ABELIAN CATEGORIES

An Introduction to the Theory of Functors

PETER FREYD

Harper's Series in Modern Mathematics

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CATEGORIES

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DEDICATION

To the National Science Foundation for paying me while I wrote part of this book.

To Columbia University for paying Sonja Levine, who typed the preliminary manuscript of the book.

To the University of Pennsylvania for paying me while I finished the book.

To Harper & Row for paying John Leahy, who proved the book.

To Pamela Freyd for typing the final manuscript and for many, many other things none of which have anything to do with pay.

P. J. F.

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INTRODUCTION

If topology were publicly defined as the study of families of sets closed under finite intersection and infinite unions a serious disservice would be perpetrated on embryonic students of topology. The mathematical correctness of such a definition reveals nothing about topology except that its basic axioms can be made quite simple. And with category theory we are confronted with the same pedagogical problem. The basic axioms, which we will shortly be forced to give, are much too simple.

A better (albeit not perfect) description of topology is that it is the study of continuous maps; and category theory is likewise better described as the theory of functors. Both descriptions are logically inadmissible as initial definitions, but they more accurately reflect both the present and the historical motivations of the subjects. It is not too misleading, at least historically, to say that categories are what one must define in order to define functors, and that functors are what one must define in order to define natural transformations.

The last notion existed in the mathematical vocabulary long before it had a definition. The fact that it could be mathematically defined was discovered by Eilenberg and MacLane [6]. They began by describing what is perhaps the best known example of a natural equivalence. Their approach seems unimprovable and therefore we imitate it:

Consider a vector space V over a field F, and let V^* be its dual space—the set of linear functionals from V into F together with the natural vector space structure. If V is finite-dimensional then so is V^*, and, indeed, V and V^* have the same dimension. The theory of vector spaces asserts, then, that V and V^* are isomorphic. There does not exist, however, any *particular* isomorphism from V to V^*. (If one is so disposed, he may say that V and V^* are *unnaturally* equivalent.)

Let V^{**} be the dual of V^*. Again the finiteness of V implies that V and V^{**} are isomorphic. But here there is a particular isomorphism, one which stands out, if you will, among all the others. Its definition requires a preliminary definition. For $x \in V$ and $(f: V \to F) \in V^*$, define $\hat{x}(f) = f(x)$. x is a linear transformation from V^* to F, that is, $\hat{x} \in V^{**}$. We define $\Phi: V \to V^{**}$ to be the function which assigns the value $\hat{x} \in V^{**}$ to each $x \in V$. Φ is a one-to-one linear transformation. The equality of dimensions in the case when V is finite thus implies that Φ is onto and hence an isomorphism.

Φ is an example of a *natural equivalence*. The analysis of "natural" starts by the observation that Φ is not just an equivalence between two vector spaces but an entire collection of such equivalences, one for each finite-dimensional vector space. But more importantly, the collection relates not just two big families of vector spaces but two *operations* on vector spaces, namely the identity operation and the second-dual operation. And most importantly, the operation not only operates on vector spaces but on the entire collection of linear transformations between them. We return momentarily to the first duals.

For $g: V_1 \to V_2$ a linear transformation between vector spaces, define $g^*: V_2^* \to V_1^*$ to be the function which assigns to $(f: V_2 \to F) \in V_2^*$ the element $(fg: V_1 \to F) \in V_1^*$ (g^* is called the *dual* of g). By iteration we obtain $g^{**}: V_1^{**} \to V_2^{**}$. The critical property of the collection of Φ's is that for every $g: V_1 \to V_2$ the following diagram commutes:

$$\begin{array}{ccc} V_1 & \xrightarrow{\Phi_1} & V_1^{**} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g^{**}} \\ V_2 & \xrightarrow[\Phi_2]{} & V_2^{**} \end{array}$$

Such operations on linear transformations will be called *functors.* A collection of maps which yield such commuting diagrams as the above will be called a *natural transformation* between functors. In the case at point, we will say that the identity functor and the second-dual functor on finite-dimensional vector spaces are naturally equivalent.

The second-dual functor assigns to each vector space a vector space and to each map between vector spaces a map between the corresponding vector spaces. The assignment has the property that the second-dual of an identity map is an identity map and that $(fg)^{**} = f^{**}g^{**}$ for any pair of composing maps f and g. The proper abstraction of these statements will become our definition of functor.

The notion of functor will be extended to operations which assign objects with different types of structure. The best early example of such is Poincaré's fundamental-group functor: to each topological space X there is assigned a group $\pi(X)$; for each continuous map $g: X_1 \to X_2$ there is assigned a group homomorphism $\pi(g): \pi(X_1) \to \pi(X_2)$.

As before, π carries identity maps into identity maps and behaves well with respect to composition. A similar example is the first-homology functor. It too assigns to a topological space X a group $H(X)$, and to continuous maps it assigns group homomorphisms. These two functors are related by a natural transformation (not an equivalence) which exhibits $H(X)$ as $\pi(X)$ "made abelian."

The precise definition of functor (and hence the precise definition of natural transformation) requires a definition of the things functors are defined on. As a first approximation, let a notion of "structure" be assumed. Let a *category* be a class of sets with structure *and* the class of structure-preserving maps between them. A functor then is a function from one category to another which assigns to the sets belonging to the first, sets belonging to the second; and which assigns to the functions between sets in the first, functions between sets in the second; and which, furthermore, carries identity functions into identity functions and behaves well with respect to composition.

As a second approximation, we eliminate the vagueness of sets-with-structure and structure-preserving functions by defining a *category of sets* as a class $\mathcal{O}$ of sets together with a class $\mathcal{M}$ of functions between them that includes the identity map of each set in $\mathcal{O}$ and the composition of any two composing maps. Thus we throw away the "structure" on the sets. If we start with a category of sets-with-structure and move to this second approximation the "structure," though missing, will have had its influence: first, in reducing the class $\mathcal{M}$ to a proper subclass of the class of all functions; second, in insuring that $\mathcal{M}$ has identity maps and is closed as much as possible with respect to composition.

For the third approximation we throw away the elements of the sets and then, necessarily, the fact that $\mathcal{M}$ is a class of *functions*. We will use the words "object" and "map" as primitives. Define a category as a class $\mathcal{O}$ of *objects*, a class of *maps* $\mathcal{M}$ and a binary operation "not everywhere defined" on $\mathcal{M}$. A list of axioms can be produced so that the class $\mathcal{O}$ is very much like a class of sets, $\mathcal{M}$ like a class of functions between the sets, and the binary operator like the composition of functions.

Among the axioms there would have to be one which insures for each object $A \in \mathcal{O}$ the existence of a map 1_A which behaves (under the binary operation) like the identity map on A. Such an axiom exhibits a redundancy among the primitives. Hence we throw away not only the elements of the objects, but the objects themselves and arrive, finally, at our definition. A **category** is a class of "maps" $\mathcal{M}$ together with a subclass $C \subset \mathcal{M} \times \mathcal{M}$ and a function $c: C \to \mathcal{M}$. If $(x,y) \in C$ we write $c(x,y) = xy$. If $(x,y) \notin C$ we say that "xy is undefined."

Category Axiom 1 (Associativity)

For $x,y,z \in \mathcal{M}$ the following are equivalent:

(a) xy and yz are defined
(b) $(xy)z$ is defined
(c) $x(yz)$ is defined
(d) $(xy)z$ and $x(yz)$ are defined and equal.

Category Axiom 2 (Enough Identities)

Define an **identity map** as an element $e \in \mathcal{M}$ such that whenever either ex or xe is defined it is equal to x. For each $x \in \mathcal{M}$ there are identity maps e_L, e_R such that e_Lx and xe_R are defined.

The recovery of the more familiar proceeds as follows:

Proposition 0.1

If e and e' are identity maps, and ex and $e'x$ are both defined, then $e = e'$.

Proof:

Let $ex = x$ and $e'x = x$. Then $e(e'x) = ex = x$; hence, by Axiom 1, ee' is defined and $e = ee' = e'$. ▌ (We shall use the sign ▌ to indicate ends of proofs.)

Proposition 0.1 together with Axiom 2 asserts the existence of a function $\mathcal{M}$ $R: \to \mathcal{M}$ such that $R(x)$ is an identity map,

$(R(x))x$ is defined, and if e is an identity map and ex is defined then $e = R(x)$. Similarly we define $D\colon \mathscr{M} \to \mathscr{M}$ such that $D(x)$ is an identity map, $xD(x)$ is defined, and if e is an identity map, xe defined, then $e = D(x)$.

Proposition 0.2

xy is defined if and only if $D(x) = R(y)$.

Proof:

$\rightarrow$ Since xy is defined and $x = xD(x)$ it follows that $(xD(x))y$ is defined. Therefore by Axiom 1, $D(x)y$ is defined, $R(y)y$ is defined, $D(x)$ and $R(y)$ are both identity maps, and $D(x) = R(y)$.

$\leftarrow$ If $D(x) = R(y) = e$, then xe and ey are defined and Axiom 1 asserts that $xy = (xe)y = x(ey)$ is defined. ∎

Proposition 0.3

If xy is defined then $D(xy) = D(y)$ and $R(xy) = R(x)$.

Proof:

Since $yD(y)$ and xy are defined, Axiom 1 asserts that $(xy)Dy$ is defined and $D(xy) = D(y)$. Similarly $R(xy) = R(y)$. ∎

"The" class of **objects** is defined to be a class $\mathscr{O}$, the elements of which are indicated by capital Latin letters, in one-to-one correspondence with the identity maps of $\mathscr{M}$. Given $A \in \mathscr{O}$ we indicate the corresponding identity maps by 1_A. We define the **range** of $x \in M$ to be the unique $B \in \mathscr{O}$ such that $1_B = R(x)$; the **domain** of x is the unique $A \in \mathscr{O}$ such that $1_A = D(x)$. Propositions 0.2 and 0.3 translate therefore to the expected statements about functions between sets. For objects $A,B \in \mathscr{O}$ we define $(A,B) \subset \mathscr{M}$ to be the set of maps with A as domain and B as range. We sometimes indicate an element $x \in (A,B)$ by the symbol $x\colon A \to B$, sometimes by $A \xrightarrow{x} B$, and sometimes just by $A \to B$ (if only one element in (A,B) is under discussion). The composition of two maps $A \to B$ and $B \to C$ will be written $A \to B \to C$. Instead of writing equations $A \to B \to C = A \to D \to C$ we shall often say that the diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ D & \to & C \end{array} \qquad \text{commutes.}$$

A **functor** from a category $\mathcal{M}_1$ to $\mathcal{M}_2$ is a function $F\colon \mathcal{M}_1 \to \mathcal{M}_2$ such that:

Functor Axiom 1

If e is an identity map in $\mathcal{M}_1$ then $F(e)$ is an identity map in $\mathcal{M}_2$.

Functor Axiom 2

If xy is defined in $\mathcal{M}_1$ then $F(x)F(y)$ is defined in $\mathcal{M}_2$ and equal to $F(xy)$.

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are classes of objects for $\mathcal{M}_1$ and $\mathcal{M}_2$ we define for $A \in \mathcal{O}_1$, $F(A) \in \mathcal{O}_2$ to be such that $1_{F(A)} = F(1_A)$.

Proposition 0.4

F(Domain (x)) = Domain $(F(x))$ and F(Range (x)) = Range $(F(x))$. ▮ (And here the sign "▮" means no proof.)

Given $x \in (A,B) \subset \mathcal{M}_1$, it follows that $F(x) \in (F(A),F(B)) \subset \mathcal{M}_2$. F will send commutative diagrams into commutative diagrams. Indeed, the functor axioms may be summarized by:

$$\text{if} \qquad \begin{tikzcd} A \arrow[r, "x"] \arrow[dr, "z"'] & B \arrow[d, "y"] \\ & C \end{tikzcd} \qquad \text{commutes,}$$

then
$$\begin{tikzcd} F(A) \arrow[r, "F(x)"] \arrow[dr, "F(z)"'] & F(B) \arrow[d, "F(y)"] \\ & F(C) \end{tikzcd}$$
commutes.

Finally, we define a **natural transformation** between two functors F and G, both from $\mathscr{M}_1$ to $\mathscr{M}_2$, to be a function $\eta\colon \mathcal{O}_1 \to \mathscr{M}_2$ such that:

Transformation Axiom 1

For $A \in \mathcal{O}_1$, $\eta(A) \in (F(A),G(A))$.

Transformation Axiom 2

For any $x \in (A,B) \subset \mathscr{M}_1$ the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(x)"] \arrow[d, "\eta(A)"'] & F(B) \arrow[d, "\eta(B)"] \\ G(A) \arrow[r, "G(x)"'] & G(B) \end{tikzcd}$$
commutes.

Once the definitions existed it was quickly noticed that functors and natural transformations *had* become a major tool in modern mathematics. In 1952 Eilenberg and Steenrod published their *Foundations of Algebraic Topology* [7], an axiomatic approach to homology theory. A *homology theory* was defined as a functor from a topological category to an algebraic category obeying certain axioms. Among the more striking results was their classification of such "theories," an impossible task without the notion of natural equivalence of functors. In a fairly explosive manner, functors and natural transformations have permeated a wide variety of subjects. Such monumental works as Cartan and Eilenberg's *Homological Algebra* [4] and Grothendieck's *Elements of Algebraic Geometry* [11] testify to the fact that functors have become an established concept in mathematics.

In 1948, MacLane drew attention to categories themselves [19]. He observed that many statements about abelian groups were equivalent to statements about the category of abelian groups. (One can prove that *all* statements about abelian groups can be so translated.) He pointed out that an advantage of the "categorical" statement was that it allowed dualization. As a quick example, we shall define a map $A \to B$ to be a *monomorphism* if $X \xrightarrow{x} A \to B = X \xrightarrow{y} A \to B$ always implies that $x = y$. The dual notion is *epimorphism*: $B \to C$ is an epimorphism if $B \to C \xrightarrow{x} X = B \to C \xrightarrow{y} X$ implies that $x = y$. (In the category of abelian groups a map is a monomorphism if and only if it is one-to-one, and it is an epimorphism if and only if it is onto.) A list may be constructed of pairs of such dual notions. The dual of a statement shall be the corresponding statement in which all the words have been replaced by their duals. MacLane found conditions on a *category* such that many of the theorems true for the category of abelian groups still held and he identified certain classes of statements that were true if and only if the dual statement was true. He called such categories *abelian*.

In 1955, Buchsbaum [2] refined the conditions and gave convincing evidence that abelian categories allowed the full development of homological algebra as in Cartan and Eilenberg's book. In 1957 Grothendieck [10] pointed out that certain categories of sheaves were abelian and proceeded to revolutionize algebraic geometry. The ubiquity of abelian categories has since become clear and their importance to mathematics has been widely accepted.

Without elements in the objects it was painfully difficult to prove even simple lemmas for abelian categories. Enough were proved, however, so that mathematicians began to recognize a class of statements, true for the category of abelian groups, which one could be confident were true for arbitrary abelian categories. A metatheorem was in order. It was provided, roughly simultaneously, by Lubkin, Heron, and the author. The proofs were entirely different. They were similar in that they proved that small abelian categories ("small" means a set of objects) were isomorphic to certain very manageable categories of abelian groups.

The aim of this work is to serve as a basis for the theory of abelian categories. The full metatheorem and embedding theorem have been chosen as targets, and indeed the book, exclusive of the exercises, assumes what is hoped to be a geodesic course to those ends. There are no prerequisites except an elementary knowledge of abelian groups and modules. (We again except the exercises.)

The full embedding theorem closes the book in more than a literal sense. Much of the theory *within* abelian categories is reduced to the theory of modules. Further investigations in the subject will necessarily be directed towards functor theory rather than category theory. It is fortunate that the attempted geodesic course of this work brings us into contact with the fundamental tools of functor theory. Chapter 6 not only serves as a vehicle for the major *constructive* part of the embedding theorems but also as an indicator of the powerful similarity of modules and functors. In Chapter 7 we not only dispatch the embedding theorems, but illustrate the principle that important statements about functors viewed as functors may follow from statements about functors viewed as objects in an abelian category.

One important area of functor theory which is not touched in the text is the theory of adjoint functors. It is too important to leave out entirely, and hence we have included a range of exercises on the subject.

Among the many people whose ideas and encouragement were necessary for this book's present existence are David Buchsbaum, Samuel Eilenberg, David Epstein, Serge Lang, Saunders MacLane, Norman Steenrod, and Charles Watts.

The writer must separately acknowledge his collaboration with Barry Mitchell. For many years Mitchell was the writer's mathematical conscience: the erroneous proofs left in this book can be explained as the result only of the writer's perversity in the presence of a master. The full embedding theorem, the target of the work, was first observed by Mitchell, and if the first rule of semantics had not prevented it, this book would be entitled *The Mitchell Theorem*.

EXERCISES ON EXTREMAL CATEGORIES

A. A category in which all maps are identity maps is a **discrete** category. Any function between discrete categories is a functor.

B. A category with only one identity map is a **monoid**. A functor from one monoid to another is a **homomorphism**.

C. A monoid in which every element has an inverse is a **group**. Let F and G be two functors, each from a group A to a group B, and let $\eta: F \rightarrow G$ be a natural transformation. There then exists $x \in B$ such that for all $y \in A$, $F(y) = xG(y)x^{-1}$ —i.e., F and G are "conjugate" homomorphisms. An *inner automorphism* is a functor naturally equivalent to the identity functor.

D. Let $\mathscr{M}$ be a category with objects $\mathcal{O}$ such that for every $A,B \in \mathcal{O}$ it is the case that $(A,B) \cup (B,A)$ has at most one element. Define the relation $\leq$ on $\mathcal{O}$ as follows:

$$\mathrm{A} \leq \mathrm{B} \leftrightarrow (\mathrm{A,B}) \neq \emptyset.$$

$\leq$ is a transitive, reflexive, asymmetric relation, i.e., $(\mathcal{O}, \leq)$ is a partially ordered class. Given two such categories $\mathscr{M}_1$ and $\mathscr{M}_2$ with classes of objects $\mathcal{O}_1$ and $\mathcal{O}_2$, a functor from $\mathscr{M}_1$ to $\mathscr{M}_2$ induces an *order-preserving* function from $\mathcal{O}_1$ to $\mathcal{O}_2$. Moreover, any order-preserving function from $\mathcal{O}_1$ to $\mathcal{O}_2$ is induced by a unique functor from $\mathcal{M}_1$ to $\mathcal{M}_2$.

Let $(\mathcal{O},\leq)$ be a partially ordered class and define $\mathcal{M} = \{[A,B] \mid A \leq B\}$. We introduce a composition on $\mathcal{M}$ as follows: $[A,B][B,C] = [A,C]$; $[A,B][B',C]$ is undefined if $B \neq B'$.

Then $\mathcal{M}$ is a category, $\mathcal{O}$ may be chosen as a class of objects for $\mathcal{M}$, and the partial ordering induced on $\mathcal{O}$ by $\mathcal{M}$ is the original.

EXERCISES ON TYPICAL CATEGORIES

1. Let $\mathcal{M}$ be a category with objects $\mathcal{O}$. Suppose $\mathcal{M}$ is a set. For every $A \in \mathcal{O}$, define $F(A) = \{x \in \mathcal{M} \mid \text{range}\,(x) = A\}$ and for $y: A \to B \in \mathcal{M}$, define $F(y): F(A) \to F(B)$ to be the function induced by composition. F is a one-to-one functor into the category of sets.

2. Let G be a semigroup (a set with an associative binary operation) with a zero element 0 ($0x = 0 = x0$, all $x \in G$). A G-set is defined to be a set S together with a "G-operation" on the set: for every $g \in G$ and $s \in S$ there is assigned $gs \in S$. More formally, a G-set is a set S together with a function $G \times S \to S$ such that for any pair $g, g' \in G$ and $s \in S$ it is the case that $g(g's) = (gg')s$. A pointed G-set is a G-set with a distinguished element $0 \in S$ such that for all $s \in S$, $0s = 0$. A G-homomorphism between two G-sets is any function $h: S_1 \to S_2$ such that for all $g \in G$ and $s \in S_1$ it is the case that $h(gs) = g(h(s))$. A G-homomorphism between pointed G-sets is said to be *passive* if it doesn't kill any element: i.e., for all $s \in S - \{0\}$, $h(s) \neq 0$.

Given any collection of pointed G-sets the collection of all passive homomorphisms between them is a category. We shall call such a category an *algebraic category*.

3. Returning to the category $\mathcal{M}$ of part 1, assume that $0 \notin \mathcal{M}$ and define $G = \mathcal{M} \cup \{0\}$. G becomes a semigroup by defining all products to be zero which are not previously defined in $\mathcal{M}$. Redefine

$F(A)$ for $A \in \mathcal{O}$ to be $\{x \in \mathcal{M} \mid \text{range}(x) = A\} \cup \{0\}$. $F(A)$ is a one-sided ideal in G. Given $y: A \to B$, the induced function, $F(y): F(A) \to F(B)$ is a passive map between pointed G-sets, and conversely, given a passive homomorphism $h: F(A) \to F(B)$ we may define $y = h(1_A)$ and obtain $h = F(y)$. Hence $\mathcal{M}$ is isomorphic to an algebraic category.

CHAPTER 1

FUNDAMENTALS

We shall work within a set-theoretic language such as that in Kelly's *General Topology* [17]. In the Introduction a category was defined as a *class* $\mathscr{M}$ together with a "composition" relation satisfying certain properties. We now explicitly impose what was then tacitly understood, the axiom that for every two objects A and B the class (A,B) is a *set*. (For heuristic purposes, a set S is a class "small enough" so that it has a *cardinality*. The class of all sets is *not* a set.) If $\mathscr{M}$ is a set we shall call it a **small** category.

We have adopted the convention of composing maps in the *linguistic* order, rather than the *diagrammatic* order. Since category theory is intended to be applied to problems concerning sets and functions, and since the linguistic order of composing functions has been generally adopted ($((fg)(x) = f(g(x)))$), the theory ought to conform. Hence $A \xrightarrow{g} B \xrightarrow{f} C$ is written $A \xrightarrow{fg} C$.

The conflict could be avoided by writing the arrows in the other direction: $C \xleftarrow{fg} A = C \xleftarrow{f} B \xleftarrow{g} A$. But here again we are confronted with the traditional precedent in older branches of mathematics, and we hesitate to declare independence (largely because we wish to avoid independence).

As often as possible we shall write "$A \xrightarrow{g} B \xrightarrow{f} C$" instead of "$fg$." We are forced to write "fg" in expressions involving *addition* of maps. The order conflict will concern us only occasionally.

1.1. CONTRAVARIANT FUNCTORS AND DUAL CATEGORIES

A **contravariant functor** from a category $\mathscr{M}_1$ to a category $\mathscr{M}_2$ is a function $F\colon \mathscr{M}_1 \to \mathscr{M}_2$ such that

CF 1. If e is an identity map in $\mathscr{M}_1$ then $F(e)$ is an identity map in $\mathscr{M}_2$.

CF 2. If xy is defined in $\mathscr{M}_1$ then $F(y)F(x)$ is defined in $\mathscr{M}_2$ and equal to $F(xy)$.

(Sometimes we modify "functor" with the word **covariant** in order to emphasize that it is not contravariant.)

For every category $\mathscr{M}$ we define the **dual category** $\mathscr{M}^* = \{x^* \mid x \in M\}$ where $x^*y^* = (yx)^*$. The function $D\colon M \to M^*$ such that $D(x) = x^*$, is a contravariant functor with a contravariant inverse $D\colon M^* \to M$, $D(x^*) = x$.

If $\mathscr{O}$ is a class of objects for $\mathscr{M}$, we may take $\mathscr{O}^* = \{A^* \mid A \in \mathscr{O}\}$ as a class of objects for $\mathscr{M}^*$. Hence $D(A \xrightarrow{x} B) = B^* \xrightarrow{x^*} A^*$.

For each property on maps or objects in categories there is a *dual property*. If P is a property on maps in categories, P^* is the property defined by "x is P^*" $\leftrightarrow$ "x^* is P." Some properties are self-dual: $P = P^*$, the most obvious example being the property of being an identity map. In the next chapter we shall list a set of axioms for abelian categories and it may be observed that if $\mathscr{M}$ is an abelian category then so is $\mathscr{M}^*$. Hence for every theorem that follows from the axioms there is a corresponding *dual theorem*; namely, the theorem in which each property is replaced by its dual property.

1.2. NOTATION

Henceforth when we say that $\mathscr{A}$ is a category we shall interpret $\mathscr{A}$ as being both the maps *and* a class of objects. Hence the statements: "let A be an object in $\mathscr{A}$," "let x be a map in $\mathscr{A}$" are legislated to be meaningful. We shall use only lower-case letters for maps, upper-case for objects. "$x \in \mathscr{A}$" means that x is a map in $\mathscr{A}$; "$A \in \mathscr{A}$" means that A is an object in $\mathscr{A}$.

The usual procedure used in defining a functor $F: \mathscr{A} \to \mathscr{B}$ will be a two-step affair. In the first step we describe, for each $A \in \mathscr{A}$, an object $F(A) \in \mathscr{B}$. In the second step we describe, for each $x \in (A,B) \subset \mathscr{A}$, a map $F(x) \in (F(A),F(B)) \subset \mathscr{A}$.

Suppose that $\mathscr{B}$ is replaced by the category of sets $\mathscr{S}$. In the first step we must, for each $A \in \mathscr{A}$, specify a set $F(A)$. In the second step we must specify, for each $A \xrightarrow{x} B \in \mathscr{A}$, a function $F(x): F(A) \to F(B)$. To do so usually requires the following initial horror:

$$\text{"For } y \in F(A), \quad [F(x)](y) = \ldots\text{."}$$

Let this be taken as a warning for the next section.

1.3. THE STANDARD FUNCTORS

Let $\mathscr{S}$ be the category of sets, $\mathscr{A}$ an arbitrary category, and A an object in $\mathscr{A}$. The functor $(A,-): \mathscr{A} \to \mathscr{S}$ is defined as follows:

For $B \in \mathscr{A}$, $(A,-)(B) = (A,B)$ (the set of maps from A to B).

For $B_1 \xrightarrow{x} B_2 \in \mathscr{A}$, $(A,-)(x)$ is the function
$(A,B_1) \xrightarrow{(A,x)} (A,B_2)$ defined by
$[(A,x)](A \xrightarrow{y} B_1) = A \xrightarrow{y} B_1 \xrightarrow{x} B_2 \in (A,B_2)$.

The contravariant functor $(-,A)\colon \mathscr{A} \to \mathscr{S}$ is defined as follows:

For $B \in \mathscr{A}$, $(-,A)(B) = (B,A)$.

For $B_1 \xrightarrow{x} B_2 \in \mathscr{A}$, $(-,A)(x)$ is the function
$(B_2,A) \xrightarrow{(x,A)} (B_1,A)$ defined by
$[(x,A)](B_2 \xrightarrow{y} A) = B_1 \xrightarrow{x} B_2 \xrightarrow{y} A \in (B_1,A)$.

1.4. SPECIAL MAPS

For the rest of this chapter and all of the next we shall be working inside categories. That is, we assume that one category is under discussion and that all maps and objects mentioned are from that one category. Three special types of maps may be mentioned:

$A \xrightarrow{a} B$ is an **isomorphism** iff there are maps
$B \xrightarrow{b_1} A$ and $B \xrightarrow{b_2} A$ such that
$B \xrightarrow{b_1} A \xrightarrow{a} B$ and $A \xrightarrow{a} B \xrightarrow{b_2} A$
are identity maps.

The property of being an isomorphism is self-dual.

$A \longrightarrow B$ is a **monomorphism** iff the only pairs
$C \xrightarrow{x} A$, $C \xrightarrow{y} A$ such that
$C \xrightarrow{x} A \longrightarrow B = C \xrightarrow{y} A \longrightarrow B$ are the obvious ones:
$x = y$.

$A \longrightarrow B$ is an **epimorphism** iff the only pairs
$B \xrightarrow{x} C$, $B \xrightarrow{y} C$ such that
$A \longrightarrow B \xrightarrow{x} C = A \longrightarrow B \xrightarrow{y} C$ are the obvious ones:
$x = y$.

Monomorphisms and epimorphisms are dual.

In the category of sets or abelian groups our definitions coincide with the old ("monomorphism" means "one-to-one," "epimorphism" means "onto"). The following propositions, obviously true in the well-known models, can be proven in general:

Proposition 1.41

If $A \to B \to C$ is a monomorphism then so is $A \to B$. If both $A \to B$ and $B \to C$ are monomorphisms then so is $A \to B \to C$. ∎

Proposition 1.42

If $A \to B \to C$ is an epimorphism then so is $B \to C$. If both $A \to B$ and $B \to C$ are epimorphisms then so is $A \to B \to C$. ∎

Proposition 1.43

An isomorphism is both a monomorphism and an epimorphism.

Proof:

If $A \xrightarrow{a} B$ is an isomorphism then there are maps such that $A \xrightarrow{a} B \xrightarrow{b_2} A$ is a monomorphism and $B \xrightarrow{b_1} A \xrightarrow{a} B$ is an epimorphism. ∎

Proposition 1.44

If $A \xrightarrow{a} B$ is an isomorphism then there is a unique map $B \xrightarrow{b} A$ such that $A \xrightarrow{a} B \xrightarrow{b} A = 1_A$ and $B \xrightarrow{b} A \xrightarrow{a} B = 1_B$ and $B \xrightarrow{b} A$ is an isomorphism.

Proof:

Let b_1 and b_2 be as in the definition of isomorphisms. $B \xrightarrow{b_1} A = B \xrightarrow{b_1} A \xrightarrow{1} A = B \xrightarrow{b_1} A \xrightarrow{a} B \xrightarrow{b_2} A = B \xrightarrow{1} B \xrightarrow{b_2} A = B \xrightarrow{b_2} A$. ∎

Proposition 1.45

The composition of isomorphisms is an isomorphism. ∎

We say that two objects are **isomorphic** if there is an isomorphism between them. The above two propositions show that the relation on objects so defined is an equivalence relation.

1.5. SUBOBJECTS AND QUOTIENT OBJECTS

Definition. Two monomorphisms $A_1 \to B$ and $A_2 \to B$ are *equivalent* if there are maps $A_1 \to A_2$ and $A_2 \to A_1$ such that

$$\begin{tikzcd} A_1 \arrow[dr] \arrow[d] & \\ A_2 \arrow[r] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_1 \arrow[dr] & \\ A_2 \arrow[r] \arrow[u] & B \end{tikzcd} \qquad \text{commute.}$$

A **subobject** of B is an equivalence class of monomorphisms into B. We define the subobject represented by $A_1 \to B$ to be **contained** in that represented by $A_2 \to B$ if there is a map $A_1 \to A_2$ such that

$$\begin{tikzcd} A_1 \arrow[dr] \arrow[d] & \\ A_2 \arrow[r] & B \end{tikzcd} \qquad \text{commutes.}$$

Note that $A_1 \to A_2$ must be a monomorphism and unique. From the uniqueness we may conclude that if it is also the case that the subobject represented by $A_2 \to B$ is contained in the subobject represented by $A_1 \to B$ it follows that the subobjects are the same and that A_1 and A_2 are isomorphic. The relation of containment is a partial ordering on subobjects.

Note that the relation "is a subobject of" is not transitive. Indeed, subobjects, as we have defined them, do not have subobjects. But this is a baroque consideration. We are initially misled, perhaps, by the transitivity of the relation "is a subset of." Such must be considered an isolated phenomenon. Consider the relation "is a quotient group of" in the classical theory of groups, and recall that "quotient group" is there defined as a set of cosets. Now a set of cosets of a set of cosets of A is not a set of cosets of A. The relation "is a quotient group of" is not transitive.

Two epimorphisms $B \to C_1$ and $B \to C_2$ are *equivalent* if there are maps $C_1 \to C_2$ and $C_2 \to C_1$ such that

$$\begin{tikzcd} & C_1 \arrow[d] \\ B \arrow[ur] \arrow[dr] & \\ & C_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C_1 \\ B \arrow[ur] \arrow[dr] & \\ & C_2 \arrow[uu] \end{tikzcd} \quad \text{commute.}$$

A **quotient object** is an equivalence class of epimorphisms. The quotient object represented by $B \to C_1$ is called smaller than the quotient object represented by $B \to C_2$ if there is a map $C_2 \to C_1$ such that

$$\begin{tikzcd} & C_1 \\ B \arrow[ur] \arrow[dr] & \\ & C_2 \arrow[uu] \end{tikzcd} \quad \text{commutes.}$$

1.6. DIFFERENCE KERNELS AND COKERNELS

Given two maps $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$ we say that $K \to A$ is a **difference kernel** of x and y if

D K 1. $K \to A \xrightarrow{x} B = K \to A \xrightarrow{y} B.$

D K 2. For all $X \to A$ such that $X \to A \xrightarrow{x} B = X \to A \xrightarrow{y} B$ there is a unique $X \to K$ such that

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ K \arrow[rr] & & A \end{tikzcd}$$ commutes.

