm{id}_V]\otimes[f_2]\otimes[\mathrm{id}_W])\circ\cdots\circ([\mathrm{id}_V]\otimes[f_k]\otimes[\mathrm{id}_W])$$

is equivalent to $[\mathrm{id}_V]\otimes([f_1]\circ[f_2]\circ\cdots\circ[f_k])\otimes[\mathrm{id}_W]$.

(c) *Any word in $\mathcal{F}$ is equivalent to a word of the form $[\mathrm{id}_V]$ or of the form*

$$([\mathrm{id}_{V_1}]\otimes[f_1]\otimes[\mathrm{id}_{W_1}])\circ([\mathrm{id}_{V_2}]\otimes[f_2]\otimes[\mathrm{id}_{W_2}])\circ\cdots\circ([\mathrm{id}_{V_k}]\otimes[f_k]\otimes[\mathrm{id}_{W_k}]).$$

PROOF. (a) By (1.3) and (1.8) we have the equivalences

$$\begin{aligned}([f]\otimes[\mathrm{id}_{b(g)}])\circ([\mathrm{id}_{s(f)}]\otimes[g]) &\sim ([f]\circ[\mathrm{id}_{s(f)}])\otimes([\mathrm{id}_{b(g)}]\circ[g])\\ &\sim [f]\otimes[g]\\ &\sim ([\mathrm{id}_{b(f)}]\circ[f])\otimes([g]\circ[\mathrm{id}_{s(g)}])\\ &\sim ([\mathrm{id}_{b(f)}]\otimes[g])\circ([f]\otimes[\mathrm{id}_{s(g)}]).\end{aligned}$$

(b) We proceed by induction on k. For $k=1$, the statement is clear. For $k>1$ the induction hypothesis, Relations (1.2), (1.4), (1.8), and Part (a) imply that

$$\begin{aligned}&([\mathrm{id}_V]\otimes[f_1]\otimes[\mathrm{id}_W])\circ([\mathrm{id}_V]\otimes[f_2]\otimes[\mathrm{id}_W])\circ\cdots\circ([\mathrm{id}_V]\otimes[f_k]\otimes[\mathrm{id}_W])\\ &\sim \Big([\mathrm{id}_V]\otimes([f_1]\circ[f_2]\circ\cdots\circ[f_{k-1}])\otimes[\mathrm{id}_W]\Big)\circ([\mathrm{id}_V]\otimes[f_k]\otimes[\mathrm{id}_W])\\ &\sim\Big(\Big([\mathrm{id}_V]\otimes([f_1]\circ[f_2]\circ\cdots\circ[f_{k-1}])\Big)\otimes[\mathrm{id}_W]\Big)\circ\Big(([\mathrm{id}_V]\otimes[f_k])\otimes[\mathrm{id}_W]\Big)\\ &\sim\Big(([\mathrm{id}_V]\otimes([f_1]\circ[f_2]\circ\cdots\circ[f_{k-1}]))\circ([\mathrm{id}_V]\otimes[f_k])\Big)\otimes([\mathrm{id}_W]\circ[\mathrm{id}_W])\\ &\sim\ [\mathrm{id}_V]\otimes([f_1]\circ[f_2]\circ\cdots\circ[f_k])\otimes[\mathrm{id}_W].\end{aligned}$$

(c) We prove the assertion by induction on the rank of words. When a word is of rank 1, then it is $[f]$ where $f = \mathrm{id}_V$ or $f \in \mathcal{F}$. In both cases it is equivalent to $[\mathrm{id}_I] \otimes [f] \otimes [\mathrm{id}_I]$. Suppose the assertion proved for all words of rank $\leq n$. Let a be a word of rank $\leq n+1$. If $a = b \circ c$, then, by the induction hypothesis, the words b and c are equivalent to identities or to a word of the above form. In both cases, a is equivalent to an identity or to a word of the desired form.

Now consider the case when $a = b \otimes c$. Let us restrict to the interesting case where b and c are not equivalent to identities. Then,

$$b \sim b_1 \circ \ldots \circ b_k \quad \text{and} \quad c \sim c_1 \circ \ldots \circ c_\ell$$

where the words $b_1 \ldots b_k, c_1, \ldots, c_\ell$ are of the form $[\mathrm{id}_V] \circ [f] \circ [\mathrm{id}_W]$ for some $f \in \mathcal{F}$. Set $S = s(\overline{b_k})$ and $T = b(\overline{c_1})$. Then by (1.3) and Part(a) we get

$$\begin{aligned} a = b \otimes c \;&\sim\; (b \circ [\mathrm{id}_S]^{\circ \ell}) \otimes ([\mathrm{id}_T]^{\circ k} \circ c) \\ &\sim\; (b_1 \otimes [\mathrm{id}_T]) \circ \ldots \circ (b_k \otimes [\mathrm{id}_T]) \circ ([\mathrm{id}_S] \otimes c_1) \circ \ldots \circ ([\mathrm{id}_S] \otimes c_\ell), \end{aligned}$$

which is equivalent to the desired form in view of

$$b_i \otimes [\mathrm{id}_T] \sim [\mathrm{id}_V] \circ [f] \circ [\mathrm{id}_{W \otimes T}] \quad \text{and} \quad [\mathrm{id}_S] \otimes c_i \sim [\mathrm{id}_{S \otimes V}] \circ [f] \circ [\mathrm{id}_W]$$

for some $f, f' \in \mathcal{F}$. The last two equivalences follow from (1.7). □

Composing and tensoring words are operations that are compatible with the above-defined equivalence relation. Denote by $\mathcal{M}(\mathcal{F})$ the class of equivalence classes of words in $\mathcal{F}$. We define a strict tensor category $\mathcal{C}(\mathcal{F})$ as follows. The objects of $\mathcal{C}(\mathcal{F})$ are the objects of $\mathcal{C}$ whereas $\mathcal{M}(\mathcal{F})$ is the class of morphisms in $\mathcal{C}(\mathcal{F})$. The identity, source, and target maps for $\mathcal{C}(\mathcal{F})$ are given by

$$\mathrm{id}_V = [\mathrm{id}_V], \quad s(a) = s(\overline{a}), \quad b(a) = b(\overline{a}).$$

The composition and the tensor product of words have already been defined.

The map sending a word a to the morphism $\overline{a}$ of $\mathcal{C}$ is a strict tensor functor from $\mathcal{C}(\mathcal{F})$ to $\mathcal{C}$. When this functor is an equivalence of categories, we say that the strict tensor category $\mathcal{C}$ is *free* on the class $\mathcal{F}$. In view of Proposition XI.1.5, this is equivalent to

$$a \sim b \quad \Longleftrightarrow \quad \overline{a} = \overline{b}$$

for any pair (a, b) of words in $\mathcal{F}$.

We also say that $\mathcal{F}$ *generates* the strict tensor category $\mathcal{C}$ if any morphism in $\mathcal{C}$ can be obtained from morphisms in $\mathcal{F}$ and from identities of $\mathcal{C}$ by applying finitely many times the operations of composing and tensoring.

If $\mathcal{F}$ generates $\mathcal{C}$, then any morphism of $\mathcal{C}$ is of the form $\overline{a}$ where a is an element of $\mathcal{M}(\mathcal{F})$.

We now wish to introduce further relations on $\mathcal{M}(\mathcal{F})$. In addition to $\mathcal{F}$, we also choose a collection $\mathcal{R}$ of pairs (c,d) of words in $\mathcal{F}$ such that $\overline{c} = \overline{d}$ in $\mathcal{C}$. Using $\mathcal{R}$ we may put a new equivalence relation on $\mathcal{M}(\mathcal{F})$. Given two elements a, a' of $\mathcal{M}(\mathcal{F})$, we say that a and a' are *congruent* modulo $\mathcal{R}$ — we write $a \sim_{\mathcal{R}} a'$ — if there exist words $a_0 = a, a_1, \ldots, a_k = a'$ such that for all i, a_{i+1} is obtained from a_i by replacing a subword c of one of them by a subword d of the other one where (c,d) is an element of $\mathcal{R}$.

We are now ready to define the presentation of a strict tensor category by generators and relations.

Definition XII.1.3. *The strict tensor category $\mathcal{C}$ is generated by $\mathcal{F}$ and by the relations $\mathcal{R}$ if*

(a) *the set $\mathcal{F}$ generates $\mathcal{C}$, and*

(b) *for any pair (a,a') of elements of $\mathcal{M}(\mathcal{F})$ we have the equivalence*

$$a \sim_{\mathcal{R}} a' \iff \overline{a} = \overline{a'}.$$

The main interest of this definition lies in the following proposition stating under which conditions one can define a functor on $\mathcal{C}$ by its restriction to the generating set $\mathcal{F}$.

Proposition XII.1.4. *Let $\mathcal{C}$ be a strict tensor category generated by the family of morphisms $\mathcal{F}$ and the relations $\mathcal{R}$. Suppose given a strict tensor category $\mathcal{D}$, a map $F_0 : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ such that $F_0(I) = I$ and*

$$F_0(V \otimes V') = F_0(V) \otimes F_0(V')$$

for all couples (V, V') of objects of $\mathcal{C}$, and a morphism g_f from $F_0(s(f))$ to $F_0(b(f))$ for any morphism $f \in \mathcal{F}$. Then there exists a unique strict tensor functor $F : \mathcal{C} \to \mathcal{D}$ such that we have $F(V) = F_0(V)$ for any object V of $\mathcal{C}$ and $F(f) = g_f$ for any morphism f in $\mathcal{F}$, if and only if for any couple (c,d) of $\mathcal{R}$ we obtain equal morphisms in $\mathcal{D}$ after replacing any subword $[f]$ $(f \in \mathcal{F})$ of c and d by g_f, and any subword $[\mathrm{id}_V]$ by $\mathrm{id}_{F_0(V)}$.

PROOF. The implication $\Rightarrow$ is clear since the words c and d represent the same morphism in $\mathcal{C}$. Therefore their images under F obtained after performing the substitutions indicated above are identical.

Let us prove the reverse implication. The uniqueness of F follows from the fact that the family $\mathcal{F}$ generates $\mathcal{C}$. It remains essentially to define F on the morphisms of $\mathcal{C}$. Now any morphism in $\mathcal{C}$ can be represented by an element $a \in \mathcal{M}(\mathcal{F})$. We define $F_1(a)$ by replacing any subword $[f](f \in \mathcal{F})$ of the word a by g_f and any subword $[\mathrm{id}_V]$ by $\mathrm{id}_{F_0(V)}$. By definition of the presentation of $\mathcal{C}$, the words a and a' represent the same morphism in $\mathcal{C}$ if and only if a and a' are congruent modulo $\mathcal{R}$. The substitution law stated in Proposition 1.4 implies that, if a and a' are congruent modulo $\mathcal{R}$, then $F_1(a) = F_1(a')$. Setting $F(\overline{a}) = F_1(a)$ defines F unambiguously on the morphisms of $\mathcal{C}$. $\square$

Proposition 1.4 will be used as an essential tool in Section 4. We end this section with a technical result. Suppose that the strict tensor category $\mathcal{C}$ is generated by a set $\mathcal{F}$ and the relations $\mathcal{R}$. Suppose also that there exists a subset $\mathcal{F}'$ of $\mathcal{F}$ such that any $f \in \mathcal{F}'$ is congruent modulo $\mathcal{R}$ to a word $a(f)$ in $\mathcal{M}(\mathcal{F}_0)$ where $\mathcal{F}_0 = \mathcal{F} \setminus \mathcal{F}'$. Denote by $\mathcal{R}_0$ the collection of pairs (c, d) of words in $\mathcal{F}_0$ obtained by replacing any $f \in \mathcal{F}'$ by the word $a(f)$ in all pairs of words of $\mathcal{R}$.

Lemma XII.1.5. *Under the previous hypothesis, the tensor category $\mathcal{C}$ is also generated by the set $\mathcal{F}_0$ and the relations $\mathcal{R}_0$.*

PROOF. Clearly, the set $\mathcal{F}_0$ generates the category, and if $a, a' \in \mathcal{M}(\mathcal{F}_0)$ are congruent modulo $\mathcal{R}_0$, then they are congruent modulo $\mathcal{R}$, which implies $\overline{a} = \overline{a'}$. Conversely, if $\overline{a} = \overline{a'}$, then by definition a and a' are congruent modulo $\mathcal{R}$. Now we may replace any $f \in \mathcal{F}'$ by $a(f) \in \mathcal{M}(\mathcal{F}_0)$ in these congruences, which yields congruences in $\mathcal{R}_0$ between a and a'. $\square$

XII.2 The Category of Tangles

We defined the concept of tangles and of isotopy classes of tangles in X.5. The reason why we introduced these one-dimensional objects is that tangles form a strict tensor category $\mathcal{T}$ as follows. By definition, the objects of $\mathcal{T}$ consist of finite sequences of $\pm$ signs, including the empty sequence $\emptyset$, and the morphisms of $\mathcal{T}$ are the isotopy classes of oriented tangles. For any oriented tangle L, the sequences $s(L)$ and $b(L)$ defined in X.5 will be the source and the target of L respectively. The identity $\mathrm{id} : \mathrm{Ob}(\mathcal{T}) \to \mathrm{Hom}(\mathcal{T})$ is defined by the following rules: $\mathrm{id}_\emptyset$ is the empty set $\emptyset$; if ε is a finite sequence of length n in $\mathrm{Ob}(\mathcal{T})$, we define id_ε as the isotopy class of the tangle L formed by the union of intervals $\{1, 2, \ldots, n\} \times \{0\} \times [0, 1]$. The orientation of these intervals is determined by the rule $s(\mathrm{id}_\varepsilon) = b(\mathrm{id}_\varepsilon) = \varepsilon$. The composition of tangles introduced in X.5 defines the composition in $\mathcal{T}$. Recall that $L' \circ L$ is obtained by placing L' on top of L. Lemmas X.5.10–5.11 imply that $\mathcal{T}$ is a category with identity maps id_ε.

We equip $\mathcal{T}$ with a tensor product. It is defined on objects by concatenation of sequences, i.e., if $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ and $\varepsilon' = (\varepsilon_{k+1}, \ldots, \varepsilon_\ell)$ are objects of $\mathcal{T}$, then their tensor product is given by

$$\varepsilon \otimes \varepsilon' = (\varepsilon_1, \ldots, \varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_\ell).$$

We also set $\emptyset \otimes \varepsilon = \varepsilon = \varepsilon \otimes \emptyset$. This operation is clearly associative. Let us now define the tensor product on the morphisms of $\mathcal{T}$. If L and L' are isotopy classes of oriented tangles, $L \otimes L'$ is the isotopy class of the oriented tangle obtained by placing L' to the right of L as in Figure 2.1.

$$\boxed{L} \otimes \boxed{L'} = \boxed{L}\ \boxed{L'}$$

Figure 2.1. The isotopy class $L \otimes L'$

This operation is well-defined up to isotopy, is associative on isotopy classes of tangles and defines a functor from $\mathcal{T} \times \mathcal{T}$ to $\mathcal{T}$. This is summarized in the next proposition.

Proposition XII.2.1. *The tangle category $\mathcal{T}$ equipped with the tensor product defined above is a strict tensor category in which the unit I is the empty set $\emptyset$.*

Observe that the endomorphisms of the unit object $\emptyset$ are exactly the tangles without boundaries, i.e., the links in the space $\mathbf{R}^2 \times]0,1[$. This observation will be crucial in Sections 4–5.

We now state the main theorem of this section. It involves the "elementary" tangles defined by (X.5.1–5.5). We shall also use the following conventions: $\downarrow = \mathrm{id}_{(+)}$ and $\uparrow = \mathrm{id}_{(-)}$, and XY is short for $X \otimes Y$ when X and Y are elements of the generating set below.

Theorem XII.2.2. *The strict tensor category $\mathcal{T}$ is generated by the six morphisms*

$$\cup, \overleftarrow{\cup}, \cap, \overleftarrow{\cap}, X_+, X_-,$$

and the relations

$$(\downarrow \cap) \circ (\cup \downarrow) = \downarrow = (\overleftarrow{\cap} \downarrow) \circ (\downarrow \overleftarrow{\cup}), \tag{2.1}$$

$$(\uparrow \overleftarrow{\cap}) \circ (\overleftarrow{\cup} \uparrow) = \uparrow = (\cap \uparrow) \circ (\uparrow \cup), \tag{2.2}$$

$$\begin{aligned} &(\cap \uparrow\uparrow) \circ (\uparrow \cap \downarrow\uparrow\uparrow) \circ (\uparrow\uparrow X_\pm \uparrow\uparrow) \circ (\uparrow\uparrow\downarrow \cup \uparrow) \circ (\uparrow\uparrow \cup) \\ &= (\uparrow\uparrow \overleftarrow{\cap}) \circ (\uparrow\uparrow\downarrow \overleftarrow{\cap} \uparrow) \circ (\uparrow\uparrow X_\pm \uparrow\uparrow) \circ (\uparrow \overleftarrow{\cup} \downarrow\uparrow\uparrow) \circ (\overleftarrow{\cup} \uparrow\uparrow), \end{aligned} \tag{2.3}$$

$$X_+ \circ X_- = X_- \circ X_+ = \downarrow\downarrow, \tag{2.4}$$

$$(X_+ \downarrow) \circ (\downarrow X_+) \circ (X_+ \downarrow) = (\downarrow X_+) \circ (X_+ \downarrow) \circ (\downarrow X_+), \tag{2.5}$$

$$(\downarrow \overleftarrow{\cap}) \circ (X_\pm \uparrow) \circ (\downarrow \cup) = \downarrow, \tag{2.6}$$

$$(\cap \downarrow\uparrow) \circ (\uparrow X_\mp \uparrow) \circ (\uparrow\downarrow \cup) \circ (\uparrow\downarrow \overleftarrow{\cap}) \circ (\uparrow X_\pm \uparrow) \circ (\overleftarrow{\cup} \downarrow\uparrow) = \downarrow\uparrow, \tag{2.7}$$

$$(\uparrow\downarrow \overleftarrow{\cap}) \circ (\uparrow X_\pm \uparrow) \circ (\overleftarrow{\cup} \downarrow\uparrow) \circ (\cap \downarrow\uparrow) \circ (\uparrow X_\mp \uparrow) \circ (\uparrow\downarrow \cup) = \uparrow\downarrow . \tag{2.8}$$

The proof of Theorem 2.2 will be given at the end of Section 3. Figures 2.2–2.9 illustrate Relations (2.1–2.8).

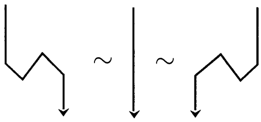


Figure 2.2. Relation (2.1)

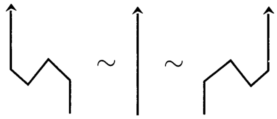


Figure 2.3. Relation (2.2)

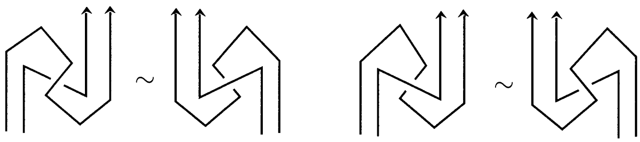


Figure 2.4. Relation (2.3)

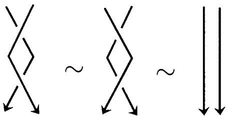


Figure 2.5. Relation (2.4)

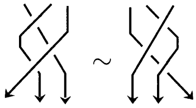


Figure 2.6. Relation (2.5)

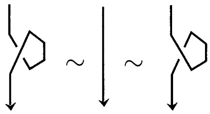


Figure 2.7. Relation (2.6)

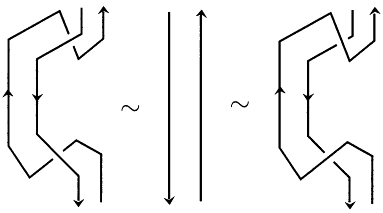


Figure 2.8. Relation (2.7)

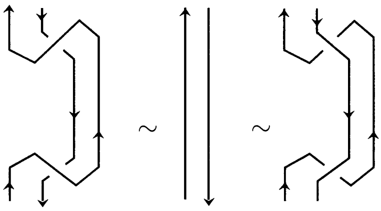


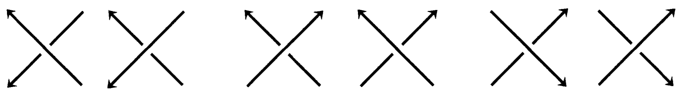
Figure 2.9. Relation (2.8)

These relations are valid in the tangle category as a consequence of Lemma X.5.7 and Theorem X.5.9. More precisely, Relations (2.1–2.3) follow purely from isotopies of diagrams, Relation (2.4), Relation (2.5), and Relation (2.6) follow from isotopies of diagrams and from Reidemeister Transformations (II), (III), and (I), respectively, whereas Relations (2.7–2.8) follow from isotopies of diagrams and from Reidemeister Transformation (II).

XII.3 The Category of Tangle Diagrams

In order to prove Theorem 2.2, we introduce a strict tensor category $\mathcal{D}$ of tangle diagrams and give a presentation of it by generators and relations. The category $\mathcal{D}$ is defined as the tangle category $\mathcal{T}$ of Section 2, but with tangles in $\mathbf{R}^2 \times [0,1]$ replaced by tangle diagrams in $\mathbf{R} \times [0,1]$. More precisely, the objects of $\mathcal{D}$ are the same as the objects of $\mathcal{T}$, namely finite sequences of $\pm$ signs. The morphisms of $\mathcal{D}$ are isotopy classes of tangle diagrams in $\mathbf{R} \times [0,1]$ as defined in Chapter X. Identity, source, target, composition, and tensor product are defined as for the tangle category. We thus obtain a strict tensor category $\mathcal{D}$. The tangle category $\mathcal{T}$ is, roughly speaking, the quotient of $\mathcal{D}$ by the Reidemeister Transformations (I–III).

Let us introduce more "elementary" tangle diagrams as in Figure 3.1. They differ from the tangles $X_\pm$ only by their orientations.



Y_+ Y_- Z_+ Z_- T_+ T_-

Figure 3.1. Six elementary tangle diagrams

Lemma XII.3.1. *The following relations hold in the category $\mathcal{D}$:*

$$Y_\pm = (\uparrow\downarrow \overleftarrow{\cap}) \circ (\uparrow X_\pm \uparrow) \circ (\overleftarrow{\cup} \downarrow\uparrow), \tag{3.1}$$

$$T_\pm = (\cap \downarrow\uparrow) \circ (\uparrow X_\pm \uparrow) \circ (\uparrow\downarrow \cup), \tag{3.2}$$

$$Z_\pm = (\cap \uparrow\uparrow) \circ (\uparrow \cap \downarrow\uparrow\uparrow) \circ (\uparrow\uparrow X_\pm \uparrow\uparrow) \circ (\uparrow\uparrow\downarrow \cup \uparrow) \circ (\uparrow\uparrow \cup), \tag{3.3}$$

$$Z_\pm = (\uparrow\uparrow \overleftarrow{\cap}) \circ (\uparrow\uparrow\downarrow \overleftarrow{\cap} \uparrow) \circ (\uparrow\uparrow X_\pm \uparrow\uparrow) \circ (\uparrow \overleftarrow{\cup} \downarrow\uparrow\uparrow) \circ (\overleftarrow{\cup} \uparrow\uparrow). \tag{3.4}$$

For a proof, see Figure X.3.10. The following statement gives a presentation by generators and relations of the category of tangle diagrams.

Proposition XII.3.2. *The strict tensor category $\mathcal{D}$ is generated by the twelve morphisms $\cup, \overleftarrow{\cup}, \cap, \overleftarrow{\cap}, X_+, X_-, Y_+, Y_-, Z_+, Z_-, T_+$ and T_-, and the Relations* (2.1), (2.2), (3.1), (3.2), (3.3) *and* (3.4).

PROOF. (a) Let $\mathcal{F}$ be the set of morphisms listed in Proposition 3.2. We first have to prove that $\mathcal{F}$ generates $\mathcal{D}$. Let Π be a generic tangle diagram. Draw a horizontal line through any vertex of Π that is not a local maximum or minimum. This divides $\mathbf{R}\times[0,1]$ into strips such that the restriction of Π to each of them involves only a crossing or a local maximum or minimum, i.e., it is of the form $\mathrm{id}\otimes f\otimes \mathrm{id}$ where f belongs to the set $\mathcal{F}$. In the category $\mathcal{D}$ the diagram Π is the composition of these restrictions. This presentation of Π is unique, i.e., to any generic diagram Π we can assign a unique word a_Π in the alphabet $\mathcal{F}$ such that $\overline{a_\Pi}=\Pi$. Since by X.5 any tangle diagram is isotopic to a generic one, we conclude that $\mathcal{F}$ generates the diagram category.

(b) According to Definition 1.3, we have to check that, given any pair (a,a') of words in $\mathcal{F}$, we have $a\sim_{\mathcal{R}} a' \Leftrightarrow \overline{a}=\overline{a'}$ where $\mathcal{R}$ is the set of relations in Proposition 3.2. By the results of Chapter X, we already know that equivalent words (as defined in Definition 1.1) represent isotopic tangle diagrams. Similarly, Relations (2.1–2.2) and (3.1–3.4) give rise to isotopic diagrams (see Figure 2.2 and Figure X.3.10).

Let now (a,a') be a pair of words in $\mathcal{F}$ such that $\overline{a}=\overline{a'}$. By Lemma 1.2 (c) we may suppose that a and a' are of the form

$$([\mathrm{id}_{S_1}]\otimes[f_1]\otimes[\mathrm{id}_{T_1}])\circ([\mathrm{id}_{S_2}]\otimes[f_2]\otimes[\mathrm{id}_{T_2}])\circ\cdots\circ([\mathrm{id}_{S_k}]\otimes[f_k]\otimes[\mathrm{id}_{T_k}]).$$

Geometrically, this means that $\overline{a}=\Pi$ and $\overline{a'}=\Pi'$ are generic tangle diagrams and that $a=a_\Pi$ and $a'=a_{\Pi'}$ where we use the notation introduced in Part (a) of this proof. By assumption, Π and Π' are isotopic diagrams. Thus, they can be obtained from each other by a finite sequence of operations taken from the Transformations (A), (B), (C), and (E) of Lemma X.5.7. In order to show that the words a_Π and $a_{\Pi'}$ are congruent modulo $\mathcal{R}$, it is therefore enough to check that the above-mentioned transformations do not change the congruence class of words. Let us verify this case by case.

(A) If Π and Π' are generically isotopic, then $a_\Pi=a_{\Pi'}$.

(B) If Π and Π' differ by a Transformation (B), then $a_\Pi\sim a_{\Pi'}$ in view of Relation (1.8).

(C) If Π differs from Π' by a Reidemeister Transformation (0), then $a_\Pi\sim_{\mathcal{R}} a_{\Pi'}$ thanks to (2.1–2.2).

(E) This case is taken care of by Relations (3.1–3.4). □

Corollary XII.3.3. *The strict tensor category $\mathcal{D}$ is generated by the six morphisms* $\cup$, $\overleftarrow{\cup}$, $\cap$, $\overleftarrow{\cap}$, X_+, X_-, *and Relations* (2.1), (2.2), (2.3).

PROOF. By Lemma 3.1, $\mathcal{D}$ is generated by the previous set of six morphisms. We now apply Lemma 1.5 to Proposition 3.2: Relations (3.1–3.2) vanish whereas Relations (3.3–3.4) give rise to Relation (2.3). □

Proof of Theorem 2.2. It will be similar to the proof of Proposition 3.2. Since any tangle may be represented up to isotopy by a generic tangle diagram, Corollary 3.3 implies that $\mathcal{T}$ is generated by the set $\mathcal{F}_0$ of the six morphisms listed in Theorem 2.2.

Let a and a' be words in $\mathcal{F}_0$ such that $\overline{a}$ and $\overline{a'}$ are isotopic tangles. By Theorem X.5.9 one can pass from $\overline{a}$ to $\overline{a'}$ by a finite number of operations consisting of isotopies of diagrams and Reidemeister Transformations (I), (II), and (III). Corollary 3.3 implies that isotopies of diagrams do not affect the congruence class modulo (2.1–2.3) of a word. In order to complete the proof of Theorem 2.2, it is therefore enough to check that Reidemeister Transformations (I), (II), and (III) also leave the congruence classes unaltered.

Let us start with Transformation (II): It suffices to check that words of type $L_\pm \circ L_\mp$ are congruent to $||$ with the right orientation where L is of the form X, Y, Z, T. When $L = X$, this follows from Relation (2.4). When $L = Z$, it follows from the operations performed in Figure 3.2: the first and last ones are isotopies of diagrams, the second one is the Reidemeister Transformation (II) represented by Relation (2.4).

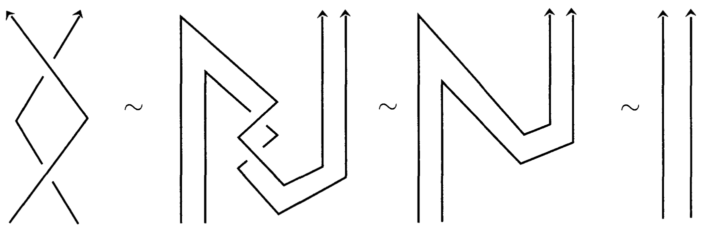


Figure 3.2. Congruence when $L = Z$

When $L = Y$ or T, it follows from Relations (2.7–2.8) as shown in Figure 3.3.

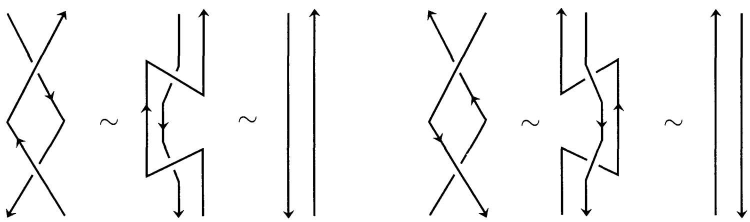


Figure 3.3. Congruence when $L = Y$ or T

We now deal with Transformation (III): When all strands are oriented downwards, it follows from Relation (2.5) and its inverse. In the remaining cases, one proceeds by reducing to the previous case as for Transformation (II) above. For details, see [Tur94], I.4.5.

Let us take care of Transformation (I). If the diagrams representing this transformation are oriented downwards, then the desired conclusion follows from Relation (2.6). We are therefore left with the same diagrams oriented upwards and we have to check that the corresponding words are congruent modulo the relations (2.1–2.8). This follows from the operations performed in Figure 3.4: the first, third, fifth, and seventh ones are isotopies of diagrams, the second one is the Reidemeister Transformation (I) already considered, the fourth one is a Reidemeister Transformation (III), the sixth one is a Reidemeister Transformation (II) applied twice. □

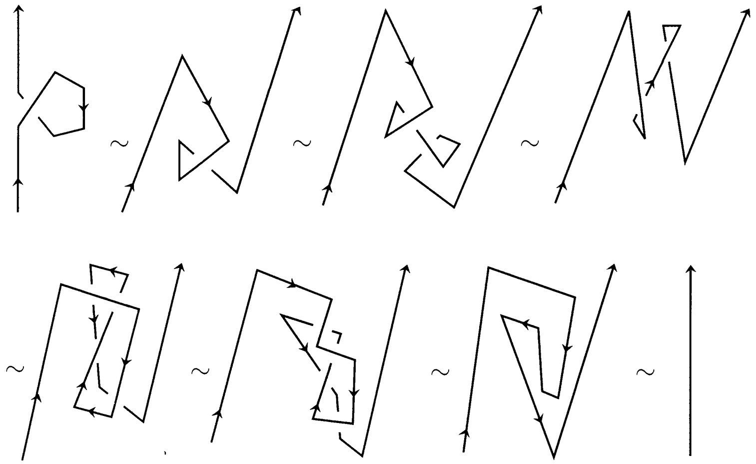


Figure 3.4. Proof for Transformation (I) "upwards"

XII.4 Representations of the Category of Tangles

In XI.5 we constructed a strict tensor category $\mathcal{C}^{str}$ from any tensor category $\mathcal{C}$. Applying this construction to the category $Vect_f(k)$ of finite-dimensional vector spaces over a field k, we get a strict tensor category $\mathcal{V}$.

We define a *representation of the tangle category* $\mathcal{T}$ to be a strict tensor functor F from the tangle category $\mathcal{T}$ into the strict tensor category $\mathcal{V}$. The main interest of this concept comes from the fact that each representation F of $\mathcal{T}$ produces an isotopy invariant for oriented links with values in the field k. Indeed, let L be an oriented link in $\mathbf{R}^2\times]0,1[$ (this space is diffeomorphic to $\mathbf{R}^3$). As we observed in Section 2, we may consider L as an endomorphism of the unit $\emptyset$ of the tangle category. Therefore, the image $F(L)$ of L under the strict tensor functor F is a k-linear endomorphism of the unit of the category $\mathcal{V}$, which is the ground field k. In other words, $F(L)$ is the multiplication by a scalar. By definition of the tangle category, this scalar depends only on the isotopy class of L.

This method of producing isotopy invariants of links is interesting in so far as we can construct representations of the tangle category in a systematic way. This will be achieved in this section using the presentation of $\mathcal{T}$ given in Theorem 2.2.

Definition XII.4.1. *Let V be a finite-dimensional vector space. An enhanced R-matrix on V is a pair (c, μ) where c is an automorphism of $V \otimes V$ satisfying the Yang-Baxter equation and μ is an automorphism of V such that*

$$c(\mu \otimes \mu) = (\mu \otimes \mu)c, \tag{4.1.a}$$

$$\mathrm{tr}_2\Big(c^{\pm 1}(\mathrm{id}_V \otimes \mu)\Big) = \mathrm{id}_V, \tag{4.1.b}$$

$$(\tau c^{\mp 1})^+(\mathrm{id}_{V^*} \otimes \mu)(c^{\pm 1}\tau)^+(\mathrm{id}_{V^*} \otimes \mu^{-1}) = \mathrm{id}_{V^* \otimes V} \tag{4.1.c}$$

where $\tau = \tau_{V,V}$.

Here we made use of the partial transpose and of the partial trace defined in II.3. We shall also use the evaluation maps ev_V, ev_{V^*} and the coevaluation maps δ_V, δ_{V^*} of II.3 where we identify any finite-dimensional vector space with its bidual. We are now ready to state the main theorem of this section.

Theorem XII.4.2. *Given an enhanced R-matrix (c, μ) on a finite-dimensional vector space V, there exists a unique strict tensor functor F from the tangle category $\mathcal{T}$ to $\mathcal{V}$ such that $F((+)) = V$, $F((-)) = V^*$, and*

$$F(X_+) = c, \quad F(\cup) = \delta_V, \quad F(\overleftarrow{\cup}) = (\mathrm{id}_{V^*} \otimes \mu^{-1})\delta_{V^*}. \tag{4.2.a}$$

Then we necessarily have

$$F(X_-) = c^{-1}, \quad F(\cap) = \mathrm{ev}_V, \quad F(\overleftarrow{\cap}) = \mathrm{ev}_{V^*}(\mu \otimes \mathrm{id}_{V^*}). \tag{4.2.b}$$

There is a converse statement to Theorem 4.2 so as to have a bijective correspondence between representations of the tangle category and enhanced R-matrices. We shall not formulate it.

Proof. Let F be a strict tensor functor from $\mathcal{T}$ to $\mathcal{V}$. Set $F((+)) = V$, $F((-)) = W$, and

$$F(\cup) = b : k \to V \otimes W, \quad F(\overleftarrow{\cup}) = b' : k \to W \otimes V, \tag{4.2.c}$$

$$F(\cap) = d : W \otimes V \to k, \quad F(\overleftarrow{\cap}) = d' : V \otimes W \to k, \tag{4.2.d}$$

$$F(X_+) = c = c^+, \quad F(X_-) = c^- : V \otimes V \to V \otimes V. \tag{4.2.e}$$

By Theorem 2.2, the above six linear maps are related by the relations obtained by applying F to (2.1–2.8), namely we have

$$(\mathrm{id}_V \otimes d)(b \otimes \mathrm{id}_V) = \mathrm{id}_V = (d' \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes b'), \tag{4.3.a}$$

$$(\mathrm{id}_W \otimes d')(b' \otimes \mathrm{id}_W) = \mathrm{id}_W = (d \otimes \mathrm{id}_W)(\mathrm{id}_W \otimes b), \tag{4.3.b}$$

$$\begin{aligned}
&(d \otimes \mathrm{id}_{W\otimes W})(\mathrm{id}_W \otimes d \otimes \mathrm{id}_{V\otimes W\otimes W})(\mathrm{id}_{W\otimes W} \otimes c^{\pm} \otimes \mathrm{id}_{W\otimes W})\\
&\qquad (\mathrm{id}_{W\otimes W\otimes V} \otimes b \otimes \mathrm{id}_W)(\mathrm{id}_{W\otimes W} \otimes b)\\
&= (\mathrm{id}_{W\otimes W} \otimes d')(\mathrm{id}_{W\otimes W\otimes V} \otimes d' \otimes \mathrm{id}_W)(\mathrm{id}_{W\otimes W} \otimes c^{\pm} \otimes \mathrm{id}_{W\otimes W})\\
&\qquad (\mathrm{id}_W \otimes b' \otimes \mathrm{id}_{V\otimes W\otimes W})(b' \otimes \mathrm{id}_{W\otimes W}),
\end{aligned} \tag{4.3.c}$$

$$c^+ c^- = c^- c^+ = \mathrm{id}_{V\otimes V}, \tag{4.3.d}$$

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c), \tag{4.3.e}$$

$$(\mathrm{id}_V \otimes d')(c^{\pm} \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes b) = \mathrm{id}_V, \tag{4.3.f}$$

$$gh = \mathrm{id}_{V\otimes W}, \quad \text{and} \quad hg = \mathrm{id}_{W\otimes V} \tag{4.3.g}$$

where the linear maps $g : W \otimes V \to V \otimes W$ and $h : V \otimes W \to W \otimes V$ are defined by

$$g = (d \otimes \mathrm{id}_{V\otimes W})(\mathrm{id}_W \otimes c^{\mp} \otimes \mathrm{id}_W)(\mathrm{id}_{W\otimes V} \otimes b) \tag{4.3.h}$$

and

$$h = (\mathrm{id}_{W\otimes V} \otimes d')(\mathrm{id}_W \otimes c^{\pm} \otimes \mathrm{id}_W)(b' \otimes \mathrm{id}_{V\otimes W}). \tag{4.3.i}$$

The data $(V, W, b, b', d, d', c, c^-)$ where V, W are finite-dimensional vector spaces and b, b', d, d', c, c^- are linear maps satisfying Relations (4.3.a–i) will be called a *representation data* for the tangle category $\mathcal{T}$.

Conversely, by Proposition 1.4 and Theorem 2.2, any representation data $(V, W, b, b', d, d', c, c^-)$ for $\mathcal{T}$ gives rise to a unique tensor functor $F : \mathcal{T} \to \mathcal{V}$ such that $F((+)) = V$, $F((-)) = W$, and such that Relations (4.2.c–e) hold.

These considerations imply that Theorem 4.2 is a consequence of the following proposition. $\square$

Proposition XII.4.3. *Let (c, μ) be an enhanced R-matrix on a finite-dimensional vector space V. Define b, b', d, d', c, c^- by*

$$b = \delta_V, \quad b' = (\mathrm{id}_{V^*} \otimes \mu^{-1})\delta_{V^*}, \quad d = \mathrm{ev}_V, \quad d' = \mathrm{ev}_{V^*}(\mu \otimes \mathrm{id}_{V^*}),$$

and $c^- = c^{-1}$. Then $(V, V^*, b, b', d, d', c, c^-)$ is a representation data for $\mathcal{T}$.

There is a converse statement whose formulation and proof are left to the reader. Before we prove Proposition 4.3, we give a corollary to Theorem 4.2, and we state two lemmas which will be used in the proof of Proposition 4.3 (they may also be used to establish the converse statement).

Let (c, μ) be an enhanced R-matrix on a finite-dimensional vector space V and F be the unique strict tensor functor from $\mathcal{T}$ to $\mathcal{V}$ satisfying Relations (4.2.a–b). Let σ be a braid with n strands. Since σ is a tangle, we may evaluate F on σ. We get an automorphism $F(\sigma)$ of $V^{\otimes n}$. Similarly, F can be evaluated on the link $\widetilde{\sigma}$ which is the closure of the braid σ (see X.8). We get an endomorphism $F(\widetilde{\sigma})$ of the ground field k, i.e., a scalar. In the following corollary, we express the automorphism $F(\sigma)$ and the scalar $F(\widetilde{\sigma})$ in terms of the representation ρ_n^c of the braid group B_n associated to the R-matrix c in Corollary X.6.9.

Corollary XII.4.4. *With the previous notation, we have*

$$F(\sigma) = \rho_n^c(\sigma) \quad \text{and} \quad F(\widetilde{\sigma}) = \operatorname{tr}(\mu^{\otimes n} \circ \rho_n^c(\sigma)) \tag{4.4}$$

for any braid σ of B_n.

PROOF. (a) It suffices to prove the first statement for the generators $\sigma_1, \ldots, \sigma_{n-1}$ of B_n. Using the notation of X.6, we have

$$\sigma_i = 1_{i-1} \otimes X_+ \otimes 1_{n-i-1}$$

in the tangle category. Applying the tensor functor F, we get by X.6.2 and by (4.2.a)

$$F(\sigma_i) = \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}} = \rho_n^c(\sigma).$$

(b) We first express the closure $\widetilde{\sigma}$ in the tangle category. We have

$$\widetilde{\sigma} = \overleftarrow{\cap}_n \circ (\sigma \uparrow \ldots \uparrow) \circ \cup_n$$

where

$$\overleftarrow{\cap}_n = \overleftarrow{\cap} \circ (\downarrow \overleftarrow{\cap} \uparrow) \circ \ldots \circ (\downarrow \ldots \downarrow \overleftarrow{\cap} \uparrow \ldots \uparrow)$$

and

$$\cup_n = (\downarrow \ldots \downarrow \cup \uparrow \ldots \uparrow) \circ \ldots \circ \ldots (\downarrow \cup \uparrow) \circ \cup.$$

Therefore

$$F(\widetilde{\sigma}) = F(\overleftarrow{\cap}_n) \circ (F(\sigma) \otimes \mathrm{id}_{V^{*\otimes n}}) \circ F(\cup_n).$$

Now, it is easy to check that (4.2.a–b) imply that

$$F(\cup_n) = \delta_{V^{\otimes n}} \quad \text{and} \quad F(\overleftarrow{\cap}_n) = \mathrm{ev}_{V^{*\otimes n}} \circ (\mu^{\otimes n} \otimes \mathrm{id}_{V^{*\otimes n}}).$$

Consequently,

$$F(\widetilde{\sigma}) = \mathrm{ev}_{V^{*\otimes n}} \circ \Big((\mu^{\otimes n} \circ F(\sigma)) \otimes \mathrm{id}_{V^{*\otimes n}} \Big) \delta_{V^{\otimes n}},$$

which is the trace of $\mu^{\otimes n} \circ F(\sigma)$ by (II.3.12). □

Let V and W be finite-dimensional vector spaces equipped with respective bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ and respective dual bases $\{v^1, \ldots, v^m\}$ and $\{w^1, \ldots, w^n\}$. Suppose also we are given four linear maps $b : k \to V \otimes W$, $b' : k \to W \otimes V$, $d : W \otimes V \to k$, and $d' : V \otimes W \to k$. Define matrices B, B', D, D' by

$$b(1) = \sum_{i,j} B_{ij}\, v_i \otimes w_j, \quad b'(1) = \sum_{i,j} B'_{ij}\, w_i \otimes v_j,$$

$$d(w_i \otimes v_j) = D_{ij}, \quad d'(v_i \otimes w_j) = D'_{ij}.$$

Lemma XII.4.5. *Under the previous hypotheses and with the previous notation, we have*

$$(\mathrm{id}_V \otimes d)(b \otimes \mathrm{id}_V) = \mathrm{id}_V \Longleftrightarrow BD = 1, \tag{4.5.a}$$

$$(d \otimes \mathrm{id}_W)(\mathrm{id}_W \otimes b) = \mathrm{id}_W \Longleftrightarrow DB = 1, \tag{4.5.b}$$

$$(d' \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes b') = \mathrm{id}_V \Longleftrightarrow D'B' = 1, \tag{4.5.c}$$

$$(\mathrm{id}_W \otimes d')(b' \otimes \mathrm{id}_W) = \mathrm{id}_W \Longleftrightarrow B'D' = 1 \tag{4.5.d}$$

where 1 *represents various matrix units.*

PROOF. Simple calculation. □

Now define linear maps $\alpha : W^* \to V$ and $\beta : V^* \to W$ by

$$\alpha(w^j) = \sum_i B_{ij}\, v_i \quad \text{and} \quad \beta(v^j) = \sum_i B'_{ij}\, w_i .$$

In the following lemma we assume that α and β are isomorphisms, which is equivalent to assuming B and B' to be invertible matrices. We also take it that the inverse matrices of B and B' are D and D' respectively. Recall that τ stands for the flip.

Lemma XII.4.6. *Let f be an endomorphism of $V \otimes V$. Under the previous hypotheses, we have*

$$\begin{aligned}(d \otimes \mathrm{id}_{W\otimes W})(\mathrm{id}_W \otimes d \otimes \mathrm{id}_{V\otimes W\otimes W})(\mathrm{id}_{W\otimes W} \otimes f \otimes \mathrm{id}_{W\otimes W})&\\ (\mathrm{id}_{W\otimes W\otimes V} \otimes b \otimes \mathrm{id}_W)(\mathrm{id}_{W\otimes W} \otimes b)&\\ = \quad \tau(\alpha^* \otimes \alpha^*) f^* (\alpha^{-1} \otimes \alpha^{-1})^* \tau,&\end{aligned} \tag{4.6.a}$$

$$\begin{aligned}(\mathrm{id}_{W\otimes W} \otimes d')(\mathrm{id}_{W\otimes W\otimes V} \otimes d' \otimes \mathrm{id}_W)(\mathrm{id}_{W\otimes W} \otimes f \otimes \mathrm{id}_{W\otimes W})&\\ (\mathrm{id}_W \otimes b' \otimes \mathrm{id}_{V\otimes W\otimes W})(b' \otimes \mathrm{id}_{W\otimes W})&\\ = \quad \tau(\beta \otimes \beta) f^* (\beta^{-1} \otimes \beta^{-1})\tau,&\end{aligned} \tag{4.6.b}$$

$$(\mathrm{id}_V \otimes d')(f \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes b) = \mathrm{tr}_2\Big(f\,(\mathrm{id}_V \otimes \mu)\Big) \tag{4.6.c}$$

where $\mu = \alpha(\beta^{-1})^*$,

$$(d\otimes\mathrm{id}_{V\otimes W})(\mathrm{id}_W\otimes f\otimes\mathrm{id}_W)(\mathrm{id}_{W\otimes V}\otimes b) = (\mathrm{id}\otimes\alpha^*)(\tau_{V,V} f)^\times \tau_{V^*,V}(\alpha^{-1}\otimes\mathrm{id})^*, \tag{4.6.d}$$

$$(\mathrm{id}_{W\otimes V}\otimes d')(\mathrm{id}_W\otimes f\otimes\mathrm{id}_W)(b'\otimes\mathrm{id}_{V\otimes W}) = (\beta\otimes\mathrm{id}_V)\tau_{V,V^*}(f\tau_{V,V})^\times(\mathrm{id}_V\otimes\beta^{-1}). \tag{4.6.e}$$

PROOF. Tedious, but easy computations. □

Proof of Proposition 4.3. Let (c,μ) be an enhanced R-matrix on the finite-dimensional vector space V. In order to apply Lemmas 4.5–4.6, we set $W = V^*$, $\alpha = \mathrm{id}_V$, and $\beta = (\mu^*)^{-1}$. Pick a basis $\{v_1, \ldots, v_m\}$ of V along with its dual basis $\{v^1, \ldots, v^m\}$. Define a matrix B' by

$$\beta(v^j) = \sum_i B'_{ij}\, v^i.$$

Since μ and β are isomorphisms, the matrix B' is invertible. Let D' be its inverse. By definition of b, b', d, d' we have

$$b(1) = \sum_i v_i \otimes v^i, \quad b'(1) = \sum_{i,j} B'_{ij}\, v^i \otimes v_j,$$

$$d(v^i \otimes v_j) = \delta_{ij}, \quad d'(v_i \otimes v^j) = D'_{ij}.$$

Let us now prove Relations (4.3.a–g).

Relations (4.3.a–b) follow from Lemma 4.5 in view of the fact that the matrices B and D are identity matrices.

Relation (4.3.c): Relation (4.1.a) implies that $f(\mu \otimes \mu) = (\mu \otimes \mu)f$ for $f = c^{\pm}$. Taking transposes, we get

$$(\mu^* \otimes \mu^*)f^* = f^*(\mu^* \otimes \mu^*),$$

which is equivalent to

$$\tau f^* \tau = \tau(\mu^* \otimes \mu^*)^{-1} f^* (\mu^* \otimes \mu^*)\tau.$$

The latter implies (4.3.c) in view of Relations (4.6.a–b), and of $\alpha = \mathrm{id}_V$ and $\beta = (\mu^*)^{-1}$.

Relation (4.3.d) holds by definition of c^- whereas Relation (4.3.e) expresses the fact that c is a solution of the Yang-Baxter equation. Relation (4.3.f) follows from (4.1.b) in view of (4.6.c).

Relation (4.3.g): In view of (4.6.d–e) and since the expressions in brackets below are isomorphisms, it is equivalent to show that

$$\Big[(\beta \otimes \mathrm{id}_V)\tau_{V,V^*}(c^{\pm}\tau_{V,V})^{\times}(\mathrm{id}_V \otimes \beta^{-1})\Big]\Big[(\mathrm{id} \otimes \alpha^*)(\tau_{V,V}c^{\mp})^{\times}\tau_{V^*,V}(\alpha^{-1} \otimes \mathrm{id}_{V^*})^*\Big]$$

is equal to $\mathrm{id}_{V^* \otimes V}$. Replacing α and β by their values, we are reduced to proving

$$((\mu^*)^{-1} \otimes \mathrm{id}_V)\tau_{V,V^*}(c^{\pm}\tau_{V,V})^{\times}(\mathrm{id}_V \otimes \mu^*)(\tau_{V,V}c^{\mp})^{\times}\tau_{V^*,V} = \mathrm{id}_{V^* \otimes V}. \tag{4.7}$$

Relation (4.7) is equivalent to

$$(\mathrm{id}_V \otimes (\mu^*)^{-1})(c^{\pm}\tau_{V,V})^{\times}(\mathrm{id}_V \otimes \mu^*)(\tau_{V,V}c^{\mp})^{\times} = \mathrm{id}_{V \otimes V^*}. \tag{4.8}$$

Taking transposes and using Lemma II.3.3, we see that (4.8) is equivalent to (4.1.c). □

XII.5 Completion of the Proof of the Existence of the Jones-Conway Polynomial

The aim of this section is to prove Proposition X.4.7 as a consequence of Theorem 4.2. Let k be a field and q be an invertible element of k. Fix an integer $m > 1$. Let V_m be a vector space over k of dimension m, equipped with a basis $\{v_1, \ldots, v_m\}$. Define a linear endomorphism c_m of $V_m \otimes V_m$ by

$$c_m(v_i \otimes v_j) = \begin{cases} \lambda_m\, q\, v_i \otimes v_i & \text{if } i = j, \\ \lambda_m\, v_j \otimes v_i & \text{if } i < j, \\ \lambda_m\, v_j \otimes v_i + \lambda_m(q - q^{-1})\, v_i \otimes v_j & \text{if } i > j \end{cases}$$

where λ_m is a non-zero scalar. The map c_m is a special case of the R-matrices described in Example 3 of VIII.1. Proposition VIII.1.4 implies that c_m is a solution of the Yang-Baxter equation satisfying the additional quadratic relation

$$\lambda_m^{-1} c_m - \lambda_m c_m^{-1} = (q - q^{-1})\, \mathrm{id}_{V_m \otimes V_m}. \tag{5.1}$$

Define an automorphism μ_m of V_m by $\mu_m(v_i) = \lambda_m^{-1} q^{-2i+1}\, v_i$. Observe that

$$\mathrm{tr}(\mu_m) = \frac{1}{\lambda_m q^m} \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{5.2}$$

Lemma XII.5.1. *If $\lambda_m = q^{-m}$, the pair (c_m, μ_m) is an enhanced R-matrix on V_m.*

PROOF. We have to check Relations (4.1.a–c). The first one is automatically verified because of the simple form of μ.

Relation (4.1.b): An immediate computation shows that

$$\begin{aligned} \mathrm{tr}_2\Big(c_m(\mathrm{id} \otimes \mu_m)\Big)(v_i) &= \Big(q^{-2(i-1)} + (1 - q^{-2}) \sum_{j<i} q^{-2(j-1)}\Big) v_i \\ &= \Big(q^{-2(i-1)} + 1 - q^{-2(i-1)}\Big) v_i = v_i. \end{aligned}$$

Therefore, $\mathrm{tr}_2(c_m(\mathrm{id} \otimes \mu_m)) = \mathrm{id}_{V_m}$. We have to check that the same relation holds when we replace c_m by its inverse. Taking advantage of (5.1), we get

$$\begin{aligned} &\mathrm{tr}_2\Big(c_m^{-1}(\mathrm{id} \otimes \mu_m)\Big) \\ &= \lambda_m^{-2}\, \mathrm{tr}_2\Big(c_m(\mathrm{id} \otimes \mu_m)\Big) - \lambda_m^{-1}(q - q^{-1})\, \mathrm{tr}_2(\mathrm{id} \otimes \mu_m) \\ &= \lambda_m^{-2}\Big(1 - \lambda_m(q - q^{-1})\, \mathrm{tr}(\mu_m)\Big)\mathrm{id}_{V_m} \\ &= \lambda_m^{-2}\Big(1 - q^{-m}(q^m - q^{-m})\Big)\mathrm{id}_{V_m} \\ &= \lambda_m^{-2} q^{-2m}\, \mathrm{id}_{V_m} = \mathrm{id}_{V_m} \end{aligned}$$

since $\lambda_m = q^{-m}$ and by (5.2).

Relation (4.1.c) is proved by a direct computation left to the reader. □

As a corollary of Theorem 4.2 and of Lemma 5.1, we get

Proposition XII.5.2. *Let $\lambda_m = q^{-m}$. There exists a unique strict tensor functor $F_{m,q}$ from the tangle category $\mathcal{T}$ into the strict tensor category $\mathcal{V}$ associated to $\mathrm{Vect}_f(k)$ such that $F_{m,q}((+)) = V_m$, $F_{m,q}((-)) = V_m^*$, and*

$$F_{m,q}(\cup)(1) = \sum_{i=1}^{m} v_i \otimes v^i, \quad F_{m,q}(\overleftarrow{\cup})(1) = \sum_{i=1}^{m} q^{2i-1-m}\, v^i \otimes v_i,$$

and $F_{m,q}(X_+) = c_m$. We also have

$$q^m F_{m,q}(X_+) - q^{-m} F_{m,q}(X_-) = (q - q^{-1}) F_{m,q}(\downarrow\downarrow), \tag{5.3}$$

and the value of $F_{m,q}$ on the trivial knot O is given by

$$F_{m,q}(O) = \mathrm{tr}(\mu_m) = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{5.4}$$

PROOF. We apply Theorem 4.2 to the pair (c_m, μ_m) of Lemma 5.1. Relations (4.2.a) imply the desired forms for $F_{m,q}(\cup)$ and for $F_{m,q}(\overleftarrow{\cup})$. Relation (5.1) translates immediately to

$$q^m F_{m,q}(X_+) - q^{-m} F_{m,q}(X_-) = (q - q^{-1}) F_{m,q}(\downarrow\downarrow).$$

For the trivial knot, we observe that it is the closure of the trivial braid in B_1. We may then appeal to Corollary 4.4, which yields $F_{m,q}(O) = \mathrm{tr}(\mu_m)$. We conclude with (5.2). □

We end this section by proving Proposition X.4.7, which completes the proof of Theorem X.4.2 on the existence of the Jones-Conway polynomial.

Proof of Proposition X.4.7. It is an application of Proposition 5.2 where $k = \mathbf{C}$ is the field of complex numbers and $q \neq 0$ is a complex number that is not a root of unity. We fix an integer $m > 1$.

Let us denote by F the restriction of the tensor functor $F_{m,q}$ to oriented links in $\mathbf{R}^2 \times]0,1[$. Since oriented links are endomorphisms of $\emptyset$ in the tangle category, F takes its values in the endomorphism ring $\mathrm{End}(\mathbf{C})$ which is canonically isomorphic to the field $\mathbf{C}$ of complex numbers. Using this isomorphism, we see that $F(L)$ is a complex number for any oriented link L in $\mathbf{R}^2 \times]0,1[$. Moreover, by definition of the tangle category, $F(L)$ depends only on the isotopy class of L. By Proposition 5.2, the value of F on the trivial knot is

$$F(O) = \frac{q^m - q^{-m}}{q - q^{-1}} \neq 0.$$

Suppose for a moment that we have proved

$$q^m F(L_+) - q^{-m} F(L_-) = (q - q^{-1}) F(L_0) \tag{5.5}$$

whenever (L_+, L_-, L_0) is a Conway triple. Then the composition $\Phi_{m,q}$ of F with a diffeomorphism of $\mathbf{R}^3$ onto $\mathbf{R}^2 \times]0,1[$ produces a complex-valued map on the oriented links in $\mathbf{R}^3$, satisfying the conditions of Proposition X.4.7. Therefore, the proof will be complete once we have checked Relation (5.5). Now, by definition of a Conway triple (L_+, L_-, L_0), there exist tangles L_i $(1 \leq i \leq 4)$ such that

$$L_+ = L_1 \circ (L_2 \otimes X_+ \otimes L_3) \circ L_4, \quad L_- = L_1 \circ (L_2 \otimes X_- \otimes L_3) \circ L_4,$$

and $L_0 = L_1 \circ (L_2 \otimes \downarrow\downarrow \otimes L_3) \circ L_4$. Since $F_{m,q}$ is a tensor functor, we get

$$\begin{aligned} & q^m F(L_+) - q^{-m} F(L_-) - (q - q^{-1}) F(L_0) \\ = \quad & F_{m,q}(L_1)\Big(F_{m,q}(L_2) \otimes S \otimes F_{m,q}(L_3)\Big) F_{m,q}(L_4) \end{aligned}$$

where

$$S = q^m F_{m,q}(X_+) - q^{-m} F_{m,q}(X_-) - (q - q^{-1}) F_{m,q}(\downarrow\downarrow).$$

The latter vanishes by (5.3). This proves Relation (5.5). □

XII.6 Exercises

1. Consider the strict tensor category whose objects are the nonnegative integers and whose morphisms are the isotopy classes of all braid diagrams in $\mathbf{R} \times [0,1]$. Show that it is generated by the morphisms X_+, X_- and the relations $X_+ \circ X_- = X_- \circ X_+ = \mathrm{id}$.

2. Let $c \in \mathrm{Aut}(V_1 \otimes V_1)$ be an R-matrix as in VIII.2, Example 2. Find all automorphisms μ of V_1 such that (c, μ) is an enhanced R-matrix.

3. Compute the trace of the automorphism $(\mu_m \otimes \mu_m) c_m$ where (c_m, μ_m) is the enhanced R-matrix of Lemma 5.1. Deduce the value of the functor $F_{m,q}$ of Proposition 5.2 on the trefoil knot and on the Hopf link (Hint: use Corollary 4.4 and (5.1)).

XII.7 Notes

The results of this chapter are essentially due to Turaev [Tur89] whose exposition we followed closely, and to Yetter [Yet88]. Enhanced R-matrices already appear in [Tur88], though in a slightly different form.

In XIV.5.1 we shall build a strict tensor category $\mathcal{R}$ out of framed tangles or ribbons (defined in X.8). A presentation of $\mathcal{R}$ by generators and relations is given in [FY89] [Tur89].

Chapter XIII
Braidings

We define the important concept of a braided tensor category due to Joyal and Street [JS93]. This concept has been introduced to formalize the characteristic properties of the tensor categories of modules over braided bialgebras as well as the idea of crossing in link and tangle diagrams. After defining braided tensor categories, we show that braids form a braided tensor category that is universal in some precise sense. We also give the "centre construction" which is the categorical version of Drinfeld's quantum double.

XIII.1 Braided Tensor Categories

XIII.1.1 Definitions and main properties

Let $\mathcal{C}$ be a category with a tensor product $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and an associativity constraint a. Denote by $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ the flip functor defined by $\tau(V, W) = (W, V)$ on any pair of objects of the category. A *commutativity constraint* c is a natural isomorphism

$$c : \otimes \to \otimes\tau.$$

This means that, for any couple (V, W) of objects of the category, we have an isomorphism

$$c_{V,W} : V \otimes W \to W \otimes V \tag{1.1}$$

such that the square

$$\begin{array}{ccc} V \otimes W & \xrightarrow{c_{V,W}} & W \otimes V \\ \Big\downarrow f\otimes g & & \Big\downarrow g\otimes f \\ V' \otimes W' & \xrightarrow{c_{V',W'}} & W' \otimes V' \end{array} \tag{1.2}$$

commutes for all morphisms f, g.

The commutativity constraint c satisfies the *Hexagon Axiom* if the two hexagonal diagrams

(H1)

$$\begin{array}{ccc} U \otimes (V \otimes W) & \xrightarrow{c_{U,V\otimes W}} & (V \otimes W) \otimes U \\ \nearrow a_{U,V,W} & & \searrow a_{V,W,U} \\ (U \otimes V) \otimes W & & V \otimes (W \otimes U) \\ \searrow c_{U,V} \otimes \mathrm{id}_W & & \nearrow \mathrm{id}_V \otimes c_{U,W} \\ (V \otimes U) \otimes W & \xrightarrow{a_{V,U,W}} & V \otimes (U \otimes W) \end{array} \tag{1.3}$$

and (H2)

$$\begin{array}{ccc} (U \otimes V) \otimes W & \xrightarrow{c_{U\otimes V,W}} & W \otimes (U \otimes V) \\ \nearrow a^{-1}_{U,V,W} & & \searrow a^{-1}_{W,U,V} \\ U \otimes (V \otimes W) & & (W \otimes U) \otimes V \\ \searrow \mathrm{id}_U \otimes c_{V,W} & & \nearrow c_{U,W} \otimes \mathrm{id}_V \\ U \otimes (W \otimes V) & \xrightarrow{a^{-1}_{U,W,V}} & (U \otimes W) \otimes V \end{array} \tag{1.4}$$

commute for all objects U, V, W of the category.

Observe that the hexagon (H2) can be obtained from (H1) by replacing the isomorphism c by its inverse c^{-1}. The following definition is due to Joyal and Street. It is central in the theory of quantum groups.

Definition XIII.1.1. *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category.*

(a) *A braiding in $\mathcal{C}$ is a commutativity constraint satisfying the Hexagon Axiom, i.e.,* (1.3–1.4).

(b) *A braided tensor category $(\mathcal{C}, \otimes, I, a, l, r, c)$ is a tensor category with a braiding.*

Note that if c is a braiding in $\mathcal{C}$, then so is the inverse c^{-1}. When the tensor category $\mathcal{C}$ is strict, the commutativity of (H1) and (H2) are equivalent to the relations

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) \tag{1.5}$$

and

$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}). \tag{1.6}$$

Let us investigate the relationship of the braiding with the unit constraints.

Proposition XIII.1.2. *For any object V of a braided tensor category with unit I, we have*

$$l_V \circ c_{V,I} = r_V, \quad r_V \circ c_{I,V} = l_V, \quad \textit{and} \quad c_{I,V} = c_{V,I}^{-1}.$$

When the category is strict, these relations simply become

$$c_{I,V} = c_{V,I} = \mathrm{id}_V. \tag{1.7}$$

PROOF. Consider the diagram

$$\begin{tikzcd}
(V\otimes I)\otimes W \arrow[r,"a"] \arrow[dr,"r_V\otimes\mathrm{id}_W"] \arrow[dd,"c\otimes\mathrm{id}_W"] & V\otimes(I\otimes W) \arrow[r,"c"] \arrow[d,"\mathrm{id}_V\otimes l_W"] & (I\otimes W)\otimes V \arrow[d,"l_W\otimes\mathrm{id}_V"] \arrow[dr,"a"] & \\
 & V\otimes W \arrow[r,"c"] & W\otimes V & I\otimes(W\otimes V) \arrow[dl,"\mathrm{id}"] \\
(I\otimes V)\otimes W \arrow[r,"a"] \arrow[ur,"l_V\otimes\mathrm{id}_W"] & I\otimes(V\otimes W) \arrow[r,"\mathrm{id}_I\otimes c"] \arrow[u,"l_{V\otimes W}"'] & I\otimes(W\otimes V) \arrow[u,"l_{W\otimes V}"'] &
\end{tikzcd}$$

The outside heptagon commutes by the commutativity of (1.3), the top square by the naturality of the braiding, the bottom square by the naturality of l, the upper left triangle by the Triangle Axiom (XI.2.9) and the lower left and the right triangles by Lemma XI.2.2. Consequently, the middle left triangle commutes, which means that

$$r_V \otimes \mathrm{id}_W = (l_V \otimes \mathrm{id}_W) \circ (c_{V,I} \otimes \mathrm{id}_W) = (l_V \circ c_{V,I}) \otimes \mathrm{id}_W.$$

Set $W = I$. Applying the naturality of r, we get $r_V = l_V \circ c_{V,I}$, which is the first equality to be proved. Replacing c by its inverse, we see that the commutativity of (1.4) implies the second relation in a similar way. The last relation is an immediate consequence of the other two. $\square$

XIII.1.2 Relation with the Yang-Baxter equation

One of the main properties of a braided tensor category is stated in the following theorem which may be considered as the categorical version of the Yang-Baxter equation.

Theorem XIII.1.3. *Let U, V, W be objects in a braided tensor category. Then the dodecagon*

$$\begin{tikzcd}
 & (U \otimes V) \otimes W \arrow[dl, "c_{U,V} \otimes \mathrm{id}_W"'] \arrow[dr, "a_{U,V,W}"] & \\
(V \otimes U) \otimes W \arrow[d, "a_{V,U,W}"] & & U \otimes (V \otimes W) \arrow[d, "\mathrm{id}_U \otimes c_{V,W}"] \\
V \otimes (U \otimes W) \arrow[d, "\mathrm{id}_V \otimes c_{U,W}"] & & U \otimes (W \otimes V) \arrow[d, "a^{-1}_{U,W,V}"] \\
V \otimes (W \otimes U) \arrow[d, "a^{-1}_{V,W,U}"] & & (U \otimes W) \otimes V \arrow[d, "c_{U,W} \otimes \mathrm{id}_V"] \\
(V \otimes W) \otimes U \arrow[d, "c_{V,W} \otimes \mathrm{id}_U"] & & (W \otimes U) \otimes V \arrow[d, "a_{W,U,V}"] \\
(W \otimes V) \otimes U \arrow[dr, "a_{W,V,U}"] & & W \otimes (U \otimes V) \arrow[dl, "\mathrm{id}_W \otimes c_{U,V}"'] \\
 & W \otimes (V \otimes U) &
\end{tikzcd}$$

commutes.

If the category $\mathcal{C}$ is strict, then the commutativity of the dodecagon is equivalent to the relation

$$\begin{aligned}
&(c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) \\
&\qquad = (\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}). \qquad (1.8)
\end{aligned}$$

This implies that the natural isomorphism $c_{V,V}$ is a solution of the Yang-Baxter equation for any object V of a braided tensor category.

PROOF. We cut the dodecagon into two hexagons of type (H2) and a square. According to (1.4), the clockwise composition of the morphisms in the dodecagon starting from $(U \otimes V) \otimes W$ and ending at $W \otimes (U \otimes V)$ is equal to $c_{U \otimes V,W}$. Similarly, the counterclockwise composition of the morphisms from $(V \otimes U) \otimes W$ to $W \otimes (V \otimes U)$ is equal to $c_{V \otimes U,W}$. It remains to check the commutativity of the square

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, "c_{U \otimes V,W}"] \arrow[d, "c_{U,V} \otimes \mathrm{id}_W"] & W \otimes (U \otimes V) \arrow[d, "\mathrm{id}_W \otimes c_{U,V}"] \\
(V \otimes U) \otimes W \arrow[r, "c_{V \otimes U,W}"] & W \otimes (V \otimes U) .
\end{tikzcd}$$

But this is a special case of the commutative square (1.2) (expressing the functoriality of the braiding) where f is replaced by $c_{U,V}$ and g by id_W. □

We give a few examples of braidings.

XIII.1.3 Braided categories of vector spaces

Example 1. (*The flip*) The flip τ is clearly a braiding in the tensor category *Vect*(k). It is also a braiding in the category $k[G]$-*Mod* of representations of a group G and, more generally, in the tensor category A-*Mod* of modules over a *cocommutative* bialgebra A (see Proposition III.5.1).

The following result relates the notions of a braided tensor category and of a braided bialgebra as defined in VIII.2 and justifies the name given to the latter.

Proposition XIII.1.4. *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. The tensor category H-Mod is braided if and only if the bialgebra H is braided.*

PROOF. Let $(H, \mu, \eta, \Delta, \varepsilon, R)$ be a braided bialgebra with universal R-matrix R. In VIII.3 we defined isomorphisms $c^R_{V,W}$ from $V \otimes W$ to $W \otimes V$ by

$$c^R_{V,W}(v \otimes w) = \tau_{V,W}\Big(R(v \otimes w)\Big)$$

where $v \in V$ and $w \in W$. Proposition VIII.3.1 implies that the family c is a braiding.

Conversely, let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Suppose that there exists a braiding c in the tensor category H-*Mod*. Define an invertible element R in $H \otimes H$ by

$$R = \tau_{H,H}(c_{H,H}(1 \otimes 1)). \tag{1.9}$$

Let us show that R is a universal R-matrix for H.

If v, w are elements of H-modules V, W, the naturality of the braiding implies the commutativity of the square

$$\begin{array}{ccc} H \otimes H & \xrightarrow{c_{H,H}} & H \otimes H \\ \Big\downarrow{\scriptstyle \bar{v}\otimes\bar{w}} & & \Big\downarrow{\scriptstyle \bar{w}\otimes\bar{v}} \\ V \otimes W & \xrightarrow{c_{V,W}} & W \otimes V \end{array}$$

where $\bar{v} : H \to V$ and $\bar{w} : H \to W$ are the H-linear maps defined by $\bar{v}(1) = v$ and $\bar{w}(1) = w$. This implies that

$$c_{V,W}(v \otimes w) = (\bar{w} \otimes \bar{v})\Big(c_{H,H}(1 \otimes 1)\Big) = \tau_{V,W}\Big((\bar{v} \otimes \bar{w})(R)\Big) = \tau_{V,W}\Big(R(v \otimes w)\Big). \tag{1.10}$$

We express the H-linearity of $c_{H,H}$: for each $a \in H$ we have

$$a\, c_{H,H}(1 \otimes 1) = c_{H,H}(a(1 \otimes 1)).$$

By (1.10) we get $\Delta(a)\tau_{H,H}(R) = \tau_{H,H}\Big(R\Delta(a)\Big)$. This is equivalent to

$$\Delta^{\mathrm{op}}(a)R = R\Delta(a)$$

for all $a \in H$.

The commutativity of the hexagons (1.3–1.4) implies the relations

$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12} \quad \text{and} \quad (\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$$

respectively. This proves that R satisfies Relations (VIII.2.1) and (VIII.2.3–2.4) defining a braided bialgebra structure on H. □

Under the correspondence set up in this proof, the commutativity of the dodecagon in Theorem 1.3 is equivalent to the equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

of Theorem VIII.2.4 (a).

XIII.1.4 Crossed G-sets

Given a group G we may form a strict braided tensor category as follows. Define a (*right*) *crossed G-set* as a set X with a right action $X \times G \to X$ of the group G and a set-theoretic map $|\ | : X \to G$ such that

$$|xg| = g^{-1}|x|g$$

for all $x \in X$ and $g \in G$. A morphism $f : X \to Y$ of crossed G-sets is a map f from X to Y such that $f(xg) = f(x)g$ and $|f(x)| = |x|$ for all $x \in X$ and $g \in G$. Crossed G-sets and their morphisms form a category $X(G)$.

We equip this category with a tensor product as follows. Given crossed G-sets X and Y, we define $X \otimes Y$ as the set-theoretic product $X \times Y$ with G-action given by $(x, y)g = (xg, yg)$ and with map $X \otimes Y \to G$ given by $|(x, y)| = |x||y|$. It is easy to check that $X \otimes Y$ belongs to $X(G)$. Similarly, given morphisms f and g, we define $f \otimes g = f \times g$. Then $X(G)$ becomes a strict tensor category with unit I equal to the crossed G-set $\{1\}$ with $|1| = 1$.

For any pair (X, Y) of crossed G-sets, define $c_{X,Y} : X \otimes Y \to Y \otimes X$ by

$$c_{X,Y}(x, y) = (y, x|y|) \tag{1.11}$$

where $x \in X$ and $y \in Y$. The proof of the following result is left to the reader.

Proposition XIII.1.5. *The maps $c_{X,Y}$ are morphisms of crossed G-sets and form a braiding for the strict tensor category $X(G)$.*

Let $X = G$ with the group acting on itself by conjugation, so that $(x, g) \mapsto g^{-1}xg$. Then $X = G$ is a crossed G-set with $|\,| = \mathrm{id}_G$. Consequently, $G^{\otimes n} = G^n$ is a crossed G-set with $|(g_1, \ldots, g_n)| = g_1 \cdots g_n$ (for $n > 0$). The full subcategory of $X(G)$ with objects $\{1, G, G^{\otimes 2}, G^{\otimes 3}, \ldots\}$ forms a braided subcategory $X_G(G)$ of $X(G)$ with braiding given by

$$c_{G^n,G^m}(g_1, \ldots, g_{n+m}) = (g_{n+1}, \ldots, g_{n+m}, g^{-1}g_1g, \ldots, g^{-1}g_ng) \tag{1.12}$$

where $g = g_{n+1} \cdots g_{n+m}$.

XIII.1.5 Symmetric tensor categories

Braided tensor categories generalize the classical concept of a symmetric tensor category introduced earlier by category theorists. A tensor category is *symmetric* if it is equipped with a braiding c such that

$$c_{W,V} \circ c_{V,W} = \mathrm{id}_{V \otimes W} \tag{1.13}$$

for all objects V, W in the category. If (1.13) holds, we call the braiding c a *symmetry* for the category. Notice that the commutativity of the hexagon (H1) and the commutativity of the hexagon (H2) are equivalent in a symmetric tensor category.

We give two examples of symmetric tensor categories.