In other words, a difference kernel of x and y is a map into A which fails to distinguish x and y, and is universal in that respect —i.e., is such that every map into A which fails to distinguish x and y factors uniquely through it.

We are not asserting here that difference kernels exist. We are only defining them.

Proposition 1.61

If $K \to A$ is a difference kernel of $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$ then it is a monomorphism and it represents the largest subobject S of A such that $S \to A \xrightarrow{x} B = S \to A \xrightarrow{y} B$.

Proof:

Let $C \xrightarrow{a} K \to A = C \xrightarrow{b} K \to A = C \xrightarrow{c} A$. Then $C \xrightarrow{c} A \xrightarrow{x} B = C \xrightarrow{c} A \xrightarrow{y} B$, by DK1. But by DK2 the factorization through K is unique and hence $a = b$. ∎

All difference kernels of $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$ represent the same subobject, and conversely, if $K \to A$ is a difference kernel of $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$ and if $K' \to A$ represents the same subobject, then $K' \to A$ is a difference kernel of $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$.

The difference kernel of $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$ is the subobject represented by any of its difference kernels and will be indicated by the notation $Ker(x\text{-}y)$. Formally, therefore, $Ker(x\text{-}y)$ is a subobject of A. But the notation $Ker(x\text{-}y) \to A$ shall be used freely to refer to *a* difference kernel.

The dual notion is difference cokernel. Given $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$ we say that $B \to F$ is a **difference cokernel** of x and y if

D C 1. $A \xrightarrow{x} B \longrightarrow F = A \xrightarrow{y} B \longrightarrow F.$

D C 2. For all $B \to X$ such that $A \xrightarrow{x} B \longrightarrow X = A \xrightarrow{y} B \longrightarrow X$ there is a unique $F \longrightarrow X$ such that

$$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$

commutes.

A difference cokernel must be epimorphic and if one exists it determines a quotient object of difference cokernels called *the* difference cokernel, symbolized by $Cok(x-y)$.

1.7. PRODUCTS AND SUMS

Given a pair of objects A, B we say that an object P is a **product** of A and B if there exists maps $P \xrightarrow{p_1} A$ and $P \xrightarrow{p_2} B$ such that for every pair of maps $X \to A$ and $X \to B$ there is a unique $X \to P$ such that

$$\begin{tikzcd} & A \\ X \arrow[ur] \arrow[r] \arrow[dr] & P \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & B \end{tikzcd}$$ commutes.

Note that in the well-known categories of sets, groups, rings, and topological spaces products can be constructed by taking Cartesian products.

Proposition 1.71

If both P and P′ are products of A and B they are isomorphic.

Proof:

Let $P \xrightarrow{p_1} A$, $P \xrightarrow{p_2} B$, $P' \xrightarrow{p_1'} A$, $P' \xrightarrow{p_2'} B$ be the maps described in the definition of products. There is a map $P \to P'$ such that the diagram

$$\begin{tikzcd} & A \\ P \arrow[ur, "p_1"] \arrow[r] \arrow[dr, "p_2"'] & P' \arrow[u, "p_1'"'] \arrow[d, "p_2'"] \\ & B \end{tikzcd}$$ commutes,

and there is a map $P' \to P$ such that the diagram

$$\begin{tikzcd} & A \\ P' \arrow[ur, "p_1'"] \arrow[r] \arrow[dr, "p_2'"'] & P \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & B \end{tikzcd}$$ commutes.

The composition $P \to P' \to P = P \xrightarrow{x} P$ shares with the map 1_P the property that

$$\begin{tikzcd} & A \\ P \arrow[ur, "p_1"] \arrow[r, "x"] \arrow[dr, "p_2"'] & P \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & B \end{tikzcd}$$ commutes.

The uniqueness condition in the definition of products then implies that $x = 1_P$. Similarly $P' \to P \to P'$ is the identity. ∎

Products are determined "up to isomorphism" and we ought not speak of *the* product. Again, this turns out to be a baroque consideration. The notation $A \times B$ is interpreted as the product of A and B, and it is assumed that

$$A \times B \xrightarrow{p_1} A \quad \text{and}$$

$$A \times B \xrightarrow{p_2} B,$$

though not uniquely determined, are fixed.

The dual of product is sum. Given a pair of objects A and B we say that an object S is a **sum** of A and B if there exist maps $A \xrightarrow{u_1} S$ and $B \xrightarrow{u_2} S$ such that for every pair of maps $A \to X$ and $B \to X$ there is a unique map $S \to X$ such that

$$\begin{tikzcd} A \arrow[d, "u_1"'] \arrow[dr] & \\ S \arrow[r] & X \\ B \arrow[u, "u_2"] \arrow[ur] & \end{tikzcd}$$ commutes.

Sums of the same objects are isomorphic; the notation $A + B$ refers to "the" sum of A and B; the maps $A \xrightarrow{u_1} A + B$ and $B \xrightarrow{u_2} A + B$ are "the" associated maps.

In the well-known categories the word "sum" is traditionally replaced by:

Categories	**Sum**
Sets	Disjoint union
Abelian groups	Direct sum (Cartesian product)
All groups	Free product
Commutative Rings	Tensor product

Given $X \xrightarrow{x_1} A$ and $X \xrightarrow{x_2} B$, the unique map $X \to A \times B$ such that

$$X \to A \times B \xrightarrow{p_1} A = X \xrightarrow{x_1} A \qquad \text{and}$$
$$X \to A \times B \xrightarrow{p_2} B = X \xrightarrow{x_2} B$$

shall be designated $X \xrightarrow{(x_1, x_2)} A \times B$.

On the other side we define $A + B \xrightarrow{\binom{x_1}{x_2}} X$ to be the unique map such that

$$A \xrightarrow{u_1} A + B \xrightarrow{\binom{x_1}{x_2}} X = A \xrightarrow{x_1} X \qquad \text{and}$$
$$B \xrightarrow{u_2} A + B \xrightarrow{\binom{x_1}{x_2}} X = B \xrightarrow{x_2} X.$$

1.8. COMPLETE CATEGORIES

Given an indexed set of objects $\{A_i\}_I$ in a category, its **product** is defined to be an object $\Pi_{i \in I} A_i$ together with maps

$$\{\Pi_{i \in I} A_i \xrightarrow{p_i} A_i\}_I$$

such that for any family $\{X \xrightarrow{x_i} A_i\}_I$ there is a unique $X \to \Pi_I A_i$ such that $X \to \Pi_I A_i \xrightarrow{p_i} A_i = X \xrightarrow{x_i} A_i$. The dual notion is **sum** and it is denoted $\{A_i \xrightarrow{u_i} \Sigma_I A_i\}$.

A category is **left-complete** if every pair of maps has a difference kernel and every indexed *set* of objects a product. Dually, a category is **right-complete** if every pair of maps has a difference cokernel and every indexed set of maps a sum. If a category is both left- and right-complete it is **complete**.

1.9. ZERO OBJECTS, KERNELS, AND COKERNELS

A **zero object** is an object with precisely one map to and from each object. We reserve the symbol O for a zero object. Hence the sets (O,A) and (A,O) have one object each, for all A. The category of sets does not have a zero object; the category of groups does: namely, the group with one element.

If the category has a zero object we define the **zero map** $A \xrightarrow{0} B$ to be the unique map $A \to O \to B$. (It does not matter which zero object is used.)

The **kernel** of $A \xrightarrow{x} B$ is defined to be the difference kernel of $A \xrightarrow{x} B$ and $A \xrightarrow{0} B$. Hence if $K \to A$ is a kernel of $A \xrightarrow{x} B$ then

K 1. $K \to A \xrightarrow{x} B = K \xrightarrow{0} B$

K 2. For all $X \to A$ such that

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow^{0} & \\ A & \xrightarrow{x} & B \end{array}$$

commutes

there is a unique $X \to K$ such that

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \\ K \arrow[r] & A \end{tikzcd} \qquad \text{commutes.}$$

The usual notation for kernel of x is $Ker(x)$. (Hence $Ker(x) = Ker(x\text{-}0)$.)

The **cokernel** of $A \xrightarrow{x} B$ is the difference cokernel of $A \xrightarrow{x} B$ and $A \xrightarrow{0} B$, and it is symbolized by $Cok(x)$.

EXERCISES

A. Epimorphisms need not be onto

1. Let R be the topological space of real numbers, $Q \subset R$ the subspace of rationals. The inclusion map $Q \to R$ is an epimorphism in the category of topological Hausdorf spaces and continuous maps. Indeed, *dense* subobjects may be defined as those represented by epimorphic monomorphisms.

2. The values of a functor need not form a subcategory, i.e., need not be closed under composition. The construction of the minimal counterexample will be useful in a later exercise.

Let [$\to$] be the category with two objects L and R and just three maps: 1_L, 1_R and a map $L \to R$.

Let [$\to\to$] be the category with objects L, M, and R and just six maps: the three identities 1_L, 1_M, 1_R, a unique map in (L,M) to be called $L \to M$, a unique map in (M,R) to be called $M \to R$, and their composition $L \to R$ the unique map in (L,R).

The sum of [$\to$] with itself in the category of small categories may be constructed as the category with objects L_1, R_1, L_2, R_2 and just six maps: the four identities and the two maps $L_1 \to R_1$, $L_2 \to R_2$.

Define the functor $[\rightarrow] + [\rightarrow] \xrightarrow{\pi} [\rightarrow\rightarrow]$ by the following:

$$\pi(L_1) = L$$

$$\pi(R_1) = \pi(L_2) = M$$

$$\pi(R_2) = R$$

$$\pi(L_1 \rightarrow R_1) = L \rightarrow M$$

$$\pi(L_2 \rightarrow R_2) = M \rightarrow R.$$

π is an epimorphism in the category of small categories. The map $L \rightarrow R$ is not a value of π. The maps $L \rightarrow M$ and $M \rightarrow R$ are values.

B. The automorphism class group

Let $\mathscr{A}$ be a category, and I the class of functors from $\mathscr{A}$ to $\mathscr{A}$ which are naturally equivalent to the identity functor. We say that $F: \mathscr{A} \rightarrow \mathscr{A}$ is an *equivalence* if there is a functor $G: \mathscr{A} \rightarrow \mathscr{A}$ such that FG and GF are in I. Let J be the class of functors from $\mathscr{A}$ to $\mathscr{A}$ which are equivalences. I and J are closed under composition. Let K be the class of natural equivalence classes of J. K, if it is a set, is a group, and is called the **automorphism class group of** $\mathscr{A}$.

1. Let $\mathscr{A}$ be the category of ordered sets and order-preserving functions. Let $D: \mathscr{A} \rightarrow \mathscr{A}$ be the functor which assigns to each ordered set the dual (opposite) ordered set. The automorphism class group of $\mathscr{A}$ has at least two elements.

2. For many interesting categories, the automorphism class group is trivial. When such is the case it is significant for roughly the same reasons that it is significant that the groups of field automorphisms of the reals is trivial. All the structure on the real numbers may be recaptured from the field structure alone; any property on real numbers may be, perhaps laborously, defined solely in terms of the properties of that number as an element of a certain field.

In essence the triviality of the automorphism class group means that all the structure on an object that can be defined anywhere can be defined "categorically"—in terms of its properties as an object in an abstract category. In throwing away everything except the way in which the maps compose, enough remains so that all the original structure may be recovered.

C. The category of sets

Let $\mathscr{S}$ be the category of sets and functions. A set D with one element is distinguished in the category by the fact that (A,D) has one element for all $A \in \mathscr{S}$. The elements of a set A are in obvious correspondence with the maps (D,A). The automorphism class group of $\mathscr{S}$ is trivial.

To prove it, let $F\colon \mathscr{S} \to \mathscr{S}$ be any automorphism and first observe that $F(D)$ still has precisely one element. Define, for each $A \in \mathscr{A}$, the function $A \to F(A)$ to be such that

$$\begin{tikzcd} D \arrow[r] \arrow[d, "x"'] & F(D) \arrow[d, "F(x)"] \\ A \arrow[r] & F(A) \end{tikzcd}$$

commutes for all $x \in (D,A)$.

D. The category of small categories

Let $\mathscr{C}$ be the category of small categories. The empty category is distinguished by the fact that there are no functors (maps) into it aside from its own identity map. The category consisting of a single identity map, which category shall be denoted by "1," is distinguished by the facts that it is not the empty category and that $(1,1)$ has a unique element. The special category $[\to]$ defined in Exercise A is distinguished, up to isomorphism, by the facts that $(1,[\to])$ has two elements and $([\to],[\to])$ has three elements. The category $[\to] + [\to]$ is distinguished by the fact that it is the sum of $[\to]$ with itself. The category $[\to\to]$ is distinguished by the fact that $(1, [\to\to])$ has three elements and $([\to], [\to\to])$ has six elements, and by the existence of an epimorphism

$$([\to] + [\to]) \to [\to\to].$$

There are two such epimorphisms. We choose one of them and call it π.

There is a unique map $[\to] \xrightarrow{\alpha} [\to\to]$ which does *not* factor through π.

1. The objects of a small category A are in obvious correspondence with (1,A).

2. The maps of A are in obvious correspondence with $([\to], A)$.

3. Given the category $\mathscr{C}$ and an object $A \in \mathscr{C}$, may we reconstruct the composition table for A? Not quite. The automorphism class group of $\mathscr{C}$ has at least two elements: the identity and the "dual" functor which assigns to each small category its dual. The choice mentioned above in selecting π will determine whether we construct the composition table or the dual composition table.

We may, however, do one or the other, as follows: Given two maps in A, represented by $[\to] \xrightarrow{x} A$ and $[\to] \xrightarrow{y} A$, their composition is defined and equal to the map in A represented by $[\to] \xrightarrow{z} A$ iff there exists a map $[\to\to] \to A$ such that

$$\begin{tikzcd}
{[\longrightarrow]} \arrow[d, "\alpha"'] \arrow[dr, "z"] & \\
{[\longrightarrow\longrightarrow]} \arrow[r] & A \\
{[\longrightarrow] + [\longrightarrow]} \arrow[u, "\pi"] \arrow[ur, "\binom{x}{y}"'] &
\end{tikzcd}$$

commutes.

4. The automorphism class group of $\mathscr{C}$ is the cyclic group of order two.

5. The automorphism group of the category of partially ordered sets and order-preserving maps is the cyclic group of order two. (By Exercise 0-D we may consider the category of partially ordered sets to be a part of the category of small categories. It contains the special objects $[\to]$, $[\to\to]$, $[\to] + [\to]$ and they are distinguished by the same facts.)

E. The category of abelian groups

Let $\mathscr{G}$ be the category of abelian groups. The group of integers Z is distinguished, up to isomorphism, by the facts that:

(1) For every $A \in \mathscr{G}$, A not a zero object, (Z,A) has more than one element.

(2) If $Z \xrightarrow{e} Z$ is such that $e^2 = e$, then either $e = 1$ or $e = 0$.

$Z + Z$ is distinguished by the fact that it is the direct sum of Z with itself in $\mathscr{G}$. Let $Z \xrightarrow{\delta} Z + Z$ be the unique map such tha

$$Z \xrightarrow{\delta} Z + Z \xrightarrow{\binom{1}{0}} Z = 1 \qquad \text{and}$$

$$Z \xrightarrow{\delta} Z + Z \xrightarrow{\binom{0}{1}} Z = 1.$$

1. The elements of $A \in \mathscr{G}$ are in obvious correspondence with (Z,A).

2. Given two elements represented by $Z \xrightarrow{x} A$ and $Z \xrightarrow{y} A$, their sum in A is represented by $Z \xrightarrow{\delta} Z + Z \xrightarrow{\binom{x}{y}} A$.

3. The automorphism class group of $\mathscr{G}$ is trivial.

F. The category of groups

Let $\mathscr{B}$ be the category of all groups, abelian or not. The group of integers is distinguished by the same facts as in Exercise E. The map $Z \xrightarrow{\delta} Z + Z$ is not distinguished. There are two maps with the following properties:

(1) $Z \xrightarrow{\delta} Z + Z \xrightarrow{\binom{1}{0}} Z = 1.$

(2) $Z \xrightarrow{\delta} Z + Z \xrightarrow{\binom{0}{1}} Z = 1.$

(3)
$$\begin{tikzcd}
Z \arrow[rr, "\delta"] \arrow[dd, "\delta"'] & & Z + Z \arrow[d, "(\delta,1)"] \\
 & & (Z + Z) + Z \arrow[d] \\
Z + Z \arrow[r, "(1,\delta)"'] & Z + (Z + Z) \arrow[r] & Z + Z + Z
\end{tikzcd}$$

commutes.

(Tedious computation is needed. Recall that $Z+Z$ is the *free* sum.)

We choose $Z \xrightarrow{\delta} Z+Z$ to be one of the two maps and as in Exercise E we recover either the multiplication table of $A \in \mathscr{B}$ or the dual multiplication table.

The automorphism class group of $\mathscr{B}$ is trivial. The two-way choice for δ suggests that there are two elements in the group. However, the functor $D: \mathscr{B} \to \mathscr{B}$ which carries each group into its dual (opposite) group is naturally equivalent to the identity.

G. Categories of topological spaces

1. Let $\mathscr{T}$ be the category of topological spaces. The space S with two elements and the nonextremal topology (S has three open sets), is distinguished by the fact that (S,S) has three elements. The space with one element, "D," is distinguished by the fact that (S,D) has one element. Choose one of the two maps in (D,S) and call it $D \xrightarrow{u} S$. There is an obvious correspondence between the elements of $A \in \mathscr{T}$ and the maps (D,A). For every map $A \xrightarrow{a} S$, let $A_a \subset (D,A)$ be defined by $A_a = \{D \to A \mid D \to A \xrightarrow{a} S = u\}$. Then one of the two following facts is always true (depending on the choice of u):

(i) For every $A \xrightarrow{a} S$, A_a corresponds to a *closed* subset of A and, conversely, every closed subset of A corresponds to A_a for some map $A \xrightarrow{a} S$.

(ii) For every $A \xrightarrow{a} S$, A_a corresponds to an *open* subset of A and conversely.

Which of these two possibilities is true may be tested by the following: Let A be any object in $\mathscr{T}$ such that for every $D \xrightarrow{x} A$ there exists $a \in (A,S)$ such that $A_a = \{x\}$. If for all such A every subset of (D,A) is of the form A_a for some $a \in (A,S)$, then (ii) is true.

The automorphism class group of $\mathscr{T}$ is trivial.

2. Let $\mathscr{T}_1$ be the category of T_1 spaces, i.e., those in which single points are closed. The space S does not live in $\mathscr{T}_1$. The space D is distinguished by the fact that (A,D) has one element for all $A \in \mathscr{T}_1$. A subset $C \subset (D,A)$ corresponds to a closed set iff there is a space

X and maps $A \to X$, $D \xrightarrow{u} X$ such that

$$C = \{D \xrightarrow{x} A \mid D \xrightarrow{x} A \to X = u\}.$$

The automorphism class group of $\mathscr{T}_1$ is trivial.

3. Let $\mathscr{T}_2$ be the category of Hausdorf spaces. The space D is distinguished by the same fact as before. $C \subset (D,A)$ corresponds to a closed set iff there is a space X and maps $A \xrightarrow{a} X$, $A \xrightarrow{b} X$ such that $G = \{D \xrightarrow{x} A \mid D \xrightarrow{x} A \xrightarrow{a} X = D \xrightarrow{x} A \xrightarrow{b} X\}$. (Every closed set is a difference kernel and conversely.) The automorphism class group of $\mathscr{T}_2$ is trivial.

H. Conjugate maps

For distinct objects A and B in a category $\mathscr{A}$ we say that $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$ are *conjugate* if there are automorphisms $\phi_1 \in (A,A)$, $\phi_2 \in (B,B)$ such that

$$A \xrightarrow{y} B = A \xrightarrow{\phi_1} A \xrightarrow{x} B \xrightarrow{\phi_2^{-1}} B.$$

We say that $A \xrightarrow{x} A$ and $A \xrightarrow{y} A$ are *conjugate* if there is an automorphism $\phi \in (A,A)$ such that

$$A \xrightarrow{y} A = A \xrightarrow{\phi} A \xrightarrow{x} A \xrightarrow{\phi^{-1}} A.$$

A functor F: $\mathscr{A} \to \mathscr{A}$ is an *inner automorphism* if:

(1) F is naturally equivalent to the identity.
(2) $F(A) = A$ for all $A \in \mathscr{A}$.

1. Two maps are conjugate iff there is an inner automorphism which carries one into the other.

2. The two δ's of Exercise F are conjugate.

I. Definition theory

Let $\mathscr{G}$ be the category of groups. Suppose $F(A)$ is a one-variable formula in the nth order language of the theory of groups (where the one free variable is understood to be a group). There exists a formula $F'(A)$ in the nth order theory of $\mathscr{G}$ such that $F'(A) \leftrightarrow F(A)$. Indeed, F' will often be in a lower order language than that of F, as is the case for $F(A) \leftrightarrow A$ is isomorphic to the infinite cyclic group.

$$F'(A) \leftrightarrow \forall_B \exists_x \{[x \in (A,B) \wedge [(x \neq 0) \vee (B = O)]\}$$
$$\wedge \forall_e \{[(e \in (A,A)) \wedge (e^2 = e) \rightarrow [(e = 0) \vee (e = 1)]\}.$$

Moreover, for each of the above mentioned categories with trivial automorphism class group the same situation occurs. In the case of the category of small categories we must take the map $[\rightarrow] + [\rightarrow] \xrightarrow{\pi} [\rightarrow\rightarrow]$ as an additional predicate.

CHAPTER 2

FUNDAMENTALS OF ABELIAN CATEGORIES

A category $\mathscr{A}$ is **abelian** if

A 0. $\mathscr{A}$ has a zero object.

A 1. For every pair of objects there is a product and
A 1*. a sum.

A 2. Every map has a kernel and
A 2*. a cokernel.

A 3. Every monomorphism is a kernel of a map.
A 3*. Every epimorphism is a cokernel of a map.

Axiom A 3 may be read as "every subobject is *normal.*" Most categories that arise in nature satisfy Axioms A 0 through A 2. Often Axiom A 0 is satisfied by using base points. Many categories satisfy one of A 3 or A 3*. Compact Hausdorf spaces with base points satisfy A 3; all groups (abelian or not) satisfy A 3*.

2.1. THEOREMS FOR ABELIAN CATEGORIES

Consider an object A. Let $\mathbf{S}$ be the family of subobjects of A, $\mathbf{Q}$ the family of quotient objects. Define $Cok\colon \mathbf{S} \to \mathbf{Q}$ to be the function which assigns to each subobject its cokernel.

Dually, define $Ker\colon \mathbf{Q} \to \mathbf{S}$ to be the function which assigns kernels. Note that *Cok* and *Ker* are order-reversing functions. Axioms A 3 and A 3* are equivalent to:

Theorem 2.11 for abelian categories

Ker and Cok are inverse functions.

Proof:

Let $A' \to A$ be a monomorphism. By Axiom A 3 it is the kernel of some map $A \to B$. Let $A \to F$ be the cokernel of $A' \to A$ and let $K \to A$ be the kernel of $A \to F$. We shall apply the definition of kernel and cokernel a number of times. For each it will be necessary to verify that a certain composition is the zero map. To begin: $A' \to A \to B = 0$ and there is a map $F \to B$ yielding a commutative diagram:

$$\begin{tikzcd}
Ker(A \to B) = A' \arrow[dr] & & F = Cok(A' \to A) \arrow[dd] \\
& A \arrow[ur] \arrow[dr] & \\
Ker(A \to F) = K \arrow[ur] & & B
\end{tikzcd}$$

$A' \to A \to F = 0$; there is a map $A' \to K$ such that

$$\begin{tikzcd}
A' \arrow[dr] \arrow[dd] & \\
& A \\
K \arrow[ur] &
\end{tikzcd}$$

commutes.

$K \to A \to B = 0$; there is a map $K \to A'$ such that

$$\begin{tikzcd} A' \arrow[dr] & \\ & A \\ K \arrow[u] \arrow[ur] & \end{tikzcd} \qquad \text{commutes.}$$

Thus the subobjects represented by $A' \to A$ and $K \to A$ are contained in each other and hence equal. $A' \to A$ is a kernel of $A \to F$. Thus $KerCok =$ Identity, and dually, $CokKer =$ Identity. ▮

Theorem 2.12 for abelian categories

A map that is both monomorphic and epimorphic is an isomorphism.

Proof:

Let $A \xrightarrow{a} B$ be monomorphic and epimorphic. $B \to O$ is clearly the cokernel of $A \xrightarrow{a} B$. $B \xrightarrow{1} B$ is clearly a kernel of $B \to O$. By the last theorem so is $A \to B$. (Already we have shown that A and B are isomorphic—they are both kernels of the same map. The theorem asserts that the *map* $A \xrightarrow{a} B$ is an isomorphism.) Hence there is a map $B \xrightarrow{b_1} A$ such that $B \xrightarrow{b_1} A \xrightarrow{a} B = B \xrightarrow{1} B$. Dually we note that $O \to A$ is a kernel of $A \xrightarrow{a} B$ and that both $A \xrightarrow{a} B$ and $A \xrightarrow{1} A$ are cokernels of $O \to A$. Hence there is a map $B \xrightarrow{b_2} A$ such that $A \xrightarrow{a} B \xrightarrow{b_2} A = A \xrightarrow{1} A$. By the definition of isomorphism, $A \xrightarrow{a} B$ is such. ▮

The **intersection** of two subobjects of A is defined to be their greatest lower bound in the family of subobjects of A.

Theorem 2.13 for abelian categories

Every pair of subobjects has an intersection.

Proof:

We shall prove a stronger property. Let $A_1 \to A$ and $A_2 \to A$ be monomorphisms, $A \to F$ a cokernel of $A_1 \to A$ and $A_{12} \to A_2$ a kernel of $A_2 \to A \to F$.

First note that since

$$\begin{array}{ccc} A_{12} & \to & A_2 \\ & & \downarrow \\ & & A \to F \end{array}$$

is zero there is a map $A_{12} \to A_1$ (necessarily monomorphic) such that

(2.131)
$$\begin{array}{ccc} A_{12} & \to & A_2 \\ \downarrow & & \downarrow \\ A_1 & \to & A \end{array} \qquad \text{commutes.}$$

(We use the fact that $A_1 = Ker(A \to Cok(x_1))$.)

Let $X \to A_1$ and $X \to A_2$ be any pair of maps such that

$$\begin{array}{ccc} X & \to & A_2 \\ \downarrow & & \downarrow \\ A_1 & \to & A \end{array} \qquad \text{commutes.}$$

We shall show that there is a unique $X \to A_{12}$ such that

$$X \to A_{12} \to A_1 = X \to A_1 \quad \text{and} \quad X \to A_{12} \to A_2 = X \to A_2$$

(when X "is a subobject" we will have proved containment in A_{12}).

The map $X \to A_{12}$ exists since $X \to A_2 \to F = X \to A_1 \to F = 0$ and $A_{12} \to A_2 = Ker(A_2 \to F)$. Thus there is a *unique* map $X \to A_{12}$ such that $X \to A_{12} \to A_2 = X \to A_2$. The other equation follows from $X \to A_{12} \to A_1 \to A = X \to A_2 \to A = X \to A_1 \to A$ and the fact that $A_1 \to A$ is a monomorphism. ∎

Dually every pair of quotient objects has a greatest lower bound. Since *Ker* and *Cok* are order-reversing and inverses of each other, every pair of subobjects has a least upper bound. Hence the family of subobjects of A is a lattice. We shall use the standard lattice symbols $\cup$ and $\cap$.

Theorem 2.14 for abelian categories

Every pair of maps $A \xrightarrow{x} B$, $A \xrightarrow{y} B$ has a difference kernel.

Proof:

We construct the difference kernel by "intersecting the graphs."

Consider the monomorphisms $A \xrightarrow{(1,x)} A \times B$ and $A \xrightarrow{(1,y)} A \times B$. Let $K \to A \times B$ represent their intersection. ($(1,x)$ is a monomorphism since when it is followed by p_1 the composition is a monomorphism.) We obtain a commutative diagram:

$$\begin{tikzcd} K \arrow[r, "k_1"] \arrow[d, "k_2"'] & A \arrow[d, "{(1,y)}"] \\ A \arrow[r, "{(1,x)}"'] & A \times B \end{tikzcd}$$

By applying p_1 we see that $k_1 = k_2$, and by applying p_2 we see that $K \xrightarrow{k} A \xrightarrow{x} B = K \xrightarrow{k} A \xrightarrow{y} B$ (where $k = k_1 = k_2$). Let $X \to A$ be such that $X \to A \xrightarrow{x} B = X \to A \xrightarrow{y} B$. Then

$$\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d, "{(1,y)}"] \\ A \arrow[r, "{(1,x)}"'] & A \times B \end{tikzcd}$$ commutes.

(to prove it apply both p_1 and p_2), and by the proof of Theorem 2.13 there is a unique factorization of $X \to A$ through $K \to A$. ∎

Dually for every pair of maps $A \xrightarrow{x} B$, $A \xrightarrow{y} B$ there is a difference cokernel.

A commutative diagram

$$\begin{array}{ccc} P & \to & B \\ \downarrow & & \downarrow \\ A & \to & C \end{array}$$

is a **pullback** diagram if for every pair of maps $X \to A$ and $X \to B$ such that

$$\begin{array}{ccc} X & \to & B \\ \downarrow & & \downarrow \\ A & \to & C \end{array} \qquad \text{commutes,}$$

there is a unique $X \to P$ such that $X \to P \to A = X \to A$ and $X \to P \to B = X \to B$. Our proof in Theorem 2.13 was actually a proof that Diagram 2.131 was a pullback diagram.

Theorem 2.15 for abelian categories

Every diagram $\begin{array}{ccc} & & B \\ & & \downarrow \\ A & \to & C \end{array}$ *can be enlarged to a pullback diagram.*

Proof:

Consider $A \times B$ and the two maps $A \times B \xrightarrow{p_1} A \to C$ and $A \times B \xrightarrow{p_2} B \to C$, and let $K \to A \times B$ be their difference kernel. Define

$$K \to A = K \to A \times B \xrightarrow{p_1} A$$

$$K \to B = K \to A \times B \xrightarrow{p_2} B.$$

It is easy to verify that

$$\begin{array}{ccc} K & \to & B \\ \downarrow & & \downarrow \\ A & \to & C \end{array}$$

is a pullback diagram. ∎

Proposition 2.151

If $\begin{array}{ccc} P & \to & B \\ \downarrow & & \downarrow \\ A & \to & C \end{array}$ *and* $\begin{array}{ccc} P' & \to & B \\ \downarrow & & \downarrow \\ A & \to & C \end{array}$ *are pullback diagrams then P and P' are isomorphic. Indeed there is a unique map $P \to P'$ such that*

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] \arrow[d] & \\ A & P' \arrow[l] \arrow[r] & B \end{tikzcd}$$

commutes, and it is an isomorphism.

Proof:

Virtually the same as for products (Prop. 1.71). To make it easy we may note that in the category whose objects are $\{(A \to C) \mid A \in \mathscr{A}\}$ (C fixed) and whose maps are described by $(A \to C, B \to C) = \{A \to B \in (A,B) \mid A \to B \to C = A \to C\}$, the product $(P \to C) = (A \to C) \times (B \to C)$ is precisely the diagonal map of the pullback diagram in $\mathscr{A}$. ∎

A commutative diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & P \end{array}$$

is a **pushout** diagram if for every pair of maps $B \to X$ and $C \to X$ such that

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & X \end{array} \qquad \text{commutes,}$$

there is a unique $P \to X$ such that $B \to P \to X = B \to X$ and $C \to P \to X = C \to X$.

Theorem 2.15* for abelian categories

Every diagram $\begin{array}{c} A \to B \\ \downarrow \\ C \end{array}$ *can be enlarged to a pushout diagram, and, up to isomorphism, uniquely so.* ∎

The **image** of a map $A \to B$ is properly defined as the smallest subobject of B such that $A \to B$ factors through the representing monomorphisms.

Theorem 2.16 for abelian categories

$A \to B$ has an image and it is equal to KerCok$(A \to B)$.

Proof:

We shall say that a monomorphism $S \to B$ *allows* $A \to B$ if $A \to B$ factors through it, i.e., if there is a map $A \to S$ such that $A \to S \to B = A \to B$. We shall say that an epimorphism $B \to F$ *kills* $A \to B$ if $A \to B \to F = 0$. These two properties are subobject and quotient object properties respectively.

Lemma. A subobject allows $A \to B$ iff its cokernel kills $A \to B$. ∎

Now *Cok*$(A \to B)$ is the largest quotient object that kills $A \to B$. Hence *KerCok*$(A \to B)$ is the smallest subobject that allows $A \to B$, i.e., it is the image of $A \to B$. ∎

Notation: $Im(A \xrightarrow{x} B)$ or $Im(x)$ is the image of $A \xrightarrow{x} B$.

Theorem 2.17 for abelian categories

$A \to B$ is epimorphic iff $Im(A \to B) = B$, and hence, iff $Cok(A \to B) = O$.

Proof:

$\to$ Clear.

$\leftarrow$ If $Cok(A \to B) = O$ then by last theorem $Im(A \to B) = B \xrightarrow{1} B$. Suppose $A \to B \xrightarrow{x} C = A \to B \xrightarrow{y} C$. Let $Ker(x\text{-}y) \to B$ be the difference kernel of x and y. Then there is $A \to Ker(x\text{-}y)$ such that $A \to B = A \to Ker(x\text{-}y) \to B$, and $Ker(x\text{-}y)$ contains the image of $A \to B$. Thus $Ker(x\text{-}y) = B$ and $x = y$. ∎

For $A \xrightarrow{x} B$ there exists a unique map $A \to Im(x)$ such that $A \to Im(x) \to B = A \xrightarrow{x} B$.

Theorem 2.18 for abelian categories

$A \xrightarrow{x} Im(x)$ is epimorphic.

Proof:

If $Cok(A \to Im(x)) \neq O$, then $A \to Im(x)$ factors through a proper subobject of $Im(x)$, which contradicts the definition of $Im(x)$. ∎

The dual of image is coimage. The **coimage** of $A \to B$ is the smallest quotient object of A through which $A \to B$ factors.

Notation: $Coim(A \to B)$, $Coim(x)$.

Theorem 2.16* for abelian categories

$Coim(A \to B) = CokKer(A \to B)$. ∎

Theorem 2.17* for abelian categories
$A \to B$ is monomorphic iff $Coim(A \to B) = A$ iff $Ker(A \to B) = O$. ∎

Let $A \to I'$ be a coimage of $A \to B$ and consider $A \to I' \to B$.

Theorem 2.18* for abelian categories
$I' \to B$ is monomorphic. ∎

"Unique factorization theorem" for abelian categories, 2.19
If $A \to B = A \to I \to B$ where $A \to I$ is epimorphic and $I \to B$ is monomorphic, then $A \to I$ is a coimage of $A \to B$ and $I \to B$ is an image of $A \to B$ and for any other such factorization $A \to \bar{I} \to B$ where $A \to \bar{I}$ is epimorphic and $\bar{I} \to B$ monomorphic, there is a unique $I \to \bar{I}$ such that

$$\begin{tikzcd} & I \arrow[dr] \arrow[dd] & \\ A \arrow[ur] \arrow[dr] & & B \\ & \bar{I} \arrow[ur] & \end{tikzcd}$$

commutes,

and $I \to \bar{I}$ is necessarily an isomorphism. ∎

2.2. EXACT SEQUENCES

Theorem 2.21 for abelian categories
For $A \to B \to C$ the following conditions are equivalent:

(*a*) $Im(A \to B) = Ker(B \to C)$
(*b*) $Coim(B \to C) = Cok(A \to B)$
(*c*) $A \to B \to C = 0$ *and* $K \to B \to F = 0$

where $K \to B$ is a kernel of $B \to C$ and $B \to F$ is a cokernel of $A \to B$.

Proof:

$(a) \rightarrow (c)$ That $A \to B \to C = 0$ is clear; we must show that $K \to B \to F = 0$. We note that $Ker(B \to C) = Im(A \to B) = KerCok(A \to B) = Ker(B \to F)$. Because $K \to B$ is a kernel of $B \to C$, it follows that $K \to B \to F = 0$.

$(c) \rightarrow (a)$ Let $I \to B$ be a kernel of $B \to F$, and thus an image of $A \to B$. Since $K \to B \to F = 0$, $Ker(B \to C) \subset Im(A \to B)$. On the other hand, since $A \to B \to C = 0$, $Im(A \to B) \subset Ker(B \to C)$.

That $(b) \leftrightarrow (c)$ is proved dually. ∎

We say that a sequence $\cdots \to A_1 \to A_2 \to A_3 \to A_4 \to \cdots$ is **exact** if for each i, $Im(A_{i-1} \to A_i) = Ker(A_i \to A_{i+1})$.

Proposition 2.22

$O \to K \to A$	*is exact iff* $K \to A$ *is monomorphic.*
$O \to K \to A \to B$	*is exact iff* $K \to A$ *is the kernel of* $A \to B$.
$B \to F \to O$	*is exact iff* $B \to F$ *is epimorphic.*
$A \to B \to F \to O$	*is exact iff* $B \to F$ *is the cokernel of* $A \to B$.
$O \to A \to B \to O$	*is exact iff* $A \to B$ *is an isomorphism.*
$A \to B \xrightarrow{1} B$	*is exact iff* $A \to B$ *is the zero map.*
$O \to A \to B \to C \to O$	*is exact iff* $A \to B$ *is a monomorphism and* $B \to C$ *is a cokernel of* $A \to B$. ∎

2.3. THE ADDITIVE STRUCTURE FOR ABELIAN CATEGORIES

Theorem 2.31 for abelian categories

The sequence $O \to A \xrightarrow{u_1} A + B \xrightarrow{\binom{0}{1}} B \to O$ *is exact.*

Proof:

$A \xrightarrow{u_1} A + B$ is clearly monomorphic since $A \xrightarrow{u_1} A + B \xrightarrow{\binom{1}{0}} A$ is. To prove that $A + B \xrightarrow{\binom{0}{1}} B$ is a cokernel of u_1, let $A + B \xrightarrow{\binom{x}{y}} X$ be a map such that $A \xrightarrow{u_1} A + B \xrightarrow{\binom{x}{y}} X = 0$. Then $x = 0$ and $A + B \xrightarrow{\binom{x}{y}} X = A + B \xrightarrow{\binom{0}{1}} B \xrightarrow{y} X$. ∎

Theorem 2.32

$O \to A \xrightarrow{(1,0)} A \times B \xrightarrow{p_2} B \to O$ is exact. ∎

Proposition 2.33 for abelian categories

The intersection of $A \xrightarrow{v_1} A + B$ and $B \xrightarrow{2} A + B$ is zero.

Proof:

The proof follows from the construction of intersections. ∎

Dually, **2.34**

The greatest lower bound of the quotient objects $A \times B \xrightarrow{p_1} A$ and $A \times B \xrightarrow{p_2} B$ is O. ∎

By *Ker–Cok* duality, the least upper bound of $A \xrightarrow{u_1} A + B$, $B \xrightarrow{u_2} A + B$ is $A + B$. Given a sum $A_1 + A_2 + \cdots + A_n$ and a product $B_1 \times \cdots \times B_m$, every map from the sum to the product is represented uniquely by a matrix (x_{ij}) where

$$A_i \xrightarrow{x_{ij}} B_j = A_i \xrightarrow{u_i} A_1 + \cdots + A_n \to B_1 \times \cdots \times B_m \xrightarrow{p_j} B_j.$$

Theorem 2.35 for abelian categories

$A_1 + A_2 \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} A_1 \times A_2$ is an isomorphism.

Proof:

Let $K \to A_1 + A_2$ be the kernel of $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. Then $K \to A_1 + A_2 \xrightarrow{\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)} A_1 \times A_2 \xrightarrow{p_2} A_2 = K \to A_1 + A_2 \xrightarrow{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} A_2$ and $K \to A_1 + A_2$ is contained in $A_1 \xrightarrow{u_1} A_1 + A_2$. Similarly it is contained in $A_2 \xrightarrow{u_2} A_1 + A_2$, and hence it is contained in their intersection, which is zero. Thus $K = O$ and $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ is monomorphic. Dually it is epimorphic and hence an isomorphism. ▌

Thus $A_1 + A_2 \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} A_1$, $A_1 + A_2 \xrightarrow{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} A_2$ may be taken as the *product* of A_1 and A_2.

Notation: $A \oplus B$ shall be used to denote the sum $A + B$ and the product $A \times B$, and shall be called the **direct sum** of A and B.

$$A \xrightarrow{\delta} A \oplus A = A \xrightarrow{(1,1)} A + A \text{ the "diagonal map."}$$

$$A \oplus A \xrightarrow{\sigma} A = A \times A \xrightarrow{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} A \text{ the "summation map."}$$

Given two maps $A \xrightarrow{x} B$, $A \xrightarrow{y} B$ we define

$$A \xrightarrow{x \underset{L}{+} y} B = A \xrightarrow{\delta} A \oplus A \xrightarrow{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)} B$$

$$A \xrightarrow{x \underset{R}{+} y} B = A \xrightarrow{(x,y)} B \times B \xrightarrow{\sigma} B.$$

Proposition 2.36

$$0 \underset{L}{+} x = x = x \underset{L}{+} 0; \quad 0 \underset{R}{+} x = x = x \underset{R}{+} 0.$$

Proof:

$$A \oplus A \xrightarrow{\binom{x}{0}} B = A + A \xrightarrow{p_1} A \xrightarrow{x} B$$

and $$A \xrightarrow{\delta} A + A \xrightarrow{\binom{x}{0}} B = A \to A + A \xrightarrow{p_1} A \xrightarrow{x} B$$
$$= A \xrightarrow{x} B. \quad \blacksquare$$

Proposition 2.37

For $B \xrightarrow{u} C$, $(ux \underset{L}{+} uy) = u(x \underset{L}{+} y)$ *and for* $C \xrightarrow{z} A$, $(xz \underset{R}{+} yz) = (x \underset{R}{+} y)z$

Proof:

$$A + A \xrightarrow{\binom{x}{y}} B \xrightarrow{u} C = A + A \xrightarrow{\binom{ux}{uy}} C. \quad \blacksquare$$

Theorem 2.38

$\underset{L}{+}$ *and* $\underset{R}{+}$ *are the same binary operations, and they are (it is) associative and commutative.*

Proof:

Consider $A \xrightarrow{\delta} A \oplus A \xrightarrow{\begin{pmatrix} w & x \\ y & z\end{pmatrix}} B \oplus B \xrightarrow{\sigma} B$. Observe that $\begin{pmatrix} w & x \\ y & z\end{pmatrix} = \left(\begin{pmatrix} w \\ y\end{pmatrix}\begin{pmatrix} x \\ z\end{pmatrix}\right)$ (i.e., if we label $A \oplus A = D$ and $\begin{pmatrix} w \\ y\end{pmatrix} = d_1$, $\begin{pmatrix} x \\ z\end{pmatrix} = d_2$, then $\begin{pmatrix} w & x \\ y & z\end{pmatrix} = (d_1, d_2)$). Thus

$$A \oplus A \xrightarrow{\begin{pmatrix} w & x \\ y & z\end{pmatrix}} B \oplus B \xrightarrow{\sigma} B = \left[\begin{pmatrix} w \\ y\end{pmatrix} \underset{R}{+} \begin{pmatrix} x \\ z\end{pmatrix}\right]$$

and

$$A \xrightarrow{\delta} A \oplus A \xrightarrow{\begin{pmatrix} w & x \\ y & z\end{pmatrix}} B \oplus B \xrightarrow{\sigma} B = \left[\begin{pmatrix} w \\ y\end{pmatrix}\delta \underset{R}{+} \begin{pmatrix} x \\ z\end{pmatrix}\delta\right]$$
$$= \left[(w \underset{L}{+} y) \underset{R}{+} (x \underset{L}{+} y)\right].$$

On the other hand, $A \xrightarrow{\delta} A \oplus A \xrightarrow{\left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)} B \oplus B = [(w,x) \underset{L}{+} (y,z)]$ and $A \xrightarrow{\delta} (A \oplus A) \xrightarrow{\left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)} (B \oplus B) \xrightarrow{\sigma} B = (w \underset{R}{+} x) \underset{L}{+} (y \underset{R}{+} z)$. Thus $(w \underset{R}{+} x) \underset{L}{+} (y \underset{R}{+} z) = (w \underset{L}{+} y) \underset{R}{+} (x \underset{L}{+} z)$. Letting $x = y = 0$ we obtain $w \underset{L}{+} z = w \underset{R}{+} z$.

Calling both $\underset{L}{+}$ and $\underset{R}{+}$ by the same name "$+$" the equation rewrites: $(u + x) + (y + z) = (u + y) + (x + z)$; letting $y = 0$, $(u + x) + z = u + (x + z)$, and letting $u = z = 0$, $x + y = y + x$. ∎

The usual rules of matrix multiplication can now be proven.

Theorem 2.39 for abelian categories
The set (A,B) with the operation $+$ is an abelian group.

Proof:

Given $A \xrightarrow{x} B$ consider the map $A \oplus B \xrightarrow{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)} A \oplus B$. Its kernel $K \xrightarrow{(a,b)} A \oplus A$ is such that $0 = K \xrightarrow{(a,b)} A \oplus B \xrightarrow{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)} A \oplus B = K \xrightarrow{(a,xa+b)} A \oplus B$ and $a = 0, b = 0$. Thus $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is monomorphic. Dually it is epimorphic and thus an isomorphism. It is easily seen that its inverse must be of the form $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ where $y + x = 0$. ∎

From now on, (A,B) shall refer to the *group* of maps from A to B. For each triple A,B,C we have a bilinear function c: $((A,B),(B,C)) \to (A,C)$ defined through composition of maps. The **endomorphisms** of an object A, that is, the maps from A to A, form a ring with unit.

2.4. RECOGNITION OF DIRECT SUM SYSTEMS

A set of four maps

$$A_1 \xrightarrow{u_1} S, \qquad A_2 \xrightarrow{u_2} S$$

$$S \xrightarrow{p_1} A_1, \qquad S \xrightarrow{p_2} A_2$$

is a **direct sum system** if S is a direct sum of A_1 and A_2 and $u_1 = (1,0), u_2 = (0,1), p_1 = \begin{pmatrix}1\\0\end{pmatrix}, p_2 = \begin{pmatrix}0\\1\end{pmatrix}$. Two useful theorems on the recognition of direct sum systems are the following:

Theorem 2.41 for abelian categories
If u_1, u_2, p_1, p_2 are such that

$$A_1 \xrightarrow{u_1} S \xrightarrow{p_1} A_1 = 1_{A_1}, \qquad A_2 \xrightarrow{u_2} S \xrightarrow{p_2} A_2 = 1_{A_2},$$

$$A_1 \xrightarrow{u_1} S \xrightarrow{p_2} A_2 = 0, \qquad A_2 \xrightarrow{u_2} S \xrightarrow{p_1} A_1 = 0,$$

and $u_1p_1 + u_2p_2 = 1_S$,
then u_1, u_2, p_1, p_2 form a direct sum system.

Proof:
Let $X \xrightarrow{x_1} A_1$ and $X \xrightarrow{x_2} A_2$ be an arbitrary pair of maps. Define $X \xrightarrow{x} S = u_1x_1 + u_2x_2$. Then $p_1x = p_1(u_1x_1 + u_2p_2) = p_1u_1x_1 + p_1u_2x_2 = x_1; p_2x = p_2(u_1x_1 + u_2x_2) = p_2u_1x_1 + p_2u_2x_2 = x_2$. We shall know, then, that $\{S \xrightarrow{p_1} A_1, S \xrightarrow{p_2} A_2\}$ is a *product*, once we know that $x = u_1x_1 + u_2x_2$ is the *only* map such that $p_1x = x_1, p_2x = x_2$. But for any such x,

$$x = 1_Sx = (u_1p_1 + u_2p_2)x = u_1x_1 + u_2x_2.$$

Dually $(A_1 \xrightarrow{u_1} S, A_2 \xrightarrow{u_2} S)$ is a sum of A_1 and A_2, and the theorem is proved. ∎

Theorem 2.42 for abelian categories

If u_1, u_2, p_1, p_2 are such that $A_1 \xrightarrow{u_1} S \xrightarrow{p_1} A_1 = 1_{A_1}$, $A_2 \xrightarrow{u_2} S \xrightarrow{p_2} A_2 = 1_{A_2}$, and $A_1 \xrightarrow{u_1} S \xrightarrow{p_2} A_2$ and $A_2 \xrightarrow{u_2} S \xrightarrow{p_1} A_1$ are exact, then u_1, u_2, p_1, p_2 form a direct sum system.

Proof:

Just as in the last proof, it may be shown that for every pair $(X \xrightarrow{x_1} A_1, X \xrightarrow{x_2} A_2)$ there is a map $X \xrightarrow{x} S$ such that $p_1x = x_1$, $p_2x = x_2$. For the uniqueness of x suppose x' is such that $p_1x' = x_1$, $p_2x' = x_2$. Let $z = x - x'$ and note that $p_1z = 0$, $p_2z = 0$. We must show that $z = 0$. $O \to A_1 \xrightarrow{u_1} S \xrightarrow{p_2} A_2$ is exact since u_1 is a monomorphism (p_1u_1 is a monomorphism). Hence there is a map $X \to A_1$ such that

$$\begin{tikzcd} & & X \arrow[dl] \arrow[d, "z"] & \\ O \arrow[r] & A_1 \arrow[r, "u_1"'] & S \arrow[r, "p_2"'] & A_2 \end{tikzcd} \qquad \text{commutes,}$$

and $X \to A_1 = X \to A_1 \xrightarrow{1} A_1 = X \to A_1 \xrightarrow{u_1} S \xrightarrow{p_1} A_2 = X \xrightarrow{z} S \xrightarrow{p_1} A_1 = 0$. Hence $X \xrightarrow{z} S = X \xrightarrow{0} A_1 \to S = 0$.

2.5. THE PULLBACK AND PUSHOUT THEOREMS

Proposition 2.51 for abelian categories
(Ker(x—y) = Ker(x—y))

Given $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$, let $z = x - y$. Then $Ker(A \xrightarrow{z} B)$ is the difference kernel of $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$. ∎

Theorem 2.52 for abelian categories

Let

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

be a pullback diagram and $K \to P$ a kernel of $P \to B$. Then $K \to P \to A$ is a kernel of $A \to C$. In particular, $P \to B$ is monomorphic iff $A \to C$ is monomorphic.

Proof:

Suppose $X \to A$ is such that $X \to A \to C = 0$. Then the diagram

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[d] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

commutes and there exists a unique map $X \to P$ such that $X \to P \to A = X \to A$ and $X \to P \to B = 0$. From the latter we obtain a unique map $X \to K$ such that $X \to K \to P \to A = X \to A$. ▮

Proposition 2.53 for abelian categories

Given a square

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d, "b"'] & A \arrow[d, "\bar{b}"] \\ B \arrow[r, "\bar{a}"'] & P \end{tikzcd}$$

consider the sequence $C \xrightarrow{(a,b)} A \oplus B \xrightarrow{\binom{\bar{b}}{-\bar{a}}} P$.

$C \to A \oplus B \to P = 0$	*iff the square commutes.*
$O \to C \to A \oplus B \to P$	*is exact iff the square is a pullback.*
$C \to A \oplus B \to P \to O$	*is exact iff the square is a pushout.*
$O \to C \to A \oplus B \to P \to O$	*is exact iff the square is both a pullback and a pushout.* ▮

In the last mentioned case the square is said to be a *Doolittle diagram*. (The apparent asymmetry of the sequence vanishes when it is observed that the minus sign could have been placed before any one of the four maps.)

Pullback theorem 2.54 for abelian categories

If

$$\begin{array}{ccc} P & \to & B \\ \downarrow & & \downarrow \\ A & \to & C \end{array}$$

is a pullback diagram and $B \to C$ is epimorphic, then so is $P \to A$.

We shall prove the dual:

Pushout theorem 2.54*

If

$$\begin{array}{ccc} C & \xrightarrow{a} & A \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle \bar{b}} \\ B & \xrightarrow[\bar{a}]{} & P \end{array}$$

is a pushout diagram and $C \xrightarrow{a} A$ is monomorphic, then so is $B \xrightarrow{\bar{a}} P$.

Proof:

By hypothesis the sequence $C \xrightarrow{(a,b)} A \oplus B \xrightarrow{\binom{\bar{b}}{-\bar{a}}} P \to O$ is exact and $C \xrightarrow{(a,b)} A \oplus B$ is a monomorphism since $C \xrightarrow{(a,b)} A \oplus B \xrightarrow{p_1} A$ is. Hence, the diagram is a Doolittle diagram, in particular it is a pullback diagram and Theorem 2.52 applies. ∎

2.6. CLASSICAL LEMMAS

We have proved all the "internal" lemmas on abelian categories that will be needed for the weak embedding theorem. Once that theorem is proved an infinite variety of lemmas become provable by checking their truth in the category of abelian groups, i.e., by the classical procedures of "chasing" elements around diagrams. This process will be elucidated in Chapter 4.

In this section we shall state and prove a number of such lemmas for abelian categories. We of course do not use the weak embedding theorem. The proofs are, however, instructive and the lemmas will be needed, albeit after the proof of the weak embedding theorem.

Throughout this section we suppose we are working in an abelian category.

Lemma 2.61 for abelian categories

Suppose that the commutative diagram

$$\begin{tikzcd} & B_{11} \arrow[r] \arrow[d] & B_{12} \arrow[d] \arrow[dr] & \\ O \arrow[r] & B_{21} \arrow[r] & B_{22} \arrow[r] & B_{23} \end{tikzcd}$$

is such that the bottom row is exact. Then the square

$$\begin{tikzcd} B_{11} \arrow[r] \arrow[d] & B_{12} \arrow[d] \\ B_{21} \arrow[r] & B_{22} \end{tikzcd}$$

is a pullback iff $O \to B_{11} \to B_{12} \to B_{23}$ is exact.

Proof:

$\to$ We shall prove that $B_{11} \to B_{12}$ is a kernel of $B_{12} \to B_{23}$. Suppose $X \to B_{12}$ is such that $X \to B_{12} \to B_{23} = 0$. Since

$X \to B_{12} \to B_{22}$ when followed by $B_{22} \to B_{23}$ is zero, we have a unique factorization $X \to B_{21}$ such that $X \to B_{21} \to B_{22} = X \to B_{12} \to B_{22}$. That is, the diagram

$$\begin{array}{ccc} X & \to & B_{12} \\ \downarrow & & \downarrow \\ B_{21} & \to & B_{22} \end{array} \qquad \text{commutes,}$$

and hence there is a unique factorization $X \to B_{11}$ such that $X \to B_{11} \to B_{12} = X \to B_{12}$.

$\Leftarrow$ Let $O \to B_{21} \to B_{22} \to B_{23}$ and $O \to B_{11} \to B_{12} \to B_{23}$ be exact and

$$\begin{array}{ccc} X & \to & B_{12} \\ \downarrow & & \downarrow \\ B_{21} & \to & B_{22} \end{array} \qquad \text{commutative.}$$

Since $X \to B_{12} \to B_{23} = X \to B_{21} \to B_{22} \to B_{23} = 0$ we have a unique factorization $X \to B_{11}$ such that $X \to B_{11} \to B_{12}$ is the given $X \to B_{12}$. We will know that B_{11} is the pullback when it is established that $X \to B_{11} \to B_{21}$ is the given $X \to B_{21}$. But $X \to B_{11} \to B_{21} \to B_{22} = X \to B_{11} \to B_{12} \to B_{22} = X \to B_{12} \to B_{22} = X \to B_{21} \to B_{22}$. Since $B_{21} \to B_{22}$ is a monomorphism it may be cancelled from the extremes of the last equation. ∎

Lemma 2.62 for abelian categories

If $B_2 \to B_3$ is a monomorphism,

$$Ker(B_1 \to B_2) = Ker(B_1 \to B_2 \to B_3).$$

Proof:

$X \to B_1 \to B_2 = 0$ iff $X \to B_1 \to B_3 = 0$. ∎

Lemma 2.63 for abelian categories
Consider the commutative diagram

$$\begin{tikzcd} & & & O \arrow[d] & \\ O \arrow[r] & B_0 \arrow[r] \arrow[d, "1"'] & B_1 \arrow[r] \arrow[d, "1"'] & B_2 \arrow[r] \arrow[d] & O \\ O \arrow[r] & B_0 \arrow[r] & B_1 \arrow[r] & B_3 & \end{tikzcd}$$

in which the top row is exact. The bottom row is exact iff the column is exact.

Proof:

$\leftarrow$ By preceding lemma.

$\rightarrow$ Consider the commutative diagram

$$\begin{tikzcd} & & & O \arrow[d] & \\ & & P \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & O \\ O \arrow[r] & B_0 \arrow[r] \arrow[d, "1"] & B_1 \arrow[r] \arrow[d, "1"'] & B_2 \arrow[r] \arrow[d] & O \\ O \arrow[r] & B_0 \arrow[r] & B_1 \arrow[r] & B_3 & \end{tikzcd}$$

in which the two bottom rows and the right hand column are exact, and the (sub)diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & K \arrow[d] \\ B_1 \arrow[r] & B_2 \end{tikzcd}$$

is a pullback diagram.

The top row is exact by the pullback theorem, 2.54. We wish to prove that $K = O$. It suffices to prove that $P \to K \to B_2 = 0$.

$P \to B_1 \xrightarrow{1} B_1 \to B_3 = 0$ implies that there is a map $P \to B_0$ such that $P \to B_1 = P \to B_0 \to B_1$. Hence $P \to K \to B_2 = P \to B_1 \to B_2 = P \to B_0 \to B_1 \to B_2 = 0$. ∎

Lemma 2.64 for abelian categories

Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & O & & O & & O \\
 & & \downarrow & & \downarrow & & \downarrow \\
O & \rightarrow & B_{11} & \rightarrow & B_{12} & \rightarrow & B_{13} \\
 & & \downarrow & & \downarrow & & \downarrow \\
O & \rightarrow & B_{21} & \rightarrow & B_{22} & \rightarrow & B_{23} \\
 & & \downarrow & & \downarrow & & \\
O & \rightarrow & B_{31} & \rightarrow & B_{32} & & \\
 & & \downarrow & & & & \\
 & & O & & & &
\end{array}$$

with exact columns and exact middle row.

The top row is exact iff the bottom row is exact.

Proof:

Since $B_{13} \rightarrow B_{23}$ is monomorphic, $O \rightarrow B_{11} \rightarrow B_{12} \rightarrow B_{13}$ is exact iff $O \rightarrow B_{11} \rightarrow B_{12} \rightarrow B_{23}$ is exact (by 2.62). $O \rightarrow B_{11} \rightarrow B_{12} \rightarrow B_{23}$ is exact iff

$$\begin{array}{ccc}
B_{11} & \rightarrow & B_{12} \\
\downarrow & & \downarrow \\
B_{21} & \rightarrow & B_{22}
\end{array}$$

is a pullback diagram (by 2.61).

Again by 2.61 (turned sideways),

$$\begin{array}{ccc}
B_{11} & \rightarrow & B_{12} \\
\downarrow & & \downarrow \\
B_{21} & \rightarrow & B_{22}
\end{array}$$

is a pullback diagram iff $O \rightarrow B_{11} \rightarrow B_{21} \rightarrow B_{32}$ is exact. Since $O \rightarrow B_{11} \rightarrow B_{21} \rightarrow B_{31}$ is exact, $O \rightarrow B_{11} \rightarrow B_{21} \rightarrow B_{32}$ is exact iff $O \rightarrow B_{31} \rightarrow B_{32}$ is exact (by 2.63). ▮

"Nine lemma"* for abelian categories, 2.65

Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & O & & O & & O & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{11} & \to & B_{12} & \to & B_{13} & \to & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{21} & \to & B_{22} & \to & B_{23} & \to & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{31} & \to & B_{32} & \to & B_{33} & \to & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & O & & O & & O & &
\end{array}$$

with exact columns and exact middle row.

The top row is exact iff the bottom row is exact.

Proof:

Simply adjoin the last lemma and its dual. ∎

The full proofs of the following are left as exercises.

2.66 Noether isomorphisms

$\left(\textit{Let } B_{11} \subset B_{21} \subset B_{22}; \textit{ then } \dfrac{B_{22}/B_{11}}{B_{21}/B_{11}} \simeq \dfrac{B_{22}}{B_{21}}.\right)$ *Let* $B_{11} \to B_{21}$ *and let* $B_{21} \to B_{22}$ *be monomorphisms. Then there exists an exact commutative diagram:*

$$\begin{array}{ccccccccc}
 & & O & & O & & O & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{11} & \to & B_{11} & \to & O & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{21} & \to & B_{22} & \to & B_{22}/B_{21} & \to & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{21}/B_{11} & \to & B_{22}/B_{11} & \to & B_{22}/B_{21} & \to & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & O & & O & & O & &
\end{array}$$

∎

* "Three-by-three lemma" would be a better name.

2.67

$\left(\dfrac{B_{12}}{B_{12} \cap B_{21}} \simeq \dfrac{B_{12} \cup B_{21}}{B_{21}}.\right)$ *Let* $B_{12} \to B_{22}$ *and* $B_{21} \to B_{22}$ *be monomorphisms such that the union (least upper bound) of their images is* B_{22}. *Then there exists an exact commutative diagram:*

$$\begin{array}{ccccccccc}
 & & O & & O & & O & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{11} & \to & B_{12} & \to & B_{12}/B_{11} & \to & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{21} & \to & B_{22} & \to & B_{22}/B_{21} & \to & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{21}/B_{11} & \to & B_{22}/B_{12} & \to & O & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & O & & O & & & &
\end{array}$$ ∎

Splitting maps, 2.68

Let $B_{21} \to B_{22}$ *be such that there is a map* $B_{22} \to B_{21}$ *such that* $B_{21} \to B_{22} \to B_{21} = 1$. *Then if* $O \to B_{21} \to B_{22} \to B_{23} \to O$ *is exact there is a map* $B_{23} \to B_{22}$ *such that* $B_{23} \to B_{22} \to B_{23} = 1$, *and* B_{22} *together with the four maps to and from* B_{21} *and* B_{23} *is a direct sum system.*

Proof:

Use the nine lemma (2.65) on the following:

$$\begin{array}{ccccccccc}
 & & & & O & & O & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & O & \to & B_{12} & \to & B_{13} & \to & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{21} & \to & B_{22} & \to & B_{23} & \to & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \to & B_{21} & \to & B_{21} & \to & O & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & O & & O & & & &
\end{array}$$

For the last part of the proposition use 2.42. ∎

EXERCISES

A. Additive categories

A *monoidal category* is a category $\mathscr{M}$ with a zero object and an operation not everywhere defined on $\mathscr{M}$ (indicated by the symbol "$+$") such that

M C 1. wxz and wyz are defined iff $w(x + y)z$ is defined iff wxz and $(x + y)$ are defined iff wyz and $(x + y)$ are defined.

M C 2. If 0 is a zero map then $(x + 0)$ and $(0 + x)$ are equal to x, whenever defined.

1. If $\mathscr{M}$ is a monoidal category and $A \times B$ exists, where A and B are objects for $\mathscr{M}$, then $A + B$ exists and is isomorphic to $A \times B$.

2. If $\mathscr{M}$ is a category with a zero object such that for every object A, $A \times A$ and $A + A$ exist and $A + A \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)} A \times A$ is an isomorphism, then there is a unique operation "$+$" which not only satisfies the data for a monoidal category but also is such that $x + y = y + x$ whenever defined and $x + (y + z) = (x + y) + z$ whenever defined.

An *additive* functor between monoidal categories is a functor which preserves the monoidal structure.

3. Let $\mathscr{M}$ be a monoidal category such that $x + y = y + x$ whenever defined and $x + (y + z) = (x + y) + z$ whenever defined, and let $\mathscr{M}^{\oplus}$ be the category of all rectilinear matrices. Prove that $\mathscr{M}^{\oplus}$ is a category under the usual composition rules for matrices.