Proposition XIII.1.6. *The strict tensor categories GL and $\mathcal{S}$ of* XI.3.2 *are symmetric.*

PROOF. We define automorphisms $s_{n,m} \in \mathrm{GL}_{n+m}(k) : n \otimes m \to m \otimes n$ as follows. If $\{e_1, \ldots, e_{n+m}\}$ is the canonical basis of the vector space k^{n+m}, we set $s_{n,m}(e_i) = e_{m+i}$ if $1 \leq i \leq n$, and $s_{n,m}(e_i) = e_{i-n}$ if $n+1 \leq i \leq n+m$. The matrix of $s_{n,m}$ in the canonical basis of k^{n+m} is the $(n+m) \times (n+m)$ matrix

$$\begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix}$$

where 1_n is the unit $n \times n$ matrix. This holds when $n > 0$ and $m > 0$. Otherwise, we have $s_{0,n} = \mathrm{id}_{k^n} = s_{n,0}$.

We claim that the family $(s_{n,m})$ is a braiding for GL. We have to check the functoriality and Relations (1.5–1.6). The functoriality is equivalent to the relation

$$s_{n,m} \circ (g \otimes h) = (h \otimes g) \circ s_{n,m}$$

for all $g \in \mathrm{GL}_n(k)$ and all $h \in \mathrm{GL}_m(k)$. This follows from the matrix relation

$$\begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 0 & h \\ g & 0 \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix}.$$

Relation (1.5) is a consequence of the relation

$$\begin{pmatrix} 0 & 1_m & 0 \\ 0 & 0 & 1_p \\ 1_n & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_m & 0 & 0 \\ 0 & 0 & 1_p \\ 0 & 1_n & 0 \end{pmatrix}\begin{pmatrix} 0 & 1_m & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & 1_p \end{pmatrix}.$$

Observe that $s_{m,n}s_{n,m} = \mathrm{id}_{n \otimes m}$. Therefore Relation (1.6) is also verified, which proves that $(s_{n,m})$ is a braiding endowing GL with the structure of a symmetric tensor category.

Since the matrix of $s_{n,m}$ is a permutation matrix, the same formulas define a symmetry on the category $\mathcal{S}$. □

XIII.2 The Braid Category

In X.6 we defined braids as a special case of tangles. When composing or tensoring two braids as we did for tangles in XII.2, we get another braid. This proves that braids form a strict tensor category $\mathcal{B}$ in which the objects are finite sequences of $+$ signs. We identify such a finite sequence of length n with the integer n under the convention that the empty sequence corresponds to the integer 0. Henceforth, we shall consider the set $\mathbf{N}$ as the set of objects of the strict tensor category $\mathcal{B}$. The purpose of this section is to show that the braid category is a strict braided tensor category.

In order to put a braiding on the braid category, we have to define isomorphisms $c_{n,m} : n \otimes m \to m \otimes n$ for any couple (n, m) of non-negative integers. This is done as follows: $c_{0,n} = \mathrm{id}_n = c_{n,0}$, and for $n, m > 0$ we set

$$c_{n,m} = (\sigma_m \sigma_{m-1} \cdots \sigma_1)(\sigma_{m+1}\sigma_m \cdots \sigma_2) \cdots (\sigma_{m+n-1}\sigma_{m+n-2} \cdots \sigma_n) \quad (2.1)$$

where $\sigma_1, \ldots, \sigma_{m+n-1}$ are the generators of B_{m+n} defined in X.6. The braid $c_{n,m}$ is represented in Figure 2.1. Observe that the permutation of the braid $c_{n,m}$ is the permutation $s_{n,m}$ of Proposition 1.6.

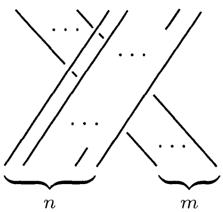


Figure 2.1. The braid $c_{n,m}$

Theorem XIII.2.1. *The family of isomorphisms $(c_{n,m})_{n,m\geq 0}$ is a braiding in the braid category $\mathcal{B}$.*

PROOF. We have to prove that the family $(c_{n,m})_{n,m\geq 0}$ is functorial with respect to all morphisms in $\mathcal{B}$ and satisfies Relations (1.5) and (1.6).

Let us start with the functoriality. Since any morphism in $\mathcal{B}$ is an element of a braid group, it is enough to check the functoriality with respect to the generators σ_i. More precisely, we must prove that for all i, j such that $1 \leq i \leq n-1$ and $1 \leq j \leq m-1$ we have

$$c_{n,m} \circ (\sigma_i \otimes \sigma_j) = (\sigma_j \otimes \sigma_i) \circ c_{n,m}.$$

Both sides of this relation are represented by the braid diagrams of Figure 2.2. It is clear that one can pass from one braid diagram to the other by repeated applications of the Reidemeister Transformation (III), which proves the equality. The reader may replace this topological proof by an algebraic one using the braid relations of Lemma X.6.4.

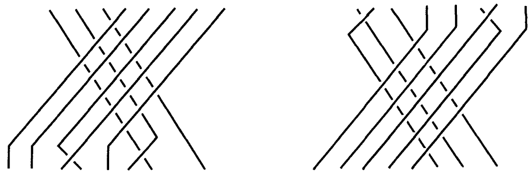


Figure 2.2.

As for Relation (1.5) [resp. Relation (1.6)], a graphical proof is given in Figure 2.3 [resp. in Figure 2.4]. □

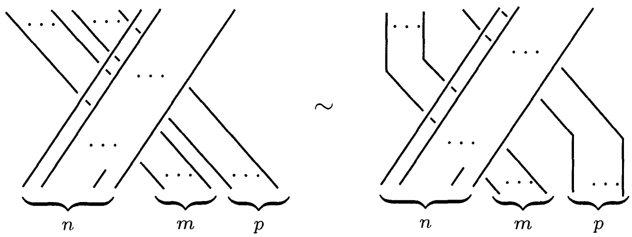


Figure 2.3. Proof of Relation 1.5

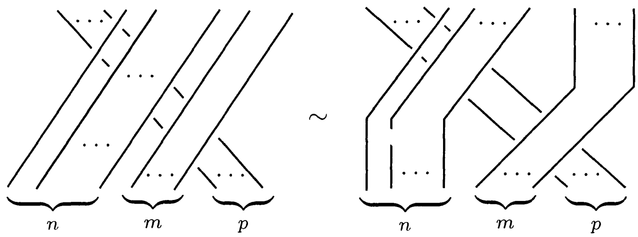


Figure 2.4. Proof of Relation 1.6

Observe that we have the relation

$$\sigma_i = \mathrm{id}_1^{\otimes(i-1)} \otimes c_{1,1} \otimes \mathrm{id}_1^{\otimes(n-i-1)} \tag{2.2}$$

in the braid category $\mathcal{B}$.

XIII.3 Universality of the Braid Category

In this section we derive two universality properties for the braid category. They imply that $\mathcal{B}$ is a model for all other braided tensor categories.

XIII.3.1 Yang-Baxter operators

We introduce Yang-Baxter operators following Joyal-Street [JS93].

Definition XIII.3.1. *If V is an object of a tensor category $(\mathcal{C}, \otimes, I, a, l, r)$, an automorphism σ of $V \otimes V$ is called a Yang-Baxter operator on V if the dodecagon*

$$
\begin{tikzcd}
 & (V \otimes V) \otimes V \arrow[dl, "\sigma \otimes \mathrm{id}_V"'] \arrow[dr, "a_{V,V,V}"] & \\
(V \otimes V) \otimes V \arrow[d, "a_{V,V,V}"] & & V \otimes (V \otimes V) \arrow[d, "\mathrm{id}_V \otimes \sigma"] \\
V \otimes (V \otimes V) \arrow[d, "\mathrm{id}_V \otimes \sigma"] & & V \otimes (V \otimes V) \arrow[d, "a^{-1}_{V,V,V}"] \\
V \otimes (V \otimes V) \arrow[d, "a^{-1}_{V,V,V}"] & & (V \otimes V) \otimes V \arrow[d, "\sigma \otimes \mathrm{id}_V"] \\
(V \otimes V) \otimes V \arrow[d, "\sigma \otimes \mathrm{id}_V"] & & (V \otimes V) \otimes V \arrow[d, "a_{V,V,V}"] \\
(V \otimes V) \otimes V \arrow[dr, "a_{V,V,V}"] & & V \otimes (V \otimes V) \arrow[dl, "\mathrm{id}_V \otimes \sigma"'] \\
 & V \otimes (V \otimes V) &
\end{tikzcd}
$$

commutes.

The commutativity of this dodecagon is equivalent to

$$(\mathrm{id} \otimes \sigma)a(\sigma \otimes \mathrm{id})a^{-1}(\mathrm{id} \otimes \sigma)a = a(\sigma \otimes \mathrm{id})a^{-1}(\mathrm{id} \otimes \sigma)a(\sigma \otimes \mathrm{id}) \tag{3.1}$$

where $a = a_{V,V,V}$ and $\mathrm{id} = \mathrm{id}_V$. In any braided tensor category, the braiding $c_{V,V}$ is a Yang-Baxter operator. This follows from Theorem 1.3. Here is a way to generate Yang-Baxter operators.

Lemma XIII.3.2. *Let $(F, \varphi_0, \varphi_2) : \mathcal{C} \to \mathcal{D}$ be a tensor functor between tensor categories. If $\sigma \in \mathrm{Aut}(V \otimes V)$ is a Yang-Baxter operator on the object V in $\mathcal{C}$, then*

$$\sigma' = \varphi_2(V, V)^{-1} \circ F(\sigma) \circ \varphi_2(V, V) \tag{3.2}$$

is a Yang-Baxter operator on $F(V)$.

Proof. Clearly σ' is an automorphism of $F(V) \otimes F(V)$. It is enough to check Relation (3.1). In other words, if we set

$$u = (\mathrm{id} \otimes \sigma')a(\sigma' \otimes \mathrm{id})a^{-1}(\mathrm{id} \otimes \sigma')a$$

and

$$v = a(\sigma' \otimes \mathrm{id})a^{-1}(\mathrm{id} \otimes \sigma')a(\sigma' \otimes \mathrm{id}),$$

we have to prove that $u = v$. Now, by Definition XI.4.1 we have

$$a = (\mathrm{id} \otimes \varphi_2^{-1})\varphi_2^{-1}F(a)\varphi_2(\varphi_2 \otimes \mathrm{id}). \tag{3.3}$$

Relations (3.2–3.3) imply that

$$\begin{aligned} u = {} & (\mathrm{id} \otimes \varphi_2^{-1})(\mathrm{id} \otimes F(\sigma))\varphi_2^{-1}F(a)\varphi_2(F(\sigma) \otimes \mathrm{id}) \\ & \varphi_2^{-1}F(a^{-1})\varphi_2(\mathrm{id} \otimes F(\sigma))\varphi_2^{-1}F(a)\varphi_2(\varphi_2 \otimes \mathrm{id}) \end{aligned}$$

where $\varphi_2 = \varphi_2(V,V)$. Now φ_2 is a natural isomorphism. Therefore the squares

$$\begin{array}{ccc} F(V \otimes V) \otimes F(V) & \xrightarrow{\varphi_2(V\otimes V,V)} & F((V \otimes V) \otimes V) \\ \big\downarrow {\scriptstyle F(\sigma)\otimes \mathrm{id}_{F(V)}} & & \big\downarrow {\scriptstyle F(\sigma\otimes \mathrm{id}_V)} \\ F(V \otimes V) \otimes F(V) & \xrightarrow{\varphi_2(V\otimes V,V)} & F((V \otimes V) \otimes V) \end{array}$$

and

$$\begin{array}{ccc} F(V) \otimes F(V \otimes V) & \xrightarrow{\varphi_2(V,V\otimes V)} & F(V \otimes (V \otimes V)) \\ \big\downarrow {\scriptstyle \mathrm{id}_{F(V)}\otimes F(\sigma)} & & \big\downarrow {\scriptstyle F(\mathrm{id}_V\otimes\sigma)} \\ F(V) \otimes F(V \otimes V) & \xrightarrow{\varphi_2(V,V\otimes V)} & F(V \otimes (V \otimes V)) \end{array}$$

commute. We rewrite them in the form

$$\varphi_2(F(\sigma)\otimes \mathrm{id}) = F(\sigma \otimes \mathrm{id})\varphi_2 \quad \text{and} \quad (\mathrm{id}\otimes F(\sigma))\varphi_2^{-1} = \varphi_2^{-1}F(\mathrm{id}\otimes\sigma). \tag{3.4}$$

Plugging (3.4) into u, we get

$$\begin{aligned} u &= (\mathrm{id} \otimes \varphi_2^{-1})\varphi_2^{-1}F(\mathrm{id} \otimes \sigma)F(a)F(\sigma \otimes \mathrm{id}) \\ & \qquad F(a^{-1})F(\mathrm{id} \otimes \sigma)F(a)\varphi_2(\varphi_2 \otimes \mathrm{id}) \\ &= (\mathrm{id} \otimes \varphi_2^{-1})\varphi_2^{-1}F\big((\mathrm{id} \otimes \sigma)a(\sigma \otimes \mathrm{id})a^{-1}(\mathrm{id} \otimes \sigma)a\big)\varphi_2(\varphi_2 \otimes \mathrm{id}). \end{aligned}$$

Similarly, we have

$$v = (\mathrm{id} \otimes \varphi_2^{-1})\varphi_2^{-1}F\Big(a(\sigma \otimes \mathrm{id})a^{-1}(\mathrm{id} \otimes \sigma)a(\sigma \otimes \mathrm{id})\Big)\varphi_2(\varphi_2 \otimes \mathrm{id}).$$

The equality $u = v$ results from the fact that σ satisfies Relation (3.1). □

We now define a new category $YB(\mathcal{C})$ out of the Yang-Baxter operators on the tensor category $(\mathcal{C}, \otimes, I, a, l, r)$. The objects of $YB(\mathcal{C})$ are pairs (V, σ) where V is an object of $\mathcal{C}$ and $\sigma : V \otimes V \to V \otimes V$ is a Yang-Baxter operator on V. A morphism $f : (V, \sigma) \to (V', \sigma')$ in $YB(\mathcal{C})$ is a morphism $f : V \to V'$ in $\mathcal{C}$ such that the square

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\sigma} & V \otimes V \\ \big\downarrow{\scriptstyle f\otimes f} & & \big\downarrow{\scriptstyle f\otimes f} \\ V' \otimes V' & \xrightarrow{\sigma'} & V' \otimes V' \end{array} \tag{3.5}$$

commutes. The identity of (V, σ) in $YB(\mathcal{C})$ is id_V.

Let us relate the category of Yang-Baxter operators to the braid category $\mathcal{B}$. Suppose that $(F, \varphi_0, \varphi_2) : \mathcal{B} \to \mathcal{C}$ is a tensor functor from $\mathcal{B}$ to the tensor category $\mathcal{C}$. By Theorems 1.3 and 2.1 we know that the automorphism $c_{1,1} = \sigma_1$ of $1 \otimes 1 = 2$ is a Yang-Baxter operator on the object 1 in $\mathcal{B}$. It follows from Lemma 3.2 that the automorphism

$$\sigma = \varphi_2^{-1} F(c_{1,1}) \varphi_2 \tag{3.6}$$

is a Yang-Baxter operator on $F(1)$ in the category $\mathcal{C}$. This defines an object $(F(1), \sigma)$ in $YB(\mathcal{C})$, which we denote by $\Theta(F)$.

We claim that Θ extends to a functor $Tens(\mathcal{B}, \mathcal{C}) \to YB(\mathcal{C})$. Let us check that, if $\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi_0', \varphi_2')$ is a natural tensor transformation, then $\eta(1) : F(1) \to F'(1)$ is a morphism in the category $YB(\mathcal{C})$. In other words, we have to show that $\eta(1)$ satisfies the following relation

$$\big(\eta(1) \otimes \eta(1)\big)\sigma = \sigma'\big(\eta(1) \otimes \eta(1)\big) \tag{3.7}$$

where $\sigma' = \varphi_2'^{-1} F'(c_{1,1}) \varphi_2'$. We have

$$\begin{aligned} \big(\eta(1) \otimes \eta(1)\big)\sigma &= \big(\eta(1) \otimes \eta(1)\big)\varphi_2^{-1} F(c_{1,1})\varphi_2 \\ &= \varphi_2'^{-1} \eta(2) F(c_{1,1}) \varphi_2 \\ &= \varphi_2'^{-1} F'(c_{1,1}) \eta(2) \varphi_2 \\ &= \varphi_2'^{-1} F'(c_{1,1}) \varphi_2' \big(\eta(1) \otimes \eta(1)\big) \\ &= \sigma'\big(\eta(1) \otimes \eta(1)\big). \end{aligned}$$

The first and last equalities follow by definition of σ and σ', the second and fourth ones by definition of a natural tensor transformation (Definition XI.4.1), and the third one by Definition XI.1.3.

We can state the first universality property of $\mathcal{B}$.

Theorem XIII.3.3. *For any tensor category $\mathcal{C}$ the functor, defined above, $\Theta : Tens(\mathcal{B}, \mathcal{C}) \to YB(\mathcal{C})$ is an equivalence of categories.*

PROOF. By Proposition XI.1.5 it suffices to check that the functor Θ is fully faithful and essentially surjective.

In order to establish that Θ is fully faithful, we have to show that Θ induces a bijection on morphisms. We build a map inverse to the map $\eta \mapsto \eta(1)$ considered above.

Let $f : (F(1), \varphi_2^{-1} F(c_{1,1}) \varphi_2) \to (F'(1), \varphi_2'^{-1} F'(c_{1,1}) \varphi_2')$ be a morphism in $YB(\mathcal{C})$ where $(F, \varphi_0, \varphi_2)$ and $(F', \varphi_0', \varphi_2')$ are tensor functors from $\mathcal{B}$ to

$\mathcal{C}$. We wish to construct a natural tensor transformation η_f from $(F, \varphi_0, \varphi_2)$ to $(F', \varphi'_0, \varphi'_2)$ such that $\eta_f(1) = f$. We proceed as follows. If $n = 0, 1$, we set $\eta_f(0) = \varphi'_0 \varphi_0^{-1}$ and $\eta_f(1) = f$. If $n > 1$ we define $\eta_f(n)$ inductively by

$$\eta_f(n) = \varphi'_2\Big(f \otimes \eta_f(n-1)\Big)\varphi_2^{-1}. \tag{3.8}$$

Lemma XIII.3.4. *The family $(\eta_f(n))_{n\geq 0}$ is a natural tensor transformation.*

PROOF. We have to check that

$$F'(g)\eta_f(n) = \eta_f(n)F(g) \tag{3.9}$$

for any integer $n \geq 0$ and any element g of the braid group B_n, that $\eta_f(0)\varphi_0 = \varphi'_0$ (this holds by definition of $\eta_f(0)$), and that for all $n, m \geq 0$

$$\eta_f(n \otimes m)\varphi_2 = \varphi'_2\Big(\eta_f(n) \otimes \eta_f(m)\Big). \tag{3.10}$$

It is enough to check Relation (3.9) when g is a generator σ_i of B_n. A computation left to the reader shows that Relation (3.9) for $g = \sigma_i$ is a consequence of Relations (2.2) and (3.8), of the Pentagon Axiom (XI.2.6), and of the definition of tensor functors.

As for Relation (3.10), one proceeds by induction on m as in the proof of Lemma XI.5.2. □

The full faithfulness of Θ follows from $\eta_f(1) = f$ and from $\eta_{\eta(1)} = \eta$. The first relation holds by definition. Let us check the second one. We shall prove

$$\eta_{\eta(1)}(n) = \eta(n) \tag{3.11}$$

by induction on n. This is clear for $n = 0, 1$. If $n > 1$, we use the fact that η and $\eta_{\eta(1)}$ are natural tensor transformations to write

$$\begin{aligned}
\eta_{\eta(1)}(n) &= \eta_{\eta(1)}((n-1) \otimes 1) \\
&= \varphi'_2\Big(\eta_{\eta(1)}(n-1) \otimes \eta_{\eta(1)}(1)\Big)\varphi_2 \\
&= \varphi'_2\Big(\eta(n-1) \otimes \eta(1)\Big)\varphi_2 \\
&= \eta((n-1) \otimes 1) \\
&= \eta(n).
\end{aligned}$$

This proves (3.11), hence the full faithfulness of Θ.

In order to complete the proof of Theorem 3.3, it remains to check that Θ is essentially surjective. By Theorem XI.5.3 we may assume that the tensor category $\mathcal{C}$ is strict. Then the essential surjectivity of Θ is a consequence of the following lemma. □

Lemma XIII.3.5. *Let $\mathcal{C}$ be a strict tensor category and (V,σ) an object of $YB(\mathcal{C})$. Then there exists a unique strict tensor functor $F:\mathcal{B}\to\mathcal{C}$ such that $F(1)=V$ and $F(c_{1,1})=\sigma$.*

PROOF. If such a functor F exists, then (2.2) implies $F(n)=V^{\otimes n}$ and

$$F(\sigma_i)=F(\mathrm{id}_1^{\otimes(i-1)}\otimes c_{1,1}\otimes \mathrm{id}_1^{\otimes(n-i-1)})=\mathrm{id}_V^{\otimes(i-1)}\otimes\sigma\otimes\mathrm{id}_V^{\otimes(n-i-1)}$$

for $1\leq i\leq n-1$. This proves the uniqueness of F in view of the fact that $\sigma_1,\sigma_2,\ldots,\sigma_{n-1}$ generate B_n as a group.

Let us prove the existence of F. Set $F(n)=V^{\otimes n}$. Define automorphisms $c_1,\ldots,c_{n-1}$ of $F(n)$ by

$$c_i=\mathrm{id}^{\otimes(i-1)}\otimes\sigma\otimes\mathrm{id}^{\otimes(n-i-1)}$$

when $1\leq i\leq n-1$. The automorphism σ being a Yang-Baxter operator, the automorphisms c_i satisfy the braid group relations (X.6.1–6.2). It follows from Theorem X.6.5 that there exists a unique morphism of groups F from the braid group B_n to $\mathrm{Aut}(F(n))$ such that $F(\sigma_i)=c_i$ for all i. The functor F is a strict tensor functor from $\mathcal{B}$ into $\mathcal{C}$ and we have $F(c_{1,1})=c_1=\sigma$.
□

XIII.3.2 Braided tensor functors

In order to state the second universality property of the braid category, we have to introduce the concept of a braided tensor functor.

Definition XIII.3.6. *A tensor functor (F,φ_0,φ_2) from a braided tensor category $\mathcal{C}$ to a braided tensor category $\mathcal{D}$ is braided if, for any pair (V,V') of objects of $\mathcal{C}$, the square*

$$\begin{array}{ccc} F(V)\otimes F(V') & \xrightarrow{\varphi_2} & F(V\otimes V') \\ \downarrow{\scriptstyle c_{F(V),F(V')}} & & \downarrow{\scriptstyle F(c_{V,V'})} \\ F(V')\otimes F(V) & \xrightarrow{\varphi_2} & F(V'\otimes V) \end{array} \tag{3.12}$$

commutes.

We denote by $Br(\mathcal{C},\mathcal{D})$ the category whose objects are the braided tensor functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are the natural tensor transformations.

Theorem XIII.3.7. *For any braided tensor category $\mathcal{C}$ the functor Θ' : $Br(\mathcal{B},\mathcal{C})\to\mathcal{C}$ defined by $\Theta'(F)=F(1)$ is an equivalence of categories.*

PROOF. By Proposition XI.1.5 again it is enough to prove that Θ' is fully faithful and essentially surjective.

Full faithfulness of Θ'. First, we claim that if $\mathcal{C}$ is a braided tensor category and F, F' are braided tensor functors, then

$$\mathrm{Hom}_{YB(\mathcal{C})}(\Theta(F), \Theta(F')) = \mathrm{Hom}_{\mathcal{C}}(\Theta'(F), \Theta'(F')).$$

It is clear that the left-hand side sits inside the right-hand side. We have to prove the opposite inclusion. Let $f : F(1) = V \to F'(1) = V'$ be an element of $\mathrm{Hom}_{\mathcal{C}}(\Theta'(F), \Theta'(F'))$. We wish to prove that f is a morphism in the category $YB(\mathcal{C})$, which means that the square (3.5) has to commute with $\sigma = \varphi_2^{-1} F(c_{1,1})\varphi_2$ and $\sigma' = {\varphi'_2}^{-1} F'(c_{1,1})\varphi'_2$. We have

$$\begin{aligned}
(f \otimes f)\sigma &= (f \otimes f)\varphi_2^{-1} F(c_{1,1})\varphi_2 \\
&= (f \otimes f)c_{V,V} \\
&= c_{V',V'}(f \otimes f) \\
&= {\varphi'_2}^{-1} F'(c_{1,1})\varphi'_2(f \otimes f) \\
&= \sigma'(f \otimes f).
\end{aligned}$$

The second and fourth equalities follow from (3.12) whereas the third one follows from the naturality of the braiding c in $\mathcal{C}$.

Now Θ' is fully faithful in view of the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{Br(\mathcal{B},\mathcal{C})}(F, F') &= \mathrm{Hom}_{\mathcal{T}ens(\mathcal{B},\mathcal{C})}(F, F') \\
&\cong \mathrm{Hom}_{YB(\mathcal{C})}(\Theta(F), \Theta(F')) \\
&= \mathrm{Hom}_{\mathcal{C}}(\Theta'(F), \Theta'(F'))
\end{aligned}$$

where the first one follows by definition, the second one from the full faithfulness of Θ (Theorem 3.3), and the last one has just been proved.

Essential surjectivity of Θ'. Let V be an object of $\mathcal{C}$. Since $\mathcal{C}$ is braided, the automorphism $c_{V,V}$ is a Yang-Baxter operator by Theorem 1.3. According to Theorem 3.3, the functor Θ is essentially surjective, which means that there exists a tensor functor $(F, \varphi_0, \varphi_2) : \mathcal{B} \to \mathcal{C}$ along with an isomorphism $\alpha : V \to F(1)$ such that

$$\varphi_2^{-1} F(c_{1,1})\varphi_2(\alpha \otimes \alpha) = (\alpha \otimes \alpha)c_{V,V}. \tag{3.13}$$

In order to prove that Θ' is essentially surjective, it is enough to prove that the functor F is braided, i.e., that for any pair (n, m) of nonnegative integers the square $(C_{n,m})$

$$\begin{array}{ccc}
F(n) \otimes F(m) & \xrightarrow{\varphi_2} & F(n+m) \\
\Big\downarrow{\scriptstyle c_{F(n),F(m)}} & & \Big\downarrow{\scriptstyle F(c_{n,m})} \\
F(n) \otimes F(m) & \xrightarrow{\varphi_2} & F(n+m)
\end{array} \tag{3.14}$$

commutes. We shall check this by induction on n and m.

The commutativity of $(C_{0,0})$, $(C_{1,0})$ and $(C_{0,1})$ is left to the reader. Let us first check that $(C_{1,1})$ commutes. We have

$$F(c_{1,1})\varphi_2 = \varphi_2(\alpha \otimes \alpha)c_{V,V}(\alpha^{-1} \otimes \alpha^{-1}) = \varphi_2\, c_{F(1),F(1)}$$

by Relation (3.13) and by naturality of the braiding c.

We now prove that the commutativity of $(C_{n,1})$ and of $(C_{n,m})$ implies the commutativity of $(C_{n,m+1})$. We have

$$\begin{aligned}
F(c_{n,m+1})\varphi_2 &= F(\mathrm{id} \otimes c_{n,1})F(c_{n,m} \otimes \mathrm{id})\varphi_2 \\
&= \varphi_2\Big(\mathrm{id} \otimes F(c_{n,1})\Big)\varphi_2^{-1}\varphi_2\Big(F(c_{n,m}) \otimes \mathrm{id}\Big)\varphi_2^{-1}\varphi_2 \\
&= \varphi_2(\mathrm{id} \otimes \varphi_2)(\mathrm{id} \otimes c_{F(n),F(1)})(\mathrm{id} \otimes \varphi_2^{-1})\varphi_2^{-1} \\
&\qquad \varphi_2(\varphi_2 \otimes \mathrm{id})(c_{F(n),F(m)} \otimes \mathrm{id})(\varphi_2^{-1} \otimes \mathrm{id})\varphi_2^{-1}\varphi_2 \\
&= \varphi_2(\mathrm{id} \otimes \varphi_2)(\mathrm{id} \otimes c_{F(n),F(1)})\, a\, (c_{F(n),F(m)} \otimes \mathrm{id}) \\
&\qquad (\varphi_2^{-1} \otimes \mathrm{id})\varphi_2^{-1}\varphi_2 \\
&= \varphi_2(\mathrm{id} \otimes \varphi_2)c_{F(n),F(m)\otimes F(1)}\, a\, (\varphi_2^{-1} \otimes \mathrm{id})\varphi_2^{-1}\varphi_2 \\
&= \varphi_2 c_{F(n),F(m+1)}(\mathrm{id} \otimes \varphi_2)\, a\, (\varphi_2^{-1} \otimes \mathrm{id})\varphi_2^{-1}\varphi_2 \\
&= \varphi_2 c_{F(n),F(m+1)}.
\end{aligned}$$

The first equality is a consequence of the fact that $(c_{n,m})_{n,m}$ is a braiding for $\mathcal{B}$, the second one follows by the naturality of φ_2, the third one by the commutativity of $(C_{n,1})$ and $(C_{n,m})$, the fourth one by Relation (XI.4.1), the fifth one by Relation (1.3), the sixth one by naturality of the braiding in $\mathcal{C}$, the seventh one again by Relation (XI.4.1).

A similar computation shows that the commutativity of $(C_{1,n})$ and of $(C_{n,m})$ implies the commutativity of $(C_{n+1,m})$. This is enough to prove that all squares $(C_{n,m})$ commute. Therefore the functor F is braided and the proof of Theorem 3.7 is complete. □

We may interpret Theorem 3.7 as saying that, given any object V in a braided tensor category $\mathcal{C}$ with braiding c, the tensor power $V^{\otimes n}$ of V (whatever parenthesizing is used) is naturally a module over the braid group B_n. If, moreover, $c_{V,V}$ is an involution, i.e., of square one, then the action of B_n factorizes through the symmetric group S_n.

The proof of the essential surjectivity of Θ' shows that the following more precise result holds for any strict braided tensor category.

Corollary XIII.3.8. *Let V be an object of a strict braided tensor category $\mathcal{C}$. Then there exists a unique strict braided tensor functor F_V from the braid category $\mathcal{B}$ to $\mathcal{C}$ such that $F_V(1) = V$.*

XIII.4 The Centre Construction

We now give a construction which assigns to any strict tensor category $(\mathcal{C}, \otimes, I)$ a braided tensor category $\mathcal{Z}(\mathcal{C})$, called the *centre* of $\mathcal{C}$. When $\mathcal{C}$ is the tensor category A-*Mod* of modules over a finite-dimensional Hopf algebra A with an invertible antipode, then $\mathcal{Z}(A\text{-}Mod)$ is tensor equivalent to the braided tensor category $D(A)$-*Mod* of modules over the quantum double $D(A)$ of A, as described in Chapter IX. In other words, this "centre construction" is the categorical version of the quantum double construction.

Definition XIII.4.1. *An object of $\mathcal{Z}(\mathcal{C})$ is a pair $(V, c_{-,V})$ where V is an object of $\mathcal{C}$ and $c_{-,V}$ is a family of natural isomorphisms*

$$c_{X,V} : X \otimes V \to V \otimes X$$

defined for all objects X in $\mathcal{C}$ such that for all objects X, Y in $\mathcal{C}$ we have

$$c_{X\otimes Y,V} = (c_{X,V} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,V}). \tag{4.1}$$

A morphism from $(V, c_{-,V})$ to $(W, c_{-,W})$ is a morphism $f : V \to W$ in $\mathcal{C}$ such that for each object X of $\mathcal{C}$ we have

$$(f \otimes \mathrm{id}_X)c_{X,V} = c_{X,W}(\mathrm{id}_X \otimes f). \tag{4.2}$$

The naturality in Definition 4.1 means that the square

$$\begin{array}{ccc} X \otimes V & \xrightarrow{c_{X,V}} & V \otimes X \\ \Big\downarrow {\scriptstyle f\otimes \mathrm{id}_V} & & \Big\downarrow {\scriptstyle \mathrm{id}_V \otimes f} \\ Y \otimes V & \xrightarrow{c_{Y,V}} & V \otimes Y \end{array} \tag{4.3}$$

commutes for any morphism $f : X \to Y$ in $\mathcal{C}$.

It is clear that the identity id_V is a morphism in $\mathcal{Z}(\mathcal{C})$ and that if f, g are composable morphisms in $\mathcal{Z}(\mathcal{C})$ then the composition $g \circ f$ in $\mathcal{C}$ is a morphism in $\mathcal{Z}(\mathcal{C})$. Consequently, $\mathcal{Z}(\mathcal{C})$ is a category in which the identity of $(V, c_{-,V})$ is id_V.

We now state the main theorem of this section.

Theorem XIII.4.2. *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category. Then $\mathcal{Z}(\mathcal{C})$ is a strict braided tensor category where*

(i) *the unit is (I, id),*

(ii) *the tensor product of $(V, c_{-,V})$ and $(W, c_{-,W})$ is given by*

$$(V, c_{-,V}) \otimes (W, c_{-,W}) = (V \otimes W, c_{-,V\otimes W})$$

where $c_{X,V\otimes W} : X \otimes V \otimes W \to V \otimes W \otimes X$ is the morphism of $\mathcal{C}$ defined for all objects X in $\mathcal{C}$ by

$$c_{X,V\otimes W} = (\mathrm{id}_V \otimes c_{X,W})(c_{X,V} \otimes \mathrm{id}_W), \tag{4.4}$$

(iii) *and the braiding is given by*

$$c_{V,W} : (V, c_{-,V}) \otimes (W, c_{-,W}) \to (W, c_{-,W}) \otimes (V, c_{-,V}).$$

PROOF. 1. Let $(V, c_{-,V})$ and $(W, c_{-,W})$ be objects in $\mathcal{Z}(\mathcal{C})$. We claim that the pair $(V \otimes W, c_{-,V\otimes W})$ defined in Theorem 4.2 (ii) is an object of $\mathcal{Z}(\mathcal{C})$.

Indeed, it follows from the properties of $(V, c_{-,V})$ and $(W, c_{-,W})$ that $c_{X,V\otimes W}$ is an isomorphism in $\mathcal{C}$ and that $c_{X,V\otimes W}$ is natural in X. We have to check Relation (4.1) for $c_{-,V\otimes W}$. For all objects X, Y of $\mathcal{C}$ we have

$$\begin{aligned}
c_{X\otimes Y,V\otimes W} &= (\mathrm{id}_V \otimes c_{X\otimes Y,W})(c_{X\otimes Y,V} \otimes \mathrm{id}_W)\\
&= (\mathrm{id}_V \otimes c_{X,W} \otimes \mathrm{id}_Y)(\mathrm{id}_{V\otimes X} \otimes c_{Y,W})\\
&\qquad (c_{X,V} \otimes \mathrm{id}_{Y\otimes W})(\mathrm{id}_X \otimes c_{Y,V} \otimes \mathrm{id}_W)\\
&= (\mathrm{id}_V \otimes c_{X,W} \otimes \mathrm{id}_Y)(c_{X,V} \otimes \mathrm{id}_{W\otimes Y})\\
&\qquad (\mathrm{id}_{X\otimes V} \otimes c_{Y,W})(\mathrm{id}_X \otimes c_{Y,V} \otimes \mathrm{id}_W)\\
&= (c_{X,V\otimes W} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,V\otimes W}).
\end{aligned}$$

The first and fourth equalities follow from (4.4), the second one from (4.1), and the third one by (XI.2.3), i.e., by the naturality of the tensor product.

2. Let $f : (V, c_{-,V}) \to (W, c_{-,W})$ and $f' : (V', c_{-,V'}) \to (W', c_{-,W'})$ be morphisms in $\mathcal{Z}(\mathcal{C})$. We claim that so is $f \otimes f'$. Let us check Relation (4.2) for $f \otimes f'$. We have

$$\begin{aligned}
(f \otimes f' \otimes \mathrm{id}_X)c_{X,V\otimes V'} &= (f \otimes \mathrm{id}_{W'} \otimes \mathrm{id}_X)(\mathrm{id}_V \otimes f' \otimes \mathrm{id}_X)\\
&\qquad (\mathrm{id}_V \otimes c_{X,V'})(c_{X,V} \otimes \mathrm{id}_{V'})\\
&= (f \otimes \mathrm{id}_{W'} \otimes \mathrm{id}_X)(\mathrm{id}_V \otimes c_{X,W'})\\
&\qquad (\mathrm{id}_V \otimes \mathrm{id}_X \otimes f')(c_{X,V} \otimes \mathrm{id}_{V'})\\
&= (\mathrm{id}_W \otimes c_{X,W'})(f \otimes \mathrm{id}_X \otimes \mathrm{id}_{W'})\\
&\qquad (c_{X,V} \otimes \mathrm{id}_{W'})(\mathrm{id}_X \otimes \mathrm{id}_V \otimes f')\\
&= (\mathrm{id}_W \otimes c_{X,W'})(c_{X,V} \otimes \mathrm{id}_{W'})\\
&\qquad (\mathrm{id}_X \otimes f \otimes \mathrm{id}_{W'})(\mathrm{id}_X \otimes \mathrm{id}_V \otimes f')\\
&= c_{X,W\otimes W'}(\mathrm{id}_X \otimes f \otimes f').
\end{aligned}$$

The first and fifth equalities follow from (4.4) and from (XI.2.3), the second and fourth ones from (4.2), and the third one from (XI.2.1).

Now it is clear that the tensor product is well-defined on the objects and on the morphisms of $\mathcal{Z}(\mathcal{C})$. It is functorial and satisfies all the required axioms because it already does so in the original category $\mathcal{C}$. Thus, the category $\mathcal{Z}(\mathcal{C})$ is a strict tensor category. We next show that it is braided.

3. Let us start by proving that $c_{V,W}$ is a morphism in $\mathcal{Z}(\mathcal{C})$. We have to check Relation (4.2) for $c_{V,W}$, namely

$$(c_{V,W} \otimes \mathrm{id}_X)c_{X,V\otimes W} = c_{X,W\otimes V}(\mathrm{id}_X \otimes c_{V,W})$$

for all objects X in $\mathcal{C}$. We have

$$\begin{aligned}
(c_{V,W} \otimes \mathrm{id}_X)c_{X,V\otimes W} &= (c_{V,W} \otimes \mathrm{id}_X)(\mathrm{id}_V \otimes c_{X,W})(c_{X,V} \otimes \mathrm{id}_W)\\
&= c_{V\otimes X,W}(c_{X,V} \otimes \mathrm{id}_W)\\
&= (\mathrm{id}_W \otimes c_{X,V})c_{X\otimes V,W}\\
&= (\mathrm{id}_W \otimes c_{X,V})(c_{X,W} \otimes \mathrm{id}_V)(\mathrm{id}_X \otimes c_{V,W})\\
&= c_{X,W\otimes V}(\mathrm{id}_X \otimes c_{V,W}).
\end{aligned}$$

The first and last equalities result from (4.4), the second and fourth one from (4.1), and the third one from the naturality of $c_{-,V}$.

4. The morphism $c_{V,W}$ is invertible by definition and it is natural with respect to all morphisms of $\mathcal{C}$, hence to those belonging to $\mathcal{Z}(\mathcal{C})$. In order for $c_{V,W}$ to qualify as a braiding, it has to satisfy both Relations (1.5) and (1.6). Now (1.6) follows from the hypothesis (4.1) and (1.5) from (4.4). Therefore the tensor category $\mathcal{Z}(\mathcal{C})$ is braided with braiding $c_{V,W}$. □

We give a universal property of the construction $\mathcal{Z}$. For any strict tensor category $\mathcal{C}$ the functor $\Pi : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ given by

$$\Pi(V, c_{-,V}) = V \tag{4.5}$$

is a strict tensor functor. It is universal in the following sense.

Proposition XIII.4.3. *Let F be a strict tensor functor from a strict braided tensor category $\mathcal{C}$ to a strict tensor category $\mathcal{C}'$. Suppose that F is bijective on objects and surjective on morphisms. Then there exists a unique strict braided functor $\mathcal{Z}(F) : \mathcal{C} \to \mathcal{Z}(\mathcal{C}')$ such that $F = \Pi \circ \mathcal{Z}(F)$.*

PROOF. Let us first prove the existence of $\mathcal{Z}(F)$. For any object V of $\mathcal{C}$ we set

$$\mathcal{Z}(F)(V) = (F(V), c_{-,F(V)})$$

where $c_{-,F(V)}$ is defined for all objects X in $\mathcal{C}'$ by $c_{X,F(V)} = F(c_{F^{-1}(X),V})$. Here $c_{-,V}$ is the braiding in $\mathcal{C}$. Relation (4.1) is satisfied because F is a tensor functor. Therefore $\mathcal{Z}(F)(V)$ is an object in $\mathcal{Z}(\mathcal{C}')$.

If $f : V \to V'$ is a morphism in $\mathcal{C}$, set $\mathcal{Z}(F)(f) = F(f)$. Relation (4.2) is satisfied because of the naturality of the braiding in $\mathcal{C}$. This proves that $\mathcal{Z}(F)$ is a functor. Clearly, $\Pi \circ \mathcal{Z}(F) = F$. Let us now check that $\mathcal{Z}(F)$ is a braided tensor functor. It preserves tensor products because of (1.5) and (4.4). It also respects braidings. Indeed, we have

$$\mathcal{Z}(F)(c_{V,W}) = F(c_{V,W}) = c_{F(V),F(W)}$$

which is the braiding of $\mathcal{Z}(\mathcal{C}')$.

The uniqueness of $\mathcal{Z}(F)$ is a consequence of the fact that it preserves braidings. □

Applying Proposition 4.3 to the identity functor of a braided tensor category, we get the following result.

Corollary XIII.4.4. *For any strict braided tensor category $\mathcal{C}$ there exists a unique braided tensor functor Z from $\mathcal{C}$ to $\mathcal{Z}(\mathcal{C})$ such that $\Pi \circ Z = \mathrm{id}_{\mathcal{C}}$.*

XIII.5 A Categorical Interpretation of the Quantum Double

We relate the centre construction of Section 4 to the quantum double constructed in Chapter IX. This will give us a simple categorical description of the quantum double.

Let $A = (A, \mu, \eta, \Delta, \varepsilon, S)$ be a finite-dimensional Hopf algebra with invertible antipode S. Under this hypothesis, we constructed a braided Hopf algebra $D(A)$ in IX.4. Let us recall that we have $D(A) = A^* \otimes A$ as a vector space, that the Hopf algebras A and $(A^{\mathrm{op}})^*$ are Hopf subalgebras of $D(A)$, and that the universal R-matrix is given by $R = \sum_i a_i \otimes a^i$ where $\{a_i\}_i$ is a basis of A and $\{a^i\}_i$ is the dual basis. Finally, in Theorem IX.5.2 we proved that a module structure over $D(A)$ is equivalent to a crossed bimodule structure over A.

We are ready to state the main theorem of this section.

Theorem XIII.5.1. *For any finite-dimensional Hopf algebra A with invertible antipode, the braided tensor categories $\mathcal{Z}(A\text{-}Mod)$ and $D(A)\text{-}Mod$ are equivalent.*

We defined the centre construction only for strict tensor categories. But there is no difficulty in extending it to A-*Mod*. We start with two preliminary results before embarking into the proof of Theorem 5.1.

Lemma XIII.5.2. *Let $(V, c_{-,V})$ be an object of $\mathcal{Z}(A\text{-}Mod)$ and Δ_V be the map from V to $V \otimes A$ defined for all $v \in V$ by $\Delta_V(v) = c_{A,V}(1 \otimes v)$. Then the map Δ_V endows the left A-module V with a crossed A-bimodule structure.*

Proof. Let $\Delta_V : V \to V \otimes A$ be defined as above. By convention, we write

$$\Delta_V(v) = \sum_{(v)} v_V \otimes v_A \in V \otimes A \tag{5.1}$$

for any $v \in V$. We call Δ_V the coaction of A on V.

The naturality of $c_{-,V}$ allows us to express $c_{X,V}$ in terms of the coaction Δ_V for any A-module X. Indeed, given x in X and $\bar{x} : A \to X$ the unique A-linear map sending 1 to x, we have the following commutative square:

$$\begin{array}{ccc} A \otimes V & \xrightarrow{c_{A,V}} & V \otimes A \\ \Big\downarrow{\scriptstyle \bar{x}\otimes \mathrm{id}_V} & & \Big\downarrow{\scriptstyle \mathrm{id}_V \otimes \bar{x}} \\ X \otimes V & \xrightarrow{c_{X,V}} & V \otimes X\,. \end{array}$$

It implies that for any $v \in V$ and $x \in X$ we have

$$c_{X,V}(x \otimes v) = \Delta_V(v)(1 \otimes x) = \sum_{(v)} v_V \otimes v_A x. \tag{5.2}$$

Let us show that the coaction Δ_V is coassociative. By (4.1) we have

$$\begin{aligned} c_{X\otimes Y,V}(x \otimes y \otimes v) &= \sum_{(v)} v_V \otimes (v_A)'x \otimes (v_A)''y \\ &= (c_{X,V} \otimes \mathrm{id}_Y)\Big((\mathrm{id}_X \otimes c_{Y,V})(x \otimes y \otimes v)\Big) \\ &= \sum_{(v)} (v_V)_V \otimes (v_V)_A\, x \otimes v_A\, y. \end{aligned}$$

Setting $X = Y = A$ and $x = y = 1$, we get

$$\sum_{(v)} v_V \otimes (v_A)' \otimes (v_A)'' = \sum_{(v)} (v_V)_V \otimes (v_V)_A \otimes v_A,$$

which proves the coassociativity of Δ_V.

We also have $c_{k,V} = \mathrm{id}_V$ because $k = I$ is the unit of the tensor category of k-modules. This implies $c_{k,V}(1 \otimes v) = \sum_{(v)} \varepsilon(v_A)v_V = v$ for all $v \in V$. This means that the coaction Δ_V is counitary. So far we have proved that the coaction Δ_V equips V with a structure of right A-comodule.

Let us now express the fact that $c_{X,V}$ is A-linear. For $a \in A, v \in V$ and $x \in X$ we have

$$a\, c_{X,V}(x \otimes v) = c_{X,V}(a(x \otimes v)).$$

Replacing $c_{X,V}$ by its expression in Δ_V, we get

$$\Delta(a)\Delta_V(v)(1 \otimes x) = \Big(\sum_{(a)} \Delta_V(a''v)(1 \otimes a')\Big)(1 \otimes x).$$

Setting $X = A$ and $x = 1$, we obtain

$$\sum_{(a)(v)} a'v_V \otimes a''v_A = \sum_{(a)(v)} (a''v)_V \otimes (a''v)_A\, a', \tag{5.3}$$

which is Relation (5.2) of Chapter IX expressing that V is a crossed A-bimodule. $\square$

By Theorem IX.5.2, we know that a crossed A-bimodule is a left $D(A)$-module. Let $R = \sum_i a_i \otimes a^i$ be the universal R-matrix of $D(A)$. We express the braiding in the braided tensor category $\mathcal{Z}(A\text{-}Mod)$ in terms of R.

Lemma XIII.5.3. *Under the previous hypotheses, if $(V, c_{-,V})$ is an object of $\mathcal{Z}(A\text{-}Mod)$ and X is an A-module, then the isomorphism $c_{X,V}$ is determined by*

$$c_{X,V}(x \otimes v) = \tau_{X,V}\Big(R(x \otimes v)\Big)$$

for all $x \in X$ and $v \in V$.

PROOF. By Relations (5.2) and (IX.5.4) we have

$$\begin{aligned} c_{X,V}(x \otimes v) &= \sum_{(v)} v_V \otimes v_A x \\ &= \sum_{(v),i} < a^i, v_A > v_V \otimes a_i x \\ &= \sum_{(v),i} a^i \cdot v \otimes a_i x \\ &= \tau_{X,V}\Big(R(x \otimes v)\Big). \end{aligned}$$

□

We prove Theorem 5.1 in five steps.

1. We first define a functor F from $\mathcal{Z}(A\text{-}Mod)$ to $D(A)\text{-}Mod$. Let $(V, c_{-,V})$ be an object of $\mathcal{Z}(A\text{-}Mod)$. By Lemma 5.2 and Theorem IX.5.2, the vector space $F(V, c_{-,V}) = V$ is a left $D(A)$-module. Recall from IX.5 that the action of $D(A)$ on V is determined by

$$(a\alpha)v = \sum_{(v)} < \alpha, v_A > av_V \tag{5.4}$$

where $a \in A$, $\alpha \in A^*$, and $v \in V$.

If f is a map in $\mathcal{Z}(A\text{-}Mod)$, then Relation (4.2) implies that f is a map of A-comodules, hence of A^*-modules. Consequently f is $D(A)$-linear. This defines F as a faithful functor.

2. Let us show that F is a strict tensor functor. The tensor product of $(V, c_{-,V})$ and of $(W, c_{-,W})$ is $(V \otimes W, c_{-,V\otimes W})$ where $c_{-,V\otimes W}$ is determined by $c_{A,V\otimes W} = (\mathrm{id}_V \otimes c_{A,W})(c_{A,V} \otimes \mathrm{id}_W)$. Therefore the coaction on $V \otimes W$ is given by

$$\Delta_{V\otimes W}(v \otimes w) = \sum_{(v)(w)} v_V \otimes w_W \otimes w_A v_A.$$

By (5.4) the action of a linear form α on a tensor $v \otimes w$ in $V \otimes W$ can be expressed as

$$\alpha \cdot (v \otimes w) = \sum_{(v)(w)} < \alpha, w_A v_A > v_V \otimes w_W,$$

which by definition of the comultiplication Δ of A^* (see IX.4) is equal to

$$\sum_{(v)(w)} < \Delta(\alpha), v_A \otimes w_A > v_V \otimes w_W = \Delta(\alpha) \cdot (v \otimes w).$$

Therefore the $D(A)$-action on $V \otimes W$ is given for $a \in A$ and $\alpha \in A^*$ by

$$(a\alpha)(v \otimes w) = \Delta(a)(\Delta(\alpha) \cdot (v \otimes w)) = \Delta(a\alpha)(v \otimes w),$$

which is exactly the action given by the comultiplication in the quantum double $D(A)$.

3. By definition of the braiding in $\mathcal{Z}(A\text{-}Mod)$, Lemma 5.3 can be reinterpreted as stating that

$$F(c_{V,W})(v \otimes w) = \tau_{V,W}\Big(R(v \otimes w)\Big),$$

which is the braiding in the category of $D(A)$-modules. Thus, the tensor functor F is braided.

4. Suppose that V is a left $D(A)$-module. For any A-module X define $c_{X,V}$ by

$$c_{X,V}(x \otimes v) = \tau_{X,V}\Big(R(x \otimes v)\Big)$$

where $v \in V$ and $x \in X$. Let us show that $(V, c_{-,V})$ is an object of $\mathcal{Z}(A\text{-}Mod)$.

The map $c_{X,V}$ is a natural isomorphism because R is invertible. Let us prove that it is A-linear. For $a \in A$ we have

$$\begin{aligned} c_{X,V}(a(x \otimes v)) &= \tau_{X,V}\Big(R\Delta(a)(x \otimes v)\Big) \\ &= \tau_{X,V}\Big(\Delta^{\mathrm{op}}(a)R(x \otimes v)\Big) \\ &= \Delta(a)\tau_{X,V}\Big(R(x \otimes v)\Big) \\ &= a\, c_{X,V}(x \otimes v) \end{aligned}$$

in view of Relation (VIII.2.1).

We also have to check Relation (4.1), namely

$$c_{X\otimes Y,V}(x \otimes y \otimes v) = (c_{X,V} \otimes \mathrm{id}_Y)\Big((\mathrm{id}_X \otimes c_{Y,V})(x \otimes y \otimes v)\Big).$$

The left-hand side is equal to

$$\tau_{X\otimes Y,V}\Big((\Delta \otimes \mathrm{id}_A)(R)(x \otimes y \otimes v)\Big)$$

whereas the right-hand side is equal to $\tau_{X\otimes Y,V}(R_{13}R_{23}(x \otimes y \otimes v))$. Both are equal in view of (VIII.2.3). Therefore $G(V) = (V, c_{V,-})$ is an object in $\mathcal{Z}(A\text{-}Mod)$.

Let $f : V \to W$ be a map of $D(A)$-modules. We have to check that $G(f) = f$ is a morphism in $\mathcal{Z}(A\text{-}Mod)$. First, it is A-linear since it is $D(A)$-linear. Next, we have

$$\begin{aligned} ((f \otimes \mathrm{id}_X)c_{X,V})(x \otimes v) &= \tau_{X,W}\Big((\mathrm{id}_X \otimes f)(R(x \otimes v))\Big) \\ &= \tau_{X,W}\Big(R(x \otimes f(v))\Big) \\ &= (c_{X,V}(\mathrm{id}_X \otimes f))(x \otimes v) \end{aligned}$$

for all $x \in X$ and $v \in V$. This proves (4.2).

5. Clearly, $FG = \mathrm{id}$. The equality $GF = \mathrm{id}$ follows from Lemma 5.3. This establishes the equivalence of $\mathcal{Z}(A\text{-}Mod)$ and of $D(A)\text{-}Mod$.

Theorem 5.1 is thus proved. Observe that the same arguments work if we restrict to finite-dimensional modules.

Remark 5.4. The natural embedding $A \subset D(A)$ of Hopf algebras induces a tensor functor $D(A)\text{-}Mod \to A\text{-}Mod$. It is easy to check that the latter corresponds to the functor $\Pi : \mathcal{Z}(A\text{-}Mod) \to A\text{-}Mod$ of (4.5) under the equivalence of Theorem 5.1.

XIII.6 Exercises

1. Let H be a braided bialgebra with universal R-matrix R. Show that the category $H\text{-}Mod$ is symmetric if and only if $\tau_{H,H}(R) = R^{-1}$.

2. Let $\mathcal{C}$ be a strict tensor category. Show that one gets a definition for a braiding equivalent to Definition 1.1 if one replaces the hexagons (H1) and H(2) by the square

$$\begin{array}{ccc} U \otimes V \otimes X \otimes Y & \xrightarrow{c_{U,V} \otimes \mathrm{id}_{X\otimes Y}} & V \otimes U \otimes X \otimes Y \\ \Big\downarrow \mathrm{id}_{U\otimes V} \otimes c_{X,Y} & & \Big\downarrow \mathrm{id}_{V\otimes U} \otimes c_{X,Y} \\ U \otimes V \otimes Y \otimes X & \xrightarrow{c_{U\otimes V} \otimes \mathrm{id}_{Y\otimes X}} & V \otimes U \otimes Y \otimes X \end{array}$$

   for all objects U, V, X, Y.

3. Resume the notation of Exercise XI.8. Define a commutativity constraint c by $c(v \otimes w) = \gamma(n,p)(w \otimes v)$ where v and w are homogeneous vectors of respective degrees n and p, and where γ is a function with values in $k \setminus \{0\}$. Show that c is a braiding if and only if the functional equations

$$\gamma(m, n+p) = \alpha(n,p,m)^{-1}\gamma(m,p)\alpha(n,m,p)\gamma(m,n)\alpha(m,n,p)^{-1}$$

   and

$$\gamma(m+n, p) = \alpha(p,m,n)\gamma(m,p)\alpha(m,p,n)^{-1}\gamma(n,p)\alpha(m,n,p)$$

   are satisfied for all integers m, n, p.

4. Given a tensor category $\mathcal{C}$, define the *reverse category* $\mathcal{C}^{\mathrm{rev}}$ as the category $\mathcal{C}$ with tensor product given by $V \otimes_{\mathrm{rev}} W = W \otimes V$. Prove that, if $\mathcal{C}$ is braided with braiding c, then $(\mathrm{id}, \varphi_0 = \mathrm{id}, \varphi_2 = c)$ is a tensor functor between $\mathcal{C}$ and $\mathcal{C}^{\mathrm{rev}}$.

5. Let $\mathcal{C}$ be a braided tensor category. Show that one can equip the strict tensor category $\mathcal{C}^{str}$ of XI.5 with a braiding such that the tensor equivalences constructed in XI.5 between both categories are braided functors.

6. (*Presentation of the braid category*) Show that the strict tensor category $\mathcal{B}$ is generated by the morphisms σ_1, σ_1^{-1} of B_2 and by the relations $\sigma_1\sigma_1^{-1} = \sigma_1^{-1}\sigma_1 = \mathrm{id}_2$ and

$$(\sigma_1 \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \sigma_1)(\sigma_1 \otimes \mathrm{id}_1) = (\mathrm{id}_1 \otimes \sigma_1)(\sigma_1 \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \sigma_1).$$

XIII.7 Notes

Braided tensor categories were introduced by Joyal and Street [JS91a] [JS93]. They generalize the concept of a symmetric tensor category which appeared in the 1960's in the work of Bénabou [Bén64] and Mac Lane [Mac63], and was extensively studied in relation to algebraic geometry and algebraic topology (see, e.g., [Del90] [DM82] [KL80] [Mac63] [SR72]).

The content of Section 3 is taken from [JS93]. Lemma 3.5 is the analogue of Theorem XII.4.2 for braids. We found the example of crossed G-sets in [FY89]. Exercise 4 is from [JS93].

The centre construction of Section 4 is due to Drinfeld (unpublished), to Joyal and Street [JS91c], and to Majid [Maj91b].

Chapter XIV

Duality in Tensor Categories

In the previous chapter we defined braided tensor categories modelled on the category of braids. We now introduce a class of tensor categories modelled on framed tangles or ribbons. These are the so-called ribbon categories. Their definition requires the concept of duality. However, when duality is involved, formulas quickly tend to become obscure and complex. To overcome this difficulty, we present a graphical calculus in which coloured tangle diagrams represent morphisms of tensor categories.

XIV.1 Representing Morphisms in a Tensor Category

We discuss a technique of presenting morphisms of a strict tensor category by planar diagrams. Let $\mathcal{C}$ be a strict tensor category. We represent a morphism $f : U \to V$ in $\mathcal{C}$ by a box with two vertical arrows oriented downwards as in Figure 1.1. Here U and V are treated as the "colours" of the arrows and f as the "colour" of the box.

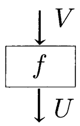


Figure 1.1. A morphism $f : U \to V$

The picture for the composition of $f : U \to V$ and of $g : V \to W$ is obtained by putting the picture of g on top of the picture of f, as shown in Figure 1.2. From now on the symbol $\doteq$ displayed in the figures means that the corresponding morphisms are equal in $\mathcal{C}$. The identity of V will be represented by the vertical arrow $\downarrow_V$ directed downwards.

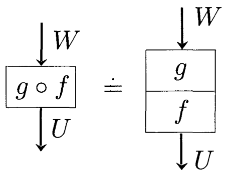


Figure 1.2. Composition of morphisms

The tensor product of two morphisms f and g is represented by boxes placed side by side as in Figure 1.3.

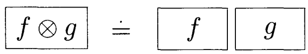


Figure 1.3. Tensor product of morphisms

If we represent a morphism $f : U_1 \otimes \cdots \otimes U_m \to V_1 \otimes \cdots \otimes V_n$ as in Figure 1.4,

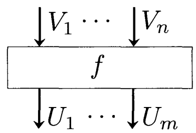


Figure 1.4. A morphism $f : U_1 \otimes \cdots \otimes U_m \to V_1 \otimes \cdots \otimes V_n$

then we have the equality of morphisms of Figure 1.5.

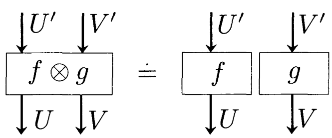


Figure 1.5.

The pictorial incarnation of the identity (XI.2.3)

$$f \otimes g = (f \circ \mathrm{id}) \otimes (\mathrm{id} \circ g) = (\mathrm{id} \circ f) \otimes (g \circ \mathrm{id})$$

is in Figure 1.6.

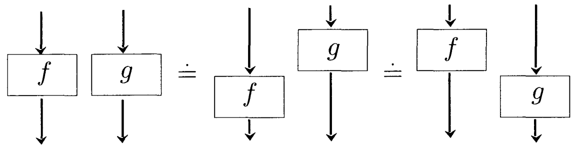


Figure 1.6. The identity (XI.2.3)

This leads to the following "partial isotopy principle": for any figure presenting a morphism of $\mathcal{C}$, the part of the figure lying to the left (or to the right) of a vertical line may be pushed up or down without changing the corresponding morphism in $\mathcal{C}$. We shall use this principle frequently and without any further explanation in the sequel.

Assume now that the tensor category is braided with a braiding c. For any pair (V, W) of objects in $\mathcal{C}$ we represent $c_{V,W}$ and its inverse $c_{V,W}^{-1}$ respectively by the pictures in Figure 1.7.

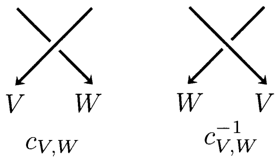


Figure 1.7. $c_{V,W}$ *and its inverse*

Figure 1.8 follows from the definitions.

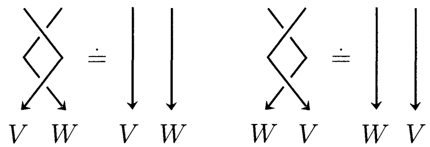


Figure 1.8. Invertibility of $c_{V,W}$

The naturality of $c_{V,W}$ is expressed in Figure 1.9.

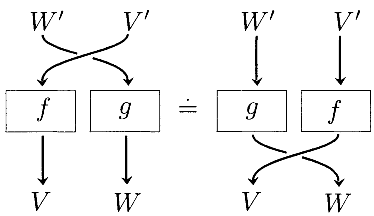


Figure 1.9. Naturality of $c_{V,W}$

It implies the naturality of $c_{V,W}^{-1}$ shown in Figure 1.10.

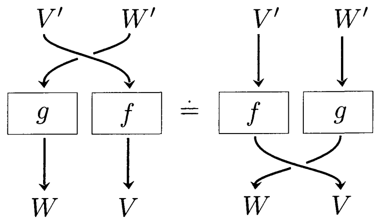


Figure 1.10. Naturality of $c^{-1}_{V,W}$

Figure 1.11 is a graphical proof of Theorem XIII.1.3.

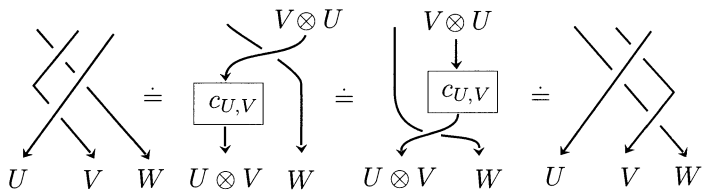


Figure 1.11. Proof of Theorem XIII.1.3

XIV.2 Duality

We now abstract the notion of duality introduced in II.3.

Definition XIV.2.1. *Let* $(\mathcal{C}, \otimes, I)$ *be a strict tensor category with tensor product* $\otimes$ *and unit* I. *It is a tensor category with left duality if for each object* V *of* $\mathcal{C}$ *there exist an object* V^* *and morphisms*

$$b_V : I \to V \otimes V^* \quad \textit{and} \quad d_V : V^* \otimes V \to I$$

in the category $\mathcal{C}$ *such that*

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V \quad \textit{and} \quad (d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}. \tag{2.1}$$

We proceed to give a graphical description of Relations (2.1) using the conventions of Section 1. Represent the identity of V^* by the vertical arrow $\big\uparrow V$ directed upwards. More generally, we use vertical arrows oriented upwards with the convention that the morphism involves not the colour of the arrow, but rather the dual object. For example, a morphism $f : U^* \to V^*$ may be represented by the four pictures of Figure 2.1.

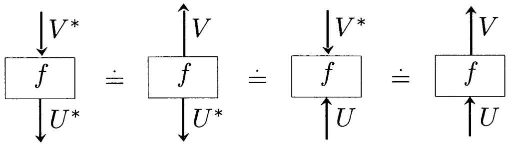


Figure 2.1. A morphism $f : U^* \to V^*$

The morphisms $b_V : I \to V \otimes V^*$ and $d_V : V^* \otimes V \to I$ are respectively represented by the pictures of Figure 2.2.

Figure 2.2. The morphisms b_V and d_V

Relations (2.1) take the graphical form of Figure 2.3.

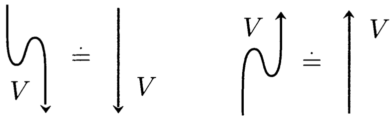


Figure 2.3. Relations (2.1)

The above data are enough to extend duality to a functor and to derive adjunction formulas of the type proved in Chapter II.

Let us first define the *transpose* $f^* : V^* \to U^*$ of a morphism $f : U \to V$ in $\mathcal{C}$ by

$$f^* = (d_V \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes b_U). \tag{2.2}$$

With our graphical conventions we can represent the transpose f^* of a morphism $f : U \to V$ as in Figure 2.4.

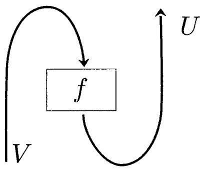


Figure 2.4. The transpose f^*

We record a few properties of left duality in the following proposition.

Proposition XIV.2.2. *Let $\mathcal{C}$ be a strict tensor category with left duality.*

(a) *If $f : V \to W$ and $g : U \to V$ are morphisms of $\mathcal{C}$, then we have $(f \circ g)^* = g^* \circ f^*$, and $(\mathrm{id}_V)^* = \mathrm{id}_{V^*}$ for any object V.*

(b) *For any family U, V, W of objects of $\mathcal{C}$, we have natural bijections*

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, W \otimes V^*)$$

and

$$\mathrm{Hom}(U^* \otimes V, W) \cong \mathrm{Hom}(V, U \otimes W).$$

(c) *For any pair V, W of objects of $\mathcal{C}$, $(V \otimes W)^*$ and $W^* \otimes V^*$ are isomorphic objects.*

This proposition implies that the map $V \mapsto V^*$ can be extended to a functor from the category $\mathcal{C}$ to the opposite category and that the functor $-\otimes V$ [resp. the functor $V^* \otimes -$] is left adjoint to the functor $-\otimes V^*$ [resp. to the functor $V \otimes -$].

PROOF. (a) Use the graphical calculus.

(b) Let $f \in \mathrm{Hom}(U \otimes V, W)$ and $g \in \mathrm{Hom}(U, W \otimes V^*)$. Define elements $f^\sharp \in \mathrm{Hom}(U, W \otimes V^*)$ and $g^\flat \in \mathrm{Hom}(U \otimes V, W)$ by

$$f^\sharp = (f \otimes \mathrm{id}_{V^*})(\mathrm{id}_U \otimes b_V) \quad \text{and} \quad g^\flat = (\mathrm{id}_W \otimes d_V)(g \otimes \mathrm{id}_V).$$

Relations (2.1) imply that $(f^\sharp)^\flat = f$ and $(g^\flat)^\sharp = g$. A similar proof works for the other adjunction formula. We invite the reader to give a graphical proof.

(c) We define a morphism $\lambda_{V,W} : W^* \otimes V^* \to (V \otimes W)^*$ by

$$\lambda_{V,W} = (d_W \otimes \mathrm{id}_{(V\otimes W)^*})(\mathrm{id}_{W^*} \otimes d_V \otimes \mathrm{id}_{W\otimes(V\otimes W)^*})(\mathrm{id}_{W^*\otimes V^*} \otimes b_{V\otimes W}) \tag{2.3}$$

and a morphism $\lambda_{V,W}^{-1} : (V \otimes W)^* \to W^* \otimes V^*$ by

$$\lambda_{V,W}^{-1} = (d_{V\otimes W} \otimes \mathrm{id}_{W^*\otimes V^*})(\mathrm{id}_{(V\otimes W)^*\otimes V} \otimes b_W \otimes \mathrm{id}_{V^*})(\mathrm{id}_{(V\otimes W)^*} \otimes b_V). \tag{2.4}$$

The morphisms $\lambda_{V,W}$ and $\lambda_{V,W}^{-1}$ are represented by the pictures of Figure 2.5.

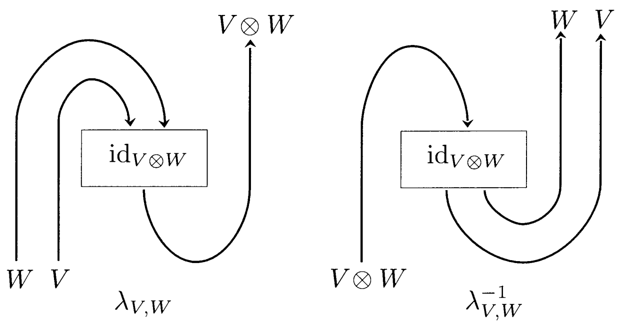


Figure 2.5. The morphisms $\lambda_{V,W}$ and $\lambda_{V,W}^{-1}$

Figures 2.6 and 2.7 show that $\lambda_{V,W}$ is an isomorphism from $W^* \otimes V^*$ onto $(V \otimes W)^*$ with inverse given by $\lambda_{V,W}^{-1}$.

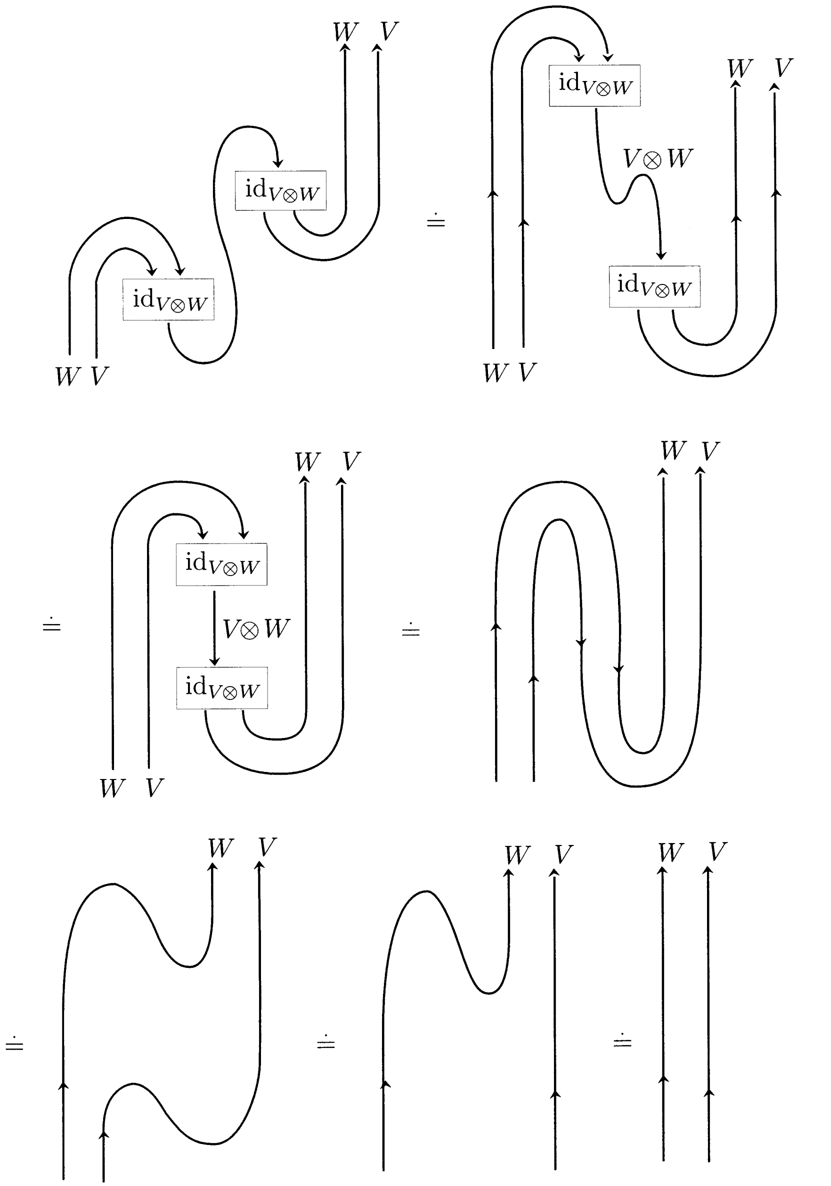


Figure 2.6. Proof of $\lambda^{-1}_{V,W} \circ \lambda_{V,W} = \mathrm{id}_{W^*\otimes V^*}$

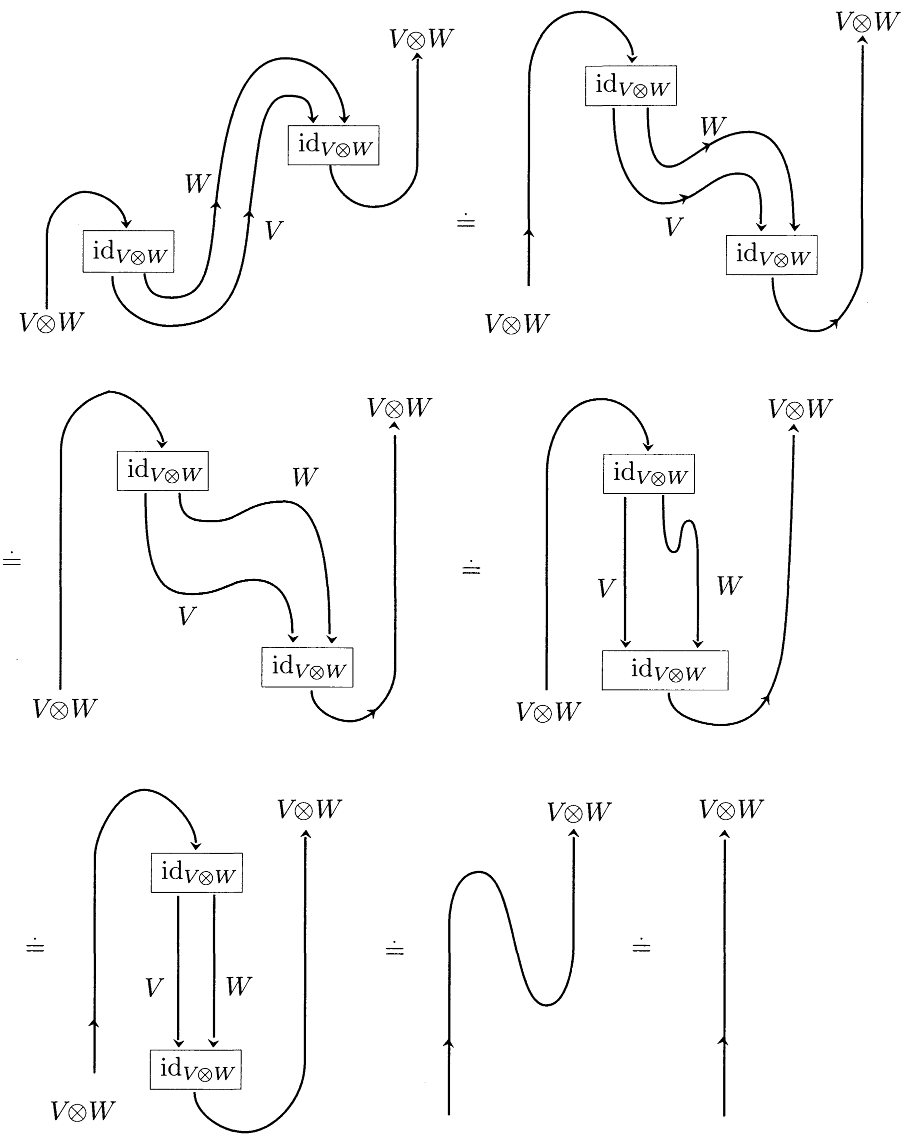


Figure 2.7. Proof of $\lambda_{V,W} \circ \lambda_{V,W}^{-1} = \mathrm{id}_{(V\otimes W)^*}$

Note that Figures 1.6 and 2.3 are used in these graphical proofs. $\square$

There is a similar notion of right duality: a strict tensor category $(\mathcal{C}, \otimes, I)$ is a *tensor category with right duality* if for each object V of $\mathcal{C}$ there exist an object ${}^*V$ and morphisms

$$b'_V : I \to {}^*V \otimes V \quad \text{and} \quad d'_V : V \otimes {}^*V \to I$$

in the category $\mathcal{C}$ such that

$$(d'_V \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes b'_V) = \mathrm{id}_V \quad \text{and} \quad (\mathrm{id}_{{}^*V} \otimes d'_V)(b'_V \otimes \mathrm{id}_{{}^*V}) = \mathrm{id}_{{}^*V}. \tag{2.5}$$

Here again $V \mapsto {}^*V$ may be extended to a functor by defining the morphism ${}^*f : {}^*W \to {}^*V$ for any $f : V \to W$ by

$${}^*f = (\mathrm{id}_{{}^*V} \otimes d'_W)(\mathrm{id}_{{}^*V} \otimes f \otimes \mathrm{id}_{{}^*W})(b'_V \otimes \mathrm{id}_{{}^*W}). \tag{2.6}$$

Right duality has properties analogous to the ones stated for left duality in Proposition 2.2. We leave their formulation to the reader. In particular, right duality implies that the functor $V \otimes -$ [resp. the functor $- \otimes {}^*V$] is left adjoint to the functor ${}^*V \otimes -$ [resp. to the functor $- \otimes V$].

In general, right duality is different from left duality unless we add extra hypotheses on $\mathcal{C}$. Nevertheless, it may happen that $\mathcal{C}$ is *autonomous*, i.e., it has left and right duality. In this case, there are isomorphisms

$${}^*(V^*) \cong V \cong ({}^*V)^*$$

for any object V. We refer to [JS93] for a proof. Hint: the first isomorphism is a consequence of the following natural isomorphisms

$$\mathrm{Hom}(U, {}^*(V^*) \otimes W) \cong \mathrm{Hom}(V^* \otimes U, W) \cong \mathrm{Hom}(U, V \otimes W),$$

the first one being implied by the right duality and the second one by the left duality.

Example 1. Let A be a Hopf algebra with antipode S. The category A-Mod_f of left A-modules that are *finite-dimensional* over the ground field k is a tensor subcategory of A-*Mod*. For any left A-module, endow the dual vector space $V^* = \mathrm{Hom}(V, k)$ with the A-action given by

$$< af, v > = < f, S(a)v > \tag{2.7}$$

where $a \in A$, $v \in V$ and $f \in V^*$. For a finite-dimensional A-module V define maps $b_V : k \to V \otimes V^*$ and $d_V : V^* \otimes V \to k$ by

$$b_V(1) = \sum_i v_i \otimes v^i \quad \text{and} \quad d_V(v^i \otimes v_j) = < v^i, v_j > \tag{2.8}$$

where $\{v_i\}_i$ is any basis of V and $\{v^i\}_i$ is the dual basis in V^*. The map b_V is the coevaluation map and the map d_V is the evaluation map of II.3. It was proved in Proposition III.5.3 that b_V and d_V are A-linear. By Proposition II.3.1 they satisfy Relations (2.1), endowing A-Mod_f with the structure of a tensor category with left duality.

Suppose, furthermore, that the antipode S is invertible. For any left A-module V, denote by ${}^*V$ the same dual vector space now equipped with the left A-action given for all $a \in A$, $v \in V$ and all linear forms f on V by

$$< af, v > = < f, S^{-1}(a)v > . \tag{2.9}$$

For a finite-dimensional V define maps $b'_V : k \to {}^*V \otimes V$ and $d'_V : V \otimes {}^*V \to k$ by

$$b'_V(1) = \sum_i v^i \otimes v_i \quad \text{and} \quad d'_V(v_i \otimes v^j) = < v^j, v_i > \tag{2.10}$$

using the same conventions as above. One checks that b'_V and d'_V are A-linear and that they satisfy Relations (2.5), endowing $A\text{-}Mod_f$ also with the structure of a tensor category with right duality. In other words, the category $A\text{-}Mod_f$ is autonomous when A is a Hopf algebra with invertible antipode.

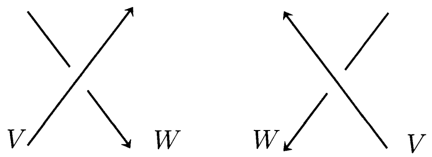
XIV.3 Ribbon Categories

Let $\mathcal{C}$ be a strict tensor category. Suppose that it is braided and has left duality at the same time. Let V and W be objects of the category. The following expresses the braiding $c_{V^*,W}$ for the dual object V^* in terms of the braiding $c_{V,W}$.

Proposition XIV.3.1. *Under the previous hypothesis, we have*

$$c_{V^*,W} = (d_V \otimes \mathrm{id}_{W \otimes V^*})(\mathrm{id}_{V^*} \otimes c^{-1}_{V,W} \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^* \otimes W} \otimes b_V).$$

By convention, we represent $c_{V^*,W}$ and its inverse by the pictures in Figure 3.1.



$c_{V^*,W}$ $c^{-1}_{V^*,W}$

Figure 3.1. $c_{V^*,W}$ *and its inverse*

The pictorial transcription of Proposition 3.1 is then in the next figure.

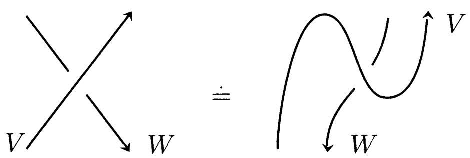


Figure 3.2. The equality of Proposition 3.1

A graphical proof of it is then:

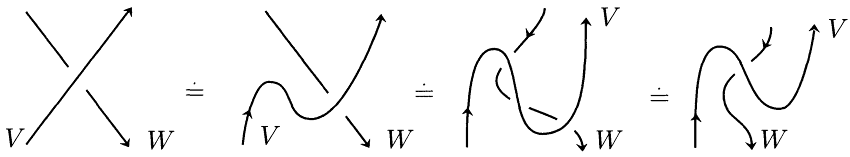


Figure 3.3. Proof of Proposition 3.1

The first equality follows from (2.1), the second one from Figure 1.8, and the last one from the naturality of the braiding $c_{-,W}$.

We go one step further by introducing the concept of a ribbon category.

Definition XIV.3.2. *Let $(\mathcal{C}, \otimes, I)$ be a strict braided tensor category with left duality.*

(a) *A twist is a family $\theta_V : V \to V$ of natural isomorphisms indexed by the objects V of $\mathcal{C}$ such that*

$$\theta_{V\otimes W} = (\theta_V \otimes \theta_W) c_{W,V} c_{V,W} \tag{3.1}$$

and

$$\theta_{V^*} = (\theta_V)^* \tag{3.2}$$

for all objects V, W in $\mathcal{C}$.

(b) *A ribbon category is a strict braided tensor category with left duality and with a twist.*

The naturality of the twist means that for any morphism $f : V \to W$ we have $\theta_W f = f\theta_V$. Using the graphical conventions of Sections 1–2, Relations (3.1–3.2) may be represented as in Figures 3.4 and 3.5.

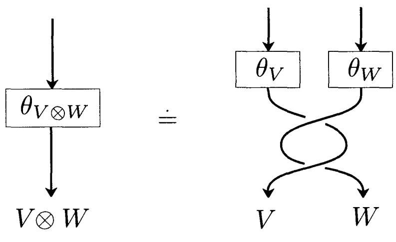


Figure 3.4. Relation (3.1)

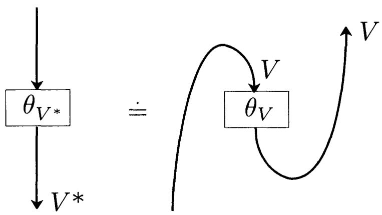


Figure 3.5. Relation (3.2)

The following gives alternative expressions for Relation (3.1).

Lemma XIV.3.3. (a) *Given objects V and W of $\mathcal{C}$ we have*

$$\theta_{V\otimes W} = c_{W,V}c_{V,W}(\theta_V \otimes \theta_W) = c_{W,V}(\theta_W \otimes \theta_V)c_{V,W}. \tag{3.3}$$

(b) *We also have $\theta_I = \mathrm{id}_I$.*

PROOF. (a) See Figure 3.6. All equalities follow by naturality of the braiding and of the twist.

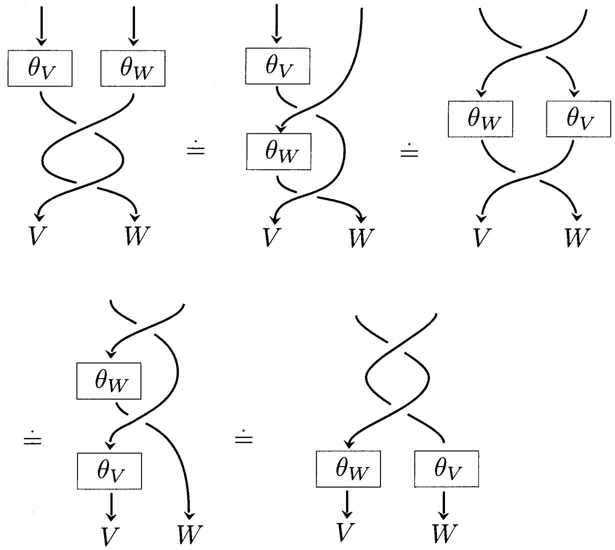


Figure 3.6. Proof of Lemma 3.3 (a)

(b) Using Relation (3.1) when $V = W = I$, we get

$$\theta_{I\otimes I} = (\theta_I \otimes \theta_I)c_{I,I}c_{I,I}.$$

Since $c_{I,I} = \mathrm{id}_I$ (by Proposition XIII.1.2) and by naturality of the identification of $V \otimes I$ with I, we get $\theta_{I\otimes I} = \theta_I \otimes \mathrm{id}_I = \theta_I \otimes \theta_I$, which implies the statement. □

Example 1. Let $\mathit{Vect}_f(k)$ be the category of finite-dimensional vector spaces over a field k. As we know, it is braided by the flip and it has left duals given by (2.7–2.8). This category is a ribbon category with trivial twist $\theta_V = \mathrm{id}_V$.

Example 2. Any symmetric tensor category $\mathcal{C}$ with left duality is a ribbon category with twist given by $\theta_V = \mathrm{id}_V$ for any object V. In this class falls the category A-Mod_f of finite-dimensional modules over a *cocommutative*

Hopf algebra or over a braided Hopf algebra A with universal R-matrix R such that $\tau_{A,A}(R) = R^{-1}$ (see Exercise 1 in Chapter XIII).

Using the braiding and the twist, we define morphisms $b'_V : I \to V^* \otimes V$ and $d'_V : V \otimes V^* \to I$ for any object V of the ribbon category $\mathcal{C}$ by

$$b'_V = (\mathrm{id}_{V^*} \otimes \theta_V) c_{V,V^*} b_V \tag{3.4}$$

and

$$d'_V = d_V c_{V,V^*} (\theta_V \otimes \mathrm{id}_{V^*}). \tag{3.5}$$

We shall agree to represent b'_V and d'_V as in Figure 3.7.

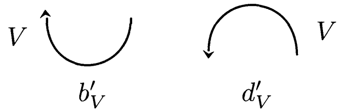


Figure 3.7. The morphisms b'_V and d'_V

Let us prove that the morphisms b'_V and d'_V equip $\mathcal{C}$ with the structure of a category with right duality, where ${}^*V = V^*$. Before we give a precise statement, we shall prove the following technical lemma.

Lemma XIV.3.4. *For any object V of a ribbon category, we have*

$$\begin{aligned} \theta_V^{-2} &= (d_V \otimes \mathrm{id}_V)(\mathrm{id}_{V^*} \otimes c_{V,V}^{-1})(c_{V,V^*} b_V \otimes \mathrm{id}_V) \\ &= (d_V c_{V,V^*} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c_{V,V^*} b_V) \\ &= (\mathrm{id}_V \otimes d_V c_{V,V^*})(c_{V,V}^{-1} \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes b_V). \end{aligned}$$

PROOF. The equalities of this lemma are represented in Figure 3.8.

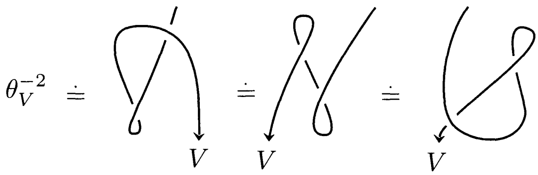


Figure 3.8. The equalities of Lemma 3.4

It is clear from the pictures that the naturality of the braiding implies the last two equalities. So it is enough to prove the first one.

By naturality of the twist and by Lemma 3.3 (b), we get

$$b_V = b_V \theta_I = \theta_{V \otimes V^*} b_V. \tag{3.6}$$

Let us denote by f the second term of the equalities in Lemma 3.4. Figure 3.9 shows that the right-hand side of (3.6) is equal to $(\theta_V^2 f \otimes \mathrm{id}_{V^*}) b_V$.

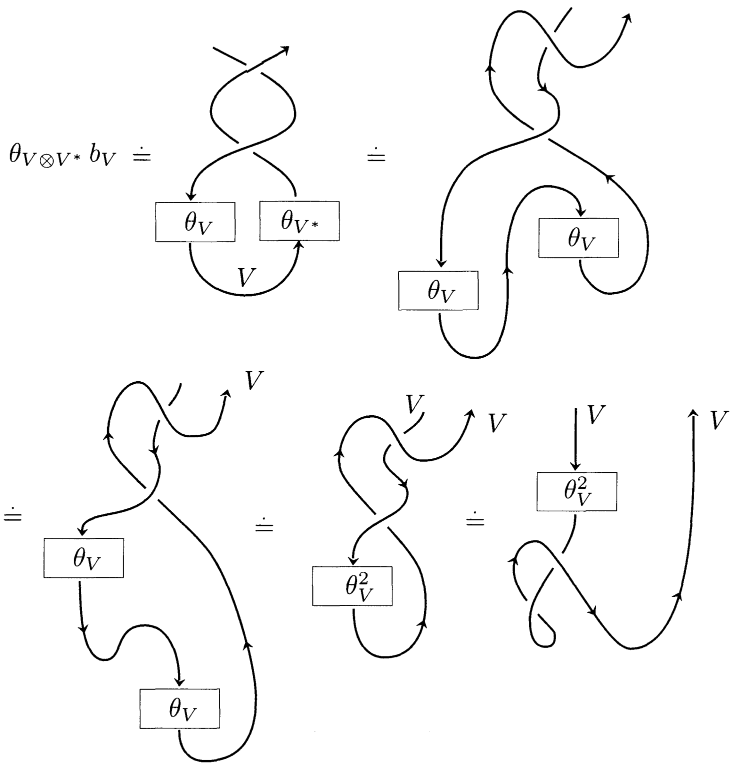


Figure 3.9. Proof of $\theta_{V\otimes V^*} b_V = (\theta_V^2 f \otimes \mathrm{id}_{V^*}) b_V$

Therefore, applying Relations (2.1) twice, we get

$$\mathrm{id}_V = (\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes d_V)\Big((\theta_V^2 f \otimes \mathrm{id}_{V^*}) b_V \otimes \mathrm{id}_V\Big) = \theta_V^2 f.$$

Therefore, we have $f = \theta_V^{-2}$, as desired.

As for Figure 3.9, the first equality follows from (3.1), the second one from Proposition 3.1 and from (3.2), the third one from the naturality of the tensor product, the fourth one from (2.1), and the last one from the naturality of the braiding. □

Proposition XIV.3.5. *Under the previous hypotheses, we have*

$$(d'_V \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes b'_V) = \mathrm{id}_V \tag{3.7}$$

and

$$(\mathrm{id}_{V^*} \otimes d'_V)(b'_V \otimes \mathrm{id}_{V^*}) = \mathrm{id}_{V^*}. \tag{3.8}$$

PROOF. (a) By (3.4–3.5) and the naturality of the tensor product, we have

$$\begin{aligned}
&(d'_V \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes b'_V)\\
&\quad = \Big(d_V c_{V,V^*}(\theta_V \otimes \mathrm{id}_{V^*}) \otimes \mathrm{id}_V\Big)\Big(\mathrm{id}_V \otimes (\mathrm{id}_{V^*} \otimes \theta_V) c_{V,V^*} b_V\Big)\\
&\quad = \theta_V g \theta_V
\end{aligned}$$

where g is the third term of the equalities in Lemma 3.4. Consequently, we have

$$(d'_V \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes b'_V) = \theta_V \theta_V^{-2} \theta_V = \mathrm{id}_V.$$

(b) The proof of (3.8) is in Figure 3.10.

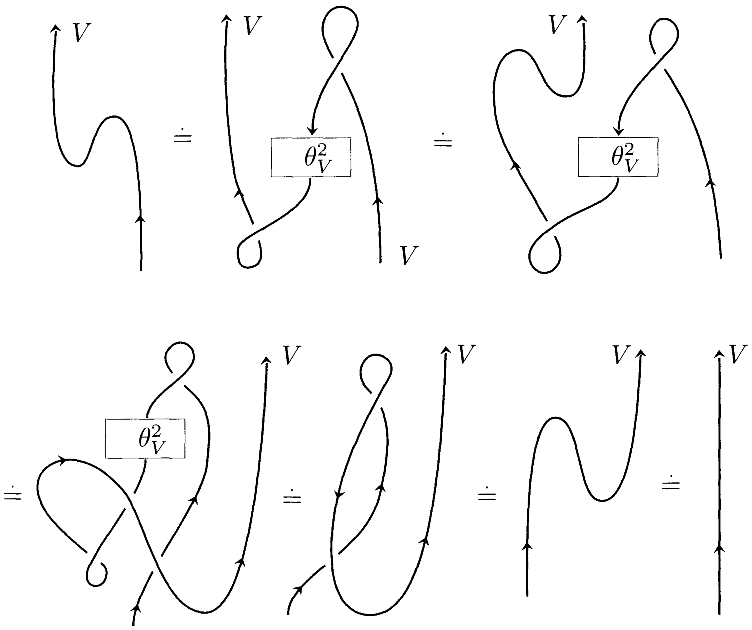


Figure 3.10. Proof of Relation (3.8)

The first equality is by definition, the second one by (2.1), the third one by naturality of the braiding, the fourth one by Lemma 3.4, the fifth one follows from Figure 1.8, and the last one from (2.1). □

It follows from Proposition 3.5 that a ribbon category is autonomous with ${}^*V = V^*$ in the sense of Section 2.

Corollary XIV.3.6. *Any object V of a ribbon category is isomorphic to its bidual $V^{**} = (V^*)^*$.*

PROOF. We saw in Section 2 that $V \cong {}^*(V^*)$ in any autonomous tensor category. □

The pictures in Figure 3.11 represent isomorphisms between V and V^{**}.

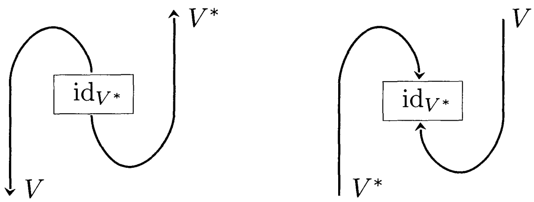


Figure 3.11. Isomorphisms between V and V^{**}

XIV.4 Quantum Trace and Dimension

By analogy with II.3 we define the concept of trace in a ribbon category.

Definition XIV.4.1. *Let $\mathcal{C}$ be a ribbon category with unit I. For any object V of $\mathcal{C}$ and any endomorphism f of V, we define the quantum trace $\mathrm{tr}_q(f)$ of f as the element*

$$\mathrm{tr}_q(f) = d'_V(f \otimes \mathrm{id}_{V^*})b_V = d_V c_{V,V^*}(\theta_V f \otimes \mathrm{id}_{V^*})b_V$$

of the monoid $\mathrm{End}(I)$*, i.e., as the composition of the morphisms*

$$I \xrightarrow{b_V} V \otimes V^* \xrightarrow{\theta_V f \otimes \mathrm{id}_{V^*}} V \otimes V^* \xrightarrow{c_{V,V^*}} V^* \otimes V \xrightarrow{d_V} I.$$

This notion coincides with the usual trace when $\mathcal{C}$ is the category $\mathit{Vect}_f(k)$ (see Proposition II.3.5). Graphical representations of $\mathrm{tr}_q(f)$ are given in Figure 4.1.

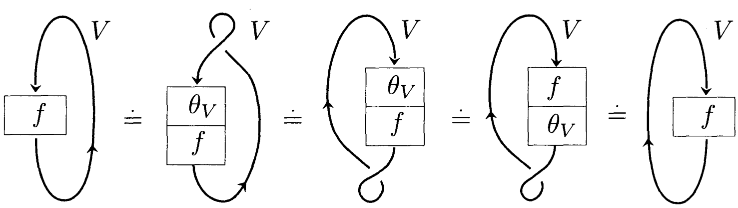


Figure 4.1. The quantum trace of f

The first equality is by definition, the second one by naturality of the braiding, the third one by naturality of the twist, and the last one by definition of b'_V.

The quantum trace enjoys the usual properties of the trace in linear algebra.

Theorem XIV.4.2. *Given endomorphisms f and g in a ribbon category, we have*

(a) $\mathrm{tr}_q(fg) = \mathrm{tr}_q(gf)$ *whenever f and g are composable,*
(b) $\mathrm{tr}_q(f \otimes g) = \mathrm{tr}_q(f)\,\mathrm{tr}_q(g)$, *and*
(c) $\mathrm{tr}_q(f) = \mathrm{tr}_q(f^*)$ *in the monoid* $\mathrm{End}(I)$.

PROOF. (a) The proof of the first relation is in Figure 4.2.

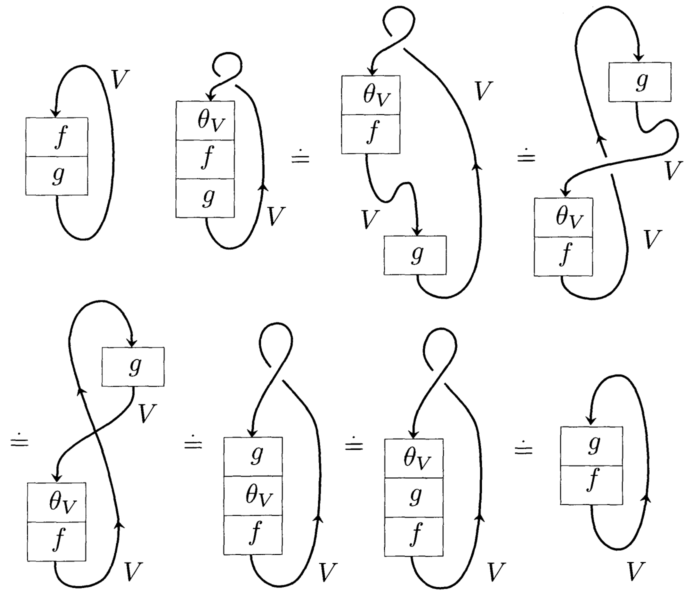


Figure 4.2. The proof of $\mathrm{tr}_q(fg) = \mathrm{tr}_q(gf)$

The first equality is by definition, the second one by (2.1), the third one by naturality of the braiding, the fourth one by (2.1), the fifth one by naturality of the braiding, the sixth one by naturality of the twist, and the last one by definition.

(b) We know from Proposition XI.2.4 that the composition in $\mathrm{End}(I)$ coincides with the tensor product. Therefore, it is equivalent to prove that $\mathrm{tr}_q(f \otimes g) = \mathrm{tr}_q(f) \otimes \mathrm{tr}_q(g)$. The proof of the latter is in Figure 4.3. The first and last equalities in that diagram on the next page follow from the definition, the second one from (3.1), the third one from (2.1), the fourth, sixth, and seventh ones by naturality of the braiding.

(c) The proof of $\mathrm{tr}_q(f) = \mathrm{tr}_q(f^*)$ is in Figure 4.4 two pages on.

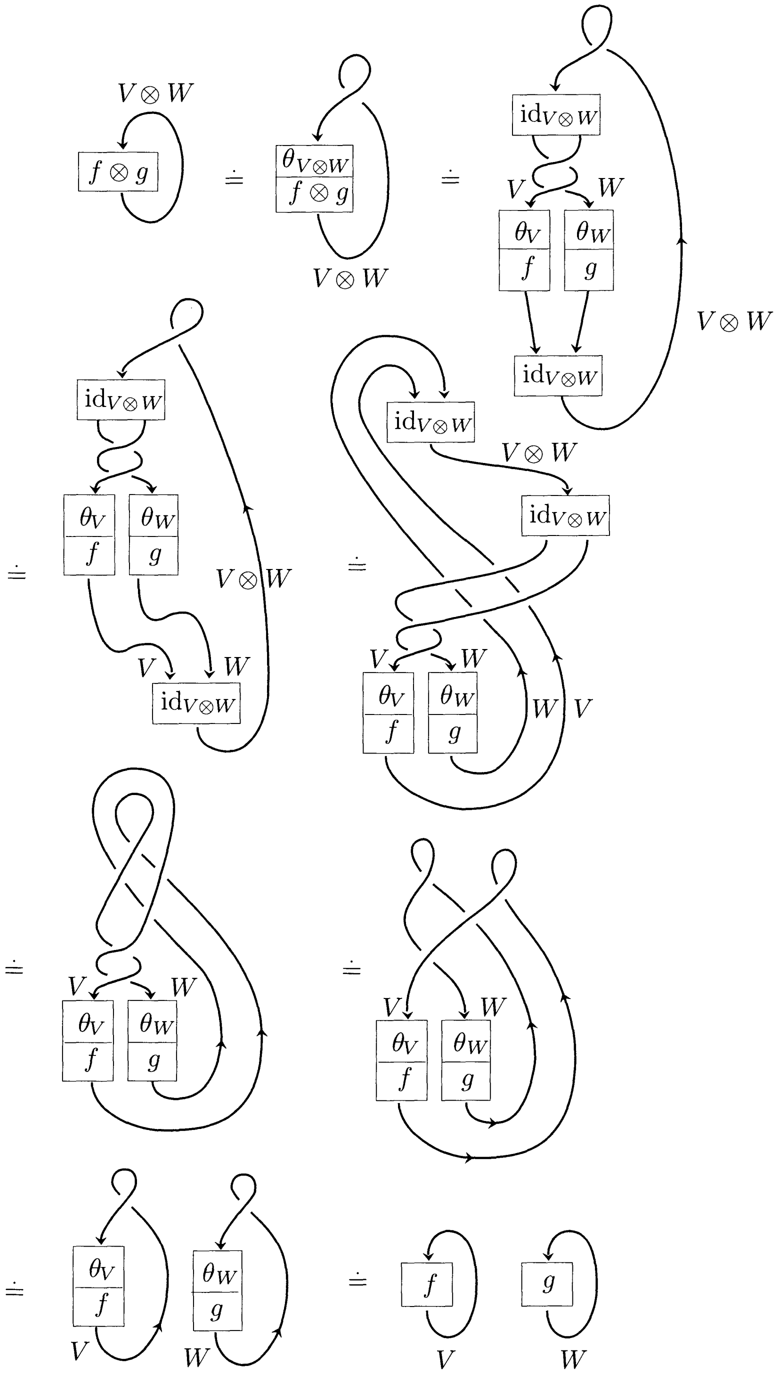


Figure 4.3. The proof of $\mathrm{tr}_q(f \otimes g) = \mathrm{tr}_q(f) \otimes \mathrm{tr}_q(g)$

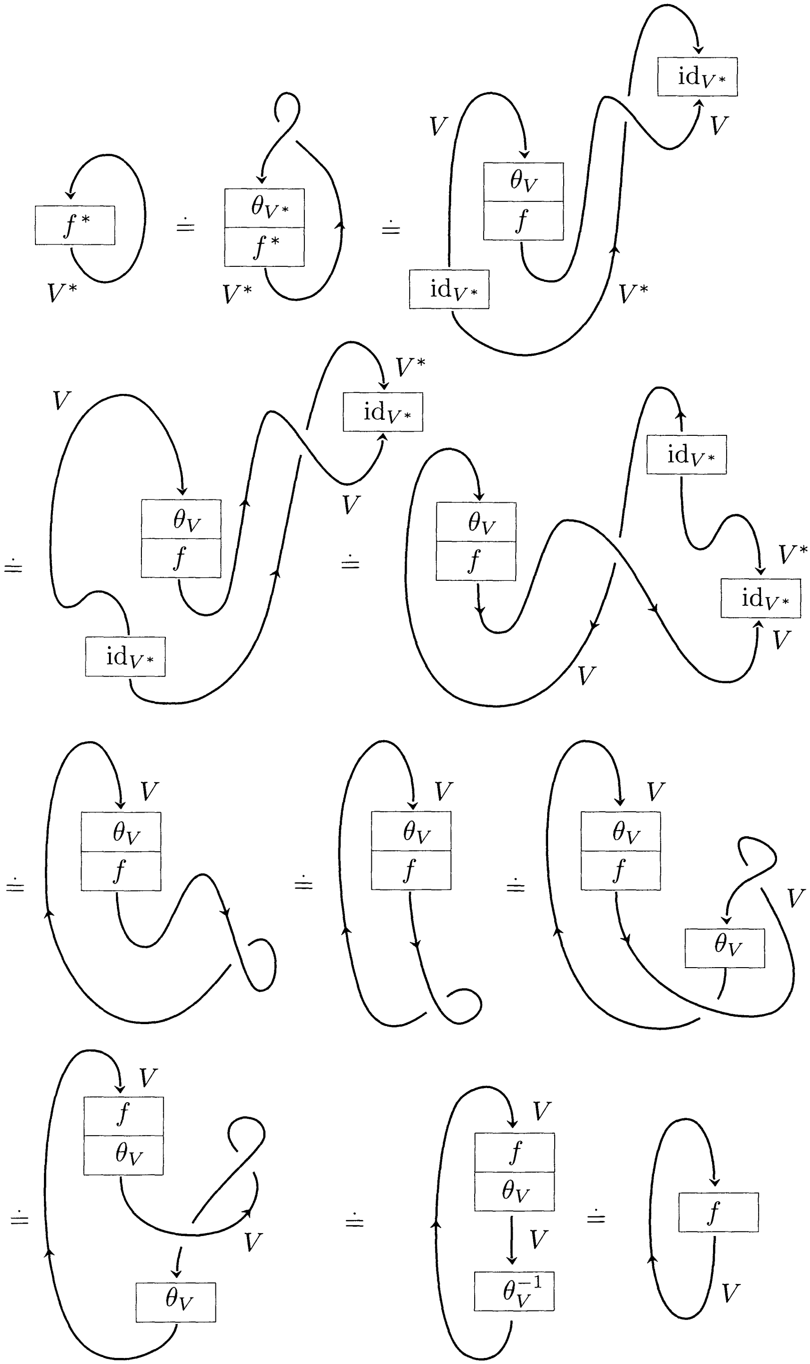


Figure 4.4. The proof of $\mathrm{tr}_q(f) = \mathrm{tr}_q(f^*)$

The first equality in the diagram on the previous page follows by definition, the second one by (3.2) and by Proposition 3.1, the third one by (3.8), the fourth one by naturality of the braiding, the fifth and the sixth ones by (2.1), the seventh one by definition of d'_V, the eighth one by naturality of the twist, the ninth one by Lemma 3.4. □

As in the classical case, we can derive a concept of dimension from the trace.

Definition XIV.4.3. *Let $\mathcal{C}$ be a ribbon category with unit I. For any object V of $\mathcal{C}$ we define the quantum dimension* $\dim_q(V)$ *as the element*

$$\dim_q(V) = \mathrm{tr}_q(\mathrm{id}_V) = d'_V b_V$$

of the monoid $\mathrm{End}(I)$.

The quantum dimension of V is represented in Figure 4.5.

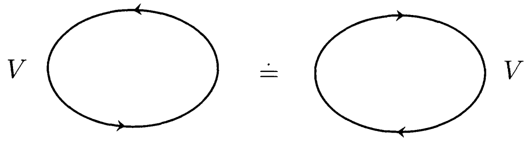


Figure 4.5. The quantum dimension of V

As an immediate consequence of Theorem 4.2, we have

Corollary XIV.4.4. *Let V, W be objects of a ribbon category. Then*

$$\dim_q(V \otimes W) = \dim_q(V)\dim_q(W) \quad \textit{and} \quad \dim_q(V^*) = \dim_q(V).$$

XIV.5 Examples of Ribbon Categories

XIV.5.1 Ribbons

In X.8 we defined the concept of a framed tangle, also called a ribbon, and we explained how ribbons could be represented by tangle diagrams. We now wish to show that ribbons allow one to build a ribbon category $\mathcal{R}$ which is universal for ribbon categories, as the category $\mathcal{B}$ of braids is universal for braided tensor categories.

The category $\mathcal{R}$ of ribbons is defined in the same way as the category $\mathcal{T}$ of tangles in XII.2: the objects of $\mathcal{R}$ are the same as the objects of $\mathcal{T}$; the morphisms of $\mathcal{R}$ are isotopy classes of framed tangles. Composition, identity, tensor product, and unit are defined as in $\mathcal{T}$. The braiding of the braid category $\mathcal{B}$ (see XIII.2) clearly defines a braiding in $\mathcal{R}$.

Let us endow the strict braided tensor category $\mathcal{R}$ with a left duality. Let ε be an object of $\mathcal{R}$, namely a finite sequence $(\varepsilon_1, \dots, \varepsilon_n)$ of $\pm$ signs. Define a dual object ε^* by the sequence $(-\varepsilon_n, \dots, -\varepsilon_1)$. The maps $b_\varepsilon : \emptyset \to \varepsilon \otimes \varepsilon^*$ and $d_\varepsilon : \varepsilon^* \otimes \varepsilon \to \emptyset$ are the framed tangles represented in Figure 5.1, the orientation of the strands being uniquely determined by the sequence of signs in ε.

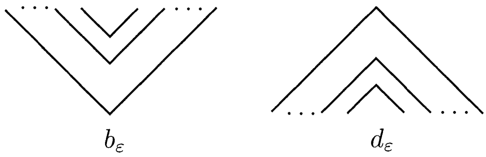


Figure 5.1. The framed tangles b_ε and d_ε

It is easy to check that the maps b_ε and d_ε satisfy Relation (2.1), thus equipping $\mathcal{R}$ with the structure of a strict braided tensor category with left duality. Observe that the transpose L^* of a ribbon L is isotopic to the ribbon obtained by rotating L through an angle π around an axis perpendicular to the plane of projection.

We define a twist on $\mathcal{R}$ as follows: $\theta_{(+)}$ is the left ribbon of Figure X.8.1 oriented downwards (also represented by the left tangle diagram of Figure X.8.2). The right ribbon of Figure X.8.1 oriented downwards defines the inverse of $\theta_{(+)}$ (it is represented by the right tangle diagram of Figure X.8.2). To define the twist for an arbitrary object, we use Relations (3.1–3.2). Check that, if ε is of length n, then θ_ε is obtained by twisting by an angle of 2π the plane containing n vertical flat ribbons.

Quantum trace and quantum dimension are defined in the ribbon category $\mathcal{R}$ by the formulas of Section 4. One can check using Reidemeister Transformations (I') and (II) that if L is a ribbon with $s(L) = b(L)$, then its quantum trace $\mathrm{tr}_q(L)$ is the closure of L drawn in Figure 5.2. Quantum dimensions are trivial links with the framing pointing to the reader.

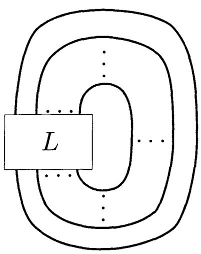


Figure 5.2. The quantum trace $\mathrm{tr}_q(L)$

The category $\mathcal{R}$ has two universal properties similar to the ones given for the category $\mathcal{B}$ in XIII.3. For the first one which corresponds to Theorem XII.4.2, we refer to [JS91c] [Tur89]. We state the second one paralleling Corollary XIII.3.8.

Theorem XIV.5.1. *Let $\mathcal{C}$ be a ribbon category and V be an object of $\mathcal{C}$. Then there exists a unique strict tensor functor F_V from $\mathcal{R}$ to $\mathcal{C}$ preserving the braiding, the left duality and the twist, such that $F_V(+) = V$.*

A proof of Theorem 5.1 can be found in several places: [FY89] [JS93] [RT90] [RT91] [Shu90] [Tur94]. This theorem produces isotopy invariants for framed links with values in the endomorphism monoid of the unit object I. Indeed, let L be a framed link. It can be viewed as an endomorphism of the unit $\emptyset$ of the category $\mathcal{R}$. Its image $F_V(L)$ is an endomorphism of I. The isotopy invariant $F_V(L)$ can be computed using the following algorithm: take a planar diagram representing the framed link L and colour each connected component with the object V. In the category $\mathcal{R}$, the framed link L is obtained by composition and tensor product of $\downarrow$, $\uparrow$, X_+, $\cup$, $\cap$, the twist, etc. Then one gets $F_V(L)$ by replacing the above diagrams respectively by id_V, id_{V^*}, $c_{V,V}$, b_V, d_V, θ_V, etc. in the categorical expression for L.

The category $\mathcal{T}$ of tangles is also a ribbon category. The only difference with $\mathcal{R}$ lies with the twist: we have $\theta_{(\pm)} = \mathrm{id}_{(\pm)}$ in $\mathcal{T}$. The twist of a general object of $\mathcal{T}$ can be defined from (3.1–3.2). There is a statement similar to Theorem 5.1 for the category $\mathcal{T}$. It suffices to replace $\mathcal{R}$ by $\mathcal{T}$ and to add the hypothesis $\theta_V = \mathrm{id}_V$ to the category $\mathcal{C}$.

XIV.5.2 Crossed G-sets

In XIII.1.4 we considered a category $X_G(G)$ of crossed G-sets where G is a group. Assume G is finite. We construct a ribbon category $\mathbf{Z}[X_G(G)]$ out of $X_G(G)$. Its objects are the same as for $X_G(G)$, namely the denumerable set of crossed sets $\{1, G, G^{\otimes 2}, \ldots\}$. A morphism $G^{\otimes n} \to G^{\otimes m}$ in $\mathbf{Z}[X_G(G)]$ is an integral matrix M indexed by all elements $(x, y) \in G^n \times G^m$ such that $M_{xg,yg} = M_{x,y}$ for all $g \in G$ and $M_{x,y} = 0$ if $|x| \neq |y|$. Composition is defined by multiplying matrices. The category $\mathbf{Z}[X_G(G)]$ is a strict tensor category. Observe that the monoid $\mathrm{End}(1)$ is the ring $\mathbf{Z}$. The braiding of $X_G(G)$ extends linearly to a braiding on $\mathbf{Z}[X_G(G)]$.

Define the duality as follows: the dual of $G^{\otimes n}$ has the same underlying G-set as $G^{\otimes n}$, but $||$ is replaced by $||^{-1}$. The maps $b_{G^{\otimes n}}$ and $d_{G^{\otimes n}}$ are defined by

$$b_{G^{\otimes n}}(1) = \sum_{g_1,\ldots,g_n \in G} (g_1, \ldots, g_n, g_1, \ldots, g_n)$$

and

$$d_{G^{\otimes n}}(g_1, \ldots, g_n, h_1, \ldots, h_n) = \delta_{g_1,h_1} \cdots \delta_{g_n,h_n}.$$

Relations (2.1) are satisfied. A twist $\theta_{G^{\otimes n}}$ is defined inductively on n by (3.1) and its initial value $\theta_G = \mathrm{id}_{\mathbf{Z}[G]}$. One checks that the quantum dimension of an object is its cardinality:

$$\dim_q(G^{\otimes n}) = \mathrm{card}\,(G)^n.$$

Apply Theorem 5.1 to the ribbon category $\mathbf{Z}[X_G(G)]$ and to the object G. We get an endomorphism $F_G(L)$ of the unit object $\{1\}$, i.e., an integer for any (framed) link L. We invite the reader to use the algorithm described above to compute this isotopy invariant for a few simple links. For instance, $F_G(L)$ is equal to the number of couples $(g_1, g_2) \in G \times G$ with $g_1 g_2 = g_2 g_1$ if L is the Hopf link, and with $g_1 g_2 g_1^{-1} = g_2^{-1} g_1 g_2$ if L is the trefoil knot. Freyd and Yetter [FY89] proved for a general link L that $F_G(L)$ is the number of group homomorphisms from the fundamental group of L to G.

XIV.6 Ribbon Algebras

We conclude this chapter by giving examples of ribbon categories consisting of modules over braided Hopf algebras (defined in VIII.2). Let D be a braided Hopf algebra with universal R-matrix $R = \sum_i s_i \otimes t_i \in D \otimes D$. Set

$$u = \sum_i S(t_i) s_i. \tag{6.1}$$

We showed in VIII.4 that u was an invertible element of D with inverse

$$u^{-1} = \sum_i t_i S^2(t_i) = \sum_i S^{-1}(t_i) S(s_i), \tag{6.2}$$

that $uS(u) = S(u)u$ was central in D, and that we had

$$\varepsilon(u) = 1 \quad \text{and} \quad \Delta(u) = (R_{21}R)^{-1}(u \otimes u). \tag{6.3}$$

Moreover, the square of the antipode is given for any x in D by

$$S^2(x) = uxu^{-1}. \tag{6.4}$$

Definition XIV.6.1. *A braided Hopf algebra D is a ribbon algebra if there exists a central element θ in D satisfying the relations*

$$\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta), \quad \varepsilon(\theta) = 1, \quad \textit{and} \quad S(\theta) = \theta. \tag{6.5}$$

Ribbon algebras produce ribbon categories.

Proposition XIV.6.2. *For any ribbon algebra D, the tensor category D-Mod$_f$ is a ribbon category with twist θ_V given on any D-module V by the multiplication by the inverse of the central element θ.*

Conversely, if D is a finite-dimensional braided Hopf algebra and the braided category D-Mod$_f$ with left duality is a ribbon category, then D is a ribbon algebra.

PROOF. (a) Let D be a ribbon algebra with the distinguished central element θ. Braiding and duality in D-Mod$_f$ are given as in XIII.1.3 and XIV.2, Example 1. Define a twist θ_V for any D-module V by $\theta_V(v) = \theta^{-1} v$ where $v \in V$.

The endomorphism θ_V is a D-linear automorphism since the element θ is central and invertible. Let us prove Relations (3.1–3.2) using (6.5). We have

$$\begin{aligned}(\theta_V \otimes \theta_W) c_{W,V} c_{V,W}(v \otimes w) &= (\theta^{-1} \otimes \theta^{-1})(R_{21}R)(v \otimes w)\\ &= \Delta(\theta^{-1})(v \otimes w)\\ &= \theta_{V \otimes W}(v \otimes w).\end{aligned}$$

As for (3.2), we have for any $v \in V$ and $\alpha \in V^*$

$$\begin{aligned}< (\theta_V)^*(\alpha), v > &= < \alpha, \theta_V(v) >\\ &= < \alpha, \theta^{-1} v >\\ &= < \alpha, S(\theta^{-1}) v >\\ &= < \theta^{-1}\alpha, v >\\ &= < \theta_{V^*}\alpha, v > .\end{aligned}$$

(b) We now assume that D is finite-dimensional and that $D\text{-}Mod_f$ is a ribbon category. By Proposition XIII.1.4, we know that D is braided. Since D is assumed to be finite-dimensional, we consider the left D-module D and the corresponding twist θ_D. Define $\theta = \theta_D(1)^{-1}$. Then by functoriality of the twist, we have for any finite-dimensional D-module

$$\theta_V(v) = \theta_D(1)\, v = \theta^{-1} v.$$

The D-linearity of θ_D implies that θ is central. From Relation (3.1) we conclude that

$$\Delta(\theta^{-1}) = (\theta^{-1} \otimes \theta^{-1})(R_{21}R)$$

whereas Relation (3.2) implies $S(\theta^{-1}) = \theta^{-1}$. Finally, $\varepsilon(\theta) = 1$ follows from Lemma 3.3 (b). □

Corollary XIV.6.3. *The central element θ^2 of a ribbon algebra acts as $uS(u)$ on any finite-dimensional module.*

As a consequence, we see that $\theta^2 = uS(u)$ if D is finite-dimensional.

PROOF. By Proposition 6.2 we know that θ^2 acts as θ_V^{-2} on V. Now by Lemma 3.4 we have in any ribbon category

$$\theta_V^{-2} = (\mathrm{id}_V \otimes d_V c_{V,V^*})(c_{V,V}^{-1} \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes b_V).$$

It is therefore enough to compute the action on V of the right-hand side of this equality. Let $\{v_i\}_i$ be a basis of V and $\{v^i\}_i$ be the dual basis. Then, using (VIII.3.1–3.2), (6.1), and (6.4), we have for any $v \in V$

$$\begin{aligned}
(\mathrm{id}_V \otimes d_V c_{V,V^*})&(c_{V,V}^{-1} \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes b_V)(v) \\
&= \sum_{i,j,k} < t_k v^i, s_k t_j v > S(s_j) v_i \\
&= \sum_{i,j,k} < v^i, S(t_k) s_k t_j v > S(s_j) v_i \\
&= \sum_{j,k} S(s_j) S(t_k) s_k t_j \, v \\
&= \sum_{j} S(s_j) u t_j \, v \\
&= \sum_{j} S(s_j) S^2(t_j) u \, v \\
&= S\Big(\sum_{j} S(t_j) s_j\Big) u \, v \\
&= S(u) u \, v = u S(u) \, v.
\end{aligned}$$

□

Quantum trace and quantum dimension are given in the category of modules over a ribbon algebra by

Proposition XIV.6.4. *Let V be any finite-dimensional module over a ribbon algebra D. We have*

$$\mathrm{tr}_q(f) = \mathrm{tr}\Big(v \mapsto \theta^{-1} u f(v)\Big)$$

for any endomorphism f of V. In particular, $\dim_q(V)$ is equal to the trace of the multiplication by $\theta^{-1}u$ on V.

PROOF. Using the definitions of d'_V and of u as well as Proposition 6.2, we immediately get

$$d'_V(v \otimes \alpha) = \sum_i < t_i \alpha, s_i \theta^{-1} v > = \sum_i < \alpha, S(t_i) s_i \theta^{-1} v > = < \alpha, u\theta^{-1} v > .$$

Therefore,

$$\mathrm{tr}_q(f) = d'_V(f \otimes \mathrm{id}_{V^*}) b_V = \sum_i < v^i, \theta^{-1} u f(v_i) >,$$

which is the trace of the linear endomorphism $v \mapsto \theta^{-1} u f(v)$. □

We end with two examples of ribbon algebras.

Example 1. (*Sweedler's four-dimensional Hopf algebra*) Consider the Hopf algebra H of Example 2 in VIII.2. It is braided. An immediate computation shows that the element u_λ corresponding to the universal R-matrix R_λ is independent of the parameter λ and is given by $u_\lambda = x = S(u_\lambda)$. Therefore $u_\lambda S(u_\lambda) = x^2 = 1$. One checks that H is a ribbon algebra with $\theta = 1$.

Example 2. This is due to Reshetikhin and Turaev [RT91]. It deals with the Hopf algebra $\overline{U}_q$, i.e., the finite-dimensional quotient of $U_q(\mathfrak{sl}(2))$ considered in VI.5 for q a root of unity. In IX.6–7 we proved that $\overline{U}_q$ was a braided Hopf algebra and we computed its universal R-matrix. We resume the notation and conventions of Chapter IX. In particular, we assume that q is a root of unity of odd order $d > 1$.

Proposition XIV.6.5. *The Hopf algebra $\overline{U}_q$ is a ribbon algebra for which $\theta = K^{-1}u = uK^{-1}$.*

PROOF. The centrality of θ follows from (6.4) and from the fact that we also have

$$S^2(x) = KxK^{-1}$$

for all $x \in \overline{U}_q$. It is immediate to check that $\varepsilon(\theta) = 1$. As for $\Delta(\theta)$, we have

$$\Delta(\theta) = \Delta(K^{-1})\Delta(u) = (K^{-1} \otimes K^{-1})(u \otimes u)(R_{21}R)^{-1} = (\theta \otimes \theta)(R_{21}R)^{-1}.$$

It remains to check that $S(\theta) = \theta$. This is equivalent to $KS(u) = K^{-1}u$. Now this can be verified directly using the formula given in IX.7 for the universal R-matrix for $\overline{U}_q$. Alternatively, there is an argument in [Dri89a], Section 5 which goes roughly as follows: let V_λ be a highest weight module with highest weight λ. Then $KS(u) = S(uK^{-1}) = S(K^{-1}u)$ acts like $K^{-1}u$ on the dual module V_λ^*. But the latter is isomorphic to V_λ. Therefore $KS(u)$ acts like the central element $K^{-1}u$ on V_λ. A general argument (see [RT91]) extends this to any module. $\square$

Let $V_n = V_{1,n}$ be the simple $\overline{U}_q$-module of VI.5. Its quantum dimension is given by

$$\dim_q(V_n) = [n+1] = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}.$$

In effect, by Proposition 6.4 it is given as the trace of the multiplication by $\theta^{-1}u$. Here $\theta^{-1}u = K$. Now K acts diagonally on V_n with eigenvalues $\{q^n, q^{n-2}, \dots, q^{-n+2}, q^{-n}\}$. Therefore,

$$\dim_q(V_n) = q^n + q^{n-2} + \cdots + q^{-n+2} + q^{-n} = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}} = [n+1].$$

XIV.7 Exercises

1. For any braided Hopf algebra D, define an algebra $D(\theta)$ as the quotient of the polynomial algebra $D[\theta]$ by the two-sided ideal generated by $\theta^2 - uS(u)$. Show that $D(\theta)$ has a unique Hopf algebra structure such that the natural inclusion of D in $D(\theta)$ is a Hopf algebra map and that $\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta)$, $\varepsilon(\theta) = 1$, and $S(\theta) = \theta$. Prove that $D(\theta)$ is a ribbon algebra.

2. Under the hypotheses of the previous exercise, show that the category of left $D(\theta)$-modules is equivalent to the category whose objects are pairs (V, θ_V) where V is a left D-module and θ_V is a D-linear automorphism of V such that for all v in V we have $\theta_V^{-2}(v) = uS(u)\, v$, and whose morphisms $(V, \theta_V) \to (W, \theta_W)$ are the D-linear maps f from V to W such that $f\theta_V = \theta_W f$.

3. Using the definitions and the notation of 5.2, compute $\theta_{G^{\otimes n}}$ for $n > 1$.

4. Given a finite abelian group A and a commutative ring K, let $K(A)$ be the commutative K-algebra of K-valued functions on A. It has a basis $\{e_a\}_{a \in A}$ over K such that the multiplication is given by $e_a e_b = \delta_{a,b}$ for all $a, b \in A$.

   (a) Show that there is a unique Hopf algebra structure on $K(A)$ such that for all $a \in A$ we have
$$\Delta(e_a) = \sum_{b \in A} e_b \otimes e_{a-b}, \quad \varepsilon(e_a) = \delta_{a,0}, \quad S(e_a) = e_{-a}.$$

   (b) Let $R = \sum_{a,b \in A} c(a,b)\, e_a \otimes e_b$ where c is a function with values in the group $K^\times$ of invertible elements of K. Prove that R equips $K(A)$ with the structure of a braided Hopf algebra if and only if
$$c(a + a', b) = c(a,b)c(a',b) \quad \text{and} \quad c(a, b + b') = c(a,b)c(a,b')$$
   for all $a, a', b, b' \in A$.

   (c) We assume that $K(A)$ with R as defined in (b) is a braided Hopf algebra. Let χ be a group homomorphism from A to $K^\times$ such that $\chi(a)^2 = 1$ for all $a \in A$. Show that $\theta = \sum_{a \in A} \chi(a) c(a,a) e_a$ endows $K(A)$ with the structure of a ribbon algebra.

   (d) Check that the quantum dimension of the $K(A)$-module $e_a K(A)$ is equal to $\chi(a)$.

5. (*Coribbon algebras*) Let $H = (H, \mu, \eta, \Delta, \varepsilon, S, r)$ be a cobraided Hopf algebra with universal R-form r (see VIII.5). It is a coribbon algebra if there exists an invertible central element ζ of the dual algebra H^* such that
$$\zeta \circ \mu = \bar{r} \star \bar{r}_{21} \star (\zeta \otimes \zeta), \quad \zeta(1) = 1, \quad \text{and} \quad \zeta \circ S = S$$

where $\bar{r}$ is the inverse of r for the convolution $\star$ in $(H \otimes H)^*$ and where $\bar{r}_{21} = r \circ \tau_{H,H}$. Prove that the category of finite-dimensional H-comodules is a ribbon category.

6. Show that the cobraided Hopf algebra $SL_q(2)$ is a coribbon algebra with central linear form ζ determined by

$$\zeta(a) = \zeta(d) = q^{-3/2} \quad \text{and} \quad \zeta(b) = \zeta(c) = 0.$$

XIV.8 Notes

The graphical calculus described in Section 1 was advocated in many papers, e.g., [FY89] [FY92] [JS91a] [Kau91] [RT90] [RT91].

The concept of duality in a tensor category appeared in the classical references quoted in Chapter XIII. The examples presented in this book require distinguishing carefully between left and right duality. In Section 2 we followed Joyal and Street's treatment of duality as proposed in [JS93] (see also [FY89]). There Joyal and Street also introduced the concept of a twist in a strict braided tensor category and the concept of a ribbon category. Actually, they called the latter tortile tensor categories. The name used here was coined by Turaev [Tur92].

Definition 4.1 is due to Turaev [Tur92] generalizing previous definitions of [KL80] and [FY89]. We devised a proof of Theorem 4.2 highlighting the power of the graphical calculus of Section 1 (a different proof can be found in [Tur94]).

Ribbon algebras were invented by Reshetikhin and Turaev [RT90] who also showed that the quantum groups of Drinfeld and Jimbo gave birth to ribbon algebras.

The construction of the ribbon algebra $D(\theta)$ of Exercise 1 is taken from [RT90]. Exercise 4 is due to Turaev: this example does not produce any interesting isotopy invariant. Exercise 5 is from [JS91b]. Exercise 6 is due to the author.

There exists an elaboration of the centre construction of XIII.4, to be found in [KT92], which assigns to any strict tensor category $\mathcal{C}$ with left duality a ribbon category $\mathcal{D}(\mathcal{C})$. It is related to the quantum double of a finite-dimensional Hopf algebra A with invertible antipode and to the construction of Exercise 1 by the equivalence of ribbon categories

$$\mathcal{D}(A\text{–}Mod_f) \cong D(A)(\theta)\text{–}Mod_f.$$

By colouring framed tangle diagrams with objects and morphisms of a strict tensor category $\mathcal{C}$, we may construct a ribbon category $\mathcal{R}(\mathcal{C})$ with the following property: the construction $\mathcal{C} \mapsto \mathcal{R}(\mathcal{C})$ is functorial and is left adjoint to the forgetful functor from the category of ribbon categories to the category of strict tensor categories. In other words, given a ribbon category $\mathcal{C}'$, there is a natural bijection between the set of strict braided tensor functors preserving duality and twist from $\mathcal{R}(\mathcal{C})$ to $\mathcal{C}'$ and the set of strict tensor functors from $\mathcal{C}$ to $\mathcal{C}'$. In particular, if $\mathcal{C}$ is a ribbon category, then the identity functor of $\mathcal{C}$ corresponds to a functor $F_{\mathcal{C}} : \mathcal{R}(\mathcal{C}) \to \mathcal{C}$ preserving tensor products, braidings, duality and twists. For more details on the category $\mathcal{R}(\mathcal{C})$, see [FY89] [JS93] [RT90] [RT91] [Tur92] [Tur94].

The existence of the functor $F_{\mathcal{C}}$ allows one to find isotopy invariants for framed links with values in the endomorphism monoid of the unit object of the ribbon category $\mathcal{C}$. Proceed as at the end of 5.1. The main difference is that we are now permitted to colour the connected components of a link with different objects of the category rather than with one single object.

Chapter XV
Quasi-Bialgebras

The aim of this chapter is to present Drinfeld's concepts of (braided) quasi-bialgebra and of gauge transformation. These concepts will be needed to express the main results of Part IV. The definitions given here are based on the formalism of tensor categories and tensor functors introduced in Chapters XI and XIII. In Section 4 we construct braid group representations for any braided quasi-bialgebra and we show that equivalent braided quasi-bialgebras give rise to equivalent representations.

We shall make frequent use of the convention of VIII.2 regarding subscripts.

XV.1 Quasi-Bialgebras

In XI.3.1 we introduced the notion of an algebra (A, Δ, ε) with comultiplication and counit: it is an associative unital k-algebra A with a morphism of algebras $\Delta : A \to A \otimes A$ (the comultiplication) and a morphism of algebras $\varepsilon : A \to k$ (the counit). We observed that the classical tensor product on *Vect*(k) restricted to a tensor product on the category A-*Mod* of left A-modules, for which $I = k$ is a unit.

Definition XV.1.1. *Let $A = (A, \Delta, \varepsilon)$ be an algebra with comultiplication and counit. It is a quasi-bialgebra if the category A-Mod equipped with the tensor product of* Vect(k) *is a tensor category.*

In other words, (A, Δ, ε) is a quasi-bialgebra if there exists an associativity constraint a, a left unit constraint l, and a right unit constraint r satisfying the Pentagon Axiom (XI.2.6) and the Triangle Axiom (XI.2.9). When these constraints are the usual ones of *Vect*(k), then A is a bialgebra by Proposition XI.3.1. We now give a characterization of quasi-bialgebras, which actually is Drinfeld's original definition in [Dri89b].

Proposition XV.1.2. *Let (A, Δ, ε) be an algebra with comultiplication and counit as above. It is a quasi-bialgebra if and only if there exist an invertible element Φ in $A \otimes A \otimes A$ and invertible elements l, r in A such that*

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \Phi\Big((\Delta \otimes \mathrm{id})(\Delta(a))\Big)\Phi^{-1}, \tag{1.1}$$

$$(\varepsilon \otimes \mathrm{id})(\Delta(a)) = l^{-1}al, \quad (\mathrm{id} \otimes \varepsilon)(\Delta(a)) = r^{-1}ar \tag{1.2}$$

for all $a \in A$,

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)\,(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) = \Phi_{234}\,(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)\,\Phi_{123}, \tag{1.3}$$

and

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = r \otimes l^{-1}. \tag{1.4}$$

Here $\Phi_{123} = \Phi \otimes 1$ and $\Phi_{234} = 1 \otimes \Phi$ according to the conventions of VIII.2. When $\Phi = 1 \otimes 1 \otimes 1$ and $l = r = 1$, we recover the usual definition of a bialgebra. From Proposition 1.2 we see that the main difference between a bialgebra and a quasi-bialgebra lies in the fact that the comultiplication of a quasi-bialgebra is no longer coassociative. Nevertheless, Relation (1.1) shows that it is almost. This situation is reminiscent of braided bialgebras whose non-cocommutativity is also controlled (see VIII.2).

The elements Φ, l and r are part of the definition of a quasi-bialgebra. Therefore, we shall denote such a quasi-bialgebra by $(A, \Delta, \varepsilon, \Phi, l, r)$. The element Φ is sometimes called the *Drinfeld associator* of A.

Proof. (a) Let Φ, l and r be elements satisfying Relations (1.1–1.4). For any triple (U, V, W) of A-modules, define an associativity constraint by

$$a_{U,V,W}\Big((u \otimes v) \otimes w\Big) = \Phi\Big(u \otimes (v \otimes w)\Big) \tag{1.5}$$

for $u \in U, v \in V$ and $w \in W$), and unit constraints

$$l_V(1 \otimes v) = lv \quad \text{and} \quad r_V(v \otimes 1) = rv. \tag{1.6}$$

The maps a, l, and r are isomorphisms because Φ, l, and r are invertible. They are A-linear thanks to Relations (1.1–1.2). Relations (1.3) and (1.4) imply the Pentagon and the Triangle Axioms respectively.

(b) Conversely, suppose that *A-Mod* is a tensor category with associativity and unit constraints a, l, and r. From the associativity constraint define an element Φ in $A \otimes A \otimes A$ by

$$\Phi = a_{A,A,A}(1 \otimes 1 \otimes 1). \tag{1.7}$$

Similarly, from the unit constraints we get elements

$$l = l_A(1 \otimes 1) \quad \text{and} \quad r = r_A(1 \otimes 1) \tag{1.8}$$

of A. Let us check that Φ, l and r satisfy the conditions of the proposition. First, these elements are invertible because the constraints are isomorphisms.

We next prove Relation (1.2). This is done as in the proof of Proposition XIII.1.4: by functoriality of the associativity constraint, for all $u \in U$, $v \in V$ and $w \in W$, we have the commutative square

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "a_{A,A,A}"] \arrow[d, "(\bar{u} \otimes \bar{v}) \otimes \bar{w}"] & A \otimes (A \otimes A) \arrow[d, "\bar{u} \otimes (\bar{v} \otimes \bar{w})"] \\
(U \otimes V) \otimes W \arrow[r, "a_{U,V,W}"] & U \otimes (V \otimes W)
\end{tikzcd}$$

where for any element u of an A-module U we denote by $\bar{u}$ the unique A-linear map from A to U sending 1 onto u. Hence

$$a_{U,V,W}\Big((u \otimes v) \otimes w\Big) = \Big(\bar{u} \otimes (\bar{v} \otimes \bar{w})\Big)(\Phi) = \Phi\Big(u \otimes (v \otimes w)\Big). \tag{1.9}$$

Let us express the A-linearity of $a_{U,V,W}$. On one hand we have

$$a_{U,V,W}\Big(a\Big((u \otimes v) \otimes w\Big)\Big) = \Phi\Big((\Delta \otimes \mathrm{id})(\Delta(a))\Big)\Big(u \otimes (v \otimes w)\Big).$$

On the other hand, we get

$$\begin{aligned}
a\Big(a_{U,V,W}\Big((u \otimes v) \otimes w\Big)\Big) &= a\Big(\Phi\Big(u \otimes (v \otimes w)\Big)\Big) \\
&= \Big((\mathrm{id} \otimes \Delta)(\Delta(a))\Phi\Big)\Big(u \otimes (v \otimes w)\Big).
\end{aligned}$$

Setting $u = v = w = 1 \in A$ yields Relation (1.1). Similarly, the functoriality of the unit constraints implies that

$$l_V(1 \otimes v) = lv \quad \text{and} \quad r_V(v \otimes 1) = rv. \tag{1.10}$$

The A-linearity of l_A and of r_A implies (1.2).

It remains to check Relations (1.3–1.4). By the Pentagon Axiom (XI.2.6) we have

$$a_{A,A,A\otimes A} \circ a_{A\otimes A,A,A} = (\mathrm{id}_A \otimes a_{A,A,A}) \circ a_{A,A\otimes A,A} \circ (a_{A,A,A} \otimes \mathrm{id}_A).$$

Applying both sides of this equation to $1 \otimes 1 \otimes 1 \otimes 1$ and using (1.9), we get Relation (1.3). A similar proof shows that the Triangle Axiom implies Relation (1.4). □

We shall see examples of quasi-bialgebras that are not bialgebras later. All of them will have trivial unit constraints, i.e., $l = r = 1$.

We also need the following concept: a *morphism of quasi-bialgebras*

$$\alpha : (A, \Delta, \varepsilon, \Phi, l, r) \to (A', \Delta', \varepsilon', \Phi', l', r')$$

is a morphism of algebras between the underlying algebras such that

$$(\alpha \otimes \alpha)\Delta = \Delta'\alpha, \quad (\alpha \otimes \alpha \otimes \alpha)(\Phi) = \Phi', \quad \alpha(l) = l', \quad \alpha(r) = r'. \tag{1.11}$$

It is an isomorphism of quasi-bialgebras if, in addition, it is invertible.

XV.2 Braided Quasi-Bialgebras

We now define the counterpart of braided algebras (VIII.2) in the context of quasi-bialgebras.

Definition XV.2.1. *A quasi-bialgebra $(A, \Delta, \varepsilon, \Phi, l, r)$ is braided if the corresponding tensor category A-Mod is braided.*

We characterize braided quasi-algebras (also called quasi-triangular quasi-bialgebras in the literature) as we did for quasi-bialgebras in Proposition 1.2.

Proposition XV.2.2. (a) *A quasi-bialgebra $(A, \Delta, \varepsilon, \Phi, l, r)$ is braided if and only if there exists an invertible element R in $A \otimes A$, called the universal R-matrix, such that for all $a \in A$ we have*

$$\Delta^{\mathrm{op}}(a) = R\Delta(a)R^{-1}, \tag{2.1}$$

$$(\mathrm{id} \otimes \Delta)(R) = (\Phi_{231})^{-1} R_{13} \Phi_{213} R_{12} (\Phi_{123})^{-1}, \tag{2.2}$$

and

$$(\Delta \otimes \mathrm{id})(R) = \Phi_{312} R_{13} (\Phi_{132})^{-1} R_{23} \Phi_{123}. \tag{2.3}$$

(b) *Moreover, the tensor category A-Mod is symmetric if and only if Relations* (2.1–2.3) *are satisfied together with the additional relation*

$$R_{21} = R^{-1}. \tag{2.4}$$

As in Section 1, we shall consider R as part of the data of a braided quasi-bialgebra and write $(A, \Delta, \varepsilon, \Phi, l, r, R)$.

PROOF. We proceed as in the proofs of Proposition 1.2 and Proposition XIII.1.4. First, given a braided quasi-bialgebra A with a universal R-matrix R, we define a braiding on the tensor category A-Mod by

$$c_{V,W}(v \otimes w) = \tau_{V,W}\Big(R(v \otimes w)\Big) \tag{2.5}$$

where v and w belong to the A-modules V and W respectively. As in the proof of Proposition VIII.3.1, Relation (2.1) implies that $c_{V,W}$ is A-linear whereas Relations (2.2–2.3) imply the Hexagon Axiom (XIII.1.3–1.4).

Conversely, if A-*Mod* is braided with braiding c, set

$$R = \tau_{A,A}\Big(c_{A,A}(1 \otimes 1)\Big). \tag{2.6}$$

The naturality of the braiding implies that for any pair V, W of A-modules, the braiding $c_{V,W}$ is of the form (2.5). As a consequence of the A-linearity of $c_{A,A}$, we get $\Delta^{\mathrm{op}}(a)R = R\Delta(a)$ for all $a \in A$, which is equivalent to Relation (2.1). The commutativity of the hexagons (H1) and (H2) in XIII.1 implies Relations (2.2) and (2.3), as follows from an easy computation using (2.5).

By (XIII.1.13) the category A-*Mod* is symmetric if $c_{W,V}c_{V,W} = \mathrm{id}_{V\otimes W}$ for all V and W. Now

$$c_{W,V}c_{V,W}(v \otimes w) = (R_{21}R)(v \otimes w).$$

Therefore, the category is symmetric if and only if $R_{21}R = 1$, which is equivalent to R_{21} being the inverse of R. □

Corollary XV.2.3. *In a braided quasi-bialgebra, the universal R-matrix satisfies the relation*

$$R_{12}\Phi_{312}R_{13}(\Phi_{132})^{-1}R_{23}\Phi_{123} = \Phi_{321}R_{23}(\Phi_{231})^{-1}R_{13}\Phi_{213}R_{12}.$$

PROOF. This counterpart of Theorem VIII.2.4 (a) follows from (1.9), (2.5), and from Theorem XIII.1.3. □

Later we shall need the following definition: a *morphism of braided quasi-bialgebras* $\alpha : (A, \Delta, \varepsilon, \Phi, l, r, R) \to (A', \Delta', \varepsilon', \Phi', l', r', R')$ is a morphism of the underlying quasi-bialgebras such that

$$(\alpha \otimes \alpha)(R) = R'. \tag{2.7}$$

XV.3 Gauge Transformations

For simplicity, all quasi-bialgebras $(A, \Delta, \varepsilon, \Phi, l, r)$ considered in the sequel will verify $l = r = 1$. In other words, the unit constraints of A-*Mod* will be the same as the unit constraints of *Vect*(k). We shall henceforth drop any reference to l and r.

The purpose of this section is to introduce an equivalence relation on quasi-bialgebras such that the categories of modules of two equivalent quasi-bialgebras are tensor equivalent.

Definition XV.3.1. *Let $A = (A, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. A gauge transformation on A is an invertible element F of A such that*

$$(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1. \tag{3.1}$$

Using a gauge transformation F on A, we can build a new quasi-bialgebra A_F as follows. Define an algebra morphism $\Delta_F : A \to A \otimes A$ by

$$\Delta_F(a) = F\Delta(a)F^{-1} \tag{3.2}$$

for any $a \in A$, and a new Drinfeld associator Φ_F by

$$\Phi_F = F_{23}(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})(F^{-1})F_{12}^{-1} \in A \otimes A \otimes A. \tag{3.3}$$

Proposition XV.3.2. *For any quasi-bialgebra $A = (A, \Delta, \varepsilon, \Phi)$ and any gauge transformation $F \in A \otimes A$ on A, the algebra $A_F = (A, \Delta_F, \varepsilon, \Phi_F)$ is a quasi-bialgebra.*

Observe that that if A happens to be a bialgebra (i.e., with $\Phi = 1$), then A_F is not in general a bialgebra. This procedure provides non-trivial examples of quasi-bialgebras.

PROOF. We must check Relations (1.1–1.4) for A_F.
Relation (1.1): We have

$$\begin{aligned}
&(\mathrm{id} \otimes \Delta_F)(\Delta_F(a))\Phi_F \\
&\quad= F_{23}(\mathrm{id} \otimes \Delta)(F\Delta(a)F^{-1})F_{23}^{-1}F_{23}(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})(F^{-1})F_{12}^{-1} \\
&\quad= F_{23}(\mathrm{id} \otimes \Delta)(F)(\mathrm{id} \otimes \Delta)(\Delta(a))\Phi(\Delta \otimes \mathrm{id})(F^{-1})F_{23}^{-1} \\
&\quad= F_{23}(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})(\Delta(a))(\Delta \otimes \mathrm{id})(F^{-1})F_{12}^{-1} \\
&\quad= F_{23}(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})(F^{-1})F_{12}^{-1}F_{12}(\Delta \otimes \mathrm{id})(F\Delta(a)F^{-1})F_{12}^{-1} \\
&\quad= \Phi_F(\Delta_F \otimes \mathrm{id})(\Delta_F(a)).
\end{aligned}$$

The first and last equalities follow by definition, the third one from Relation (1.1).
Relation (1.2): For all $a \in A$ we have

$$(\varepsilon \otimes \mathrm{id})(\Delta_F(a)) = (\varepsilon \otimes \mathrm{id})(F)\Big((\varepsilon \otimes \mathrm{id})(\Delta(a))\Big)(\varepsilon \otimes \mathrm{id})(F^{-1}) = a$$

in view of the counit axiom and of Relation (3.1). We similarly obtain $(\mathrm{id} \otimes \varepsilon)\Delta_F = \mathrm{id}_A$.
Relation (1.3): We have to verify the pentagonal relation

$$(\mathrm{id}\otimes\mathrm{id}\otimes\Delta_F)(\Phi_F)\,(\Delta_F\otimes\mathrm{id}\otimes\mathrm{id})(\Phi_F) = (\Phi_F)_{234}\,(\mathrm{id}\otimes\Delta_F\otimes\mathrm{id})(\Phi_F)\,(\Phi_F)_{123}. \tag{3.4}$$

Now,

$$\begin{aligned}
&(\mathrm{id}\otimes\mathrm{id}\otimes\Delta_F)(\Phi_F)\,(\Delta_F\otimes\mathrm{id}\otimes\mathrm{id})(\Phi_F)\\
&=\; F_{34}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\Big(F_{23}(\mathrm{id}\otimes\Delta)(F)\Phi(\Delta\otimes\mathrm{id})(F^{-1})F_{12}^{-1}\Big)F_{34}^{-1}\\
&\qquad F_{12}(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\Big(F_{23}(\mathrm{id}\otimes\Delta)(F)\Phi(\Delta\otimes\mathrm{id})(F^{-1})F_{12}^{-1}\Big)F_{12}^{-1}\\
&=\; F_{34}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(F_{23})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\Big((\mathrm{id}\otimes\Delta)(F)\Big)(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\\
&\qquad (\Delta\otimes\Delta)(F^{-1})F_{12}^{-1}F_{34}^{-1}F_{12}F_{34}(\Delta\otimes\Delta)(F)\\
&\qquad (\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\Big((\Delta\otimes\mathrm{id})(F^{-1})\Big)\\
&\qquad (\Delta\otimes\mathrm{id}\otimes\mathrm{id})(F_{12}^{-1})F_{12}^{-1}\\
&=\; F_{34}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(F_{23})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\Big((\mathrm{id}\otimes\Delta)(F)\Big)(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\\
&\qquad (\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\Big((\Delta\otimes\mathrm{id})(F^{-1})\Big)\\
&\qquad (\Delta\otimes\mathrm{id}\otimes\mathrm{id})(F_{12}^{-1})F_{12}^{-1}\\
&=\; F_{34}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(F_{23})\Phi_{234}(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\Big((\mathrm{id}\otimes\Delta)(F)\Big)\\
&\qquad \Phi_{234}^{-1}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)\Phi_{123}^{-1}\\
&\qquad (\mathrm{id}\otimes\Delta\otimes\mathrm{id})\Big((\Delta\otimes\mathrm{id})(F^{-1})\Big)\Phi_{123}(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(F_{12}^{-1})F_{12}^{-1}\\
&=\; F_{34}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(F_{23})\Phi_{234}(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\Big((\mathrm{id}\otimes\Delta)(F)\Big)\\
&\qquad (\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\Big((\Delta\otimes\mathrm{id})(F^{-1})\Big)\\
&\qquad \Phi_{123}(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(F_{12}^{-1})F_{12}^{-1}\\
&=\; F_{34}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(F_{23})\Phi_{234}(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(F_{23}^{-1})F_{23}^{-1}\\
&\qquad F_{23}(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\Big(F_{23}(\mathrm{id}\otimes\Delta)(F)\Phi(\Delta\otimes\mathrm{id})(F^{-1})F_{12}^{-1}\Big)F_{23}^{-1}\\
&\qquad F_{23}(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(F_{12})\Phi_{123}(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(F_{12}^{-1})F_{12}^{-1}\\
&=\; (\Phi_F)_{234}\,(\mathrm{id}\otimes\Delta_F\otimes\mathrm{id})(\Phi_F)\,(\Phi_F)_{123},
\end{aligned}$$

which proves (3.4). The first and last equalities follow from (3.2–3.3), the second and sixth ones from the fact that Δ is an algebra morphism, the third one holds because F_{12} and F_{34} commute, the fourth one follows by applying (1.1) to $a = F$ and F^{-1}, and the fifth one from (1.3).