4. Every pair of objects in $\mathscr{M}^{\oplus}$ has a product and a sum, and they are isomorphic.

5. The obvious functor $\mathscr{M} \to \mathscr{M}^{\oplus}$ has the property that, for every monoidal category $\mathscr{B}$ with products for every pair of objects and additive functor $\mathscr{M} \to \mathscr{B}$, there is an additive functor $\mathscr{M}^{\oplus} \to \mathscr{B}$ such that

$$\begin{tikzcd} \mathscr{M} \arrow[r] \arrow[dr] & \mathscr{M}^{\oplus} \arrow[d] \\ & \mathscr{B} \end{tikzcd} \quad \text{commutes,}$$

and, moreover, $\mathscr{M}^{\oplus} \to \mathscr{B}$ is unique up to natural equivalence.

B. Idempotents

An *idempotent* is a map e such that $ee = e$. We say that *idempotents split* in a category $\mathscr{A}$ if for every $A \xrightarrow{e} A$ such that $e^2 = e$ there is an object B and maps $A \to B$, $B \to A$ such that $A \to B \to A = e$ and $B \to A \to B = 1$.

1. If every idempotent may be factored into an epimorphism followed by a monomorphism, then idempotents split.

2. Let $\mathscr{A}$ be any category. Let $\mathscr{S}$ be the category whose objects are pairs (A,e) where $A \in \mathscr{A}$ and e is an idempotent on A. The maps from (A_1,e_1) to (A_2,e_2) are defined to be those maps $A_1 \to A_2$ such that $A_1 \xrightarrow{e} A_1 \to A_2 \xrightarrow{e_2} A_2 = A_1 \to A_2$. Prove that $\mathscr{S}$ is a category in which idempotents split.

Letting $\mathscr{A} \to \mathscr{S}$ be the functor which sends A to $(A,1)$, prove that, for every category $\mathscr{B}$ in which idempotents split and every functor $\mathscr{A} \to \mathscr{B}$, there is a functor $\mathscr{S} \to \mathscr{B}$ such that

$$\begin{tikzcd} \mathscr{A} \arrow[r] \arrow[dr] & \mathscr{S} \arrow[d] \\ & \mathscr{B} \end{tikzcd} \qquad \text{commutes}$$

and moreover the functor $\mathscr{S} \to \mathscr{B}$ is unique up to natural equivalence.

3. If every pair of objects in $\mathscr{A}$ has a product (sum) then every pair of objects in $\mathscr{S}$ has a product (sum).

C. Groups in categories

1. In the category of sets with base points, a *group* is an object A together with a map $A \times A \xrightarrow{m} A$ such that:

(1) $A \times (A \times A) \xrightarrow{1 \times m} A \times A \xrightarrow{m} A =$
$(A \times A) \times A \xrightarrow{m \times 1} A \times A \xrightarrow{m} A.$

(2) $A \xrightarrow{(0,1)} A \times A \xrightarrow{m} A = 1$

(3) There exists a map $A \xrightarrow{r} A$ such that
$A \xrightarrow{(r,1)} A \times A \xrightarrow{m} A = 0$

The group is *commutative* if:

(4) For $A \times A \xrightarrow{t} A \times A$ the map which "twists components," i.e., is such that

$$A \times A \xrightarrow{t} A \times A \xrightarrow{p_i} A = \begin{cases} p_2 & \text{if } i = 1 \\ p_1 & \text{if } i = 2, \end{cases}$$

it is the case that $A \times A \xrightarrow{t} A \times A \xrightarrow{m} A = m$.

Given two groups $A \times A \xrightarrow{m} A$ and $B \times B \xrightarrow{m'} B$, a *homomorphism* from A to B is a map $A \xrightarrow{x} B$ such that

$$\begin{array}{ccc} A \times A & \xrightarrow{m} & A \\ {\scriptstyle (xp_1, xp_2)}\downarrow & & \downarrow{\scriptstyle x} \\ B \times B & \xrightarrow{m'} & B \end{array} \qquad \text{commutes.}$$

2. Let $\mathscr{A}$ be a category with finite products and a zero object. A **group in** $\mathscr{A}$ may be defined precisely as above and so may homomorphisms between groups in $\mathscr{A}$.

If $A \times A \xrightarrow{m} A$ is a group in $\mathscr{A}$, then the contravariant functor $(-,A)\colon \mathscr{A} \to \mathscr{S}$ may be factored through the *forgetful functor* from the category of all groups to the category of sets with base points (the functor forgets the group structure). This is simply the observation that for any $B \in \mathscr{A}$, $(B,A) \times (B,A) \to (B,A \times A) \xrightarrow{(B,m)} (B,A)$ is a group in $\mathscr{S}$, and that for any $B \to B' \in \mathscr{B}$, the induced map from (B',A) to (B,A) will satisfy the requirement for a homomorphism.

3. Let A be an object in $\mathscr{A}$ and let F be a contravariant functor from $\mathscr{A}$ to the category of all groups $\mathscr{G}$ such that when followed by the forgetful functor into the category of sets the composition results in the functor $(-,A)$. Define $m \in (A \times A,A)$ to be the image of (p_1,p_2) under the map $(A \times A,A) \times (A \times A,A) \to (A \times A,A)$ which results from group multiplication. Then $A \times A \xrightarrow{m} A$ is a group in $\mathscr{A}$ and the given functor F is the same as described in part 2 above.

4. A **cogroup in $\mathscr{A}$** is an object A together with a map $A \to A + A$ which satisfies the duals of the axioms for a group. If A is a cogroup and B is a group then the set (A,B) enjoys group structures inherited from either A or B. They are, in fact, the same, and regardless of the commutativity of either the given group or cogroup structures, (A,B) is a commutative group. (2.38.)

5. A **topological group** is a group in the category of topological spaces. Let $\mathscr{C}$ be the category of commutative groups in the category of compact Hausdorf spaces. $\mathscr{C}$ is an abelian category.

CHAPTER 3

SPECIAL FUNCTORS AND SUBCATEGORIES

It has been said that categories were invented in order to eliminate the inside theory and thus concentrate on the outside. Thus far we have been inside a given, but unspecified, category. But as is usually the case (wherefore categories), it is necessary to go outside in order to see the inside. Hence our first chapter on functors.

3.1. ADDITIVITY AND EXACTNESS

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. Given a functor $F\colon \mathscr{A} \to \mathscr{B}$ and any two objects $A_1,A_2 \in \mathscr{A}$, F induces a function

$$(A_1,A_2) \to (F(A_1),F(A_2)).$$

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. F is **additive** if the function $(A_1,A_2) \to (F(A_1),F(A_2))$ is a group homomorphism for every $A_1,A_2 \in \mathscr{A}$.

Example. Let $\mathscr{A}$ be an abelian category, A an object in $\mathscr{A}$ and $(A,-)\colon \mathscr{A} \to \mathscr{G}$ the functor from A to the category of abelian groups $\mathscr{G}$, defined by $(A,-)(B) = (A,B)$ the group of maps from A to B.

Theorem 3.11

For abelian categories $\mathscr{A}$ and $\mathscr{B}$ a functor $F\colon \mathscr{A} \to \mathscr{B}$ is additive iff it carries direct sum systems into direct sum systems.

Proof:

$\rightarrow$ The conditions in the hypothesis of Theorem 2.41 are preserved by additive functors.

$\leftarrow$ Let $A \xrightarrow{u_1} A \oplus A$, $A \xrightarrow{u_2} A \oplus A$, $A \oplus A \xrightarrow{p_1} A$, $A \oplus A \xrightarrow{p_2} A$ be a direct sum system in $\mathscr{A}$. By hypothesis it is the case that $F(u_1)$, $F(u_2)$, $F(p_1)$, $F(p_2)$ is a direct sum system in $\mathscr{B}$. Let $x,y \in (A,B)$. Then by the definition of $+$ in 2.3 we obtain $A \xrightarrow{x+y} B = A \xrightarrow{(1,1)} A \oplus A \xrightarrow{\binom{x}{y}} B$. Hence $F(A \xrightarrow{x+y} B) = F(A) \xrightarrow{F(1,1)} F(A \oplus A) \xrightarrow{F\binom{x}{y}} F(B) = F(A) \xrightarrow{(1,1)} F(A) \oplus F(A) \xrightarrow{\binom{F(x)}{F(y)}} F(B) = F(x) + F(y)$. ∎

A **left-exact** sequence is an exact sequence of the form $O \to A_1 \to A_2 \to A_3$. A **left-exact functor** between abelian categories is a functor which carries left-exact sequences into left-exact sequences. (Equivalently, it is a functor which preserves *kernels.*)

Theorem 3.12

A left-exact functor is additive.

Proof:

The conditions of the hypothesis of Theorem 2.42 are preserved by left-exact functors. Indeed, we use only the fact that for every exact $O \to A' \to A \to A'' \to O$ it is the case that $F(A') \to F(A) \to F(A'')$ is exact. Such a functor is called **half-exact** or **middle-exact.** ∎

Example. $(A,-)\colon \mathscr{A} \to \mathscr{G}$ is left-exact.

A **right-exact** sequence is an exact sequence of the form $A_1 \to A_2 \to A_3 \to O$. A **right-exact functor** is a functor between abelian categories which carries right-exact sequences into right-exact sequences.

Theorem 3.12*

A right-exact functor is additive. ∎

An **exact functor** is a functor between abelian categories which carries exact sequences into exact sequences.

Proposition 3.13

A functor is exact iff it is both right-exact and left-exact. ∎

Henceforth all functors between abelian categories will be additive.

3.2. EMBEDDINGS

A functor $F: \mathscr{A} \to \mathscr{B}$ is an **embedding** if for any two $A_1,A_2 \in \mathscr{A}$ the function $(A_1,A_2) \to (F(A_1),F(A_2))$ is one-to-one.

Theorem 3.21

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, $F: \mathscr{A} \to \mathscr{B}$ an additive functor. Then the following are equivalent:

(a) *F is an embedding*
(b) *F carries noncommutative diagrams into noncommutative diagrams*
(c) *F carries nonexact sequences into nonexact sequences.*

Proof:

(a) $\leftrightarrow$ (b) Trivial.

(c) $\to$ (a) Let $A_1 \xrightarrow{x} A_2 \neq 0$. Then $A_1 \xrightarrow{1} A_1 \xrightarrow{x} A_2$ is not exact. Hence $F(A_1) \xrightarrow{1} F(A_1) \xrightarrow{F(x)} F(A_2)$ is not exact and $F(x) \neq 0$.

(a) $\Rightarrow$ (c) Let $A' \to A \to A''$ be a nonexact sequence. Let $O \to K \to A \to A''$ and $A' \to A \to G \to O$ be exact.

By proposition 2.21 then either $A' \to A \to A'' \neq 0$ or $K \to A \to G \neq 0$.

Hence either $F(A') \to F(A) \to F(A'') \neq 0$ or $F(K) \to F(A) \to F(G) \neq 0$.

In the first situation it is clear that $F(A') \to F(A) \to F(A'')$ is not exact. Assume that $F(K) \to F(A) \to F(G) \neq 0$. Let $O \to B' \to F(A) \to F(A'')$ and $F(A') \to F(A) \to B'' \to O$ be exact in $\mathscr{B}$. $K \to A \to A'' = 0$ implies that $F(K) \to F(A) \to F(A'') = 0$, and there is a map $F(K) \to B'$ such that $F(K) \to B' \to F(A) = F(K) \to F(A)$ and a map $B'' \to F(G)$ such that $F(A) \to B'' \to F(G) = F(A) \to F(G)$. Hence if $F(A') \to F(A) \to F(A'')$ were exact then $B' \to F(A) \to B'' = 0$ and $F(K) \to B' \to F(A) \to B'' \to F(G) = 0$, a contradiction. ∎

If a functor F: $\mathscr{A} \to \mathscr{B}$ is an exact embedding, the exactness and commutativity of a diagram in $\mathscr{A}$ is equivalent to the exactness and commutativity of the F-image of the diagram.

3.3. SPECIAL OBJECTS

A phenomenon in category theory is that an interesting property on functors may be used to define what is usually an interesting property on objects in categories. As an example we define an object P in an abelian category $\mathscr{A}$ to be **projective** iff the functor $(P,-)$: $\mathscr{A} \to \mathscr{G}$ is exact. (For any $A \in \mathscr{A}$ it is the case that $(A,-)$ is left-exact; hence P is projective iff $(P,-)$ is right-exact.) The easiest example of a projective is the ring itself in the category of its modules.

Proposition 3.31

P is projective iff for every epimorphism $A \twoheadrightarrow A''$ and map $P \to A''$ there is a map $P \to A$ such that $P \to A \to A'' = P \to A''$. ∎

Proposition 3.32

If $\{P_i\}$ is a family of projectives in an abelian category, then the direct sum ΣP_i (if it exists in $\mathscr{A}$) is projective. ∎

An object $G \in \mathscr{A}$ is a **generator** iff the functor $(G,-)\colon \mathscr{A} \to \mathscr{G}$ is an embedding. Again the ring itself in the category of its modules is an example.

Proposition 3.33

G is a generator iff for every $A \to B \neq 0$ there is a map $G \to A$ such that $G \to A \to B \neq 0$.

G is a generator iff for every proper subobject of A there is a map $G \to A$ whose image is not contained in the given subobject. ∎

Proposition 3.34

If P is projective then it is a generator iff (P,A) is nontrivial for all nontrivial A. ∎

It may also be shown that an exact functor is an embedding iff it fails to kill nonzero objects.

The curious contrary relation of exact and embedding functors exhibited by Theorem 3.21 (part c) is reflected among projectives and generators and may be seen most strikingly in the category of modules over a ring R where:

A is projective iff A appears as a direct summand of a direct sum (possibly infinite) of copies of R.

A is a generator iff R appears as a direct summand of a direct sum (possibly infinite) of copies of A.

Proposition 3.35

If an abelian category has a generator then the family of subobjects of any object is a set.

Proof:

If G is a generator and A is any object, then a subobject $A' \to A$ is distinguished by the subset $(G,A') \subset (G,A)$. ∎

Proposition 3.36

G is a generator in a right-complete abelian category $\mathscr{A}$ iff for every $A \in \mathscr{A}$ the obvious map $\Sigma_{(G,A)}G \to A$ is epimorphic. (The "obvious" map is such that for all $x \in (G,A)$,

$$G \xrightarrow{u_x} \Sigma_{(G,A)}G \to A = G \xrightarrow{x} A.) \quad ∎$$

The dual notions are as follows: An object Q is **injective** if the contravariant functor $(-,Q)$ carries exact sequences into exact sequences, albeit with a reversal in direction. (Q is injective in $\mathscr{A}$ iff Q^* is projective in $\mathscr{A}^*$.) An object C is a **cogenerator** if the contravariant functor $(-,C)$ is an embedding. (C is a cogenerator for $\mathscr{A}$ iff C^* is a generator for $\mathscr{A}^*$.)

Proposition 3.37

Let $\mathscr{A}$ be a left-complete abelian category with a generator. Every object in $\mathscr{A}$ may be embedded in an injective object iff $\mathscr{A}$ has an injective cogenerator.

Proof:

$\leftarrow$ Let C be an injective cogenerator for $\mathscr{A}$, and $A \in \mathscr{A}$ an arbitrary object. The obvious (or perhaps "co-obvious") map $A \to \Pi_{(A,C)}C$ is a monomorphism and $\Pi_{(A,G)} C$ is injective. (We are using 3.36*.)

$\rightarrow$ Let G be a generator for $\mathscr{A}$, and let P be the product of all the quotient objects of G (Prop. 3.35 says there are only a set of quotient objects of G). Let $P \to E$ be a monomorphism with E injective. Then E is a cogenerator. To prove it, let $A \to B$ be a nonzero map. Since G is a generator there exists a map $G \to A$ such that $G \to A \to B \neq 0$. Let $I \to B$ be the image of $G \to A \to B$, and $I \to P \to E$ a monomorphism. Since E is injective, there exists a map $B \to E$ such that $I \to B \to E = I \to P \to E$. $A \to B \to E \neq 0$ because $G \to A \to B \to E = G \to A \to I \to B \to E \neq 0$. ▌

3.4. SUBCATEGORIES

Recalling the original definition of a category as a class of maps $\mathscr{M}$ together with a composition relation, we define a subclass $\mathscr{M}'$ to be a **subcategory** if (1) for every $x,y \in \mathscr{M}'$ such that xy is defined in $\mathscr{M}$ it is the case that $xy \in \mathscr{M}'$, and (2) if e is an identity map in $\mathscr{M}$, $x \in \mathscr{M}'$, and either ex or xe is defined in $\mathscr{M}$, then $e \in \mathscr{M}'$.

$\mathscr{M}'$ is easily seen to be a category and the inclusion function, an embedding functor.

Let $\mathscr{A}$ be an abelian category and $\mathscr{A}'$ a subcategory. We say that $\mathscr{A}'$ is an **exact subcategory** if $\mathscr{A}'$ is abelian and the inclusion functor is exact. The inclusion functor is automatically an embedding and all questions relating to the exactness of diagrams in $\mathscr{A}'$ can therefore be answered by considering their exactness in $\mathscr{A}$.

$F: \mathscr{A} \to \mathscr{B}$ is a **full functor** if for every $A_1,A_2 \in \mathscr{A}$ the induced function $(A_1,A_2) \to (F(A_1),F(A_2))$ is onto.

A **full subcategory** is a subcategory whose inclusion functor is full. Given a category $\mathscr{A}$ and a collection of objects, $\mathscr{O} \subset \mathscr{A}$, the subcategory consisting of all the maps between the objects in $\mathscr{O}$ is a full subcategory (said to be that which is generated by $\mathscr{O}$), and every full subcategory can be so obtained.

When we are considering a subcategory $\mathscr{A}$ of a category $\mathscr{B}$ the statement "$A_1 \to A_2$ is an $\mathscr{A}$-monomorphism" means that $A_1 \to A_2$ is a monomorphism in $\mathscr{A}$. "$A_1 \to A_2$ is a $\mathscr{B}$-monomorphism" means that $A_1 \to A_2$, considered as a map in $\mathscr{B}$, is a monomorphism. Similarly we may say that K is an "$\mathscr{A}$-kernel of $A_1 \to A_2$," "$\mathscr{B}$-kernel of $A_1 \to A_2$." In general the prefixes "$\mathscr{A}$-" and "$\mathscr{B}$-" qualify a property or description relative to $\mathscr{A}$ or $\mathscr{B}$.

Theorem 3.41

Let $\mathscr{B}$ be an abelian category, and $\mathscr{A}$ a nonempty full subcategory. Then $\mathscr{A}$ is an exact subcategory iff for every $A_1 \xrightarrow{x} A_2 \in \mathscr{A}$ there is a $\mathscr{B}$-kernel of x, a $\mathscr{B}$-cokernel of x, and a $\mathscr{B}$-direct-sum of A_1 and A_2, all lying in $\mathscr{A}$.

Proof:

$\to$ Let $\mathscr{A}$ be an exact full subcategory of $\mathscr{B}$. In particular, then, $\mathscr{A}$ is abelian and $A_1 \xrightarrow{x} A_2$ has an $\mathscr{A}$-kernel, K, and an $\mathscr{A}$-cokernel, F, in $\mathscr{A}$. The exactness of the inclusion functor implies that K is a $\mathscr{B}$-kernel of x and F is a $\mathscr{B}$-cokernel of x. Similarly, if S is an $\mathscr{A}$-direct-sum of A_1 and A_2, then the additivity of the inclusion functor implies that it is a $\mathscr{B}$-direct-sum.

$\leftarrow$ Let $\mathscr{A}$ be a nonempty full subcategory closed under the operations (defined in $\mathscr{B}$) of kernel, cokernel, and direct sum. We must first show that $\mathscr{A}$ is abelian. We consider half of the axioms (the other half are dual).

Axiom 0. $\mathscr{A}$ is nonempty; let $A \xrightarrow{1} A \in \mathscr{A}$ and let $O \to A \in \mathscr{A}$ be a $\mathscr{B}$-kernel of 1_A. Then O is a zero object for $\mathscr{A}$.

Axiom 1. Let $A_1, A_2 \in \mathscr{A}$, and $S \xrightarrow{p_1} A_1$, $S \xrightarrow{p_2} A_2$ a $\mathscr{B}$-direct-sum, where $S \in \mathscr{A}$. The fullness of $\mathscr{A}$ implies that S is an $\mathscr{A}$-direct-sum.

Axiom 2. Let $A_1 \to A_2 \in \mathscr{A}$ and $O \to K \to A_1 \to A_2$ be exact in $\mathscr{B}$, $K \in \mathscr{A}$. Again the fullness of $\mathscr{A}$ implies that K is an $\mathscr{A}$-kernel of $A_1 \to A_2$.

Axiom 3. A map $A_1 \to A_2$ is an $\mathscr{A}$-monomorphism iff it is a $\mathscr{B}$-monomorphism (in each case the kernel must be trivial). Hence if $A_1 \to A_2$ is an $\mathscr{A}$-monomorphism we let $O \to A_1 \to A_2 \to A_3 \to O$ be exact in $\mathscr{B}$, $A_3 \in \mathscr{A}$. Then $A_1 \to A_2$ is an $\mathscr{A}$-kernel of $A_2 \to A_3$.

The exactness of the inclusion functor is straightforward. ∎

3.5. SPECIAL CONTRAVARIANT FUNCTORS

A contravariant functor $F: \mathscr{A} \to \mathscr{B}$ induces for each pair of objects $A_1,A_2 \in \mathscr{A}$ a function $(A_1,A_2) \to (F(A_2),F(A_1))$.

If $\mathscr{A}$ and $\mathscr{B}$ are abelian we say that F is additive if these induced functions are group homomorphisms; F is an embedding if they are one-to-one, F is full if they are onto. An exact contravariant functor carries exact sequences into exact sequences (with an order reversal, of course).

Proposition 3.51

The additive functor $(-,A): \mathscr{A} \to \mathscr{G}$ where $\mathscr{A}$ is abelian, $A \in \mathscr{A}$, and $\mathscr{G}$ is the category of abelian groups, carries right-exact sequences into left-exact sequences. ∎

3.6. BIFUNCTORS

Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be categories, i.e., classes of maps with composition relations. The Cartesian product $\mathscr{M}_1 \times \mathscr{M}_2$ enjoys a natural category structure. If $\mathscr{O}_1$ and $\mathscr{O}_2$ are classes of objects for $\mathscr{M}_1$ and $\mathscr{M}_2$ then $\mathscr{O}_1 \times \mathscr{O}_2$ may be taken as a class of objects for $\mathscr{M}_1 \times \mathscr{M}_2$.

A functor from $\mathscr{M}_1 \times \mathscr{M}_2$ is said to be a functor on two variables, one from $\mathscr{M}_1$ and the other from $\mathscr{M}_2$.

Proposition 3.61

Let $F\colon \mathscr{M}_1 \times \mathscr{M}_2 \to \mathscr{M}_3$ be a function. F is a functor iff:

(1) *For each identity $1_A \in \mathscr{M}_1$, the function $F(1_A,-)\colon \mathscr{M}_2 \to \mathscr{M}_3$ is a functor.*

(2) *For each identity $1_B \in \mathscr{M}_2$, the function $F(-,1_B)\colon \mathscr{M}_1 \to \mathscr{M}_3$ is a functor.*

(3) *For any $A \xrightarrow{x} A' \in \mathscr{M}_1$, $B_1 \xrightarrow{y} B_2 \in \mathscr{M}_2$ the diagram*

$$\begin{tikzcd} F(A,B) \arrow[r, "F(x{,}1_B)"] \arrow[d, "F(1_A{,}y)"'] \arrow[dr, "F(x{,}y)"] & F(A',B) \arrow[d, "F(1_{A'}{,}y)"] \\ F(A,B') \arrow[r, "F(x{,}1_{B'})"'] & F(A',B') \end{tikzcd}$$

commutes. ∎

We complicate matters by allowing functors to be covariant on one variable, contravariant on the other. In so doing, we obtain for any category $\mathscr{A}$ the functor $Hom\colon \mathscr{A} \times \mathscr{A} \to \mathscr{S}$ ($\mathscr{S}$ is the category of sets). $Hom(A,B)$ = the set of maps (A,B). (We could take $\mathscr{A}^* \times \mathscr{A}$ as domain.)

A natural transformation from $F\colon \mathscr{M}_1 \times \mathscr{M}_2 \to \mathscr{M}_3$ to $G\colon \mathscr{M}_1 \times \mathscr{M}_2 \to \mathscr{M}_3$ is precisely what it must be: a function $\eta\colon \mathscr{O}_1 \times \mathscr{O}_2 \to \mathscr{M}_3$ which satisfies the requirements of natural equivalences.

Proposition 3.62

$\eta\colon \mathscr{M}_1 \times \mathscr{M}_2 \to \mathscr{M}_3$ is a natural transformation from F to G iff:

(1) $\eta(A,B) \in (F(A,B), G(A,B))$.

(2) *For each $A \in \mathscr{O}_1$, $\eta(A,-)\colon \mathscr{M}_2 \to \mathscr{M}_3$ is a natural transformation from $F(A,-)$ to $G(A,-)$.*

(3) *For each $B \in \mathscr{O}_2$, $\eta(-,B)\colon \mathscr{M}_1 \to \mathscr{M}_3$ is a natural transformation from $F(-,B)$ to $G(-,B)$.* ∎

Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be abelian categories and F a functor from $\mathscr{A} \times \mathscr{B}$ to $\mathscr{C}$. F is a **bifunctor** if:

(1) For each $A_1 \in \mathscr{A}$, $F(A_1,-)\colon \mathscr{B} \to \mathscr{C}$ is additive.

(2) For each $A_2 \in \mathscr{B}$, $F(-,A_2)\colon \mathscr{A} \to \mathscr{C}$ is additive.

Proposition 3.63

Hom: $\mathscr{A} \times \mathscr{A} \to \mathscr{G}$ *is a bifunctor where Hom*(A,B) *is the group of maps* (A,B). ▮

EXERCISES

A. Equivalence of categories

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. They are **isomorphic** if there exist functors $F_1\colon \mathscr{A} \to \mathscr{B}$, $F_2\colon \mathscr{B} \to \mathscr{A}$ such that F_1F_2 and F_2F_1 are identity functors. $\mathscr{A}$ and $\mathscr{B}$ are **equivalent** if there exist functors $F_1\colon \mathscr{A} \to \mathscr{B}$, $F_2\colon \mathscr{B} \to \mathscr{A}$ such that F_1F_2 and F_2F_1 are *naturally equivalent* to the identity functors; in this case F_1 and F_2 are called **equivalences.** There are few properties or categories of any consequence which are not preserved by equivalences. Besides the property of smallness, perhaps the only two are the following:

$\mathscr{A}$ is a **skeletal** category if every isomorphism of objects in $\mathscr{A}$ implies equality (i.e., all isomorphisms in $\mathscr{A}$ are automorphisms).

$\mathscr{A}$ is a **replete** category if for every $A \in \mathscr{A}$ the class of objects in $\mathscr{A}$ isomorphic to A is not a set, or, equivalently, enjoys a one-to-one correspondence with the universal class.

Every category is equivalent to a skeletal category and to a replete category.

Equivalent skeletal categories are isomorphic and equivalent replete categories are isomorphic. If $\mathscr{A}$ and $\mathscr{B}$ are skeletal and F_1: $\mathscr{A} \to \mathscr{B}$ and $F_2\colon \mathscr{B} \to \mathscr{A}$ are such that F_1F_2 and F_2F_1 are naturally equivalent to the identities then both F_1 and F_2 are isomorphisms (which is not to say that F_1F_2 and F_2F_1 are equal to the identities). The same statement for replete categories is false.

If $\mathscr{B}$ is replete and $F: \mathscr{A} \to \mathscr{B}$ is any functor, then F is naturally equivalent to a functor which is one-to-one on objects.

Two properties on subcategories are as follows:

A subcategory $\mathscr{A} \subset \mathscr{B}$ is a **replete subcategory** in $\mathscr{B}$ if for every $B \in \mathscr{B}$ isomorphic to an object in $\mathscr{A}$ it is the case that $B \in \mathscr{A}$.

A subcategory $\mathscr{A} \subset \mathscr{B}$ is **representative in** $\mathscr{B}$ if for every $B \in \mathscr{B}$ there is an object $A \in \mathscr{A}$ which is isomorphic to B.

If $\mathscr{A}$ is a replete representative full subcategory of $\mathscr{B}$ then $\mathscr{A} = \mathscr{B}$.

If $\mathscr{A}$ is a full representative subcategory of $\mathscr{B}$ then $\mathscr{A}$ is equivalent to $\mathscr{B}$.

Every category has a full representative skeletal subcategory (often called its **skeleton**). Skeletons of equivalent categories are isomorphic.

The image of a full functor or of a functor which is one-to-one on objects is a subcategory. A functor is an equivalence iff it is a full embedding whose image is representative.

Any number of baroque considerations may be obviated by adopting the convention that the categories and functors under discussion can always be replaced by equivalent categories and functors. This convention, of course, makes sense only when properties invariant under such substitutions are being discussed.

B. Roots

Let $\mathscr{D}$ and $\mathscr{A}$ be categories and $F: \mathscr{D} \to \mathscr{A}$ a functor. The **left root** (if it exists) of F is a constant functor $L: \mathscr{D} \to \mathscr{A}$ which "best approximates" F via a transformation $L \to F$. To wit: for any constant functor $C: \mathscr{D} \to \mathscr{A}$ and transformation $C \to F$ there exists a unique $C \to L$ such that $C \to L \to F = C \to F$. Bear in mind that the constant functors into $\mathscr{A}$ are in obvious correspondence with the objects of $\mathscr{A}$, and the transformations between constant functors with the maps of $\mathscr{A}$. If we use L to represent both the functor and its unique value we note that $L \to F$ is a collection of maps $\{L \to F(D) \mid D \in \mathscr{D}\}$ with the condition that for any $D \xrightarrow{x} D'$, the triangle

$$\begin{tikzcd} & F(D) \arrow[d, "F(x)"] \\ L \arrow[ur] \arrow[r] & F(D') \end{tikzcd}$$

commutes.

L is a left root therefore if for any such family $\{C \to F(D) \mid D \in \mathscr{D}\}$ (which satisfies the same sort of "consistency" requirement) there is a unique map $C \to L$ such that

$$C \to L \to F(D) = C \to F(D) \text{ for all } D \in \mathscr{D}.$$

If L and L' are both left roots of F they are naturally equivalent.

Let $\mathscr{D}$ be the category with two objects A and B and two non-identity maps $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$. For $F\colon \mathscr{D} \to \mathscr{A}$, the left root of F is the difference kernel of $F(x)$ and $F(y)$.

Let $\mathscr{D}$ be the category with two objects A and B and no maps besides the two identities (the discrete category with two objects). For $F\colon \mathscr{D} \to \mathscr{A}$ the left root of F is the product of $F(A)$ and $F(B)$.

Let $\mathscr{D}$ be a category with only identity maps (any discrete category). For $F\colon \mathscr{D} \to \mathscr{A}$ the left root of F is the product of $\{F(D)\}_{D \in \mathscr{D}}$.

Let A be an object in $\mathscr{A}$ and $\mathscr{F}$ a family of monomorphisms into A together with all the inclusion maps between them. The left root of the inclusion functor $\mathscr{F} \to \mathscr{A}$ is the intersection of the subobjects in $\mathscr{F}$; that is, the left root is a subobject of A and it is the greatest lower bound of the subobjects in $\mathscr{F}$.

The dual notion is as follows. The **right root** of a functor $F\colon \mathscr{D} \to \mathscr{A}$ is a constant functor $R\colon \mathscr{D} \to \mathscr{A}$ together with a natural transformation $F \to R$ such that for any constant functor $C\colon \mathscr{D} \to \mathscr{A}$ and transformation $F \to C$ there exists a unique transformation $R \to C$ such that $F \to R \to C = F \to C$. As examples of right roots we may obtain difference cokernels, sums, and the dual of intersections, namely greatest lower bounds in the families of quotient objects.