Relation (1.4): Using the definition of Φ_F and Relations (1.4) and (3.1), we immediately get $(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})(\Phi_F) = FF^{-1} = 1\otimes 1$. □

When F is an gauge transformation on A, then so is F^{-1} and we have

$$(A_F)_{F^{-1}} = A = (A_{F^{-1}})_F. \tag{3.5}$$

If F' is another gauge transformation, then so is the product FF' and we have

$$(A_{F'})_F = A_{FF'}. \tag{3.6}$$

Definition XV.3.3. *Two quasi-bialgebras* $(A, \Delta, \varepsilon, \Phi)$ *and* $(A', \Delta', \varepsilon', \Phi')$ *are equivalent if there exist a gauge transformation* F *on* A' *and an isomorphism* $\alpha : A \to A'_F$ *of quasi-bialgebras.*

Relations (3.5–3.6) imply that this is an equivalence relation. We now prove that equivalent quasi-bialgebras have equivalent tensor categories of modules. We start with a preliminary result. Let $A = (A, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra and $F \in A \otimes A$ be a gauge transformation. Define

$$\varphi_2^F(V, W)(v \otimes w) = F^{-1}(v \otimes w) \tag{3.7}$$

where v and w belong to the A-modules V and W respectively.

Lemma XV.3.4. *Under the previous hypothesis, the triple* $(\mathrm{id}, \mathrm{id}, \varphi_2^F)$ *is a tensor functor from the tensor category A-Mod to the tensor category A_F-Mod.*

PROOF. Recall Definition XI.4.1. We have to check Relations (XI.4.1–4.3), namely $\varphi_2(k, V) = \varphi_2(V, k) = \mathrm{id}_V$ and

$$\varphi_2(U, V \otimes W)(\mathrm{id}_U \otimes \varphi_2(V, W))\, a^F_{U,V,W} = a_{U,V,W}\, \varphi_2(U \otimes V, W)(\varphi_2(U, V) \otimes \mathrm{id}_W) \tag{3.8}$$

where a^F is the associativity constraint induced by Φ_F. The first set of equalities follows from (3.1) and (3.7). Let us prove (3.8). For all $u \in U$, $v \in V$ and $w \in W$ we have

$$\begin{aligned}
&\Big(\varphi_2(U, V \otimes W)(\mathrm{id}_U \otimes \varphi_2(V, W))\, a^F_{U,V,W}\Big)(u \otimes v \otimes w) \\
&= (\mathrm{id} \otimes \Delta)(F^{-1})F_{23}^{-1}\Phi_F(u \otimes v \otimes w) \\
&= \Phi(\Delta \otimes \mathrm{id})(F^{-1})F_{12}^{-1}(u \otimes v \otimes w) \\
&= \Big(a_{U,V,W}\, \varphi_2(U \otimes V, W)(\varphi_2(U, V) \otimes \mathrm{id}_W)\Big)(u \otimes v \otimes w).
\end{aligned}$$

The first and last equalities follow from (1.5) and (3.7), and the second one from the definition of Φ_F. □

We state the first main result of this section. Let A and A' be equivalent quasi-bialgebras with a gauge transformation F on A' and an isomorphism $\alpha : A \to A'_F$ of quasi-bialgebras. The map α induces a strict tensor functor $(\alpha^*, \mathrm{id}, \mathrm{id})$ from A'_F-*Mod* to A-*Mod* as explained in Example 2 of XI.4. Since α is an isomorphism, α^* is a tensor equivalence.

Theorem XV.3.5. *The tensor functor* $(\alpha^*, \mathrm{id}, \varphi_2^F)$ *is a tensor equivalence between A'-Mod and A-Mod.*

PROOF. Replacing F by F^{-1} which is another gauge transformation, we get a tensor functor $(\mathrm{id}, \mathrm{id}, \varphi_2^{F^{-1}})$ from A'_F-*Mod* to A'-*Mod* which turns out to be an inverse to $(\mathrm{id}, \mathrm{id}, \varphi_2^F)$. The tensor functor $(\alpha^*, \mathrm{id}, \varphi_2^F)$ is the composition of the tensor equivalence $(\mathrm{id}, \mathrm{id}, \varphi_2^F) : A'\text{-}Mod \to A'_F\text{-}Mod$ and of the tensor equivalence $(\alpha^*, \mathrm{id}, \mathrm{id})$. □

We now extend the gauge transformations to braided quasi-bialgebras. Consider a braided quasi-bialgebra $(A, \Delta, \varepsilon, \Phi, R)$ with a universal R-matrix R. For any gauge transformation F on A, define Δ_F and Φ_F as above. Also set

$$R_F = F_{21} R F^{-1}. \tag{3.9}$$

Proposition XV.3.6. *The algebra $A_F = (A, \Delta_F, \varepsilon, \Phi_F, R_F)$ is a braided quasi-bialgebra.*

PROOF. One may check Relations (2.1–2.3) directly for R_F. Alternatively, one may also proceed as follows. Let c be the braiding of A-*Mod* corresponding to the universal R-matrix R. Define $c^F_{V,W} : V \otimes W \to W \otimes V$ by $c^F_{V,W}(v \otimes w) = \tau_{V,W}(R_F(v \otimes w))$. An immediate computation using (3.9) shows that

$$c^F_{V,W} = (\varphi_2^F(V, W))^{-1} \circ c_{V,W} \circ \varphi_2^F(V, W).$$

One then checks that c^F is a braiding on A_F-*Mod* as in the proof of Lemma XIII.3.2. Finally, apply Proposition 2.2. □

Let $(A, \Delta, \varepsilon, \Phi, R)$ again be a braided quasi-bialgebra and F be a gauge transformation on A.

Lemma XV.3.7. *Under this hypothesis, the tensor functor* $(\mathrm{id}, \mathrm{id}, \varphi_2^F)$ *is a braided tensor equivalence from A-Mod to A_F-Mod.*

PROOF. In view of Theorem 3.5, it is enough to show that $(\mathrm{id}, \mathrm{id}, \varphi_2^F)$ is braided in the sense of Definition XIII.3.6. We must check that we have $\varphi_2^F \circ c^F_{U,V} = c_{U,V} \circ \varphi_2^F$. The latter is equivalent to $F^{-1}(R_F)_{21} = (RF^{-1})_{21}$, which follows from (3.9). □

We adapt Definition 3.3 to braided quasi-bialgebras.

Definition XV.3.8. *The two braided quasi-bialgebras $(A, \Delta, \varepsilon, \Phi, R)$ and $(A', \Delta', \varepsilon', \Phi', R')$ are equivalent if there exist a gauge transformation F on A' and an isomorphism $\alpha : A \to A'_F$ of braided quasi-bialgebras.*

Suppose we are in the situation of equivalent braided quasi-bialgebras A and A' with F and α as in the previous definition. As a consequence of Theorem 3.5 and Lemma 3.7, we get the second main result of the section.

Theorem XV.3.9. *In the situation just considered, the tensor functor $(\alpha^*, \mathrm{id}, \varphi_2^F)$ is a braided tensor equivalence between the braided tensor categories A'-Mod and A-Mod.*

XV.4 Braid Group Representations

Let $(A, \Delta, \varepsilon, \Phi, R)$ be a braided quasi-bialgebra with trivial unit constraints, V a left A-module, and n an integer > 1. We recall from Chapters X and XIII how to define a representation of the braid group B_n on $V^{\otimes n}$. Since Φ is not necessarily trivial, we have to make precise what we mean by $V^{\otimes n}$.

Let us place ourselves in the general situation where we have a braided tensor category $\mathcal{C}$ with associativity constraint a and braiding c. In XI.5 we constructed a strict tensor category $\mathcal{C}^{str}$ which is tensor equivalent to $\mathcal{C}$. The tensor product of $\mathcal{C}^{str}$ is denoted $*$. By definition, we set $V^{\otimes n} = V^{*n}$, which means that $V^{\otimes n}$ is equipped with the unique system of parentheses opening only at the extreme left. For instance, we have $V^{\otimes 4} = ((V \otimes V) \otimes V) \otimes V$. Define automorphisms $c_1, \ldots, c_{n-1}$ of $V^{\otimes n} = V^{*n}$ in $\mathcal{C}^{str}$ by

$$c_i = \mathrm{id}_{V^{*(i-1)}} * c_{V,V} * \mathrm{id}_{V^{*(n-i-1)}}.$$

By Theorem XIII.1.3 and Corollary X.6.9 we know that there exists a unique morphism of groups $\rho_n^{\mathcal{C}} : B_n \to \mathrm{Aut}(V^{\otimes n})$ sending the generator σ_i of B_n to c_i for any $i = 1, \ldots, n-1$. The representation $\rho_n^{\mathcal{C}}$ will be called the *braid group representation associated to the braided tensor category* $\mathcal{C}$. We now make c_i explicit in terms of the original category $\mathcal{C}$.

Lemma XV.4.1. *We have*

$$\begin{aligned} c_i = \ & \Big(a^{-1}_{V^{\otimes(i-1)},V,V} \otimes \mathrm{id}_V^{\otimes(n-i-1)}\Big)\Big(\mathrm{id}_{V^{\otimes(i-1)}} \otimes c_{V,V} \otimes \mathrm{id}_V^{\otimes(n-i-1)}\Big) \\ & \quad \Big(a_{V^{\otimes(i-1)},V,V} \otimes \mathrm{id}_V^{\otimes(n-i-1)}\Big). \end{aligned}$$

PROOF. This follows from Relation (XI.5.4) which expresses the tensor product of morphisms in the strict category $\mathcal{C}^{str}$ in terms of the tensor product of morphisms and of the associativity constraint in $\mathcal{C}$. We also use Relation (XI.5.3) in the following special cases:

$$\varphi(S, (V)) = \mathrm{id}_{F(S)\otimes V} \quad \text{and} \quad \varphi(S, (V, V)) = a^{-1}_{F(S),V,V}.$$

□

When $\mathcal{C} = A\text{-}Mod$ is the braided category of left modules over the braided quasi-bialgebra A, the braiding c is given by

$$c_{V,V}(v_1 \otimes v_2) = \Big(R(v_1 \otimes v_2)\Big)_{21}.$$

Consequently, by Lemma 4.1 we have

$$c_1(v_1 \otimes \cdots \otimes v_n) = \Big(R_{12}(v_1 \otimes \cdots \otimes v_n)\Big)_{21} \tag{4.1}$$

and if $i > 1$

$$c_i(v_1 \otimes \cdots \otimes v_n) = \Phi_i^{-1}\Big((R_{i,i+1}\Phi_i)(v_1 \otimes \cdots \otimes v_n)\Big)_{i+1,i} \tag{4.2}$$

where we used the subscript convention of VIII.2 and where

$$\Phi_i = \Delta^{(i+1)}(\Phi) \otimes 1^{\otimes(n-i-1)}$$

is the invertible element of $A^{\otimes n}$ expressed in terms of the map $\Delta^{(i+1)} : A^{\otimes 3} \to A^{\otimes(i+1)}$ defined inductively by $\Delta^{(3)} = \mathrm{id}_{A^{\otimes 3}}$ and by the relations $\Delta^{(i+1)} = (\Delta \otimes \mathrm{id}_A^{\otimes(i-1)})\Delta^{(i)}$. The corresponding representation $\rho_n^A = \rho_n^{A\text{-}Mod}$ will be called the *braid group representation associated to the braided quasi-bialgebra A*.

Let $A' = (A', \Delta', \varepsilon', \Phi', R')$ be another braided bialgebra. We assume that A and A' are equivalent braided quasi-bialgebras in the sense of Definition 3.8, i.e., there exist a gauge transformation F on A' and an isomorphism $\alpha : A \to (A')_F$ of braided quasi-bialgebras. In particular, we have

$$F_{21} R' F^{-1} = (\alpha \otimes \alpha)(R). \tag{4.3}$$

Let V be a left A'-module. By α it becomes an A-module. For any integer $n > 1$ we have two braid group representations $\rho_n^A, \rho_n^{A'} : B_n \to \mathrm{Aut}(V^{\otimes n})$ associated to A and A' respectively and acting on the same space. The main result of this section asserts that these representations are equivalent.

Theorem XV.4.2. *Let A and A' be equivalent braided quasi-bialgebras. With the previous notation, we have*

$$\rho_n^{A'}(g)(w) = F_{12}^{-1} \rho_n^A(g)(F_{12}\, w)$$

for all $g \in B_n$ and $w \in V^{\otimes n}$.

PROOF. By Theorem 3.9, we know that both braid group representations are equivalent. It is therefore enough to compute the equivalence on one special element of the braid group. We choose the generator σ_1. By (4.1) and (4.3) we have for $v_1, \ldots, v_n \in V$

$$\begin{aligned}
\rho_n^{A'}(\sigma_1)(v_1 \otimes \cdots \otimes v_n) &= \Big(R'_{12}(v_1 \otimes \cdots \otimes v_n)\Big)_{21} \\
&= \Big((F_{21}^{-1}(\alpha \otimes \alpha)(R)F_{12})(v_1 \otimes \cdots \otimes v_n)\Big)_{21} \\
&= F_{12}^{-1}\Big((\alpha \otimes \alpha)(R)(F_{12}(v_1 \otimes \cdots \otimes v_n))\Big)_{21} \\
&= F_{12}^{-1} \rho_n^A(\sigma_1)\Big(F_{12}(v_1 \otimes \cdots \otimes v_n)\Big).
\end{aligned}$$

□

It should be clear that the statement of Theorem 4.2 depends on the way we put parentheses on $V^{\otimes n}$. Other systems of parenthesizing give rise to different, but equivalent braid group representations.

XV.5 Quasi-Hopf Algebras

For the sake of completeness we introduce quasi-Hopf algebras as defined by Drinfeld [Dri89b]. As above, all quasi-bialgebras considered here have trivial unit constraints, i.e., $l = r = 1$.

Definition XV.5.1. *A quasi-bialgebra $(A, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra if there exist an invertible anti-automorphism S of the algebra A and elements α and β of A such that for all elements a in A we have*

$$\sum_{(a)} S(a')\alpha a'' = \varepsilon(a)\alpha, \quad \sum_{(a)} a'\beta S(a'') = \varepsilon(a)\beta, \tag{5.1}$$

and

$$\sum_i X_i \beta S(Y_i)\alpha Z_i = 1, \quad \sum_i S(\bar{X}_i)\alpha \bar{Y}_i \beta S(\bar{Z}_i) = 1 \tag{5.2}$$

where $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$ and $\Phi^{-1} = \sum_i \bar{X}_i \otimes \bar{Y}_i \otimes \bar{Z}_i$. A quasi-Hopf algebra is braided if the underlying quasi-bialgebra is.

We shall write $(A, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ to express the complete data of a quasi-Hopf algebra. As in XIV.2 consider the category $A\text{-}Mod_f$ of left A-modules that are finite-dimensional vector spaces over the ground field k. Equip it with the tensor category structure induced by Δ and Φ. For any object V of $A\text{-}Mod_f$ consider the objects V^* and ${}^*V$ as defined in Example 1 of XIV.2. We define maps $b_V : k \to V \otimes V^*$, $d_V : V^* \otimes V \to k$, $b'_V : k \to {}^*V \otimes V$, and $d'_V : V \otimes {}^*V \to k$ by

$$b_V(1) = \sum_i \beta v_i \otimes v^i, \quad d_V(v^i \otimes v_j) = < v^i, \alpha v_j >,$$

$$b'_V(1) = \sum_i v^i \otimes S^{-1}(\beta)v_i, \quad d'_V(v_i \otimes v^j) = < S^{-1}(\alpha)v_i, v^j >$$

where $\{v_i\}_i$ is a basis of V and $\{v^i\}_i$ the corresponding dual basis.

Proposition XV.5.2. *The maps b_V, d_V, b'_V and d'_V are A-linear and the composite maps*

$$V \cong k \otimes V \xrightarrow{b_V \otimes \mathrm{id}} (V \otimes V^*) \otimes V \xrightarrow{a} V \otimes (V^* \otimes V) \xrightarrow{\mathrm{id} \otimes d_V} V \otimes k \cong V$$

$$V^* \cong V^* \otimes k \xrightarrow{\mathrm{id} \otimes b_V} V^* \otimes (V \otimes V^*) \xrightarrow{a^{-1}} (V^* \otimes V) \otimes V^* \xrightarrow{d_V \otimes \mathrm{id}} k \otimes V^* \cong V^*$$

$$V \cong V \otimes k \xrightarrow{\mathrm{id} \otimes b'_V} V \otimes ({}^*V \otimes V) \xrightarrow{a^{-1}} (V \otimes {}^*V) \otimes V \xrightarrow{d'_V \otimes \mathrm{id}} k \otimes V \cong V$$

$${}^*V \cong k \otimes {}^*V \xrightarrow{b'_V \otimes \mathrm{id}} ({}^*V \otimes V) \otimes {}^*V \xrightarrow{a} {}^*V \otimes (V \otimes {}^*V) \xrightarrow{\mathrm{id} \otimes d'_V} {}^*V \otimes k \cong {}^*V$$

are all identities.

PROOF. The first statement follows from Relations (5.1), and the second one from Relations (5.2). □

Consider the braided tensor category $A\text{-}Mod_f$ associated to the quasi-Hopf algebra A as well as the strict braided tensor category $(A\text{-}Mod_f)^{str}$. Proposition 5.2 can be interpreted as follows: the strict braided tensor category $(A\text{-}Mod_f)^{str}$ is autonomous, i.e., has left and right dualities given by the maps b_V, d_V, b'_V, and d'_V.

We end this section with an example of a non-trivial braided quasi-Hopf algebra close to the quantum double of the algebra $k[G]$ of a finite group G. Suppose given a *normalized* 3-*cocycle* on the group G, i.e., a function $\omega : G \times G \times G \to k \setminus \{0\}$ such that

$$\omega(x,y,z)\omega(tx,y,z)^{-1}\omega(t,xy,z)\omega(t,x,yz)^{-1}\omega(t,x,y) = 1 \tag{5.3}$$

for all $t, x, y, z \in G$, and such that $\omega(x,y,z) = 1$ whenever x, y, or $z = 1$. Consider a finite-dimensional vector space $D^\omega(G)$ with a basis $\{e_g x\}_{(g,x)\in G\times G}$ indexed by $G \times G$. Define a product on $D^\omega(G)$ by

$$(e_g x)(e_h y) = \delta_{g,xhx^{-1}}\theta(g,x,y)\, e_g(xy) \tag{5.4}$$

where $\theta(g,x,y) = \omega(g,x,y)\omega(x,y,(xy)^{-1}gxy)\omega(x,x^{-1}gx,y)^{-1}$. It is easy to check that this product is associative and has the element $1 = \sum_{g\in G} e_g 1$ as a left and right unit. Observe that when the cocycle ω is trivial, i.e., $\omega(x,y,z) = 1$ for all x, y, z, then $D^\omega(G)$ is isomorphic to the quantum double $D(k[G])$ (see IX.4.3). In contrast to the trivial case, the map sending x to $\sum_{g\in G} e_g x$ is not a morphism of algebras from $k[G]$ to $D^\omega(G)$ in general, but the map $e_g \mapsto e_g 1$ is, which will allow us to identify $e_g 1$ with e_g.

We define morphisms of algebras $\Delta : D^\omega(G) \to D^\omega(G) \otimes D^\omega(G)$ and $\varepsilon : D^\omega(G) \to k$ by

$$\Delta(e_g x) = \sum_{uv=g} \gamma(x,u,v)\, e_u x \otimes e_v x \quad \text{and} \quad \varepsilon(e_g x) = \delta_{g,1} \tag{5.5}$$

where $\gamma(x,u,v) = \omega(u,v,x)\omega(x,x^{-1}ux,x^{-1}vx)\omega(u,x,x^{-1}vx)^{-1}$. Set also

$$\Phi = \sum_{x,y,z\in G} \omega(x,y,z)^{-1} e_x \otimes e_y \otimes e_z, \quad R = \sum_{g\in G} e_g \otimes (\sum_h e_h)g, \tag{5.6}$$

$\alpha = 1$, and $\beta = \sum_{g\in G} \omega(g,g^{-1},g)e_g$. We define an anti-automorphism S of the algebra $D^\omega(G)$ by

$$S(e_g x) = \theta(g^{-1},x,x^{-1})^{-1}\gamma(x,g,g^{-1})^{-1}\, e_{x^{-1}gx}x^{-1}. \tag{5.7}$$

Then $(D^\omega(G), \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ is a braided quasi-Hopf algebra with universal R-matrix R in the sense of Definitions 1.1, 2.1 and 5.1.

XV.6 Exercises

1. Let $(A, \Delta, \varepsilon, S)$ be a Hopf algebra and $F = \sum_i f_i \otimes g_i \in A \otimes A$ be a gauge transformation such that
$$F_{23}(\mathrm{id}_A \otimes \Delta)(F) = F_{12}(\Delta \otimes \mathrm{id}_A)(F).$$
Consider the element $x = \sum_i f_i S(g_i)$ of A. Show that it is invertible and that $(A, \Delta_F, \varepsilon, S_F)$ is a Hopf algebra where $S_F(a) = xS(a)x^{-1}$ for all $a \in A$.

2. Show that if $(A, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra each of $(A^{\mathrm{op}}, \Delta, \varepsilon, \Phi^{-1})$, $(A, \Delta^{\mathrm{op}}, \varepsilon, (\Phi_{321})^{-1})$ and $(A^{\mathrm{op}}, \Delta^{\mathrm{op}}, \varepsilon, \Phi_{321})$ is a quasi-bialgebra.

3. Let $A = (A, \Delta, \varepsilon, \Phi, l, r)$ and $A' = (A', \Delta', \varepsilon', \Phi', l', r')$ be quasi-bialgebras. Let $\alpha : A \to A'$ be a morphism between the underlying algebras. Suppose that the induced functor $\alpha^* : A'$-*Mod* $\to A$-*Mod* extends to a tensor functor $(\alpha^*, \mathrm{id}, \varphi_2)$. Show that there exists an invertible element F in $A' \otimes A'$ such that $\varphi_2(u \otimes v) = F^{-1}(u \otimes v)$. Prove that necessarily $\varepsilon\alpha = \varepsilon'$,
$$(\alpha\otimes\alpha)\Delta(a)F = F\Delta'(\alpha(a)), \quad \Phi' F_{12}(\Delta\otimes\mathrm{id})(F) = F_{23}(\mathrm{id}\otimes\Delta)(F)\alpha(\Phi),$$
$$l' = \alpha(l)(\varepsilon' \otimes \mathrm{id})(F), \quad \text{and} \quad r' = \alpha(r)(\mathrm{id} \otimes \varepsilon')(F).$$

4. (*Gauge transformation of a quasi-Hopf algebra*) Let $(A, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ be a quasi-Hopf algebra and $F = \sum_i f_i \otimes g_i$ be a gauge transformation on A with inverse $F^{-1} = \sum_i \bar{f}_i \otimes \bar{g}_i$. Set
$$\alpha_F = \sum_i S(\bar{f}_i)\alpha\bar{g}_i \quad \text{and} \quad \beta_F = \sum_i f_i \beta S(g_i).$$
Prove that $(A, \Delta_F, \varepsilon, \Phi_F, S, \alpha_F, \beta_F)$ is a quasi-Hopf algebra.

5. Let $(A, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ be a braided quasi-Hopf algebra with universal R-matrix R. Suppose that $\Phi^{-1} = \sum_i \bar{X}_i \otimes \bar{Y}_i \otimes \bar{Z}_i$ and $R = \sum_j s_j \otimes t_j$. Set
$$u = \sum_{i,j} S(\bar{Y}_i \beta S(\bar{Z}_i)) S(t_j) \alpha s_j \bar{X}_i.$$
Prove that u is an invertible element in A such that $\varepsilon(u) = 1$ and $S^2(a) = uau^{-1}$ for all $a \in A$.

XV.7 Notes

Quasi-bialgebras, quasi-Hopf algebras, and gauge transformations were invented by Drinfeld [Dri89b][Dri90][Dri89c] in relation with his treatment of the monodromy of the Knizhnik-Zamolodchikov equations (to be considered in Chapter XIX). Drinfeld used the term "quasi-triangular quasi-bialgebra" for a braided quasi-bialgebra. In [Dri89b], Section 1, Drinfeld showed that one always could reduce a general quasi-Hopf algebra to a quasi-bialgebra with $l = r = 1$.

Altschuler and Coste [AC92] proved (see Exercise 5) that in any braided quasi-Hopf algebra the square of the antipode is an inner automorphism (just as for braided Hopf algebras, c.f. VIII.4). They also defined ribbon quasi-Hopf algebras generalizing the ribbon algebras of XIV.6. The braided quasi-Hopf algebra $D^\omega(G)$ of Section 5 is due to Dijkgraaf, Pasquier, and Roche [DPR90] and was shown in [AC92] to be a ribbon quasi-Hopf algebra. Let us remark that when the 3-cocycle ω is changed by a coboundary, then $D^\omega(G)$ is changed by a gauge transformation, so that the tensor category of modules of $D^\omega(G)$ depends on the cohomology class of ω.

Exercise 1 is taken from [Res90] while Exercise 4 is from [Dri89b].

Part Four

Quantum Groups and Monodromy

Chapter XVI
Generalities on Quantum Enveloping Algebras

In order to state the main results of Part IV, we need the concept of a quantum enveloping algebra. This requires the use of formal series and of h-adic topology. The chapter is completed by an appendix on inverse limits.

XVI.1 The Ring of Formal Series and h-Adic Topology

Consider the complex algebra $K = \mathbf{C}[[h]]$ of complex formal series in one variable h. Any element of K is of the form

$$f = \sum_{n \geq 0} a_n h^n \tag{1.1}$$

where $(a_0, a_1, \ldots)$ is a family of complex numbers indexed by the set $\mathbf{N}$ of non-negative integers. If $f' = \sum_{n\geq 0} a'_n h^n$ is another formal series, then the sum $f + f'$ and the product ff' of f and f' in K are given by

$$f + f' = \sum_{n\geq 0} (a_n + a'_n) h^n \quad \text{and} \quad ff' = \sum_{n\geq 0} \Big(\sum_{p+q=n} a_p a'_q \Big) h^n. \tag{1.2}$$

Any polynomial in h may be considered as an element of K. In particular, the constant polynomial 1 is an element of K where it acts as a unit for the product as can easily be seen from (1.2). The following characterizes invertible elements in K.

Lemma XVI.1.1. *A formal series $f = \sum_{n\geq 0} a_n h^n$ is invertible in $\mathbf{C}[[h]]$ if and only if $a_0 \neq 0$ in $\mathbf{C}$.*

PROOF. The formal series f is invertible if and only if there exists another series $g = \sum_{n\geq 0} b_n h^n$ such that $fg = 1$. From (1.1) we see that this is equivalent to the existence of an infinite family $(b_0, b_1, \ldots)$ of complex numbers such that $a_0 b_0 = 1$ and

$$a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = 0 \tag{1.3}$$

for all $n > 0$. The relation $a_0 b_0 = 1$ shows that the invertibility of a_0 is a necessary condition for f to be invertible. This condition is also sufficient since the family $(b_0, b_1, \ldots)$ can be determined inductively from $b_0 = a_0^{-1}$ and Relations (1.3). □

Lemma 1.1 may be interpreted as saying that the ring K is a local ring whose maximal ideal is the ideal (h) generated by h.

For any integer $n > 0$ consider the algebra $K_n = \mathbf{C}[h]/(h^n)$ of truncated polynomials obtained as the quotient of the algebra of complex polynomials in one variable by the ideal generated by h^n. There is a morphism of algebras π_n from K to K_n sending a formal series $f = \sum_{n\geq 0} a_n h^n$ to the class of $\sum_{k=0}^{n-1} a_k h^k$ modulo (h^n). This map is surjective and its kernel is the ideal $h^n K$ generated by h^n in the ring of formal series. Consequently, π_n induces an isomorphism of algebras

$$\mathbf{C}[[h]]/(h^n) \cong \mathbf{C}[h]/(h^n). \tag{1.4}$$

For $n > 0$ there is also a surjective morphism of algebras p_n from K_n to K_{n-1} induced by the inclusion of ideals $(h^n) \subset (h^{n-1})$. Consider the inverse system of algebras $(K_n, p_n)_n$ and its inverse limit $\varprojlim_n K_n$ as defined in the Appendix. We have $p_n \circ \pi_n = \pi_{n-1}$ for all n. It follows from Proposition 9.1 that there exists a unique morphism of algebras π from K to $\varprojlim_n K_n$ whose composition with the projection of the inverse limit onto K_n equals π_n.

Proposition XVI.1.2. *The map $\pi : \mathbf{C}[[h]] \to \varprojlim_n \mathbf{C}[h]/(h^n)$ is an isomorphism of algebras.*

PROOF. The map π is injective since its kernel, which is the intersection of all ideals (h^n), is zero in view of (1.2).

In order to prove the surjectivity of π we construct a right inverse to it as follows. Let $(f_n)_{n>0}$ be an element of the inverse limit (see the Appendix for a definition). By definition f_n belongs to K_n, which allows us to represent it as

$$f_n = \sum_{k=0}^{n-1} a_k^{(n)} h^k,$$

and we have $p_n(f_n) = f_{n-1}$ for all $n > 0$. Hence $a_k^{(n)} = a_k^{(n-1)}$ for k running from 0 to $n-2$. We can therefore define a formal series $f = \sum_{n\geq 0} a_n h^n$ by $a_n = a_n^{(n+2)} = a_n^{(n+3)} = \cdots$. We have $\pi(f) = (f_n)_n$. $\square$

Proposition 1.2 allows us to equip K with the inverse limit topology described in the Appendix. This topology is called the *h-adic topology.* By definition, the family of subsets $\pi_n^{-1}(U_n)$, where $n > 0$ and U_n is any subset of K_n, is a basis of open sets of K. Since $\{0\}$ is a family of open neighbourhoods in the discrete set K_n, the family $\pi_n^{-1}(0) = (h^n)$ is a family of open neighbourhoods of 0 in K for the h-adic topology. From this it is easy to see that the h-adic is a metric topology where the metric may be defined as follows. For any non-zero formal series $f = \sum_{n\geq 0} a_n h^n$, let $\omega(f)$ be the unique non-negative integer such that $a_{\omega(f)} \neq 0$ and $a_k = 0$ for all $k < \omega(f)$. When $f = 0$, set $\omega(0) = +\infty$. Extend the natural order of $\mathbf{N}$ to $\mathbf{N} \cup \{+\infty\}$ by requiring that $+\infty > n$ for all $n \in \mathbf{N}$. We see that

$$(h^n) = \Big\{ f \in K \;\Big|\; \omega(f) > n-1 \Big\}. \tag{1.5}$$

As a consequence, we get

$$\bigcap_{n>0} (h^n) = \{0\}, \tag{1.6}$$

a trivial fact already used in the proof of Proposition 1.2. We also have

$$\omega(f+g) \geq \min\Big(\omega(f), \omega(g)\Big) \tag{1.7}$$

for all $f, g \in K$. Define a map $|\ |$ from K to the set of non-negative real numbers by

$$|f| = 2^{-\omega(f)} \tag{1.8}$$

if $f \neq 0$ and $|0| = 0$ if $f = 0$. The next result is an immediate consequence of the previous considerations.

Lemma XVI.1.3. *For all f and g in K we have*

$$|f| = 0 \iff f = 0, \quad |-f| = |f|, \quad |f+g| \leq \max(|f|, |g|).$$

As a corollary we get a distance on K.

Corollary XVI.1.4. *Define $d(f,g) = |f-g|$ for any $f, g \in \mathbf{C}[[h]]$. Then d is an ultrametric distance on $\mathbf{C}[[h]]$, i.e., we have*

(i) *$d(f,g) = 0 \iff f = g$,*
(ii) *$d(f,g) = d(g,f)$, and*
(iii) *$d(f,h) \leq \max\Big(d(f,g), d(g,h)\Big)$ for any triple (f,g,h) of formal series.*

The distance d puts a metric on $\mathbf{C}[[h]]$. From its definition and from (1.5) it is clear that the family of ideals (h^n) is also a set of open neighbourhoods of 0 for the metric topology. Therefore the latter is equivalent to the h-adic topology.

XVI.2 Topologically Free Modules

Let M be a left module over the algebra $K = \mathbf{C}[[h]]$. Consider the family $(h^n M)_{n>0}$ of submodules and the canonical K-linear projections

$$p_n : M_n = M/h^n M \to M_{n-1} = M/h^{n-1} M.$$

They form an inverse system of K-modules, and we may consider the inverse limit

$$\widetilde{M} = \lim_{\substack{\longleftarrow \\ n}} M_n \tag{2.1}$$

which has a natural structure as a K-module. The inverse limit $\widetilde{M}$ has a natural topology, the inverse limit topology, for which it is easy to see as in Section 1 that the family of submodules $(h^n \widetilde{M})_n$ is a family of open neighbourhoods. The module $\widetilde{M}$ is called the *h-adic completion* of M.

The projections $i_n : M \to M_n$ induce a unique K-linear map $i : M \to \widetilde{M}$ such that $\pi_n \circ i = i_n$ for all n. The kernel of i is given by

$$\mathrm{Ker}\,(i) = \bigcap_{n>0} h^n M.$$

Definition XVI.2.1. *A K-module M is separated if $\bigcap_{n>0} h^n M = \{0\}$. It is complete if the map i is surjective.*

For any module M the module $M/(\bigcap_{n>0} h^n M)$ is separated and the completion $\widetilde{M}$ is complete. Indeed, consider the projection $\pi_n : \widetilde{M} \to M_n$. Its kernel is $h^n \widetilde{M}$, which implies the isomorphism of modules

$$\widetilde{M}/h^n \widetilde{M^n} \cong M/h^n M. \tag{2.2}$$

Taking inverse limits, we get $\widetilde{\widetilde{M}} = \widetilde{M}$, which proves that $\widetilde{M}$ is complete.

Any separated, complete K-module will be equipped with the topology, called the h-adic topology, coming from the inverse limit topology on $\widetilde{M}$ via the isomorphism $M \cong \widetilde{M}$.

We now describe an important class of separated, complete K-modules. It includes K itself, viewed as a K-module by left multiplication. Take any complex vector space V. Define $V[[h]]$ as the set of all formal series

$$\sum_{n\geq 0} v_n h^n \tag{2.3}$$

where $(v_0, v_1, \ldots)$ is an infinite family of elements of V. Using Formulas (1.2) we can put a left K-module structure on $V[[h]]$. Any left K-module of this form will be called a *topologically free* module. We recover K by taking $V = \mathbf{C}$.

Proposition XVI.2.2. *Any topologically free module is separated and complete.*

PROOF. From the definition we see that the submodule $h^nV[[h]]$ is the set of all elements $\sum_{n\geq 0} v_n h^n$ such that $v_0 = \cdots = v_{n-1} = 0$. It follows that the intersection of all submodules $h^nV[[h]]$ is zero. This implies that $V[[h]]$ is separated.

It is also complete: a proof parallel to the proof of Proposition 1.2 shows $V[[h]]$ isomorphic to the inverse limit of the family $\Big(V[[h]]/h^nV[[h]]\Big)_{n>0}$. □

As in the case $V = \mathbf{C}$, the h-adic topology on $V[[h]]$ induced by the inverse limit topology can be defined by a metric built in the same way as for $\mathbf{C}[[h]]$ (see Section 1).

A topologically free module $V[[h]]$ has the following additional properties.

Proposition XVI.2.3. (a) *Let $\{e_i\}_{i\in I}$ be a basis of the vector space V. Then the K-submodule generated by the set $\{e_i\}_{i\in I}$ is dense in $V[[h]]$ for the h-adic topology.*

(b) *For any separated, complete K-module N, there is a natural bijection*

$$\mathrm{Hom}_K(V[[h]], N) \cong \mathrm{Hom}(V, N)$$

where Hom_K *denotes the space of K-linear maps.*

Observe that a K-linear map $f : M \to N$ between separated, complete K-modules is always continuous for the h-adic topology since $f(h^nM)$ is contained in h^nN by K-linearity.

PROOF. (a) Let W be the submodule of $V[[h]]$ generated by the set $\{e_i\}_{i\in I}$. Take any element $f = \sum_{n\geq 0} v_n h^n$ of $V[[h]]$. We have to show that for any integer $n > 0$, there exists an element $f_n \in W$ such that $f - f_n$ belongs to $h^nV[[h]]$. The element f_n is constructed as follows: $f_n = \sum_{k=0}^{n-1} v_k h^k$. Clearly, the difference $f - f_n$ lies in $h^nV[[h]]$. It remains to check that f_n belongs to W. Indeed, since $\{e_i\}_{i\in I}$ is a basis we have

$$f_n = \sum_{k=0}^{n-1} \Big(\sum_{i\in I} \lambda_i^{(k)} e_i\Big) h^k = \sum_{i\in I} \Big(\sum_{k=0}^{n-1} \lambda_i^{(k)} h^k\Big) e_i \in W$$

where $\lambda_i^{(k)}$ is a family of complex numbers, all but finitely many equal to zero.

(b) Let f be a continuous K-linear map from $V[[h]]$ to N. Considering V as the space of constant formal series in $V[[h]]$, we may restrict f to a $\mathbf{C}$-linear map from V to N. Conversely, let g be a K-linear map from V to N. Extend it to a K_n-linear map g_n from $V[[h]]/h^nV[[h]]$ to N/h^nN by

$$g_n(\sum_{k=0}^{n-1} v_k h^k) = \sum_{k=0}^{n-1} g(v_k) h^k \mod h^nN.$$

Taking inverse limits yields a K-linear map g_∞ between the corresponding inverse limits. Since $V[[h]$ and N are separated and complete, we get a map, still denoted g_∞, from $V[[h]]$ to N. This map restricts to g on V. □

Topologically free modules can be characterized in a simple way. Recall that a K-module M is *torsion free* if $hm \neq 0$ when m is any non-zero element of M.

Proposition XVI.2.4. *A left K-module is topologically free if and only if it is separated, complete, and torsion-free.*

PROOF. By Proposition 2.2 we know that any topologically free module is separated and complete. It has no torsion in view of (1.2).

Conversely, let M be a separated, complete, and torsion-free module. We have to show that M is of the form $V[[h]]$. Choose a vector subspace of M which is supplementary to hM. Because of the torsion-free assumption, we get $h^n M = h^n V \oplus h^{n+1} M$ for all $n \geq 0$. Hence,

$$M/h^n M = V \oplus hV \oplus \cdots \oplus h^{n-1} V = V[[h]]/h^n V[[h]].$$

Taking inverse limits and using the fact that M and $V[[h]]$ are separated and complete, we get

$$M \cong \varprojlim_n M/h^n M = \varprojlim_n V[[h]]/h^n V[[h]] \cong V[[h]].$$

□

We end with a caveat. We have $V[[h]] \cong V \otimes \mathbf{C}[[h]]$ only if V is a finite-dimensional vector space. There is no such isomorphism when V is infinite-dimensional, in which case $V[[h]]$ is strictly bigger than $V \otimes \mathbf{C}[[h]]$. Indeed, take an infinite family $(e_n)_{n \in \mathbf{N}}$ of linearly independent vectors; then the element $\sum_{n \geq 0} e_n h^n$ of $V[[h]]$ does not belong to $V \otimes \mathbf{C}[[h]]$.

XVI.3 Topological Tensor Product

Let M and N be left modules over the algebra $K = \mathbf{C}[[h]]$. Consider the K-module $M \otimes_K N$ obtained as the quotient of the vector space $M \otimes N$ by the subspace spanned by all elements of the form $fm \otimes n - m \otimes fn$ where f belongs to K, m to M, and n to N.

Definition XVI.3.1. *The topological tensor product $M \widetilde{\otimes} N$ of M and N is the h-adic completion of $M \otimes_K N$:*

$$M \widetilde{\otimes} N = (M \otimes_K N)^{\sim} = \varprojlim_{n>0} (M \otimes_K N)/h^n (M \otimes_K N).$$

Since it is defined as a completion, the topological tensor product of two modules is always complete. Given $m \in M$ and $n \in N$ we denote by $m\tilde{\otimes}n$ the image of $m \otimes n$ under the natural maps $M \otimes N \to M \otimes_K N \to M\tilde{\otimes}N$. The subspace of the topological tensor product spanned by all elements of this form is dense in $M\tilde{\otimes}N$. The usual associativity and commutativity constraints induce the following K-linear isomorphisms

$$(M\tilde{\otimes}N)\tilde{\otimes}P \cong M\tilde{\otimes}(N\tilde{\otimes}P), \tag{3.1}$$

$$M\tilde{\otimes}N \cong N\tilde{\otimes}M. \tag{3.2}$$

We also have

$$K\tilde{\otimes}M \cong \widetilde{M} \cong M\tilde{\otimes}K, \tag{3.3}$$

which means that K serves as a unit for completions.

The topological tensor product is functorial as can be seen from the definition: if $f : M \to M'$ and $g : N \to N'$ are K-linear maps, then there exists a K-linear map

$$f\tilde{\otimes}g : M\tilde{\otimes}N \to M'\tilde{\otimes}N'$$

enjoying the formal properties of the algebraic tensor product.

Proposition XVI.3.2. *If M and N are topologically free modules, then so is $M\tilde{\otimes}N$. More precisely, we have*

$$V[[h]]\tilde{\otimes}W[[h]] = (V \otimes W)[[h]].$$

PROOF. For any K-module M, the natural maps

$$M \otimes_K K_n \to M \otimes_K K/h^nK \to M/h^nM$$

are isomorphisms, where the first one is induced by (1.4) and the second one is given by $m\otimes f \mapsto fm$ (the inverse map being induced by $m \mapsto m\otimes 1$). Applying this to $M \otimes_K N$ where $M = V[[h]]$ and $N = W[[h]]$ we get

$$\begin{aligned}
(M \otimes_K N)/h^n(M \otimes_K N) &\cong (M \otimes_K N) \otimes_K K_n \\
&\cong \Big(M \otimes_K K_n\Big) \otimes_{K_n} \Big(N \otimes_K K_n\Big) \\
&\cong M/h^nM \otimes_{K_n} N/h^nN \\
&\cong \Big(V \otimes K_n\Big) \otimes_{K_n} \Big(W \otimes K_n\Big) \\
&\cong (V \otimes W) \otimes K_n \\
&\cong (V \otimes W)[[h]]/h^n(V \otimes W)[[h]].
\end{aligned}$$

Passing to the inverse limit yields the desired result. □

XVI.4 Topological Algebras

We extend the definitions of algebras, quasi-bialgebras, etc. to the setting of $\mathbf{C}[[h]]$-modules. This is done by replacing the algebraic tensor product of II.1 by the topological tensor product of Section 3.

A *topological algebra* is a triple (A, μ, η) where A is a module over the ring $K = \mathbf{C}[[h]]$, $\mu : A\widetilde{\otimes}A \to A$ and $\eta : K \to A$ are K-linear maps such that

$$\mu \circ (\mu\widetilde{\otimes}\mathrm{id}_A) = \mu \circ (\mathrm{id}_A\widetilde{\otimes}\mu) \tag{4.1}$$

and

$$\mu \circ (\eta\widetilde{\otimes}\mathrm{id}_A) = \mathrm{id}_A = \mu \circ (\mathrm{id}_A\widetilde{\otimes}\eta). \tag{4.2}$$

As in the algebraic case, we use the convention

$$aa' = \mu(a\widetilde{\otimes}a') \tag{4.3}$$

for the product of two elements a, a' of a topological algebra (A, μ, η). We also write 1 for the image under η of the unit element 1 of K.

Let (A, μ, η) be a topological algebra and $f(h) = \sum_{n\geq 0} c_n h^n$ be a formal series with complex coefficients. For an element $a \in A$, the formula

$$f(ha) = \sum_{n\geq 0} c_n a^n h^n \tag{4.4}$$

defines a unique element in the inverse limit $\widetilde{A} = \varprojlim_n A/h^nA$. Therefore if A is separated and complete, it defines an element, still denoted $f(ha)$, in $A \cong \widetilde{A}$. This procedure can be applied to the classical exponential function $e^h = \sum_{n\geq 0} \frac{h^n}{n!}$, yielding elements of the form

$$e^{ha} = \sum_{n\geq 0} \frac{a^n h^n}{n!} \tag{4.5}$$

in any separated complete topological algebra A. If a' is another element in A *commuting with* a, then

$$e^{ha}e^{ha'} = e^{h(a+a')}. \tag{4.6}$$

As a consequence, we see that e^{ha} is invertible in A with inverse equal to e^{-ha}.

A *morphism* $f : (A, \mu, \eta) \to (A', \mu', \eta')$ *of topological algebras* is a K-linear map $f : A \to A'$ such that

$$f \circ \mu = \mu' \circ (f\widetilde{\otimes}f) \quad \text{and} \quad f \circ \eta = \eta'. \tag{4.7}$$

Using the convention of (4.3), Relations (4.7) can be rewritten as

$$f(a_1a_2) = f(a_1)f(a_2) \quad \text{and} \quad f(1) = 1 \tag{4.8}$$

where a_1 and a_2 are elements of A.

Example 1. Let $A = K = \mathbf{C}[[h]]$. We identify $K\widetilde{\otimes}K$ with K. Then we see $(K, \mathrm{id}_K, \mathrm{id}_K)$ is a topological algebra. Moreover, the map $\eta : K \to A$ is a morphism of topological algebras for any topological algebra (A, μ, η).

Example 2. Let (A, μ, η) and (A', μ', η') be topological algebras. Then so is

$$(A\widetilde{\otimes}A', (\mu\widetilde{\otimes}\mu') \circ (\mathrm{id}_A\widetilde{\otimes}\widetilde{\tau}_{A,A'}\widetilde{\otimes}\mathrm{id}_A), \eta\widetilde{\otimes}\eta')$$

where $\widetilde{\tau}_{A,A'} : A\widetilde{\otimes}A' \to A'\widetilde{\otimes}A$ is the flip. In other words, the product in the *tensor product algebra* $A\widetilde{\otimes}A'$ is given by

$$(a_1\widetilde{\otimes}a_1')(a_2\widetilde{\otimes}a_2') = a_1a_2\widetilde{\otimes}a_1'a_2' \tag{4.9}$$

and the unit is $1\widetilde{\otimes}1$.

A *topological quasi-bialgebra* is a sextuple $(A, \mu, \eta, \Delta, \varepsilon, \Phi)$ where (A, μ, η) is a topological algebra, $\Delta : A \to A\widetilde{\otimes}A$ and $\varepsilon : A \to K$ are K-linear maps, and Φ is an invertible element of the tensor product algebra $A\widetilde{\otimes}A\widetilde{\otimes}A$ such that

$$(\mathrm{id}_A\widetilde{\otimes}\Delta)(\Delta(a)) = \Phi\Big((\Delta\widetilde{\otimes}\mathrm{id}_A)(\Delta(a))\Big)\Phi^{-1} \tag{4.10}$$

for all $a \in A$,

$$(\varepsilon\widetilde{\otimes}\mathrm{id}_A)\Delta = \mathrm{id}_A = (\mathrm{id}_A\widetilde{\otimes}\varepsilon)\Delta, \tag{4.11}$$

$$(\mathrm{id}_A\widetilde{\otimes}\mathrm{id}_A\widetilde{\otimes}\Delta)(\Phi)\,(\Delta\widetilde{\otimes}\mathrm{id}_A\widetilde{\otimes}\mathrm{id}_A)(\Phi) = \Phi_{234}\,(\mathrm{id}_A\widetilde{\otimes}\Delta\widetilde{\otimes}\mathrm{id}_A)(\Phi)\,\Phi_{123}, \tag{4.12}$$

and

$$(\mathrm{id}_A\widetilde{\otimes}\varepsilon\widetilde{\otimes}\mathrm{id}_A)(\Phi) = 1 \otimes 1. \tag{4.13}$$

When $\Phi = 1\widetilde{\otimes}1\widetilde{\otimes}1$, we call A a *topological bialgebra.*

A *morphism* $f : (A, \mu, \eta, \Delta, \varepsilon, \Phi) \to (A', \mu', \eta', \Delta', \varepsilon', \Phi')$ *of topological quasi-bialgebras* is a morphism f between the underlying topological algebras such that

$$(f\widetilde{\otimes}f)\Delta = \Delta' f \quad \text{and} \quad (f\widetilde{\otimes}f\widetilde{\otimes}f)(\Phi) = \Phi'. \tag{4.14}$$

A topological quasi-bialgebra $(A, \mu, \eta, \Delta, \varepsilon, \Phi)$ is a *topological braided quasi-bialgebra* if there exists an invertible element R of the tensor product algebra $A\widetilde{\otimes}A$ such that

$$\Delta^{\mathrm{op}}(a) = R\Delta(a)R^{-1}, \tag{4.15}$$

$$(\mathrm{id}_A\widetilde{\otimes}\Delta)(R) = (\Phi_{231})^{-1}R_{13}\Phi_{213}R_{12}(\Phi_{123})^{-1}, \tag{4.16}$$

and

$$(\Delta\widetilde{\otimes}\mathrm{id}_A)(R) = \Phi_{312}R_{13}(\Phi_{132})^{-1}R_{23}\Phi_{123}. \tag{4.17}$$

As before, R is called the universal R-matrix of A. It is part of the data of a topological braided quasi-bialgebra. A morphism of topological braided quasi-bialgebras is a morphism of the underlying topological quasi-bialgebras sending the universal R-matrix of the first one to the universal R-matrix of the second one.

We also need the concept of a *gauge transformation* on a topological quasi-bialgebra A: it is an invertible element F of $A\tilde{\otimes}A$ such that

$$(\varepsilon\tilde{\otimes}\mathrm{id}_A)(F) = (\mathrm{id}_A\tilde{\otimes}\varepsilon)(F) = 1. \tag{4.18}$$

From a topological (braided) quasi-bialgebra A and a gauge transformation F one can form a new topological (braided) quasi-bialgebra A_F by (XV.3.2–3.3) and (XV.3.9).

Example 3. Let $A_0 = (A_0, \mu_0, \eta_0, \Delta_0, \varepsilon_0, \Phi_0, R_0)$ be a braided quasi-bialgebra over the field $\mathbf{C}$ of complex numbers. Using Proposition 2.3 (b) and Proposition 3.2, one may define the topological braided quasi-bialgebra

$$A_0[[h]] = (A_0[[h]], \mu, \eta, \Delta, \varepsilon, \Phi_0, R_0)$$

on the space of formal series with coefficients in A_0 where μ, η, Δ and ε are the unique K-linear maps such that $\eta(f) = f\eta_0(1) = f1$ for all $f \in K$,

$$\mu(a\tilde{\otimes}a') = \mu_0(a \otimes a'), \quad \Delta(a) = \Delta_0(a), \quad \varepsilon(a) = \varepsilon_0(a)$$

for all $a, a' \in A_0$. We call $A_0[[h]]$ the *trivial topological braided quasi-bialgebra* associated to A_0.

Example 4. Let $A = (A, \mu, \eta, \Delta, \varepsilon, \Phi, R)$ be a topological braided quasi-bialgebra. Since $(A\tilde{\otimes}A)/h(A\tilde{\otimes}A) \cong A/hA \otimes A/hA$, the K-linear maps $\mu, \eta, \Delta, \varepsilon$ induce $\mathbf{C}$-linear maps

$$\bar{\mu} : A/hA \otimes A/hA \to A/hA, \quad \bar{\eta} : \mathbf{C} \to A/hA,$$

$$\bar{\Delta} : A/hA \to A/hA \otimes A/hA, \quad \bar{\varepsilon} : A/hA \to \mathbf{C}.$$

Define $\bar{\Phi}$ as the class of Φ modulo $(A/hA)^{\otimes 3}$ and $\bar{R}$ as the class of R modulo $(A/hA)^{\otimes 2}$. Then $\bar{A} = (A/hA, \bar{\mu}, \bar{\eta}, \bar{\Delta}, \bar{\varepsilon}, \bar{\Phi}, \bar{R})$ is a braided quasi-bialgebra.

Another concept we need to adapt is the concept of a *topological A-module M* over a topological algebra $A = (A, \mu, \eta)$. It is a left K-module with a K-linear map $\mu_M : A\tilde{\otimes}M \to M$ such that

$$\mu_M \circ (\mu\tilde{\otimes}\mathrm{id}_M) = \mu_M \circ (\mathrm{id}_A\tilde{\otimes}\mu_M) \quad \text{and} \quad \mu_M \circ (\eta\tilde{\otimes}\mathrm{id}_M) = \mathrm{id}_M. \tag{4.19}$$

We shall write $\mu_M(a\tilde{\otimes}m) = am$ for $a \in A$ and $m \in M$. The definition of a morphism of topological A-modules is left to the reader.

Let M and N be topological A-modules. Then their topological tensor product $M\tilde{\otimes}N$ is a topological $A\tilde{\otimes}A$-module. If A has a comultiplication $\Delta : A \to A\tilde{\otimes}A$, we can pull back the $A\tilde{\otimes}A$-module structure on $M\tilde{\otimes}N$ to a topological A-module structure given by

$$a(m\tilde{\otimes}n) = \Delta(a)(m\tilde{\otimes}n) \tag{4.20}$$

for all $a \in A$, m in M, and n in N.

If A is a topological braided bialgebra with universal R-matrix R, then for any topological A-module M the K-linear automorphism $c^R_{M,M}$ defined as in VIII.3 by

$$c^R_{M,M}(m_1 \tilde{\otimes} m_2) = \Big(R(m_1 \tilde{\otimes} m_2)\Big)_{21} \tag{4.21}$$

is a solution of the Yang-Baxter equation in $M \tilde{\otimes} M \tilde{\otimes} M$.

Proceeding as in XV.4, we can show that the universal R-matrix of a topological braided quasi-bialgebra gives rise to a representation of the braid group B_n on the topological A-module $M^{\tilde{\otimes} n}$ where n is any integer >1 and M is any topological A-module. Theorem XV.4.2 can be reformulated in the present context.

Mimicking IX.5, we say that a topological A-module M over a topological bialgebra $A = (A, \mu, \eta, \Delta, \varepsilon)$ with left action $\mu_M : A \tilde{\otimes} M \to M$ is a *topological crossed A-bimodule* if there exists a K-linear map $\Delta_M : M \to M \tilde{\otimes} A$ such that

$$(\mathrm{id}_M \tilde{\otimes} \Delta)\Delta_M = (\Delta_M \tilde{\otimes} \mathrm{id}_A)\Delta_M, \quad (\mathrm{id}_M \tilde{\otimes} \varepsilon)\Delta_M = \mathrm{id}_M \tag{4.22}$$

and

$$\begin{aligned} &(\mu_M \tilde{\otimes} \mu)(\mathrm{id}_A \tilde{\otimes} \tilde{\tau}_{A,M} \tilde{\otimes} \mathrm{id}_A)(\Delta \tilde{\otimes} \Delta_M) \\ &= (\mathrm{id}_M \tilde{\otimes} \mu)(\Delta_M \tilde{\otimes} \mathrm{id}_A)\tilde{\tau}_{A,M}(\mathrm{id}_A \tilde{\otimes} \mu_M)(\Delta \tilde{\otimes} \mathrm{id}_M). \end{aligned} \tag{4.23}$$

XVI.5 Quantum Enveloping Algebras

Let $\mathfrak{g}$ be a complex Lie algebra. In V.2 we defined its enveloping algebra $U(\mathfrak{g})$ and proved that it had a natural bialgebra structure determined by

$$\Delta(x) = 1 \otimes x + x \otimes 1 \quad \text{and} \quad \varepsilon(x) = 0$$

for all x belonging to $\mathfrak{g}$. We equip it with a trivial braided quasi-bialgebra structure with $\Phi = 1 \otimes 1 \otimes 1$ and $R = 1 \otimes 1$.

Definition XVI.5.1. *A quantum enveloping algebra (QUE) for the Lie algebra $\mathfrak{g}$ is a topological braided quasi-bialgebra $A = (A, \mu, \eta, \Delta, \varepsilon, \Phi, R)$ such that A is a topologically free module, the induced braided quasi-bialgebra $\bar{A} = (A/hA, \bar{\mu}, \bar{\eta}, \bar{\Delta}, \bar{\varepsilon}, \bar{\Phi}, \bar{R})$ as in Example 4 of Section 4 coincides with the trivial braided quasi-bialgebra structure of $U(\mathfrak{g})$ and the map η is trivially extended from $\bar{\eta}$.*

Let us be more explicit about this definition. First, a QUE is topologically free. This means that $A = (A/hA)[[h]]$ as a left K-module. By hypothesis, we also have $A/hA = U(\mathfrak{g})$.

Therefore

$$A = U(\mathfrak{g})[[h]] \tag{5.1}$$

as a K-module. From Proposition 3.2 we derive

$$A^{\widetilde{\otimes} n} = \left(U(\mathfrak{g})^{\otimes n}\right)[[h]] \tag{5.2}$$

for all $n > 0$. By Proposition 2.3 (b) we know that the maps μ, η, Δ and ε are determined by their restrictions to $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $\mathbf{C}$, $U(\mathfrak{g})$ and $U(\mathfrak{g})$ respectively. For elements $a, a' \in U(\mathfrak{g})$, we have

$$\mu(a \otimes a') = \sum_{n \geq 0} \mu_n(a \otimes a')h^n \tag{5.3}$$

where $(\mu_n)_{n\geq 0}$ is a family of linear maps from $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ to $U(\mathfrak{g})$ such that μ_0 is the product in the enveloping algebra. Similarly,

$$\Delta(a) = \sum_{n \geq 0} \Delta_n(a)h^n \tag{5.4}$$

where $(\Delta_n)_{n\geq 0}$ is a family of linear maps from $U(\mathfrak{g})$ to $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ such that Δ_0 is the comultiplication of the enveloping algebra described above. We also have

$$\varepsilon(a) = \sum_{n \geq 0} \varepsilon_n(a)h^n \tag{5.5}$$

where $(\varepsilon_n)_{n\geq 0}$ is a family of linear maps from $U(\mathfrak{g})$ to $\mathbf{C}$ such that ε_0 is the counit of the enveloping algebra. The last part of Definition 5.1 means that the unit η of A is given by

$$\eta(f) = f1 \tag{5.6}$$

for all $f \in \mathbf{C}[[h]]$. Finally, by Proposition 3.2 again, the elements Φ and R can be written

$$\Phi = \sum_{n \geq 0} \Phi_n h^n \tag{5.7}$$

and

$$R = \sum_{n \geq 0} R_n h^n \tag{5.8}$$

where $(\Phi_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ are families of elements of $U(\mathfrak{g})^{\otimes 3}$ and $U(\mathfrak{g})^{\otimes 2}$ respectively such that

$$\Phi_0 = 1 \otimes 1 \otimes 1 \quad \text{and} \quad R_0 = 1 \otimes 1. \tag{5.9}$$

It is clear from Lemma 1.1 that (5.9) ensures the invertibility of Φ and R.

By definition, a quantum enveloping algebra A is associated to a Lie algebra $\mathfrak{g}$. One recovers $\mathfrak{g}$ from A by

$$\mathfrak{g} = \left\{x \in A/hA \;\middle|\; \Delta_0(a) = 1 \otimes a + a \otimes 1\right\} \tag{5.10}$$

in view of the fact (stated in V.9) that the subspace of primitive elements in $U(\mathfrak{g})$ is $\mathfrak{g}$ provided that the ground field is of characteristic zero.

We now associate another invariant to a QUE A. If R is its universal R-matrix and R_{21} is the image of R under the flip, the formula

$$R_{21}R \equiv 1 \otimes 1 + ht \mod h^2 \tag{5.11}$$

defines a unique element $t \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Expressing t using (5.8), we get $t = R_1 + (R_1)_{21}$, which immediately proves that

$$t_{21} = t. \tag{5.12}$$

Proposition XVI.5.2. *The element t is an invariant symmetric element of $\mathfrak{g} \otimes \mathfrak{g}$, i.e., we have $t_{21} = t$ and $[\Delta(x), t] = 0$ for all $x \in \mathfrak{g}$. It remains unchanged under any gauge transformation.*

The element $t \in \mathfrak{g} \otimes \mathfrak{g}$ will be referred to as *the canonical* 2*-tensor* of the quantum enveloping algebra A. Drinfeld calls the pair $(\mathfrak{g}, t)$ the *classical limit* of A and the quantum enveloping algebra A a *quantization* of the pair $(\mathfrak{g}, t)$.

PROOF. We have already observed that t is symmetric. Let us prove that it belongs to the subspace $\mathfrak{g} \otimes \mathfrak{g}$. We again use the fact that $\mathfrak{g}$ is the subspace of primitive elements in the enveloping algebra. Let us compute $(\Delta \widetilde{\otimes} \mathrm{id}_A)(R_{21}R)$ using (4.16–4.17). We have

$$\begin{aligned}(\Delta \widetilde{\otimes} \mathrm{id}_A)(R_{21}R) &= \Big((\mathrm{id}_A \widetilde{\otimes} \Delta)(R)\Big)_{312} (\Delta \widetilde{\otimes} \mathrm{id}_A)(R) \\ &= \Phi^{-1} R_{32} \Phi_{132} R_{31} (\Phi_{312})^{-1} \Phi_{312} R_{13} (\Phi_{132})^{-1} R_{23} \Phi.\end{aligned}$$

Identifying the coefficients of h, we get

$$(\Delta_0 \otimes \mathrm{id})(t) = t_{13} + t_{23}. \tag{5.13}$$

Now write $t = \sum_i x_i \otimes y_i$ where $(y_i)_i$ is a family of linearly independent elements of $U(\mathfrak{g})$. Thus (5.13) becomes

$$\sum_i \Delta_0(x_i) \otimes y_i = \sum_i (x_i \otimes 1 + 1 \otimes x_i) \otimes y_i, \tag{5.14}$$

which implies that $\Delta_0(x_i) = x_i \otimes 1 + 1 \otimes x_i$ for all i. Since the element x_i is primitive for the comultiplication Δ_0 of $U(\mathfrak{g})$, it belongs to $\mathfrak{g}$. Consequently, t belongs to $\mathfrak{g} \otimes U\mathfrak{g}$. Relation (5.12) implies actually that t is in $\mathfrak{g} \otimes \mathfrak{g}$.

Let us check the invariance of t. By (4.15) applied twice, we get

$$\Delta(a) R_{21} R = R_{21} R \Delta(a) \tag{5.15}$$

for all $a \in A$. Identifying the coefficients of h, we obtain $\Delta_0(x)t = t\Delta_0(x)$ for all $x \in \mathfrak{g}$.

Finally, let us apply a gauge transformation F to A. Then by (XV.3.9) we have

$$(R_F)_{21}R_F = FR_{21}F_{21}^{-1}F_{21}RF^{-1} = FR_{21}RF^{-1}. \tag{5.16}$$

Taking the coefficients of h, we get $t_F = t$ where t_F is the canonical 2-tensor of the QUE A_F obtained from A by the gauge-transformation F. □

At this point, the only explicit quantum enveloping algebras for a given Lie algebra $\mathfrak{g}$ we know are the trivial QUE $U(\mathfrak{g})[[h]]$ constructed from $U(\mathfrak{g})$ as explained in Example 3 of Section 4 and their gauge-transforms. Since the universal R-matrix of such a QUE is $1 \otimes 1$, the corresponding canonical 2-tensor vanishes: $t = 0$.

We now present an example of a QUE with a non-zero canonical 2-tensor. We shall see more non-trivial examples in Chapters XVII and XIX.

Example 1. (*A quantum enveloping algebra associated to the Heisenberg Lie algebra*) We consider the 3-dimensional Lie algebra $\mathfrak{g}$ with the set $\{x, y, z\}$ as a basis and with Lie bracket determined by

$$[x, y] = z \quad \text{and} \quad [x, z] = [y, z] = 0.$$

The symmetric 2-tensor $t = z \otimes z$ is invariant because z is central in the Lie algebra. We claim that there exists a QUE whose classical limit is $(\mathfrak{g}, t)$. Indeed, take the trivial bialgebra $A = U(\mathfrak{g})[[h]]$ as in Example 3 of Section 4, except that we set $R = e^{ht/2}$ and $\Phi = 1\tilde{\otimes}1\tilde{\otimes}1$. In order to make sure that A is a topological braided bialgebra, we have to check Relations (4.15–4.17). The first one follows from the fact that t is invariant. Relations (4.16–4.17) with $\Phi = 1\tilde{\otimes}1\tilde{\otimes}1$ are equivalent to

$$e^{h(t_{13}+t_{12})/2} = e^{ht_{13}/2}e^{ht_{12}/2} \quad \text{and} \quad e^{h(t_{13}+t_{23})/2} = e^{ht_{13}/2}e^{ht_{23}/2}. \tag{5.17}$$

Relations (5.17) hold because the elements t_{12}, t_{13} and t_{23} commute with one another, due to the centrality of z. Now,

$$R_{21}R = e^{ht} \equiv 1 \otimes 1 + ht \mod h^2$$

shows that $t = z \otimes z$ is the canonical 2-tensor of A.

XVI.6 Symmetrizing the Universal R-Matrix

The aim of this section is to prove that the universal R-matrix R of a quantum enveloping algebra will always satisfy $R = R_{21}$ after a suitable gauge transformation. We start with the following technical result.

Lemma XVI.6.1. *Let A be a topological algebra which is a topologically free module. Given an element $a \in A$, there exists a unique family $(c_n)_{n>0}$ of complex numbers such that*

$$\Big(1 + \sum_{n>0} c_n a^n h^n\Big)^2 = 1 + ah. \tag{6.1}$$

PROOF. Any formal series $1 + \sum_{n>0} c_n a^n h^n$ of the above form defines an element of the inverse limit $\widetilde{A} = \varprojlim_n A/h^n A$, hence of A since $A \cong \widetilde{A}$ by hypothesis. Equation (6.1) is equivalent to the system of equations

$$2c_1 = 1 \quad \text{and} \quad 2c_n + \sum_{p=1}^{n-1} c_p c_{n-p} = 0 \tag{6.2}$$

if $n > 1$. This system has a unique solution as can be seen by an easy induction. □

The unique element $1 + \sum_{n>0} c_n a^n h^n$ satisfying (6.1) is called the *square root* of the element $1 + ah$ and is denoted by $(1 + ah)^{1/2}$. Its inverse will be denoted by $(1 + ah)^{-1/2}$.

Proposition XVI.6.2. *Let $\mathfrak{g}$ be a complex Lie algebra and A be a quantum enveloping algebra for $\mathfrak{g}$. Then there exists a gauge transformation $F \in A\widetilde{\otimes}A$ with $F \equiv 1 \otimes 1$ modulo h such that, if we set $R' = F_{21}RF^{-1}$, then $R'_{21} = R'$. Moreover, if A is cocommutative, there exists such an F satisfying the additional relation $F\Delta(a) = \Delta(a)F$ for all $a \in A$.*

PROOF. For any element $u \in A\widetilde{\otimes}A$, define $\overline{u} = u_{21}$. If $R' = \overline{F}RF^{-1}$, then $\overline{R'} = F\overline{R}\,\overline{F}^{-1}$. We look for an element F such that $R' = \overline{R'}$. In other words we must solve the equation

$$F\overline{R}\,\overline{F}^{-1} = \overline{F}RF^{-1}, \tag{6.3}$$

which can also be written in the form

$$\overline{R}R = F^{-1}\overline{F}RF^{-1}\overline{F}R = (F^{-1}\overline{F}R)^2. \tag{6.4}$$

We claim that

$$F = \left(R(\overline{R}R)^{-1/2}\right)^{1/2} \tag{6.5}$$

is a solution of (6.4) where we use the notation defined after Lemma 6.1. The element F is invertible and congruent to $1 \otimes 1$ modulo h since R is. In order to prove the claim, we observe that $R(\overline{R}R) = (R\overline{R})R$ implies $R(\overline{R}R)^n = (R\overline{R})^n R$ for all $n \geq 1$, hence

$$Rf(\overline{R}R) = f(R\overline{R})R \tag{6.6}$$

for any complex formal series f in the variable h; in particular, we have

$$R(\overline{R}R)^{-1/2} = (R\overline{R})^{-1/2}R. \tag{6.7}$$

Let us compute $F^2\overline{F}^2$. By (6.7) we have

$$F^2\overline{F}^2 = R(\overline{R}R)^{-1/2}\overline{R}(R\overline{R})^{-1/2} = (R\overline{R})^{-1/2}R\overline{R}(R\overline{R})^{-1/2} = 1 \otimes 1.$$

Consequently, $\overline{F}^2 = F^{-2}$. By uniqueness of the square root, we obtain $\overline{F} = F^{-1}$. Using (6.7) again, we derive

$$(F^{-1}\overline{F}R)^2 = (\overline{F}^2R)^2 = (\overline{R}(R\overline{R})^{-1/2}R)^2 = (\overline{R}R(\overline{R}R)^{-1/2})^2 = \overline{R}R,$$

which proves that (6.4) has a solution. This takes care of the first part of the proposition.

As for the second one, observe that Relation (XV.2.1) and the cocommutativity of Δ imply that $\Delta(a)$ commutes with R and with $\overline{R}$ for all $a \in A$. Consequently, $\Delta(a)$ commutes with F in view of (6.5). $\square$

XVI.7 Exercises

1. Show that
$$1 + \frac{h}{2} + \sum_{n\geq 2} (-1)^{n-1}\frac{(2n-3)!!}{2^n n!}\,h^n$$
is a square root of $1 + h$ in the algebra $\mathbf{C}[[h]]$ of formal series where $(2n-3)!! = \prod_{k=1}^{n-1}(2k-1)$.

2. Let $M = V[[h]]$ and $N = W[[h]]$ be topologically free modules. Show that $\mathrm{Hom}_K(M, N)$ is a topologically free module isomorphic to $\mathrm{Hom}(V, W)[[h]]$. Deduce that if P is a third topologically free module, then
$$\mathrm{Hom}_K(M\widetilde{\otimes}N, P) \cong \mathrm{Hom}_K(M, \mathrm{Hom}_K(N, P)).$$

3. Let $\mathfrak{g}$ be a Lie algebra and $t \in \mathfrak{g}\otimes\mathfrak{g}$ such that $[t_{12}, t_{13}] = [t_{13}, t_{23}] = 0$ in $U(\mathfrak{g})^{\otimes 3}$. Consider the gauge transformation $F = e^{ht}$. Show that $(U\mathfrak{g}[[h]])_F$ is a topological bialgebra.

4. Show that the inverse systems of abelian groups and the maps of inverse systems form a category *Inv* such that $\varprojlim_n$ is a functor from *Inv* to the category *Ab* of abelian groups. Prove that $\varprojlim_n$ is left adjoint to the functor assigning to each abelian group A the constant inverse system (A_n, p_n) where $A_n = A$ and $p_n = \mathrm{id}_A$ for all n.

5. Let $(C_n)_{n\geq 0}$ be a denumerable family of abelian groups. Consider the inverse system (A_n, p_n) where $A_n = C_0 \times \cdots \times C_n$ and p_n is the natural projection. Prove that the inverse limit of this system is isomorphic to the direct product of all groups C_n.

6. Let (A_n, p_n) be an inverse system of abelian groups. Use the fact that its inverse limit can be expressed as the kernel of an endomorphism of $\prod_n A_n$ to prove that for any abelian group C there is a natural isomorphism
$$\mathrm{Hom}(C, \varprojlim_n A_n) \cong \varprojlim_n \mathrm{Hom}(C, A_n).$$

7. (*The ring of p-adic integers*) Given a prime p consider the inverse system of rings $(\mathbf{Z}/p^n\mathbf{Z})$ equipped with the natural projections induced by the inclusions of ideals $(p^n) \subset (p^{n-1})$. Show that the inverse limit $\mathbf{Z}_p$ is a ring with a unique maximal ideal. Prove that the inverse limit topology on $\mathbf{Z}_p$ can be defined by an ultrametric distance and that the ring of natural integers $\mathbf{Z}$ forms a dense subring of $\mathbf{Z}_p$ in which all integers prime to p are invertible.

XVI.8 Notes

The material of Sections 1–4 is standard. For details on h-adic topology and completions, read [Bou61], III and [Mat70], Chap. 9. The concept of a quantum enveloping algebra and the content of Sections 5–6 are due to Drinfeld (see [Dri87] and [Dri89b], Section 3). Exercise 3 is taken from [Enr92].

XVI.9 Appendix. Inverse Limits

An *inverse system of abelian groups* (A_n, p_n) is a family $(A_n)_{n\in\mathbf{N}}$ of abelian groups and of morphisms of groups $(p_n : A_n \to A_{n-1})_{n>0}$. Given such a system we can define its *inverse limit* $\varprojlim_n A_n$ by

$$\varprojlim_n A_n = \Big\{(x_n)_{n\geq 0} \in \prod_{n\geq 0} A_n \;\Big|\; p_n(x_n) = x_{n-1} \text{ for all } n > 0\Big\}. \tag{9.1}$$

The inverse limit has an abelian group structure as a subset of the direct product $\prod_{n\geq 0} A_n$ whose group structure is defined component-wise. The natural projection from $\prod_{n\geq 0} A_n$ to A_k restricts to a morphism of groups $\pi_k : \varprojlim_n A_n \to A_k$. It is defined by $\pi_k((x_n)_n) = x_k$. If all maps p_n are surjective, then so are the maps π_n.

By definition of the inverse limit, we have

$$p_n \circ \pi_n = \pi_{n-1} \tag{9.2}$$

for all $n > 0$. The inverse limit has the following universal property.

Proposition XVI.9.1. *For any abelian group C and any given family $(f_n : C \to A_n)_{n\geq 0}$ of morphims of groups such that $p_n \circ f_n = f_{n-1}$ for all $n > 0$, there exists a unique morphism of groups*

$$f : C \to \varprojlim_n A_n$$

such that $\pi_n \circ f = f_n$ for all $n \geq 0$.

PROOF. The family $(f_n)_n$ defines a unique morphism of groups f from C into the direct product of all groups A_n. The hypothesis $p_n \circ f_n = f_{n-1}$ implies that the image of f lies in the subgroup $\varprojlim_n A_n$. This proves the existence of f. The condition $\pi_n \circ f = f_n$ implies its uniqueness. $\square$

The inverse limit is functorial. Define a map from the inverse system (A_n, p_n) to the inverse system (A'_n, p'_n) as a family $(f_n : A_n \to A'_n)_{n \geq 0}$ of morphisms of groups such that $p'_n \circ f_n = f_{n-1} \circ p_n$ for all $n > 0$.