What we have called a left root is sometimes called an **inverse limit**, and what we have called a right root is sometimes called a **direct limit**. We prefer to reserve the word "limit" for the case in which the domain category is "directed." In Exercise 0-D we defined a *partially ordered category*. A **directed category** is a partially ordered category such that for every pair of objects A and B there exists an object C such that neither (A,C) nor (B,C) is empty (in terms of the partial ordering on the objects: $A \leq C$ and $B \leq C$). If $\mathscr{D}$ is a directed category and $F\colon \mathscr{D} \to \mathscr{A}$ a functor, F is sometimes called a **direct system** in $\mathscr{A}$, and its right root is what we call a direct limit.

The best known example of a direct limit is the following: Let G be an abelian group and $\mathscr{F}$ the family of finitely generated subgroups of G, together with all the inclusion maps between them. $\mathscr{F}$ is a directed category. The direct limit of its inclusion functor is G, or, as is usually said, G is the direct limit of its finitely generated subgroups.

If $\mathscr{D}$ is the dual of a directed category then $F: \mathscr{D} \to \mathscr{A}$ is an **inverse system** in $\mathscr{A}$ and its left root is its inverse limit.

We insist upon the word "root" because there are too many important theorems special to limits to justify the destruction of the word "limit" in that use. (For an example see Exercise 5-E). There are important functors which preserve all direct limits but do not preserve all right roots. The phrase **directly continuous** has been used to describe such functors. The stronger condition, that all right roots are preserved, we shall describe by the phrase **right-root-preserving.**

The classical notation for the direct limit of a functor F is $\varinjlim F$, and for the inverse limit, $\varprojlim F$. This notation we shall use for all roots. Hence $\varinjlim F$ is the right root of F, whether the domain of F is directed or not, and $\varprojlim F$ is the left root of F.

C. Construction of roots

It is tempting to call $\mathscr{A}$ *left-complete* if for every small category $\mathscr{D}$ and functor $F: \mathscr{D} \to \mathscr{A}$ it is the case that F has a left root. We are prevented from doing so only by our definition in Chapter 1 of a left-complete category as one which has difference kernels and infinite products. Luckily the two definitions are coextensive.

The classical construction of left roots is as follows:

Given a functor $F: \mathscr{D} \to \mathscr{S}$ into the category of sets, consider the product $P = \Pi_{D \in \mathscr{D}} F(D)$ and let $L \subset P$ be the subset of all elements $y \in P$ such that for each $D \xrightarrow{x} D' \in \mathscr{D}$, $[P \xrightarrow{p} F(D) \xrightarrow{F(x)} F(D')](y) = [P \xrightarrow{p} F(D')](y)$. L is the left root of F.

Theorem: If $\mathscr{A}$ is a left-complete category (that is, it has difference kernels and products), then every functor into $\mathscr{A}$ from a small category has a left root. (And, obviously, conversely.)

Given $F: \mathscr{D} \to \mathscr{A}$, $\mathscr{D}$ small, let $P = \Pi_{D \in \mathscr{D}} F(D)$. For each $D \xrightarrow{x} D'$, let $K_x \to P$ be the difference kernel of $P \xrightarrow{p} F(D) \xrightarrow{F(x)} F(D')$ and

$P \xrightarrow{p} F(D')$. The intersection of all such difference kernels is the left root of F.

But we don't know yet that we have intersections.

On the other hand, we do not need the intersection of just any old family of subobjects, *but only of families of difference kernels*, and such intersections may be constructed as follows: Let $\{(P \xrightarrow{x_i} A_i, P \xrightarrow{y_i} A_i)\}_{i \in I}$ be a family of pairs of maps. The difference kernel of the two maps $P \xrightarrow{x} \Pi_I A_i$ and $P \xrightarrow{y} \Pi_I A_i$, where

$$P \xrightarrow{x} \Pi_I A_i \xrightarrow{p_i} A_i = x_i, P \xrightarrow{y} \Pi_I A_i \xrightarrow{p_i} A_i = y_i,$$

is the intersection of the family $\{Ker(x_i - y_i)\}_{i \in I}$.

The proof of the above theorem yields a proof of the fact that a functor from a left-complete category is left-root-preserving iff it preserves difference kernels and products. A slight modification yields that a category with difference kernels and *finite* products possesses left roots for every functor from a finite domain, as is the case with abelian categories. And in the case of abelian categories, a functor is *finite-left-root-preserving* iff it is left-exact.

D. Small complete categories are lattices

Suppose that $\mathscr{A}$ is a small left-complete category and that for some pair of objects $A,B \in \mathscr{A}$ it is the case that (A,B) has more than one element. Let K be an indexing set of cardinality larger than that of the category $\mathscr{A}$. Then if $\Pi_K B$ existed in $\mathscr{A}$ we could reach a contradiction since $(A,\Pi_K B)$ must have at least 2^K elements. We conclude therefore that for every $A,B \in \mathscr{A}$ it is the case that (A,B) has at most one element.

Let $\mathscr{A}'$ be a skeleton of $\mathscr{A}$. It follows that $\mathscr{A}'$ is a partially ordered category. The completeness of $\mathscr{A}'$ implies that the partial ordering is complete; in other words, $\mathscr{A}$ is equivalent to a complete lattice category.

The moral: If one insists upon simplifying the language so as to exclude categories that are not small, then all interesting complete categories will have been excluded.

E. The standard functors

Let $\mathscr{A}$ be any category and $A \in \mathscr{A}$. The functor $(A,-)\colon \mathscr{A} \to \mathscr{S}$ preserves all left roots; formally speaking, for any $F\colon \mathscr{D} \to \mathscr{A}$ such that $\underleftarrow{\mathrm{Lim}}\, F$ exists, it is the case that $(A, \underleftarrow{\mathrm{Lim}}\, F)$ is the left root of $\mathscr{D} \xrightarrow{F} \mathscr{A} \xrightarrow{(A,-)} \mathscr{S}$.

The functor $(-,A)\colon \mathscr{A} \to \mathscr{S}$ carries right roots into left roots.

Given any constant functor $C\colon \mathscr{D} \to \mathscr{A}$ and transformation $C \to F$, we may test whether C is a left root of F by applying all the functors of the form $(A,-)$. For $\mathscr{A}$ an additive category, we may replace $\mathscr{S}$ with $\mathscr{G}$ and obtain the same statements.

The functor $(A,-)\colon \mathscr{G} \to \mathscr{G}$ preserves *direct limits* (is directly continuous) iff A is a finitely generated group.

F. Reflections

Let $\mathscr{A}$ be a subcategory of $\mathscr{B}$. Given an object $B \in \mathscr{B}$ we define its **reflection** in $\mathscr{A}$ (if it exists) to be an object $\bar{B} \in \mathscr{A}$ which "best approximates" B via a map $B \to \bar{B}$. To be precise, for any $A \in \mathscr{A}$ and map $B \to A$ there is a unique map $\bar{B} \to A \in \mathscr{A}$ such that

$$\begin{tikzcd} & \bar{B} \arrow[dd] \\ B \arrow[ur] \arrow[dr] & \\ & A \end{tikzcd} \qquad \text{commutes.}$$

Reflections are unique up to isomorphism. If every object in $\mathscr{B}$ has a reflection in $\mathscr{A}$ we say that $\mathscr{A}$ is a **reflective** subcategory. In this case we obtain a functor $R\colon \mathscr{B} \to \mathscr{A}$ which assigns to each object $B \in \mathscr{B}$ a reflection in $\mathscr{A}$. R is called a **reflector**. If we consider R to be a functor from $\mathscr{B}$ to $\mathscr{B}$ we obtain a natural transformation from the identity functor on $\mathscr{B}$ to R. This transformation establishes a natural equivalence from $(R(B),A)_{\mathscr{A}}$ to $(B,A)_{\mathscr{B}}$ for all $B \in \mathscr{B}$ and $A \in \mathscr{A}$.

The dual notion of reflection is **coreflection.**

Among the best known examples of reflective subcategories are: the category of compact spaces in the category of normal Hausdorf spaces; the category of abelian groups in the category of all groups; the category of torsion-free groups in the category of abelian groups; the category of complete metric spaces in the category of all metric spaces and uniformly continuous maps. The category of torsion groups in the category of abelian groups is an example of a coreflective subcategory.

If $\mathscr{A}$ is a reflective subcategory of $\mathscr{B}$, then:

The inclusion functor $\mathscr{A} \to \mathscr{B}$ preserves left roots.

The reflector $R\colon \mathscr{B} \to \mathscr{A}$ preserves right roots.

If $\mathscr{B}$ is right-complete and $\mathscr{A}$ is full then $\mathscr{A}$ is right-complete. (First obtain the right root in $\mathscr{B}$, then reflect.)

If $\mathscr{A}$ is a full subcategory then the inclusion functor of $\mathscr{A}$ followed by the reflector is naturally equivalent to the identity on $\mathscr{A}$.

If $\mathscr{B}$ is left-complete and $\mathscr{A}$ is full then $\mathscr{A}$ is left-complete.

Let $r\colon I \to R$ be the associated transformation from the identity to the reflector. By iteration we obtain a transformation $R \to R^2$ which splits; i.e., there exists a transformation $R^2 \to R$ such that $R \to R^2 \to R$ is the identity transformation of R. $\mathscr{A}$ is a full subcategory iff $R \to R^2$ is an isomorphism.

Let $\mathscr{A}$ be an arbitrary subcategory of $\mathscr{B}$, $R\colon \mathscr{B} \to \mathscr{B}$ a functor whose image lies in $\mathscr{A}$, and $r\colon I \to R$ a transformation such that $r \mid \mathscr{A}\colon I \mid \mathscr{A} \to R \mid \mathscr{A}$ splits in $\mathscr{A}$; i.e., such that the inverse $s\colon R \mid \mathscr{A} \to I \mid \mathscr{A}$ assumes all of its values in $\mathscr{A}$. Then $\mathscr{A}$ is a reflective subcategory and R is its reflector. (Prove that for any $B \in \mathscr{B}$ and $A \in \mathscr{A}$

$$(B,A)_{\mathscr{B}} \xrightarrow{R} (R(B),R(A))_{\mathscr{A}} \xrightarrow{(R(B),{}^{s}A)} (R(B),A)$$

is an isomorphism and is equal to

$$(B,A)_{\mathscr{B}} \xrightarrow{(r_B,A)} (R(B),A)_{\mathscr{A}} \qquad .)$$

G. Adjoint functors

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, and $S\colon \mathscr{A} \to \mathscr{B}$ and $T\colon \mathscr{B} \to \mathscr{A}$ covariant functors. We say that S is the **left-adjoint** of T (and T is the **right-adjoint** of S) if $(S(A),B)_{\mathscr{B}}$ and $(A,T(B))_{\mathscr{A}}$ are naturally equivalent; more formally, if there exists a natural equivalence between the two functors

$$\mathscr{A} \times \mathscr{B} \xrightarrow{S \times I} \mathscr{B} \times \mathscr{B} \xrightarrow{Hom} \mathscr{S}$$

$$\mathscr{A} \times \mathscr{B} \xrightarrow{I \times T} \mathscr{A} \times \mathscr{A} \xrightarrow{Hom} \mathscr{S}.$$

If $\mathscr{A}$ and $\mathscr{B}$ are additive categories we replace $\mathscr{S}$ with $\mathscr{G}$, and require, of course, that the equivalence preserve group structure.

Some examples of adjoint functors are the following:

Let $\mathscr{A}$ be a reflective subcategory of $\mathscr{B}$. Then its reflector is the left-adjoint of the inclusion functor $\mathscr{A} \to \mathscr{B}$. Indeed, a subcategory is reflective iff its inclusion functor has a left-adjoint, and is co-reflective iff its inclusion functor has a right-adjoint.

If $\mathscr{A}$ is a complete category then the functor $(A,-)\colon \mathscr{A} \to \mathscr{S}$ has a left-adjoint, thus: Define $F\colon \mathscr{S} \to \mathscr{A}$ by $F(S) = \Sigma_S A$. Then $(F(S),A')$ is naturally equivalent to $(S,(A,A'))$.

The functor $(A,-)\colon \mathscr{G} \to \mathscr{G}$ has a left-adjoint, namely the tensor product. $(B \otimes A,A')$ is naturally equivalent to $(B,(A,A'))$. We have not defined tensor products in this book, nor need we now give any other definition save the one just given: - $\otimes A$ is the left-adjoint of $(A,-)$. The proof of its existence is another matter.

The contravariant cases:

Let $S\colon \mathscr{A} \to \mathscr{B}$ and $T\colon \mathscr{B} \to \mathscr{A}$ be contravariant functors. S and T are **adjoint on the left** if $(S(A),B)_{\mathscr{B}}$ is naturally equivalent to $(T(B),A)_{\mathscr{A}}$, and they are **adjoint on the right** if $(B,S(A))_{\mathscr{A}}$ is naturally equivalent to $(A,T(B))_{\mathscr{B}}$.

For a complete category $\mathscr{A}$ the functor $(-,A)\colon \mathscr{A} \to \mathscr{S}$ has an adjoint on the right, thus: Define $F\colon \mathscr{S} \to \mathscr{A}$ by $F(S) = \Pi_S A$. The functor $(-,A)\colon \mathscr{G} \to \mathscr{G}$ has an adjoint on the right: itself!

Some facts about adjoint functors are the following:

If S is the left-adjoint of T and T is the right-adjoint of S then T preserves left roots and S preserves right roots.

If S and T are adjoint on the left then they both carry left roots into right roots. If S and T are adjoint on the right then they both carry right roots into left roots.

If a covariant functor $S\colon \mathscr{A} \to \mathscr{S}$ has a left-adjoint then there exists $A \in \mathscr{A}$ such that S is naturally equivalent to $(A,-)$. (In which case we say that S is a **representable functor**; in particular, it is **represented** by A.) To find A, simply evaluate the left-adjoint of S on a set with a single element.

In the additive case the same statement is true. If a covariant functor $S\colon \mathscr{A} \to \mathscr{G}$ has a left-adjoint then it is representable. To find the object which represents it, evaluate the left-adjoint on the infinite cyclic group.

A contravariant functor $S: \mathscr{A} \to \mathscr{S}$ which has an adjoint on the right is representable, which in this case means that there is an object $A \in \mathscr{A}$ such that S is naturally equivalent to $(-,A)$. And the same statement is true in the additive case.

H. Transformation adjoints

Let $T_1,T_2: \mathscr{A} \to \mathscr{B}$ be covariant functors and $\eta: T_1 \to T_2$ a natural transformation. For every $A \in \mathscr{A}$, $B \in \mathscr{B}$, η induces a function $(B,T_1(A)) \to (B,T_2(A))$. If we define $(-,T_i(-)): \mathscr{B} \times \mathscr{A} \to \mathscr{S}$ to be the composition $\mathscr{B} \times \mathscr{A} \xrightarrow{I \times T_i} \mathscr{B} \times \mathscr{B} \xrightarrow{Hom} \mathscr{S}$ we obtain a natural transformation $\bar{\eta}: (-,T_1(-)) \to (-,T_2(-))$. Conversely, given any such $\bar{\eta}$ define the natural transformation $\eta: T_1 \to T_2$ by $\eta_A = \bar{\eta}_{T_1(A),A}(1_{T_1(A)})$. These two processes take us around in a circle.

Similarly, given $S_1, S_2: \mathscr{B} \to \mathscr{A}$ and a natural transformation $\eta: S_2 \to S_1$ we obtain $\bar{\eta}: (S_1(B),A) \to (S_2(B),A)$. (The interchanging of the indices is not a misprint.)

If S_i is a left-adjoint of T_i and $\eta: T_1 \to T_2$ is a natural transformation then there is a unique $\eta^*: S_2 \to S_1$ such that

$$\begin{array}{ccc} (B,T_1(A)) & \xrightarrow{(B,\eta_A)} & (B,T_2(A)) \\ \downarrow & & \downarrow \\ (S_1(B),A) & \xrightarrow[(\eta^*_B,A)]{} & (S_2(B),A) \end{array}$$

commutes for all $A \in \mathscr{A}$, $B \in \mathscr{B}$.

If further, $\beta: T_2 \to T_3$ is a natural transformation, then $(\beta\eta)^* = \eta^*\beta^*$. Set theoretical difficulties prevent us from saying that the category of functors from $\mathscr{A}$ to $\mathscr{B}$ with left-adjoints is dual to the category of functors from $\mathscr{B}$ to $\mathscr{A}$ with right-adjoints.

Adjoints are unique up to isomorphism.

Given abelian categories $\mathscr{A}$ and $\mathscr{B}$, covariant functors T_1,T_2,T_3: $\mathscr{A} \to \mathscr{B}$, and transformations $T_1 \to T_2$, $T_2 \to T_3$ such that for all $A \in \mathscr{A}$, $O \to T_1(A) \to T_2(A) \to T_3(A)$ is exact in $\mathscr{B}$, then if S_1,S_2,S_3 are left-adjoints of T_1,T_2,T_3 the induced transformations $S_3 \to S_2$,

$S_2 \to S_1$ are such that $S_3(B) \to S_2(B) \to S_1(B) \to O$ is exact for all $B \in \mathscr{B}$.

Suppose that $S: \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ is a covariant functor such that for every $B \in \mathscr{B}$, $S(-,B): \mathscr{A} \to \mathscr{C}$ has a right-adjoint $T_B: \mathscr{C} \to \mathscr{A}$. We obtain then a functor $T: \mathscr{B} \times \mathscr{C} \to \mathscr{A}$ contravariant on $\mathscr{B}$, covariant on $\mathscr{C}$. The adjointness yields isomorphisms $(S(A,B),C) \to (A,T(B,C))$. (For the foundational example let S: $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ be the tensor product and $T(B,C)$ the group of maps from B to C.)

Because $S(-,B)$ and $T(B,-)$ are adjoint, $S(-,B)$ is right-exact and $T(B,-)$ is left-exact. If furthermore $S(A,-)$ is right-exact then $T(-,C)$ carries right-exact sequences into left-exact sequences and conversely.

I. The reflectivity of images of adjoint functors

Let $S: \mathscr{A} \to \mathscr{B}$ be the left-adjoint of $T: \mathscr{B} \to \mathscr{A}$. Suppose that T is one-to-one on objects. Let $\mathscr{A}'$ be the image of T. For each $A \in \mathscr{A}$ define $r_A: A \to TS(A)$ to be the map which corresponds to $1_{S(A)}$ under the natural equivalence $(S(A),S(A)) \to (A,TS(A))$. The collection $\{r_A\}$ forms a natural transformation from the identity on $\mathscr{A}$ to TS. Similarly the isomorphisms $(ST(B),B) \to (T(B),T(B))$ establish a transformation r' from ST to the identity on $\mathscr{B}$. (r'_B corresponds to $1_{T(B)}$.)

For each $A \in \mathscr{A}'$ define $s_A: TS(A) \to A$ to be the map $T(r'_B)$ for any B such that $T(B) = A$. The collection $\{s_A\}$ forms a natural transformation from $TS \mid \mathscr{A}'$ to the identity of $\mathscr{A}'$. The composition

$$I_{\mathscr{A}'} \xrightarrow{r} TS \mid \mathscr{A}' \xrightarrow{s} I_{\mathscr{A}'}$$

may be seen to be the identity.

By Exercise 3-F, therefore, TS is the reflector of $\mathscr{A}'$ and dually ST is the correflector of the subcategory of $\mathscr{B}$ generated by S. We may say, therefore, that the images of right-adjoints generate reflective subcategories, and the images of left-adjoints generate correflective subcategories.

If we consider the functor $T: \mathscr{B} \to \mathscr{A}'$ (that is, if we redefine the range of T to be $\mathscr{A}'$) then it is clear that the composition $\mathscr{A}' \subset \mathscr{A} \xrightarrow{S} \mathscr{B}$ is the left-adjoint of T.

J. The adjoint functor theorem

A category is **well-powered** if it shares with the category of sets the property that the family of subobjects of any object is a set. (Prop. 3.35 says, then, that an abelian category with a generator is well-powered. Electrifying.)

Let $\mathscr{A}$ be a well-powered, left-complete category, and $T: \mathscr{A} \to \mathscr{B}$ any covariant functor. Then T has a left-adjoint iff

(0) *For every $B \in \mathscr{B}$ there is $A \in \mathscr{A}$ and a map $B \to T(A) \in \mathscr{B}$.*

(1) *T preserves left roots.*

(2) *(The solution set condition.) For every $B \in \mathscr{B}$ there exists a* set *$S_B \subset \mathscr{A}$ such that for every $A \in \mathscr{A}$ and map $B \to T(A) \in \mathscr{B}$ there is an object $A' \in S_B$ and maps $A' \xrightarrow{x} A \in \mathscr{A}$, $B \to T(A') \in \mathscr{B}$ such that*

$$\begin{tikzcd} & T(A') \arrow[d, "T(x)"] \\ B \arrow[ur] \arrow[r] & T(A) \end{tikzcd}$$

commutes.

One direction has almost been established: If T has a left-adjoint S then condition (1) appeared in Exercise 3-G, and for the solution set take $S_B = \{TS(B)\}$.

For the other direction, let $B \in \mathscr{B}$ and let S_B be a solution set as described in the second condition. Define $\bar{A} = \Pi_{S_B}\Pi_{(B,A')}A'$ and note that there is a map $B \to T(\bar{A})$ such that for any $A \in \mathscr{A}$ and $B \to T(A) \in \mathscr{B}$ there is a map $\bar{A} \xrightarrow{x} A \in \mathscr{A}$ such that

$$\begin{tikzcd} & T(\bar{A}) \arrow[d, "T(x)"] \\ B \arrow[ur] \arrow[r] & T(A) \end{tikzcd}$$

commutes. (No uniqueness.)

A few definitions which not only simplify the statement of the rest of the proof, but will be needed in the next few exercises, are the following: Given a map $B \xrightarrow{y} T(A)$, we shall say that a subobject $A' \to A$ *allows* y if $B \xrightarrow{y} T(A)$ may be factored through $T(A') \to T(A)$. We shall say that y *generates* A if no proper subobject of A allows y. (The word "generates" here is best appreciated by letting $\mathscr{A}$ be the category of groups and T the forgetful functor into the category of sets.)

The left-completeness of $\mathscr{A}$ together with the left-root-preservation of T implies that for every map $B \xrightarrow{y} T(A)$ there is a minimal subobject of A which allows y. Thus there exists a factorization $B \xrightarrow{y} T(A) = B \xrightarrow{y'} T(A') \to T(A)$ such that y' generates A'. We shall call the subobject A' the *subobject generated by* y.

If $B \xrightarrow{y} T(A)$ generates A, then if $B \xrightarrow{y} T(A) \xrightarrow{T(a)} T(C) = B \xrightarrow{y} T(A) \xrightarrow{T(b)} T(C)$ it is the case that $Ker(a - b) \to A$ allows y and hence that $Ker(a - b) = A$ and that $a = b$.

Starting with the map defined above, $B \to T(\bar{A})$, we let $\bar{\bar{A}}$ be the subobject of $\bar{A}$ generated by $B \to T(\bar{A})$. The map $B \to T(\bar{\bar{A}})$ has the property that for every $B \xrightarrow{z} T(A)$ there exists a unique $\bar{\bar{A}} \xrightarrow{x} A$ such that

$$\begin{tikzcd} & T(\bar{\bar{A}}) \arrow[dd, "T(x)"] \\ B \arrow[ur] \arrow[dr, "z"'] & \\ & T(A) \end{tikzcd} \qquad \text{commutes.}$$

We define $S: \mathscr{B} \to \mathscr{A}$ by, first, letting $S(B) = \bar{\bar{A}}$; second, doing the same for all the other objects of $\mathscr{B}$; third, for a map $B_1 \xrightarrow{z} B_2$, letting $S(z) = x$, where x is the unique map from $S(B_1)$ to $S(B_2)$ such that

$$\begin{tikzcd} B_1 \arrow[r] \arrow[d, "z"'] & T(S(B_1)) \arrow[d, "T(x)"] \\ B_2 \arrow[r] & T(S(B_2)) \end{tikzcd} \qquad \text{commutes.}$$

The stipulation in condition two, that S_B be a *set*, is not baroque. Because mathematics has progressed for a long time without having had to take the set–class distinction seriously does not mean that the distinction is spurious. The requirement that there be a set such as S_B is of the same nature as a requirement that a group be generated by a finite set. Both requirements can be very difficult to fulfill, and both can have powerful consequences.

Whereas the set–class distinction first appeared in order to solve certain puzzles in the formulation of a language for mathematics, the distinction must be considered more than a linguistic accident. True, there are languages for mathematics which do not admit the distinction; and it is likewise true that such languages either do not admit any interesting examples of complete categories (Exercise 3-D), or, if they do, have simply renamed the distinction (usually in terms of accessibility of cardinals or of level of type). Many of the classic results of algebraic number theory may be stated in a language which does not admit *infinite* sets. Indeed, theorems such as the Dirichlet unit theorem become much more obviously the deep theorems which they are when so stated, for they become in such languages existence theorems. (There is a group of units.) It is another question whether the theorems may be proved in such languages.

K. Some immediate applications of the adjoint functor theorem

Let $\mathscr{A}$ be a complete well-powered and co-well-powered additive category and $A \in \mathscr{A}$. Then the functor $(A,-)\colon \mathscr{A} \to \mathscr{G}$ has a left-adjoint.

The functor $(A,-)$ preserves left roots and we need only verify the solution set condition. Let $G \in \mathscr{G}$ and define S_G to be a representative set of all the quotient objects of $\Sigma_G A$. For any $G \xrightarrow{f} (A,B) \in \mathscr{G}$ let $B' \xrightarrow{x} B$ be the image of the map $\Sigma_G A \to B$ where $A \xrightarrow{u_g} \Sigma_G A \to B = f(g)$ for all $g \in G$. Then $B' \in S_G$ and the image of f lies in $(A,B') \xrightarrow{(A,x)} (A,B)$.

The adjoint of $(A,-)$ we shall call $-\otimes A\colon \mathscr{G} \to \mathscr{A}$. Hence for $G \in \mathscr{G}$, $A,A' \in \mathscr{A}$, $(G \otimes A,A')$ is naturally equivalent to $(G,(A,A'))$. By Exercise 3-H we may obtain a functor $\otimes\colon \mathscr{G} \times \mathscr{A} \to \mathscr{A}$ which is right-exact in both variables. We call this functor the **tensor product**.

Dually, the contravariant functor $(-,A)$: $\mathscr{A} \to \mathscr{G}$ has an adjoint on the right which we shall indicate by the symbol $\overline{(-,A)}$. For $G \in \mathscr{G}$, $A \in \mathscr{A}$, $\overline{(G,A)}$ is an object in $\mathscr{A}$. For $A' \in \mathscr{A}$, $(G,(A',A))$ is naturally equivalent to $(A',\overline{(G,A)})$. Exercise 3-H leads to the definition of $\overline{(-,-)}$: $\mathscr{G} \times \mathscr{A} \to \mathscr{A}$ a functor on two variables, contravariant on the first and covariant on the second. We call it the **symbolic hom** functor.

The tensor product and symbolic hom functors are related through duality as follows:

$$G \otimes A = \overline{(G,A^*)}^*, \qquad \overline{(G,A)} = (G \otimes A^*)^*.$$

There is a natural equivalence between $(A,\overline{(G,A')})$ and $(G \otimes A,A')$.

The solution set condition is often guaranteed to hold by certain other hypotheses. For instance, we may obtain the old theorem:

Let $\mathscr{B}$ be a complete well-powered and co-well-powered category and $\mathscr{A}$ a full subcategory replete in $\mathscr{B}$ such that $\mathscr{A}$ is closed under the formation of products and subobjects. Then $\mathscr{A}$ is a reflective subcategory of $\mathscr{B}$.

For $B \in \mathscr{B}$ let S_B be a representative set of quotient objects of B which lie in $\mathscr{A}$.

As immediate applications one may obtain the reflectivity of Hausdorf spaces in all spaces, torsion-free groups in all groups (abelian or not), and countless similar well-known cases.

Let $\mathscr{A}$ be a well-powered left-complete category and let T: $\mathscr{A} \to \mathscr{B}$ be a left-root-preserving full functor whose image is all of $\mathscr{B}$. Then T has a left-adjoint.

For $B \in \mathscr{B}$, $\{A\}$ is a solution set if $T(A) = B$.

As a consequence, a left-root-preserving functor from a left-complete well-powered category has a left-adjoint iff its image generates a reflective subcategory of the range.

L. How to find solution sets

Let $\mathscr{A}$ be a left-complete well-powered category, and T: $\mathscr{A} \to \mathscr{B}$ a left-root-preserving functor. Fix an object $B \in \mathscr{B}$. Given an object

$A \in \mathscr{A}$ we shall say that B *generates* A *through* T if there exists a map $B \xrightarrow{y} T(A)$ such that y generates A (as defined in Exercise 3-J).

Let S_B be a solution set for B and let $B \xrightarrow{y} T(A)$ generate A. There exists an object $A' \in S_B$ and $A' \xrightarrow{x} A \in \mathscr{A}$ such that $B \xrightarrow{y} T(A) = B \to T(A') \xrightarrow{T(x)} T(A)$. $A' \xrightarrow{x} A$ must be an epimorphism, for if $A' \xrightarrow{x} A \xrightarrow{a} C = A' \xrightarrow{x} A \xrightarrow{b} C$ then $Ker(a - b) \to A$ allows x and $Ker(a - b) = A$ and $a = b$.

If $\mathscr{A}$ is co-well-powered and if T has a left-adjoint then each object in $\mathscr{B}$ generates at most a set of nonisomorphic objects in $\mathscr{A}$.

Conversely, if B generates at most a set of nonisomorphic objects in $\mathscr{A}$ then B has a solution set. Indeed, if we let S_B be a representative set of objects in $\mathscr{A}$ which may be generated by B it is easy to verify that S_B is a solution set.

Let $\mathscr{A}$ be a left-complete well-powered category and $T: \mathscr{A} \to \mathscr{B}$ a covariant functor. Then T has a left-adjoint if (and, in the case that $\mathscr{A}$ is also co-well-powered, only if)

(0) *For every $B \in \mathscr{B}$ there is $A \in \mathscr{A}$ and $B \to T(A) \in \mathscr{B}$.*
(1) *T preserves left roots.*
(2) *Every object in $\mathscr{B}$ generates through T at most a set of nonisomorphic objects in $\mathscr{A}$.*

As an immediate application (see Exercises 5-D, F, and I for more), let $\mathscr{A}$ be the category of lattices and functions between lattices that preserve finite unions and intersections. Let $T: \mathscr{A} \to \mathscr{S}$ be the forgetful functor into the category of sets. For $B \in \mathscr{S}$ the only objects in $\mathscr{A}$ which may be generated by B are of cardinality less than or equal to that of B (unless B is finite, in which case, B generates only denumerably infinite lattices). The left-adjoint of T carries B into what is usually called the *free lattice* generated by B. We can complicate the example by defining $\mathscr{A}$ to be the category of countably complete lattices and then replacing "countable" with any cardinal.

M. The special adjoint functor theorem

The chief failing of the adjoint functor theorem is that it involves not only the (unavoidable) continuity condition on the functor but also a (generally necessary) smallness condition relating the domain category, the functor, and the range category. The special adjoint functor theorem below says in effect that the smallness condition will always be satisfied by left-root-preserving functors if the domain category is "small enough" to have a cogenerator.

Let $\mathscr{A}$ be a well-powered, left-complete category with a cogenerator and $T\colon \mathscr{A} \to \mathscr{B}$ any covariant functor. Then T has a left-adjoint iff T preserves left roots and for all $B \in \mathscr{B}$ there is $A \in \mathscr{A}$ and $B \to T(A) \in \mathscr{B}$.

Let C be a cogenerator for $\mathscr{A}$ and suppose that $B \xrightarrow{y} T(A)$ generates A. The function $(A,C) \xrightarrow{T} (T(A),T(C)) \xrightarrow{(y,T(C))} (B,T(C))$ is one-to-one. Hence $A \to \Pi_{(A,C)}C \to \Pi_{(B,T(C))}C$ is monomorphic.

If B generates A through T (see last exercise) then A is isomorphic to a subobject of $\Pi_{(B,T(d))}C$.

As an immediate application, we note that the full subcategory of compact spaces in the category of Hausdorf spaces is reflective. The Urysohn lemma asserts that the unit interval is a cogenerator for the category of compact Hausdorf spaces, and the Tychonoff theorem implies that the inclusion functor preserves left roots.

N. The special adjoint functor theorem at work

By dualizing the range and domain we obtain three other theorems, in which we omit the "zero" condition:

Let $\mathscr{A}$ be a well-powered, left-complete category with a cogenerator and $T\colon \mathscr{A} \to \mathscr{B}$ a contravariant functor. Then T has an adjoint on the left iff T carries left roots into right roots.

(Dualize $\mathscr{B}$.) ∎

Let $\mathscr{A}$ be a co-well-powered, right-complete category with a generator and $T\colon \mathscr{A} \to \mathscr{B}$ a contravariant functor. Then T has an adjoint on the right iff T carries right roots into left roots.

(Dualize $\mathscr{A}$.) ∎

Let $\mathscr{A}$ be a co-well-powered, right-complete category with a generator and $T\colon \mathscr{A} \to \mathscr{B}$ a covariant functor. Then T has a right-adjoint iff T preserves right roots.

(Dualize both $\mathscr{A}$ and $\mathscr{B}$.)

Let R be a ring and $\mathscr{G}^R$ the category of left R-modules. Let $T\colon \mathscr{G}^R \to \mathscr{G}$ be any contravariant functor which carries right roots into left roots. Then T is representable.

We may easily determine that T is represented by a module whose underlying abelian group is $T(R)$. The module structure of $T(R)$ is determined by $r\colon T(R) \to T(R) = T(r)$.

If we are allowed to use the fact that the group of rational numbers modulo the subgroup of integers, which group we shall call Q/Z, is an injective cogenerator for $\mathscr{G}$, then we may construct an injective cogenerator for $\mathscr{G}^R$. The forgetful functor $\mathscr{G}^R \xrightarrow{F} \mathscr{G}$ preserves all roots, and hence $\mathscr{G}^R \xrightarrow{F} \mathscr{G} \xrightarrow{(-,Q/Z)} \mathscr{G}$ is an exact contravariant embedding which carries right roots into left roots. Since it is representable, it must be represented by an injective cogenerator.

Now that $\mathscr{G}^R$ has a cogenerator we may obtain **Watts' theorem:**

A covariant functor $T\colon \mathscr{G}^R \to \mathscr{G}$ is representable iff it preserves left roots.

Finally, we obtain the local representation theorem:

Given an arbitrary left-complete category $\mathscr{A}$, a small subcategory $\mathscr{A}'$, and a covariant left-root-preserving functor $T\colon \mathscr{A} \to \mathscr{G}$, there exists an object $A \in \mathscr{A}$ such that $(A,-) \mid \mathscr{A}'$ is naturally equivalent to $T \mid \mathscr{A}'$.

Let $\mathscr{A}''$ be the smallest full subcategory replete in $\mathscr{A}$ which contains $\mathscr{A}'$ and is closed under the formation of products and difference kernels. Then $\Pi_{\mathscr{A}'} A$ is a cogenerator for $\mathscr{A}''$, and $T \mid \mathscr{A}''$ is left-root-preserving.

O. Exercise for model theorists

An n-ary predicate on a set S is a subset of the n-fold product of S. Given an indexed collection of finite numbers $\{n_1, n_2, \cdots, n_j\}$, a first-order statement is a well-formed formula obtained by combining the atomic formulas, $P_1(x_1, x_2, \cdots, x_{n_1}), \cdots, P_j(x_1, x_2, \cdots, x_{n_j})$ using conjunction, disjunction, implication, negation and then quantifying the lower-case variables. Examples:

$$\forall_x \forall_y \forall_z [P(x,y) \wedge P(y,z) \rightarrow P(x,z)],$$

$$\forall_x \forall_y [P(x,y) \wedge P(y,x) \rightarrow x = y], \qquad \exists_x \forall_y [P(y,x)]$$

A *theory* $\mathbf{T}$ is any set of first-order statements. The above list of examples is a theory of partial orderings with maximal elements. A *model* for $\mathbf{T}$ is a set S together with a designated set of predicates on S such that all the statements in $\mathbf{T}$ become true. We shall notationally confuse the model with its underlying set.

We may start with a theory and consider its class of models; conversely we may start with a model (for the empty theory) and consider its complete theory. Two models are said to be *elementarily equivalent* if they have the same complete theories. A function between the underlying sets of two models $A \xrightarrow{f} B$ is said to be an *elementary extension*, if for every formula F (not all the lower-case letters need be quantified) that can be built from the original predicates and for every $x_1, x_2, \cdots, x_n \in A$ it is the case that

$$F(x_1, \cdots, x_n) \leftrightarrow F(f(x_1), \cdots, f(x_n)).$$

If f is an inclusion function, A is an *elementary submodel* of B. The Lowenheim-Skolem theorem says that every model B has a countable elementary submodel in the case that the original list of predicates is finite or countable and otherwise of cardinality equal to that of the original list of predicates.

Godel's completeness theorems say that every logically consistent theory has a model (and it is an article of faith that the complete theory of a model is consistent). A corollary is the *compactness theorem:* If every finite subset of **T** has a model then so does **T**. Finally, using Jonson's amalgamation, every set of elementarily equivalent models has a common elementary extension.

In order to define a *category of models* it is necessary to specify what we mean by maps. Categories of elementary extensions do not seem to be interesting as categories. Suppose **F** is a set of formulas made up from the original list of predicates. We shall say that a function between models $A \xrightarrow{f} B$ is an **F**-map if every formula in **F** is "preserved," in the positive sense, by f. That is, for $F \in \mathbf{F}$ and $x_1, x_2, \cdots, x_n \in A, \mathbf{F}(x_1, \cdots, x_n) \to F(f(x_1), \cdots, f(x_n))$. If **F** is empty, any function is an **F**-map; if **F** is the set of all possible formulas then only elementary extensions are **F**-maps. (Note that if the formula $x \neq y$ is in **F**, then every **F**-map is one-to-one.) Given a theory **T** and a set of formulas **F**, a category of models is determined. As familiar examples we can obtain the category of groups and group homomorphisms, the category of lattices and lattice homomorphisms, the category of small categories and functors.

If **F** is empty and **T** has models of every cardinality (and one infinite model implies a model of every infinite cardinality) then the corresponding category of models is equivalent to the category of sets. We shall tacitly assume this to be the case throughout.

A category of models is well-powered. Suppose $f: A \to B$ is an **F**-map and that $|A|$ (the cardinality of A) is greater than $2^{|B|}$ and $2^{|\mathbf{T}|}$. We shall show that f is *not* a monomorphism. For each $y \in B$ let U_y be a new unary predicate: $U_y(x)$ is true for A iff $f(x) = y$. Let $\mathbf{T}_2$ be the complete theory of A with respect to the original predicates and the new. Let E be the set of elementary (with respect to the original predicates and the new) submodels of A of cardinality $|\mathbf{T}_2| = |B| + |\mathbf{T}_1|$. The union of the models in E is all of A because for each $x \in A$ we could have added another unary predicate insuring that elementary submodels contain x. Hence E contains at least $|A|$ distinct subsets of A and there are only $2^{|B|+|\mathbf{T}_1|}$ isomorphism classes. Necessarily, then, there is a model A' and distinct elementary extensions $A' \xrightarrow{g_1} A$, $A' \xrightarrow{g_2} A$ which when followed by f agree. g_1 and g_2 are certainly **F**-maps.

A category of models is co-well-powered. Let $f: A \to B$ be an **F**-map and suppose that $|B|$ is greater than $2^{|A|+|\mathbf{T}|}$. We shall show that f is *not* an epimorphism. For each $x \in A$ let U_x be a new unary predicate: $U_x(y)$ is true for B iff $f(x) = y$. Let $\mathbf{F}_2$ be the set of formulas involving both the original and the new predicates. There must be distinct $y_1, y_2 \in B$ such that for any unary formula $F \in \mathbf{F}_2$ $F(y_1) \leftrightarrow F(y_2)$. Let V be another unary predicate and consider the two models B_1 and B_2 defined by: $V(x)$ is true in B_i iff $x = y_i$. B_1 and B_2 are elementarily equivalent with respect to all the predicates. Let B' be a common elementary extension. The two embeddings $B_1 \xrightarrow{g_1} B'$ and $B_2 \xrightarrow{g_2} B'$ must be different, for in the complete theories of B_1 and B_2 is to be found the statement

$$\forall_{x,y}[V(x) \wedge V(y) \to x = y].$$

g_1 and g_2 are both **F**-maps and when preceded by f are the same.

A left-complete category has a generator: Let $\{A_i\}$ be a set which represents every countable isomorphism class of models. ΣA_i is a generator (regardless of **F**).

Let $\mathscr{A}$ be a category of models. The forgetful functor $\mathscr{A} \to \mathscr{S}$ into the category of sets always satisfies the solution set condition. (For infinite $S \in \mathscr{S}$ define **S** to be a representative set of models of cardinality no greater than $|S| + |\mathbf{T}_1|$.) The zero condition is easy, and hence the forgetful functor has an adjoint iff it preserves left roots, which is equivalent to saying that the standard constructions of products (cartesian) and difference kernels (subsets) work. The adjoint of the forgetful functor has for values what would normally be called **free models**. The situation may be generalized by letting $\mathbf{T}_1 \subset \mathbf{T}_2$ and $\mathbf{F}_1 \subset \mathbf{F}_2$ considering the forgetful functor $\mathscr{A}_2 \to \mathscr{A}_1$ where $\mathscr{A}_i$ is determined by $\mathbf{T}_i$, $\mathbf{F}_i$.

CHAPTER 4

METATHEOREMS

In Chapter 7 we shall prove that for every small abelian category $\mathscr{A}$ there is an exact embedding $\mathscr{A} \to \mathscr{G}$.

To illustrate the usefulness of the existence of exact embeddings let us consider the "five lemma":

Let $\mathscr{A}$ be an abelian category and

$$\begin{array}{ccccccc}
 & & & & O & & \\
 & & & & \downarrow & & \\
 & & O & & K & & O \\
 & & \downarrow & & \downarrow & & \downarrow \\
A_{11} & \to & A_{12} & \to & A_{13} & \to & A_{14} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_{21} & \to & A_{22} & \to & A_{23} & \to & A_{24} \\
\downarrow & & & & & & \\
O & & & & & &
\end{array}$$

a commutative diagram in $\mathscr{A}$ with exact rows and columns. We wish to prove that $K = O$. Let $F\colon \mathscr{A} \to \mathscr{G}$ be an exact embedding. F sends the diagram into a similar exact commutative diagram of groups and homomorphisms and $K = O$ iff $F(K) = O$.

The verification that the five lemma is true in $\mathscr{G}$ may be effected by classical diagram-chasing techniques such as the following, in which we will write $x_{ij} \to x_{kl}$ instead of

$$(A_{ij} \to A_{kl})x_{ij} = x_{kl}.$$

Let $x_{13} \in A_{13}$ be such that $x_{13} \to 0_{23}$. We wish to show that $x_{13} = 0_{13}$. Let $x_{13} \to x_{14}$ and observe that $x_{14} \to 0_{24}$, and hence that $x_{14} = 0_{14}$. By exactness there is $x_{12} \in A_{12}$ such that $x_{12} \to x_{13}$. Let $x_{12} \to x_{22}$ and observe that $x_{22} \to 0_{23}$, and hence, by exactness, there is $x_{21} \in A_{21}$ such that $x_{21} \to x_{22}$. Let $x_{11} \in A_{11}$ be such that $x_{11} \to x_{21}$. Because $A_{12} \to A_{22}$ is one-to-one, $x_{11} \to x_{12}$ and then $x_{12} \to 0 = x_{13}$.

4.1. VERY ABELIAN CATEGORIES

For expository purposes we say that an abelian category $\mathscr{B}$ is **very abelian** if for every small exact subcategory $\mathscr{A} \subset \mathscr{B}$ there is an exact embedding $\mathscr{A} \to \mathscr{G}$. The weak embedding theorem of Chapter 7 will prove that every abelian category is very abelian.

We wish to describe a class of statements which are true in every very abelian category iff they are true in $\mathscr{G}$. As a first approximation we may consider the following. Define a *simple diagrammatic* statement to be a statement about the exactness and commutativity of a diagram. A *compound diagrammatic* statement shall be of the form $P \to Q$ where P and Q are simple diagrammatic statements. A compound diagrammatic statement is true in every very abelian category iff it is true in $\mathscr{G}$.

The formalization of the matter starts by defining "diagram." A **diagram scheme** is a small category, and a **diagram** in a category $\mathscr{A}$ is a functor from a diagram scheme into $\mathscr{A}$. A set of **exactness conditions** on a scheme is a set of ordered pairs of maps in the scheme. Given a scheme (category) S, a set of exactness conditions E, and a diagram D (functor) on S into an abelian category $\mathscr{A}$, we say that D *satisfies the exactness conditions* if for every $(x,y) \in E$, it is the case that $(D(x),D(y))$ is an exact sequence in $\mathscr{A}$.

A surprising amount may be said about a diagram by imposing exactness conditions. Let $D\colon S \to \mathscr{A}$ be a diagram which satisfies a set of exactness conditions E. Then

$$D(A) = O \quad \text{if} \quad (A \xrightarrow{1} A, A \xrightarrow{1} A) \in E.$$

$$D(A \to B) = O \quad \text{if} \quad (A \longrightarrow B, B \xrightarrow{1} B) \in E$$

$$\left.\begin{array}{l} D(A_1 \xrightarrow{u_1} S), D(A_2 \xrightarrow{u_2} S) \\ D(S \xrightarrow{p_1} A_1), D(S \xrightarrow{p_2} A_2) \\ \text{is a direct-sum system} \end{array}\right\} \quad \text{if} \quad \left\{\begin{array}{l} A_1 \xrightarrow{u_1} S \xrightarrow{p_1} A_1 = 1 \\ A_2 \xrightarrow{u_2} S \xrightarrow{p_2} A_2 = 1 \\ (A_1 \xrightarrow{u_1} S, S \xrightarrow{p_2} A_2) \in E \\ (A_2 \xrightarrow{u_2} S, S \xrightarrow{p_1} A_1) \in E \\ \text{(See Prop. 2.42.)} \end{array}\right.$$

By extending these "ifs" one may see that commutativity conditions may be imposed through exactness conditions.

Given a scheme S, and two sets of exactness conditions E_1, E_2, we say that the compound diagrammatic statement (S, E_1, E_2) is true in $\mathscr{A}$ if every diagram $D\colon S \to \mathscr{A}$ which satisfies the exactness conditions E_1, also satisfies the conditions E_2.

We observe that if $\mathscr{A} \to \mathscr{B}$ is an exact embedding then if (S, E_1, E_2) is true in $\mathscr{B}$ it is true in $\mathscr{A}$.

4.2. FIRST METATHEOREM

To finish off the metatheorem we need the following:

Proposition 4.21

For every set $\{A_i\}_I$ of objects in an abelian category, there is a full small exact subcategory $\bar{\mathscr{A}} \subset \mathscr{A}$ such that $A_i \in \bar{\mathscr{A}}$ for all i.

Proof:

Let

K: (Maps in $\mathscr{A}$) $\to$ (Objects in $\mathscr{A}$)
F: (Maps in $\mathscr{A}$) $\to$ (Objects in $\mathscr{A}$), and
S: (Pairs of objects in $\mathscr{A}$) $\to$ (Objects in $\mathscr{A}$)

be functions such that

$K(x)$ is a kernel of x
$F(x)$ is a cokernel of x
$S(A,B)$ is a direct sum of A and B.

Given a full subcategory $\mathscr{B} \subset \mathscr{A}$ define $C(\mathscr{B})$ to be the full subcategory generated by $\mathscr{B}$, $K(\mathscr{B})$, $F(\mathscr{B})$ and $S(\mathscr{B} \times \mathscr{B})$.

If $\mathscr{B}$ is small then so is $C(\mathscr{B})$. Define $C^{n+1}(\mathscr{B}) = C(C^n(\mathscr{B}))$. $C^\infty(\mathscr{B}) = \bigcup_{n=1}^{\infty} C^n(\mathscr{B})$ is, by Theorem 3.41, a full exact subcategory. $C^\infty(\mathscr{B})$ is small if $\mathscr{B}$ is small. ▮

Metatheorem 4.22

Every compound diagrammatic statement true in $\mathscr{G}$ is true in every very abelian category.

Proof:

Suppose (S, E_1, E_2) is true in $\mathscr{G}$. Let $D: S \to \mathscr{A}$ be a diagram in a very abelian $\mathscr{A}$ satisfying the exactness conditions E_1. Let $\bar{\mathscr{A}}$ be a small exact subcategory of $\mathscr{A}$ such that the image of D lies in $\bar{\mathscr{A}}$. Then D satisfies E_1 in $\bar{\mathscr{A}}$, and it satisfies E_2 in $\bar{\mathscr{A}}$ iff it satisfies E_2 in $\mathscr{A}$. Let $F: \bar{\mathscr{A}} \to \mathscr{G}$ be an exact embedding. $FD: S \to \mathscr{G}$ satisfies E_1 and it satisfies E_2 iff $D: S \to \bar{\mathscr{A}}$ satisfies E_2. ▮

4.3. FULLY ABELIAN CATEGORIES

The important *connecting homomorphism theorem* is stated as follows:

If

$$\begin{array}{ccccccc} & & O & & O & & O \\ & & \downarrow & & \downarrow & & \downarrow \\ & & A_{11} & \to & A_{12} & \to & A_{13} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & A_{21} & \to & A_{22} & \to & A_{23} & \to & O \\ & & \downarrow & & \downarrow & & \downarrow \\ O & \to & A_{31} & \to & A_{32} & \to & A_{33} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & A_{41} & \to & A_{42} & \to & A_{43} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & O & & O & & O \end{array}$$

is a commutative diagram in an abelian category with exact rows and columns then there is a map $A_{13} \to A_{41}$ such that $A_{12} \to A_{13} \to A_{41} \to A_{42}$ is exact.

The first metatheorem does not shed light on the *existence* of maps. The connecting homomorphism theorem was classically proved for modules over a ring R, as follows: Given $x_{13} \in A_{13}$ let $x_{13} \to x_{23}$ and choose $x_{22} \in A_{22}$ such that $x_{22} \to x_{23}$. Let $x_{22} \to x_{32}$. Since $x_{32} \to 0_{33}$ there is $x_{31} \in A_{31}$ such that $x_{31} \to x_{32}$. Let $x_{31} \to x_{41}$ and define $f(x_{13}) = x_{41}$. The definition is invariant under the choice of x_{22} since if x'_{22} is such that $x_{22} \to x_{23}$ then $(x_{22} - x'_{22}) \to 0_{23}$ and there is $x_{21} \in A_{21}$ such that $x_{21} \to (x_{22} - x'_{22})$. Letting $x'_{22} \to x'_{32}$ and $x'_{31} \to x'_{32}$ we see that $(x_{31} - x'_{31}) \to (x_{32} - x'_{32})$ and that $x_{21} \to (x_{22} - x'_{22})$. $x_{21} \to (x_{31} - x'_{31})$ since $A_{31} \to A_{32}$ is monomorphic, and $x_{31} - x'_{31} \to 0_{41}$; hence $x'_{31} \to x_{41}$. f is a homomorphism since it is a composition of additive correspondences. To show that $A_{12} \to A_{13} \xrightarrow{f} A_{41}$ is exact we suppose that $f(x_{13}) = 0_{41}$ and let $x_{13} \to x_{23}$, $x_{22} \to x_{23}$, $x_{22} \to x_{32}$, $x_{31} \to x_{32}$, $x_{31} \to 0_{41}$. There is $x_{21} \in A_{21}$ such that $x_{21} \to x_{31}$. Let $x_{21} \to x'_{22}$ and note that

$$(x'_{22} - x_{22}) \to x_{23}, \qquad (x'_{22} - x_{22}) \to 0_{32}.$$

Hence there is $x_{12} \in A_{12}$ such that $x_{12} \to (x_{22} - x'_{22})$ and $x_{12} \to x_{13}$.

To prove that $A_{13} \xrightarrow{f} A_{41} \to A_{42}$ is exact let $x_{41} \in A_{41}$ be such that $x_{41} \to 0_{42}$. Choose $x_{31} \in A_{31}$ such that $x_{31} \to x_{41}$ and let $x_{31} \to x_{32}$. Note that $x_{32} \to 0_{42}$. Hence there is $x_{22} \in A_{22}$ such that $x_{22} \to x_{32}$ and we let $x_{22} \to x_{23}$. Since $x_{23} \to 0_{33}$ there is $x_{13} \in A_{13}$ such that $x_{13} \to x_{23}$. $f(x_{13}) = x_{41}$.

The *full* embedding theorem which will be proved in the last chapter says that for every small abelian category there is a ring R and an exact *full* embedding into the category of R-modules. The full embedding theorem allows us to dispatch certain existential questions in abelian categories exemplified by the connecting homomorphism theorem.

Define a *map extension* of a scheme S to be a scheme $\bar{S}$ together with a one-to-one functor $G: S \to \bar{S}$ such that all the objects of $\bar{S}$ appear as values of G (i.e., G establishes a one-to-one correspondence between the objects of S and the objects of $\bar{S}$).

Given a scheme S, a map extension $S \to \bar{S}$, and sets of exactness conditions E for S and $\bar{E}$ for $\bar{S}$, we say that the full compound diagrammatic statement $(S \to \bar{S}, E, \bar{E})$ is true for $\mathscr{A}$ if for every diagram $D: S \to \mathscr{A}$ which satisfies the conditions E, there is a diagram $\bar{D}: \bar{S} \to \mathscr{A}$ which satisfies the condition $\bar{E}$ and $\bar{D} = DG$.

We say that an abelian category $\mathscr{A}$ is **fully abelian** if for every full small exact subcategory $\bar{\mathscr{A}} \subset \mathscr{A}$ there is a ring R and a full exact embedding of $\bar{\mathscr{A}}$ into the category of R-modules. (We shall show in Chapter 7 that every abelian category is fully abelian.)

The full metatheorem, 4.31

If a full compound diagrammatic statement is true for all categories of R-modules then it is true for all fully abelian categories.

The proof is similar to that of the first metatheorem. ▮

4.4. MITCHELL'S THEOREM

Let R be a ring and $\mathscr{G}^R$ the category of left R-modules. Then R is a projective generator in $\mathscr{G}^R$. Indeed the functor

$$(R, -): \mathscr{G}^R \to \mathscr{G}$$

is the "forgetful" functor—it assigns to each R-module M the underlying abelian group M (it forgets that M is an R-module). If we were consistent category theorists we would not speak of elements of an R-module M but of maps from R to M. The element-chasing proof of the five lemma could be replaced by a map-chasing proof. Instead of starting with an element $x_{13} \in A_{13}$ such that $x_{13} \to 0_{23}$, we could start with a map $R \to A_{13}$ such that $R \to A_{13} \to A_{23} = 0$. We would prove that $R \to A_{13} \to A_{14} = 0$, and using the exactness of $A_{12} \to A_{13} \to A_{14}$ and the projectiveness of R obtain a map $R \to A_{12}$ such that $R \to A_{12} \to A_{13} = R \to A_{13}$. We could continue chasing until we reached a commutative diagram of the form

$$\begin{tikzcd}
R \arrow[dr] \arrow[drr] \arrow[drrr] \arrow[drrrr, "0"] \arrow[ddr] & & & & \\
& A_{11} \arrow[r] \arrow[d] & A_{12} \arrow[r] \arrow[d] & A_{13} \arrow[r] \arrow[d] & A_{14} \arrow[d] \\
& A_{21} \arrow[r] & A_{22} \arrow[r] & A_{23} \arrow[r] & A_{24}
\end{tikzcd}$$

Finally, then, $R \to A_{13} = R \to A_{11} \to A_{12} \to A_{13} = 0$.

All that was used in the chasing process was the projectiveness of R. We conclude that $A_{13} \to A_{23}$ is a monomorphism because R is a generator. Hence the entire proof of the five lemma could have been effected in any abelian category with a projective generator. This fact, that projective generators are as good as elements, was a part of the folklore of the subject from the beginning. We can formalize with

Proposition 4.43

An abelian category with a projective generator is very abelian. ∎

But far better is

Theorem 4.44 (Mitchell)

A complete abelian category with a projective generator is fully abelian.

Proof:

Let $\mathscr{A}'$ be a small full exact subcategory of a complete abelian category $\mathscr{A}$, and $\bar{P}$ a projective generator for $\mathscr{A}$. For each $A \in \mathscr{A}'$ we consider the epimorphism

$$\sum_{(\bar{P},A)} \bar{P} \to A.$$

By taking $I = \bigcup_{A \in \mathscr{A}'} (\bar{P},A)$ and defining $P = \Sigma_I \bar{P}$, we obtain a projective generator P such that for each $A \in \mathscr{A}'$ there is an epimorphism $P \to A$.

Define R to be the ring of endomorphisms of P. For every $A \in \mathscr{A}$, the abelian group (P,A) has a canonical R-module structure: for $P \xrightarrow{x} A \in (P,A)$ and $P \xrightarrow{r} P \in R$ define $rx \in (P,A)$ to be $P \xrightarrow{r} P \xrightarrow{x} A$.

Given a map $A \xrightarrow{y} B \in \mathscr{A}$, the induced map $(P,A) \xrightarrow{\bar{y}} (P,B)$ is an R-homomorphism ($\bar{y}(rx) = P \xrightarrow{r} P \xrightarrow{x} A \xrightarrow{y} B = r(\bar{y}(x))$). We define, therefore, F: $\mathscr{A} \to \mathscr{G}^R$ ($\mathscr{G}^R$ is the category of R-modules) by $F(A) = (P,A)$ with the canonical R-module structure. F is an exact embedding since P is a projective generator. $F \mid \mathscr{A}'$ is known to be an exact full embedding, therefore, once it is known to be full. Given $A,B \in \mathscr{A}'$ and a map $F(A) \xrightarrow{\bar{y}} F(B) \in \mathscr{G}^R$ we wish to find a map $A \xrightarrow{y} B \in \mathscr{A}'$

such that $F(y) = \bar{y}$. Let $O \to K \to P \to A \to O$ and $P \to B \to O$ be exact sequences in $\mathscr{A}$. Notice that $F(P) = R$. We obtain the commutative diagram in $\mathscr{G}^R$:

$$\begin{tikzcd} O \arrow[r] & F(K) \arrow[r] & R \arrow[r] \arrow[d, "f"] & F(A) \arrow[r] \arrow[d, "\bar{y}"] & O \\ & & R \arrow[r] & F(B) \arrow[r] & O \end{tikzcd}$$

where the existence of the map f is insured by the projectiveness of R in $\mathscr{G}^R$. Since R is a ring, any automorphism on R must be equivalent to multiplication on the right by an R-element. We assume then that $f(s) = sr$ for all $s \in R$, where $P \xrightarrow{r} P \in R$. Returning to $\mathscr{A}$, the diagram

$$\begin{tikzcd} O \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d, "r"'] & A \arrow[r] & O \\ & & P \arrow[r] & B \arrow[r] & O \end{tikzcd}$$

is such that $K \to P \xrightarrow{r} P \to B = 0$, since $F(K) \to R \xrightarrow{f} R \to F(B) = 0$ and F is an embedding. Hence there is a map $A \xrightarrow{y} B$ such that

$$\begin{tikzcd} P \arrow[r] \arrow[d, "r"'] & A \arrow[d, "y"] \\ P \arrow[r] & B \end{tikzcd} \qquad \text{commutes.}$$

Hence

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"'] & F(A) \arrow[d, "F(y)"] \\ R \arrow[r] & F(B) \end{tikzcd} \qquad \text{commutes}$$

and since $R \to F(A)$ is epimorphic, $F(y) = \bar{y}$. ∎

This last theorem reduces the problem of proving that every abelian category is fully abelian to the following: Given a small abelian category $\mathscr{A}$, find a complete abelian category $\mathscr{B}$ with a projective generator and an exact full embedding $\mathscr{A} \to \mathscr{B}$.

EXERCISES

A. Abelian lattice theory

Let $\mathscr{A}$ be a very abelian category and $A \in \mathscr{A}$. The lattice of subobjects of A is a modular lattice. (If $A_1 \subset A_2$, then $A_1 \cup (B \cap A_2) = (A_1 \cup B) \cap A_2$.)

B. Functor metatheory

One may state (or at least feel) a metatheorem concerning *functors* between very and fully abelian categories. It may be strong enough to handle connected sequences of functors and, as a test, Proposition III.4.1 of Cartan & Eilenberg [4, page 44].

C. Correspondences in categories

Let $\mathscr{A}$ be any category. For $A,B \in \mathscr{A}$ define a **pam** from A to B to be an element of (B,A). Given a finite sequence

$$A_1, A_2, \cdots, A_n \in \mathscr{A}$$

define a **cword** from A to B to be a sequence of maps and pams running through $A_1, A_2, \cdots, A_n$, or, more precisely, an element in the set $(A,A_1) \times (A_2,A_1) \times (A_2,A_3) \times \cdots \times (A_n,B)$. The composition of two cwords, one from A to B, the other from B to C, is defined to be their concatenation.

A map from A to B induces a function from (X,A) to (X,B) for every X, and a pam from A to B induces a correspondence from (X,A) to (X,B) (that is, a set of ordered pairs in $(X,A) \times (X,B)$). A cword from A to B likewise induces a correspondence from (X,A) to (X,B). Dually it induces a correspondence from (B,Y) to (A,Y) for every Y. We define two cwords from A to B to be equivalent if they always induce the same correspondences from (X,A) to (X,B) and from (B,Y) to (A,Y). An equivalence class of cwords from A to B will be called a **correspondence** in $\mathscr{A}$. If a correspondence in $\mathscr{A}$ is such that all the induced correspondences are functions then it will be called a **function** in $\mathscr{A}$.

In the classical construction of the connecting homomorphism a cword was defined and then shown to represent a function.

In a category of R-modules every function is represented by a map. If $\mathscr{A}$ is fully abelian then every function in $\mathscr{A}$ is represented (obviously uniquely) by a map in $\mathscr{A}$. More generally, every correspondence from A to B may be represented by a map from a subobject of A to a quotient object of B.

D. A specialized embedding theorem

The proof of Theorem 4.44 proved a stronger statement than that of the theorem: If $\mathscr{A}$ is a small full exact subcategory of a complete abelian category $\mathscr{B}$ with a projective generator, then $\mathscr{A}$ is isomorphic to a full exact subcategory of *cyclic* modules over some ring R. We may go a step further. Assume, $\mathscr{B}$ is a category of modules and replace the projective generator P in the proof by $\Sigma_K P$, where K is an infinite indexing set at least as large as P. Then the ring R is such that for every $A \in \mathscr{A}$ there is an exact sequence $R \to R \to A \to O$. By iteration we may finally obtain a ring R big enough so that for every $A \in \mathscr{A}$ there is an infinite exact sequence $\cdots \to R \to R \to R \to A \to O$.

But instead of making the ring larger we may make it smaller. There is a ring R such that R and $\mathscr{A}$ have the same cardinality and such that $\mathscr{A}$ is isomorphic to a full exact subcategory of cyclic modules over R. To obtain such, assume that $\mathscr{A}$ is a full exact subcategory of cyclic modules over a ring S. Let $\mathbf{F}$ be a minimal family of ideals such that for every $A \in \mathscr{A}$ there is $\mathfrak{A} \in \mathbf{F}$ and an exact sequence $O \to \mathfrak{A} \to S \to A \to O$. Let T be a subset of S such that for every $\mathfrak{A},\mathfrak{L} \in \mathbf{F}$ and $s \in S$ with $\mathfrak{A}s \subset \mathfrak{L}$ there exists $t \in T$ with $s - t \in \mathfrak{L}$. The cardinality of T need be no larger than that of $\mathscr{A}$.

For any ring R, $T \subset R \subset S$, $\mathscr{A}$ is isomorphic to a full subcategory of cyclic modules over R. $(S/\mathfrak{A} \to R/R \cap \mathfrak{A})$, but not necessarily an exact subcategory. However, if R has the further property that for every $t, t' \in T$, $\mathfrak{A} \in \mathbf{F}$, $s \in S$ such that $st - t' \in \mathfrak{A}$ there is $r \in R$ such that $rt - t' \in \mathfrak{A}$, then $\mathscr{A}$ is isomorphic to a full *exact* subcategory of cyclic modules over R.