Proposition XVI.9.2. *Under the previous hypothesis, there exists a unique morphism of groups*

$$f = \varprojlim_n f_n : \varprojlim_n A_n \to \varprojlim_n A'_n$$

such that $\pi'_n \circ f = f_n \circ \pi_n$ *for all* $n \geq 0$.

PROOF. The family $(f_n \circ \pi_n : \varprojlim_n A_n \to A'_n)_n$ satisfies the hypothesis of Proposition 9.1. It follows that there exists a unique morphism f such that $\pi'_n \circ f = f_n \circ \pi_n$ for all n. $\square$

For composable maps of inverse systems, we have

$$\left(\varprojlim_n f_n\right) \circ \left(\varprojlim_n g_n\right) = \varprojlim_n (f_n \circ g_n).$$

The inverse limit of any inverse system (A_n, p_n) possesses a natural topology called the *inverse limit topology.* It is obtained as follows. Put the discrete topology on each A_n, i.e., the topology for which each subset is an open set. The inverse limit topology on $\varprojlim_n A_n$ is the restriction of the direct product topology on $\prod_{n \geq 0} A_n$. In other words, a basis of open sets of the inverse limit is given by the family of all subsets $\pi_n^{-1}(U_n)$ where n runs over the non-negative integers and U_n is any subset of A_n. By definition of this topology, the structural maps π_n from $\varprojlim_n A_n$ to A_n are continuous. Moreover, a map f from a topological set to $\varprojlim_n A_n$ is continuous with respect to the inverse limit topology if and only if the map $\pi_n \circ f$ into A_n is continuous for all $n \geq 0$.

One may replace the word "abelian group" by "ring", "module"... in the above definition. The statements of the Appendix remain true in this case, a fact we have consistently used in this chapter without further explanation.

Chapter XVII

Drinfeld and Jimbo's Quantum Enveloping Algebras

In Part I we have investigated at length the quantum enveloping algebra of $\mathfrak{sl}(2)$. In this chapter we give a brief presentation of the algebras $U_h(\mathfrak{g})$ associated by Drinfeld [Dri85][Dri87] and Jimbo [Jim85] to the other semisimple Lie algebras $\mathfrak{g}$. The algebras $U_h(\mathfrak{g})$ provide non-trivial examples of quantum enveloping algebras as defined in XVI.5 as well as examples of isotopy invariants of links. We shall also need $U_h(\mathfrak{g})$ in Chapter XIX to state the Drinfeld-Kohno theorem on the monodromy of the Knizhnik-Zamolodchikov systems. Finally, in Section 4 we shall determine an explicit universal R-matrix for the quantum enveloping algebra of $\mathfrak{sl}(2)$, using the crossed bimodules of IX.5.

XVII.1 Semisimple Lie Algebras

Before we present Drinfeld and Jimbo's quantum enveloping algebras, we recall a few facts from the theory of complex semisimple Lie algebras.

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. For any finite-dimensional representation ρ of $\mathfrak{g}$, we can define a bilinear form on $\mathfrak{g}$ by

$$< x, y >_\rho = \mathrm{tr}(\rho(x)\rho(y)) \tag{1.1}$$

where x, y are elements of $\mathfrak{g}$. From the properties of the trace, we immediately see that this bilinear form is symmetric and invariant, i.e., we have

$$< y, x >_\rho = < x, y >_\rho \quad \text{and} \quad < [x, y], z >_\rho = < x, [y, z] >_\rho \tag{1.2}$$

for all elements x, y, and z of $\mathfrak{g}$. When $\rho = \mathrm{ad}$ is the adjoint representation of $\mathfrak{g}$, the bilinear form $<\,,\,>_{\mathrm{ad}}$ is called the *Killing form* of $\mathfrak{g}$.

A *semisimple Lie algebra* is a finite-dimensional complex Lie algebra whose Killing form is non-degenerate. For any basis $\{x_i\}_i$ of $\mathfrak{g}$, there exists a unique basis $\{x^i\}_i$ called the dual basis of $\{x_i\}_i$ and determined by

$$< x_i, x^j >_{\mathrm{ad}} = \delta_{ij}$$

for all i, j. Define linear forms α_{ij} and β_{ij} on $\mathfrak{g}$ by

$$[x_i, x] = \sum_j \alpha_{ij}(x) x_j \quad \text{and} \quad [x^i, x] = \sum_j \beta_{ij}(x) x^j. \tag{1.3}$$

Lemma XVII.1.1. *We have $\beta_{ij} = -\alpha_{ji}$ for all i, j.*

PROOF. Applying (1.2) to the Killing form, we get

$$< [x_i, x], x^j >_{\mathrm{ad}} = < x_i, [x, x^j] >_{\mathrm{ad}} . \tag{1.4}$$

Expanding the left-hand side of (1.4) gives $< [x_i, x], x^j >_{\mathrm{ad}} = \alpha_{ij}(x)$ whereas we have $< x_i, [x, x^j] >_{\mathrm{ad}} = -\beta_{ji}(x)$ for the right-hand side. □

We now define the *Casimir element* C of $\mathfrak{g}$ as the element

$$C = \sum_i x_i x^i \tag{1.5}$$

of the enveloping algebra $U(\mathfrak{g})$.

Proposition XVII.1.2. *The Casimir element C is independent of the basis $\{x_i\}_i$ and belongs to the centre of $U(\mathfrak{g})$.*

PROOF. The first assertion follows from a well-known fact in linear algebra: if $\{y_i\}_i$ is a basis related to the basis $\{x_i\}_i$ by $y_i = \sum_j A_{ij} x_j$ where the scalars $(A_{ij})_{ij}$ form an invertible matrix A, then the dual basis $\{y^i\}_i$ of the basis $\{y_i\}_i$ is related to $\{x^i\}_i$ by

$$y^i = \sum_j B_{ji} x^j$$

where $B = (B_{ij})_{ij}$ is the inverse of the matrix A. Now,

$$\begin{aligned} \sum_i y_i y^i &= \sum_{j,k} \Big(\sum_i B_{ki} A_{ij} \Big) x_j x^k \\ &= \sum_{j,k} \delta_{kj} x_j x^k \\ &= \sum_j x_j x^j = C, \end{aligned}$$

which proves that C is independent of the choice of bases.

In order to prove that C is central, it suffices to check that C commutes with any element x of $\mathfrak{g}$. We have

$$\begin{aligned}
[C,x] &= \sum_i [x_i x^i, x] \\
&= \sum_i x_i [x^i, x] + \sum_i [x_i, x] x^i \\
&= \sum_{i,j} \Big(\beta_{ij}(x) x_i x^j + \alpha_{ij}(x) x_j x^i\Big) \\
&= 0
\end{aligned}$$

by Lemma 1.1. □

Using the comultiplication Δ of the enveloping algebra, we derive the element

$$t = \frac{\Delta(C) - 1 \otimes C - C \otimes 1}{2} = \sum_i \frac{x_i \otimes x^i + x^i \otimes x_i}{2} \tag{1.6}$$

of $\mathfrak{g} \otimes \mathfrak{g}$. This element will play a central rôle in Chapter XIX. It enjoys the following properties.

Proposition XVII.1.3. *The element t is a symmetric $\mathfrak{g}$-invariant element of $\mathfrak{g} \otimes \mathfrak{g}$, i.e., we have*

$$t_{21} = t \quad \text{and} \quad [\Delta(x), t] = 0 \tag{1.7}$$

for all $x \in \mathfrak{g}$, where $t_{21} = \tau_{\mathfrak{g},\mathfrak{g}}(t)$.

PROOF. The symmetry of t is clear from its definition. As for the $\mathfrak{g}$-invariance, it is enough to prove that $\Delta(x)$ commutes with $\Delta(C)$ and with $1 \otimes C + C \otimes 1$. For the first condition, we have $[\Delta(x), \Delta(C)] = \Delta([x, C]) = 0$ since Δ is a morphism of algebras and C is central. We also have

$$[\Delta(x), 1 \otimes C + C \otimes 1] = 1 \otimes [x, C] + [x, C] \otimes 1 = 0$$

again because C is central. □

Example 1. Consider the 3-dimensional simple Lie algebra $\mathfrak{sl}(2)$ of Chapter V. It is easy to check that its Killing form is non-degenerate, and that the dual of the basis $\{X, Y, H\}$ considered in V.3 is the basis $\{Y/4, X/4, H/8\}$. Consequently, for $\mathfrak{sl}(2)$ we get

$$t = \frac{1}{4}\Big(X \otimes Y + Y \otimes X + \frac{H \otimes H}{2}\Big). \tag{1.8}$$

Elie Cartan characterized every semisimple Lie algebra by its *Cartan matrix*, which is a square matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ with the following properties:

(i) its coefficients a_{ij} are non-positive integers when $i \neq j$, and $a_{ii} = 2$,

(ii) there exists a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ with entries, called the *root lengths*, in the set $\{1, 2, 3\}$ such that the matrix DA is symmetric positive definite.

According to a theorem of Serre's [Ser65], the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is isomorphic to the algebra generated by $3n$ generators $\{X_i, Y_i, H_i\}_{1 \leq i \leq n}$ and the relations

$$[H_i, H_j] = 0, \quad [X_i, Y_j] = \delta_{ij} H_i, \tag{1.9}$$

$$[H_i, X_j] = a_{ij} X_j, \quad [H_i, Y_j] = -a_{ij} Y_j, \tag{1.10}$$

and if $i \neq j$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} X_i^k X_j X_i^{1-a_{ij}-k} = 0 \tag{1.11}$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} Y_i^k Y_j Y_i^{1-a_{ij}-k} = 0. \tag{1.12}$$

The Cartan matrix for $\mathfrak{sl}(2)$ is the 1×1-matrix $A = (2)$ with $D = (1)$. In this case, the presentation above reduces to the formulas (V.3.2).

We end this summary by a few words on the representation theory of a semisimple Lie algebra $\mathfrak{g}$. Any finite-dimensional $\mathfrak{g}$-module is semisimple, i.e., is the direct sum of simple modules. The finite-dimensional simple $\mathfrak{g}$-modules are classified by the set of *dominant weights*: a dominant weight is a linear form λ on the subspace $\mathfrak{h}$ of $\mathfrak{g}$ spanned by $H_1, \ldots, H_n$ such that $\lambda(H_i)$ is a non-negative integer for all $i = 1, \ldots, n$. For every dominant weight λ, there exists a unique finite-dimensional simple $\mathfrak{g}$-module V_λ generated by an element v_λ, called a highest weight vector, such that

$$H_i v_\lambda = \lambda(H_i) v_\lambda \quad \text{and} \quad X_i v_\lambda = 0 \tag{1.13}$$

for all $i = 1, \ldots, n$. All finite-dimensional simple $\mathfrak{g}$-modules are of this form. The Casimir element C acts by a positive scalar on every simple $\mathfrak{g}$-module V_λ of dimension > 1, i.e., with $\lambda \neq 0$. We have proved these facts for $\mathfrak{sl}(2)$ in Chapter V. In the case of $\mathfrak{sl}(2)$ the set of dominant weights is in bijection with $\mathbf{N}$, the dominant weight λ corresponding to the integer n being defined by $\lambda(H) = n$.

XVII.2 Drinfeld-Jimbo Algebras

Before we describe the quantum enveloping algebra $U_h(\mathfrak{g})$ attached by Drinfeld and Jimbo to any complex semisimple Lie algebra $\mathfrak{g}$, we introduce the notion of a topological algebra presented by generators and relations. Recall the notation $K = \mathbf{C}[[h]]$ and $K_n = \mathbf{C}[h]/(h^n)$ from XVI.1.

Given a set X, we define the *topologically free algebra generated by* X as the algebra of formal series over the free complex algebra generated by the set X:

$$K\langle X\rangle = (\mathbf{C}\langle X\rangle)[[h]].$$

We equip $K\langle X\rangle$ with the h-adic topology. It has the following universal property which is the topological counterpart of Proposition I.2.1.

Proposition XVII.2.1. *Let $f : X \to A$ be a map from a set X to a separated complete K-algebra A. Then there exists a unique continuous K-linear map $\bar{f} : K\langle X\rangle \to A$ such that $\bar{f}(x) = f(x)$ for all $x \in X$.*

PROOF. Clearly, f extends to a unique K_n-linear algebra morphism

$$f_n : K_n\langle X\rangle \to A/h^n A.$$

We then take the inverse limit of the maps f_n. The uniqueness of $\bar{f}$ results from the fact that the K-subalgebra generated by X is dense in $K\langle X\rangle$. □

Definition XVII.2.2. *Let X be a set and R be a subset of the topologically free algebra $K\langle X\rangle$ generated by X. A K-algebra A is said to be the K-algebra topologically generated by the set X of generators and the set R of relations if A is isomorphic to the quotient of $K\langle X\rangle$ by the closure (for the h-adic topology) of the two-sided ideal generated by R.*

As an immediate consequence of Proposition 2.1 and of Definition 2.2, we see that the space of morphisms of K-algebras from A to a separated complete K-algebra A' is in bijection with the set of maps $f : X \to A'$ such that $\bar{f}$ vanishes on R.

We also recall the definition of the following symbols already considered in VI.1. We added a subscript q in order to stress the dependence on the parameter q. For any invertible element q and any integer n, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

If r is a non-negative integer, set $[0]_q! = 1$ and if $r > 0$

$$[r]_q! = [1]_q[2]_q \cdots [n]_q \quad \text{and} \quad \begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q[n-1]_q \ldots [n-r+1]_q}{[r]_q!}.$$

We now turn to the definition of the Drinfeld-Jimbo algebra $U_h(\mathfrak{g})$. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $A = (a_{ij})_{1\leq i,j\leq n}$ be its Cartan matrix, with the diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ of root lengths.

Definition XVII.2.3. *The algebra $U_h(\mathfrak{g})$ is defined to be the K-algebra topologically generated by the set of generators $\{X_i, Y_i, H_i\}_{1\leq i\leq n}$ and the relations*

$$[H_i, H_j] = 0, \quad [X_i, Y_j] = \delta_{ij}\frac{\sinh(hd_iH_i/2)}{\sinh(hd_i/2)}, \tag{2.1}$$

$$[H_i, X_j] = a_{ij}X_j, \quad [H_i, Y_j] = -a_{ij}Y_j \tag{2.2}$$

and if $i \neq j$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} X_i^k X_j X_i^{1-a_{ij}-k} = 0 \tag{2.3}$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} Y_i^k Y_j Y_i^{1-a_{ij}-k} = 0 \tag{2.4}$$

where $q_i = e^{hd_i/2}$ and where sinh *is the formal series*

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}.$$

Note that, although $\sinh(hd_i/2)$ is not invertible, it is the product of h with a unique invertible element, so that $\sinh(hd_iH_i/2)/\sinh(hd_i/2)$ is a well-defined element of $K\langle\{X_i, Y_i, H_i\}_{1\leq i\leq n}\rangle$. We have

$$\frac{\sinh(hd_iH_i/2)}{\sinh(hd_i/2)} \equiv H_i \mod h.$$

Observe also that Relations (2.2) imply that

$$e^{\lambda hH_i}X_j = e^{\lambda a_{ij}}X_je^{\lambda hH_i} \quad \text{and} \quad e^{\lambda hH_i}Y_j = e^{-\lambda a_{ij}}Y_je^{\lambda hH_i} \tag{2.5}$$

for all i, j and any complex number λ.

We now state the main result of this section.

Theorem XVII.2.4. *The topological algebra $U_h(\mathfrak{g})$ is a quantum enveloping algebra*

$$(U_h(\mathfrak{g}), \mu_h, \eta_h, \Delta_h, \varepsilon_h, \Phi_h, R_h)$$

for the Lie algebra $\mathfrak{g}$ with $\Phi_h = 1\tilde{\otimes}1\tilde{\otimes}1$ and comultiplication Δ_h and counit ε_h determined by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i, \tag{2.6}$$

$$\Delta_h(X_i) = X_i \otimes e^{hd_iH_i/4} + e^{-hd_iH_i/4} \otimes X_i, \tag{2.7}$$

$$\Delta_h(Y_i) = Y_i \otimes e^{hd_iH_i/4} + e^{-hd_iH_i/4} \otimes Y_i, \tag{2.8}$$

and

$$\varepsilon_h(H_i) = \varepsilon_h(X_i) = \varepsilon_h(Y_i) = 0. \tag{2.9}$$

We refer to [Dri87] for a proof. Let us make a few remarks. First, if we set $h = 0$ in Relations (2.1–2.4) and (2.6–2.9), we recover the enveloping algebra of $\mathfrak{g}$ in Serre's presentation. In other words, we have an isomorphism of algebras

$$U_h(\mathfrak{g})/hU_h(\mathfrak{g}) \cong U(\mathfrak{g}). \tag{2.10}$$

The fact that $U_h(\mathfrak{g})$ is a topologically free K-module is not straightforward. It can be proved by constructing a Poincaré-Birkhoff-Witt-type basis. One has also to check that (2.6–2.9) define morphisms of algebras Δ_h and ε_h. For Δ_h this follows from (2.5) and the q-binomial formula of Proposition IV.2.2.

The topological bialgebra $U_h(\mathfrak{g})$ has an antipode S_h determined by

$$S_h(H_i) = -H_i, \quad S_h(X_i) = -e^{hd_i/2}X_i, \quad S_h(Y_i) = -e^{-hd_i/2}Y_i. \tag{2.11}$$

Note that the comultiplication of $U_h(\mathfrak{g})$ is not cocommutative and that the antipode is not involutive. Nevertheless, for all $a \in U_h(\mathfrak{g})$ we have

$$S_h^2(a) = e^{h\rho}ae^{-h\rho} \tag{2.12}$$

where $\rho = \sum_{i=1}^n \mu_i H_i$, the scalars μ_i being determined from the inverse A^{-1} of the Cartan matrix by $\mu_i = \sum_{j=1}^n (A^{-1})_{ji} d_j$.

More importantly, Theorem 2.4 implicitly states that $U_h(\mathfrak{g})$ has a universal R-matrix, which we denote by R_h. Drinfeld proved that R_h is of the form

$$R_h = \sum_{\ell \in \mathbf{N}^n} e^{h\left(\frac{t_0}{2} + \frac{1}{4}(H_\ell \otimes 1 - 1 \otimes H_\ell)\right)} P_\ell \tag{2.13}$$

where $H_\ell = \sum_{1 \leq \ell \leq n} \ell_i H_i$ for $\ell = (\ell_1, \ldots, \ell_n)$, t_0 is the element

$$t_0 = \sum_{1 \leq i,j \leq n} (DA)_{ij}^{-1} H_i \otimes H_j \tag{2.14}$$

of $\mathfrak{g} \otimes \mathfrak{g}$, and P_ℓ is a polynomial in the variables $X_1 \otimes 1, \ldots, X_n \otimes 1$ and in $1 \otimes Y_1, \ldots, 1 \otimes Y_n$ (homogeneous of degree ℓ_i in $X_i \otimes 1$ and $1 \otimes Y_i$). We have $P_0 = 1 \otimes 1$ and

$$R_h \equiv 1 \otimes 1 \mod h. \tag{2.15}$$

The polynomials P_ℓ can be determined by induction on ℓ using Relations (XVI.4.15–4.17). Explicit expressions for R_h can be found in [KR90] [LS90] [Ros89] [Ros92].

The representation theory of $U_h(\mathfrak{g})$ is parallel to that of the Lie algebra $\mathfrak{g}$. Indeed, for any dominant weight λ of $\mathfrak{g}$, there exists a unique topologically free $U_h(\mathfrak{g})$-module $\widetilde{V_\lambda}$ satisfying

$$\widetilde{V_\lambda}/h\widetilde{V_\lambda} = V_\lambda \tag{2.16}$$

and generated by an element v_λ, called a highest weight vector, such that

$$H_i v_\lambda = \lambda(H_i) v_\lambda \quad \text{and} \quad X_i v_\lambda = 0 \tag{2.17}$$

for all $i = 1, \ldots, n$, as in the classical case. Rosso [Ros88] proved that any topologically free $U_h(\mathfrak{g})$-module W with $\dim(W/hW) < \infty$ was a direct sum of modules of the form $\widetilde{V_\lambda}$. We shall give an explanation of this fact in XVIII.4.

XVII.3 Quantum Group Invariants of Links

We now show how to construct an isotopy invariant $Q_{\mathfrak{g},V}$ out of any complex semisimple Lie algebra $\mathfrak{g}$ and of any finite-dimensional simple $\mathfrak{g}$-module V.

Consider the category $U_h(\mathfrak{g})$-Mod_{fr} of finite-rank topologically free $U_h(\mathfrak{g})$-modules, i.e., of topological modules of the form $V[[h]]$ where V is a finite-dimensional vector space. This category is a tensor category for the topological tensor product $\widetilde{\otimes}$ of XVI.3, the associativity and unit constraints being the canonical isomorphisms (XVI.3.1) and (XVI.3.3). Actually, the category $U_h(\mathfrak{g})$-Mod_{fr} is a braided tensor category with left duality: the braiding is induced by the universal R-matrix R_h while the duality is given on objects by $V[[h]]^* = V^*[[h]]$. The structure maps b and d of duality are $\mathbf{C}[[h]]$-linearly extended from the evaluation and coevaluation maps of II.2–3.

We claim that $U_h(\mathfrak{g})$-Mod_{fr} is a *ribbon category*. To sustain the claim, it suffices to exhibit a twist as defined in XIV.3. We proceed as in XIV.6. Let u be the invertible element of $U_h(\mathfrak{g})$ defined by Formula (VIII.4.1), which still makes sense in the present context. We have $u \equiv 1$ modulo h. By Proposition VIII.4.1 and by (2.12), we get

$$S_h^2(a) = uau^{-1} = e^{h\rho} a e^{-h\rho}$$

for all $a \in U_h(\mathfrak{g})$. This implies that

$$\theta = e^{-h\rho} u = u e^{-h\rho} \tag{3.1}$$

belongs to the centre of $U_h(\mathfrak{g})$.

Proposition XVII.3.1. *The central element θ satisfies the relations*

$$\Delta(\theta) = ((R_h)_{21} R_h)^{-1} (\theta \otimes \theta), \quad \varepsilon(\theta) = 1, \quad S(\theta) = \theta.$$

PROOF. We claim that

$$\theta^2 = uS(u) = S(u)u. \tag{3.2}$$

Then Proposition 3.1 follows from Proposition VIII.4.5 and from the fact that $uS(u)$ has a unique square root whose constant term is 1.

Relation (3.2) is reduced in [Dri89a], Proposition 5.1 to showing that both terms have the same action on all modules of the form $\widetilde{V_\lambda}$. It is enough to evaluate the central elements $S(u)u$ and $\theta^2 = e^{-2h\rho}u^2$ on a highest weight vector of $\widetilde{V_\lambda}$. Since u can be expressed in such a way that the generators X_i killing the highest weight vector appear to the right of Y_i, we see that the actions of $S(u)u$ and of θ^2 are the same as the actions of the elements obtained from the part of R_h corresponding to $\ell = 0$ in Formula (2.13). A simple computation shows then that $S(u)u$ and of θ^2 act by the same scalar on $\widetilde{V_\lambda}$. For more details, see [Dri89a], Section 5. □

Combining Propositions 3.1 and XIV.6.2, we conclude that the action by θ^{-1} induces a twist on the category $U_h(\mathfrak{g})$-Mod_{fr}, thus turning it into a ribbon category.

By Section 2, any finite-dimensional simple $\mathfrak{g}$-module V gives rise to a unique object $\widetilde{V}$ of $U_h(\mathfrak{g})$-Mod_{fr} such that $\widetilde{V}/h\widetilde{V} = V$. Applying Theorem XIV.5.1 to our ribbon category and to the object $\widetilde{V}$, we get a tensor functor $F_{\widetilde{V}}$ from the category $\mathcal{R}$ of framed tangles to $U_h(\mathfrak{g})$-Mod_{fr}, sending the object $(+)$ to $\widetilde{V}$. Restricting $F_{\widetilde{V}}$ to framed links yields an isotopy invariant $Q_{\mathfrak{g},V}$ for framed links with values in $\mathbf{C}[[h]]$. It is easy to check that

$$Q_{\mathfrak{g},V}(L) \equiv \dim(V)^d \mod h \tag{3.3}$$

for any link of order d. Since $U_h(\mathfrak{g})$-Mod_{fr} is a ribbon category, we have a quantum dimension for any object. Actually, by definition of the functor $F_{\widetilde{V}}$ above, $\dim_q(\widetilde{V})$ coincides with the value of $Q_{\mathfrak{g},V}$ on the trivial knot. Let us explain how one may determine $\dim_q(\widetilde{V})$ when $\widetilde{V} = \widetilde{V_\lambda}$ for a dominant weight λ. By Proposition XIV.6.4, $\dim_q(\widetilde{V_\lambda})$ is the trace of the action of $\theta^{-1}u = e^{h\rho}$ on $\widetilde{V_\lambda}$. Since $\widetilde{V_\lambda}$ has the same weight decomposition as the $\mathfrak{g}$-module $V_\lambda = \widetilde{V_\lambda}/h\widetilde{V_\lambda}$ and since $e^{h\rho}$ acts on a vector of weight μ by $e^{h<\mu,\rho>}$, we may compute $\dim_q(\widetilde{V_\lambda})$ as follows. Use Weyl's character formula to determine the character

$$\mathrm{ch}(V_\lambda) = \sum_\mu d_\mu\, e^\mu \tag{3.4}$$

of the simple module V_λ (as defined, e.g., in [Bou60], Chap. 8, §9) where μ runs over the weights of V_λ. Then

$$\dim_q(\widetilde{V_\lambda}) = \sum_\mu d_\mu\, e^{h<\mu,\rho>}. \tag{3.5}$$

When $h = 0$, we recover the dimension of V_λ. Therefore, the quantum dimension of $\widetilde{V_\lambda}$ may be viewed as a q-analogue of the dimension of V_λ.

We end this section by stating a special property of the universal R-matrix R_h of $U_h(\mathfrak{g})$. In the next chapter (see Corollary XVIII.4.2) we shall establish the existence of a *unique* isomorphism of topological algebras α from the centre of $U_h(\mathfrak{g})$ to the algebra $Z(\mathfrak{g})[[h]]$ of formal series over the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ such that $\alpha \equiv \mathrm{id}_{Z(\mathfrak{g})}$ modulo h. The Casimir element C in $Z(\mathfrak{g}) \subset Z(\mathfrak{g})[[h]]$, as defined by (1.5), can be pulled back to a unique central element, the *quantum Casimir element*,

$$C_h = \alpha^{-1}(C) \tag{3.6}$$

of $U_h(\mathfrak{g})$ satisfying

$$C_h \equiv C \mod h. \tag{3.7}$$

Proposition 5.1 of [Dri89a] asserts that

$$\theta = e^{hC_h/2}. \tag{3.8}$$

This is proved along the same lines as Proposition 3.1 above. In order to determine the action of C_h on $\widetilde{V_\lambda}$, it is enough to evaluate the action of the classical Casimir operator C on V_λ, which is well-known.

Combining (3.8) and Proposition 3.1, we get the following property for the universal R-matrix of $U_h(\mathfrak{g})$.

Proposition XVII.3.2. *The universal R-matrix R_h of $U_h(\mathfrak{g})$ satisfies the relation*

$$(R_h)_{21}R_h = \Delta_h(e^{hC_h/2})(e^{-hC_h/2} \otimes e^{-hC_h/2}).$$

As an immediate consequence, we see that the canonical 2-tensor of $U_h(\mathfrak{g})$ (as defined in XVI.5) is the symmetric invariant 2-tensor t of (1.6).

XVII.4 The Case of $\mathfrak{sl}(2)$

When the Lie algebra $\mathfrak{g}$ is the three-dimensional Lie algebra $\mathfrak{sl}(2)$, then Definition 2.3 implies that $U_h = U_h(\mathfrak{sl}(2))$ is the K-algebra topologically generated by the three variables X, Y, H and the relations

$$[H, X] = 2X, \quad [H, Y] = -2Y \tag{4.1}$$

and

$$[X, Y] = \frac{\sinh(hH/2)}{\sinh(h/2)} = \frac{e^{hH/2} - e^{-hH/2}}{e^{h/2} - e^{-h/2}}. \tag{4.2}$$

The following relates the Hopf algebra $U_q = U_q(\mathfrak{sl}(2))$ of Chapters VI and VII with U_h. We assume that the ground field k on which U_q is defined is the field of fractions of the algebra K of complex formal series.

Proposition XVII.4.1. *There exists a map of Hopf algebras $i : U_q \to U_h$ such that*

$$i(E) = Xe^{hH/4}, \quad i(F) = e^{-hH/4}Y, \quad i(K) = e^{hH/2}, \quad i(K^{-1}) = e^{-hH/2}, \tag{4.3}$$

and $i(q) = e^{h/2}$.

The proof is left as an exercise (use Relation (2.5) among others). Actually, the map i is injective, which allows one to identify U_q with the subalgebra of U_h generated by $q = e^{h/2}$, $E = Xe^{hH/4}$, $F = e^{-hH/4}Y$, $K = e^{hH/2}$, and $K^{-1} = e^{-hH/2}$.

We now describe the universal R-matrix R_h of the topological braided bialgebra U_h.

Theorem XVII.4.2. *The element*

$$\begin{aligned} R_h &= e^{\frac{h(H\otimes H)}{4}} \left(\sum_{\ell \geq 0} \frac{(q-q^{-1})^\ell}{[\ell]_q!} q^{\ell(\ell-1)/2} \, (E^\ell \otimes F^\ell) \right) \\ &= \sum_{\ell \geq 0} \frac{(q-q^{-1})^\ell}{[\ell]_q!} q^{-\ell(\ell+1)/2} \, e^{\frac{h}{2}\left(\frac{H\otimes H}{2} + \frac{1}{2}(\ell H \otimes 1 - 1 \otimes \ell H)\right)} \, (X^\ell \otimes Y^\ell) \end{aligned}$$

of $U_h \widetilde{\otimes} U_h$ is a universal R-matrix for $U_h(\mathfrak{sl}(2))$.

Observe that it is because of the factor $e^{\frac{h(H\otimes H)}{4}}$ that R_h is not well-defined on the subalgebra U_q, thus preventing U_q from being a braided Hopf algebra in the purely algebraic sense of VIII.2.

PROOF. The second equality is easy: it follows from the definitions of E and F. We leave it to the reader and concentrate on the first one. There are several proofs for it. The first one follows from a direct checking of Relations (XVI.4.15–4.17). Another method consists in adapting Drinfeld's theory of the quantum double (as developed in Chapter IX) to the topological setting and then in proceeding along the lines of the proof of Theorem IX.7.1. This second method has been used by Rosso in [Ros89]. We shall sketch a third way using topological crossed bimodules as defined at the end of XVI.4.

This proof goes as follows. As in IX.6 we start by defining a subbialgebra B_h of U_h. It is the closure of the K-submodule of U_h generated by the linearly independent set $\{H^m E^n\}_{m,n\geq 0}$ where $E = Xe^{hH/4}$ as above. From (4.1) and (4.3) it is clear that B_h is closed under the product and the coproduct in U_h, and that the multiplication in B_h is determined by the relation

$$[H, E] = 2E, \tag{4.4}$$

and the comultiplication by

$$\Delta(H) = 1 \otimes H + H \otimes 1 \quad \text{and} \quad \Delta(E) = 1 \otimes E + E \otimes K \tag{4.5}$$

where $K = e^{hH/2}$. Observe also that

$$KE = q^2 EK \tag{4.6}$$

for $q = e^{h/2}$. We now characterize topological crossed B_h-bimodules.

Proposition XVII.4.3. *Let M be a topological crossed B_h-bimodule with coaction Δ_M. Then for any element $x \in M$ we have*

$$\Delta_M(x) = \sum_{m,n\geq 0} \frac{q^{n(n-1)/2}}{m![n]_q!} (\Delta_1^m \Delta_2^n)(x) \widetilde{\otimes} H^m E^n$$

where Δ_1 and Δ_2 are h-adically locally nilpotent K-linear endomorphisms of M such that

$$[\Delta_1, \Delta_2] = -\frac{h}{2}\Delta_2, \tag{4.7}$$

$$[H, \Delta_1] = 0, \quad [H, \Delta_2] = -2\Delta_2, \tag{4.8}$$

$$[E, \Delta_1] = -\frac{h}{2}E, \quad [E, \Delta_2] = K - e^{2\Delta_1}. \tag{4.9}$$

A K-linear endomorphism Δ of M is said to be *h-adically locally nilpotent* if for all $x \in M$ there exists an integer n such that $\Delta^i(x) \subset hM$ for all $i \geq n$. The h-adic local nilpotence condition on Δ_1 and Δ_2 ensures that the infinite sum in the statement above converges in the h-adic topology. We identified E, H, and K with their actions on M in Relations (4.7–4.9). Proposition 4.3 will be proved later.

Let us denote by D_h the K-algebra topologically generated by E, H, Δ_1, Δ_2 and Relations (4.4) and (4.7–4.9). Proposition 4.3 can be interpreted as saying that a topological crossed B_h-bimodule is the same as a topological D_h-module with an h-adic local nilpotence condition. The algebra D_h can be considered as a kind of quantum double for B_h in view of Theorem IX.5.2.

The next step in the proof of Theorem 4.2 is the following.

Proposition XVII.4.4. *There exists a morphism of topological algebras $\chi : D_h \to U_h$ such that*

$$\chi(E) = E, \quad \chi(H) = H, \quad \chi(\Delta_1) = \frac{h}{4}H, \quad \chi(\Delta_2) = (q - q^{-1})F.$$

Observe that $\chi(\Delta_1) \equiv \chi(\Delta_2) \equiv 0 \bmod h$. Consequently, $\chi(\Delta_1)$ and $\chi(\Delta_2)$ are also h-adically locally nilpotent.

PROOF. It is essentially enough to check that $\chi(E)$, $\chi(H)$, $\chi(\Delta_1)$ and $\chi(\Delta_2)$ satisfy Relations (4.4) and (4.7–4.9). This is straightforward, except possibly for the second formula in (4.9).

Let us check it: we have

$$\begin{aligned}[\chi(E), \chi(\Delta_2)] &= (q - q^{-1})[E, F] \\ &= K - K^{-1} \\ &= K - e^{-hH/2} \\ &= \chi(K - e^{-2\Delta_1}).\end{aligned}$$

□

The final step of the proof of Theorem 4.2 goes as follows: By Propositions 4.3–4.4 we know that any topological U_h-module becomes a topological crossed B_h-bimodule via χ. In view of Relation (IX.5.5) a universal R-matrix for U_h is given by

$$R_h = \sum_{m,n\geq 0} \frac{q^{n(n-1)/2}}{m![n]_q!} \chi(H^m E^n) \widetilde{\otimes} \chi(\Delta_1^m \Delta_2^n). \tag{4.10}$$

By definition of χ we get

$$\begin{aligned} R_h &= \sum_{m,n\geq 0} \frac{(q-q^{-1})^n}{m![n]_q!} q^{n(n-1)/2} \frac{h^m}{4^m} H^m E^n \otimes H^m F^n \\ &= \Big(\sum_{m\geq 0} \frac{h^m}{4^m m!} H^m \otimes H^m\Big)\Big(\sum_{n\geq 0} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} E^n \otimes F^n\Big) \\ &= e^{h(H\otimes H)/4}\Big(\sum_{n\geq 0} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} E^n \otimes F^n\Big). \end{aligned}$$

□

We now prove Proposition 4.3.

Proof of Proposition 4.3. For any x in M, the element $\Delta_M(x)$ is of the form

$$\Delta_M(x) = \sum_{m,n\geq 0} \Delta_M^{m,n,p}(x)\, H^m E^n \mod h^p \tag{4.11}$$

for all $p > 0$. In the inverse limit the family $(\Delta_M^{m,n,p})_p$ assembles to form a K-linear endomorphism $\Delta_M^{m,n}$ of M. Now the sum in (4.11) is finite, which implies that $\Delta_M^{m,n}(x)$ vanishes modulo h for m and n large enough.

The counitarity of Δ_M yields $\Delta_M^{0,0} = \mathrm{id}_M$ whereas the coassociativity gives

$$\Delta_M^{i,j}\Delta_M^{m,n} = q^{-2nj}\binom{j+n}{n}_{q^2} \sum_{t\geq 0} \frac{(jh)^t}{2^t t!}\binom{i+m-t}{i}\Delta_M^{i+m-t,j+n} \tag{4.12}$$

for all i, j, m, n after using the classical binomial formula as well as the q-binomial formula of Proposition IV.2.2. Here we agree that $\Delta_M^{m,n} = 0$

when m or $n < 0$. Set $\Delta_1 = \Delta_M^{1,0}$ and $\Delta_2 = \Delta_M^{0,1}$. By (4.12) and (VI.1.7) we get

$$\Delta_M^{m,0} = \frac{1}{m!}(\Delta_M^{1,0})^m = \frac{1}{m!}\Delta_1^m, \tag{4.13}$$

$$\Delta_M^{0,n} = \frac{q^{n(n-1)}}{(n)_{q^2}!}(\Delta_M^{0,1})^n = \frac{q^{n(n-1)/2}}{[n]_q!}\Delta_2^n, \tag{4.14}$$

$$\Delta_M^{m,n} = \Delta_M^{m,0}\Delta_M^{0,n} = \frac{q^{n(n-1)/2}}{m![n]_q!}\Delta_1^m\Delta_2^n. \tag{4.15}$$

From (4.12) and (4.15) we derive

$$\Delta_2\Delta_1 = \sum_{t\geq 0}\frac{h^t}{2^t t!}\Delta_M^{1-t,1} = \Delta_M^{1,1} + \frac{h}{2}\Delta_M^{0,1} = \Delta_1\Delta_2 + \frac{h}{2}\Delta_2,$$

which is equivalent to Relation (4.7).

Let us prove that Δ_1 and Δ_2 are h-adically locally nilpotent. Indeed, we know that for any $x \in M$ we have $\Delta_M^{m,0}(x) \equiv 0$ modulo h for m large enough. Now, $\Delta_M^{m,0}(x) = \Delta_1^m(x)$. Therefore, $\Delta_1^m(x) \subset hM$ for m large enough. The h-adic local nilpotence of Δ_2 is proved similarly.

So far we have expressed the fact that M is a comodule. Now we deal with Relation (XVI.4.23). Lengthy, but easy, computations using

$$EH^m = (H-2)^m E \quad \text{and} \quad E^n H = (H-2n)E^n$$

show that (XVI.4.23) is equivalent to the two relations

$$\Delta_M^{m,n} H = H\Delta_M^{m,n} + 2n\Delta_M^{m,n} \tag{4.16}$$

and

$$\Delta_M^{m,n}E + \Delta_M^{m,n-1}K = \sum_{t\geq 0}\frac{h^t}{2^t t!}E\Delta_M^{m-t,n} + \sum_{r\geq 0}(-2)^r\binom{M+r}{r}\Delta_M^{m+r,n-1} \tag{4.17}$$

for all $m, n \geq 0$. Specializing the exponents m and n to 0 and 1 in (4.16) gives Relations (4.8) whereas setting $m = 1$ and $n = 0$ in (4.17) gives $[E, \Delta_1] = -\frac{h}{2}E$. When we set $m = 0$ and $n = 1$ in (4.17), then necessarily $t = 0$ and we get

$$\Delta_2 E + K = E\Delta_2 + \sum_{r\geq 0}(-2)^r\Delta_M^{r,0}.$$

Therefore,

$$[E, \Delta_2] = K - \sum_{r\geq 0}(-2)^r\frac{\Delta_1^r}{r!} = K - e^{-2\Delta_1},$$

which is the second formula in (4.9). This completes the proof of Proposition 4.3. $\square$

We end this section by an explicit description of the topologically free U_h-modules extending the simple U_q-modules $V(n)$ of V.4. For any nonnegative integer n, consider a $(n+1)$-dimensional complex vector space $V(n)$ with a basis $\{v_0, \dots, v_n\}$ and the free K-module $\widetilde{V_n} = V(n)[[h]] = V(n) \otimes K$. Consider the three $(n+1) \times (n+1)$-matrices

$$\rho_n(X) = \begin{pmatrix} 0 & [n]_q & 0 & \cdots & 0 \\ 0 & 0 & [n-1]_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\rho_n(Y) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & [2]_q & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & [n]_q & 0 \end{pmatrix},$$

and

$$\rho_n(H) = \begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ 0 & n-2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -n+2 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}$$

where $q = e^{h/2}$ and $[n]_q = \frac{\sinh(nh/2)}{\sinh(h/2)}$. The matrices $\rho_n(X)$, $\rho_n(Y)$, and $\rho_n(H)$ satisfy Relations (4.1–4.2) and, hence, define a topological U_h-module structure on $\widetilde{V_n}$. Observe that

$$\widetilde{V_n}/h\widetilde{V_n} = V(n) \tag{4.18}$$

as $\mathfrak{g}$-modules and that, when viewed as a U_q-module via the injection i of Proposition 4.1, $\widetilde{V_n}$ is isomorphic to the simple U_q-module $V_{1,n}$ of VI.3.

Let us check by a direct computation that the quantum dimension of $\widetilde{V_n}$ as defined in Section 3 is given by

$$\dim_q \widetilde{V_n} = [\dim(V(n))]_q = [n+1]_q \tag{4.19}$$

where $q = e^{h/2}$. Indeed, the element ρ defined in Section 2 is $\rho = \frac{H}{2}$ in the case of $\mathfrak{sl}(2)$. By Section 3, $\dim_q(\widetilde{V_n})$ is equal to the trace of the action of $e^{h\rho}$.

Therefore,

$$\begin{aligned}\dim_q(\widetilde{V_n}) &= e^{nh/2} + e^{(n-2)h/2} + \cdots + e^{-(n-2)h/2} + e^{-nh/2} \\ &= q^n + q^{n-2} + \cdots + q^{-(n-2)} + q^{-n} \\ &= \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}} = [n+1]_q.\end{aligned}$$

The elements X and Y, and hence, E and F act nilpotently on $\widetilde{V_n}$ so that it makes sense to apply the universal R-matrix R_h to it. This allows us to build a K-linear automorphism c_n by

$$c_n(v_1 \otimes v_2) = \Big(R_h(v_1 \otimes v_2)\Big)_{21}$$

where $v_1, v_2 \in \widetilde{V_n}$ as in VIII.3. The automorphisms c_n are solutions of the Yang-Baxter equation. In the case of $\widetilde{V_1}$, an immediate application of Theorem 4.2 shows that c_1 is defined in the basis consisting of the vectors $v_0 \otimes v_0$, $v_1 \otimes v_1$, $v_0 \otimes v_1$, $v_1 \otimes v_0$ by the matrix

$$q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & q - q^{-1} \end{pmatrix}, \tag{4.20}$$

an R-matrix already encountered in VIII.7 where it allowed us to define the bialgebra $M_q(2)$ and its quotient $SL_q(2)$ using the FRT construction.

This completes our study of the "quantum groups" associated to $SL(2)$ and of their relations to the R-matrix of (4.20).

XVII.5 Exercises

1. Compute P_ℓ in Formula (2.13) for R_h when $\ell = (0, \ldots, 1, \ldots, 0)$ where 1 occurs exactly once.

2. Show that $U_h(\mathfrak{sl}(2))$ is topologically free.

3. Let $\mathfrak{g} = \mathfrak{sl}(2)$ and $V = V(1)$. Relate the isotopy invariant $Q_{\mathfrak{g},V}$ to the invariant $\Phi_{2,q}$ of Proposition X.4.7.

XVII.6 Notes

A full account of the theory of semisimple Lie algebras can be found, for instance, in [Bou60][Dix74][Hum72][Jac79][Ser65][Var74]. See [Bou60] for the complete list of Cartan matrices.

The presentation we gave in Section 2 for $U_h(\mathfrak{g})$ as an algebra over $\mathbf{C}[[h]]$ is due to Drinfeld [Dri85] [Dri87]. The version considered by Jimbo in [Jim85] is the algebraic variant $U_q(\mathfrak{g})$ already discussed in VI.7. The latter can be viewed as the Hopf subalgebra of $U_h(\mathfrak{g})$ generated by $q = e^{h/2}$, X_i, Y_i, $K_i = e^{hd_iH_i/2}$, and K_i^{-1} for $i = 1, \ldots, n$.

In the special case $\mathfrak{g} = \mathfrak{sl}(2)$, the algebra $U_h(\mathfrak{sl}(2))$ had previously been constructed by Kulish and Reshetikhin [KR81] with the Hopf algebra structure found by Sklyanin [Skl85].

Drinfeld devised the quantum double construction precisely in order to find a universal R-matrix for $U_h(\mathfrak{g})$. This method was applied by Drinfeld [Dri87] himself to give an explicit form of R_h in the case $\mathfrak{sl}(2)$ and by Rosso [Ros89] in the case $\mathfrak{sl}(n)$. Expressions of the universal R-matrix in the general case are due to Kirillov-Reshetikhin [KR90] and to Levendorsky-Soibelman [LS90].

The representation theory of $U_h\mathfrak{g}$ was elucidated by Lusztig [Lus88] and Rosso [Ros88].

Chapter XVIII
Cohomology and Rigidity Theorems

In this chapter we prove two rigidity theorems, both needed in Chapter XIX. The first one is classical: it asserts that any formal deformation of the enveloping algebra of a semisimple Lie algebra is trivial. The proof is based on the vanishing of certain cohomology groups. The second rigidity result is due to Drinfeld [Dri89b] [Dri90]. It states that if A and A' are quantum enveloping algebras with the same underlying cocommutative bialgebras and the same universal R-matrices, then there exists a gauge transformation from A to A'. The proof again relies on some cohomological considerations, this time involving the cobar complex of a symmetric coalgebra.

The ground field is assumed to be the field of complex numbers.

XVIII.1 Cohomology of Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra and M be a left $\mathfrak{g}$-module, i.e., a vector space with a bilinear map $\mathfrak{g} \times M \to M$ such that

$$[x, y]m = x(ym) - y(xm) \tag{1.1}$$

for all $x, y \in \mathfrak{g}$ and $m \in M$. It was shown in V.2 that a left $\mathfrak{g}$-module is the same as a left module over the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

For $n > 0$, let $C^n(\mathfrak{g}, M) = \mathrm{Hom}(\Lambda^n\mathfrak{g}, M)$ be the space of all antisymmetric n-linear maps from $\mathfrak{g}$ to M. An n-linear map f is *antisymmetric* if $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \varepsilon(\sigma)f(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in \mathfrak{g}$ and all permutations σ of the set $\{1, \ldots, n\}$. If $n = 0$, we set $C^0(\mathfrak{g}, M) = M$.

For $f \in C^n(\mathfrak{g}, M)$ we define a $(n+1)$-linear map δf by

$$
\begin{aligned}
(\delta f)(x_1, \ldots, x_{n+1}) \\
= \sum_{1 \le i < j \le n+1} & (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n+1}) \\
& + \sum_{1 \le i \le n+1} (-1)^{i+1} x_i f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}) \qquad (1.2)
\end{aligned}
$$

for all $x_1, \ldots, x_{n+1} \in \mathfrak{g}$. The hat $\widehat{\ }$ on a letter means that it has been omitted. If f belongs to $C^0(\mathfrak{g}, M) = M$, we set $(\delta f)(x) = xf$. A classical computation using the Jacobi identity and the definition of a $\mathfrak{g}$-module gives the following.

Lemma XVIII.1.1. *If f is in $C^n(\mathfrak{g}, M)$, then δf is in $C^{n+1}(\mathfrak{g}, M)$. Moreover, $\delta \circ \delta = 0$.*

Let us denote the kernel and the image of δ in $C^n(\mathfrak{g}, M)$ by $Z^n(\mathfrak{g}, M)$ and $B^n(\mathfrak{g}, M)$ respectively. An element of $Z^n(\mathfrak{g}, M)$ is called an *n-cocycle* whereas an element of $B^n(\mathfrak{g}, M)$ is called a *n-coboundary.* Lemma 1.1 implies that $B^n(\mathfrak{g}, M)$ is a vector subspace of $Z^n(\mathfrak{g}, M)$. This allows us to consider the quotient space

$$H^n(\mathfrak{g}, M) = Z^n(\mathfrak{g}, M)/B^n(\mathfrak{g}, M) \qquad (1.3)$$

which is called the *n-th cohomology group* of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module M.

Let us describe $H^n(\mathfrak{g}, M)$ in degree $n = 0, 1, 2$. In degree 0 we have

$$H^0(\mathfrak{g}, M) = Z^0(\mathfrak{g}, M) = \{m \in M \mid \mathfrak{g}m = 0\}.$$

A linear map $f : \mathfrak{g} \to M$ is a 1-cocycle if and only if

$$f([x, y]) = xf(y) - yf(x) \qquad (1.4)$$

for all $x, y \in \mathfrak{g}$. In other words, a 1-cocycle is a *derivation* from $\mathfrak{g}$ to M. It is a 1-coboundary if and only if it is an *inner derivation*, i.e., there exists an element m in M such that $f(x) = xm$ for all x in $\mathfrak{g}$. Thus, the cohomology group $H^1(\mathfrak{g}, M)$ classifies all derivations up to inner derivations.

In degree 2 an antisymmetric bilinear map $f : \mathfrak{g} \times \mathfrak{g} \to M$ is a cocycle if and only if

$$xf(y,z) + yf(z,x) + zf(x,y) - f([x,y],z) - f([y,z],x) - f([z,x],y) = 0 \quad (1.5)$$

for all $x, y, z \in \mathfrak{g}$. It is a coboundary if and only if there exists a linear map $\alpha : \mathfrak{g} \to M$ such that for all x, y we have

$$f(x, y) = x\alpha(y) - y\alpha(x) - \alpha([x, y]). \qquad (1.6)$$

We shall see in the next section that 2-cocycles appear when we "deform" Lie algebras and their enveloping algebras.

The second cohomology group $H^2(\mathfrak{g}, M)$ has also an interpretation in terms of extensions of $\mathfrak{g}$. These are defined as follows. Let $\mathfrak{g}$ be a Lie algebra and M be a left $\mathfrak{g}$-module. An *extension* of the Lie algebra $\mathfrak{g}$ with kernel M is a Lie algebra $\widetilde{\mathfrak{g}}$ together with a surjective morphism $p : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ of Lie algebras such that

(i) the kernel of p (which is a Lie ideal in $\widetilde{\mathfrak{g}}$) is M, and

(ii) for any $x \in \widetilde{\mathfrak{g}}$ and $m \in M$, we have

$$[x, m] = -[m, x] = p(x)m. \tag{1.7}$$

Such an extension is *split* if there exists a morphism $s : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ of Lie algebras such that $p \circ s = \mathrm{id}_{\mathfrak{g}}$. The following relates extensions of $\mathfrak{g}$ to cohomology.

Proposition XVIII.1.2. *If $H^2(\mathfrak{g}, M) = 0$, then any extension of $\mathfrak{g}$ with kernel M is split.*

PROOF. Let us decompose the vector space $\widetilde{\mathfrak{g}}$ as $\mathfrak{g} \bigoplus M$. By definition of an extension, the Lie bracket on $\widetilde{\mathfrak{g}} = \mathfrak{g} \bigoplus M$ is necessarily of the form

$$[(x, m), (y, n)] = \Big([x, y], xn - ym + f(x, y)\Big) \tag{1.8}$$

where $x, y \in \mathfrak{g}$, $m, n \in M$, and f is a bilinear map from $\mathfrak{g} \times \mathfrak{g}$ into M. Since a Lie bracket is antisymmetric, f has to be antisymmetric. The Jacobi identity for the bracket (1.8) forces another condition on f which is nothing else than Relation (1.5). In other words, f is a 2-cocycle with values in M. By hypothesis, f is a 2-coboundary, which means that there exists a linear map $\alpha : \mathfrak{g} \to M$ such that (1.6) holds. Define the linear map $s = (\mathrm{id}, -\alpha)$ from $\mathfrak{g}$ to $\widetilde{\mathfrak{g}} = \mathfrak{g} \bigoplus M$. We have $p \circ s = \mathrm{id}_{\mathfrak{g}}$. Let us check that s is a morphism of Lie algebras, which will show that the extension is split. We have

$$\begin{aligned}
[s(x), s(y)] &= [(x, -\alpha(x)), (y, -\alpha(y))] \\
&= \Big([x, y], -x\alpha(y) + y\alpha(x) + f(x, y)\Big) \\
&= \Big([x, y], -\alpha([x, y])\Big) \\
&= s([x, y]).
\end{aligned}$$

We used (1.6) in the third equality. □

In the next section we shall need the following corollary to Proposition 1.2. Let us consider a Lie algebra $\mathfrak{g}$ and a $U(\mathfrak{g})$-bimodule M, i.e., a vector space M with a left and a right action of $U(\mathfrak{g})$ such that $(u_1 m)u_2 = u_1(mu_2)$ for all $u_1, u_2 \in U(\mathfrak{g})$ and $m \in M$. We denote by $\overline{M}$ the vector space M equipped with the left $\mathfrak{g}$-module structure defined by $x \cdot m = xm - mx$ for all $x \in \mathfrak{g}$ and $m \in M$.

Corollary XVIII.1.3. *Let $f : U(\mathfrak{g}) \times U(\mathfrak{g}) \to M$ be a bilinear map such that for all x, y, z in $U(g)$, we have $f(1, x) = f(x, 1) = 0$ and*

$$xf(y,z) - f(xy,z) + f(x,yz) - f(x,y)z = 0. \tag{1.9}$$

Then, if $H^2(\mathfrak{g}, \overline{M}) = 0$, there exists a linear map $\alpha : U(\mathfrak{g}) \to M$ such that $\alpha(1) = 0$ and

$$f(x,y) = x\alpha(y) - \alpha(xy) + \alpha(x)y \tag{1.10}$$

for all $x, y \in U(\mathfrak{g})$.

PROOF. We define a product on $U(\mathfrak{g}) \bigoplus M$ by

$$(x,m)(y,n) = \Big(xy, xn + my + f(x,y)\Big) \tag{1.11}$$

where $x, y \in U(\mathfrak{g})$ and $m, n \in M$. Relation (1.9) implies that this product is associative. It has a unit which is $(1, 0)$. We get a Lie bracket on the same space by taking the commutator

$$\begin{aligned} [(x,m),(y,n)] &= (x,m)(y,n) - (y,n)(x,m) \\ &= \Big([x,y], xn - nx + my - ym + f(x,y) - f(y,x)\Big). \end{aligned}$$

The subspace $\widetilde{\mathfrak{g}} = \mathfrak{g} \bigoplus M$ is a Lie subalgebra of $U(\mathfrak{g}) \bigoplus M$. The first projection p from $\widetilde{\mathfrak{g}}$ onto $\mathfrak{g}$ is a surjective morphism of Lie algebras. An easy computation shows that the kernel of this extension is the $\mathfrak{g}$-module $\overline{M}$. Since $H^2(\mathfrak{g}, \overline{M}) = 0$, we know by Proposition 1.2 that the extension $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ is split. Thus, there exists a morphism of Lie algebras $s : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ such that $p \circ s = \mathrm{id}_{\mathfrak{g}}$. Composing it with the inclusion of $\widetilde{\mathfrak{g}}$ into $U(\mathfrak{g}) \bigoplus M$, we obtain a morphism of Lie algebras $s' : \mathfrak{g} \to U(\mathfrak{g}) \bigoplus M$ which, when composed with the first projection, is the inclusion of $\mathfrak{g}$ into its enveloping algebra. By Theorem V.2.1, s' extends to a morphism of algebras σ from $U(\mathfrak{g})$ into $U(\mathfrak{g}) \bigoplus M$ splitting the first projection. This map is necessarily of the form $\sigma = (\mathrm{id}, -\alpha)$ where α is a linear map from $U(\mathfrak{g})$ to M. Let us express the fact that σ is a morphism of algebras. First, we have $(1, 0) = \sigma(1) = (1, -\alpha(1))$, which implies that $\alpha(1) = 0$. Next,

$$\begin{aligned} \sigma(x)\sigma(y) &= (x, -\alpha(x))(y, -\alpha(y)) \\ &= \Big(xy, -x\alpha(y) - \alpha(x)y + f(x,y)\Big) \\ &= \sigma(xy) \\ &= (xy, -\alpha(xy)), \end{aligned}$$

from which we derive $f(x, y) = x\alpha(y) - \alpha(xy) + \alpha(x)y$. □

XVIII.2 Rigidity for Lie Algebras

We now use the cohomology groups introduced in Section 1 to derive two classical theorems on topological algebras. The first one is a uniqueness theorem.

Theorem XVIII.2.1. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras. Suppose given two morphisms α and α' of topological algebras from $U(\mathfrak{g})[[h]]$ to $U(\mathfrak{g}')[[h]]$ such that $\alpha \equiv \alpha'$ modulo h. If $H^1(\mathfrak{g}, U(\mathfrak{g}')) = 0$, there exists an invertible element $F \in U(\mathfrak{g}')[[h]]$ with $F \equiv 1$ modulo h such that $\alpha'(x) = F\alpha(x)F^{-1}$ for all $x \in U(\mathfrak{g})[[h]]$.*

The class modulo h of α (and of α') is an algebra morphism α_0 from $U(\mathfrak{g})$ to $U(\mathfrak{g}')$. We give $U(\mathfrak{g}')$ a left $\mathfrak{g}$-module structure by setting $x \cdot u = [\alpha_0(x), u]$ where $x \in \mathfrak{g}$ and $u \in U(\mathfrak{g}')$. The cohomological condition in Theorem 2.1 refers precisely to this module structure.

PROOF. Since α is $\mathbf{C}[[h]]$-linear, it is determined by its restriction on $U(\mathfrak{g})$. Write the latter in the form

$$\alpha(x) = \sum_{n \geq 0} \alpha_n(x) h^n \tag{2.1}$$

where $(\alpha_n)_n$ is a family of linear maps from $U(\mathfrak{g})$ to $U(\mathfrak{g}')$. The map α preserves the unit, which implies that $\alpha_0(1) = 1$ and $\alpha_n(1) = 0$ if $n > 0$. It also preserves the product, which is equivalent to the relations

$$\alpha_0(xy) = \alpha_0(x)\alpha_0(y) \tag{2.2}$$

and

$$\alpha_n(xy) = \sum_{p+q=n} \alpha_p(x)\alpha_q(y) \tag{2.3}$$

if $n > 0$. In particular, we have

$$\alpha_1(xy) = \alpha_0(x)\alpha_1(y) + \alpha_1(x)\alpha_0(y). \tag{2.4}$$

Suppose now that x and y are elements of $\mathfrak{g}$. Then Relation (2.4) implies that

$$\alpha_1([x,y]) = [\alpha_0(x), \alpha_1(y)] - [\alpha_0(y), \alpha_1(x)]. \tag{2.5}$$

In view of our definition of the $\mathfrak{g}$-action on $U(\mathfrak{g}')$ and of (1.4), we see that α_1 is a 1-cocycle of $\mathfrak{g}$ with values in $U(\mathfrak{g}')$. Since $H^1(\mathfrak{g}, U(\mathfrak{g}')) = 0$, the map α is a 1-coboundary, which means that there exists an element $u_1 \in U(\mathfrak{g}')$ such that

$$\alpha_1(x) = [\alpha_0(x), u_1] \tag{2.6}$$

for all $x \in \mathfrak{g}$. Set

$$\alpha^{(1)}(x) = (1 + u_1 h)\alpha(x)(1 + u_1 h)^{-1} \tag{2.7}$$

where $x \in U(\mathfrak{g})$. This extends $\mathbf{C}[[h]]$-linearly to a new morphism of topological algebras from $U(\mathfrak{g})[[h]]$ to $U(\mathfrak{g}')[[h]]$. Modulo h^2, we have

$$\alpha^{(1)}(x) \equiv \alpha_0(x) + \Big(u_1\alpha_0(x) - \alpha_0(x)u_1 + \alpha_1(x)\Big)h \equiv \alpha_0(x)$$

in view of (2.6–2.7). This holds for all $x \in \mathfrak{g}$. Since α_0 and $\alpha^{(1)}$ are algebra morphisms defined on $U(\mathfrak{g})$, it holds for all elements of the enveloping algebra. Set $\alpha^{(1)} = \sum_{n\geq 0} \alpha_n^{(1)} h^n$. The previous computation shows that $\alpha_0^{(1)} = \alpha_0$ and $\alpha_1^{(1)} = 0$.

Now apply (2.3) to $\alpha^{(1)}$ and $n = 2$. We have

$$\alpha_2^{(1)}(xy) = \alpha_0(x)\alpha_2^{(1)}(y) + \alpha_2^{(1)}(x)\alpha_0(y),$$

which shows that the restriction of $\alpha_2^{(1)}$ to $\mathfrak{g}$ again is a 1-cocycle with values in $U(\mathfrak{g}')$. For the same reasons as above, there exists an element $u_2 \in U(\mathfrak{g}')$ such that $\alpha_2^{(1)}(x) = [\alpha_0(x), u_2]$ for all $x \in \mathfrak{g}$. Set $\alpha^{(2)}(x) = (1 + u_2h^2)\alpha^{(1)}(x)(1 + u_2h^2)^{-1}$. A computation as above shows that

$$\alpha^{(2)} \equiv \alpha_0 \mod h^3. \tag{2.8}$$

Proceeding by induction, we similarly construct elements $u_3, u_4, \ldots$ in $U(\mathfrak{g}')$ such that

$$U_n\alpha(x)U_n^{-1} \equiv \alpha_0(x) \mod h^{n+1} \tag{2.9}$$

for all $n > 0$ and all $x \in U(\mathfrak{g})$. Here U_n is defined by

$$U_n = (1 + u_nh^n)(1 + u_{n-1}h^{n-1})\ldots(1 + u_1h).$$

When we pass to the inverse limit, we see that the family $(U_n)_n$ defines an invertible element $U \in U(\mathfrak{g}')[[h]]$ such that $U \equiv 1$ modulo h and

$$\alpha(x) = U^{-1}\alpha_0(x)U \tag{2.10}$$

for all $x \in U(\mathfrak{g})$.

Now let us prove Theorem 2.1. Proceeding as for α, we get an element U' in $U(\mathfrak{g}')[[h]]$ such that $U' \equiv 1$ modulo h and $\alpha'(x) = U'^{-1}\alpha'_0(x)U'$. By hypothesis, $\alpha'_0 = \alpha_0$. From this and from (2.10) we need only to set $F = U'^{-1}U$ in order to complete the proof. □

We now consider a topological algebra (A, μ, η), as defined in XVI.4, satisfying the following conditions:

(i) as an algebra, A/hA is the enveloping algebra of a complex Lie algebra $\mathfrak{g}$:

$$A/hA = U(\mathfrak{g}), \tag{2.11}$$

(ii) as a $\mathbf{C}[[h]]$-module, A is topologically free, i.e.,

$$A = U(\mathfrak{g})[[h]], \tag{2.12}$$

(iii) the unit $\eta(1)$ of A is equal to the constant formal series 1 in $U(\mathfrak{g})[[h]]$ under the identification (2.12).

The second theorem of this section is an existence theorem. Since it states that any such topological algebra is isomorphic to the trivial topological algebra associated to $U(\mathfrak{g})$, Theorem 2.2 is called a *rigidity theorem.*

Theorem XVIII.2.2. *Under the previous hypotheses, if $H^2(\mathfrak{g}, U(\mathfrak{g}))$ is zero, there exists an isomorphism $\alpha : A \to U(\mathfrak{g})[[h]]$ of topological algebras inducing the identity $A/hA \to U(\mathfrak{g})$ modulo h.*

Here $\mathfrak{g}$ acts on $U(\mathfrak{g})$ by the adjoint representation.

PROOF. We first proceed as in XVI.5 by identifying the $\mathbf{C}[[h]]$-module A with $U(\mathfrak{g})[[h]]$ and by expanding in a formal series the $\mathbf{C}[[h]]$-linear map μ from $A\tilde{\otimes}A = (U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]]$ to $A = U(\mathfrak{g})[[h]]$, i.e.,

$$\mu = \sum_{n \geq 0} \mu_n h^n \tag{2.13}$$

as in (XVI.5.3), where $(\mu_n)_n$ is a family of bilinear maps from $U(\mathfrak{g}) \times U(\mathfrak{g})$ to $U(\mathfrak{g})$ such that μ_0 is the multiplication of the enveloping algebra of $\mathfrak{g}$. Condition (iii) above can be reformulated as

$$\mu_n(1, x) = \mu_n(x, 1) = 0 \tag{2.14}$$

for all $x \in U(\mathfrak{g})$ and all $n > 0$. The associativity of the product μ is expressed by

$$\mu(\mu(x, y), z) = \mu(x, \mu(y, z)) \tag{2.15}$$

for all $x, y, z \in U(\mathfrak{g})$. Expanding μ with (2.13), we obtain the equivalent system of equations

$$\sum_{p+q=n} \mu_p(\mu_q(x, y), z) = \sum_{p+q=n} \mu_p(x, \mu_q(y, z)) \tag{2.16}$$

for all $x, y, z \in U(\mathfrak{g})$ and all $n \geq 0$. Let N be the smallest integer $n > 0$ (if it exists) such that $\mu_n \neq 0$. If no such integer exists, we have $\mu = \mu_0$, which means that A coincides with $U(\mathfrak{g})[[h]]$ as a topological algebra and the theorem is proved. If N exists, let us rewrite (2.16) for $n = N$. Using the customary notation for the product in $U(\mathfrak{g})$, we get

$$\mu_N(xy, z) + \mu_N(x, y)z = \mu_N(x, yz) + x\mu_N(y, z) \tag{2.17}$$

for all $x, y, z \in U(\mathfrak{g})$. In other words, μ_N satisfies Condition (1.9) of Corollary 1.3 with $M = U(\mathfrak{g})$. Since $H^2(\mathfrak{g}, U(\mathfrak{g}))$ vanishes, we may apply Corollary 1.3, which yields a linear endomorphism α_N of $U(\mathfrak{g})$ with $\alpha_N(1) = 0$ and

$$\mu_N(x, y) = x\alpha_N(y) - \alpha_N(xy) + \alpha_N(x)y \tag{2.18}$$

for all $x, y \in U(\mathfrak{g})$. Define a $\mathbf{C}[[h]]$-linear automorphism α of $U(\mathfrak{g})[[h]]$ by

$$\alpha = \mathrm{id} + \alpha_N h^N,$$

its inverse being given by $\sum_{n\geq 0}(-1)^n\alpha_N^n h^{nN}$. We have $\alpha(1)=1$. Define a new product $\mu'=\sum_{n\geq 0}\mu'_n h^n$ on A by

$$\mu'(x,y)=\alpha\bigl(\mu(\alpha^{-1}(x),\alpha^{-1}(y))\bigr). \tag{2.19}$$

Since $\alpha\equiv \mathrm{id} \bmod h^N$, we have $\mu'\equiv\mu \bmod h^N$. Let us compute μ' modulo h^{N+1}. Relation (2.18) implies that

$$\begin{aligned}
\mu'(x,y) &\equiv xy+\mu'_N(x,y)h^N\\
&\equiv (\mathrm{id}+\alpha_N h^N)\bigl((\mu_0+\mu_N h^N)(x-\alpha_N(x)h^N, y-\alpha_N(y)h^N)\bigr)\\
&\equiv xy+\bigl(\alpha_N(xy)+\mu_N(x,y)-\alpha_N(x)y-x\alpha_N(y)\bigr)h^N\\
&\equiv xy \mod h^{N+1}.
\end{aligned}$$

Consequently, $\mu'_0=\mu_0$ is the multiplication of $U(\mathfrak{g})$ whereas

$$\mu'_1=\cdots=\mu'_N=0. \tag{2.20}$$

We use this procedure to construct an isomorphism of algebras between A and $U(\mathfrak{g})[[h]]$. In effect, applying the above considerations to the case $N=1$, we get an isomorphism of the form $\mathrm{id}+\alpha_1 h^1$ from the algebra A to A equipped with a new product $\mu^{(1)}$ such that $\mu_1^{(1)}=0$. Applying now to $\mu^{(1)}$ and $N=2$, we get an isomorphism $\mathrm{id}+\alpha_2 h^2$ from $(A,\mu^{(1)})$ to $(A,\mu^{(2)})$ where $\mu^{(2)}$ is a product with $\mu_1^{(2)}=\mu_2^{(2)}=0$. Repeating this infinitely many times and composing all the isomorphisms, we get an isomorphism α from A to A endowed with a product $\mu^{(\infty)}$ satisfying $\mu_n^{(\infty)}=0$ for all $n>0$. In other words, $\mu^{(\infty)}=\mu_0$ is the usual product of $U(\mathfrak{g})[[h]]$. □

XVIII.3 Vanishing Results for Semisimple Lie Algebras

We shall use Theorems 2.1 and 2.2 in Section 4 in case $\mathfrak{g}$ is a finite-dimensional complex semisimple Lie algebra. To apply them, we have to prove the vanishing of $H^i(\mathfrak{g},U(\mathfrak{g}))$ for $i=1,2$. We start with the following result.

Proposition XVIII.3.1. *If $\mathfrak{g}$ is a finite-dimensional complex semisimple Lie algebra and M is a finite-dimensional non-trivial simple left $\mathfrak{g}$-module, then $H^n(\mathfrak{g},M)=0$ for all $n\geq 0$.*

Here non-trivial means that M is not the one-dimensional $\mathfrak{g}$-module on which the Lie algebra acts by zero.

PROOF. We need the Casimir element $C=\sum_k x_k x^k$ defined by (XVII.1.5). We know that C acts on any finite-dimensional non-trivial simple $\mathfrak{g}$-module by a non-zero scalar. In order to prove the proposition, we construct for all n a map $h : C^n(\mathfrak{g}, M) \to C^{n-1}(\mathfrak{g}, M)$ such that

$$Cf = \delta hf + h\delta f \tag{3.1}$$

for all $f \in C^n(\mathfrak{g}, M)$. By Cf we mean the n-linear map defined by

$$(Cf)(y_1, \ldots, y_n) = C\Big(f(y_1, \ldots, y_n)\Big)$$

where $y_1, \ldots, y_n$ belong to $\mathfrak{g}$. Let f be an n-cocycle with values in M, i.e., such that $\delta f = 0$. By (3.1) we get $Cf = \delta(hf)$, which means that Cf is a n-coboundary. Since C acts by a non-zero scalar on M, we see that f too is a coboundary. This proves the vanishing of $H^n(\mathfrak{g}, M)$.

We are left with building a map h satisfying (3.1). Given $f \in C^n(\mathfrak{g}, M)$ with $n > 0$ and the Casimir element, we define an antisymmetric $(n-1)$-linear map hf with values in M by

$$(hf)(y_1, \ldots, y_{n-1}) = \sum_k x_k f(x^k, y_1, \ldots, y_{n-1}) \tag{3.2}$$

for all $y_1, \ldots, y_{n-1} \in \mathfrak{g}$. If $f \in C^0(\mathfrak{g}, M)$, set $hf = 0$. Using (3.2) and (1.2), we get

$$(\delta hf + h\delta f)(y_1, \ldots, y_n) = Cf(y_1, \ldots, y_n) + \sum_{1 \leq i \leq n} (-1)^i Z_i$$

where

$$Z_i = \sum_k \Big([x_k, y_i] f(x^k, y_1, \ldots, \widehat{y_i}, \ldots, y_n) + x_k f([x^k, y_i], y_1, \ldots, \widehat{y_i}, \ldots, y_n)\Big).$$

Relation (3.1) will be proved if we show that all Z_i vanish. Using the linear forms $\alpha_{k\ell}$ and $\beta_{k\ell}$ of XVII.1, we get

$$\begin{aligned} Z_i \quad = \quad & \sum_{k,\ell} \Big(\alpha_{k\ell}(y_i)\, x_\ell f(x^k, y_1, \ldots, \widehat{y_i}, \ldots, y_n) \\ & \qquad + \beta_{k\ell}(y_i)\, x_k f(x^\ell, y_1, \ldots, \widehat{y_i}, \ldots, y_n)\Big). \end{aligned}$$

Exchanging k and ℓ in the second summand, we obtain

$$Z_i = \sum_{k,\ell} \Big(\alpha_{k\ell}(y_i) + \beta_{\ell k}(y_i)\Big)\, x_\ell f(x^k, y_1, \ldots, \widehat{y_i}, \ldots, y_n),$$

which vanishes in view of Lemma XVII.1.1. $\square$

As a consequence, we get the so-called "Whitehead lemmas".

Corollary XVIII.3.2. *If $\mathfrak{g}$ is a semisimple Lie algebra and M is any finite-dimensional left $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, M) = H^2(\mathfrak{g}, M) = 0$.*

PROOF. We know that any finite-dimensional module M over a semisimple Lie algebra is a direct sum $M = \bigoplus_i M_i$ of simple modules M_i. Since the complex $C^*(\mathfrak{g}, M)$ is the direct sum of the subcomplexes $C^*(\mathfrak{g}, M_i)$, we have $H^n(\mathfrak{g}, M) = \bigoplus_i H^n(\mathfrak{g}, M_i)$. In view of Proposition 3.1, it is enough to prove Corollary 3.2 when M is the trivial one-dimensional $\mathfrak{g}$-module $\mathbf{C}$.

(a) We first prove the vanishing of $H^1(\mathfrak{g}, \mathbf{C})$, which will imply the vanishing of $H^1(\mathfrak{g}, M)$ for all finite-dimensional modules M. Let f be a 1-cocycle with values in the trivial module $\mathbf{C}$. Relation (1.4) reduces to $f([x, y]) = 0$ for all $x, y \in \mathfrak{g}$. Now Serre's relations (XVII.1.9–1.10) show that the elements $[x, y]$ span the vector space $\mathfrak{g}$. Therefore $f = 0$ on the whole space $\mathfrak{g}$.

(b) The argument for the vanishing of $H^2(\mathfrak{g}, \mathbf{C})$ is slightly more involved. We first claim that if f is a 2-cocycle with values in $\mathbf{C}$, then the linear map $\widetilde{f}$ given by $\widetilde{f}(x)(y) = f(x, y)$ for all $x, y \in \mathfrak{g}$, is a 1-cocycle of $\mathfrak{g}$ with values in the dual vector space $\mathfrak{g}^*$. Such a statement presupposes that we have defined a left action of $\mathfrak{g}$ on $\mathfrak{g}^*$. This is done by taking the coadjoint representation given by

$$(x\alpha)(y) = \alpha([y, x]) \tag{3.3}$$

where $x, y \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$. Indeed, if f is a 2-cocycle with values in $\mathbf{C}$, we have by (1.5)

$$\widetilde{f}([x, y])(z) = \widetilde{f}(y)([z, x]) - \widetilde{f}(x)([z, y])$$

for all $x, y, z \in \mathfrak{g}$. Reformulating this with (3.3), we get

$$\widetilde{f}([x, y]) = x\widetilde{f}(y) - y\widetilde{f}(x),$$

which shows that $\widetilde{f}$ is a 1-cocycle with values in the finite-dimensional $\mathfrak{g}$-module $\mathfrak{g}^*$. By the first part of Corollary 3.2, the cocycle $\widetilde{f}$ is a coboundary, i.e., there exists a linear form $\alpha \in \mathfrak{g}^*$ such that $\widetilde{f}(x) = x\alpha$. We thus get

$$f(x, y) = \widetilde{f}(x)(y) = (x\alpha)(y) = \alpha([y, x]) = -\alpha([x, y]).$$

In other words, the 2-cocycle f is the coboundary of α. This completes the proof of the vanishing of H^2. Observe that, incidentally, we proved that $H^2(\mathfrak{g}, \mathbf{C}) \cong H^1(\mathfrak{g}, \mathfrak{g}^*)$. □

Let us equip $U(\mathfrak{g})$ with the adjoint representation of $\mathfrak{g}$ for which the Lie algebra acts on $U(\mathfrak{g})$ on the left by $x \cdot u = xu - ux = [x, u]$ where $x \in \mathfrak{g}$ and $u \in U(\mathfrak{g})$. If $u = x_1 \dots x_n$ with $x_1, \dots, x_n$ belonging to $\mathfrak{g}$, an easy induction shows that

$$x \cdot u = \sum_{i=1}^{n} x_1 \dots x_{i-1}[x, x_i]x_{i+1} \dots x_n. \tag{3.4}$$

We record the following corollary.

Corollary XVIII.3.3. *Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra acting on $U(\mathfrak{g})$ as above. Then $H^1(\mathfrak{g}, U(\mathfrak{g})) = H^2(\mathfrak{g}, U(\mathfrak{g})) = 0$.*

PROOF. We use the symmetrization map $\eta : S(\mathfrak{g}) \to U(\mathfrak{g})$ defined in V.2 by

$$\eta(x_1 \dots x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}$$

where $x_1, \dots, x_n \in \mathfrak{g}$. We know that η is a linear isomorphism. Moreover, if we equip $S(\mathfrak{g})$ with the left $\mathfrak{g}$-module structure given by

$$x \cdot (x_1 \dots x_n) = \sum_{i=1}^{n} x_1 \dots x_{i-1}[x, x_i]x_{i+1} \dots x_n \tag{3.5}$$

the map η becomes an isomorphism of $\mathfrak{g}$-modules. Now, as can easily be seen from (3.5), the action of $\mathfrak{g}$ respects the decomposition of $S(\mathfrak{g})$ into its homogeneous components $S^n(\mathfrak{g})$. We thus obtain an isomorphism

$$U(\mathfrak{g}) \cong \bigoplus_{n \geq 0} S^n(\mathfrak{g})$$

of $\mathfrak{g}$-modules. Consequently, for $i = 1, 2$ we have

$$H^i(\mathfrak{g}, U(\mathfrak{g})) = \bigoplus_{n \geq 0} H^i(\mathfrak{g}, S^n(\mathfrak{g})) = 0$$

by application of Corollary 3.2 to the finite-dimensional modules $S^n(\mathfrak{g})$.
□

XVIII.4 Application to Drinfeld-Jimbo Quantum Enveloping Algebras

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and let $U_h(\mathfrak{g})$ be the Drinfeld-Jimbo quantum enveloping algebra of XVII.2. The first three sections of this chapter culminate in the following result.

Theorem XVIII.4.1. *There exists an isomorphism $\alpha : U_h(\mathfrak{g}) \to U(\mathfrak{g})[[h]]$ of topological algebras which is congruent to the identity modulo h. If α' is another such isomorphism, there exists an element F in $U(\mathfrak{g})[[h]]$ such that $F \equiv 1$ modulo h and $\alpha'(a) = F\alpha(a)F^{-1}$ for all elements a of $U_h(\mathfrak{g})$.*

PROOF. The first statement is a direct consequence of Theorem 2.2 and of the vanishing of $H^2(\mathfrak{g}, U(\mathfrak{g}))$ proved in Corollary 3.3.

As for the second one, observe that $\alpha' \circ \alpha^{-1}$ and the identity are two automorphisms of the topological algebra $U(\mathfrak{g})[[h]]$ inducing the identity on

$U(\mathfrak{g})$. By Theorem 2.1 and by the vanishing of $H^1(\mathfrak{g}, U(\mathfrak{g}))$ (see Corollary 3.3), there exists an element $F \equiv 1 \bmod h$ in $U(\mathfrak{g})[[h]]$ such that we have $(\alpha' \circ \alpha^{-1})(u) = FuF^{-1}$ for all $u \in U(\mathfrak{g})[[h]]$. Replacing u by $\alpha(a)$ yields the conclusion. $\square$

Since conjugated morphisms restrict to the same map on the centre, we get the following important result which we have already used in XVII.3 in order to define the quantum Casimir element C_h in the Drinfeld-Jimbo algebra $U_h(\mathfrak{g})$.

Corollary XVIII.4.2. *There exists a unique isomorphism α of topological algebras from the centre of $U_h(\mathfrak{g})$ to the centre of $U(\mathfrak{g})[[h]]$ such that $\alpha \equiv \mathrm{id}$ modulo h.*

We may use the isomorphism α between $U_h(\mathfrak{g})$ and $U(\mathfrak{g})[[h]]$ to assign a topologically free $U_h(\mathfrak{g})$-module to any finite-dimensional $\mathfrak{g}$-module. Indeed, let V be such a $\mathfrak{g}$-module. We equip $V[[h]]$ with the extended $U(\mathfrak{g})[[h]]$-module structure. Define $\widetilde{V}$ as $V[[h]]$ equipped with the $U_h(\mathfrak{g})$-module structure given by $a \cdot v = \alpha(a)v$ where $a \in U_h(\mathfrak{g})$ and $v \in V[[h]]$. Since α is congruent to the identity modulo h, we see that $\widetilde{V}/h\widetilde{V}$ is isomorphic to V as a $\mathfrak{g}$-module. It can be shown that, when V_λ is the simple $\mathfrak{g}$-module associated to the dominant weight λ, then $\widetilde{V_\lambda}$ is the highest weight $U_h(\mathfrak{g})$-module alluded to at the end of XVII.2.

XVIII.5 Cohomology of Coalgebras

In order to prove the second rigidity theorem of the chapter, we set up a cohomology theory for coalgebras. Let $(C, \Delta, \varepsilon, 1)$ be a coalgebra over a field k with an element 1 in C such that $\Delta(1) = 1 \otimes 1$ (which implies that $\varepsilon(1) = 1$). Clearly, any bialgebra satisfies these conditions with 1 equal to the unit of its multiplication.

We set $T^n(C) = C^{\otimes n}$ if $n > 0$ and $T^0(C) = k$. Define linear maps $\delta_n^0, \ldots, \delta_n^{n+1}$ from $T^n(C)$ to $T^{n+1}(C)$ by

$$\begin{aligned}
\delta_n^0(x_1 \otimes \cdots \otimes x_n) &= 1 \otimes x_1 \otimes \cdots \otimes x_n, \\
\delta_n^{n+1}(x_1 \otimes \cdots \otimes x_n) &= x_1 \otimes \cdots \otimes x_n \otimes 1, \\
\delta_n^i(x_1 \otimes \cdots \otimes x_n) &= x_1 \otimes \cdots \otimes x_{i-1} \otimes \Delta(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_n
\end{aligned}$$

if $1 \le i \le n$. If $n = 0$, we set $\delta_0^0(1) = \delta_0^1(1) = 1$.

Lemma XVIII.5.1. *We have $\delta_{n+1}^j \delta_n^i = \delta_{n+1}^i \delta_n^{j-1}$ for all integers i, j such that $0 \le i < j \le n+2$.*

PROOF. If $j \ge i+2$, this is straightforward. Let us concentrate on the case $j = i+1$. If $1 \le i \le n$, we have

$$\delta_{n+1}^{i+1}\delta_n^i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{i-1} \otimes (\mathrm{id}_C \otimes \Delta)(\Delta(x_i)) \otimes x_{i+1} \otimes \cdots \otimes x_n.$$

On the other hand, we have

$$\delta^i_{n+1}\delta^i_n(x_1\otimes\cdots\otimes x_n) = x_1\otimes\cdots\otimes x_{i-1}\otimes(\Delta\otimes \mathrm{id}_C)(\Delta(x_i))\otimes x_{i+1}\otimes\cdots\otimes x_n.$$

Both are equal in view of the coassociativity of Δ.

If $i = 0$, we have

$$\begin{aligned}\delta^1_{n+1}\delta^0_n(x_1\otimes\cdots\otimes x_n) &= \Delta(1)\otimes x_1\otimes\cdots\otimes x_n\\ &= 1\otimes 1\otimes x_1\otimes\cdots\otimes x_n\\ &= \delta^0_{n+1}\delta^0_n(x_1\otimes\cdots\otimes x_n).\end{aligned}$$

Similar computations prove the desired relations in the remaining cases.
□

As a consequence we can equip $T^\bullet(C)$ with a differential of degree $+1$.

Corollary XVIII.5.2. *Define the differential $\delta : T^n(C) \to T^{n+1}(C)$ by $\delta = \sum_{i=0}^{n+1}(-1)^i\delta^i_n$. Then $\delta\circ\delta = 0$.*

The cochain complex $(T^\bullet(C), \delta)$ is called the *cobar complex* of the coalgebra C.

PROOF. In degree n, we have

$$\begin{aligned}\delta\circ\delta &= \sum_{j=0}^{n+2}\sum_{i=0}^{n+1}(-1)^{i+j}\delta^j_{n+1}\delta^i_n\\ &= \sum_{i<j}(-1)^{i+j}\delta^j_{n+1}\delta^i_n + \sum_{j\le i}(-1)^{i+j}\delta^j_{n+1}\delta^i_n\\ &= \sum_{i<j}(-1)^{i+j}\left(\delta^j_{n+1}\delta^i_n - \delta^i_{n+1}\delta^{j-1}_n\right)\\ &= 0\end{aligned}$$

by Lemma 5.1. □

The natural isomorphisms $T^n(C)\otimes T^m(C)\cong T^{n+m}(C)$ induce an associative graded product on $T(C) = \bigoplus_{n\ge 0} T^n(C)$. This product is compatible with the differential δ in the following sense.

Lemma XVIII.5.3. *If $\omega \in T^n(C)$ and $\omega' \in T^m(C)$, then*

$$\delta(\omega\omega') = \delta(\omega)\omega' + (-1)^n\omega\delta(\omega') \tag{5.1}$$

for the product $\omega\omega'$ in $T^{n+m}(C)$.

We shall use the term *differential graded algebra* for a graded algebra with a differential satisfying Relation (5.1). It follows from Lemma 5.3 that the product on $T^\bullet(C)$ induces an associative graded product on the cohomology $H^\bullet(T^\bullet(C), \delta)$ of the cobar complex.

PROOF. This results from the relations $\delta^i_{n+m}(\omega\omega') = \delta^i_n(\omega)\omega'$ if $i \leq n$, from $\delta^i_{n+m}(\omega\omega') = \omega\delta^{i-n}_m(\omega')$ if $i \geq n+1$, and from

$$\delta^{n+1}_n(\omega)\omega' = \omega \otimes 1 \otimes \omega' = \omega\delta^0_m(\omega').$$

□

Suppose that C possesses an involution $x \mapsto \bar{x}$ such that $\bar{1} = 1$ and $\overline{\Delta(x)} = \Delta^{\mathrm{op}}(\bar{x})$, e.g., C is a *cocommutative coalgebra* with involution equal to the identity. Then we can put an involution on the complex $(T^\bullet(C), \delta)$ as well. Define an automorphism σ_n of $T^n(C)$ by $\sigma_0 = \mathrm{id}_k$ and by

$$\sigma_n(x_1 \otimes \cdots \otimes x_n) = (-1)^{n(n+1)/2}\, \bar{x}_n \otimes \cdots \otimes \bar{x}_1 \tag{5.2}$$

if $n > 0$. The automorphism σ_n is an involution.

Lemma XVIII.5.4. *We have $\delta\sigma_n = \sigma_{n+1}\delta$.*

PROOF. It is enough to prove that $\delta^i_n\sigma_n = (-1)^{n+1}\sigma_{n+1}\delta^{n+1-i}_n$. If $i = 0$, we have

$$\delta^0_n\sigma_n(x_1 \otimes \cdots \otimes x_n) = (-1)^{n(n+1)/2}\, 1 \otimes \bar{x}_n \otimes \cdots \otimes \bar{x}_1.$$

On the other hand,

$$(-1)^{n+1}\sigma_{n+1}\delta^{n+1}_n(x_1 \otimes \cdots \otimes x_n) = (-1)^{\frac{(n+1)(n+2)}{2} - (n+1)}\, \bar{1} \otimes \bar{x}_n \otimes \cdots \otimes \bar{x}_1,$$

which is the same since $\frac{(n+1)(n+2)}{2} - (n+1) = \frac{n(n+1)}{2}$. If $1 \leq i \leq n$, we have

$$\begin{aligned}
&\delta^i_n\sigma_n(x_1 \otimes \cdots \otimes x_n)\\
&\quad= (-1)^{n(n+1)/2}\, \bar{x}_n \otimes \cdots \otimes \Delta(\bar{x}_{n+1-i}) \otimes \cdots \otimes \bar{x}_1\\
&\quad= (-1)^{(n+1)(n+2)/2-(n+1)}\, \bar{x}_n \otimes \cdots \otimes \overline{\Delta^{\mathrm{op}}(x_{n+1-i})} \otimes \cdots \otimes \bar{x}_1\\
&\quad= (-1)^{n+1}\sigma_{n+1}\delta^{n+1-i}_n(x_1 \otimes \cdots \otimes x_n).
\end{aligned}$$

The second equality holds by the assumption on Δ. □

As a consequence of Lemma 5.4, the cochain complex $(T^\bullet(C), \delta)$ is the direct sum

$$T^\bullet(C) = T^\bullet_+(C) \oplus T^\bullet_-(C) \tag{5.3}$$

of the subcomplexes $(T^\bullet_+(C), \delta)$ and $(T^\bullet(C)_-, \delta)$ defined for all n by

$$T^n_\pm(C) = \Big\{\omega \in T^n(C) \;\Big|\; \sigma_n(\omega) = \pm\omega\Big\}. \tag{5.4}$$

XVIII.6 Action of a Semisimple Lie Algebra on the Cobar Complex

We return to the situation of a *cocommutative* coalgebra (C, Δ, ε) with an element 1 such that $\Delta(1) = 1 \otimes 1$. Assume we also have a Lie algebra $\mathfrak{g}$ acting on C such that, if we denote by $x \cdot c$ the action of an element $x \in \mathfrak{g}$ on an element $c \in C$, we have $x \cdot 1 = 0$ and

$$\Delta(x \cdot c) = \Delta(x) \cdot \Delta(c) = \sum_{(c)} \Big(x \cdot c' \otimes c'' + c' \otimes x \cdot c'' \Big) \tag{6.1}$$

in Sweedler's sigma notation. The examples we have in mind are the coalgebras $U(\mathfrak{g})$ and $S(\mathfrak{g})$, on which $\mathfrak{g}$ acts by the adjoint representation.

Equipping the tensor powers of C with the induced $\mathfrak{g}$-module structures, we get the following result.

Lemma XVIII.6.1. *The cobar complex $(T^\bullet(C), \delta)$ is a complex made up of $\mathfrak{g}$-modules.*

Proof. It suffices to check that the maps δ_n^i of Section 5 are maps of $\mathfrak{g}$-modules. Let $c_1, \ldots, c_n$ be elements of C and x be in $\mathfrak{g}$. For δ_n^0, we get

$$\begin{aligned}
\delta_n^0\Big(x \cdot (c_1 \otimes \ldots \otimes c_n)\Big) &= \sum_{k=1}^{n} \delta_n^0(c_1 \otimes \ldots \otimes x \cdot c_k \otimes \ldots \otimes c_n) \\
&= \sum_{k=1}^{n} 1 \otimes c_1 \otimes \ldots \otimes x \cdot c_k \otimes \ldots \otimes c_n \\
&= x \cdot \delta_n^0(c_1 \otimes \ldots \otimes c_n)
\end{aligned}$$

since $x \cdot 1 = 0$. There is a similar proof for δ_n^{n+1}. If $1 \leq i \leq n$, we have

$$\begin{aligned}
\delta_n^i\Big(x \cdot (c_1 \otimes \ldots \otimes c_n)\Big) &= \sum_{k=1}^{n} \delta_n^i(c_1 \otimes \ldots \otimes x \cdot c_k \otimes \ldots \otimes c_n) \\
&= \sum_{k \neq i} c_1 \otimes \ldots \otimes x \cdot c_k \otimes \ldots \otimes \Delta(c_i) \otimes \ldots \otimes c_n \\
&\qquad + c_1 \otimes \ldots \otimes \Delta(x \cdot c_i) \otimes \ldots \otimes c_n \\
&= x \cdot \Big(c_1 \otimes \ldots \otimes \Delta(c_i) \otimes \ldots \otimes c_n\Big) \\
&= x \cdot \delta_n^i(c_1 \otimes \ldots \otimes c_n)
\end{aligned}$$

by (6.1). □

Observe also that the subcomplexes $(T_\pm^\bullet(C), \delta)$ are preserved by the $\mathfrak{g}$-action where C is equipped with the identity involution.

We next restrict to the case when $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra acting on C such that C is a direct sum of finite-dimensional

$\mathfrak{g}$-modules. This is the case for $C = S(\mathfrak{g})$ and, hence, for the isomorphic coalgebra $U(\mathfrak{g})$. For any $\mathfrak{g}$-module V, we define a $\mathfrak{g}$-submodule $V^{\mathfrak{g}}$ by

$$V^{\mathfrak{g}} = \{v \in V \mid x \cdot v = 0 \quad \forall x \in \mathfrak{g}\}.$$

Elements of $V^{\mathfrak{g}}$ are called $\mathfrak{g}$-*invariant.* The linear span $\mathfrak{g}V$ of the elements $x \cdot v$ where x runs over $\mathfrak{g}$ and v over V is also a $\mathfrak{g}$-submodule of V.

Proposition XVIII.6.2. *Under the previous hypotheses, each of the complexes $(T^\bullet_\pm(C), \delta)$ is the direct sum of the respective subcomplexes $(T^\bullet_\pm(C)^{\mathfrak{g}})$ and $\mathfrak{g}T^\bullet_\pm(C)$:*

$$T^\bullet_\pm(C) = T^\bullet_\pm(C)^{\mathfrak{g}} \oplus \mathfrak{g}T^\bullet_\pm(C).$$

PROOF. Since the constructions $V \mapsto V^{\mathfrak{g}}$ and $V \mapsto \mathfrak{g}V$ are functorial, it is clear that $T^\bullet_\pm(C)^{\mathfrak{g}}$ and $\mathfrak{g}T^\bullet_\pm(C)$ are subcomplexes of $T^\bullet_\pm(C)$. Therefore, in order to prove the proposition, it suffices to check that

$$V = V^{\mathfrak{g}} \oplus \mathfrak{g}V \tag{6.2}$$

for all $\mathfrak{g}$-modules V that are direct sums of finite-dimensional $\mathfrak{g}$-modules. Since $\mathfrak{g}$ is semisimple and Equality (6.2) is preserved by the direct sum of $\mathfrak{g}$-modules, it is enough to check (6.2) when V is finite-dimensional and simple. If V is the trivial one-dimensional module, then $\mathfrak{g}$ acts by zero, which implies that $V = V^{\mathfrak{g}}$ and $\mathfrak{g}V = 0$. If V is a non-trivial simple module, it corresponds to a dominant weight $\lambda \neq 0$. Let v be a highest weight vector for V. Since $\lambda \neq 0$, there exists an element H_i of $\mathfrak{g}$ such that $H_i \cdot v = \lambda(H_i)v \neq 0$. Consequently, v does not belong to $V^{\mathfrak{g}}$ and $V^{\mathfrak{g}} \neq V$. Since V is simple, the submodule $V^{\mathfrak{g}}$ has to be zero. On the other hand, the same relation shows that $\mathfrak{g}V \neq 0$. We again appeal to the simplicity of V, now obtaining $\mathfrak{g}V = V$. In both cases, we get (6.2). □

XVIII.7 Computations for Symmetric Coalgebras

We assume in this section that k is a field of characteristic zero. We now compute the cohomology of the complexes $(T^\bullet(C), \delta)$ and $(T^\bullet_\pm(C), \delta)$ in the special case when C is the symmetric bialgebra $C = (S(V), \Delta, \varepsilon)$ where V is a finite-dimensional vector space over k and

$$\Delta(v) = v \otimes 1 + 1 \otimes v \quad \text{and} \quad \varepsilon(v) = 0 \tag{7.1}$$

for any element v of V.

Theorem XVIII.7.1. *Under the previous hypotheses,*

(a) *there exists a unique map $\mu : (T^\bullet(S(V)), \delta) \to (\Lambda^\bullet(V), 0)$ of differential graded algebras where the exterior algebra $\Lambda^\bullet(V)$ is given the zero* *differential, such that the restriction of* μ *to* $T^1(S(V)) = S(V)$ *is the projection onto the direct summand* $S^1(V) = V = \Lambda^1(V)$*, and the induced map* $\mu^\bullet : H^\bullet(T^\bullet(S(V)), \delta) \to \Lambda^\bullet(V)$ *on cohomology is an isomorphism.*

(b) *The antisymmetrization map* $\alpha : \Lambda^n(V) \to T^n(S(V))$ *given by*

$$\alpha(v_1 \wedge \ldots \wedge v_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

for all $v_1, \ldots, v_n$*, is a map of complexes, i.e.,* $\delta \circ \alpha = 0$*. Furthermore, we have* $\mu(\alpha(\omega)) = n!\,\omega$ *for all* $\omega \in \Lambda^n(V)$*.*

(c) *The map* μ *induces isomorphisms*

$$H^{2n}(T_+(S(V)), \delta) \cong \Lambda^{2n}(V), \quad H^{2n+1}(T_+(S(V)), \delta) \cong 0,$$

and

$$H^{2n}(T_-(S(V)), \delta) \cong 0, \quad H^{2n+1}(T_-(S(V)), \delta) \cong \Lambda^{2n+1}(V)$$

in cohomology for all $n \geq 0$*.*

The rest of the section is devoted to the proof of Theorem 7.1. The idea is to dualize the complex $(T^\bullet(S(V)), \delta)$ and to compute the homology of the dual complex. For a definition of the exterior algebra, see II, Exercise 6.

We first need the concept of a graded dual vector space: if $V = \bigoplus_{n\geq 0} V_n$ is a vector space with a positive grading, we define the graded dual vector space of V by

$$V^*_{gr} = \bigoplus_{n\geq 0} V^*_n. \tag{7.2}$$

We can apply this to the vector spaces $T(V)$, $S(V)$, $\Lambda(V)$ and $T(S(V))$ with their natural gradings. If $V = \bigoplus_{n\geq 0} V_n$ and $W = \bigoplus_{n\geq 0} W_n$ are vector spaces with gradings, we may consider their tensor product $V \otimes W$ graded by

$$(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q. \tag{7.3}$$

Lemma XVIII.7.2. *Suppose* $V = \bigoplus_{n\geq 0} V_n$ *and* $W = \bigoplus_{n\geq 0} W_n$ *have gradings for which* V_n *and* W_n *are finite-dimensional for all* n*. Then there is a canonical isomorphism*

$$V^*_{gr} \otimes W^*_{gr} \cong (V \otimes W)^*_{gr}.$$

Proof. This is straightforward. It uses the fact proved in II.2 that this isomorphism holds for finite-dimensional vector spaces. □

Let $(C, \Delta, \varepsilon, 1)$ be a graded coalgebra with unit, meaning that the underlying vector space $C = \bigoplus_{n\geq 0} C_n$ has a grading, that Δ and ε are graded maps (we equip k with the trivial grading concentrated in degree zero), and that 1 belongs to C_0. If C_n is finite-dimensional for all n — which we assume henceforth —, then it is clear that $A = C^*_{\mathrm{gr}}$ is an algebra graded by $A_n = C^*_n$, with multiplication given by the transpose Δ^* of Δ, with unit given by ε^*, and with a map of algebras $\varepsilon : A \to k$ defined by $\varepsilon(a) = a(1)$. The map ε will be called the augmentation of A.

From the formula for δ, we see that the complex $(T^\bullet(C), \delta)$ of Section 5 is a complex with a grading. Applying Lemma 7.2, we get

$$T^n(C)^*_{\mathrm{gr}} = T^n(C^*_{\mathrm{gr}}) = T^n(A). \tag{7.4}$$

Lemma XVIII.7.3. *The transpose $d = \delta^*$ of δ is given by*

$$\begin{aligned} d(a_1 \otimes \cdots \otimes a_n) = \varepsilon(a_1)a_2 \otimes \cdots \otimes a_n \\ + \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n \\ + (-1)^n a_1 \otimes \cdots \otimes a_{n-1}\varepsilon(a_n) \end{aligned}$$

for all elements $a_1, \ldots, a_n$ of A.

The chain complex $(T^\bullet(A), d)$ is called the *bar complex* of the augmented algebra A.

Proof. Apply both sides to $x_1 \otimes \cdots \otimes x_n$ where $x_1, \ldots, x_n$ belong to C. □

Lemma XVIII.7.4. *Under the previous hypotheses, the cohomology of $(T^\bullet(C), \delta)$ is the graded dual of the homology of $(T^\bullet(A), d)$.*

Proof. This is a consequence of the fact that the duality functor is exact and that biduality is a natural isomorphism on finite-dimensional vector spaces. □

In order to prove Theorem 7.1, it therefore suffices to compute the homology of the chain complex $(T^\bullet(S(V)^*_{\mathrm{gr}}), d)$. We first identity the algebra $S(V)^*_{\mathrm{gr}}$.

Lemma XVIII.7.5. *If V is a finite-dimensional vector space over a field k of characteristic zero, then the graded dual of the graded coalgebra with unit $S(V)$ is the graded augmented algebra $S(W)$ where $W = V^*$ is the dual vector space of V.*

Proof. Let $\{v_1, \ldots, v_N\}$ be a basis of V. Then $\{v_1^{\alpha_1} \ldots v_N^{\alpha_N}\}_{\alpha_1+\cdots+\alpha_N=n}$ is a basis of $S^n(V)$. We define a basis $\{w_1^{\alpha_1} \ldots w_N^{\alpha_N}\}_{\alpha_1+\cdots+\alpha_N=n}$ of $S^n(V)^*$ by

$$< w_1^{\alpha_1} \ldots w_N^{\alpha_N}, v_1^{\beta_1} \ldots v_N^{\beta_N} > = \delta_{\alpha_1,\beta_1} \cdots \delta_{\alpha_N,\beta_N}\, \alpha_1! \ldots \alpha_N!. \tag{7.5}$$

The product map $\star : S^n(V)^* \otimes S^m(V)^* \to S^{n+m}(V)^*$ on $S(V)^*_{gr}$ is by definition the transpose of the comultiplication Δ. We have

$$\begin{aligned}
&< (w_1^{\alpha_1} \dots w_N^{\alpha_N}) \star (w_1^{\beta_1} \dots w_N^{\beta_N}), v_1^{\gamma_1} \dots v_N^{\gamma_N} > \\
&= \; < w_1^{\alpha_1} \dots w_N^{\alpha_N} \otimes w_1^{\beta_1} \dots w_N^{\beta_N}, \Delta(v_1^{\gamma_1} \dots v_N^{\gamma_N}) > \\
&= \; < w_1^{\alpha_1} \dots w_N^{\alpha_N} \otimes w_1^{\beta_1} \dots w_N^{\beta_N}, (v_1 \otimes 1 + 1 \otimes v_1)^{\gamma_1} \dots \\
&\qquad (v_N \otimes 1 + 1 \otimes v_N)^{\gamma_N} > \\
&= \sum_{i_1,\dots,i_N} \binom{\gamma_1}{i_1} \cdots \binom{\gamma_N}{i_N} < w_1^{\alpha_1} \dots w_N^{\alpha_N}, v_1^{i_1} \dots v_N^{i_N} > \\
&\qquad < w_1^{\beta_1} \dots w_N^{\beta_N}, v_1^{\gamma_1 - i_1} \dots v_N^{\gamma_N - i_N} > \\
&= \delta_{\alpha_1+\beta_1,\gamma_1} \cdots \delta_{\alpha_N+\beta_N,\gamma_N} \, \alpha_1!\beta_1! \dots \alpha_N!\beta_N! \\
&\qquad \binom{\alpha_1+\beta_1}{\alpha_1} \cdots \binom{\alpha_N+\beta_N}{\alpha_N} \\
&= \delta_{\alpha_1+\beta_1,\gamma_1} \cdots \delta_{\alpha_N+\beta_N,\gamma_N} (\alpha_1+\beta_1)! \dots (\alpha_N+\beta_N)! \\
&= \; < w_1^{\alpha_1+\beta_1} \dots w_N^{\alpha_N+\beta_N}, v_1^{\gamma_1} \dots v_N^{\gamma_N} > .
\end{aligned}$$

This proves that

$$(w_1^{\alpha_1} \dots w_N^{\alpha_N}) \star (w_1^{\beta_1} \dots w_N^{\beta_N}) = w_1^{\alpha_1+\beta_1} \dots w_N^{\alpha_N+\beta_N}, \tag{7.6}$$

which shows that the product on $S(V)^*_{gr}$ is the product of the symmetric algebra $S(V^*)$. The rest of the proof is left to the reader. □

We next compute the homology of the chain complex $(T^\bullet(S(W)), d)$. Define a linear map α from $\Lambda^\bullet(W)$ to $T^\bullet(S(W))$ by

$$\alpha(w_1 \wedge \dots \wedge w_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)} \tag{7.7}$$

where $w_1, \dots, w_n \in W$ and $\varepsilon(\sigma)$ is the sign of the permutation σ.

Proposition XVIII.7.6. *We have $d \circ \alpha = 0$ and also that the induced map $\alpha_\bullet : \Lambda^n(W) \to H_n(T^\bullet(S(W)), d)$ is an isomorphism for all $n \geq 0$. If $p_1 : S(W) \to W$ is the natural projection onto the direct summand $W = S^1(W)$ and μ is defined as the composite map*

$$\mu : T^\bullet(S(W)) \xrightarrow{p_1^{\otimes n}} W^{\otimes n} \to \Lambda^n(W),$$

then μ is a chain map. We have $(\mu \circ \alpha)(\omega) = n!\,\omega$ for all elements ω belonging to $\Lambda^n(W)$.

Proof. We proceed in six steps using the terminology and the results of the Appendix.

1. We first observe that the bar complex $(T^\bullet(A), d)$ is obtained from a complex $(T'_\bullet(A), d')$ of left A-modules by

$$T^\bullet(A) = k \otimes_A T'_\bullet(A) \quad \text{and} \quad d = \mathrm{id}_k \otimes_A d' \tag{7.8}$$

where $T'_n(A) = A \otimes A^{\otimes n}$ and the left A-linear differential d' is given by

$$d'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n$$
$$+(-1)^n a_0 \otimes \cdots \otimes a_{n-1}\varepsilon(a_n) \tag{7.9}$$

for $a_0, \ldots, a_n \in A$. The reader may easily check that $d' \circ d' = 0$.

2. We claim that the complex $(T'(A), d')$ is a resolution of k by free left A-modules. It suffices to prove that the complex

$$\cdots \xrightarrow{d'} T'_2(A) \xrightarrow{d'} T'_1(A) \xrightarrow{d'} T'_0(A) \xrightarrow{\varepsilon} k \to 0$$

is acyclic. Define $s : T'_n(A) \to T'_{n+1}(A)$ by

$$s(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n \tag{7.10}$$

and $s : k \to T'_0(A)$ by $s(1) = 1$. An easy computation shows that

$$d's + sd' = \mathrm{id} \tag{7.11}$$

on all $T'_n(A)$, which proves the acyclicity of $(T'(A), d')$.

3. In case A is the symmetric algebra $S(W)$, there is another resolution of k by free left A-modules: it is the Koszul resolution $(K_\bullet(W), \partial)$ defined by $K_n(W) = S(W) \otimes \Lambda^n(W)$ and

$$\partial(a \otimes w_1 \wedge \cdots \wedge w_n) = \sum_{i=1}^{n} (-1)^{i+1} a w_i \otimes w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_n \tag{7.12}$$

where $a \in S(W)$, $w_1, \ldots, w_n \in W$ and where the hat on w_i again means that we omit this element. Check that $\partial \circ \partial = 0$.

We claim that $(K_\bullet(W), \partial)$ is a resolution of k. This again is due to the existence of a homotopy: define a map $h : K_n(W) \to K_{n+1}(W)$ by

$$h(w_1 \ldots w_m \otimes \omega) = \sum_{i=0}^{m} w_1 \ldots \widehat{w_i} \ldots w_m \otimes w_i \wedge \omega \tag{7.13}$$

where $w_1, \ldots, w_m$ in W and ω in $\Lambda^n(W)$. Then we have

$$(\partial h + h\partial)(P \otimes \omega) = (m+n)(P \otimes \omega) \tag{7.14}$$

for all P in $S^m(W)$ and ω in $\Lambda^n(W)$. Relation (7.14) shows the acyclicity of the Koszul resolution in degree > 0. As for degree 0, observe that the cokernel of the map $\partial : K_1(W) \to K_0(W)$ given by $\partial(a \otimes w) = aw$ is isomorphic to k.

Since $\varepsilon(w) = 0$ for all $w \in W$, $(k \otimes_{S(W)} K_\bullet(W), \mathrm{id}_k \otimes_{S(W)} \partial)$, the induced complex, is isomorphic to the complex $\Lambda^\bullet(W)$ with zero differential:

$$(k \otimes_{S(W)} K_\bullet(W), \mathrm{id}_k \otimes_{S(W)} \partial) \cong (\Lambda^\bullet(W), 0). \tag{7.15}$$

4. We compare the resolutions $(K_\bullet(W), \partial)$ and $(T'_\bullet(S(W)), d)$.

Lemma XVIII.7.7. *The map* $\mathrm{id}_{S(W)} \otimes \alpha : K_\bullet(W) \to T'_\bullet(S(W))$ *is a chain map over the identity.*

PROOF. We have to prove that

$$d' \circ (\mathrm{id}_{S(W)} \otimes \alpha) = (\mathrm{id}_{S(W)} \otimes \alpha) \circ \partial. \tag{7.16}$$

All maps in (7.16) being $S(W)$-linear, it is enough to check this relation on elements of the form $1 \otimes w_1 \wedge \cdots \wedge w_n$ where $w_1, \ldots, w_n$ belong to W. By definition, we have

$$d'\Big((\mathrm{id}_{S(W)} \otimes \alpha)(1 \otimes w_1 \wedge \cdots \wedge w_n)\Big) = Z_1 + Z_2 + Z_3$$

where

$$Z_1 = \sum_{\sigma \in S_n} \varepsilon(\sigma) w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)},$$

$$Z_2 = \sum_{i=1}^{n-1} (-1)^i \sum_{\sigma \in S_n} \varepsilon(\sigma) w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(i)} w_{\sigma(i+1)} \otimes \cdots \otimes w_{\sigma(n)}$$

and

$$Z_3 = (-1)^n \sum_{\sigma \in S_n} \varepsilon(\sigma) w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n-1)} \varepsilon(w_{\sigma(n)}).$$

Let us first deal with Z_1. We have

$$Z_1 = \sum_{i=1}^{n} \sum_{\substack{\sigma \in S_n, \\ \sigma(1)=i}} \varepsilon(\sigma) w_i \otimes w_{\sigma(2)} \otimes \cdots \otimes w_{\sigma(n)}.$$

By applying the permutation $(12 \ldots i)$ of sign $(-1)^{i+1}$, we get

$$\begin{aligned} Z_1 &= (\mathrm{id}_{S(W)} \otimes \alpha)\Big(\sum_{i=1}^{n} (-1)^{i+1} w_i \otimes w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_n\Big) \\ &= (\mathrm{id}_{S(W)} \otimes \alpha)\big(\partial(1 \otimes w_1 \wedge \cdots \wedge w_n)\big) \end{aligned}$$

by (7.12). Relation (7.16) will be proved once we have checked the vanishing of Z_2 and of Z_3. Concerning Z_2, we have

$$\begin{aligned}\sum_{\sigma\in S_n} & \varepsilon(\sigma)w_{\sigma(1)}\otimes\cdots\otimes w_{\sigma(i)}w_{\sigma(i+1)}\otimes\cdots\otimes w_{\sigma(n)} \\ = & \sum_{\substack{\sigma\in S_n,\\ \sigma(i)<\sigma(i+1)}} \varepsilon(\sigma)w_{\sigma(1)}\otimes\cdots\otimes w_{\sigma(i)}w_{\sigma(i+1)}\otimes\cdots\otimes w_{\sigma(n)} \\ & + \sum_{\substack{\sigma\in S_n,\\ \sigma(i+1)<\sigma(i)}} \varepsilon(\sigma)w_{\sigma(1)}\otimes\cdots\otimes w_{\sigma(i)}w_{\sigma(i+1)}\otimes\cdots\otimes w_{\sigma(n)}.\end{aligned}$$

By exchanging i and $i+1$, we see that the second summand is the opposite of the first one. This proves the vanishing of

$$\sum_{\sigma\in S_n} \varepsilon(\sigma)w_{\sigma(1)}\otimes\cdots\otimes w_{\sigma(i)}w_{\sigma(i+1)}\otimes\cdots\otimes w_{\sigma(n)},$$

hence of Z_2. Eventually, $Z_3=0$ because $\varepsilon(w)=0$ for all $w\in W=S^1(W)$. □

5. By Corollary 11.2, there exist $S(W)$-linear maps $\beta : T'_n(S(W)) \to K_n(W)$, $h_1 : T'_n(S(W)) \to T'_{n+1}(S(W))$, and $h_2 : K_n(W) \to K_{n+1}(W)$ such that $\partial\beta = \beta d'$,

$$(\mathrm{id}_{S(W)}\otimes\alpha)\beta = \mathrm{id} + d'h_1 + h_1d' \quad\text{and}\quad \beta(\mathrm{id}_{S(W)}\otimes\alpha) = \mathrm{id}+\partial h_2 + h_2\partial. \tag{7.17}$$

Tensoring both resolutions on the left with k over $S(W)$, we see that

$$\mathrm{id}_k\otimes_{S(W)}(\mathrm{id}_{S(W)}\otimes\alpha) = \alpha : (\Lambda^\bullet(W),0)\to(T^\bullet(S(W)),d)$$

induces an isomorphism in homology: indeed, by (7.17) the chain maps $\alpha\circ(\mathrm{id}_k\otimes_{S(W)}\beta)$ and $(\mathrm{id}_k\otimes_{S(W)}\beta)\circ\alpha$ are homotopic to the identities.

6. It is easy to check that μ is a chain map, i.e., it annihilates all elements of the form

$$\varepsilon(a_1)a_2\otimes\cdots\otimes a_n + \sum_{i=1}^{n-1}(-1)^i a_1\otimes\cdots\otimes a_{i-1}\otimes a_ia_{i+1}\otimes a_{i+2}\otimes\cdots\otimes a_n$$
$$+(-1)^{n+1}a_1\otimes\cdots\otimes a_{n-1}\varepsilon(a_n).$$

The rest of Proposition 7.6 is obvious. □

We now prove Theorem 7.1.

Proof of Theorem 7.1. (a–b) Dualize Proposition 7.6 using Lemmas 7.3–7.5. We still have to prove that the map $\mu : T^n(S(W)) \to \Lambda^n(W)$ is the transpose of the antisymmetrization map $\alpha : \Lambda^n(V)\to T^n(S(V))$. Indeed, let us check that

$$<\mu(X),\omega> = < X,\alpha(\omega)> \tag{7.18}$$

for all homogeneous $X \in T^n(S(W))$ and $\omega \in \Lambda^n(W)$. If X belongs to $S^{d_1}(W) \otimes \cdots \otimes S^{d_n}(W)$ where $(d_1, \ldots, d_n) \neq (1, \ldots, 1)$, then both sides of (7.18) vanish. If $X = w_{i_1} \otimes \cdots \otimes w_{i_n}$ where $w_{i_1}, \ldots, w_{i_n} \in W$, then

$$< \mu(w_{i_1} \otimes \cdots \otimes w_{i_n}), v_{j_1} \wedge \ldots \wedge v_{j_n} > = < w_{i_1} \wedge \ldots \wedge w_{i_n}, v_{j_1} \wedge \ldots \wedge v_{j_n} >$$

vanishes when $(i_1, \ldots, i_n)$ is not a permutation of $(j_1, \ldots, j_n)$. If it is, the right-hand side is equal to the sign of this permutation. On the other hand,

$$\begin{aligned} &< w_{i_1} \otimes \ldots \otimes w_{i_n}, \alpha(v_{j_1} \wedge \ldots \wedge v_{j_n}) > \\ &\quad = \sum_{\sigma \in S_n} \varepsilon(\sigma) < w_{i_1} \otimes \ldots \otimes w_{i_n}, v_{\sigma(j_1)} \otimes \ldots \otimes v_{\sigma(j_n)} > \end{aligned}$$

yields the same result. This proves (7.18).

(c) Using μ and α, we determine the action of the involution σ on the cohomology of $(T^\bullet(S(V)), \delta)$. By (5.2) we have

$$\begin{aligned} (\mu \circ \sigma_n \circ \alpha)(v_1 \wedge \ldots \wedge v_n) &= (-1)^{n(n+1)/2} \sum_{\sigma \in S_n} \varepsilon(\sigma)\, \mu(v_{\sigma(n)} \otimes \ldots \otimes v_{\sigma(1)}) \\ &= (-1)^{n(n+1)/2} \sum_{\sigma \in S_n} \varepsilon(\sigma)\, v_{\sigma(n)} \wedge \cdots \wedge v_{\sigma(1)}. \end{aligned}$$

Apply the change of variables effected by $\sigma = \sigma'\tau$ where τ is the permutation $(1, n)(2, n-1) \cdots$ with sign $(-1)^{n(n-1)/2}$. It follows that

$$(\mu \circ \sigma_n \circ \alpha)(v_1 \wedge \cdots \wedge v_n) = (-1)^n n!\, v_1 \wedge \cdots \wedge v_n. \tag{7.19}$$

From Parts (a–b) of Theorem 7.1 and from (7.19), we conclude that σ_{2n} acts as the identity on the cohomology while σ_{2n+1} acts as $-\mathrm{id}$. □

XVIII.8 Uniqueness Theorem for Quantum Enveloping Algebras

In this section we deal only with a quantum enveloping algebra whose underlying quasi-bialgebra is the trivial topological bialgebra $U(\mathfrak{g})[[h]]$ of formal series over the enveloping algebra of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$. We state the second rigidity theorem.

Theorem XVIII.8.1. *Assume we have $A = (U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi, R)$ and $A' = (U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi', R)$ which are quantum enveloping algebras for the same finite-dimensional semisimple Lie algebra $\mathfrak{g}$ and have the same universal R-matrix R satisfying the conditions $R_{21} = R$, $R \equiv 1 \otimes 1$ modulo h and $R = \Delta(u)V$, where $u \in U(\mathfrak{g})[[h]]$ and V is a central element of $(U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]]$. Then there exists a gauge transformation F in the algebra $(U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$ with $F_{21} = F$, $F \equiv 1 \otimes 1$ modulo h, and $[F, \Delta(a)] = 0$ for all $a \in A$ such that $A' = A_F$.*

The data Δ and R remain unchanged under a gauge transformation F verifying the properties of Theorem 8.1. Indeed, by definition of Δ_F (see XV.3) and by the last condition on F, we have $\Delta_F(a) = F\Delta(a)F^{-1} = \Delta(a)$ for all $a \in A$. As for R, by (XV.3.9) we have $R_F = F_{21}RF^{-1} = FRF^{-1}$. Now, in view of the third assumption on R, we have

$$[R, F] = [\Delta(u)V, F] = [\Delta(u), F]V + \Delta(u)[V, F] = 0.$$

Consequently, $R_F = R$.

Observe that we may (and shall) apply Theorem 8.1 to the case when $R = e^{ht/2}$ where t is the 2-tensor in (XVII.1.6). Indeed, we have $R_{21} = R$ and $R \equiv 1 \otimes 1$ modulo h. Moreover, (XVII.1.6) implies that

$$R = \Delta(e^{hC/4})\,(e^{-hC/4} \otimes e^{-hC/4}).$$

Before we prove Theorem 8.1, we establish the following lemma.

Lemma XVIII.8.2. *Let $(A, \Delta, \varepsilon, \Phi, R)$ be a cocommutative quantum enveloping algebra such that $R_{21} = R$ and $R \equiv 1 \otimes 1$ modulo h. Then $\Phi_{321} = \Phi^{-1}$.*

PROOF. We first claim that

$$R_{12}(\Delta \otimes \mathrm{id})(R) = \Phi_{321}R_{23}(\mathrm{id} \otimes \Delta)(R)\Phi. \tag{8.1}$$

Indeed, we have

$$\begin{aligned} R_{12}(\Delta \otimes \mathrm{id})(R) &= R_{12}\Phi_{312}R_{13}(\Phi_{132})^{-1}R_{23}\Phi \\ &= \Phi_{321}R_{23}(\Phi_{231})^{-1}R_{13}\Phi_{213}R_{12} \\ &= \Phi_{321}R_{23}(\mathrm{id} \otimes \Delta)(R)\Phi. \end{aligned}$$

The first and last equalities follow from Proposition XV.2.2 while the middle one follows from Corollary XV.2.3. Next, apply the involution τ_{13} to $A \otimes A \otimes A$. Since $\Delta = \Delta^{\mathrm{op}}$ and $R = R_{21}$, Relation (8.1) becomes

$$R_{23}(\mathrm{id} \otimes \Delta)(R) = \Phi R_{12}(\Delta \otimes \mathrm{id})(R)\Phi_{321}. \tag{8.2}$$

Combining Relations (8.1–8.2), we get

$$\begin{aligned} \Big(R_{12}(\Delta \otimes \mathrm{id})(R)\Big)^2 &= \Phi^{-1}R_{23}(\mathrm{id} \otimes \Delta)(R)(\Phi_{321})^{-1}\Phi_{321}R_{23}(\mathrm{id} \otimes \Delta)(R)\Phi \\ &= \Phi^{-1}\Big(R_{23}(\mathrm{id} \otimes \Delta)(R)\Big)^2\Phi. \end{aligned}$$

By uniqueness of the square roots of elements congruent to the unit modulo h, we get

$$R_{12}(\Delta \otimes \mathrm{id})(R) = \Phi^{-1}R_{23}(\mathrm{id} \otimes \Delta)(R)\Phi. \tag{8.3}$$

Comparing (8.1) and (8.3), we conclude that $\Phi_{321} = \Phi^{-1}$. $\square$

Proof of Theorem 8.1. We have to show that we can find a gauge transformation taking Φ to Φ'. Suppose Φ and Φ' are equal modulo h^n for some $n \geq 1$. This always holds for $n = 1$ since Φ and Φ' are congruent to $1\otimes 1\otimes 1$ modulo h. Define $\varphi \in U(\mathfrak{g})^{\otimes 3}$ by

$$\Phi' \equiv \Phi + h^n \varphi \quad \text{modulo } h^{n+1}.$$

Let $\mathrm{Ant}(\varphi)$ be the element

$$\mathrm{Ant}(\varphi) = \varphi - \varphi_{213} - \varphi_{132} - \varphi_{321} + \varphi_{231} + \varphi_{312} \tag{8.4}$$

of $U(\mathfrak{g})^{\otimes 3}$. The first step in the proof of Theorem 8.1 is the following lemma with the same hypotheses.

Lemma XVIII.8.3. *The element φ is $\mathfrak{g}$-invariant and satisfies the relations $\varphi_{321} = -\varphi$, $\mathrm{Ant}(\varphi) = 0$, and*

$$1 \otimes \varphi - (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\varphi) + (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\varphi) - (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\varphi) + (\varphi \otimes 1) = 0.$$

PROOF. (a) Since Δ is coassociative, Relation (XV.1.1) may be rewritten in the form

$$[(\Delta \otimes \mathrm{id})(\Delta(x)), \Phi] = 0$$

for all $x \in \mathfrak{g}$, which means that Φ and Φ' are $\mathfrak{g}$-invariant, namely that they belong to the subspace $(U(\mathfrak{g})^{\otimes 3})^{\mathfrak{g}}[[h]]$. Consequently, φ is $\mathfrak{g}$-invariant too.

(b) Lemma 8.2 implies that

$$\Phi'_{321} \equiv \Phi_{321} + h^n \varphi_{321} \equiv (\Phi')^{-1} \equiv (\Phi + h^n\varphi)^{-1} \equiv \Phi - h^n \varphi$$

modulo h^{n+1}. It follows that $\varphi_{321} = -\varphi$.

(c) We now prove that $\mathrm{Ant}(\varphi) = 0$. Consider Relation (XV.2.3) for Φ and Φ':

$$(\Delta \otimes \mathrm{id})(R) = \Phi_{312} R_{13} (\Phi_{132})^{-1} R_{23} \Phi = \Phi'_{312} R_{13} (\Phi'_{132})^{-1} R_{23} \Phi'.$$

Reducing the latter modulo h^{n+1} implies that

$$\varphi_{312} - \varphi_{132} + \varphi = 0. \tag{8.5}$$

Since $\varphi_{321} = -\varphi$, Relation (8.5) yields

$$-\varphi_{213} + \varphi_{231} - \varphi_{321} = 0. \tag{8.6}$$

Adding Relations (8.5–8.6), we get

$$\mathrm{Ant}(\varphi) = \varphi - \varphi_{213} - \varphi_{132} - \varphi_{321} + \varphi_{231} + \varphi_{312} = 0.$$

(d) Using the pentagonal relations (XV.1.3) for Φ and Φ', we get

$$(1\otimes\Phi')(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi')(\Phi'\otimes 1)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi')^{-1}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi')^{-1}$$

$$= (1\otimes\Phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)(\Phi\otimes 1)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)^{-1}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)^{-1} = 1.$$

Reducing these equalities modulo h^{n+1}, we obtain the desired 5-term functional equation for φ. □

The next step is the following one.

Lemma XVIII.8.4. *There exists a $\mathfrak{g}$-invariant element $f \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ such that*

$$\varphi = 1\otimes f - (\Delta\otimes\mathrm{id})(f) + (\mathrm{id}\otimes\Delta)(f) - f\otimes 1 \quad \textit{and} \quad f_{21} = f.$$

PROOF. Using the cohomological language of Section 5, we can paraphrase Lemma 8.3 by saying that φ is a $\mathfrak{g}$-invariant element of $U(\mathfrak{g})^{\otimes 3}$ satisfying

$$\delta(\varphi) = 0, \quad \sigma_3(\varphi) = -\varphi, \quad \text{and} \quad \mathrm{Ant}(\varphi) = 0. \tag{8.7}$$

The first two relations in (8.7) mean that φ is a 3-cocycle in the cobar complex $(T_-^\bullet(U(\mathfrak{g})), \delta)$. We claim that φ is a coboundary. Using the isomorphism $\eta : S(\mathfrak{g}) \to U(\mathfrak{g})$ of coalgebras, it suffices to check that $\psi = (\eta^{-1}\otimes\eta^{-1}\otimes\eta^{-1})(\varphi)$ is a coboundary in $(T_-^\bullet(S(\mathfrak{g})), \delta)$. We also have $\mathrm{Ant}(\psi) = 0$. By Theorem 7.1 (c) we have $H^3(T_-^\bullet(S(\mathfrak{g})), \delta) \cong \Lambda^3(\mathfrak{g})$, the isomorphism being induced by the map μ. It is therefore enough to check that $\mu(\psi) = 0$. Now $\mathrm{Ant}(\psi) = 0$ implies $\mathrm{Ant}((\mu\otimes\mu\otimes\mu)(\psi)) = 0$. An immediate computation shows that $\alpha(\mu(\psi)) = \mathrm{Ant}((\mu\otimes\mu\otimes\mu)(\psi))$. Therefore, by Theorem 7.1 we have

$$\mu(\psi) = \frac{1}{6}\,\mu(\alpha(\mu(\psi))) = \frac{1}{6}\,\mu(\mathrm{Ant}((\mu\otimes\mu\otimes\mu)(\psi))) = 0,$$

which tells us that the cohomology class of ψ is zero.

Since the 3-cocycle φ is a coboundary, there is an element $f \in T_-^2(U(\mathfrak{g}))$, i.e., an element $f \in U(\mathfrak{g})^{\otimes 2}$ with $f_{21} = f$ such that $\varphi = \delta(f)$, i.e.,

$$\varphi = 1\otimes f - (\Delta\otimes\mathrm{id})(f) + (\mathrm{id}\otimes\Delta)(f) - f\otimes 1. \tag{8.8}$$

We have so far proved Lemma 8.4 up to the fact that we can choose f to be $\mathfrak{g}$-invariant.

This last fact is a consequence of Proposition 6.2 applied to the coalgebra $U(\mathfrak{g})$ on which $\mathfrak{g}$ acts by the adjoint representation (this is where we use the assumption that $\mathfrak{g}$ is semisimple). Since $(T_-^\bullet(U(\mathfrak{g})), \delta)$ splits into the direct sum of $(T_-^\bullet(U(\mathfrak{g}))^{\mathfrak{g}}, \delta)$ and of $(\mathfrak{g}T_-^\bullet(U(\mathfrak{g})), \delta)$ and since φ belongs to $T_-^3(U(\mathfrak{g}))^{\mathfrak{g}}$, then f necessarily belongs to $T_-^2(U(\mathfrak{g}))^{\mathfrak{g}}$. □

From the element f whose existence is asserted by Lemma 8.4, we deduce the gauge transformation

$$F = 1 \otimes 1 + h^n f. \tag{8.9}$$

The $\mathfrak{g}$-invariance of f implies that $[F, \Delta(x)] = 0$ for all $x \in U(\mathfrak{g})$. We also have $F_{21} = F$. We already know that Δ and R remain unaffected by such a gauge transformation. Let us compute Φ_F modulo h^{n+1}. From (8.8–8.9) and from (XV.3.3) we get

$$\Phi' - \Phi_F \equiv h^n \Big(\varphi - (1 \otimes f - (\Delta \otimes \mathrm{id})(f) + (\mathrm{id} \otimes \Delta)(f) - f \otimes 1) \Big) \equiv 0,$$

which can be reexpressed as $\Phi' \equiv \Phi_F$ modulo h^{n+1}. We now define an element φ_1 of $U(\mathfrak{g})^{\otimes 3}$ by

$$\Phi' \equiv \Phi_F + h^{n+1}\varphi_1 \quad \text{modulo } h^{n+2}$$

and start the whole procedure all over again. By composing all the gauge transformations obtained in this way, we obtain a gauge transformation between the quasi-bialgebras A and A'. This completes the proof of Theorem 8.1. □

XVIII.9 Exercises

1. Compute $H^2(\mathfrak{g}, \mathbf{C})$ for all complex Lie algebras of dimension ≤ 3.

2. Show that the space of primitive elements of a coalgebra C can be realized as the cohomology group $H^1(T^\bullet(C), \delta)$.

3. Give a direct proof of Theorem 7.1 when V is one-dimensional.

4. Let $n \geq 2$ be an integer. Consider the algebra $A = k[t]/(t^n - 1)$. Let N be the element $N = 1 + t + \cdots t^{n-1}$ of A. Show that

$$\cdots \xrightarrow{1-t} A \xrightarrow{N} A \xrightarrow{1-t} A \xrightarrow{N} A \xrightarrow{1-t} A \to 0$$

   is a resolution of k by free left A-modules.

5. Prove the assertions of the Appendix.

XVIII.10 Notes

The content of Sections 1-7 is classical. The cohomology of Lie algebras was introduced by Chevalley and Eilenberg in [CE48] following ideas of E. Cartan. See [Ger64] for a general deformation theory for algebras.

The cobar complex appeared independently in papers by Adams [Ada56] and Cartier [Car57]. Theorem 7.1 is due to the latter. The reader is advised to take a look at Cartier's elegant proof using resolutions of comodules in [Car57]. We gave here a more pedestrian proof based on Eilenberg and Mac Lane's bar complex [EM53] (see also [SC56]). Drinfeld gives a third one in [Dri89b], Prop. 2.2 and Prop. 3.11.

The content of Section 8 is entirely due to Drinfeld (see [Dri90], Section 3). Observe that Theorems 4.1 and 8.1 are non-constructive. It would be interesting to find an explicit isomorphism $\alpha : U_h(\mathfrak{g}) \to U(\mathfrak{g})[[h]]$ and an explicit gauge transformation F, even in the case $\mathfrak{g} = \mathfrak{sl}(2)$.

XVIII.11 Appendix. Complexes and Resolutions

We recall some facts from homological algebra. For details and proofs, see, e.g., Cartan-Eilenberg's book [CE56].

Let A be an algebra. A *chain complex of left A-modules* $(C_\bullet, d)$ is a family $(C_n)_{n\geq 0}$ of left A-modules together with A-linear maps $d : C_n \to C_{n-1}$, defined for all $n \geq 1$, such that $d \circ d = 0$. The last condition implies that the image of d sits inside its kernel. We can therefore define the *homology groups* $H_\bullet(C_\bullet, d)$ of the chain complex by

$$H_n(C_\bullet, d) = \frac{\mathrm{Ker}\,(d : C_n \to C_{n-1})}{\mathrm{Im}\,(d : C_{n+1} \to C_n)}. \tag{11.1}$$

A chain complex is *acyclic* if all its homology groups vanish.

One similarly defines a *cochain complex of left A-modules* $(C^\bullet, d)$ as a family $(C^n)_{n\geq 0}$ of left A-modules with A-linear maps $\delta : C^n \to C^{n+1}$ such that $\delta \circ \delta = 0$. The *cohomology groups* $H^\bullet(C^\bullet, d)$ are defined by

$$H^n(C^*, d) = \frac{\mathrm{Ker}\,(d : C^n \to C^{n+1})}{\mathrm{Im}\,(d : C^{n-1} \to C^n)}. \tag{11.2}$$

In both cases, we agree that $C_{-1} = C^{-1} = 0$.

Let M be a left A-module. A *resolution* of M by free left A-modules is a chain complex $(C_\bullet, d)$ of free left A-modules together with an A-linear map $\varepsilon : C_0 \to M$ such that the chain complex

$$\cdots \xrightarrow{d} C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} M \to 0 \tag{11.3}$$

is acyclic. Any A-module has a resolution by free left A-modules.

A *chain map* $f : (C_\bullet, d) \to (C'_\bullet, d')$ between chain complexes of A-modules is a family $(f_n : C_n \to C'_n)_{n\geq 0}$ of A-linear maps such that

$$f_n \circ d = d' \circ f_{n+1} \tag{11.4}$$

for all n. A chain map f induces a map $f_\bullet : H_\bullet(C_\bullet, d) \to H_\bullet(C'_\bullet, d)$ between the corresponding homology groups.

A *homotopy* h between two chain maps $f, f' : (C_\bullet, d) \to (C'_\bullet, d')$ is a family $(h_n : C_n \to C'_{n+1})_{n\geq 0}$ of A-linear maps such that

$$f_n - f'_n = d' \circ h_n + h_{n-1} \circ d \tag{11.5}$$

for all n (by convention $h_{-1} = 0$). If there exists a homotopy between f and f', we say that the two chain maps are *homotopic*. Homotopy is an equivalence relation. Homotopic chain maps f, f' induce the same map on homology: $f_\bullet = f'_\bullet$.

One of the basic results in homological algebra is the following comparison theorem for resolutions.

Theorem XVIII.11.1. *Let $(C_\bullet, d)$ [resp. $(C'_\bullet, d')$] be a resolution of an A-module M [resp. of M'] by free left A-modules. Suppose we also have an A-linear map f_{-1} from M to M'. Then there exists a chain map f from $(C_\bullet, d)$ to $(C'_\bullet, d')$ over f_{-1}, i.e., such that $f_{-1} \circ \varepsilon = \varepsilon' \circ f_{-1}$. If f' is another chain map over f_{-1}, then f and f' are homotopic.*

Corollary XVIII.11.2. *Let $(C_\bullet, d)$ and $(C'_\bullet, d')$ be resolutions of an A-module M by free left A-modules. There exist chain maps*

$$f : (C_\bullet, d) \to (C'_\bullet, d') \quad \textit{and} \quad g : (C'_\bullet, d') \to (C_\bullet, d)$$

such that $g \circ f$ and $f \circ g$ are homotopic to the identities.

Proof. Applying the above theorem to $f_{-1} = \mathrm{id}$, we get chain maps f, g such that $\varepsilon = \varepsilon' \circ f_0$ and $\varepsilon' = \varepsilon \circ g_0$. Now $g \circ f$ is a chain map from $(C_\bullet, d)$ to itself with $\varepsilon \circ (g_0 \circ f_0) = \varepsilon$. So is the identity on $C_\bullet$. By the second part of the theorem, we see that $g \circ f$ is necessarily homotopic to the identity. A similar argument works for $f \circ g$. $\square$

Chapter XIX
Monodromy of the Knizhnik-Zamolodchikov Equations

The purpose of this chapter is twofold:

(i) For any complex Lie algebra $\mathfrak{g}$ and any invariant symmetric 2-tensor $t \in \mathfrak{g}\otimes\mathfrak{g}$, construct a quantum enveloping algebra $A_{\mathfrak{g},t}$ for $\mathfrak{g}$ whose canonical 2-tensor is t.

(ii) Give Drinfeld's reformulation and proof of an important result of Kohno's which asserts that, if $\mathfrak{g}$ is semisimple, the monodromy of a certain system of differential equations, called the Knizhnik-Zamolodchikov system, is equivalent to the braid group representation provided by the universal R-matrix of the quantum enveloping algebra $U_h(\mathfrak{g})$ introduced in Chapter XVII. In terms of categories, Drinfeld's proof amounts to showing that the braided tensor category $U_h(\mathfrak{g})$-Mod_{fr} of modules over the Drinfeld-Jimbo algebra (as defined in XVII.3) is equivalent to a braided category of modules over the trivial deformation $U(\mathfrak{g})[[h]]$, equipped with a non-trivial associativity constraint.

We shall use some elementary differential geometry in this chapter, but, as was the case with knot theory earlier in this book, we shall focus on Drinfeld's ideas and therefore skip the details that are not essential to their understanding.

XIX.1 Connections

We assume some standard knowledge of differential geometry. Let us nevertheless recall a few facts. For more details, the reader may consult [KN63].

Let X be a complex analytic variety of dimension n and $p : E \to X$ be a complex analytic vector bundle of rank d over X. If x is a point of X, we denote by F_x the fibre at x: $F_x = p^{-1}(x)$. A *connection* on E is a linear map ∇ from the space $\Gamma(X, E)$ of sections of the vector bundle E into the space $\Omega^1(X, E)$ of differential 1-forms with values in E such that for any section s and any complex analytic function f on X we have

$$\nabla(fs) = (df)s + f\nabla(s). \tag{1.1}$$

If ∇_1 and ∇_2 are two connections on E, then the difference $\nabla_1 - \nabla_2$ is $\mathcal{O}(X)$-linear, where $\mathcal{O}(X)$ is the ring of complex analytic functions on X. Locally, we can write a section s in the form

$$s = f_1 e_1 + \cdots + f_d e_d \tag{1.2}$$

where $f_1, \ldots, f_d$ are complex analytic functions on X and $\{e_1, \ldots, e_d\}$ is a basis of the fibre. Any connection ∇ on E can be written locally as

$$\nabla s = ds - \Gamma s \tag{1.3}$$

where d is the de Rham differential and Γ is a differential 1-form on X with values in the endomorphism ring of E.

A section s of the bundle is *horizontal* for the connection ∇ if $\nabla s = 0$, i.e., if locally s is a solution of the system

$$ds = \Gamma s. \tag{1.4}$$

Let $\gamma : [0, 1] \to X$ be a smooth path in X from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. We may pull back the matrix Γ of differential forms on X along γ to a matrix $A(\theta)d\theta = \gamma^*\Gamma$ of differential forms on the interval $[0, 1]$. By the theory of ordinary differential equations, there exists a unique smooth map A_γ from $[0, 1]$ into the group of linear automorphisms of the fibre bundle such that $A_\gamma(0) = \mathrm{id}$ and $w(\theta) = A_\gamma(\theta)w(0)$ is a solution of the differential equation

$$\frac{dw(\theta)}{d\theta} = A(\theta)w(\theta). \tag{1.5}$$

The automorphism $A_\gamma(1)$ defines a linear isomorphism $T_\gamma : F_{x_0} \to F_{x_1}$, called the *parallel transport* along the path γ. When γ' is a path from x_1 to x_2 we may consider the composed path $\gamma\gamma'$, as in the Appendix to Chapter X. The uniqueness theorem on systems of first order linear differential equations implies that

$$T_{\gamma\gamma'} = T_{\gamma'} \circ T_\gamma. \tag{1.6}$$

The *holonomy group* at x_0 is defined as the subgroup of $\mathrm{Aut}(F_{x_0})$ generated by T_γ for all loops γ based at x_0 at X. In general, the holonomy depends on the local as well as on the global structure of X. In other words,

T_γ may change as γ varies (even infinitesimally). Let us give a condition on the connection ∇ under which the parallel transport depends only on the homotopy class of the path.

We need the notions of covariant derivative and of curvature of a connection. It is not difficult to show that the connection ∇ extends to a unique endomorphism of degree 1, still denoted ∇, and called the *covariant derivative*, of $\Omega^\bullet(X, E)$ such that

$$\nabla(\omega\omega') = (d\omega)\omega' + (-1)^p \omega \nabla(\omega')$$

for any pair (ω, ω') of differential forms where p is the degree of ω.

Lemma XIX.1.1. *The curvature $K = \nabla \circ \nabla$ is $\mathcal{O}(X)$-linear.*

PROOF. Let ω be a differential form and f be a function on X. We have

$$\begin{aligned} K(f\omega) &= \nabla((df)\omega + f\nabla(\omega)) = (d^2 f)\omega - (df)\nabla(\omega) + (df)\nabla(\omega) + fK(\omega) \\ &= fK(\omega) \end{aligned}$$

since $d^2 f = 0$. □

Locally, the curvature can be expressed in terms of Γ by

$$K(s) = d(ds - \Gamma s) - \Gamma(ds - \Gamma s) = (-d\Gamma + \Gamma \wedge \Gamma)s,$$

which leads to the formula

$$K = -d\Gamma + \Gamma \wedge \Gamma. \tag{1.7}$$

When $K = 0$ we say that the connection is *flat*. In this case, $\Omega^\bullet(X, E)$ becomes a cochain complex with differential ∇.

Proposition XIX.1.2. *Given a connection ∇, we have $T_\gamma = T_{\gamma'}$ for any pair (γ, γ') of homotopic paths in X if and only if the connection is flat.*

This statement implies that, if $K = 0$, then for any point x_0 in X the parallel transport induces a group morphism T from the fundamental group $\pi_1(X, x_0)$ to $\mathrm{Aut}(F_{x_0})$. It is called the *monodromy representation* of the fundamental group acting on the fibre. We shall not prove Proposition 1.2 for which we refer to the classical literature.

XIX.2 Braid Group Representations from Monodromy

We apply the generalities of Section 1 to the following situation. Suppose given a finite-dimensional complex vector space W, an integer $n > 1$ and a family $\{A_{ij}\}_{1 \le i < j \le n}$ of endomorphisms of W satisfying the conditions

$$[A_{ij}, A_{k\ell}] = 0 \tag{2.1}$$

whenever i, j, k, ℓ are distinct integers $\leq n$, and

$$[A_{ij}, A_{ik} + A_{jk}] = [A_{jk}, A_{ij} + A_{ik}] = 0 \tag{2.2}$$

whenever i, j, k are distinct integers. Consider the differential system

$$dw = \sum_{1 \leq i < j \leq n} \frac{A_{ij}}{z_i - z_j}(dz_i - dz_j)w. \tag{2.3}$$

According to Section 1, this defines a connection $\nabla = d - \Gamma$ on the trivial bundle $Y_n \times W$ where

$$\Gamma = \sum_{1 \leq i < j \leq n} \frac{A_{ij}}{z_i - z_j}(dz_i - dz_j) \tag{2.4}$$

and Y_n is the complex variety

$$Y_n = \left\{(z_1, \ldots, z_n) \in \mathbf{C}^n \;\middle|\; i \neq j \Rightarrow z_i \neq z_j\right\}$$

already considered in X.6.

Proposition XIX.2.1. *The connection $\nabla = d - \Gamma$ is flat.*

PROOF. By Proposition 1.2 it suffices for us to show that the curvature $K = -d\Gamma + \Gamma \wedge \Gamma$ vanishes. Since the endomorphisms A_{ij} do not depend on the variables $z_1, \ldots, z_n$, we have $d\Gamma = 0$. It is therefore enough to check that $\Gamma \wedge \Gamma = 0$. Set

$$u_{ij} = d\log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}.$$

We have

$$\Gamma \wedge \Gamma = \sum_{i<j,\, k<\ell} A_{ij} A_{k\ell}\, u_{ij} \wedge u_{k\ell}. \tag{2.5}$$

The right-hand side of (2.5) is equal to $K_2 + K_3 + K_4$ where

$$K_p = \sum_{i<j,\, k<\ell} A_{ij} A_{k\ell}\, u_{ij} \wedge u_{k\ell},$$

the set of indices $\{i < j, k < \ell\} \subset \{1, \ldots, n\}$ running over all such subsets of cardinality p. We now show that K_2, K_3 and K_4 vanish separately due to Relations (2.1–2.2). For K_2, this results from $u_{ij} \wedge u_{ij} = 0$. Let us deal with K_4: exchanging (i, j) and (k, ℓ), we get

$$K_4 = \sum_{i<j,\, k<\ell} A_{k\ell} A_{ij}\, u_{k\ell} \wedge u_{ij}.$$

The right-hand side is equal to $-K_4$ because $u_{k\ell} \wedge u_{ij} = -u_{ij} \wedge u_{k\ell}$ and $A_{ij}A_{k\ell} = A_{k\ell}A_{ij}$, the latter following from (2.1). Consequently, $2K_4 = 0$, which proves the vanishing of K_4.

Before we prove that $K_3 = 0$, we record the following well-known lemma (sometimes called Arnold's lemma):

Lemma XIX.2.2. *For any triple (i, j, k) of distinct integers, we have*

$$u_{ij} \wedge u_{jk} + u_{jk} \wedge u_{ki} + u_{ki} \wedge u_{ij} = 0.$$

PROOF. Let i, j, k be distinct indices. Then

$$u_{ij} \wedge u_{jk} + u_{jk} \wedge u_{ki} + u_{ki} \wedge u_{ij} = \oint \frac{(dz_i - dz_j) \wedge (dz_j - dz_k)}{(z_i - z_j)(z_j - z_k)}$$

where the symbol $\oint$ means that we take the sum of the term under the integral with the other two obtained by circular permutations of the indices. We have

$$\begin{aligned} \text{l.h.s} &= \oint \frac{dz_i \wedge dz_j + dz_j \wedge dz_k + dz_k \wedge dz_i}{(z_i - z_j)(z_j - z_k)} \\ &= \Big(\oint \frac{1}{(z_i - z_j)(z_j - z_k)}\Big)(dz_i \wedge dz_j + dz_j \wedge dz_k + dz_k \wedge dz_i) \\ &= \Big(\oint (z_k - z_i)\Big) \frac{dz_i \wedge dz_j + dz_j \wedge dz_k + dz_k \wedge dz_i}{(z_i - z_j)(z_j - z_k)(z_k - z_i)} = 0. \end{aligned}$$

□

Let us resume the proof of Proposition 2.1. We still have to prove that $K_3 = 0$. We break the sum K_3 into three smaller pieces $K_3 = K_5 + K_6 + K_7$. The first piece is

$$K_5 = \sum_{i<j\neq k} A_{ij}A_{ik}\, u_{ij} \wedge u_{ik}.$$

Exchanging j and k, we get

$$K_5 = \sum_{i<j<k} [A_{ij}, A_{ik}]\, u_{ij} \wedge u_{ik}.$$

Similarly,

$$K_6 = \sum_{i\neq j<k} A_{ik}A_{jk}\, u_{ik} \wedge u_{jk} = \sum_{i<j<k} [A_{ik}, A_{jk}]\, u_{ik} \wedge u_{jk}.$$

The last piece is

$$\begin{aligned} K_7 &= \sum_{i<j<k} \Big(A_{ij}A_{jk}\, u_{ij} \wedge u_{jk} + A_{jk}A_{ij}\, u_{jk} \wedge u_{ij}\Big) \\ &= \sum_{i<j<k} [A_{ij}, A_{jk}]\, u_{ij} \wedge u_{jk}. \end{aligned}$$

Therefore $K_3 = \sum_{i<j<k} Z_{ijk}$ where

$$Z_{ijk} = [A_{ij}, A_{ik}]\, u_{ij} \wedge u_{ik} + [A_{ik}, A_{jk}]\, u_{ik} \wedge u_{jk} + [A_{ij}, A_{jk}]\, u_{ij} \wedge u_{jk}.$$

Using Lemma 2.2, we get

$$\begin{aligned} Z_{ijk} &= [A_{ij}, A_{ik}]\, u_{ij} \wedge u_{ik} + [A_{ik}, A_{jk}]\, u_{ik} \wedge u_{jk} \\ &\qquad + [A_{ij}, A_{jk}]\, (u_{ik} \wedge u_{jk} + u_{ij} \wedge u_{ik}) \\ &= [A_{ij}, A_{ik} + A_{jk}]\, u_{ij} \wedge u_{ik} + [A_{ij} + A_{ik}, A_{jk}]\, u_{ik} \wedge u_{jk} \\ &= 0 \end{aligned}$$

by (2.2). This implies the vanishing of K_3 and completes the proof of Proposition 2.1. □

Since the connection associated to the differential system (2.3) is flat, there exists a monodromy representation of the fundamental group of Y_n on the vector space W.

Remark 2.3. It can be proved that the fundamental group of Y_n is the pure braid group P_n defined as the kernel of the natural surjection of the braid group B_n onto the symmetric group S_n. Let $\mathfrak{p}_n$ be the Lie algebra generated by a set $\{X_{ij}\}_{1 \leq i < j \leq n}$ of generators and Relations (2.1–2.2). A $\mathfrak{p}_n$-module W is nothing but a vector space W with a family $(A_{ij})_{1 \leq i < j \leq n}$ of endomorphisms satisfying (2.1–2.2). For any such module, the connection corresponding to the differential system (2.3) is flat, therefore inducing a monodromy representation of the group P_n. It thus makes sense to view the Lie algebra $\mathfrak{p}_n$ as the analogue of the Lie algebra of a Lie group for the pure braid group P_n and monodromy as the analogue of integrating a representation from the Lie algebra to the Lie group. For these reasons, Relations (2.1–2.2) are sometimes called the *infinitesimal braid group relations.* For more details on the relationship between P_n and $\mathfrak{p}_n$, see [Aom78] [Hai86] [Koh85].