Using the Lowenheim-Skolem theorem from the theory of models it suffices for metatheoretic purposes to test any theorem on just countable abelian categories. Joining that fact with the observation that an onto ring homomorphism $V \to R$ induces an exact full embedding $\mathscr{G}^R \to \mathscr{G}^V$ and assuming the final theorem of the book, 7.34, we may improve Theorem 4.31 to:

A full compound diagrammatic statement is true for all abelian categories if and only if it is true for the category of countable modules over the ring freely generated by a countable set of (noncommuting) indeterminants.

E. Small projectives

Let $\mathscr{A}$ be a right-complete abelian category. A projective object $P \in \mathscr{A}$ is a **small projective** if the functor $(P,-)\colon \mathscr{A} \to \mathscr{G}$ preserves all roots, or equivalently if it preserves sums.

(1) A projective object is a small projective iff for every map $P \to \Sigma_I A_i$ there is a finite $J \subset I$ such that $P \to \Sigma_I A_i = P \to \Sigma_J A_j \to \Sigma_I A_i$.

(2) Every ascending chain of proper subobjects in a small projective is bounded by a proper subobject and every family of proper subobjects closed under finite union is bounded by a proper subobject. (Let $\{P_i \to P\}_I$ be an ascending family of subobjects which is not bounded by a proper subobject. It follows that $\Sigma_I P_i \to P$ is epimorphic. Now use the fact that P is projective.)

(3) If the category $\mathscr{A}$ is such that for $x\colon P \to A$ and ascending family of subobjects $\{A_i \to A\}_I$ it is the case that $\cup\, x^{-1}(A_i) = x^{-1}(\cup A_i)$ then the property of small projectives in (2) characterizes them. (Given $P \to \Sigma_I A_i$ consider the inverse image of $\Sigma_J A_j$ for all finite $J \subset I$.)

(4) A projective module is small iff it is finitely generated.

F. Categories representable as categories of modules

Let $\mathscr{A}$ be a right-complete abelian category with a small projective generator P. Let R be the ring of endomorphisms of P and define $F\colon \mathscr{A} \to \mathscr{G}^R$ as in 4.44. $F(A)$ is the left R-module (P,A). Then F is an exact embedding which preserves all roots. Its image contains R and all free modules. Moreover, any map between free modules comes from a map in $\mathscr{A}$. Since the image of F is closed on the right we may conclude that it is a full representative subcategory. By Exercise 3-A, F is an equivalence of categories.

A category is equivalent to a category of modules iff it is a right-complete abelian category with a small projective generator.

G. Compact abelian groups

Let $\mathscr{C}$ be the category of compact abelian groups, advertised in Exercise 2-C as being an abelian category. Let $C \in \mathscr{C}$ be the "circle group," defined as the multiplicative group of complex numbers of modulus one, or additively, as the group of reals modulo the subgroup of integers. We shall treat C as an additive group. The only proper closed subgroup of C are finite and cyclic. The only automorphisms of C are rigid (the metric structure of C may be defined via the group structure and topology and a continuous automorphism must be an isometry). The only rigid automorphisms on C are the identity and the map which results by multiplying by -1. The last three sentences combine to prove that the only endomorphisms of C are those which result by multiplying by integers. That is, the ring of endomorphisms of C is the ring of integers.

C may be proven to be a cogenerator for $\mathscr{C}$. The most efficient proof is beyond the scope of this book. It involves among other things the fact that the space of complex numbers is a cogenerator for the category of Banach algebras. But granted that C is a cogenerator we may prove the Pontrjagin duality theorem:

First, C is injective in $\mathscr{C}$. Indeed, any cogenerator for any abelian category whose ring of endomorphisms is a principal ideal domain is an injective cogenerator. (Given a monomorphism $C \to A$ let $I \subset (C,C)$ be the set of maps of the form $C \to A \xrightarrow{x} C$. I is an ideal and if it is generated by $C \xrightarrow{n} C$ then every map in I kills $Ker(n)$. Now using the fact that C is a cogenerator we conclude that $Ker(n) = O$ and that I is generated by the identity.)

For any $x: A \to B \in \mathscr{C}$ and descending family of subobjects $\{A_i \to A\}_I$ it is the case that $x(\cap A_i) = \cap x(A_i)$. Hence C^* is a small projective generator for $\mathscr{C}^*$ (Exercise E). The Tychonoff theorem implies that $\mathscr{C}$ is a left-complete category and hence that $\mathscr{C}^*$ is right-complete. By the last exercise $\mathscr{C}^*$ is equivalent to $\mathscr{G}$. More particularly $(-,C): \mathscr{C} \to \mathscr{G}$ is a contravariant equivalence. An inverse of $(-,C)$ may be described as the symbolic hom functor $\overline{(-,C)}: \mathscr{G} \to \mathscr{C}$ and computed to be such that $\overline{(G,C)}$ is the space of homomorphisms from G to C topologized by pointwise convergence.

H. Fully is more than very

1. The fact that not every small abelian category enjoys a full embedding into $\mathscr{G}$ is easily established, thus,

(1) If G is an abelian group whose ring of endomorphisms is a field of characteristic zero then G is isomorphic to the group of rational numbers.

(2) Let F be a field of characteristic zero, not isomorphic to the field of rational numbers, and let $\mathscr{A}$ be the category of finite-dimensional vector spaces over F. Then $\mathscr{A}$ does not enjoy a full embedding into $\mathscr{G}$.

2. The statement of the full metatheorem cannot be simplified by replacing the arbitrary ring R with the ring of integers. For,

(1) If $O \to A \to B$ is an exact sequence in $\mathscr{G}$ and $B \xrightarrow{2} B = 0$, then the map $A \to B$ splits, i.e., there is a map $B \to A$ such that $A \to B \to A = 1$.

(2) Let Z_2 be the ring of integers modulo two and let R be the ring $\{(a,b) \mid a,b \in Z_2\}$ whose multiplication is defined by $(a,b)(a',b') = (aa',ab' + a'b)$. ($R$ is isomorphic to $Z_2[X]/(X^2)$ and $Z[X]/(2,X^2)$.) Let $A = \{(0,a) \mid a \in Z_2\} \subset R$. The inclusion map $A \to R$ does not split in the category of R-modules.

I. Unembeddable categories

Not every category may be embedded in the category of sets. What seems to be the simplest counterexample may be described as follows:

For objects let there be for each ordinal number α an object named A_α; let there be a zero object O; and let there be a special object S.

Let there be maps named $A_\alpha \xrightarrow{x^\alpha_\beta} S$, $S \xrightarrow{y^\alpha_\beta} A_\alpha$, and $A_\alpha \xrightarrow{z^\alpha_\beta} A_\alpha$ for every pair of ordinal numbers $\beta < \alpha$, and let there be a zero map between any two objects, and let there be an identity map for every object.

For the composition of maps let $A_\alpha \xrightarrow{x^\alpha_\beta} S \xrightarrow{y^\alpha_{\beta'}} A = \xrightarrow{z^\alpha_{\beta''}} A$, where $\beta'' = max(\beta,\beta')$. Let all other compositions of nonidentity maps be zero maps (which makes the verification of associativity downright trivial), and finally, let the composition of maps with identity maps be what it must.

Calling the above-described category $\mathscr{A}$, suppose that $F\colon \mathscr{A} \to \mathscr{S}$ is an embedding into the category of sets. Let α be an ordinal number of cardinality greater than that of the family of subsets of $F(S)$. There must exist $\beta < \beta' < \alpha$ such that $Im(F(x^\alpha_\beta)) = Im(F(x^\alpha_{\beta'}))$. On the other hand the image of $F(x^\alpha_\beta)$ is not in the difference kernel of $F(y^\alpha_\beta)$ and $F(y^\alpha_{\beta'})$, whereas the image of $F(x^\alpha_{\beta'})$ is. A contradiction.

(Every category may be embedded in an abelian category (using techniques not to be covered in this book) and the above counterexample leads to an example of an abelian category which cannot be embedded, exactly or not, in the category of abelian groups. The presence of a projective generator or an injective cogenerator, of course, implies the existence of an exact embedding. The only embedding theorem for large abelian categories that we know of besides the just named triviality is, that if an abelian category, small or not, has both a generator and a cogenerator, then it has a group-valued exact embedding. The proof is, in light of the special nature of the result, too long for inclusion.)

CHAPTER 5

FUNCTOR CATEGORIES

We began this book with the observation that to describe topology as the study of continuous maps is more to the point than to describe it as the study of the models of the axioms for a topological space. It has often been said that most of mathematics is concerned with functions rather than the things functions are defined on. The axioms for a category stand as an embodiment of such a viewpoint. But the same viewpoint leads one to study not categories but functors; and then not functors but natural transformations. And happily this returns us to categories.

5.1. ABELIANNESS

Let $\mathscr{A}$ be a small abelian category, and $\mathscr{G}$ the category of abelian groups. $(\mathscr{A},\mathscr{G})$ shall denote the category of additive functors from $\mathscr{A}$ to $\mathscr{G}$. The objects are functors, the maps are natural transformations.

Theorem 5.11
$(\mathscr{A},\mathscr{G})$ is an abelian category.

Proof:
We indicate the verification of half of the axioms:

Axiom 0. The constantly zero functor is a zero object.

Axiom 1. Given $F_1,F_2 \in (\mathscr{A},\mathscr{G})$ define $F_1 \oplus F_2$ to be a functor such that $(F_1 \oplus F_2)(A) = F_1(A) \oplus F_2(A)$ and

$$(F_1 \oplus F_2)(x) = \begin{pmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{pmatrix}.$$

Axiom 2. Let $F_1 \to F_2 \in (\mathscr{A},\mathscr{G})$. For each $A \in \mathscr{A}$ let $O \to K(A) \to F_1(A) \to F_2(A)$ be exact. Given $A \xrightarrow{x} B \in \mathscr{A}$ there is a unique map $K(x)\colon K(A) \to K(B)$ such that

$$\begin{array}{ccc} K(A) & \longrightarrow & F_1(A) \\ {\scriptstyle K(x)}\downarrow & & \downarrow{\scriptstyle F_1(x)} \\ K(B) & \longrightarrow & F_1(B) \end{array} \qquad \text{commutes.}$$

Then K is a functor and $K \to F_1$ is a natural transformation.

Axiom 3. The above construction shows that a transformation $F_1 \to F_2$ is a monomorphism in $(\mathscr{A},\mathscr{G})$ iff $F_1(A) \to F_2(A)$ is a monomorphism in $\mathscr{A}$ for each A. The dual construction needed for Axiom 2* indicates that if $F_1 \to F_2$ is a monomorphism then it is a kernel of its cokernel. ∎

The constructions above indicate that a sequence $F' \to F \to F''$ is exact in $(\mathscr{A},\mathscr{G})$ iff the sequences $F'(A) \to F(A) \to F''(A)$ are exact in $\mathscr{A}$ for all $A \in \mathscr{A}$. More formally the **evaluation functor** $E_A\colon (\mathscr{A},\mathscr{G}) \to \mathscr{G}$ defined by $E_A(F_1 \xrightarrow{\eta} F_2) = F_1(A) \xrightarrow{\eta(A)} F_2(A)$ is an exact functor for each $A \in \mathscr{A}$. The product

$$(\Pi_{\mathscr{A}}E_A)\colon (\mathscr{A},\mathscr{G}) \to \mathscr{G}$$

defined by $(\Pi_{\mathscr{A}}E_A)(F) = \Pi_{\mathscr{A}}E_A(F) = \Pi_{\mathscr{A}}F(A)$ is an exact embedding.

Proposition 5.12

$(\mathscr{A},\mathscr{G})$ is a complete abelian category.

Proof:

Let $\{F_i\}_I$ be an indexed family of functors in $(\mathscr{A},\mathscr{G})$. $\Pi_I F_i$ and $\Sigma_I F_i$ are constructed "pointwise" (just as were finite direct sums):

$$(\Pi_I F_i)(A) = \Pi_I F_i(A)$$
$$(\Sigma_I F_i)(A) = \Sigma_I F_i(A). \quad \blacksquare$$

5.2. GROTHENDIECK CATEGORIES

$\mathscr{G}$ and $(\mathscr{A},\mathscr{G})$ enjoy a critical property with respect to certain infinite operations. Note that if G is an abelian group and $\{G_i\}_I$ is a *linearly ordered* family of subgroups, and H is any subgroup of G, then $H \cap \bigcup G_i = \bigcup (H \cap G_i)$. A complete well-powered category in which this same statement is always true for the lattice of subobjects of any object is called a **Grothendieck** category (the property is elsewhere referred to as **AB5**). Just one of the many consequences of the Grothendieck property is explored in the next chapter. Among the many properties equivalent to the Grothendieck property is the following: for all $x: A \to B$ and ascending families $\{B_i \to B\}_I$ it is the case that $x^{-1}(\cup B_i) = \cup x^{-1}(B_i)$. For any category such is the case (purely lattice theoretically) for epimorphic x. In the case that x is a monomorphism the two properties are immediately equivalent.

Proposition 5.21

$(\mathscr{A},\mathscr{G})$ is a Grothendieck category.

Proof:

We simply observe that given a collection $\{F_i\}_I$ of subfunctors, their union and intersection may be constructed "pointwise": $(\bigcup F_i)(A) = \bigcup (F_i(A)) \subset F(A)$. Hence given a linearly ordered family $\{F_i\}$ and a subfunctor $H \subset F$, $(H \cap \bigcup F_i)(A) = H(A) \cap \bigcup F_i(A) = \bigcup [H(A) \cap F_i(A) = [(\bigcup (H \cap F_i))](A)$. $\blacksquare$

5.3. THE REPRESENTATION FUNCTOR

We define the **representation functor** as the contravariant functor $\mathscr{A} \xrightarrow{H} (\mathscr{A},\mathscr{G})$ such that $H(A) = (A,-) \in (\mathscr{A},\mathscr{G})$, $H(A \xrightarrow{x} B) = (B,-) \xrightarrow{(x,-)} (A,-)$. When $(A,-)$ is being considered as an *object* in $(\mathscr{A},\mathscr{G})$ we shall denote it by H^A. Given $A \xrightarrow{x} B \in \mathscr{A}$ it is convenient to denote the corresponding transformation by $H^B \xrightarrow{H^x} H^A$.

***Proposition* 5.31**

$\mathscr{A} \xrightarrow{H} (\mathscr{A},\mathscr{G})$ carries right-exact sequences into left-exact sequences. ∎

Given $A \in \mathscr{A}$, $F \in (\mathscr{A},\mathscr{G})$ we consider the group of natural transformations (H^A,F). Let $\eta \in (H^A,F)$. By evaluating at A we obtain a group homomorphism $\eta_A \in (H^A(A),F(A))$. By evaluating at $1_A \in (A,A) = H^A(A)$ we obtain an element $\eta_A(1_A) \in F(A)$. We define the **Yoneda** function $y\colon (H^A,F) \to F(A)$ by $y(\eta) = \eta_A(1_A)$. It is clear that y is a group homomorphism. Moreover, it is a natural transformation: a statement which needs clarification.

We define two group-valued functors D,E each on two variables, one variable from $\mathscr{A}$, the other from $(\mathscr{A},\mathscr{G})$. D is defined to be the composition

$$\mathscr{A} \times (\mathscr{A},\mathscr{G}) \xrightarrow{(H\times I)} (\mathscr{A},\mathscr{G}) \times (\mathscr{A},\mathscr{G}) \xrightarrow{Hom} \mathscr{G}.$$

Hence $D(A,F) = (H^A,F) \in \mathscr{G}$.

$E\colon \mathscr{A} \times (\mathscr{A},\mathscr{G}) \to \mathscr{G}$, the "evaluating functor," is defined by

$$E(A,F) = F(A)$$

$$E(A,F_1 \xrightarrow{\eta} F_2) = F_1(A) \xrightarrow{\eta_A} F_2(A)$$

$$E(A_1 \xrightarrow{x} A_2,F) = F(A_1) \xrightarrow{F(x)} F(A_2).$$

(Prop. 3.61 on the recognition of functors on two variables is useful here. Condition three of that proposition is here equivalent to the defining condition for natural transformations.)

Theorem 5.32

The Yoneda functions $y\colon (H^A,F) \to F(A)$, $y(\eta) = \eta_A(1_A)$, *provide a natural transformation from D to E.*

Proof:

By proposition 3.62 it suffices to show that

(1) for $F_1 \xrightarrow{\alpha} F_2 \in (\mathscr{A},\mathscr{G})$,

$$\begin{array}{ccc} (H^A,F_1) & \xrightarrow{(H^A,\alpha)} & (H^A,F_2) \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle y} \\ F_1(A) & \xrightarrow{\alpha_A} & (F_2A) \end{array}$$

commutes,

and

(2) for $A_1 \xrightarrow{x} A_2$,

$$\begin{array}{ccc} (H^{A_1},F) & \xrightarrow{(H^x,F)} & (H^{A_2},F) \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle y} \\ F(A_1) & \xrightarrow{F(x)} & F(A_2) \end{array}$$

commutes.

(1) is easy: starting with $\eta \in (H^A,F_1)$ and traveling clockwise we obtain $\eta \to \alpha\eta \to (\alpha\eta)_A(1_A)$; traveling counterclockwise, $\eta \to \eta_A(1_A) \to (\alpha_A\eta_A(1_A))$. But, of course, $(\alpha\eta)_A$ is the composition of α_A and η_A and we obtain the same element in $F_2(A)$ regardless of direction of travel.

For condition (2) we start with $\alpha \in (H^{A_1},F)$, and traveling clockwise we obtain

$$\alpha \to \alpha H^x \to (\alpha H^x)_{A_2}(1_{A_2}) = \alpha_{A_2}(x,A_2)(1_{A_2}) = \alpha_{A_2}(x).$$

Traveling counterclockwise we obtain

$$\alpha \to \alpha_{A_1}(1_{A_1}) \to F(x)[\alpha_{A_1}(1_{A_1})].$$

To see that $\alpha_{A_2}(x) = F(x)[\alpha_{A_1}(1_{A_1})]$ we use the fact that α is a natural transformation and that the diagram

$$\begin{array}{ccc} (A_1,A_1) & \xrightarrow{(A_1,x)} & (A_1,A_2) \\ {\scriptstyle \alpha(A_1)}\downarrow & & \downarrow{\scriptstyle \alpha(A_2)} \\ F(A_1) & \longrightarrow & F(A_2) \end{array} \qquad \text{commutes.}$$

Starting with $1_{A_1} \in (A_1,A_1)$ and traveling clockwise:

$$1_{A_1} \to x \to \alpha_{A_2}(x);$$

traveling counterclockwise,

$$1_{A_1} \to \alpha_{A_1}(1_{A_1}) \to F(x)(\alpha_{A_1}(1_{A_1})). \quad \blacksquare$$

Theorem 5.34

The Yoneda transformation y: $D \to E$ is a natural equivalence. ((H^A,F) is naturally equivalent to $F(A)$.)

Proof:

First, y is one-to-one. Let $\alpha \in (H^A,F)$ and $0 = y(\alpha) = \alpha_A(1_A)$. We must show that α is the zero transformation. Let $A_2 \in \mathscr{A}$ and $x \in (A,A_2) = H^A(A_2)$. In the last step in the last proof it was shown that $\alpha_{A_2}(x) = F(x)(\alpha_A(1_A))$. Hence if $y(\alpha) = \alpha_A(1_A) = 0$ then $\alpha_{A_2}(x) = 0$ and $\alpha = 0$.

To show that y is onto, we let $z \in F(A)$. For each $B \in \mathscr{A}$ we define the function α_B: $(A,B) \to F(B)$ by $\alpha_B(x) = (F(x))(z)$ for $x \in (A,B)$. The additivity of F implies that α_B is a group homomorphism. If the collection of α_B's produces a natural transformation α it is clear that y is onto since $y(\alpha) = z$.

To prove that α is natural we must show that for any $B_1 \xrightarrow{w} B_2$ the diagram

$$\begin{array}{ccc} (A,B_1) & \xrightarrow{(A,w)} & (A,B_2) \\ {\scriptstyle \alpha_{B_1}}\downarrow & & \downarrow{\scriptstyle \alpha_{B_2}} \\ F(B_1) & \xrightarrow{F(w)} & F(B_2) \end{array} \qquad \text{commutes.}$$

Starting with $x \in (A,B_1)$ and traveling clockwise,

$$x \to wx \to \alpha_{B_2}(wx) = [F(wx)](z);$$

counterclockwise,

$$x \to \alpha_{B_1}(x) \to [F(w)](\alpha_{B_1}(x)) = F(w)[F(x)(z)].$$

Since F is a functor, $F(wx) = F(w)F(x)$ and α is natural. ∎

***Theorem* 5.35**
$\Sigma_{\mathscr{A}} H^A$ is a projective generator for $(\mathscr{A},\mathscr{G})$.

Proof:
$(\Sigma H^A,-)(\mathscr{A},\mathscr{G}) \to \mathscr{G}$ is naturally equivalent to

$$(\Pi E_A)\colon (\mathscr{A},\mathscr{G}) \to \mathscr{G}. \quad ∎$$

***Theorem* 5.36**
The representation functor $\mathscr{A} \xrightarrow{H} (\mathscr{A},\mathscr{G})$ is a contravariant full embedding.

Proof:
$(H^A,H^B) = (B,A)$. ∎

EXERCISES

A. Duals of functor categories

Let $\mathscr{A}$ be a small category, $\mathscr{B}$ any category, $\mathscr{A}^*$ and $\mathscr{B}^*$ their duals.

Both $(\mathscr{A}^*,\mathscr{B})$ and $(\mathscr{A},\mathscr{B}^*)$ may be interpreted as the category of contravariant functors from $\mathscr{A}$ to $\mathscr{B}$. However, $(\mathscr{A}^*,\mathscr{B})$ and $(\mathscr{A},\mathscr{B}^*)$ are dual.

$(\mathscr{A},\mathscr{B})$ is dual to $(\mathscr{A}^*,\mathscr{B}^*)$.

B. Co-Grothendieck categories

1. If the dual of an abelian category $\mathscr{A}$ is a Grothendieck category, then the lattice of subobjects of each object $A \in \mathscr{A}$ has the property:

$$\text{if } \{A_i\} \text{ is a descending family then}$$
$$B \cup \bigcap A_i = \bigcap (B \cup A_i).$$

2. The category of abelian groups is not the dual of a Grothendieck category.

3. If the abelian category $\mathscr{A}$ and its dual both were Grothendieck categories, then for every $A \in \mathscr{A}$ the natural map $\sum_{i=1}^{\infty} A \to \prod_{i=1}^{\infty} A$ is an isomorphism and $A = O$. (Let $x = 1_A + 1_A + 1_A + \cdots$. Then $x = 1_A + x$.)

C. Categories of modules

Let $\mathscr{A}$ be any monoidal category and $(\mathscr{A},\mathscr{G})$ the category of additive functors.

1. $(\mathscr{A},\mathscr{G})$ is abelian.

2. Consider a ring R as a monoidal category. $(R,\mathscr{G})$ is isomorphic to the category of R-modules.

3. If $\mathscr{C}$, the category of compact abelian groups, has been identified as the dual of the category of groups, then the dual of the category of left R-modules may be identified as the category of compact right R-modules.

D. Projectives and injectives in functor categories

The functor $\Sigma_{\mathscr{A}} E_A : (\mathscr{A},\mathscr{G}) \to \mathscr{G}$ preserves all right roots and if followed by $(-,Q/Z) : \mathscr{G} \to \mathscr{G}$ results in a contravariant exact embedding which carries right roots into left roots. (Exercise 3-G.) It must be representable, and therefore $(\mathscr{A},\mathscr{G})$ has an injective cogenerator.

More generally: If $\mathscr{B}$ has a projective generator then so does $(\mathscr{A},\mathscr{B})$. Each evaluation functor $E_A : (\mathscr{A},\mathscr{B}) \to \mathscr{B}$ preserves all roots. That it satisfies the further condition of Exercise 3-J for functors with left-adjoints may be directly verified. Letting $E_A^* : \mathscr{B} \to (\mathscr{A},\mathscr{B})$ be the left-adjoint of E_A, and P a (projective) generator for $\mathscr{B}$, it follows that $\Sigma_{\mathscr{A}} E_A^*(P)$ is a (projective) generator for $(\mathscr{A},\mathscr{B})$.

For arbitrary $B \in \mathscr{B}$, the functor $E_A^*(B)$ may be identified as the functor from $\mathscr{A}$ to $\mathscr{B}$ which sends A' into $(A,A') \otimes B$, where $\otimes$ refers to the functor defined in Exercise 3-K. The right-adjoint of $E_A: (\mathscr{A},\mathscr{B}) \to \mathscr{B}$, evaluated at $B \in \mathscr{B}$, is the functor which sends A' into $\overline{((A',A),B)}$.

E. Grothendieck categories

Let $\mathscr{B}$ be a Grothendieck category, $\mathscr{D}$ a directed category (see Exercise 3-B), $F,G: \mathscr{D} \to \mathscr{B}$ two functors, and $F \to G$ a monomorphic transformation. The induced map $\varinjlim F \to \varinjlim G$ is a monomorphism. ("The direct limit of monomorphisms is a monomorphism.") If such is always the case in a complete abelian category then the category is a Grothendieck category.

Let A be an object in a Grothendieck category, $\{A_i\}$ an ascending family of subobjects of A the union of which is all of A. Then A may be identified as the direct limit of the system $\{A_i\}$. The statement remains true for Grothendieck categories if we require only that $\{A_i\}$ be *directed* (i.e., that every pair of subobjects in $\{A_i\}$ have an upper bound in $\{A_i\}$), and becomes another characterization of Grothendieck categories among complete categories.

F. Left-completeness almost implies completeness

Let $\mathscr{A}$ be any category, and $\mathscr{D}$ any small category. Define $\mathscr{C}$ to be the full subcategory of constant functors in the category of all functors $(\mathscr{D},\mathscr{A})$. Given $F \in (\mathscr{D},\mathscr{A})$, F has a reflection in $\mathscr{C}$ iff F has a left root, and, in fact, the two are the same. On the other side, F has a coreflection in $\mathscr{C}$ iff F has a right root, and, again, the two are equal.

Suppose that $\mathscr{A}$ is a left-complete, well-powered category with a cogenerator and a "right zero object" $O_R \in \mathscr{A}$ with the property that for all $A \in \mathscr{A}$, (A,O_R) has precisely one element. Then the same is true for $\mathscr{C}$ (they are isomorphic categories), and the inclusion functor $\mathscr{C} \to (\mathscr{D},\mathscr{A})$ is left-root-preserving. By Exercise 3-M, therefore, $\mathscr{C}$ is reflective, and since this is true for any small $\mathscr{D}$, we conclude that $\mathscr{A}$ is *right-complete.*

Suppose that $\mathscr{A}$ does not have a cogenerator but that it is left-complete, well-powered, and co-well-powered. The right-completeness of $\mathscr{A}$ is implied by the existence of difference cokernels and sums. By the left-completeness of $\mathscr{A}$ every map has an image (the partially ordered family of subobjects of any object is a complete lattice; the image of $A \xrightarrow{x} B$ is the least subobject which allows x). Because $\mathscr{A}$ has difference kernels one may prove that if the image of x is all of B then x is an epimorphism.

Given an object A and a family of quotient objects $\{A \to A''_i\}$, let A'' be the image of $A \to \Pi A''_i$. Then $A \to A''$ represents the least upper bound of all the quotient objects $\{A \to A''_i\}$. Hence, the family of quotient objects of A is a complete lattice.

Necessary and sufficient conditions for the existence of sums are best expressed by expanding the language of Exercise 3-J as follows: Given a family $\mathscr{F} = \{A_i \xrightarrow{x_i} B\}$ we shall say that a subobject $B' \to B$ **allows** $\mathscr{F}$ if it allows every $x_i \in \mathscr{F}$. We shall say that $\mathscr{F}$ generates B is no proper subset of B allows $\mathscr{F}$. We shall say that $\{A_i\}$ generates B if there exists a family $\{A_i \xrightarrow{x_i} B\}$ which generates B. Finally, then, *if $\mathscr{A}$ is a left-complete, well-powered and co-well-powered category with a right zero object then it is right-complete iff every set of objects generates at most a set of nonisomorphic objects.*

If $\mathscr{A}$ does not have a right zero object we may easily adjoin one. In that case, the right root of $T\colon \mathscr{D} \to \mathscr{A}$ is the right zero object iff T has no transformations into any constant functor into the original category. The ideal right zero object plays a role analogous to $+\infty$ for the real numbers and indeed $+\infty$ is a right zero object in the category that is associated with the ordering type of the real numbers.

If we were to relax our definition of completeness in categories in the analogous way (sets of real numbers with any upper bound have a least upper bound) then we could leave out the ideal zero objects. In particular, we could prove that categories of models [Exercise 3-O] are left-complete iff they are right-complete, where the notion of completeness is understood to be the relaxed notion.

Let $\mathscr{A}$ be a small abelian category and define $\mathscr{L}(\mathscr{A})$ to be the full subcategory of left-exact functors in the category of all additive functors $(\mathscr{A},\mathscr{G})$. In the next chapter $\mathscr{L}(\mathscr{A})$ will be shown to be a reflective category (but not via the adjoint functor theorem). Let $\mathscr{R}(\mathscr{A})$ be the full subcategory of right-exact functors. The only proof that we know of that $\mathscr{R}(\mathscr{A})$ is a coreflective subcategory (or, in classical language, that 0th left-derived functors always exist), is via the special adjoint functor theorem and the statement that the set $\{T \in \mathscr{R}(\mathscr{A}) \mid$ the cardinality of $\cup_{\mathscr{A}} T(A)$ is less than that of $\mathscr{A}\}$, is a generating set for $\mathscr{R}(\mathscr{A})$.

The result may be generalized as follows: Instead of specifying right-exactness, consider any class of functors into $\mathscr{A}$, and then consider the full subcategory of all those functors which preserve their right roots. It is coreflective.

On the other side, the full subcategory of functors which preserve the left roots of some specified class is reflective. These two results do not have a common proof, and both depend on the special nature of the range category $\mathscr{G}$. (It does not depend on the abelianness of $\mathscr{A}$, or for that matter on anything about $\mathscr{A}$ save its smallness, and $\mathscr{G}$ may be replaced with the category of sets.)

G. Small projectives in functor categories

Let $\mathscr{A}$ be a small additive category, and $(\mathscr{A},\mathscr{G})$ the category of additive functors from $\mathscr{A}$ to $\mathscr{G}$. By the Yoneda theorem H^A is a small projective in $(\mathscr{A},\mathscr{G})$, and the family of all such small projectives generates $(\mathscr{A},\mathscr{G})$.

Suppose that $\mathscr{A}$ not only is additive but also has finite direct sums and that idempotents split in $\mathscr{A}$ (see Exercise 2-B). Such a category is called **amenable**. Let P be a small projective in $(\mathscr{A},\mathscr{G})$. Then P is isomorphic to H^A for some $A \in \mathscr{A}$. To prove it, first find $\{A_i\}_I$ and an epimorphism $\Sigma_I H^{A_i} \to P$ (the H^A's generate $(\mathscr{A},\mathscr{G})$); second, let $P \to \Sigma H^{A_i} \to P = 1$ (P is projective); third, let $J \subset I$ be a finite subset such that $P \to \Sigma_J H^{A_i} \to \Sigma_I H^{A_i} = P \to \Sigma_I H^{A_i}$ (P is small); fourth, let $A = \oplus_J A_i$ ($\mathscr{A}$ has finite direct sums) and simplify to the maps $P \to H^A \to P = 1$; fifth, find $x \in (A,A) \subset \mathscr{A}$ such that $H^A \to P \to H^A = H^x$ ($\mathscr{A}$ is additive) and observe that $x^2 = x$; sixth, let $A \to B$ and $B \to A$ be such that $A \to B \to A = x$ and $B \to A \to B = 1$ (idempotents split in $\mathscr{A}$); seventh, conclude from the factorization $H^x = H^A \to P \to H^A = H^A \to H^B \to H^A$ that P is isomorphic to H^B ($(\mathscr{A},\mathscr{G})$ is abelian).

The moral is that any property of $F\colon \mathscr{A} \to \mathscr{G}$ which may be stated in terms of its behavior as a functor may be stated in terms of its behavior as an object in $(\mathscr{A},\mathscr{G})$.

H. Categories representable as functor categories

Let $\mathscr{B}$ be a right-complete abelian category with a generating set of small projectives $\mathscr{P}$. That is, for any $A \to B \neq 0$ there exists a small projective $P \in \mathscr{P}$ and a map $P \to A$ such that $P \to A \to B \neq 0$.