What we actually would like to have is a monodromy representation of the full braid group B_n, not only of the subgroup P_n. This can be achieved as follows. Suppose we have a left action of the symmetric group S_n on the vector space W. Then there exists a right action of S_n on the trivial vector bundle $Y_n \times W$ given by

$$(z_1, \ldots, z_n; w)\sigma = (z_{\sigma(1)}, \ldots, z_{\sigma(n)}; \sigma^{-1} w) \tag{2.6}$$

for $\sigma \in S_n$, $(z_1, \ldots, z_n) \in Y_n$, and $w \in W$. The composition

$$Y_n \times W \xrightarrow{p} Y_n \to X_n = Y_n / S_n$$

factors through the quotient space $E = (Y_n \times W)/S_n$ of S_n-orbits on $Y_n \times W$. The topological space E thus becomes a non-trivial vector bundle over X_n with fibre W. In the space E we have

$$(z_1, \ldots, z_n; \sigma w) = (z_{\sigma(1)}, \ldots, z_{\sigma(n)}; w). \tag{2.7}$$

Now, if the differential system (2.3) is invariant under the action of S_n, then the connection $\nabla = d - \Gamma$ descends to a connection on E. If Relations (2.1–2.2) are satisfied, then it has a monodromy representation on the fundamental group of X_n which, by Proposition X.6.14, is the full braid group B_n.

XIX.3 The Knizhnik-Zamolodchikov Equations

We consider a differential system that is a special case of the systems considered in Section 2 and that depends on the following data:

(i) a finite-dimensional complex Lie algebra $\mathfrak{g}$,

(ii) an invariant symmetric 2-tensor t on $\mathfrak{g}$, i.e., an element $t = \sum_r x_r \otimes y_r$ of $\mathfrak{g} \otimes \mathfrak{g}$ such that

$$t_{21} = t \quad \text{and} \quad [\Delta(x), t] = 0 \tag{3.1}$$

for all $x \in \mathfrak{g}$,

(iii) a complex parameter h,

(iv) an integer $n > 1$, and

(v) a finite-dimensional $\mathfrak{g}$-module V.

Definition XIX.3.1. *The Knizhnik-Zamolodchikov differential system associated to the above data is the system*

$$dw = \frac{h}{2\pi\sqrt{-1}} \sum_{1 \leq i < j \leq n} \frac{t_{ij}}{z_i - z_j}(dz_i - dz_j)w \tag{KZ_n}$$

where $w = w(z_1, \ldots, z_n)$ is a function on Y_n with values in $W = V^{\otimes n}$ and where t_{ij} is the element of $U(\mathfrak{g})^{\otimes n}$ defined for all $i \neq j$ by

$$t_{ij} = \sum_r x_r^{(1)} \otimes \cdots \otimes x_r^{(n)}$$

where $x_r^{(i)} = x_r$, $x_r^{(j)} = y_r$ and $x_r^{(k)} = 1$ otherwise.

Lemma XIX.3.2. *The elements $(t_{ij})_{1 \leq i < j \leq n}$ induce endomorphisms of $V^{\otimes n}$ satisfying Relations* (2.1–2.2).

PROOF. Relations (2.1) hold by definition of t_{ij}. Relations (2.2) follow from the $\mathfrak{g}$-invariance of t. We show this when $i = 1$, $j = 2$, and $k = 3$. We have

$$\begin{aligned} [t_{12}, t_{13} + t_{23}] &= \sum_{r,s} [x_r \otimes y_r \otimes 1, x_s \otimes 1 \otimes y_s + 1 \otimes x_s \otimes y_s] \\ &= \sum_s \Big[\sum_r x_r \otimes y_r, x_s \otimes 1 + 1 \otimes x_s\Big] \otimes y_s \\ &= \sum_s [t, \Delta(x_s)] \otimes y_s = 0 \end{aligned}$$

by (3.1). □

Lemma 3.2 and Proposition 2.1 imply that the system (KZ_n) defines a flat connection on the trivial bundle over Y_n with fibre $V^{\otimes n}$ and, consequently, determines a monodromy representation

$$\rho_n^{\mathrm{KZ}} : \pi_1(Y_n) \to \mathrm{Aut}(V^{\otimes n}).$$

Let us prove that this monodromy can be extended to a representation of the full braid group B_n. As explained at the end of Section 2, we have to specify a left action of the symmetric group S_n on the space $W = V^{\otimes n}$. We choose the action given by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \tag{3.2}$$

for $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$. The 2-tensor t being symmetric by hypothesis, we have $t_{ij} = t_{ji}$ for any pair (i,j) of distinct integers. Consequently, we can rewrite the system (KZ_n) as

$$dw = \frac{h}{4\pi\sqrt{-1}} \sum_{1\leq i\neq j\leq n} \frac{t_{ij}}{z_i - z_j}(dz_i - dz_j)w. \tag{3.3}$$

It is clear from (3.3) that the system is invariant under the action of the symmetric group. We thus obtain a monodromy representation

$$\rho_n^{\mathrm{KZ}} : B_n = \pi_1(X_n, p) \to \mathrm{Aut}(V^{\otimes n}). \tag{3.4}$$

Here we take the point $p = (1, 2, \ldots, n)$ as the base-point of X_n and we identify the fibre of the bundle $E = (Y_n \times V^{\otimes n})/S_n$ at the point p with $V^{\otimes n}$.

The main objective of this chapter is to compute the monodromy representation ρ_n^{KZ} as explicitly as possible from the above data. This is a difficult task. To begin with, let us consider the following special cases.

(a) Case when $h = 0$: the differential system reduces to $dw = 0$ which has constant solutions over Y_n. The corresponding monodromy is the representation of B_n on $V^{\otimes n}$ coming from the action (3.2) of the symmetric group.

(b) Case when $n = 2$: the system (KZ_2) reduces to

$$dw = \frac{h}{2\pi\sqrt{-1}} \frac{t}{z_1 - z_2}(dz_1 - dz_2)w. \tag{3.5}$$

In order to determine its monodromy, we represent the generator σ_1 of the braid group B_2 by the loop $z(s) = (z_1(s), z_2(s))$ where s varies in $[0,1]$ and

$$z_1(s) = \frac{1}{2}(3 - e^{\pi\sqrt{-1}s}) \quad \text{and} \quad z_2(s) = \frac{1}{2}(3 + e^{\pi\sqrt{-1}s}).$$

We have $z(0) = z(1) = (1,2) = (2,1)$ in X_2. Pulling back (3.5) along this loop leads to the ordinary differential equation

$$\frac{dw}{ds} = \frac{ht}{2}w(s) \tag{3.6}$$

whose unique solution is given by

$$w(s) = e^{hts/2} w(0), \tag{3.7}$$

where $e^{hts/2}$ is the classical entire series $e^{hts/2} = \sum_{n \geq 0} \frac{h^n t^n s^n}{2^n n!}$ converging in $\mathrm{Aut}(V \otimes V)$ for all values of the complex parameter h. We get the monodromy action of the generator σ_1 by setting $s = 1$ in (3.7), namely

$$\rho_2^{\mathrm{KZ}}(\sigma_1)(v_1 \otimes v_2) = \tau_{V,V}\Big(e^{ht/2}(v_1 \otimes v_2)\Big) \tag{3.8}$$

for all $v_1, v_2 \in V$. The flip appears in (3.8) as a consequence of the equality

$$(2, 1; v_1 \otimes v_2) = (1, 2; v_2 \otimes v_1)$$

in the non-trivial bundle $E = (Y_2 \times V^{\otimes 2})/S_2$.

(c) Suppose that $n \geq 2$ and that the condition

$$[t_{ij}, t_{k\ell}] = 0 \tag{3.9}$$

holds for all i, j, k, ℓ. We claim that in this case the monodromy action of B_n is given for each of the generators $\sigma_1, \ldots, \sigma_{n-1}$ by

$$\rho_n^{\mathrm{KZ}}(\sigma_i)(v_1 \otimes \cdots \otimes v_n) = \tau^{i,i+1}\Big(e^{ht_{i,i+1}/2}(v_1 \otimes \cdots \otimes v_n)\Big) \tag{3.10}$$

where $\tau^{i,i+1} = \mathrm{id}_{V^{\otimes(i-1)}} \otimes \tau_{V,V} \otimes \mathrm{id}_{V^{\otimes(n-i-1)}}$. To prove the claim, represent the generator σ_i of the braid group by the loop $z(s) = (z_1(s), \ldots, z_n(s))$ where

$$z_i(s) = \frac{1}{2}(2i + 1 - e^{\pi\sqrt{-1}s}), \quad z_{i+1}(s) = \frac{1}{2}(2i + 1 + e^{\pi\sqrt{-1}s}) \tag{3.11}$$

and $z_j(s) = j$ for $j \neq i, i+1$. Then the system obtained by pulling back (KZ_n) along this loop can be solved as in Case (b) because the t_{ij} commute with one another.

We end this section with some observations based on the three special cases we have just considered. Firstly, in all cases above, the monodromy is an analytic function in the complex parameter h viewed as a variable. This holds for any (KZ)-system by the general theory of ordinary differential equations: the differential system depending linearly on h, its solutions are analytic in h. We shall henceforth consider the monodromy as an analytic function in h or rather, as a formal series in the variable h. Denote by $(U(\mathfrak{g})[[h]], \Delta, \varepsilon)$ the trivial topological bialgebra associated to the bialgebra $(U(\mathfrak{g}), \Delta, \varepsilon)$ as in Example 3 of XVI.4. For any $\mathfrak{g}$-module V, put on $V[[h]]$ the $U(\mathfrak{g})[[h]]$-module structure induced from the action of $U(\mathfrak{g})$ on V. Recall that $(V[[h]])^{\widetilde{\otimes} n} = V^{\otimes n}[[h]]$ by Proposition XVI.3.2. We can then express the analytical dependence on h of the monodromy of (KZ_n) by a group morphism

$$\rho_n^{\mathrm{KZ}} : B_n \to \mathrm{Aut}_{\mathbf{C}[[h]]}(V^{\otimes n}[[h]]). \tag{3.12}$$

From Case (a) we see that ρ_n^{KZ} is congruent modulo h to the representation coming from the symmetric group action (3.2).

We next observe that the monodromy is independent of the $\mathfrak{g}$-module V. This again holds in full generality because the system has coefficients in the tensor powers of $U(\mathfrak{g})$. One can prove this using Chen's theory of formal connections and formal monodromy [Che73] [Che75] [Che77a].

The last remark is the following: in Case (c) above, the monodromy can be derived from a *topological braided bialgebra* structure. Indeed, assume that $(U(\mathfrak{g})[[h]], \Delta, \varepsilon)$ is the trivial topological bialgebra as above. Set $R = e^{ht/2}$. When t satisfies Relations (3.9), the element R of $U(\mathfrak{g})[[h]]\widetilde{\otimes}U(\mathfrak{g})[[h]]$ satisfies Relations (XVI.4.15–4.17) with $\Phi = 1\widetilde{\otimes}1\widetilde{\otimes}1$. The proof of this claim is similar to the one used in XVI.5, Example 1. Therefore,

$$A_{\mathfrak{g},t} = (U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi = 1\widetilde{\otimes}1\widetilde{\otimes}1, R = e^{ht/2})$$

is a topological braided bialgebra whose universal R-matrix is symmetric: $R_{21} = R$. The $\mathfrak{g}$-module V extends to the $A_{\mathfrak{g},t}$-module $V[[h]]$ defined above. The universal R-matrix R gives rise to a representation

$$\rho_n^R : B_n \to \mathrm{Aut}_{\mathbf{C}[[h]]}(V^{\otimes n}[[h]])$$

of the braid group B_n following the procedure explained in XV.4. A comparison with (3.10) gives the following.

Proposition XIX.3.3. *When t satisfies* (3.9), *the monodromy of the system* (KZ_n) *coincides with the braid group representation induced by the universal R-matrix $R = e^{ht/2}$ of the topological braided bialgebra $A_{\mathfrak{g},t}$, i.e., we have*

$$\rho_n^{\mathrm{KZ}} = \rho_n^R.$$

In the next section, we shall extend this result to the case of an arbitrary invariant symmetric 2-tensor t in spite of the fact that $A_{\mathfrak{g},t}$ may no longer be a topological braided bialgebra.

XIX.4 The Drinfeld-Kohno Theorem

In addition to the hypotheses of Section 3, we assume that the Lie algebra $\mathfrak{g}$ is semisimple. In this situation, there exists a topological braided bialgebra

$$(U_h(\mathfrak{g}), \Delta_h, \varepsilon_h, \Phi_h = 1\widetilde{\otimes}1\widetilde{\otimes}1, R_h)$$

quantizing the enveloping algebra of $\mathfrak{g}$ (see XVII.2). Any finite-dimensional $\mathfrak{g}$-module V can be extended to a canonical $U_h(\mathfrak{g})$-module $\widetilde{V}$ such that $\widetilde{V}/h\widetilde{V} = V$. Indeed, if $V = V_\lambda$ is the simple $\mathfrak{g}$-module associated to a dominant weight λ, then we define $\widetilde{V_\lambda}$ as the $U_h(\mathfrak{g})$-module whose existence is asserted in XVII.2. If $V = \bigoplus_\lambda V_\lambda$ is a direct sum of simple $\mathfrak{g}$-modules, we define $\widetilde{V}$ to be $\widetilde{V} = \bigoplus_\lambda \widetilde{V_\lambda}$ (see also XVIII.4). The universal R-matrix R_h induces a morphism of groups

$$\rho_n^{R_h} : B_n \to \mathrm{Aut}_{\mathbf{C}[[h]]}(\widetilde{V}^{\widetilde{\otimes} n}) = \mathrm{Aut}_{\mathbf{C}[[h]]}(V^{\otimes n}[[h]]) \tag{4.1}$$

into the automorphism group of $V^{\otimes n}$ with formal series coefficients. On the other hand, the monodromy of the system (KZ_n) is a morphism of groups

$$\rho_n^{\mathrm{KZ}} : B_n \to \mathrm{Aut}_{\mathbf{C}[[h]]}(V^{\otimes n}[[h]]). \tag{4.2}$$

We now state the Drinfeld-Kohno theorem.

Theorem XIX.4.1. *If $\mathfrak{g}$ is a semisimple Lie algebra and $t \in \mathfrak{g} \otimes \mathfrak{g}$ is the invariant symmetric 2-tensor given by* (XVII.1.6), *then the braid group representations ρ_n^{KZ} and $\rho_n^{R_h}$ are equivalent for any $n > 1$ and any $\mathfrak{g}$-module V. In other words, there exists a $\mathbf{C}[[h]]$-linear automorphism u of $V^{\otimes n}[[h]]$ such that*

$$\rho_n^{\mathrm{KZ}}(g) = u\rho_n^{R_h}(g)u^{-1}$$

for all elements g of the braid group B_n.

The rest of this chapter is devoted to the proof of this important theorem which expresses a geometrical problem in terms of quantum groups and produces an explicit expression for the monodromy of the Knizhnik-Zamolodchikov equations in the semisimple case.

Drinfeld's proof of Theorem 4.1 relies on two main ideas. The first one is to consider the trivial topological bialgebra $(U(\mathfrak{g})[[h]], \Delta, \varepsilon)$ equipped with the invertible element $R_{\mathrm{KZ}} = e^{ht/2} \in U(\mathfrak{g})^{\otimes 2}[[h]]$ as at the end of Section 3. In general, R_{KZ} does not induce a topological braided bialgebra structure on $U(\mathfrak{g})[[h]]$. Nevertheless, it does induce the structure of a topological braided quasi-bialgebra, making $U(\mathfrak{g})[[h]]$ into a quantum enveloping algebra. Moreover, this structure contains all the information on the monodromy of all (KZ_n)-systems. These assertions are summarized in the following theorem which generalizes Proposition 3.3.

Theorem XIX.4.2. *For any complex Lie algebra and any element t of $\mathfrak{g} \otimes \mathfrak{g}$ satisfying Conditions* (3.1), *there exists an element*

$$\Phi_{\mathrm{KZ}} \in U(\mathfrak{g})^{\otimes 3}[[h]] = \Big(U(\mathfrak{g})[[h]]\Big)^{\widetilde{\otimes} 3}$$

with $\Phi_{\mathrm{KZ}} \equiv 1 \otimes 1 \otimes 1 \bmod h$ such that

(i) *the topological (quasi-)bialgebra*

$$A_{\mathfrak{g},t} = \Big(U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi_{\mathrm{KZ}}, R_{\mathrm{KZ}} = e^{ht/2}\Big)$$

is a quantum enveloping algebra, and

(ii) *for any integer $n > 1$ and any $\mathfrak{g}$-module V, the monodromy representation ρ_n^{KZ} of B_n on $V[[h]]^{\widetilde{\otimes} n}$ coincides with the braid group representation $\rho_n^{R_{\mathrm{KZ}}}$ induced by the universal R-matrix $R_{\mathrm{KZ}} = e^{ht/2}$ as in* XV.4.

Since $(R_{\mathrm{KZ}})_{21}R_{\mathrm{KZ}} = e^{ht} \equiv 1\otimes 1 + ht \bmod h$, we see that the canonical 2-tensor of $A_{\mathfrak{g},t}$ coincides with t. Thus, $A_{\mathfrak{g},t}$ provides a solution to the quantization of $(\mathfrak{g}, t)$. Notice that this solution is obtained without deforming the multiplication or the comultiplication of the enveloping algebra. This is the first major property of the QUE $A_{\mathfrak{g},t}$.

The second one is expressed in Part (ii) of Theorem 4.2: loosely speaking, it means that the algebra $A_{\mathfrak{g},t}$ is universal for the monodromy of all Knizhnik-Zamolodchikov differential systems. Observe also that the solution of (KZ_2) as given by Formula (3.8) forces the universal R-matrix of $A_{\mathfrak{g},t}$ to be equal to $e^{ht/2}$.

Drinfeld's second idea for the proof of Theorem 4.1 can be expressed as follows.

Theorem XIX.4.3. *If, furthermore, the Lie algebra $\mathfrak{g}$ is semisimple and t is the 2-tensor considered in Theorem 4.1, then there exist a gauge transformation $F \in A_{\mathfrak{g},t}\widetilde{\otimes}A_{\mathfrak{g},t}$ and a $\mathbf{C}[[h]]$-linear isomorphism*

$$\alpha : U_h(\mathfrak{g}) \to \left(A_{\mathfrak{g},t}\right)_F$$

of topological braided quasi-bialgebras.

As a consequence of Theorem 4.3 and of Theorem XV.3.9, we get the following important categorical interpretation of Drinfeld-Jimbo's algebras. A more precise statement can be found in Corollary XX.6.2.

Corollary XIX.4.4. *The tensor functor $(\alpha^*, \mathrm{id}, \varphi_2^F)$ is a braided tensor equivalence from a braided tensor category of topologically free $U(\mathfrak{g})[[h]]$-modules equipped with associativity constraint induced by Φ_{KZ} and braiding induced by R_{KZ}, to the category $U_h(\mathfrak{g})$-Mod_{fr} of* XVII.3.

Theorem 4.3 will be proved in the next section as a consequence of results of Chapter XVIII. The construction of $A_{\mathfrak{g},t}$ and an indication of the proof of Theorem 4.2 will be given in Sections 7–8. We now prove the Drinfeld-Kohno theorem.

Proof of Theorem 4.1. It follows immediately from Theorem 4.2, Theorem 4.3 and Theorem XV.4.2. The latter implies, furthermore, that the automorphism u is given by the action of the element F_{12} of $U(\mathfrak{g})^{\otimes n}[[h]]$. $\square$

XIX.5 Equivalence of $U_h(\mathfrak{g})$ and $A_{\mathfrak{g},t}$

We start with a semisimple Lie algebra $\mathfrak{g}$ and the 2-tensor

$$t = \frac{\Delta(C) - 1 \otimes C - C \otimes 1}{2} \tag{5.1}$$

associated to the Casimir element C of $U(\mathfrak{g})$. The aim of this section is to prove Theorem 4.3. We shall do this in three steps.

Step 1. Since $\mathfrak{g}$ is semisimple, we may apply Theorem XVIII.4.1. It gives us a $\mathbf{C}[[h]]$-linear isomorphism of algebras $\alpha : U_h(\mathfrak{g}) \to U(\mathfrak{g})[[h]]$ with $\alpha \equiv \mathrm{id}$ modulo h, sending the Drinfeld-Jimbo QUE to the trivial deformation of $U(\mathfrak{g})$. Using α, we may transfer all structure maps of $U_h(\mathfrak{g})$ to $U(\mathfrak{g})[[h]]$. In particular, define

$$\Delta_h^\alpha = (\alpha \otimes \alpha)\Delta_h \alpha^{-1} \quad \text{and} \quad \varepsilon_h^\alpha = \varepsilon_h \alpha^{-1}. \tag{5.2}$$

The map α becomes an isomorphism of topological braided bialgebras from $U_h(\mathfrak{g})$ to

$$(U(\mathfrak{g})[[h]], \Delta_h^\alpha, \varepsilon_h^\alpha, (\alpha \otimes \alpha)(R_h)).$$

The maps Δ_h^α and Δ are algebra morphisms, both congruent to Δ modulo h. Recall from V.2 that $U(\mathfrak{g}) \otimes U(\mathfrak{g}) = U(\mathfrak{g} \times \mathfrak{g})$. Now apply Theorem XVIII.2.1 with $\mathfrak{g}' = \mathfrak{g} \times \mathfrak{g}$. We get an invertible element F' in $(U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$ such that $F' \equiv 1 \otimes 1$ modulo h and

$$\Delta_h^\alpha(x) = F'^{-1}\Delta(x)F' \tag{5.3}$$

for all $x \in U(\mathfrak{g})[[h]]$.

Lemma XIX.5.1. *We have $\varepsilon_h^\alpha = \varepsilon$.*

PROOF. Since ε_h is a counit for Δ_h, it follows that ε_h^α is a counit for Δ_h^α. Therefore

$$\mathrm{id} = (\varepsilon_h^\alpha \otimes \mathrm{id})\Delta_h^\alpha = (\varepsilon_h^\alpha \otimes \mathrm{id})(F'^{-1}\Delta F'),$$

which means that

$$(\varepsilon_h^\alpha \otimes \mathrm{id})\Delta(x) = \ell x \ell^{-1} \tag{5.4}$$

where $\ell = (\varepsilon_h^\alpha \otimes \mathrm{id})(F')$. Using Sweedler's sigma notation, we get

$$\begin{aligned} \varepsilon_h^\alpha(x) &= \varepsilon_h^\alpha\Big(\sum_{(x)} x'\varepsilon(x'')\Big) \\ &= \varepsilon\Big(\sum_{(x)} \varepsilon_h^\alpha(x')x''\Big) \\ &= \varepsilon(\ell x \ell^{-1}) \\ &= \varepsilon(\ell)\varepsilon(x)\varepsilon(\ell)^{-1} \\ &= \varepsilon(x) \end{aligned}$$

by the counit axiom and (5.4). □

We have proved so far that the QUE $(U_h(\mathfrak{g}), \Delta_h, \varepsilon_h, 1 \otimes 1 \otimes 1, R_h)$ is isomorphic to the topological braided quasi-bialgebra

$$(U(\mathfrak{g})[[h]], F'^{-1}\Delta F', \varepsilon, 1 \otimes 1 \otimes 1, (\alpha \otimes \alpha)(R_h)). \tag{5.5}$$

Using F' as a gauge transform, we get the following, which concludes Step 1.

Proposition XIX.5.2. *The QUE $(U_h(\mathfrak{g}), \Delta_h, \varepsilon_h, 1 \otimes 1 \otimes 1, R_h)$ is isomorphic to the gauge transform by F' of the topological braided quasi-bialgebra $(U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi, R)$ where*

$$\Phi = (\mathrm{id} \otimes \Delta)(F')(1 \otimes F')(F'^{-1} \otimes 1)(\Delta \otimes \mathrm{id})(F'^{-1})$$

and

$$R = F'_{21}(\alpha \otimes \alpha)(R_h)F'^{-1}.$$

Observe that since Δ is coassociative, Φ has to be $\mathfrak{g}$-invariant, i.e.,

$$[(\Delta \otimes \mathrm{id})\Delta(x), \Phi] = 0$$

for all $x \in \mathfrak{g}$. Similarly, R is $\mathfrak{g}$-invariant since Δ is cocommutative.

Step 2. We apply the symmetrization procedure of XVI.6. By Proposition XVI.6.2, there exists a gauge transformation F'' on $U(\mathfrak{g})^{\otimes 2}[[h]]$ such that $[\Delta(x), F''] = 0$ for all $x \in \mathfrak{g}$ and such that, if we set $R' = F''_{21} R F''^{-1}$, then $R'_{21} = R'$. We make the following capital claim.

Lemma XIX.5.3. *Under the previous hypotheses, we have $R' = e^{ht/2}$.*

PROOF. By Proposition 5.2 we have

$$\begin{aligned} R'^2 &= R'_{21} R' \\ &= F''F'(\alpha \otimes \alpha)((R_h)_{21})F'^{-1}_{21} F''^{-1}_{21} F''_{21} F'_{21}(\alpha \otimes \alpha)(R_h)F'^{-1}F''^{-1} \\ &= F''F'(\alpha \otimes \alpha)((R_h)_{21}R_h)F'^{-1}F''^{-1}. \end{aligned}$$

Now by Proposition XVII.3.2 and by Relations (5.2–5.3) and (XVII.3.6) we have

$$\begin{aligned} R'^2 &= F''F'(\alpha \otimes \alpha)\Big(\Delta_h(e^{hC_h/2})(e^{-hC_h/2} \otimes e^{-hC_h/2})\Big)F'^{-1}F''^{-1} \\ &= F''F'\Delta_h^{\alpha}(e^{h\alpha(C_h)/2})\Big(e^{-h\alpha(C_h)/2} \otimes e^{-h\alpha(C_h)/2}\Big)F'^{-1}F''^{-1} \\ &= F''F'F'^{-1}\Delta(e^{hC/2})F'(e^{-hC/2} \otimes e^{-hC/2})F'^{-1}F''^{-1} \\ &= F''\Delta(e^{hC/2})F''^{-1}(e^{-hC/2} \otimes e^{-hC/2}) \\ &= \Delta(e^{hC/2})(e^{-hC/2} \otimes e^{-hC/2}) \end{aligned}$$

since C is central and F'' is $\mathfrak{g}$-invariant. Finally, in view of the relationship between the Casimir element and the 2-tensor t, and of the centrality of C, we get

$$\begin{aligned} R'^2 &= e^{h\Delta(C)/2}(e^{-hC/2} \otimes e^{-hC/2}) \\ &= e^{h(\Delta(C) - 1\otimes C - C\otimes 1)/2} \\ &= e^{ht}. \end{aligned}$$

Since $R' \equiv 1 \otimes 1 \bmod h$, it follows that $R' = e^{ht/2}$. □

Remark 5.4. The only property peculiar to $U_h(\mathfrak{g})$ used so far in this proof is the one stated in Proposition XVII.3.2. Drinfeld [Dri90] actually proves more: if A is *any* QUE for a complex semisimple Lie algebra $\mathfrak{g}$, then A is necessarily isomorphic to the gauge transform of a topological braided quasi-bialgebra of the form $(U\mathfrak{g}[[h]], \Delta, \varepsilon, \Phi, R)$ where $R = R_{21} = e^{h\theta/2}$ for some invariant symmetric 2-tensor θ on $\mathfrak{g}$. The trivial deformation $(U\mathfrak{g}[[h]], \Delta, \varepsilon, 1\tilde{\otimes}1\tilde{\otimes}1, 1\tilde{\otimes}1)$ corresponds to $\theta = 0$. This concludes Step 2.

Step 3. Summing up Steps 1 and 2, we see that Drinfeld and Jimbo's QUE $U_h(\mathfrak{g})$ is isomorphic as a topological braided bialgebra to the gauge transform of a topological braided quasi-bialgebra of the form

$$\Big(U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi', R' = e^{ht/2}\Big). \tag{5.6}$$

Now the QUE $A_{\mathfrak{g},t}$ of Theorem 4.2 is of the same form, except that Φ' may differ from the element Φ_{KZ} of $A_{\mathfrak{g},t}$. This discrepancy is taken care of by Theorem XVIII.8.1 which implies the existence of a gauge transformation F''' on $U(\mathfrak{g})^{\otimes 2}[[h]]$ such that

$$\Big(U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi', R' = e^{ht/2}\Big) = \Big(A_{\mathfrak{g},t}\Big)_{F'''}. \tag{5.7}$$

Setting $F = F'(F'')^{-1}F'''$, we obtain an isomorphism between $U_h(\mathfrak{g})$ and $(A_{\mathfrak{g},t})_F$, which proves Theorem 4.3.

XIX.6 Drinfeld's Associator

In order to construct the element Φ_{KZ} of Theorem 4.2, we investigate the linear differential equation

$$G'(z) = \frac{h}{2\pi\sqrt{-1}}\Big(\frac{A}{z} + \frac{B}{z-1}\Big)G(z) \tag{6.1}$$

where $G(z)$ is a formal series in two non-commuting variables A and B with coefficients which are analytic functions in the complex variable z. As above, h is a formal parameter.

Equation (6.1) has a singularity at 0 and 1. Changing z into $1/z$ shows that it also has a singularity at ∞. These singularities are regular or Fuchsian. The theory of such equations is classical (see [Was87]).

Let $\mathbf{C}'$ be the simply-connected, connected complement of the union of the real half-lines $]-\infty, 0]$ and $[1, +\infty[$ in the complex plane. By the fundamental theorem of linear differential equations, Equation (6.1) has a unique analytical solution on $\mathbf{C}'$ with specified value at any given point in $\mathbf{C}'$. Since the equation depends linearly on the parameter h, the solutions have an analytic dependence on h. They can thus be considered as formal series in h. Observe that, when $h = 0$, the equation reduces to $G'(z) = 0$ whose solutions are constant.

We first examine the asymptotic behaviour of the solutions of Equation (6.1) at the singularities 0 and 1. Set $\bar{h} = \frac{h}{2\pi\sqrt{-1}}$.

Proposition XIX.6.1. *There exist unique solutions G_0 and G_1 of Equation* (6.1) *such that*

$$G_0(z) \sim_{z\to 0} z^{\bar{h}A} \quad \textit{and} \quad G_1(z) \sim_{z\to 1} (1-z)^{\bar{h}B}.$$

By this we mean that $G_0(z)z^{-\bar{h}A}$ [resp. $G_1(z)(1-z)^{-\bar{h}B}$] has an analytic continuation in a neighbourhood of 0 [resp. of 1] with value at 0 [resp. at 1] equal to 1. Here $z^{\bar{h}A}$ and $(1-z)^{\bar{h}B}$ are well-defined on the simply-connected subspace $\mathbf{C}'$.

PROOF. We shall give the proof for G_0. Let us look for a solution of the form

$$G_0(z) = P(z)z^{\bar{h}A} \tag{6.2}$$

with $P(z) = \sum_{r\geq 0} P_r z^r$. Suppose we can find such a series, that it is convergent, and that $P(0) = P_0 = 1$. Then the function G_0 satisfies the requirements of Proposition 6.1. The uniqueness property follows by uniqueness of the solutions.

Let us now prove that there exists a family $(P_r)_{r\geq 0}$ such that $P_0 = 1$ and $G_0(z)$ is a formal solution of (6.1). We have

$$G_0'(z) = \Big(P'(z) + \bar{h}P(z)\frac{A}{z}\Big)z^{\bar{h}A} = \bar{h}\Big(\frac{A}{z} + \frac{B}{z-1}\Big)P(z)z^{\bar{h}A}.$$

This can be rewritten under the form

$$zP'(z) - \bar{h}[A, P(z)] = -\bar{h}B\frac{zP(z)}{1-z}. \tag{6.3}$$

Expanding (6.3) in powers of z, we get $[A, P_0] = 0$, and, for $r > 0$,

$$rP_r - \bar{h}[A, P_r] = -\bar{h}B(P_0 + \cdots + P_{r-1}). \tag{6.4}$$

Equations (6.4) have a solution. Indeed, take $P_0 = 1$; then P_r is uniquely determined by $P_0, \dots, P_{r-1}$ due to the fact that the operator $r\,\mathrm{id} - \bar{h}\,\mathrm{ad}(A)$ is invertible with inverse equal to

$$\frac{1}{r}\sum_{i\geq 0}\frac{\bar{h}^i}{r^i}\mathrm{ad}(A)^i.$$

The convergence of $P(z)$ results from the general fact that a formal solution of a regular singular equation is necessarily convergent. We refer to [Was87], II.5 for details.

Similarly, one proves that there exists an analytic function $Q(z)$ defined in a neighbourhood of 0 such that

$$G_1(z) = Q(1-z)\,(1-z)^{\bar{h}B} \tag{6.5}$$

when z is close to 1. □

Since G_0 and G_1 are both non-zero solutions of Equation (6.1), they have to differ by an invertible element.

Definition XIX.6.2. *We define $\Phi(A,B)$ by $G_0(z) = G_1(z)\Phi(A,B)$.*

The element $\Phi = \Phi(A,B)$ is by definition an element of the algebra S of formal series in the non-commuting variables A and B with coefficients in $\mathbf{C}[[h]]$ (the variable h commuting with A and B). It is called *Drinfeld's associator*, a term which will be justified by the results of Sections 7–8 and of XX.6.

We end this section by giving an expression for Φ in terms of the iterated integrals and the multiple zeta values defined in the Appendix to this chapter. For any real number a such that $0 < a < 1$, let $G_a(z)$ be the unique solution of Equation (6.1) such that $G_a(a) = 1$. The element Φ is related to the solutions G_a as follows.

Lemma XIX.6.3. *We have*

$$\Phi(A,B) = \lim_{a\to 0} a^{-\bar{h}B} G_a(1-a)\,a^{\bar{h}A}.$$

Proof. Let a be a positive real number sufficiently close to 0 so that $P(a)$ in (6.2) is defined. Since G_0 and G_a are both solutions of Equation (6.1), they differ by a constant which one gets by evaluating both solutions at $z = a$. We have

$$G_a(z) = G_0(z)G_0(a)^{-1} = G_0(z)a^{-\bar{h}A}P(a)^{-1} \tag{6.6}$$

for all z. When z is close to 1, we have, by (6.5),

$$G_1(z) = Q(1-z)\,(1-z)^{\bar{h}B}. \tag{6.7}$$

Setting $z = 1 - a$ and using (6.6–6.7) and Definition 6.2, we get

$$\begin{aligned} a^{-\bar{h}B} G_a(1-a)\, a^{\bar{h}A} &= a^{-\bar{h}B} G_0(1-a)\, a^{-\bar{h}A} P(a)^{-1} a^{\bar{h}A} \\ &= a^{-\bar{h}B} Q(a)\, a^{\bar{h}B}\, \Phi\, a^{-\bar{h}A} P(a)^{-1} a^{\bar{h}A}. \end{aligned}$$

When a tends to 0, then $Q(a)$ and $P(a)$ tend to 1. Consequently, the right-hand side of the last equation tends to Φ. □

Let $b \in]0,1[$. By Picard's method of approximation (see Appendix), the value $G_a(b)$ of the solution G_a can be computed in terms of iterated integrals. More precisely, we have

$$G_a(b) = 1 + \sum_M \Big(\int_a^b \Omega(M) \Big) M \tag{6.8}$$

where M runs over all monomials in A and B and $\int_a^b \Omega(M)$ is the iterated integral obtained by replacing each occurrence A in M by the 1-form $h\Omega_0$ and each occurrence B by $h\Omega_1$ where

$$\Omega_0 = \frac{1}{2\pi\sqrt{-1}} \frac{ds}{s} \quad \text{and} \quad \Omega_1 = \frac{1}{2\pi\sqrt{-1}} \frac{ds}{s-1}.$$

By (11.15) of the Appendix we know that, if the monomial M starts with A and ends with B, i.e., is of the form $A^{p_1} B^{q_1} \ldots A^{p_k} B^{q_k}$, the limit

$$\lim_{a \to 0} \int_a^{1-a} \Omega(M)$$

exists and is equal to

$$\lim_{a \to 0} \int_a^{1-a} h^{p_1 + \cdots + q_k}\, \Omega_0^{p_1} \Omega_1^{q_1} \ldots \Omega_0^{p_k} \Omega_1^{q_k} = h^{p_1 + \cdots + q_k} \tau(p_1, q_1, \ldots, p_k, q_k) \tag{6.9}$$

where the complex numbers $\tau(p_1, q_1, \ldots, p_k, q_k)$ have been defined and computed in the Appendix in terms of multiple zeta values. If the monomial M begins with B or ends with A, then the integral $\int_a^{1-a} \Omega(M)$ diverges as a tends to 0. In order to get rid of such "diverging" monomials, we consider the $\mathbf{C}[[h]]$-submodule $\overline{S}$ of formal series in S spanned by all monomials beginning with A and ending with B. Let $\pi : S \to \overline{S}$ be the projection which is the identity on $\overline{S}$ and sends the "diverging" monomials to 0. Clearly, $\pi(G_a(1-a))$ has a limit Γ in $\overline{S}$ when a tends to 0. By (6.8–6.9) we get the following explicit expression for the limit Γ, namely

$$\Gamma = 1 + \sum_{k \geq 1} \sum_{p_1, q_1, \ldots, p_k, q_k \geq 1} h^{p_1 + \cdots + q_k} \tau(p_1, q_1, \ldots, p_k, q_k)\, A^{p_1} B^{q_1} \ldots A^{p_k} B^{q_k}. \tag{6.10}$$

We now compute Φ in terms of Γ. Consider the algebra $S[\alpha, \beta]$ of polynomials in two commuting variables α and β with coefficients in S. Any monomial in $S[\alpha, \beta]$ can be written uniquely as $\beta^p M \alpha^q$ where M is a monomial in S. Define a $\mathbf{C}[[h]]$-linear map $f' : S[\alpha,\beta] \to S$ by $f'(\beta^p M \alpha^q) = B^p M A^q$. This allows us to build a $\mathbf{C}[[h]]$-linear endomorphism f of S by the formula

$$f(\Gamma(A,B)) = f'(\Gamma(A-\alpha, B-\beta)) \tag{6.11}$$

where $\Gamma(A,B)$ is any element of S. Observe that if M is a "diverging" monomial in S, i.e., starting with B or ending with A, then $f(M) = 0$. Moreover, if M is any monomial of S, then $f(M) = M+N$ where N is a sum of "diverging" monomials killed by f. Therefore $f^2 = f$ is an idempotent endomorphism of S. The following result gives an explicit expression for $\Phi = \Phi(A,B)$ in terms of the multiple zeta values of the Appendix.

Proposition XIX.6.4. *We have*

$$\begin{aligned}\Phi(A,B) &= f(\Gamma)\\ &= 1 + \sum_{k\geq 1} \sum_{p_1,q_1,\ldots,p_k,q_k \geq 1} h^{p_1+\cdots q_k} \tau(p_1,q_1,\ldots,p_k,q_k) f(A^{p_1}B^{q_1}\ldots A^{p_k}B^{q_k}).\end{aligned}$$

Proof. Applying f to both sides of the relation in Lemma 6.3, we get

$$f(\Phi) = f\Big(\lim_{a\to 0} a^{-\bar{h}B} G_a(1-a)\, a^{\bar{h}A}\Big).$$

Since $f(BM) = f(MA) = 0$, we have

$$f(\Phi) = f\Big(\lim_{a\to 0} G_a(1-a)\Big) = f(\Gamma).$$

In order to complete the proof, it suffices to check that $f(\Phi) = \Phi$. Let H_0 [resp. H_1] be obtained from the solution G_0 [resp. from G_1] of (6.1) by replacing A by $A-\alpha$ [resp. by replacing B by $B-\beta$]. Clearly, H_0 and H_1 are solutions of the differential equation

$$G'(z) = \frac{h}{2\pi\sqrt{-1}}\Big(\frac{A-\alpha}{z} + \frac{B-\beta}{z-1}\Big)G(z). \tag{6.12}$$

Moreover, H_0 is asymptotic to $z^{\bar{h}(A-\alpha)}$ in a neighbourhood of 0. Now the function $z^{-\bar{h}\alpha}(1-z)^{-\bar{h}\beta}G_0(z)$ is another solution of Equation (6.12) with the same asymptotic behaviour as H_0. By uniqueness, we get

$$H_0(z) = z^{-\bar{h}\alpha}(1-z)^{-\bar{h}\beta}G_0(z). \tag{6.13}$$

Similarly, one has

$$H_1(z) = z^{-\bar{h}\alpha}(1-z)^{-\bar{h}\beta}G_1(z). \tag{6.14}$$

It follows from (6.13–6.14) that

$$\Phi(A-\alpha, B-\beta) = H_1^{-1}H_0 = G_1^{-1}G_0 = \Phi(A,B).$$

Therefore,

$$f(\Phi(A,B)) = f'(\Phi(A-\alpha, B-\beta)) = f'(\Phi(A,B)) = \Phi(A,B). \qquad \square$$

As a consequence of Proposition 6.4 and of Formulas (11.21–11.23) of the Appendix, we get the following expression for the first terms of $\Phi(A,B)$.

Corollary XIX.6.5. *Modulo h^4 we have*

$$\Phi(A,B) \equiv 1 - \frac{\zeta(2)}{(2\pi\sqrt{-1})^2}[A,B]\,h^2 + \frac{\zeta(3)}{(2\pi\sqrt{-1})^3}\Big([[A,B],B] - [A,[A,B]]\Big)\,h^3.$$

Here ζ is Riemann's zeta function. By Euler's formula $\zeta(2) = \pi^2/6$, we see that the coefficient of h^2 in the expansion of $\Phi(A,B)$ is $\frac{1}{24}[A,B]$.

Remarks 6.6. (a) If $AB = BA$, any monomial $M \neq 1$ in (6.10) can be rewritten in the form of a "diverging" monomial, hence is killed by f. It follows that $\Phi(A,B) = 1$ in this case.

(b) In [Dri90], §2, Drinfeld showed that $\Phi(A,B)$ was the exponential of a Lie series. He obtained the following formula for the logarithm $\ln\Phi$ of $\Phi(A,B)$ modulo $L'' = [[L,L],[L,L]]$ where L is the completion of the free Lie algebra generated by A and B, namely

$$\ln\Phi \equiv \sum_{k,\ell\geq 0} c_{k\ell}\,\mathrm{ad}(B)^{\ell}\mathrm{ad}(A)^{k}[A,B]\,h^{k+\ell+2} \mod L''. \tag{6.15}$$

The complex numbers $c_{k\ell}$ are given by the generating function

$$1 + \sum_{k,\ell\geq 0} c_{k\ell}u^{k+1}v^{\ell+1} = \exp\left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n(2\pi\sqrt{-1})^n}\Big(u^n + v^n - (u+v)^n\Big)\right). \tag{6.16}$$

From (6.16) we get $c_{k\ell} = c_{\ell k}$ and $c_{k0} = c_{0k} = -\zeta(k+2)/(2\pi\sqrt{-1})^{k+2}$ for all $k \geq 0$.

XIX.7 Construction of the Topological Braided Quasi-Bialgebra $A_{\mathfrak{g},t}$

In order to construct $A_{\mathfrak{g},t}$, we have to find an element $\Phi = \Phi_{\mathrm{KZ}}$ in $U(\mathfrak{g})^{\otimes 3}[[h]]$ verifying Relations (XVI.4.10–4.13) and Relations (XVI.4.15–4.17) with $R = R_{\mathrm{KZ}} = e^{ht/2}$. The element Φ_{KZ} we are looking for has also to induce the monodromy representations of the (KZ)-systems. We proceed as in [Dri89b], pp. 1453–1455, [Dri90], Section 2, and [Dri89c].

We observed in Section 3 that the universal R-matrix $R_{\mathrm{KZ}} = e^{ht/2}$ of $A_{\mathfrak{g},t}$ was forced upon us by the monodromy of the system (KZ_2). The element Φ_{KZ} will now come out of the system (KZ_3). In Section 6 we introduced a formal series $\Phi(A, B)$ in two non-commuting variables A and B. We use it to define Φ_{KZ}.

Definition XIX.7.1. *We set* $\Phi_{\mathrm{KZ}} = \Phi(t_{12}, t_{23})$.

We claim that $\Phi_{\mathrm{KZ}} \in U(\mathfrak{g})^{\otimes 3}[[h]]$ satisfies all requirements for

$$A_{\mathfrak{g},t} = \Big(U(\mathfrak{g})[[h]], \Delta, \varepsilon, \Phi_{\mathrm{KZ}}, R_{\mathrm{KZ}} = e^{ht/2}\Big) \tag{7.1}$$

to be a topological braided quasi-bialgebra. The proof of this claim will be sketched in Section 8. We also claim that $A_{\mathfrak{g},t}$ provides the monodromy of all Knizhnik-Zamolodchikov systems. We have already checked this for (KZ_2) in Section 3.

Consider a solution $w(z_1, \ldots, z_n)$ of the system (KZ_n). By definition, it satisfies the system of partial differential equations

$$\frac{\partial w}{\partial z_i} = \bar{h} \sum_{j=1, j\neq i}^{n} \frac{t_{ij}}{z_i - z_j}\, w(z_1, \ldots, z_n) \qquad (i = 1, \ldots, n) \tag{7.2}$$

where $\bar{h} = h/(2\pi\sqrt{-1})$.

Lemma XIX.7.2. *If* $w(z_1, \ldots, z_n)$ *is a solution of* (7.2), *then it also satisfies the relations*

$$\sum_{i=1}^{n} \frac{\partial w}{\partial z_i} = 0 \quad \textit{and} \quad \sum_{i=1}^{n} z_i \frac{\partial w}{\partial z_i} = \sum_{1\leq i<j\leq n} t_{ij}\, w(z_1, \ldots, z_n).$$

Proof. This follows from (7.2) using $t_{ij} = t_{ji}$. □

As a consequence of this lemma, a solution w of (7.2) depends on $n - 2$ variables. In particular, a solution $w(z_1, z_2, z_3)$ of (KZ_3) depends on one variable z. Let us from now on focus on (KZ_3) and make the change of variables

$$w(z_1, z_2, z_3) = (z_3 - z_1)^{\bar{h}(t_{12}+t_{23}+t_{13})} G(z) \tag{7.3}$$

where $z = (z_2 - z_1)/(z_3 - z_1)$.

Lemma XIX.7.3. *With the above notation,* $w(z_1, z_2, z_3)$ *is a solution of* (KZ_3) *if and only if* $G(z)$ *is a solution of the ordinary differential equation*

$$G'(z) = \bar{h}\Big(\frac{t_{12}}{z} + \frac{t_{23}}{z-1}\Big) G(z). \tag{7.4}$$

PROOF. Relation (7.3) and

$$\frac{\partial w}{\partial z_2} = \bar{h}\Big(\frac{t_{12}}{z_2 - z_1} + \frac{t_{23}}{z_2 - z_3}\Big) w(z_1, z_2, z_3)$$

imply that

$$\begin{aligned}(z_3 - z_1)^{\bar{h}(t_{12}+t_{23}+t_{13})} & \frac{G'(z)}{z_3 - z_1} \\ &= \bar{h}\Big(\frac{t_{12}}{z_2 - z_1} + \frac{t_{23}}{z_2 - z_3}\Big)(z_3 - z_1)^{\bar{h}(t_{12}+t_{23}+t_{13})} G(z).\end{aligned}$$

Since $(z_2 - z_3)/(z_3 - z_1) = z - 1$, we get

$$(z_3 - z_1)^{\bar{h}(t_{12}+t_{23}+t_{13})}\Big(G'(z) - \bar{h}\Big(\frac{t_{12}}{z} + \frac{t_{23}}{z-1}\Big)G(z)\Big) = 0.$$

Consequently, $G(z)$ satisfies Equation (7.4). Conversely, one checks easily that, if $G(z)$ is a solution of (7.4), then $w(z_1, z_2, z_3)$ is a solution of (7.2). □

Equation (7.4) has been studied at length in Section 6. Let $G_0(z)$ and $G_1(z)$ be the solutions of (7.4) obtained from the solutions $G_0(z)$ and $G_1(z)$ of (6.1) by replacing A by t_{12} and B by t_{23}. From Proposition 6.1 we get unique solutions of (7.2)

$$W_i(z_1, z_2, z_3) = (z_3 - z_1)^{\bar{h}(t_{12}+t_{23}+t_{13})} G_i(z) \qquad (i = 0, 1) \tag{7.5}$$

whose asymptotic behaviours are given by

$$W_0(z_1, z_2, z_3) \sim (z_2 - z_1)^{\bar{h}t_{12}} (z_3 - z_1)^{\bar{h}(t_{23}+t_{13})} \tag{7.6}$$

when $|z_2 - z_1| \ll |z_3 - z_1|$, i.e., when $|z_2 - z_1|/|z_3 - z_1|$ tends to 0, and

$$W_1(z_1, z_2, z_3) \sim (z_3 - z_2)^{\bar{h}t_{23}} (z_3 - z_1)^{\bar{h}(t_{12}+t_{13})} \tag{7.7}$$

when $|z_2 - z_3| \ll |z_1 - z_3|$. In view of Definitions 6.2 and 7.1, W_0 and W_1 are related by

$$W_0(z_1, z_2, z_3) = W_1(z_1, z_2, z_3)\, \Phi_{\mathrm{KZ}}. \tag{7.8}$$

Let us determine the monodromy of (KZ_3). The change of variables (7.3) has the following property: z_1 is close to z_2 if and only if z is close to 0. Similarly, z_3 is close to z_1 or to z_2 if and only if z is close to ∞ or to 1 respectively. Now consider the generator σ_1 of the braid group B_3 with the parametrization given by (3.11). An immediate computation shows that

$$z(s) = \frac{z_1(s) - z_2(s)}{z_1(s) - z_3(s)} = \frac{2e^{\pi\sqrt{-1}s}}{3 + e^{\pi\sqrt{-1}s}}. \tag{7.9}$$

In particular, $z(0) = 1/2$, $z(1/2) = (1 + 3\sqrt{-1})/5$ and $z(1) = -1$, which shows that under the change of variables, the generator σ_1 corresponds to a counterclockwise half-turn around 0 in the complex plane. Similarly, the generator σ_2 of B_3 corresponds to a counterclockwise half-turn around 1. Choose a base-point in the configuration space X_3 corresponding under the change of variables to a point close to 0 in the complex plane. By definition of the solution $G_0(z)$, it is multiplied by $e^{ht_{12}}$ when z makes a complete positive turn around the singularity 0. Consequently, the value of the monodromy of (KZ_3) on the generator σ_1 is $e^{ht_{12}/2}$. As for σ_2, we first have to move from a neighbourhood of 0 to a neighbourhood of 1 with the help of Φ_{KZ}, then turn around the singularity 1 and come back to a vicinity of 0. This sets the value of the monodromy for σ_2 at $\Phi_{\mathrm{KZ}}^{-1}\, e^{ht_{12}/2}\, \Phi_{\mathrm{KZ}}$. These are the values of the monodromy exactly predicted by Formula (XV.4.2). These considerations prove Part (ii) of Theorem 4.2 when $n = 3$.

We leave the remaining cases $n > 3$ to the reader. Let us only note that pulling back the general system (KZ_n) along the loop σ_i of B_n parametrized by (3.11) leads to the linear differential equation

$$\begin{aligned}
\frac{dw}{ds} &= \frac{\partial w}{\partial z_i}\frac{dz_i}{ds} + \frac{\partial w}{\partial z_{i+1}}\frac{dz_{i+1}}{ds} \\
&= \frac{h}{2}\Bigg(t_{i,i+1} + \sum_{j \neq i,i+1} \Big(\frac{e^{\pi\sqrt{-1}s}}{2(j-i) - 1 + e^{\pi\sqrt{-1}s}}\, t_{ij} \\
&\qquad - \frac{e^{\pi\sqrt{-1}s}}{2(j-i) - 1 - e^{\pi\sqrt{-1}s}}\, t_{i+1,j} \Big) \Bigg) w(s).
\end{aligned}$$

This equation can be solved in terms of iterated integrals using Picard's method of approximation recalled in the Appendix.

XIX.8 Verification of the Axioms

In order to complete the proof of Part (i) of Theorem 4.2, we are left with showing that $A_{\mathfrak{g},t}$ is a topological braided quasi-bialgebra. Set $\Phi = \Phi_{\mathrm{KZ}}$ and $R = R_{\mathrm{KZ}}$. We have to check Relations (XVI.4.10–4.13) and (XVI.4.15–4.17). Let us write down the as yet unproved relations, namely

$$(\mathrm{id} \otimes \Delta)\Delta(a) = \Phi(\Delta \otimes \mathrm{id})\Delta(a)\Phi^{-1} \quad \text{and} \quad \Delta^{\mathrm{op}}(a) = R\Delta(a)R^{-1} \qquad (8.1)$$

for all $a \in A_{\mathfrak{g},t} = U(\mathfrak{g})[[h]]$,

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = 1 \otimes 1, \qquad (8.2)$$

$$(\Delta \otimes \mathrm{id})(R) = \Phi_{312} R_{13} (\Phi_{132})^{-1} R_{23} \Phi, \qquad (8.3)$$

$$(\mathrm{id} \otimes \Delta)(R) = (\Phi_{231})^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1}, \qquad (8.4)$$

and

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)\,(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) = (1 \otimes \Phi)\,(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)\,(\Phi \otimes 1). \quad (8.5)$$

Relations (8.1). Since the comultiplication Δ is coassociative and cocommutative and since the Lie algebra $\mathfrak{g}$ generates $A_{\mathfrak{g},t}$, Relations (8.1) are equivalent to the relations

$$[\Delta^{(2)}(x), \Phi] = 0 \quad \text{and} \quad [\Delta(x), R] = 0 \quad (8.6)$$

for all $x \in \mathfrak{g}$, where $\Delta^{(2)} = (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. By hypothesis, $\Delta(x)$ commutes with the 2-tensor t. By the following special case of Leibniz's formula, we have

$$[\Delta(x), t^n] = [\Delta(x), t^{n-1}]t + t^{n-1}[\Delta(x), t],$$

which implies $[\Delta(x), t^n] = 0$ by induction on n. Consequently,

$$[\Delta(x), R] = [\Delta(x), e^{ht/2}] = \sum_{n \geq 0} \frac{h^n}{2^n n!}\,[\Delta(x), t^n] = 0.$$

This proves the second relation in (8.1).

Let us deal with the first one. We claim that

$$[\Delta^{(2)}(x), t_{12}] = [\Delta^{(2)}(x), t_{23}] = 0 \quad (8.7)$$

for all $x \in \mathfrak{g}$. Indeed, for any element x of $\mathfrak{g}$, we have

$$\Delta^{(2)}(x) = \Delta(x)_{12} + 1 \otimes 1 \otimes x = x \otimes 1 \otimes 1 + \Delta(x)_{23},$$

implying

$$[\Delta^{(2)}(x), t_{12}] = [\Delta(x), t]_{12} + [1 \otimes 1, t] \otimes x = 0.$$

One shows that $\Delta^{(2)}(x)$ commutes with t_{23} in a similar way. Now, by repeated application of the Leibniz rule to (8.7), we see that $\Delta^{(2)}(x)$ commutes with all (non-commutative) monomials in the variables t_{12} and t_{23}. In particular, $\Delta^{(2)}(x)$ commutes with Φ in view of Proposition 6.4 and Definition 7.1. This proves Relations (8.1).

Relation (8.2). The element $t \in \mathfrak{g}$ is annihilated by $\mathrm{id} \otimes \varepsilon$ and $\varepsilon \otimes \mathrm{id}$. Therefore

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(t_{12}) = (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(t_{23}) = 0. \quad (8.8)$$

Since $\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}$ is a $\mathbf{C}[[h]]$-linear morphism of algebras, it kills all non-trivial monomials in t_{12} and t_{23}. Therefore, again by definition of Φ, we have

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(1 \otimes 1 \otimes 1) = 1 \otimes 1.$$

Relation (8.3). We shall be sketchy. For more details, see [Dri89b], Section 3. Recall the solutions W_0 and W_1 of the system (KZ_3) described with their asymptotic behaviour in (7.5–7.7). By permuting z_1, z_2, z_3 we get four other solutions W_2, W_3, W_4, W_5 of (KZ_3), uniquely determined by the following asymptotic behaviour:

$$W_2(z_1, z_2, z_3) \sim (z_2 - z_3)^{\bar{h}t_{23}}(z_2 - z_1)^{\bar{h}(t_{12}+t_{13})} \quad \text{when } |z_3 - z_2| \ll |z_2 - z_1|,$$

$$W_3(z_1, z_2, z_3) \sim (z_3 - z_1)^{\bar{h}t_{13}}(z_2 - z_1)^{\bar{h}(t_{12}+t_{23})} \quad \text{when } |z_3 - z_1| \ll |z_2 - z_1|,$$

$$W_4(z_1, z_2, z_3) \sim (z_1 - z_3)^{\bar{h}t_{13}}(z_2 - z_3)^{\bar{h}(t_{12}+t_{23})} \quad \text{when } |z_3 - z_1| \ll |z_2 - z_3|,$$

$$W_5(z_1, z_2, z_3) \sim (z_2 - z_1)^{\bar{h}t_{12}}(z_2 - z_3)^{\bar{h}(t_{13}+t_{23})} \quad \text{when } |z_1 - z_2| \ll |z_2 - z_3|.$$

We observe that W_2 is obtained from W_1 by exchanging z_2 and z_3. Letting z_3 pass in front of z_2 following the loop σ_2 of the braid group B_3 yields

$$W_1 = W_2\, e^{ht_{23}/2} = W_2 R_{23}. \tag{8.9}$$

We next remark that W_2 and W_3 are solutions of (KZ_3) where t_{12} and t_{13} have been exchanged. Therefore, by definition of Φ, we have

$$W_3 = W_2\, \Phi_{132}. \tag{8.10}$$

Similarly, W_4 has been obtained from W_3 by having z_3 pass in front of z_1. Consequently,

$$W_3 = W_4\, e^{ht_{13}/2} = W_4 R_{13}. \tag{8.11}$$

An argument analogous to the one applied to W_2 and W_3 shows that

$$W_4 = W_5\, \Phi_{312}. \tag{8.12}$$

Relations (7.8) and (8.9–8.12) imply

$$W_0 = W_5\, \Phi_{312} R_{13} (\Phi_{132})^{-1} R_{23} \Phi. \tag{8.13}$$

Finally, W_5 is obtained from W_0 by letting z_3 pass in front of z_1 and of z_2. Hence,

$$W_0 = W_5 e^{h(t_{13}+t_{23})/2} = W_5 (\Delta \otimes \mathrm{id})(R) \tag{8.14}$$

since $t_{13} + t_{23} = (\Delta \otimes \mathrm{id})(t)$. By uniqueness of the solutions, Relations (8.13–8.14) imply Relation (8.3). Figure 8.1 on the next page summarizes the movements of z_1, z_2 and z_3 considered in the previous argument.

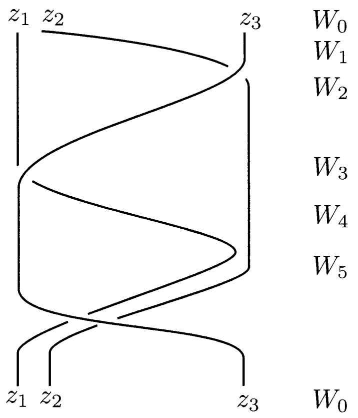


Figure 8.1. The movements of z_1, z_2, *and* z_3

Relation (8.4). One may proceed as for (8.3). An alternate proof consists in first showing that $\Phi_{321} = \Phi^{-1}$, which is done by replacing z by $1 - z$ in Equation (6.1). Then, as in the proof of Lemma XVIII.8.2, apply the involution τ_{13} to Relation (8.3) and use the fact that $\Delta = \Delta^{\mathrm{op}}$ and $R = R_{21}$ to derive Relation (8.4).

Relation (8.5). In order to prove the "pentagonal" relation we now consider the system (KZ_4). The following lemma is due to Drinfeld [Dri90], Section 2, to which the reader is referred for a proof.

Lemma XIX.8.1. *There exist solutions* X_1, X_2, X_3, X_4 *and* X_5 *of* (KZ_4) *uniquely determined by*

$$X_1(z_1, z_2, z_3, z_4) \sim (z_2 - z_1)^{\bar{h}t_{12}}(z_3 - z_1)^{\bar{h}(t_{13}+t_{23})}(z_4 - z_1)^{\bar{h}(t_{14}+t_{24}+t_{34})},$$

$$X_2(z_1, z_2, z_3, z_4) \sim (z_3 - z_2)^{\bar{h}t_{23}}(z_3 - z_1)^{\bar{h}(t_{12}+t_{13})}(z_4 - z_1)^{\bar{h}(t_{14}+t_{24}+t_{34})},$$

$$X_3(z_1, z_2, z_3, z_4) \sim (z_3 - z_2)^{\bar{h}t_{23}}(z_4 - z_2)^{\bar{h}(t_{24}+t_{34})}(z_4 - z_1)^{\bar{h}(t_{12}+t_{13}+t_{14})},$$

$$X_4(z_1, z_2, z_3, z_4) \sim (z_4 - z_3)^{\bar{h}t_{34}}(z_4 - z_2)^{\bar{h}(t_{23}+t_{24})}(z_4 - z_1)^{\bar{h}(t_{12}+t_{13}+t_{14})},$$

and

$$X_5(z_1, z_2, z_3, z_4) \sim (z_2 - z_1)^{\bar{h}t_{12}}(z_4 - z_3)^{\bar{h}t_{34}}(z_4 - z_1)^{\bar{h}(t_{13}+t_{14}+t_{23}+t_{24})}.$$

For X_1 the sign $\sim$ means that there exists an analytic function $f(u, v)$ such that $f(0, 0) = 1$ and

$$X_1(z_1, z_2, z_3, z_4) = f(u, v)(z_2 - z_1)^{\bar{h}t_{12}}(z_3 - z_1)^{\bar{h}(t_{13}+t_{23})}(z_4 - z_1)^{\bar{h}(t_{14}+t_{24}+t_{34})}$$

where $u = (z_2 - z_1)/(z_4 - z_1)$ and $v = (z_3 - z_1)/(z_4 - z_1)$. The reader will be able to give a precise meaning to $\sim$ in the remaining cases.

The "pentagonal" relation (8.5) is an immediate consequence of the following lemma. This completes the proof of Part (i) of Theorem 4.2.

Lemma XIX.8.2. *Under the previous hypotheses, we have*

$$X_1 = X_2(\Phi \otimes 1), \quad X_2 = X_3(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi), \quad X_3 = X_4(1 \otimes \Phi),$$

$$X_4 = X_5(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi^{-1}), \quad X_5 = X_1(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi^{-1}).$$

PROOF. (a) We start with the proof of the first relation $X_1 = X_2(\Phi \otimes 1)$. Set

$$V_1(z_1, z_2, z_3, z_4) = X_1(z_1, z_2, z_3, z_4)(z_4 - z_1)^{-\hbar(t_{14}+t_{24}+t_{34})}$$

and

$$V_2(z_1, z_2, z_3, z_4) = X_2(z_1, z_2, z_3, z_4)(\Phi \otimes 1)(z_4 - z_1)^{-\hbar(t_{14}+t_{24}+t_{34})}.$$

It is enough to prove that $V_1 = V_2$. By Lemma 3.2 we have

$$[t_{12}, t_{14} + t_{24} + t_{34}] = [t_{12}, t_{14} + t_{24}] + [t_{12}, t_{34}] = 0.$$

A similar computation shows that t_{23} commutes with $t_{14}+t_{24}+t_{34}$. Hence, $\Phi\otimes 1$, which is a formal series in t_{12} and in t_{23}, commutes with $t_{14}+t_{24}+t_{34}$. Therefore, V_2 can be rewritten as

$$V_2(z_1, z_2, z_3, z_4) = X_2(z_1, z_2, z_3, z_4)(z_4 - z_1)^{-\hbar(t_{14}+t_{24}+t_{34})}(\Phi \otimes 1).$$

A simple computation shows that V_1 and V_2 both satisfy the following system of partial differential equations:

$$\frac{\partial V}{\partial z_1} = \hbar \sum_{j\neq 1} \frac{t_{1j}}{z_1 - z_j} V(z_1, z_2, z_3, z_4) + \hbar V(z_1, z_2, z_3, z_4) \frac{t_{14} + t_{24} + t_{34}}{z_4 - z_1}, \tag{8.15}$$

$$\frac{\partial V}{\partial z_i} = \hbar \sum_{j\neq i} \frac{t_{ij}}{z_i - z_j} V(z_1, z_2, z_3, z_4) \qquad \text{for } i = 2, 3, \tag{8.16}$$

and

$$\frac{\partial V}{\partial z_4} = \hbar \sum_{j\neq 4} \frac{t_{4j}}{z_4 - z_j} V(z_1, z_2, z_3, z_4) - \hbar V(z_1, z_2, z_3, z_4) \frac{t_{14} + t_{24} + t_{34}}{z_4 - z_1}. \tag{8.17}$$

We set $z_4 = \infty$ in Equations (8.15–8.16) (this is possible since the equations are actually defined on the complex projective line). Then $V_1(z_1, z_2, z_3, \infty)$ and $V_2(z_1, z_2, z_3, \infty)$ become solutions of the system (KZ_3). Moreover, by Lemma 8.1, $V_1(z_1, z_2, z_3, \infty)$ and $V_2(z_1, z_2, z_3, \infty)(\Phi^{-1} \otimes 1)$ have the same asymptotic behaviour as the solutions W_0 and W_1 of (KZ_3) respectively. By uniqueness of these solutions, we get

$$V_1(z_1, z_2, z_3, \infty) = W_0(z_1, z_2, z_3) \otimes 1$$

and

$$V_2(z_1, z_2, z_3, \infty) = (W_1(z_1, z_2, z_3)\Phi) \otimes 1.$$

By definition of Φ, this implies that

$$V_1(z_1, z_2, z_3, \infty) = V_2(z_1, z_2, z_3, \infty) \tag{8.18}$$

for all z_1, z_2, z_3. Relation (8.18) and Equation (8.17) imply that V_1 and V_2 coincide everywhere.

(b) To prove the second relation of Lemma 8.2, it is enough to check that the functions U_1 and U_2 coincide when we set

$$U_1(z_1, z_2, z_3, z_4) = X_2(z_1, z_2, z_3, z_4)\,(z_3 - z_2)^{-\hbar t_{23}}$$

and

$$U_2(z_1, z_2, z_3, z_4) = X_3(z_1, z_2, z_3, z_4)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)\,(z_3 - z_2)^{-\hbar t_{23}}.$$

The element $(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)$ is a formal series in the variables

$$(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(t_{12}) = t_{12} + t_{13} \quad \text{and} \quad (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(t_{23}) = t_{24} + t_{34}.$$

By Lemma 3.2 again, $t_{24} + t_{34}$ commutes with t_{23}. Therefore U_2 can be rewritten as

$$U_2(z_1, z_2, z_3, z_4) = X_3(z_1, z_2, z_3, z_4)(z_3 - z_2)^{-\hbar t_{23}}(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi).$$

Both U_1 and U_2 are solutions of the system

$$\frac{\partial U}{\partial z_i} = \hbar \sum_{j \neq i} \frac{t_{ij}}{z_i - z_j} U(z_1, z_2, z_3, z_4) \qquad \text{for } i = 1, 4, \tag{8.19}$$

$$\frac{\partial U}{\partial z_2} = \hbar \sum_{j \neq 2,3} \frac{t_{2j}}{z_2 - z_j} U(z_1, z_2, z_3, z_4) + \hbar \frac{[t_{23}, U(z_1, z_2, z_3, z_4)]}{z_2 - z_3}, \tag{8.20}$$

and

$$\frac{\partial U}{\partial z_3} = \hbar \sum_{j \neq 2,3} \frac{t_{3j}}{z_3 - z_j} U(z_1, z_2, z_3, z_4) - \hbar \frac{[t_{23}, U(z_1, z_2, z_3, z_4)]}{z_2 - z_3}. \tag{8.21}$$

When $z_2 = z_3$, we claim that

$$U_1(z_1, z_2, z_2, z_4) = U_2(z_1, z_2, z_2, z_4)$$

for all z_1, z_2, z_4. Define $T_i(z_1, z_2, z_4) = U_i(z_1, z_2, z_2, z_4)$ for $i = 1, 2$. Equations (8.19–8.21) imply that T_1 and T_2 are solutions of the system

$$\frac{\partial T}{\partial z_1} = \hbar\Big(\frac{t_{12} + t_{13}}{z_1 - z_2} + \frac{t_{14}}{z_1 - z_4}\Big)T(z_1, z_2, z_4), \tag{8.22}$$

$$\frac{\partial T}{\partial z_2} = \hbar\Big(\frac{t_{12} + t_{13}}{z_2 - z_1} + \frac{t_{24} + t_{34}}{z_2 - z_4}\Big)T(z_1, z_2, z_4), \tag{8.23}$$

and

$$\frac{\partial T(z_1, z_2, z_4)}{\partial z_4} = \bar{h}\Big(\frac{t_{14}}{z_4 - z_1} + \frac{t_{24} + t_{34}}{z_4 - z_2}\Big) T(z_1, z_2, z_4). \tag{8.24}$$

Now,

$$t_{12} + t_{13} = (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(t_{12}), \quad t_{14} = (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(t_{13})$$

and

$$t_{24} + t_{34} = (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(t_{23}).$$

Therefore Equations (8.22–8.24) imply that T_1 and T_2 are solutions of the system (KZ_3) in which the coefficients t_{ij} have been replaced by new coefficients $(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(t_{ij})$. By the results of Section 7, there exist solutions H_0 and H_1 of this modified (KZ_3)-system such that

$$H_0(z_1, z_2, z_4) = H_1(z_1, z_2, z_4)\,(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi) \tag{8.25}$$

with the asymptotic behaviour

$$H_0(z_1, z_2, z_4) \sim (z_2 - z_1)^{\bar{h}(t_{12}+t_{13})}(z_4 - z_1)^{\bar{h}(t_{14}+t_{24}+t_{34})}$$

when $|z_2 - z_1| \ll |z_4 - z_1|$ and

$$H_1(z_1, z_2, z_4) \sim (z_4 - z_2)^{\bar{h}(t_{24}+t_{34})}(z_4 - z_1)^{\bar{h}(t_{12}+t_{13}+t_{14})}$$

when $|z_2 - z_4| \ll |z_1 - z_4|$. It follows from this, from Lemma 8.1, and from the fact that t_{23} commutes with $t_{12} + t_{13}$, $t_{14} + t_{24} + t_{34}$, $t_{24} + t_{34}$ and with $t_{12} + t_{13} + t_{14}$, that T_1 and $T_2(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)^{-1}$ have the same asymptotic behaviours as H_0 and H_1 respectively. Consequently, $T_1 = H_0$ and $T_2(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)^{-1} = H_1$. Combining these relations with (8.25), we conclude that T_1 and T_2 coincide. Therefore,

$$U_1(z_1, z_2, z_2, z_4) = U_2(z_1, z_2, z_2, z_4) \tag{8.26}$$

for all z_1, z_2, z_4. Relation (8.26) and Equation (8.20) imply that the functions U_1 and U_2 coincide everywhere.

(c) The remaining relations of Lemma 8.2 are proved in a similar fashion: in the case of the third relation, we send z_1 to ∞ whereas for the last two, we have to identify z_3 with z_4, and z_1 with z_2, respectively. The movements of z_1, z_2, z_3 and z_4 in this proof can be represented as a system of four particles moving as in Figure 8.2. $\square$

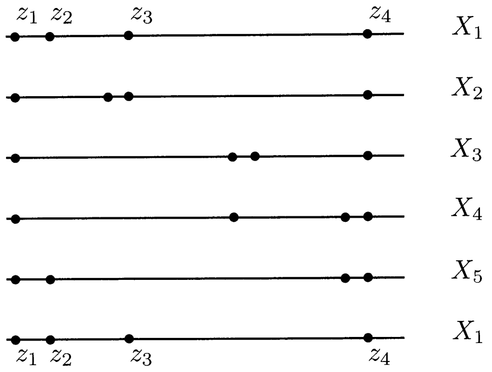


Figure 8.2. Five configurations of four particles

Remark 8.3. Let us consider variables $(t_{ij})_{1\leq i<j\leq 4}$ satisfying the infinitesimal braid group relations (2.1–2.2). Reviewing the proofs of Relations (8.1–8.5), we see that we have actually established the existence of a formal series $\Phi(A, B)$ in two non-commuting variables A and B with constant term 1, belonging to the algebra S of Section 6 and satisfying the three relations

$$\Phi(t_{12}, t_{23}+t_{24})\Phi(t_{13}+t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\Phi(t_{12}+t_{13}, t_{24}+t_{34})\Phi(t_{12}, t_{23}), \tag{8.27}$$

$$e^{\frac{h}{2}(t_{13}+t_{23})} = \Phi(t_{13}, t_{12})e^{\frac{ht_{13}}{2}}\Phi(t_{13}, t_{23})^{-1}e^{\frac{ht_{23}}{2}}\Phi(t_{12}, t_{23}), \tag{8.28}$$

$$e^{\frac{h}{2}(t_{13}+t_{12})} = \Phi(t_{23}, t_{13})^{-1}e^{\frac{ht_{13}}{2}}\Phi(t_{12}, t_{13})e^{\frac{ht_{12}}{2}}\Phi(t_{12}, t_{23})^{-1}. \tag{8.29}$$

Relation (8.27) is the translation of (8.5) while Relations (8.28–8.29) correspond to (8.3–8.4) in view of $R_{ij} = e^{\frac{ht_{ij}}{2}}$ for $1 \leq i < j \leq 3$ and of

$$(\Delta\otimes \mathrm{id})(R) = e^{\frac{h}{2}(t_{13}+t_{23})} \quad \text{and} \quad (\mathrm{id}\otimes\Delta)(R) = e^{\frac{h}{2}(t_{13}+t_{12})}.$$

An element $\Phi(A, B)$ of S, with constant term 1 and satisfying Relations (8.27–8.29), will be called a *Drinfeld series.* Drinfeld's associator Φ_{KZ} is the only explicit Drinfeld series constructed so far. Drinfeld actually established the existence of a Drinfeld series with *rational* coefficients (see [Dri90], Theorems A, A′, A″). It would be interesting to have a description of it, especially in view of the constructions of XX.6–7. In case the elements t_{ij} all come from the invariant symmetric 2-tensor t of a semisimple Lie algebra, Drinfeld also showed in [Dri89b], Theorem 3.15 that Φ was unique up to gauge transformation by a symmetric invariant element F.

XIX.9 Exercises

1. Let $\mathfrak{g}$ be a semisimple Lie algebra and $t \in \mathfrak{g} \otimes \mathfrak{g}$ be the 2-tensor (XVII.1.6). Show that $[t_{12}, t_{13}] \neq 0$.

2. (a) Compute Drinfeld's associator $\Phi(A, B)$ in the case that A and B commute with the commutator $[A, B]$.

   (b) Show that $\Phi(A, B)$ is the exponential of a Lie series (Hint: prove that $\Delta\Phi = \Phi \otimes \Phi$).

3. Prove that the projector f defined by (6.11) coincides with the convolution $\nu_B \star \mathrm{id} \star \nu_A$ where ν_A and ν_B are the algebra endomorphisms of S determined by

$$\nu_A(A) = -A, \quad \nu_A(B) = 0, \quad \nu_B(A) = 0, \quad \nu_B(B) = -B.$$

4. Prove that there exists an analytic function $V(z)$ defined in a neighbourhood of $[0, 1]$ such that $z^{\hbar A}(1-z)^{\hbar B}V(z)$ is a solution of Equation (6.1). Show that $\Phi(A, B) = V(1)V(0)^{-1}$.

5. Let $v_1, \ldots, v_n$ be analytic functions. Consider the differential equation

$$G'(z) = \Big(\sum_{i=1}^{n} A_i u_i\Big) G(z) \tag{9.1}$$

where $A_1, \ldots, A_n$ are non-commuting variables and $u_i = v_i'/v_i$ for $i = 1, \ldots, n$. Set $V(z) = v_n^{-A_n} \ldots v_1^{-A_1} G(z)$ where $G(z)$ is a solution of Equation (9.1). Establish that $V(z)$ is a solution of the equation $V'(z) = Q(z)V(z)$ where

$$Q = \sum_{i=2}^{n} u_i v_n^{-\mathrm{ad}(A_n)} \ldots v_i^{-\mathrm{ad}(A_i)} \Big(v_{i-1}^{-\mathrm{ad}(A_{i-1})} \ldots v_1^{-\mathrm{ad}(A_1)} - 1\Big)(A_i).$$

XIX.10 Notes

The material in Sections 1–2 is standard. For more on the configuration space X_n, see [Aom78] [Hai86] [Koh85].

The equations (KZ_n) were introduced by Knizhnik and Zamolodchikov [KZ84] in connection with the Wess-Zumino-Witten model in conformal field theory.

Theorem 4.1, which is the main result of this chapter, first appeared in [Koh87] [Koh88]. For the proof we followed Drinfeld [Dri89b] [Dri90] [Dri89c] closely. As a matter of fact, most results of Sections 4–8 are due to Drinfeld. There is an exception in Section 6 where Proposition 6.4 is due to Le and Murakami [LM93b]. We also used [LM93a]. As was shown in Section 8, Drinfeld's proof that $A_{\mathfrak{g},t}$ is a braided quasi-bialgebra relies on the asymptotic behaviour of certain solutions of the systems (KZ_3) and (KZ_4). Kapranov [Kap93] discussed the asymptotic zones used by Drinfeld and related them to all possible bracketings of the permutations of a finite set of letters.

In the Appendix we collected several facts on iterated integrals which we found in [Aom78] [Che61] [Che73] [Che75] [Che77a] [Che77b] [Gol80] [Lap53] [Ree58] [Reu93] [Was87] [Zag93].

XIX.11 Appendix. Iterated Integrals

Let $\omega_1, \ldots, \omega_n$ be complex-valued differential 1-forms defined on a real interval $[a, b]$. We have $\omega_i = f_i(s)ds$ where $f_1, \ldots, f_n$ are complex functions. Define the iterated integral $\int_a^b \omega_1 \ldots \omega_n$ inductively by

$$\int_a^b \omega_1 = \int_a^b f_1(s)ds \tag{11.1}$$

and

$$\int_a^b \omega_1 \ldots \omega_n = \int_a^b f_1(s)\Big(\int_a^s \omega_2 \ldots \omega_n\Big)ds \tag{11.2}$$

if $n > 1$. Iterated integrals enjoy the following formal properties:

$$\int_a^b \omega_1 \ldots \omega_n = (-1)^n \int_b^a \omega_n \ldots \omega_1, \tag{11.3}$$

$$\int_a^c \omega_1 \ldots \omega_n = \int_a^b \omega_1 \ldots \omega_n + \sum_{k=1}^{n-1} \int_a^b \omega_1 \ldots \omega_k \int_b^c \omega_{k+1} \ldots \omega_n + \int_b^c \omega_1 \ldots \omega_n \tag{11.4}$$

for $a < b < c$, and

$$\int_a^b \omega_1 \ldots \omega_n \int_a^b \omega_{n+1} \ldots \omega_{n+m} = \sum_\sigma \int_a^b \omega_{\sigma(1)} \ldots \omega_{\sigma(n+m)} \tag{11.5}$$

where σ runs over all (n, m)-shuffles of the symmetric group S_{n+m}.

Iterated integrals occur in the solution of certain linear differential equations. Let us consider an equation of the form

$$\frac{dY}{ds} = A(s)Y(s) \tag{11.6}$$

where $Y(s)$ is a differentiable function defined on the real interval $[a, b]$, with values in the endomorphism ring of some complex vector space and where $A(s)$ is a linear endomorphism for each $s \in [a,b]$. The differential equation (11.6) has a unique solution $Y(s)$ such that its initial value $Y(a)$ is the identity. Picard's method of approximation leads to the following formal expression for $Y(s)$:

$$Y(s) = \mathrm{id} + Q_1(s) + Q_2(s) + \cdots \tag{11.7}$$

where the family $(Q_p)_{p\geq 0}$ is defined inductively by $Q_0 = \mathrm{id}$ and for $p > 0$ by

$$Q_p(s) = \int_a^s A(s_1)Q_{p-1}(s_1)\, ds_1. \tag{11.8}$$

Equivalently, Q_p can be defined as an integral over the real p-simplex

$$\Delta_p(a;s) = \Big\{(s_1,\ldots,s_p) \;\Big|\; s \geq s_1 \geq s_2 \geq \cdots \geq s_p \geq a\Big\}$$

by

$$Q_p(s) = \int_{\Delta_p(a;s)} A(s_1)A(s_2)\ldots A(s_p)\, ds_1 ds_2 \ldots ds_p. \tag{11.9}$$

We now wish to apply Picard's method to the differential equation

$$\frac{dY}{ds} = \sum_{j=1}^{n} \frac{A_j\, Y(s)}{s - a_j} \tag{11.10}$$

where $A_1,\ldots,A_n$ are constant linear endomorphisms and $a_1,\ldots,a_n$ are distinct complex numbers lying outside the real interval $[a,b]$. By (11.1–11.2) and (11.7–11.9) the unique formal solution $Y(s)$ of (11.10) with $Y(a) = \mathrm{id}$ is given by the formal series

$$Y(s) = \mathrm{id} + \sum_{r>0} \sum_{1\leq j_1,\ldots,j_r\leq n} L_a(a_{j_1},\ldots,a_{j_r}|s)\, A_{j_1}\ldots A_{j_r} \tag{11.11}$$

where the complex functions $L_a(a_{j_1},\ldots,a_{j_r}|s)$ are defined as the following iterated integrals

$$L_a(a_{j_1},\ldots,a_{j_r}|s) = \int_a^s \frac{ds}{s - a_{j_1}} \cdots \frac{ds}{s - a_{j_r}}. \tag{11.12}$$

Functions of this kind already appeared in [Poi84], III and were investigated at length by Lappo-Danilevsky in [Lap53], Mémoire II under the name "hyperlogarithms".

We now concentrate on the hyperlogarithms built on the particular 1-forms

$$\Omega_0 = \frac{1}{2\pi\sqrt{-1}}\frac{ds}{s} \quad \text{and} \quad \Omega_1 = \frac{1}{2\pi\sqrt{-1}}\frac{ds}{s-1}. \tag{11.13}$$

Iterated integrals of Ω_0 (or of Ω_1) are easily computed. For instance, by induction on k, we get

$$\int_a^b \Omega_0^k = \frac{1}{(2\pi\sqrt{-1})^k k!}\left(\log\frac{b}{a}\right)^k \tag{11.14a}$$

and

$$\int_a^b \Omega_1^k = \frac{1}{(2\pi\sqrt{-1})^k k!}\left(\log\frac{1-b}{1-a}\right)^k \tag{11.14b}$$

when $0 < a < b < 1$.

Now, as in (11.11–11.12) we wish to consider "mixed" iterated integrals $\int_0^1 \omega_1 \ldots \omega_n$ where each of $\omega_1, \ldots, \omega_n$ may be either Ω_0 or Ω_1. If $\omega_1 = \Omega_1$ or if $\omega_n = \Omega_0$, the integral $\int_0^1 \omega_1 \ldots \omega_n$ does not converge. However, it does in the remaining cases. Set

$$\tau(p_1, q_1, \ldots, p_k, q_k) = \int_0^1 \Omega_0^{p_1}\Omega_1^{q_1} \ldots \Omega_0^{p_k}\Omega_1^{q_k} \tag{11.15}$$

where $p_1, q_1, \ldots, p_k, q_k$ are integers > 0. We shall now compute the iterated integrals (11.15) in terms of series reminiscent of Riemann's zeta function.

To this end, we introduce the convergent series

$$L(i_1, \ldots, i_k; x) = \sum_{0<m_1<\cdots<m_k} \frac{x^{m_k}}{m_1^{i_1} \ldots m_k^{i_k}} \tag{11.16}$$

where $i_1, \ldots, i_k$ are positive integers, x is a real number such that $0 < x < 1$, and $m_1, \ldots, m_k$ run over the set of positive integers. The special case $L(n; x)$ is the n-th polylogarithm which appears in number theory, geometry and algebraic K-theory. When $n = 1$, we have

$$L(1; x) = \sum_{0<m} \frac{x^m}{m} = -\log(1-x) = -2\pi\sqrt{-1}\int_0^1 \Omega_1. \tag{11.17}$$

Taking the derivative of $L(i_1, \ldots, i_{k-1}, i_k; x)$, we get

$$\frac{dL(i_1, \ldots, i_{k-1}, i_k; x)}{dx} = \frac{L(i_1, \ldots, i_{k-1}, i_k - 1; x)}{x}.$$

when $i_k > 1$. Hence,

$$L(i_1, \ldots, i_{k-1}, i_k; x) = \int_0^x \frac{L(i_1, \ldots, i_{k-1}, i_k - 1; s)}{s} ds. \tag{11.18}$$

If $i_k = 1$ we have

$$\frac{dL(i_1, \ldots, i_{k-1}, 1; x)}{dx} = \sum_{0<m_1<\cdots<m_{k-1}<m_k} \frac{x^{m_k - 1}}{m_1^{i_1} \ldots m_{k-1}^{i_{k-1}}}.$$

Set $m = m_k - m_{k-1} - 1 \geq 0$. Then

$$\begin{aligned}\frac{dL(i_1,\ldots,i_{k-1},1;x)}{dx} &= \sum_{m\geq 0} x^m \sum_{0<m_1<\cdots<m_{k-1}} \frac{x^{m_{k-1}}}{m_1^{i_1}\ldots m_{k-1}^{i_{k-1}}} \\ &= \frac{L(i_1,\ldots,i_{k-1};x)}{1-x}.\end{aligned}$$

It follows that

$$L(i_1,\ldots,i_{k-1},1;x) = \int_0^x \frac{L(i_1,\ldots,i_{k-1};s)}{1-s}ds. \tag{11.19}$$

We define the *multiple zeta value* $\zeta(i_1,\ldots,i_k)$ by

$$\zeta(i_1,\ldots,i_k) = L(i_1,\ldots,i_k;1) = \sum_{0<m_1<\cdots<m_k} \frac{1}{m_1^{i_1}\ldots m_k^{i_k}}. \tag{11.20}$$

The special case $\zeta(i_1)$ coincides with the value of Riemann's zeta function at the positive integer i_1. An easy induction using (11.15–11.20) expresses the mixed iterated integrals $\tau(p_1,q_1,\ldots,p_k,q_k)$ in terms of multiple zeta values. To be precise, we get

$$\begin{aligned}\tau(p_1,q_1,\ldots,p_k,q_k) &= \frac{(-1)^{q_1+\cdots+q_k}}{(2\pi\sqrt{-1})^{p_1+q_1+\cdots+p_k+q_k}} \\ &\times \zeta(1,\ldots,1,p_k+1,1,\ldots,1,p_{k-1}+1,\ldots,1,\ldots,1,p_1+1)\end{aligned} \tag{11.21}$$

where the first set of 1's is of length $q_k - 1$, the second one of length $q_{k-1} - 1$, ... and the last one is $q_1 - 1$ long. In particular, if $q_1 = \cdots = q_k = 1$, we get an expression for the multiple zeta values in terms of the mixed iterated integrals (11.15), namely

$$\zeta(i_1,\ldots,i_k) = (-1)^k(2\pi\sqrt{-1})^{i_1+\cdots+i_k}\,\tau(i_k-1,1,i_{k-1}-1,1,\ldots,i_1-1,1). \tag{11.22}$$

As a consequence of (11.3), we get the inversion formula

$$\tau(p_1,q_1,\ldots,p_k,q_k) = (-1)^{p_1+q_1+\cdots+p_k+q_k}\,\tau(q_k,p_k,\ldots,q_1,p_1). \tag{11.23}$$

Relations (11.21–11.23) imply $\zeta(1,2) = \zeta(3)$. We used the last equality to derive Corollary 6.5.

Chapter XX

Postlude. A Universal Knot Invariant

In Section 1 we present the concept of a knot invariant of finite type and prove that all quantum group invariants are of finite type. Then we construct a universal knot invariant $Z(K)$ of finite type, with values in a commutative algebra built on pairs of points on a circle. We also show that the quantum group invariants of XVII.3 can be recovered from $Z(K)$ in a simple combinatorial way. The proof of this fact, as well as the construction of $Z(K)$, use the formalism of the KZ-equations and Drinfeld's results stated in XIX.4.

These new, fascinating developments have now reached a state of clarity and simplicity which allows us to conclude this book with a brief account.

XX.1 Knot Invariants of Finite Type

We start with singular knots. Consider an immersion f of the circle S^1 into the 3-dimensional oriented Euclidean space $\mathbf{R}^3$. Assume that for any m in the image of f, the cardinality of $f^{-1}(m)$ is 1 or 2. If it is 1, the point m will be called an ordinary point; if it is 2, the point m will be called a *double point*. We restrict to immersions with finitely many double points, and such that locally at any double point both branches meet transversally. We also assume that the image of f comes with an orientation. These conditions define a *singular knot*. If the singular knot is equipped with a framing (as defined in X.8), we say that the singular knot is framed. Singular links and framed singular links are defined as immersions of a finite number of circles with similar restrictions on the singularities. Singular knots and links are represented by planar singular knot and singular link diagrams defined in the same way as ordinary knot and link diagrams are (see X.3). One also has an obvious notion of isotopy generalizing the one introduced in X.1.

Now any double point in a singular link diagram can be "desingularized" by locally replacing the pattern X formed by the double point and the two downwards oriented branches, by the patterns X_+ and X_- described in X.4. This observation allows us to extend any isotopy invariant of links to any singular link. Indeed, let P be such an invariant with values in some complex vector space V. Then the rule

$$P(L) = P(L_+) - P(L_-) \tag{1.1}$$

defines the invariant P on the set of isotopy classes of all singular links with one double point. Here L is a link diagram with one double point and the ordinary link diagrams L_+ and L_- are obtained from L by replacing a neighbourhood X of the double point by X_+ and X_-, respectively. By induction on the number of double points we may extend P to an isotopy invariant for all singular links (with a finite number of double points).

Definition XX.1.1. *Let m be a non-negative integer. An isotopy invariant of oriented links is an invariant of degree $\leq m$ if it vanishes on all singular links with more than m double points.*

There is a similar definition for framed links. Observe that an invariant P is of degree 0 if and only if we have $P(L_+) = P(L_-)$ on all link diagrams, which means that the invariant P does not distinguish between undercrossings and overcrossings. Therefore it depends only on the number of connected components of the link. In particular, P is constant on the space of all ordinary knots.

The first question we wish to address is the following: Are there any non-trivial examples of finite-degree invariants of higher degree? Before we answer this question, let us say that a (framed) link isotopy invariant $P(L) = \sum_{m\geq 0} P_m(L)\, h^m$ with values in $V[[h]]$, where V is a complex vector space, is of *finite type* if, for all $m \geq 0$, the V-valued invariant P_m is of degree $\leq m$. We now state the main source of invariants of finite type.

Proposition XX.1.2. *Let P be a link isotopy invariant with values in $V[[h]]$ where V is a complex vector space. If, for any link L, we have $P(L_+) \equiv P(L_-)$ modulo h at any crossing point of a link diagram of L, then P is of finite type.*

PROOF. Define V-valued invariants P_m by $P(L) = \sum_{m\geq 0} P_m(L)\, h^m$. We have to show that each P_m is of degree $\leq m$. Let L be a singular link diagram with one double point. By Relation (1.1) and by the assumption on P, the series $P(L)$ is divisible by h. An easy induction on m shows that $P(L)$ is divisible by h^{m+1} whenever L is a singular link with $m+1$ (or more) double points. Consequently, the coefficient $P_m(L)$ is zero on such a singular link. $\square$

As a consequence of Proposition 1.2, all quantum group invariants are of finite type. Recall from XVII.3 that a quantum group invariant is associated to any semisimple Lie algebra $\mathfrak{g}$ and any finite-dimensional simple $\mathfrak{g}$-module V.

Corollary XX.1.3. *The isotopy framed link invariant $Q_{\mathfrak{g},V}$ of* XVII.3 *is of finite type.*

PROOF. By construction of $Q_{\mathfrak{g},V}$, the congruence $R_h \equiv 1 \otimes 1$ implies that $Q_{\mathfrak{g},V}(L_+) \equiv Q_{\mathfrak{g},V}(L_-)$ modulo h at any crossing point (one may also use (XVII.3.3)). Then use Proposition 1.2. $\square$

Proposition 1.2 may also be applied to the Jones-Conway polynomial and hence to the Alexander and the Jones polynomials which are specializations thereof. Recall from X.4 that the Jones-Conway polynomial satisfies the skein relation

$$xP(L_+) - x^{-1}P(L_-) = yP(L_0) \tag{1.2}$$

for any Conway triple (L_+, L_-, L_0). Replace x and y by formal series $x(h)$ and $y(h)$ in h such that

$$x(h) - 1 \equiv y(h) \equiv 0 \qquad \mod h.$$

Then Relation (1.2) simplifies to $P(L_+) \equiv P(L_-)$ modulo h, which allows us to apply Proposition 1.2. In particular, using Taylor expansions, we see that the m-th derivative of the Jones polynomial [resp. of the Conway polynomial] at the point 1 [resp. at the point 0] is an invariant of degree $\leq m$. The invariants $\Phi_{m,q}$ of Proposition X.4.7 give also rise to invariants of finite degree.

For some open questions on finite-degree invariants, see [BN92], Section 7 and [Bir93], Section 8.

XX.2 Chord Diagrams and Kontsevich's Theorem

Let us restrict to knots in this section. Given a complex vector space V and a non-negative integer m, we denote by $\mathbf{V}^{(m)}(V)$ the vector space of all knot invariants of degree $\leq m$ with values in V. Since

$$\mathbf{V}^{(m)}(V) = \mathbf{V}^{(m)} \otimes V \tag{2.1}$$

where $\mathbf{V}^{(m)} = \mathbf{V}^{(m)}(\mathbf{C})$, it is enough to consider complex-valued invariants. We have the following inclusion of vector spaces

$$\mathbf{V}^{(0)} \subset \mathbf{V}^{(1)} \subset \ldots \subset \mathbf{V}^{(m-1)} \subset \mathbf{V}^{(m)} \subset \ldots.$$

One similarly defines the vector space $\mathbf{V}_{fr}^{(m)}$ of complex-valued framed knot invariants of degree $\leq m$. In this section we shall show that the spaces $\mathbf{V}^{(m)}$ and $\mathbf{V}_{fr}^{(m)}$ are all finite-dimensional and give a combinatorial description of the quotients $\mathbf{V}^{(m)}/\mathbf{V}^{(m-1)}$ and $\mathbf{V}_{fr}^{(m)}/\mathbf{V}_{fr}^{(m-1)}$.

To this end, we need the notion of a *chord diagram* on a circle: it is a finite set of unordered pairs of distinct points on the circle considered up to homeomorphisms preserving the orientation. To specify a pair one draws a dashed line, called a *chord*, between the two points. Given $2m$ distinct points there are

$$(2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2m-1)$$

different ways to pair them. Indeed, given one point among $2m$, we may pair it with $(2m-1)$ points. Take another point among the remaining $(2m-2)$; it may be paired to $(2m-3)$ other points, etc.

There is a relationship between invariants of finite degree and chord diagrams on a circle which we explain now. Let D be a chord diagram on the circle with m chords (i.e., with $2m$ points paired two by two). By an *embedding* of D into $\mathbf{R}^3$ we mean any singular knot $f : S^1 \to \mathbf{R}^3$ with exactly m double points such that $f(s) = f(s')$ if and only if $s = s'$ or s and s' are the two endpoints of a chord in D. There always exists an embedding K_D of D. If K'_D is another embedding of D, then it can be obtained from K_D by a series of operations consisting in replacing an undercrossing by an overcrossing and vice-versa. Suppose we are given a complex-valued knot invariant P of degree $\leq m$. Since P vanishes on singular knots with at least $m+1$ double points, P remains constant by Rule (1.1) under the operations transforming K_D into K'_D, which means that $P(K_D)$ is independent of the embedding of D chosen to compute it.

Define E_m as the complex vector space with a basis given by all chord diagrams on the circle with m chords. The dimension of E_m is finite and $\leq (2m-1)!!$. Then the evaluation of an invariant of degree $\leq m$ on an embedding of a chord diagram with m chords gives rise to a pairing

$$< \, , \, > : \mathbf{V}^{(m)} \otimes E_m \to \mathbf{C}. \tag{2.2}$$

Suppose that $< P, D > = 0$ for all chord diagram with m chords. Since any singular knot with m double points can be represented as an embedding of a chord diagram, we see that P vanishes on all singular knots with m double points, which means that P is an invariant of degree $\leq m-1$. Consequently, the map $P \mapsto < P, - >$ induces an injection

$$Y_m : \mathbf{V}^{(m)}/\mathbf{V}^{(m-1)} \to \mathrm{Hom}(E_m, \mathbf{C}) \tag{2.3}$$

of the quotient $\mathbf{V}^{(m)}/\mathbf{V}^{(m-1)}$ into a finite-dimensional space. A similar argument works for $\mathbf{V}_{fr}^{(m)}/\mathbf{V}_{fr}^{(m-1)}$. We get the following result.

Proposition XX.2.1. *The spaces $\mathbf{V}^{(m)}$ and $\mathbf{V}_{fr}^{(m)}$ are finite-dimensional.*

PROOF. We have already noted that knot invariants of degree 0 are constant. Therefore, $\mathbf{V}^{(0)} = \mathbf{V}_{fr}^{(0)} = \mathbf{C}$, which proves the assertion for $m = 0$. The rest follows by an easy induction on m using the injection (2.3). □

Actually, the proof of Proposition 2.1 shows that the dimensions of $\mathbf{V}^{(m)}$ and $\mathbf{V}_{fr}^{(m)}$ are bounded by $1 + \sum_{k=1}^{m} (2m-1)!!$.

What we next aim to, is to restrict the size of the image of the map Y_m. More precisely, we shall show that any linear form in the image of Y_m satisfies an important four-term relation. Let D be a chord diagram with $m-2$ chords. Consider the four pictures in Figure 2.1 involving each 2 chords.

Figure 2.1. The chord diagrams defining the four-term relation

Denote by D_1, D_2, D_3 and D_4 the chord diagrams obtained by adding to D successively the pictures of Figure 2.1 at the same place (the vertical lines in the pictures represent portions of the circle carrying the chord diagram). We claim the following.

Proposition XX.2.2. (a) *If P is an element of $\mathbf{V}^{(m)}$ or of $\mathbf{V}_{fr}^{(m)}$, we have*

$$< P, D_1 > - < P, D_2 > + < P, D_3 > - < P, D_4 > = 0 \qquad (2.4)$$

for any chord diagram D with $m-2$ chords.

(b) *Any element P of $\mathbf{V}^{(m)}$ vanishes on any chord diagram with an isolated chord, i.e., a chord that does not intersect any other one in the diagram.*

Relation (2.4) is called the *four-term relation* for invariants of finite degree.

PROOF. (a) Let K_1, K_2, K_3 and K_4 be singular knots differing locally by the pictures in Figure 2.2. They are embeddings of chord diagrams D_1, D_2, D_3 and D_4 as described above.

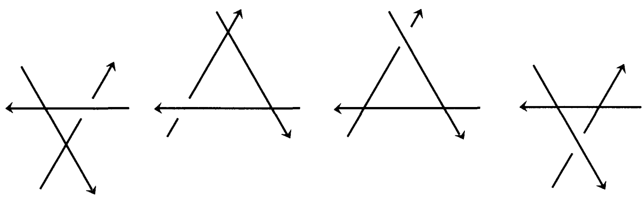


Figure 2.2. The local differences of K_1, K_2, K_3, and K_4

Then the left-hand side of (2.4) equals

$$P(K_1) - P(K_2) + P(K_3) - P(K_4). \tag{2.5}$$

To compute (2.5) we apply Relation (1.1) to all double points in Figure 2.2. Then (2.5) becomes a sum of 16 terms where all $2^3 = 8$ possible configurations of three crossing points occur twice with opposite signs.

(b) Let K be an embedding of the chord diagram D. Let us focus on the double point of K corresponding to the isolated chord. It separates the knot into two distinct unrelated parts. From this we see that K_+ and K_- are isotopic. It follows from Relation (1.1) that

$$P(K) = P(K_+) - P(K_-) = 0. \qquad \square$$

The argument in Part (b) of the proof above does not work for framed knot invariants since K_+ and K_- are not necessarily isotopic as framed knots.

The relations in Proposition 2.2 appear as universal relations satisfied by all invariants of finite degree. We may wonder whether there are more such relations. The answer is negative. In order to make this more precise, we define a vector space A_m as the quotient of E_m by the subspace generated by all elements of the form

$$D_1 - D_2 + D_3 - D_4 = 0 \tag{2.6}$$

where D is any chord diagram on the circle with $m-2$ chords and D_1, D_2, D_3, D_4 have been defined above. If $m = 0, 1$ we set $A_m = E_m$. Moreover, we define $\overline{A}_m$ as the quotient of A_m by all chord diagrams with isolated chords.

As a consequence of this definition and of Proposition 2.2, we see that Y_m embeds $\mathbf{V}_{fr}^{(m)}/\mathbf{V}_{fr}^{(m-1)}$ in $\mathrm{Hom}(A_m, \mathbf{C})$ and also embeds $\mathbf{V}^{(m)}/\mathbf{V}^{(m-1)}$ in $\mathrm{Hom}(\overline{A}_m, \mathbf{C})$. The following deep theorem due to Kontsevich [Kon93] signifies that A_m and $\overline{A}_m$ capture all universal relations for knot invariants of finite degree.

Theorem XX.2.3. *The maps Y_m*

$$\mathbf{V}_{fr}^{(m)}/\mathbf{V}_{fr}^{(m-1)} \to \mathrm{Hom}(A_m, \mathbf{C}) \quad \textit{and} \quad \mathbf{V}^{(m)}/\mathbf{V}^{(m-1)} \to \mathrm{Hom}(\overline{A}_m, \mathbf{C})$$

are isomorphisms.

Let us explain Kontsevich's proof of Theorem 2.3. Define

$$A = \bigoplus_{m \geq 0} A_m \quad \text{and} \quad \overline{A} = \bigoplus_{m \geq 0} \overline{A}_m. \tag{2.7}$$

We also need the direct products

$$\widehat{A} = \prod_{m \geq 0} A_m \quad \text{and} \quad \widehat{\overline{A}} = \prod_{m \geq 0} \overline{A}_m. \tag{2.8}$$

We shall treat an element of $\widehat{A}$ or of $\widehat{\overline{A}}$ as a formal series $\sum_{m \geq 0} D_m h^m$ where D_m belongs to A_m or to $\overline{A}_m$. With this notation fixed, Kontsevich [Kon93] assigns to each framed knot K an element

$$Z(K) = \sum_{m \geq 0} Z_m(K)\, h^m \in \widehat{A} \tag{2.9}$$

with the following properties:

(i) the map $K \mapsto Z(K)$ is a framed knot invariant of finite type, i.e., Z_m is of degree $\leq m$ for all $m \geq 0$.

(ii) for each chord diagram D with m chords and each embedding K_D of D, we have

$$Z(K_D) \equiv D h^m \mod h^{m+1}. \tag{2.10}$$

The invariant $Z(K)$ and its image $\overline{Z}(K)$ in $\widehat{\overline{A}}$ are called the *Kontsevich universal (framed) knot invariants.* We now use $Z(K)$ to prove Theorem 2.3.

Proof of Theorem 2.3. We assume the existence of such an invariant Z. Let w be a linear form on A_m. By Property (i) above, $P_w(K) = w(Z_m(K))$ is an element of $\mathbf{V}_{fr}^{(m)}$. Define a map X_m from $\mathrm{Hom}(A_m, \mathbf{C})$ to $\mathbf{V}_{fr}^{(m)}/\mathbf{V}_{fr}^{(m-1)}$ by composing $w \mapsto P_w$ with the projection onto $\mathbf{V}_{fr}^{(m)}/\mathbf{V}_{fr}^{(m-1)}$. By (2.10), we have

$$\begin{aligned} (Y_m \circ X_m)(w)(D) &= \ < P_w, D > \\ &= P_w(K_D) \\ &= w(Z_m(K_D)) = w(D) \end{aligned}$$

for any linear form w on A_m and any chord diagram D with m chords. This shows that $Y_m \circ X_m = \mathrm{id}$. Consequently, Y_m is surjective. We already know it is injective. Therefore, both Y_m and X_m are isomorphisms. There is a similar proof for (unframed) knot invariants. $\square$

As a consequence of the proof of Theorem 2.3, we see that $X_m \circ Y_m = \mathrm{id}$, which means that, if P is any framed knot invariant of degree $\leq m$, then there exists a unique linear form $w = Y_m(P)$ on A_m such that $P - w \circ Z_m$ is an invariant of degree $\leq m - 1$. It follows by induction that

$$P = w_m \circ Z_m + w_{m-1} \circ Z_{m-1} + \cdots + w_0 \circ Z_0$$

for a unique family of linear maps $(w_i : A_i \to \mathbf{C})_{0 \leq i \leq m}$. Consequently, for any framed knot invariant $P = \sum_{m \geq 0} P_m h^m$ of finite type, with values in the formal series algebra $\mathbf{C}[[h]]$, there exists a unique linear map w from $\widehat{A}$ to $\mathbf{C}[[h]]$ such that

$$P(K) = w(Z(K))$$

for all framed knots K. The bijection set up by $Z(K)$ between framed knot isotopy invariants of finite type and linear maps defined on $\widehat{A}$ justifies the qualifier "universal" for $Z(K)$. We have a similar formulation for knot invariants after replacing $\widehat{A}$ by $\widehat{\overline{A}}$ and $Z(K)$ by $\overline{Z}(K)$. In particular, any quantum group invariant can be obtained in this way (we shall give details in Section 8).

Kontsevich's original definition of the universal knot invariant $Z(K)$ used complicated multiple integrals depending on the realization of the knot as a smooth curve in the three-dimensional space. In Section 7 we shall give a combinatorial construction of $Z(K)$ using a planar diagram of the knot and category theory in the spirit of what we did in Chapters X, XII, XIV.5.1 and XVII.3. The combinatorial construction is due to Cartier [Car93], Le-Murakami [LM93c] and Piunikhin [Piu93].

XX.3 Algebra Structures on Chord Diagrams

We extend the notion of a chord diagram. Let T be an oriented (framed) tangle as defined in X.5. A *chord diagram* on T is a finite set of unordered pairs of distinct points of $T \setminus \partial T$ (where ∂T is the boundary of the tangle). Again, as in the previous section, the pairs, called chords and represented by dashed lines, are considered up to homeomorphisms preserving each connected component and the orientation of the tangle.

Let $E(T)$ be the complex vector space with a basis given by all chord diagrams on T. The vector space $E(T)$ has a grading

$$E(T) = \bigoplus_{m \geq 0} E_m(T) \tag{3.1}$$

where $E_m(T)$ is spanned by all chord diagrams with m chords. The subspace $E_0(T)$ is the one-dimensional subspace spanned by the unique chord diagram without chords. If $f : T \to T'$ is a homeomorphism of tangles, then f sends any chord diagram on T to a chord diagram on T', thus inducing an isomorphism $E(T) \cong E(T')$. In particular, since any tangle is homeomorphic to an "unknotted" tangle, the isomorphism class of $E(T)$ depends only on the number of circles and segments composing T.

Let T and T' be tangles such that $s(T) = b(T')$ in the notation of X.5 and XII.2. Under this condition, the composition $T \circ T'$ is defined. Placing a chord diagram of T on top of a chord diagram of T', we get a chord diagram of $T \circ T'$. This construction extends to a linear map

$$E(T) \otimes E(T') \to E(T \circ T') \tag{3.2}$$

sending $E_m(T) \otimes E_{m'}(T')$ into $E_{m+m'}(T \circ T')$.

We now define the vector space $A(T)$ as the quotient of $E(T)$ by the four-term relation (2.6) that already served us to define A_m in Section 2. Here again the pictures in Figure 2.1 have to be understood as local modifications of a chord diagram with the vertical full lines representing portions of the tangle. The graded structure of $E(T)$ passes to $A(T)$ and we have

$$A(T) = \bigoplus_{m \geq 0} A_m(T) \tag{3.3}$$

where m counts the number of chords.

Similarly, one defines $\overline{A}(T) = \bigoplus_{m \geq 0} \overline{A}_m(T)$ as the quotient of $A(T)$ by all chord diagrams with isolated chords. When T is a circle, we have isomorphisms

$$A_m(T) \cong A_m \quad \text{and} \quad \overline{A}_m(T) \cong \overline{A}_m. \tag{3.4}$$

Since the four-term relation is local, the composition (3.2) induces linear maps

$$A(T) \otimes A(T') \to A(T \circ T') \quad \text{and} \quad \overline{A}(T) \otimes \overline{A}(T') \to \overline{A}(T \circ T'), \tag{3.5}$$

defined when $s(T) = b(T')$. The maps (3.5) preserve the gradings.

Next put a graded algebra structure on the vector space $A = \bigoplus_{m \geq 0} A_m$. Consider the braid 1_n with $n > 0$ vertical segments oriented downwards (defined in X.6). In the tangle category, 1_n is the identity of the sequence consisting of n +-signs. Since $1_n \circ 1_n = 1_n$ the maps (3.2) and (3.5) yield algebra structures on $E(1_n)$, $A(1_n)$ and $\overline{A}(1_n)$ whose units are the chord diagrams without chords.

We use these algebra structures to produce a family of elements of $A(1_n)$ satisfying the infinitesimal braid group relations (XIX.2.1–2.2). For integers $1 \leq i \neq j \leq n$, let t^{ij} be the unique chord diagram on 1_n with a single chord between the i-th and the j-th strands. We have $t^{ji} = t^{ij}$ by definition. Using the algebra structure of $E(1_n)$ we also have

$$[t^{ij}, t^{k\ell}] = t^{ij}t^{k\ell} - t^{k\ell}t^{ij} = 0 \tag{3.6}$$

whenever i, j, k, ℓ are distinct. In view of the definition of the product on $E(1_n)$ and on $A(1_n)$, and of the four-term relation (2.6), we have

$$[t^{ij}, t^{ik} + t^{jk}] = t^{ij}t^{ik} - t^{ik}t^{ij} + t^{ij}t^{jk} - t^{jk}t^{ij} = 0 \tag{3.7}$$

in the quotient algebras $A(1_n)$ and $\overline{A}(1_n)$ when i, j, k are distinct integers. Consequently, the classes of the elements $(t^{ij})_{1 \leq i < j \leq n}$ satisfy the infinitesimal braid group relations in $A(1_n)$ and in $\overline{A}(1_n)$.

The algebras $A(1_n)$ and $\overline{A}(1_n)$ also have bialgebra structures. The comultiplication Δ is given by the formula

$$\Delta(D) = \sum_{\emptyset \subseteq D' \subseteq D} D' \otimes D'' \tag{3.8}$$

where D' runs over all subdiagrams of D including the chordless diagram and D'' is the subdiagram complementary to D' in D. The counit is zero on all chord diagrams with at least one chord and is 1 on the chordless diagram $\emptyset$. The reader may check that the comultiplication and the counit are well-defined on $A(1_n)$ and $\overline{A}(1_n)$ and satisfy all the required axioms. Observe that Δ is cocommutative. Actually, these bialgebras are Hopf algebras as are all graded bialgebras whose zero-th part is equal to $\mathbf{C}$. The antipode S is defined inductively on the number of chords by $S(\emptyset) = \emptyset$ and

$$S(D) = -D - \sum_{\emptyset \neq D' \neq D} S(D')D''. \tag{3.9}$$

We now consider the special case $n = 1$ and denote $A(1_1)$ and $\overline{A}(1_1)$ by $A(\downarrow)$ and $\overline{A}(\downarrow)$ respectively. The following lemma holds in $A(\downarrow)$.

Lemma XX.3.1. *Let D be a chord diagram on $\downarrow = 1_1$ with at least two chords. Let p be the highest point of D and $\{p, q\}$ be the corresponding chord. Let p' be a point of $\downarrow$ not in D and lower than all points of D. Define a new chord diagram D' by $D' = (D \setminus \{p, q\}) \bigcup \{p', q\}$. Then D and D' define the same element in $A(\downarrow)$.*

PROOF. We first reformulate the four-term relation (2.6). Let D be a chord diagram on $\downarrow$ with at least one chord $\alpha = \{y, z\}$ and let q be a point of $\downarrow$ not in D. Define four points x_1, x_2, x_3, x_4 by their heights $\mathrm{ht}(x_i)$ as follows:

$$\mathrm{ht}(x_1) = \mathrm{ht}(y) + \varepsilon, \quad \mathrm{ht}(x_2) = \mathrm{ht}(y) - \varepsilon,$$

$$\mathrm{ht}(x_3) = \mathrm{ht}(z) + \varepsilon, \quad \mathrm{ht}(x_4) = \mathrm{ht}(z) - \varepsilon$$

where ε is a positive, small enough, real number. Consider the diagrams $D_i^{\alpha,q} = D \cup \{q, x_i\}$ for $i = 1, \ldots, 4$. With this notation the four-term relation (2.6) translates into the relation

$$D_1^{\alpha,q} - D_2^{\alpha,q} + D_3^{\alpha,q} - D_4^{\alpha,q} = 0. \tag{3.10}$$

The proof of the lemma now follows from (3.10) and from the equality

$$D - D' = \sum \left(\bar{D}_1^{\alpha,q} - \bar{D}_2^{\alpha,q} + \bar{D}_3^{\alpha,q} - \bar{D}_4^{\alpha,q} \right)$$

where the sum is taken over all chords α of $\bar{D} = D \setminus \{p, q\}$. □

Corollary XX.3.2. (a) *The algebra $A(\downarrow)$ is commutative.*

(b) *Closing the braid $\downarrow$ induces an isomorphism*

$$A(\downarrow) \cong A = \oplus_{m \geq 0} A_m.$$

PROOF. (a) This results from a repeated application of Lemma 3.1.

(b) Let D be a chord diagram on the circle. By slitting the latter at some point distinct of the endpoints of the chords of D we get a chord diagram on 1_1. Lemma 3.1 shows that this is independent of the place where the circle is slit. □

We already know that $A(\downarrow)$ has a cocommutative Hopf algebra structure. It is also commutative by the previous result. We use the isomorphism of Corollary 3.2 to transport this structure on A. Now, by a well-known result of Milnor and Moore [MM65], any commutative cocommutative Hopf algebra A over a field of characteristic zero is a symmetric algebra over the subspace $\mathrm{Prim}(A) = \bigoplus_{m \geq 0} \mathrm{Prim}(A)_m$ of primitive elements. This applies to the isomorphic Hopf algebras A and $A(\downarrow)$. Despite the fact that the algebras A and $A(\downarrow)$ are polynomial algebras, not much is known about their generators, not even $d_m = \dim(\mathrm{Prim}(A)_m)$, which is the (finite) number of generators of A in degree m. The dimension d_m has been computed in degrees up to $m = 8$. According to [BN92], Section 6 we have the following table for d_m.

m	0	1	2	3	4	5	6	7	8
d_m	1	1	1	1	2	3	5	8	12

A final observation is in order: denote by C the image in A of the unique chord diagram with one single chord and by (C) the two-sided ideal it generates. We have $\overline{A} = A/(C)$ and $A \cong \overline{A}[C]$.

XX.4 Infinitesimal Symmetric Categories

Let $\mathcal{S} = (\mathcal{S}, \otimes, I)$ be a strict tensor category whose sets of morphisms $\mathrm{Hom}_{\mathcal{S}}(V, W)$ are all complex vector spaces and where the composition and the tensor product of morphisms are $\mathbf{C}$-bilinear maps. We assume that $\mathcal{S}$ is symmetric with an involutive braiding $(\sigma_{V,W})_{V,W}$.

Definition XX.4.1. *Under the previous hypotheses, define an infinitesimal braiding on $\mathcal{S}$ as a family of functorial endomorphisms in $\mathcal{S}$*

$$t_{V,W} : V \otimes W \to V \otimes W, \tag{4.1}$$

defined for all pairs (V, W) of objects of $\mathcal{S}$, such that

$$\sigma_{V,W} \circ t_{V,W} = t_{W,V} \circ \sigma_{V,W}, \tag{4.2}$$

and

$$t_{U,V\otimes W} = t_{U,V} \otimes \mathrm{id}_W + (\sigma_{U,V} \otimes \mathrm{id}_W)^{-1} \circ (\mathrm{id}_V \otimes t_{U,W}) \circ (\sigma_{U,V} \otimes \mathrm{id}_W) \quad (4.3)$$

for all objects U, V, W in $\mathcal{S}$.

A symmetric category as above equipped with an infinitesimal braiding is called an infinitesimal symmetric category.

Observe that in view of (4.2), Relation (4.3) is equivalent to

$$t_{U\otimes V,W} = \mathrm{id}_U \otimes t_{V,W} + (\mathrm{id}_U \otimes \sigma_{V,W})^{-1} \circ (t_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \sigma_{V,W}). \quad (4.4)$$

Relations (4.3–4.4) are infinitesimal versions of the relations (XIII.1.5–1.6) defining a braiding in a braided tensor category. Indeed, suppose we have a braided tensor category in which the morphisms depend on a formal parameter h and, in particular, the braiding $c_{V,W}$ is of the form

$$c_{V,W} = \sigma_{V,W}\Big(\mathrm{id}_{V\otimes W} + ht_{V,W} + \text{ terms of higher degree in } h\Big)$$

for some symmetry $\sigma_{V,W}$. An immediate computation shows that if $c_{V,W}$ satisfies Relations (XIII.1.5–1.6), then the endomorphisms $t_{V,W}$ satisfy Relations (4.3–4.4).

If, in addition, the infinitesimal symmetric category $\mathcal{S}$ has a left duality $V \mapsto V^*$ with structure maps $b_V^0 : I \to V \otimes V^*$ and $d_V^0 : V^* \otimes V \to I$ (as defined in XIV.2), then the infinitesimal braiding is of the form

$$t_{V,W} = \frac{1}{2}\Big(C_{V\otimes W} - C_V \otimes \mathrm{id}_W - \mathrm{id}_V \otimes C_W\Big) \quad (4.5)$$

where $(C_V : V \to V)_V$ is a natural family of endomorphisms of $\mathcal{S}$ defined by

$$C_V = -\Big(\mathrm{id}_V \otimes (d_V^0 \circ t_{V^*,V})\Big) \circ (b_V^0 \otimes \mathrm{id}_V). \quad (4.6)$$

Let us give an example of an infinitesimal braiding. We know that if $H = (H, \Delta, \varepsilon, S)$ is a *cocommutative* Hopf algebra, then the category H-*Mod* of H-modules is a symmetric tensor category, with the flip as symmetry. Let $\mathrm{Prim}(H)$ be the vector space of primitive elements in H. We have the following characterization of infinitesimal braidings on H-*Mod*.

Proposition XX.4.2. (a) *Let t be an element of* $\mathrm{Prim}(H) \otimes \mathrm{Prim}(H)$ *satisfying the conditions $t_{21} = t$ and $[\Delta(a), t] = 0$ for all $a \in H$. For any pair (V, W) of H-modules define the endomorphism $t_{V,W}$ of $V \otimes W$ by*

$$t_{V,W}(v \otimes w) = t(v \otimes w) \quad (4.7)$$

where $v \in V$ and $w \in W$. Then $(t_{V,W})_{V,W}$ is an infinitesimal braiding on the category H-Mod.

(b) *Conversely, any infinitesimal braiding $(t_{V,W})_{V,W}$ on H-Mod is of the form* (4.7) *with $t = t_{H,H}(1 \otimes 1) \in H \otimes H$. The element t belongs to* $\mathrm{Prim}(H) \otimes \mathrm{Prim}(H)$ *and satisfies the two conditions of Part* (a).

PROOF. Part (a) follows by direct checking. To prove Part (b) we proceed as in the proof of Proposition XIII.1.4. The functoriality of the infinitesimal braiding forces it to be of the form (4.7) with $t = t_{H,H}(1 \otimes 1)$. The H-linearity of the infinitesimal braiding implies that $[\Delta(a), t] = 0$ for all a in H. Conditions (4.2) and (4.3) yield $t_{21} = t$ and $(\mathrm{id} \otimes \Delta)(t) = t_{12} + t_{13}$ respectively. The fact that t belongs to the subspace generated by primitive elements follows from an argument already used in the proof of Proposition XVI.5.2. □

Let us restrict to the subcategory H-Mod_f of finite-dimensional H-modules. It has left duality. If H-Mod has an infinitesimal braiding induced by the element $t = \sum_i x_i \otimes y_i$ where x_i, y_i are primitive, then the endomorphisms C_V of (4.6) are induced by the action of a single element, namely $C = \sum_i x_i y_i \in H$. This follows from (4.6) and the fact that the antipode of a primitive element is equal to its opposite.

We may apply Proposition 4.2 to the enveloping algebra $H = U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ with $t \in \mathfrak{g} \otimes \mathfrak{g}$ being equal to the symmetric invariant 2-tensor (XVII.1.6). In this case, C is the Casimir element (XVII.1.5).

We shall need the following result in Section 5.

Lemma XX.4.3. *If U, V, W are objects of an infinitesimal symmetric category with symmetry $(\sigma_{V,W})_{V,W}$ and infinitesimal braiding $(t_{V,W})_{V,W}$, then we have*

$$[t_{U,V} \otimes \mathrm{id}_W, \sigma^{-1}(t_{U,W} \otimes \mathrm{id}_V)\sigma + \mathrm{id}_U \otimes t_{V,W}] = 0 \tag{4.8}$$

where $\sigma = \mathrm{id}_U \otimes \sigma_{V,W}$.

PROOF. The square

$$\begin{array}{ccc} U \otimes V \otimes W & \xrightarrow{t_{U\otimes V,W}} & U \otimes V \otimes W \\ \Big\downarrow{\scriptstyle t_{U,V}\otimes \mathrm{id}_W} & & \Big\downarrow{\scriptstyle t_{U,V}\otimes \mathrm{id}_W} \\ U \otimes V \otimes W & \xrightarrow{t_{U\otimes V,W}} & U \otimes V \otimes W \end{array}$$

commutes by functoriality of the infinitesimal braiding. In other words, we have

$$[t_{U,V} \otimes \mathrm{id}_W, t_{U\otimes V,W}] = 0.$$

Replacing $t_{U\otimes V,W}$ by its expression in Relation (4.4) yields (4.8). □

XX.5 A Universal Category for Infinitesimal Braidings

We now construct an infinitesimal symmetric category $\mathcal{AB}$ of special interest. The objects of the category $\mathcal{AB}$ are the objects of the braid category $\mathcal{B}$ of XIII.2, namely nonnegative integers. A morphism in $\mathcal{AB}$ is an element of the complex vector space $A^{hor}(T)$ for some braid T where $A^{hor}(T)$ has the same definition as $A(T)$ (see Section 3), except that we allow only *horizontal* chords. The source [resp. the target] of such a chord diagram is the sequence $s(T)$ [resp. the sequence $b(T)$] defined in X.5. The composition of morphisms is given by the map (3.5). The identity of an integer n is the chordless diagram on the braid 1_n (defined in X.6).

We put the same tensor product on $\mathcal{AB}$ as the one we put on the braid category, namely, we have $n \otimes m = n + m$ on objects while the tensor product of morphisms is defined by placing chord diagrams side by side. The tensor product is well-defined and strictly associative with unit $I = 0$.

The braiding (XIII.2.1) of the braid category induces a braiding on the category $\mathcal{AB}$: it suffices to take the chordless diagrams on the corresponding braids. Since we are considering braid chord diagrams up to homeomorphisms, we see that this braiding is symmetric in $\mathcal{AB}$ although it is not in the braid category.

Given objects n, m of $\mathcal{AB}$, define an endomorphism $t_{n,m}$ of $n \otimes m = n+m$ as follows. If n or $m = 0$, set $t_{n,m} = 0$. Otherwise, set

$$t_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} t^{i,n+j} \tag{5.1}$$

where t^{ij} is the chord diagram (already defined in Section 3) with a unique chord between the i-th and the j-th strands.

Proposition XX.5.1. *The family $(t_{n,m})_{n,m\geq 0}$ is an infinitesimal braiding on the category $\mathcal{AB}$.*

PROOF. Relations (4.2–4.3) are easy to check. It remains to prove that the family $(t_{n,m})_{n,m}$ is functorial with respect to all morphisms of $\mathcal{AB}$. Since the category is symmetric, it is enough to show that the square

$$\begin{array}{ccc} n \otimes m & \xrightarrow{t_{n,m}} & n \otimes m \\ \big\downarrow {\scriptstyle f\otimes \mathrm{id}_m} & & \big\downarrow {\scriptstyle f\otimes \mathrm{id}_m} \\ n \otimes m & \xrightarrow{t_{n,m}} & n \otimes m \end{array} \tag{5.2}$$

commutes for all morphisms f. Now the endomorphisms of n in $\mathcal{AB}$ are clearly generated by the generators $\sigma_1, \ldots, \sigma_{n-1}$ of the braid group B_n and by the chord diagrams $(t^{ij})_{1\leq i<j\leq n}$ of $A^{hor}(1_n)$. Therefore, it suffices to check the commutativity of (5.2) when f is of type σ_i and when it is of type t^{ij}. This is easy in the first case. In the second case, using Relations (4.3–4.4), we see it is enough to consider the case $n = 2$, $m = 1$ and $f = t^{12}$. We have

$$t_{2,1} \circ (t^{12} \otimes \mathrm{id}_1) = t^{13}t^{12} + t^{23}t^{12}$$

and

$$(t^{12} \otimes \mathrm{id}_1) \circ t_{2,1} = t^{12}t^{13} + t^{12}t^{23}.$$

Consequently, the commutativity of (5.2) in this special case is equivalent to the relation

$$[t^{12}, t^{13} + t^{23}] = 0. \tag{5.3}$$

The latter follows from (3.7). □

This proof shows that the four-term relation is imposed by the naturality of the family $(t_{n,m})_{n,m\geq 0}$ in the category $\mathcal{AB}$. We now state a universality property for $\mathcal{AB}$ which is the infinitesimal analogue of Corollary XIII.3.8.

Proposition XX.5.2. *Let $\mathcal{S}$ be an infinitesimal symmetric category with symmetry $(\sigma_{V,W})_{V,W}$ and infinitesimal braiding $(t_{V,W})_{V,W}$. For any object V of $\mathcal{S}$ there exists a unique braided strict tensor functor $F_V : \mathcal{AB} \to \mathcal{S}$ such that*

$$F_V(1) = V \quad \text{and} \quad F_V(t^{12} : 1 \otimes 1 \to 1 \otimes 1) = t_{V,V}. \tag{5.4}$$

PROOF. One proceeds as for Lemma XIII.3.5. Define F_V on the generators $(t^{ij})_{1\leq i<j\leq n}$ by

$$F_V(t^{ij}) = (\sigma^{ij})^{-1}(\mathrm{id}_{V^{\otimes(i-1)}} \otimes t_{V,V} \otimes \mathrm{id}_{V^{\otimes(n-i-1)}})\sigma^{ij} \tag{5.5}$$

where $\sigma^{ij} = \mathrm{id}_{V^{\otimes i}} \otimes \sigma_{V^{\otimes(j-i-1)},V} \otimes \mathrm{id}_{V^{\otimes(n-j)}}$. We have to check the relations defining the morphisms of $\mathcal{AB}$, including (3.6–3.7). Relation (3.6) is clear while Relation (3.7) follows from (4.8). □

Using Proposition 5.2, one may derive an equivalence between the category $\mathcal{S}$ and a category of braided tensor functors preserving infinitesimal braidings from $\mathcal{AB}$ to $\mathcal{S}$.

XX.6 Formal Integration of Infinitesimal Symmetric Categories

We review a categorical construction due to Cartier [Car93]. Let Φ be a Drinfeld series as defined in Remark XIX.8.3, for instance Drinfeld's associator Φ_{KZ}.

Given an infinitesimal symmetric category $\mathcal{S}$ with symmetry $(\sigma_{V,W})_{V,W}$ and infinitesimal braiding $(t_{V,W})_{V,W}$, we construct a braided tensor category $\mathcal{S}[[h]]$ as follows. The objects of $\mathcal{S}[[h]]$ are the same as the objects of $\mathcal{S}$. A morphism from V to W in $\mathcal{S}[[h]]$ is a formal series $\sum_{n\geq 0} f_n h^n$ where $f_0, f_1, f_2, \ldots$ are morphisms from V to W in $\mathcal{S}$. The composition in $\mathcal{S}[[h]]$ extends the composition in $\mathcal{S}$ and the multiplication of formal series. The identity of V in $\mathcal{S}[[h]]$ is the constant formal series id_V.

Theorem XX.6.1. *Under the previous hypotheses, there exists a unique structure of braided tensor category on $\mathcal{S}[[h]]$ such that the tensor product on objects and the unit are the same as in $\mathcal{S}$, the tensor product on morphisms* *extends* $\mathbf{C}[[h]]$*-linearly the tensor product in* $\mathcal{S}$*, the associativity constraint* a *is given by*

$$a_{U,V,W} = \Phi(t_{U,V}, t_{V,W}), \tag{6.1}$$

and the braiding c *is given by*

$$c_{V,W} = \sigma_{V,W} \circ e^{ht_{V,W}/2}. \tag{6.2}$$

PROOF. We have to check the Pentagon Axiom (XI.2.6) and the Hexagon Axioms (XIII.1.3–1.4). Now this follows from Relations (XIX.8.27–8.29) satisfied by Φ. □

Applying this construction to the category $\mathcal{S} = \mathcal{AB}$ of Section 5, we get a braided tensor category $\mathcal{AB}[[h]]$. Choose the object 1 in it. Then by Corollary XIII.3.8 there exists a unique strict braided tensor functor Z from the braid category $\mathcal{B}$ to the category $\mathcal{AB}[[h]]$ which sends 1 to 1 and, therefore, is the identity on objects. Restricting to the endomorphisms of n in $\mathcal{B}$, namely to the braid group B_n, we get a group morphism

$$Z : B_n \to S_n \times \Big(1 + \sum_{m\geq 1} A_m^{hor}(1_n)h^m\Big)$$

which by Lemma XV.4.1 is defined on the generators of the braid group by

$$Z(\sigma_1) = \sigma_1 e^{ht^{12}/2} \tag{6.3}$$

and

$$Z(\sigma_i) = \Big(a^{-1}_{V^{\otimes(i-1)},V,V} \otimes \mathrm{id}_V^{\otimes(n-i-1)}\Big)\sigma_i e^{ht^{i,i+1}/2}\Big(a_{V^{\otimes(i-1)},V,V} \otimes \mathrm{id}_V^{\otimes(n-i-1)}\Big) \tag{6.4}$$

when $2 \leq i \leq n-1$. The associativity isomorphisms $a_{V^{\otimes(i-1)},V,V}$ have to be computed from the Drinfeld series Φ using (6.1) and (4.3–4.4). The composition of Z with the projection onto S_n is the surjection sending each braid to its permutation. In the next section, we shall extend the map Z to all tangles.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $t \in \mathfrak{g}\otimes\mathfrak{g}$ be the invariant symmetric 2-tensor given by (XVII.1.6). Consider the infinitesimal symmetric category $U(\mathfrak{g})$-Mod_f. We can reformulate precisely Drinfeld's Theorem XIX.4.3 and Corollary XIX.4.4 as follows.

Corollary XX.6.2. *In case* $\Phi = \Phi_{\mathrm{KZ}}$*, there is a braided tensor equivalence between the braided tensor category* $U_h(\mathfrak{g})$-Mod_{fr} *of* XVII.3 *and the braided tensor category* $(U(\mathfrak{g})$-$Mod_f)[[h]]$.

XX.7 Construction of Kontsevich's Universal Invariant

We first state a complement to Theorem 6.1 in the case when the infinitesimal symmetric category $\mathcal{S}$ has a left duality $V \mapsto V^*$ with structure maps

$b_V^0 : I \to V \otimes V^*$ and $d_V^0 : V^* \otimes V \to I$. In Section 6 we constructed a non-strict braided tensor category $\mathcal{S}[[h]]$. Let $\mathcal{S}[[h]]^{str}$ be the strict braided tensor category associated to $\mathcal{S}[[h]]$ by the procedure of XI.5. We keep the notations of Section 6.

Theorem XX.7.1. *Under these hypotheses, the strict braided tensor category $\mathcal{S}[[h]]^{str}$ is a ribbon category with twist θ_V given by*

$$\theta_V = e^{hC_V/2} \tag{7.1}$$

and with left duality defined as follows: for any object V the dual object V^* is the same as in the category $\mathcal{S}$; the structure maps b_V and d_V are defined by

$$b_V = b_V^0 \quad \text{and} \quad d_V = d_V^0 \circ (\lambda_{V^*}^{-1} \otimes \mathrm{id}_V) \tag{7.2}$$

where λ_{V^*} is the automorphism of V^* defined by

$$\lambda_{V^*} = (d_V^0 \otimes \mathrm{id}_{V^*}) \circ \Phi(t_{V^*,V}, t_{V,V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V^0). \tag{7.3}$$

Ribbon categories were defined in XIV.3 and the endomorphisms C_V by (4.6).

PROOF. The axioms (XIV.2.1) for the duality follow from a computation and the axioms (XIV.3.1–3.2) for the twist essentially from Relation (4.5). $\square$

The importance of this theorem lies in the fact explained in XIV.5.1 that, by colouring links with any object of $\mathcal{S}$, the ribbon category $\mathcal{S}[[h]]^{str}$ provides us with a framed link invariant with values in the endomorphism ring $\mathrm{End}_{\mathcal{S}}(I)[[h]]$ of the unit object I in $\mathcal{S}[[h]]^{str}$. This fact will now be used to construct Kontsevich's universal invariant. From now on, we assume that the Drinfeld series we want Theorems 6.1 and 7.1 to work with is Drinfeld's associator Φ_{KZ}.

We first define an infinitesimal symmetric category $\mathcal{A}$ with left duality. It is built in the same way as the category $\mathcal{AB}$ of Section 5, except that braids are now replaced by framed tangles and chords are no longer assumed to be horizontal. More precisely, the objects of the category $\mathcal{A}$ are the objects of the tangle category $\mathcal{T}$, namely finite sequences of $+$ and $-$, including the empty sequence $\emptyset$. A morphism in $\mathcal{A}$ is an element of the complex vector space $A(T)$ for some framed tangle T (as defined in Section 3). Its source [resp. its target] is the sequence $s(T)$ [resp. the sequence $b(T)$] defined in X.5. The composition of morphisms is given by the map (3.5). The identity of a sequence S is the chordless diagram on the tangle id_S.

We define a strictly associative tensor product on $\mathcal{A}$ as on $\mathcal{AB}$. Its unit is the empty sequence: $I = \emptyset$. Remember that the monoid of endomorphisms of $\emptyset$ in the framed tangle category is the set of all isotopy classes of framed links in $\mathbf{R}^2 \times]0,1[$. Here, the monoid of endomorphisms of $\emptyset$ in the category $\mathcal{A}$ is a complex associative algebra since the sets of morphisms are complex vector spaces. This algebra is bigraded by the number of chords and the number of connected components of the link. We have

$$\mathrm{End}_{\mathcal{A}}(I) \cong \bigoplus_{m,n\geq 0} A_m(O^{\otimes n}) \tag{7.4}$$

where $O^{\otimes n}$ denotes the disjoint union of n circles.

The symmetry and the infinitesimal braiding of the category $\mathcal{AB}$ define a symmetry and an infinitesimal braiding on $\mathcal{A}$. The latter also has a left duality induced by the left duality of the category $\mathcal{R}$ of ribbons (see XIV.5.1).

The category $\mathcal{A}$ satisfying the hypotheses of Theorem 7.1, we get a ribbon category $\mathcal{A}[[h]]^{str}$. Take the object $(+)$ in it. By Theorem XIV.5.1 there exists a unique strict braided tensor functor

$$Z : \mathcal{R} \to \mathcal{A}[[h]]^{str}$$

preserving the duality and the twist, sending the object $(+)$ of the category $\mathcal{R}$ of ribbons to the object $(+)$ of $\mathcal{A}[[h]]^{str}$. Consequently, the functor Z is the identity on objects. The restriction of Z to braids is the morphism defined by (6.3–6.4).

Let K be a framed link. It can be viewed as an endomorphism of the unit object in the category $\mathcal{R}$. Its image $Z(K)$ is an isotopy invariant living in

$$\mathrm{End}_{\mathcal{A}[[h]]^{str}}(\emptyset) = \bigoplus_{n>0} \widehat{A}(O^n).$$

When K is a framed *knot*, the invariant $Z(K)$ lies in $\widehat{A} = \prod_{m\geq 0} A_m$. This is the universal invariant we are after. Indeed, by the definition of the braiding (6.2) in $\mathcal{A}[[h]]$, the invariant $Z(K) = \sum_{m\geq 0} Z_m(K)h^m$ satisfies the hypotheses of Proposition 1.2. It results that Z is an invariant of finite type. In order to check Relation (2.10), we have to extend the invariant Z to singular knots. This is done using (1.1). At each double point of a singular knot, we have for Z a local contribution of the form $c_{S,S'} - c_{S,S'}^{-1}$. By (6.2) this looks like

$$e^{ht_{S,S'}/2} - e^{-ht_{S,S'}/2} = ht_{S,S'} + \text{ terms of degree } > 1. \tag{7.5}$$

Relation (7.5) and an induction on the number of double points imply Relation (2.10).

Remarks 7.2. (a) Le and Murakami [LM93c] showed that $Z(K)$ coincided with the invariant originally constructed by Kontsevich with multiple integrals. The reader is advised to read [LM93c] where $Z(K)$ is defined in a slightly different way using the concept of quasi-tangles.

(b) The appearance of Φ_{KZ} in the definition of $Z(K)$ makes it difficult to compute for any framed knot. Nevertheless, the first terms of the formal series $Z(K)$ may be determined using Corollary XIX.6.5.

XX.8 Recovering Quantum Group Invariants

The aim of this section is to show how one recovers the quantum group invariants of XVII.3 from Kontsevich's universal invariant Z. We first state a universal property for the category $\mathcal{A}$ of Section 7, parallel to Proposition 5.2.

Proposition XX.8.1. *Let $\mathcal{S}$ be an infinitesimal symmetric category with left duality, with symmetry $(\sigma_{V,W})_{V,W}$ and infinitesimal braiding $(t_{V,W})_{V,W}$. For any object of $\mathcal{S}$ there exists a unique functor F_V from the category $\mathcal{A}$ to the category $\mathcal{S}$ such that*

$$F_V(S \otimes S') = F_V(S) \otimes F_V(S'), \;\; F_V(\emptyset) = I, \;\; F_V(+) = V, \;\; F_V(-) = V^*, \tag{8.1}$$

$$F_V(\sigma_{S,S'}) = \sigma_{F_V(S),F_V(S')}, \quad F_V(b_S) = b_{F_V(S)}, \quad F_V(d_S) = d_{F_V(S)} \tag{8.2}$$

and

$$F_V(t_{S,S'}) = t_{F_V(S),F_V(S')} \tag{8.3}$$

for all objects S and S' of $\mathcal{S}$.

PROOF. One proceeds as for Proposition 5.2. The main difference lies in the existence of general chord diagrams in $\mathcal{A}$. In order to show that Relation (8.3) determines F_V on any chord diagram, we observe that any chord may be arranged so as to be horizontal after possibly adding some maxima and minima to the diagram. □

We wish to illustrate Proposition 8.1 in the case when $\mathcal{S} = H\text{-}Mod_f$ where $H = (H, \Delta, \varepsilon)$ is a complex Hopf algebra along with an element $t = \sum_j x_i \otimes y_i$ in $\mathrm{Prim}(H) \otimes \mathrm{Prim}(H)$ such that $t_{21} = t$ and $[t, \Delta(a)] = 0$ for all $a \in H$. By Proposition 4.2 we know that $\mathcal{S}$ is an infinitesimal symmetric category with left duality, the symmetry being the flip and the infinitesimal braiding given by (4.7). Fix a finite-dimensional left H-module V. By Proposition 8.1, there exists a well-defined functor $F_V : \mathcal{A} \to H\text{-}Mod_f$ such that $F_V(+) = V$. Consequently, if D is a chord diagram on $\downarrow = \mathrm{id}_+$, then $F_V(D)$ is an H-linear endomorphism of V. We now determine this endomorphism.

Let D be a chord diagram on $\downarrow$ with $m > 0$ chords. Define an element C_D of H by the following combinatorial rule. Running down along the strand $\downarrow$, write x_{j_k} whenever you come across the k-th upper endpoint of a chord and write y_{j_k} when you meet its lower endpoint. In this way one gets a word w_D. Suppose the word is

$$w_D = x_{j_1} x_{j_2} x_{j_3} y_{j_2} x_{j_4} y_{j_1} y_{j_4} x_{j_5} y_{j_3} y_{j_5}$$

(here $m = 5$). Then the element C_D is by definition

$$C_D = (-1)^m \sum_{j_1,\ldots,j_5} x_{j_1} x_{j_2} x_{j_3} y_{j_2} x_{j_4} y_{j_1} y_{j_4} x_{j_5} y_{j_3} y_{j_5}. \tag{8.4}$$

Proposition XX.8.2. *For any chord diagram D on $\downarrow$, the element C_D is central in H and depends only on the class of D in $A(\downarrow)$. Moreover, the endomorphism $F_V(D)$ is the action of the central element C_D of H on V.*

PROOF. We first deform D into a chord diagram whose chords are horizontal. We claim that (8.1–8.3) imply that $F_V(D)$ is equal to the action of C_D on V. Now F_V is defined on the equivalence classes of chord diagrams. Therefore C_D depends only on the equivalence class of D. Finally, the endomorphism $F_V(D)$ being H-linear, C_D is central.

Let us prove the claim in the special case when D is the unique chord diagram on $\downarrow$ with two intersecting chords. This diagram can be expressed as

$$D = (\mathrm{id}_+ \otimes d_+)(t_{+,-} \otimes \mathrm{id}_+)(\mathrm{id}_+ \otimes t_{-,+})(b_+ \otimes \mathrm{id}_+) \tag{8.5}$$

in the category $\mathcal{A}$. Its image under the functor F_V is the endomorphism

$$F_D(V) = (\mathrm{id}_V \otimes d_V)(t_{V,V^*} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes t_{V^*,V})(b_V \otimes \mathrm{id}_V). \tag{8.6}$$

Let v be an element of V and $\{v_i\}_i$ be a basis of V. We denote the dual basis by $\{v^i\}_i$. We have

$$\begin{aligned}
F_V(D)(v) &= \sum_{i,j,k} x_j v_i < y_j x_k v^i, y_k v > \\
&= \sum_{i,j,k} x_j v_i < v^i, S(y_j x_k) y_k v > \\
&= \sum_{i,j,k} x_j v_i < v^i, x_k y_j y_k v > \\
&= \Big(\sum_{j,k} x_j x_k y_j y_k\Big) v \\
&= C_D\, v.
\end{aligned}$$

The third equality follows from $S(y_j x_k) = S(x_k)S(y_j) = (-1)^2 x_k y_j$, which holds because x_k and y_j are primitive elements of H. □

Proposition 8.2 provides an interesting way of constructing central elements of H. For instance, if H is the enveloping algebra of a complex semisimple Lie algebra $\mathfrak{g}$ with its canonical 2-tensor t, then we recover the Casimir operator (XVII.1.5) $C = C_D$ from the chord diagram D with one chord. It would be interesting to characterize the subspace of the centre of $U(\mathfrak{g})$ spanned by all elements C_D.

We are now ready to indicate how one recovers the quantum group invariant $Q_{\mathfrak{g},V}$ from Kontsevich's universal invariant Z. Recall that $Q_{\mathfrak{g},V}$ is defined for a semisimple Lie algebra $\mathfrak{g}$ and a finite-dimensional simple $\mathfrak{g}$-module V. To the data $(\mathfrak{g}, V)$, we associate a linear map $w_{\mathfrak{g},V}$ on the space of all chord diagrams as follows. Let D be a chord diagram on the circle. To it corresponds a uniquely defined chord diagram, still denoted D, on the line $\downarrow$. By Proposition 8.1 we know how to build a central element C_D of $U(\mathfrak{g})$. Since V is simple, C_D acts as a scalar μ_D on V. We define $w_{\mathfrak{g},V}(D)$ as

$$w_{\mathfrak{g},V}(D) = \mu_D. \tag{8.7}$$

The relationship between the quantum group invariant $Q_{\mathfrak{g},V}$ and Kontsevich's universal knot invariant Z is given by the following statement.

Theorem XX.8.3. *Under the previous hypotheses, for all framed knots K we have*

$$Q_{\mathfrak{g},V}(K) = \dim_q(\widetilde{V}) \sum_{m\geq 0} w_{\mathfrak{g},V}(Z_m(K))h^m \tag{8.8}$$

where $\dim_q(\widetilde{V})$ *is the quantum dimension defined in* XVII.3.

PROOF. Applying Proposition 8.1 to $\mathcal{S} = U(\mathfrak{g})\text{-}Mod_f$ and to the given simple module V, we get a functor F_V from $\mathcal{A}$ to $U(\mathfrak{g})\text{-}Mod_f$ with $F_V(+) = V$. By Theorem 7.1, F_V extends to a ribbon functor $\widetilde{F}_V$ from $\mathcal{A}[[h]]^{str}$ to $(U(\mathfrak{g})\text{-}Mod_f)[[h]]^{str}$ such that $\widetilde{F}_V(+) = V$. By a ribbon functor, we mean a strict braided tensor functor preserving left duality and twist. Composing $\widetilde{F}_V$ with the functor Z of Section 7, we get the ribbon functor $\widetilde{F}_V \circ Z$ from the category $\mathcal{R}$ of framed tangles to $(U(\mathfrak{g})\text{-}Mod_f)[[h]]^{str}$ such that $(\widetilde{F}_V \circ Z)(+) = V$. Now by Corollary 6.2, the latter category is equivalent to the category $U_h(\mathfrak{g})\text{-}Mod_{fr}$ of XVII.3 via a strict braided tensor functor E sending the simple $\mathfrak{g}$-module V to the topologically free $U_h(\mathfrak{g})$-module $\widetilde{V}$. Actually, the equivalence E preserves also the duality and the twist (see [Dri89b]). Therefore, $E\circ\widetilde{F}_V\circ Z$ is a ribbon functor from $\mathcal{R}$ to $U_h(\mathfrak{g})\text{-}Mod_{fr}$, sending $(+)$ to V. By the uniqueness statement in Theorem XIV.5.1, we have

$$E \circ \widetilde{F}_V \circ Z = F_{\widetilde{V}} \tag{8.9}$$

where $F_{\widetilde{V}}$ is the ribbon functor introduced in XVII.3. Let K be a framed knot. By construction of the invariant $Q_{\mathfrak{g},V}$, we get

$$Q_{\mathfrak{g},V}(K) = (E \circ \widetilde{F}_V)(Z(K)). \tag{8.10}$$

Let us evaluate $E\circ\widetilde{F}_V$ on a chord diagram D on the circle. By Proposition 8.2, by (8.2–8.3), (8.7) and by XIV.4, we have

$$\begin{aligned}(E \circ \widetilde{F}_V)(D) &= E(\operatorname{tr}_q(C_{D|V})) = \mu_D\, E(\dim_q(V)) \\ &= \mu_D \dim_q(E(V)) = \dim_q(\widetilde{V})\, w_{\mathfrak{g},V}(D)\end{aligned} \tag{8.11}$$

where the quantum trace and dimension are taken first in the ribbon category $(U(\mathfrak{g})\text{-}Mod_f)[[h]]^{str}$, then in the equivalent category $U_h(\mathfrak{g})\text{-}Mod_{fr}$. Combining the last set of equalities with (8.10) yields Theorem 8.3. $\square$

XX.9 Exercises

1. Find all primitive elements of degree ≤ 4 in the Hopf algebra A of Section 3.

2. Let $O(N)$ be the framed trivial knot whose framing twists the knot by $2\pi N$. Compute its Kontsevich invariant $Z(O(N))$ modulo h^4. (Hint: use Corollary XIX.6.5.)

3. Compute the Kontsevich invariant of the closure of the braid σ_1^{2N+1} of B_2 modulo h^4.

XX.10 Notes

The concept of a knot invariant of finite degree (also called "Vassiliev invariant" in the literature) was introduced by Gusarov [Gus91] and Vassiliev [Vas90] [Vas92] around 1989–90. Vassiliev's approch was based on the theory of singularities. Soon after, a number of mathematicians made substantial contributions to this new theory such as D. Bar-Natan, J. Birman, P. Cartier, M. Kontsevich, Le T.Q.T., X.S. Lin, J. Murakami, S. Piunikhin, T. Stanford (see [BN92] [Bir93] [BL93] [Car93] [Kon93] [LM93b] [LM93a] [LM93c] [Lin91] [Piu92] [Piu93] [Sta92] [Sta93]). One will find a review of their results in [Vog93]. A major step forward was undertaken by Kontsevich who constructed the universal knot invariant $Z(K)$ and proved Theorem 2.3. Kontsevich's definition of $Z(K)$ used complicated multiple integrals. It was proved by Cartier [Car93], Le-Murakami [LM93c] and Piunikhin [Piu93] that it could be defined in a simpler way using tangle diagrams. Theorems 6.1 and 7.1 are due to Cartier [Car93].

The contents of Sections 5 and 8 seem to be new. For a generalization, see [KT94].

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