Let $\mathscr{A}$ be the full subcategory of $\mathscr{B}$ generated by $\mathscr{P}$ and let $(\mathscr{A}^*,\mathscr{G})$ be the category of contravariant additive functors from $\mathscr{A}$ to $\mathscr{G}$. Define $F: \mathscr{B} \to (\mathscr{A}^*,\mathscr{G})$ to be the covariant functor which sends B into the contravariant functor $(-,B) \mid \mathscr{A}$. Regardless of the special nature of $\mathscr{A}$, F preserves left roots. The fact that the objects of $\mathscr{A}$ are small projectives in $\mathscr{B}$ implies that F preserves right roots. And the fact that the objects of $\mathscr{A}$ generate $\mathscr{B}$ implies that F is an embedding.

As in Exercise 4-F it may now be shown that F is an equivalence of categories. *A category is equivalent to a category of group-valued functors iff it is a right-complete abelian category with a generating set of small projectives.*

I. Tensor products of additive functors

Let $\mathscr{A}$ be a small additive category, $\mathscr{B}$ any additive category and $(\mathscr{A}^*,\mathscr{G})$ the category of contravariant group-valued additive functors from $\mathscr{A}$. Given any covariant $F: \mathscr{A} \to \mathscr{B}$ define $\boldsymbol{F}: \mathscr{B} \to (\mathscr{A}^*,\mathscr{G})$ to be such that B is sent into the contravariant functor $(F(-),B) \in (\mathscr{A}^*,\mathscr{G})$. We obtain a diagram

$$\begin{tikzcd}
 & \mathscr{B} \arrow[d, "\boldsymbol{F}"] \\
\mathscr{A} \arrow[ur, "F"] \arrow[r, "H"'] & (\mathscr{A}^*,\mathscr{G})
\end{tikzcd}$$

where $H: \mathscr{A} \to (\mathscr{A}^*,\mathscr{G})$ is the covariant functor which sends A into the contravariant functor $(-,A)$. (If $\mathscr{B} = \mathscr{A}$ and F is the identity then $H = \boldsymbol{F}$.)

If $\mathscr{B}$ is left-complete and well-powered and has a cogenerator, then $\boldsymbol{F}$ has a left-adjoint $\boldsymbol{F}^*: (\mathscr{A}^*,\mathscr{G}) \to \mathscr{B}$. Somewhat surprisingly it suffices to assume that $\mathscr{B}$ is *right*-complete, well-powered, and co-well-powered. (This is a *weaker* assumption by Exercise 5-F.)

Define $\mathscr{B}' \subset \mathscr{B}$ to be the smallest full subcategory which contains the image of F and is closed under the formation of sums and quotient objects. $\mathscr{B}'$ is a coreflective subcategory and we define $R\colon \mathscr{B} \to \mathscr{B}'$ to be its coreflector. By the isomorphisms $(F(-),B) \to (F(-),R(B))$ we obtain a commutative diagram

$$\begin{tikzcd}
 & & \mathscr{B} \arrow[dl, "R"] \arrow[dd, "F"] \\
\mathscr{A} \arrow[urr, "F"] \arrow[r, "F"] \arrow[drr, "H"'] & \mathscr{B} \arrow[dr, "F"] & \\
 & & (\mathscr{A}^*,\mathscr{G}).
\end{tikzcd}$$

Because $\mathscr{B}$, is right-complete and co-well-powered and has a generator, namely $\Sigma_{\mathscr{A}}\, F(A)$, it is also left-complete. It is clear that if $F\colon \mathscr{B}' \to (\mathscr{A}^*,\mathscr{G})$ has a left-adjoint then so does $F\colon \mathscr{B} \to (\mathscr{A}^*,\mathscr{G})$. We thus reduce to the case that $\mathscr{B}$ is left-complete, well-powered, and co-well-powered.

Let $T \in (\mathscr{A}^*,\mathscr{G})$ and suppose that $B \in \mathscr{B}$ is generated by T through F, i.e., there is a transformation $\eta\colon T \to F(B) \in (\mathscr{A}^*,\mathscr{G})$ such that η generates B. It follows that we obtain an epimorphism

$$\Sigma_{\mathscr{A}}\Sigma_{T(A)}\, F(A) \xrightarrow{y} B$$

where y is such that for $x \in T(A)$ $F(A) \xrightarrow{u_x} \Sigma_{\mathscr{A}}\Sigma_{T(A)}\, F(A) \xrightarrow{y} B = \eta_{\mathscr{A}}(x)$ (the image of y allows η). Hence T generates B only if B is a quotient object of $\Sigma_{\mathscr{A}}\Sigma_{T(A)}\, F(A)$ and by Exercise 3-K F has a left-adjoint $F^*\colon (\mathscr{A}^*,\mathscr{G}) \to \mathscr{B}$. We obtain a commutative diagram

$$\begin{tikzcd}
 & (\mathscr{A}^*,\mathscr{G}) \arrow[dd, "F^*"] \\
\mathscr{A} \arrow[ur, "H"] \arrow[dr, "F"'] & \\
 & \mathscr{B}
\end{tikzcd} ,$$

that is, $F(H_A) = F(A)$. This fact together with the fact that F preserves right roots characterizes F up to isomorphism.

Given a transformation $\eta\colon F_1 \to F_2$ we easily obtain $\bar{\eta}\colon F_2 \to F_1$ and then by Exercise 3-H a transformation $\eta^*\colon F_1^* \to F_2^*$. Define for $T \in (\mathscr{A}^*,\mathscr{G})$, $F \in (\mathscr{A},\mathscr{B})$ $T \otimes F = F^*(T)$. We obtain a bifunctor

$(\mathscr{A}^*,\mathscr{G}) \times (\mathscr{A},\mathscr{B}) \to \mathscr{B}$ which preserves right roots on both variables separately. (This fact together with $H_A \otimes F = F(A)$ characterizes it.)

Define for $B \in \mathscr{B}$ $F \in (\mathscr{A}, \mathscr{B})$ $(\overline{F,B}) = (F(-),B) \in (\mathscr{A}^*,\mathscr{G})$. We obtain a bifunctor $(\mathscr{A},\mathscr{B}) \times \mathscr{B} \to (\mathscr{A}^*,\mathscr{G})$, contravariant on the first variable, covariant on the second. The adjointness yields isomorphisms $(T \otimes F,B) \to (T,(\overline{F,B}))$.

When $\mathscr{A}$ is the category consisting only of the group of integers we obtain the previously described tensor product and symbolic hom functors.

If we view these bifunctors as operations and replace $\mathscr{B}$ with $(\mathscr{C},\mathscr{B})$ we obtain a long list of associativity and commutativity statements which generalize the classical list on tensor products and the hom functors on modules.

CHAPTER 6

INJECTIVE ENVELOPES

We have shown that the category $(\mathscr{A},\mathscr{G})$ is a Grothendieck category with a generator. In this chapter we prove that such conditions insure the existence of injective envelopes. In the next chapter we shall return to $(\mathscr{A},\mathscr{G})$ and put the injectives to work.

All categories in this chapter are abelian.

6.1. EXTENSIONS

We recall that an object E in an abelian category $\mathscr{A}$ is injective if the contravariant functor $(-,E)$: $\mathscr{A} \to \mathscr{G}$ is exact.

Given an object $A \in \mathscr{A}$ we shall call a monomorphism $A \to B$ an **extension** of A, and sometimes B itself will be called an extension.

A **trivial extension** of an object is a monomorphism $A \to B$ which "splits," i.e., which is such that there is a map $B \to A$

such that $A \to B \to A = A \xrightarrow{1} A$. [Equivalently, $A \to B$ is a trivial extension if there is an object C such that $B = A \oplus C$ and $A \to B = A \xrightarrow{u_1} A \oplus C$. (See 2.68.)]

Proposition 6.12
An object E in $\mathscr{A}$ is injective iff it has only trivial extensions.

Proof:
$\rightarrow$ From the dual of 3.31.
$\leftarrow$ Let $A \to B$ be a monomorphism and $A \to E$ any map. Consider the pushout diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ E & \to & P . \end{array}$$

The pushout theorem, 2.54*, asserts that $E \to P$ is monomorphic; hence by hypothesis P is a trivial extension of E. Let $P \to E$ be such that $E \to P \to E = E \xrightarrow{1} E$ and define

$$B \to E = B \to P \to E.$$

Then

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & E & \end{tikzcd}$$

commutes. ∎

An **essential extension** is a monomorphism $A \to B$ such that for every nonzero monomorphism $B' \to B$, the intersections (of the images) of $A \to B$ and $B' \to B$ are nonzero.

Equivalently, $A \to B$ is essential if for every $B \to F$ such that $A \to B \to F$ is monomorphic it is the case that $B \to F$ is monomorphic.

Theorem 6.13

In a Grothendieck category an object is injective iff it has no proper essential extensions.

Proof:

$\rightarrow$ If E is injective, its only proper extensions are trivial and thus clearly not essential.

$\leftarrow$ Let E have no proper essential extensions and consider an extension $E \to B$. We wish to show that the extension is trivial.

Let $\mathscr{F}$ be the partially ordered family of subobjects of B which have zero intersections with (the image of) $E \to B$. The following lemma is provable directly from the definition of the Grothendieck property.

Lemma 6.131. *If* $\{B_i\}_I$ *is an ascending chain in* $\mathscr{F}$ *then* $\bigcup B_i$ *is in* $\mathscr{F}$.

By Zorn's lemma, then, $\mathscr{F}$ has a maximal element $B' \subset B$. The corresponding family $\mathscr{F}^*$ of quotient objects of B ($B \to F \in \mathscr{F}^*$ iff $E \to B \to F$ is monomorphic) likewise has a minimal element: $B \to B''$. Certainly then $E \to B \to B''$ is monomorphic. Moreover the minimal nature of B'' insures that $E \to B''$ is essential, since if $B'' \to F$ is such that $E \to B \to B'' \to F$ is monomorphic, then the coimage of $B \to B'' \to F$ yields an element in $\mathscr{F}^*$ not smaller than B'' and hence equal to B''.

By hypothesis E has no *proper* essential extensions: $E \to B \to B''$ is an isomorphism and $E \to B$ is a trivial extension. ▮

The next theorem is a classic characterization of injective modules. We have included it, not because it will be directly needed, but because its proof, suitably modified, will become the proof of the main theorem of this chapter.

Theorem 6.14

Let R be a ring. If a left R-module A has the property that for every left ideal $I \subset R$ it is the case that $(R,A) \to (I,A)$ is epimorphic, then A is injective in the category of left R-modules.

Proof:

By the last theorem it suffices to show that A has no proper essential extensions. Assume then that $A \subset B$ and that $x \in B$ and $x \notin A$. Let $R \xrightarrow{x} B$ be the map which sends 1 into x and let

$$\begin{array}{ccc} I & \rightarrow & R \\ \downarrow & & \downarrow \\ A & \rightarrow & B \end{array}$$

be a pullback diagram. Let $y \in A$ be such that $I \rightarrow R \xrightarrow{y} A = I \rightarrow A$. The element $x-y$ is not trivial and it generates a submodule of B which meets A only trivially. B is not essential. ∎

6.2. ENVELOPES

An **injective envelope** of A is an injective essential extension. It is, therefore, a maximal essential extension and a minimal injection extension. The latter follows easily since $A \rightarrow E$ is an essential extension of every proper subobject between (the image of) A and E and thus none could be injective.

The construction of injective envelopes for arbitrary objects in Grothendieck categories proceeds from the following propositions:

Lemma 6.21

An essential extension of an essential extension is essential. ∎

Lemma 6.22

Let $\{A \rightarrow E\}$ be an extension of A in a Grothendieck category, and $\{E_i\}$ an ascending chain of subobjects between (the image of) A and E. If E_i is an essential extension of A for each i, then $\bigcup E_i$ is an essential extension of A.

Proof:

Let S be an arbitrary nonzero subobject of $\bigcup E_i$. Then $S = S \cap \bigcup E_i = \bigcup(S \cap E_i)$ and $S \cap E_i \neq O$ for some i. Because E_i is an essential extension of A it follows that $A \cap S \neq O$. ▮

We show next that in a Grothendieck category every ascending chain of extensions may be embedded in a common extension. Lemma 6.22 says, then, that every ascending chain of essential extensions is bounded by an essential extension.

Theorem 6.23

Let $\mathscr{B}$ be a Grothendieck category, I an ordered set, and $\{E_i \to E_j\}_{i<j}$ a family of monomorphisms such that for $i < j < h$ $E_i \to E_j \to E_k = E_i \to E_k$. Then there is an object $E \in \mathscr{B}$ and a family of monomorphisms $\{E_i \to E\}$, such that for $i < j$,

$$E_i \to E_j \to E = E_i \to E.$$

Proof:

Let $S = \Sigma_I E_i$ and for each $i \in I$ let $E_i \xrightarrow{u_i} S$ be the associated map. For each $j \in I$ define $h_j \colon S \to S$ to be the unique map such that

$$E_i \to S \xrightarrow{h_j} S = \begin{cases} E_i \to E_j \xrightarrow{u_j} S & \text{if } i \leq j \\ E_i \xrightarrow{u_i} S & \text{if } j \leq i. \end{cases}$$

Let $S \xrightarrow{h} E$ be an epimorphism such that $Ker(h) = \bigcup Ker(h_j)$. Note that $\{Ker(h_j)\}$ is an ascending family since for $j \leq j'$

$$S \xrightarrow{h_j'} S = S \xrightarrow{h_j} S \xrightarrow{h_j'} S.$$

To conclude that $E_i \xrightarrow{u_i} S \xrightarrow{h} E$ is a monomorphism it suffices to establish that $Im(E_i \to S) \cap \bigcup(Ker(h_j)) = O$. By the

Grothendieck property, therefore, it suffices to establish that $Im(E_i \to S) \cap Ker(h_j) = O$ for all j, i.e., that $E_i \to S \xrightarrow{h_j} S$ is a monomorphism. But this last statement follows immediately from the definition of h_j. ∎

Let $\mathscr{B}$ be a Grothendieck category and using the axiom of choice let E: (objects of $\mathscr{B}$) $\to$ (monomorphism in $\mathscr{B}$) be such that $E(A) = (A \to B)$, where B is a proper essential extension of A, unless, of course, A is injective, in which case $B = A$. We define $E^\gamma(A)$ for all ordinal numbers γ by

$$E^{\gamma+1}(A) = E \to E^\gamma(A) \to E(E^\gamma(A)),$$

and for α, a limit ordinal, we let $E^\alpha(A)$ be a minimal essential extension for all $E^\gamma(A)$, $\gamma < \alpha$ as insured by the last theorem.

Then the sequence $\{E^\gamma(A)\}$ becomes stationary only when it reaches an injective envelope of A.

We need only show that $\{E^\gamma(A)\}$ becomes stationary and we will know that—

Theorem 6.25

If $\mathscr{B}$ is a Grothendieck category with a generator, then every object has an injective envelope.

The presence of the generator in $\mathscr{B}$ is necessary: without it the sequence $\{E^\gamma(A)\}$ might very well continue to grow through the entire sequence of ordinal numbers (see Exercise 6-A).

But in the presence of a generator G we show that any sequence of essential extensions becomes stationary at some ordinal number.

We shall indicate three proofs. The first two use results which have appeared only in the exercises.

First Proof, *in which it is assumed that $\mathscr{B}$ has a cogenerator C (which by Exercise* 5-D *is good for $(\mathscr{A},\mathscr{G})$):*

Let $A \to E$ be an essential extension. Letting G be a generator choose for every $x \in (G,A)$ a map $f(x) \in (E,C)$ such that $G \xrightarrow{x} A \to E \xrightarrow{f(x)} C \neq 0$. Then $A \to E \xrightarrow{y} \Pi_{(G,A)}C$ is a monomorphism $(E \xrightarrow{y} \Pi_{(G,A)}C \xrightarrow{p_x} C = f(x))$. Since $A \to E$ is essential it follows that y is a monomorphism. Hence every essential extension of A is isomorphic to a subobject of $\Pi_{(G,A)}C$. To finish things off let Ω be an ordinal number of greater cardinality than that of the family of subobjects of $\Pi_{(G,A)}C$. Then any sequence of essential proper extension must terminate before Ω.

Second Proof *(Mitchell), in which it is assumed that modules may be embedded in injectives (Exercise* 5-D)*:*

Let R be the ring of endomorphisms of the generator G and define the functor $F\colon \mathscr{B} \to \mathscr{G}^R$ to be that which sends B into the R-module (G,B). (The endomorphisms of G operate obviously on the group (G,B).)

Lemma. If $A \to E$ is an essential extension in $\mathscr{B}$ then $F(A) \to F(E)$ is an essential extension in $\mathscr{G}^R$.

Proof of lemma. Let $M \subset F(E)$ be a nontrivial submodule and $x \in M$ a nontrivial element. We shall construct a nontrivial element in $M \cap Im[F(A) \to F(E)]$. Remembering that $x \in (G,E)$ we let

$$\begin{array}{ccc} P & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle x} \\ A & \longrightarrow & E \end{array}$$

be a pullback diagram. Since $A \to E$ is essential, $P \neq O$ and there exists $G \to P$ such that $G \to P \to G \xrightarrow{x} E \neq 0$. $G \to P \to G \xrightarrow{x} E$ is an element of M (M is a submodule) and in the image of $F(A \to E)$.

The lemma implies the theorem by a cardinality argument similar to that in the first proof. Using the fact that $F(A)$ has an injective extension $F(A) \to Q$ it follows that there exists a map $F(E) \to Q$ such that $F(A) \to F(E) \to Q = F(A) \to Q$ and that $F(E)$ is isomorphic to a subobject of Q. If Ω is an ordinal of cardinality larger than that of the family of subobjects of Q then the fact that F is an embedding implies that any sequence of proper essential extensions of A must terminate before Ω.

If we were to have made this second proof independent of the exercises we would have had to include in the text the proof that $\mathscr{G}$ has an injective cogenerator, then the proof that modules have injective extensions, and then this proof of a theorem which has those two results as special cases.

Third Proof:

In analysing the proof of Theorem 6.14 two points may be made. The first is that the projectiveness of the ring R is not used. The fact that it is a generator is sufficient. The second point is more subtle. In proving that $A \to B$ is not essential we did not use the fact that every map into A from a subobject of the generator extends to a map from the generator into A. We need only to extend those maps *which allow an extension into B.*

We suppose that $\mathscr{B}$ is a Grothendieck category with a generator G and that $\{E_\gamma\}$ is a sequence of essential extensions throughout the entire sequence of ordinal numbers. We wish to show that the sequence eventually becomes stationary.

For any monomorphism $G' \to G$ and ordinal number γ we note that $\{Im[(G,E_\alpha) \to (G',E_\gamma)]\}_{\alpha>\gamma}$ is an ascending family of subsets of (G',E_γ). This family therefore must stabilize (there is only a *set* of subsets of (G',E)), and since there is only a set of subobjects of G it follows that there is an ordinal number $F(\gamma)$ such that $Im[(G,E_{F(\gamma)}) \to (G',E_\gamma)] \supset Im[(G,E_\alpha) \to (G'E_\gamma)]$ for all $\alpha > \gamma$ and $G' \subset G$. Because it suffices to prove that any cofinal subsequence of $\{E_\gamma\}$ is eventually stationary we may suppose that the sequence is such that $F(\gamma) = \gamma + 1$.

Now let Ω be the first ordinal of cardinality greater than that of the family of subobjects of G. We shall prove that $E_{\Omega+1} = E_\Omega$.

Supposing otherwise, we let $G \xrightarrow{x} E$ be a map whose image is not contained in the image of $E_\Omega \to E_{\Omega+1}$.

For all $\gamma < \Omega + 1$, we shall identify E_γ with the image of $E_\gamma \to E_{\Omega+1}$—that is, we shall suppose that it is a subobject of $E_{\Omega+1}$. The family of subobjects of G, $\{x^{-1}(E_\gamma)\}$, is an ascending family, and by the choice of Ω it must stabilize before Ω. There exists, then, an ordinal $\gamma < \Omega$ such that $x^{-1}(E_\gamma) = x^{-1}(E_\Omega)$. (We use here the fact that in a Grothendieck category the inverse images of ascending unions behave well.) By our assumption that $F(\gamma) = \gamma + 1$ we obtain a map $G \xrightarrow{y} E_{\Omega+1}$ such that $Im(y) \subset E_\Omega \subset E_{\Omega+1}$, $x \neq y$, $Ker(x-y) = x^{-1}(E_\Omega)$.

We conclude that the map $x - y$ has a nontrivial image which meets E_Ω only trivially. If such were not the case there would exist a map $G \xrightarrow{z} G$ such that $O \neq Im((x-y)\,z) \subset E_\Omega$. But then $Im(z) \subset x^{-1}(E_\Omega)$ and $(x-y)\,z \neq 0$. This involves a contradiction. ∎

EXERCISES

A. A very large Grothendieck category

Define $\mathscr{B}$ to be the category whose objects are pairs $(G, f\colon S \to (G,G))$ where G is an abelian group, S is a set, and f is a function from S into the set of endomorphisms on G. We adopt the convention that $f(y) = 0$ for all $y \notin S$. A homomorphism $G \xrightarrow{h} G'$ is a *map* from $(G, f\colon S \to (G,G))$ to $(G', f'\colon S' \to (G',G'))$ iff

$$\begin{array}{ccc} G & \xrightarrow{f(x)} & G \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ G' & \xrightarrow{f'(x)} & G' \end{array}$$

commutes for all $x \in S \cup S'$.

1. $\mathscr{B}$ is a Grothendieck category.
2. $\mathscr{B}$ is well-powered.
3. Let Z be the group of integers, $A_0 = (\mathrm{Z},\emptyset : \emptyset \to (\mathrm{Z},\mathrm{Z})) \in \mathscr{B}$. For every x define $A_x = (Z \oplus Z, f_x : \{x\}) \in \mathscr{B}$ by

$$Z \xrightarrow{u_i} Z \oplus Z \xrightarrow{f_x(x)} Z \oplus Z \xrightarrow{p_j} Z = \begin{cases} 1 & \text{if } i = 2, j = 1. \\ 0 & \text{otherwise} \end{cases}$$

$Z \xrightarrow{u_1} Z \oplus Z$ and $Z \oplus Z \xrightarrow{p_2} Z$ yield maps $A_0 \xrightarrow{u_1} A_x$, $A_x \xrightarrow{p_2} A_0$. $O \to A_0 \xrightarrow{u_1} A_x \xrightarrow{p_2} A_0 \to O$ is exact.

For $x \neq y$, A_x and A_y are not isomorphic. Hence the class of isomorphism types of objects B such that $O \to A_0 \to B \to A_0 \to O$ is exact, is *not* a set.

4. If $\mathscr{B}'$ is an abelian category, $A \in \mathscr{B}'$, and $A \to E$ is an injective extension, $O \to A \to B \to C \to O$ exact, then there is a monomorphism $B \to E \oplus C$.

5. $\mathrm{A}_0 \in \mathscr{B}$ does not have an injective extension. In fact, no nontrivial object in $\mathscr{B}$ is injective or projective.

6. Construct a sequence $\{E_\alpha\}$ of proper essential extensions running through the entire range of ordinal numbers.

7. Let $\mathscr{A}$ be any small category. Construct an exact full embedding $(\mathscr{A},\mathscr{G}) \to \mathscr{B}$.

B. Divisible groups

Let R be a principal ideal domain. The characterization of injective modules of Theorem 6.14 reduces, for modules over R, to the condition that $A \xrightarrow{r} A$ is epimorphic for all nonzero $r \in R$. This property is clearly inherited by quotient modules of A. Finally, then, we may prove that Q/Z is an injective object in $\mathscr{G}$. (Q/Z is the group of rationals modulo the subgroup integers.) A direct argument now suffices for the fact that Q/Z is a cogenerator.

The exact contravariant embedding $\mathscr{G} \xrightarrow{(-,Q/Z)} \mathscr{G}$ may be used to prove a duality metatheorem for very abelian categories.

C. Modules over principal ideal domains

1. In the last exercise it was learned that if R is a principal ideal domain and if $O \to R \to E \to E/R \to O$ is exact, where E is an

injective envelope of R, then E/R is injective. Let $r \neq 0$ and consider an exact commutative diagram:

$$\begin{array}{ccccccccc} O & \to & R & \xrightarrow{r} & R & \to & R/(r) & \to & O \\ & & \downarrow & & \downarrow & & \downarrow & & \\ O & \to & R & \longrightarrow & E & \to & E/R & \to & O \end{array}$$

All three vertical maps are monomorphisms. Hence every proper cyclic module is embeddable in E/R.

Let $A \subset E$ be a finitely generated submodule. Because E is essential over R and R is a domain, A is isomorphic to a submodule of R, hence to R itself. Every finitely generated submodule of E is cyclic and therefore every finitely generated submodule of E/R is cyclic.

2. Let A be a finitely generated module. The family of all ideals that appear in the form $Ker(R \to A)$ is a finite family with (r) as a minimal member. Let $R/(r) \to A$ be an embedding. If $(r) = O$ let $A \to E$ be such that $R/(r) \to A \to E$ is a monomorphism. If $(r) \neq O$ let $A \to E/R$ be such that $R/(r) \to A \to E/R$ is a monomorphism. In either case the map from A has a cyclic image and we obtain a monomorphism $R/(r) \to A \to R/(s)$. Note that $(s) \subset (r)$.

There exists $R \to A$ such that $R \to A \to R/(s)$ is onto.

$$Ker(R \to A) \subset (s) \subset (r),$$

hence $Ker(R \to A) = (s) = (r)$ and we obtain a splitting

$$R/(r) \to A \to R/(r) = 1.$$

By iteration, $A \simeq R/(r_1) \oplus \cdots \oplus R/(r_n)$, where $(r_1) \subset (r_2) \subset \cdots \subset (r_n)$.

3. The uniqueness of any such representation of A may be obtained from the following: For any prime $p \in R$, the number of (r_i)'s such that $(r_i) \subset (p^m)$ is equal to the dimension of $(p^{m-1}A)/(p^m A)$ as a vector space over $R/(p)$.

In particular if (p) and (q) are distinct nonzero prime ideals then

$R/(p^m) \oplus R/(q^n) \simeq R/(p^m q^n)$, which when read backwards yields a representation of A as a sum of indecomposable cyclic modules, that is, of the form $R/(p^m)$ where (p) is a prime ideal.

D. Injectives over acc rings

A ring R obeys the ascending chain condition for left ideals iff the class of injective left R-modules is closed under infinite sums. For one direction, assume R is an ascending chain ring and use Theorem 6.14, recalling that a map from a finitely generated module into an infinite sum must factor through a finite subsum. For the other direction consider an ascending chain $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots$ and let $\mathfrak{A} = \cup \mathfrak{A}_i$. For each i, let E_i be an injective envelope of $\mathfrak{A}/\mathfrak{A}_i$. Define $f\colon \mathfrak{A} \to \Pi_i E_i$ to be such that $Ker(p_i f) = \mathfrak{A}_i$. For any $x \in \mathfrak{A}$, $p_i f(x) = 0$ for almost all i and $Im(f) \subset \Sigma_i E_i \subset \Pi_i E_i$. Since f extends to R and any map from R factors through a finite subsum we conclude that $p_i f = 0$ for almost all i, that is, $\mathfrak{A}_i = \mathfrak{A}$ for almost all i.

Define a module to be *absolutely indecomposable* if it contains no decomposable submodules (a module is *decomposable* if it is isomorphic to the sum of two nonzero modules). An indecomposable injective is absolutely indecomposable. A module is absolutely indecomposable iff it is an essential extension of an absolutely indecomposable module iff its injective envelope is indecomposable. Two absolutely indecomposable modules A and B have isomorphic injective envelopes iff there exist nonzero $A' \subset A$, $B' \subset B$ such that A' is isomorphic to B'.

Every module contains an absolutely indecomposable submodule. To prove it, it clearly suffices to start with a finitely generated module A. If A is not absolutely indecomposable, there exist nonzero submodules $B_1, C_1, B_1 \cap C_1 = O$. If C_1 is not absolutely indecomposable there would exist nonzero B_2, C_2 in $C_1, B_2 \cap C_2 = O$. If this process did not stop we would obtain an ascending chain $C_1, C_1 \oplus C_2, \cdots$

Given an injective E we shall say that a set of indecomposable injective submodules $\{E_i \subset E\}$ is independent if none of them overlaps nontrivially the submodule generated by the others. By Zorn's lemma choose a maximal independent family of indecomposable injective submodules. They generate in E a module E' isomorphic to a sum of indecomposables. If E' were not all of E then $E = E' \oplus E''$

and by the last paragraph E'' contains an indecomposable injective, hence contradicting the maximality of the family used to construct E'. Every injective is a sum of indecomposable injectives.

The injective envelope of a finitely generated module is a *finite* sum of indecomposables. Moreover, if $E_1, \cdots, E_n, E'_1, \cdots, E'_m$ are indecomposable injectives and $f\colon E_1 \oplus \cdots \oplus E_n \to E'_1 \oplus \cdots \oplus E'_m$ is an isomorphism then $n = m$ and there is a one-to-one correspondence between the indexed sets $\{E_i\}$ and $\{E'_j\}$ pairing isomorphic injectives. In other words, a unique factorization theorem holds. To prove it note that $\bigcap_i Ker(p_i f u_n) = O$, thus there is an i such that $Ker(p_i f u_n) = O$, hence $p_i f u_n$ is an isomorphism. If we let $i = m$ and use standard matrix manipulations we obtain an isomorphism $E_1 \oplus \cdots \oplus E_{n-1} \to E'_1 \oplus \cdots \oplus E'_{m-1}$.

E. Semisimple rings and the Wedderburn theorems

1. Let K be a skew field (a division ring). Every K-module is injective. The only indecomposable injective is K itself.

If V is an n-dimensional vector space over K ($V \simeq K \oplus \cdots \oplus K$, n times) and R is the ring of endomorphisms of V, then $\mathscr{G}^K \xrightarrow{(V,-)} \mathscr{G}^R$ is an equivalence of categories by Exercise 4-F. (All exact sequences in $\mathscr{G}^K$ split, hence every object is projective.) R, of course, is simply the ring of $n \times n$ matrices. If $R_1, \cdots, R_m$ are all matrix rings over skew fields $K_1, \cdots, K_m$ then $\mathscr{G}^{R_1 \times \cdots \times R_n} \simeq \mathscr{G}^{R_1} \times \cdots \times \mathscr{G}^{R_n} \simeq \mathscr{G}^{K_1} \times \cdots \times \mathscr{G}^{K_n}$. All modules over $S = R_1 \times \cdots \times R_m$ are injective.

The uniqueness of the skew fields and of the dimensions of the matrix rings in such representations of the ring S may be seen as follows: The number m is equal to the size of a maximal set of nonisomorphic simple S-modules (simple modules have no proper nonzero submodules). Letting $\{A_1, \cdots, A_m\}$ be such a set of simple modules, it follows that the dimensions of R_i may be obtained from the number of components of S, when decomposed into simple modules, that are isomorphic to A_i (we are assuming that the numbering has been arranged to our advantage).

2. Let R be a ring such that all left R-modules are injective. Because a sum of injective R-modules is injective, R obeys the ascending chain condition. R as an R-module is a finite sum of indecomposable modules which must be simple modules. Any map

between simple modules is either zero or an isomorphism and R is isomorphic, as a ring, to a product of matrix rings over skew fields.

3. Let R be a **semisimple** ring, that is, a ring which obeys the descending chain condition and has no nilpotent ideals ($\mathfrak{A}^n = O$ implies $\mathfrak{A} = O$). Every ideal in R is a direct summand, as an R-module, of R. To prove it let $\mathfrak{A}$ be a minimal counterexample. If $\mathfrak{A}$ is not minimal in the family of all nonzero ideals there exist $\mathfrak{B} \subset \mathfrak{A}$ and a map $R \to \mathfrak{B}$ such that $\mathfrak{B} \to \mathfrak{A} \to R \to \mathfrak{B} = 1$. Letting $\mathfrak{C} = Ker(\mathfrak{A} \to R \to \mathfrak{B})$, we obtain $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$. Hence $\mathfrak{A} \to R \to \mathfrak{B} \oplus \mathfrak{C} \to \mathfrak{A} = 1$. If $\mathfrak{A}$ is minimal in the family of all nonzero ideals there must exist $x \in \mathfrak{A}$ such that $\mathfrak{A} \to R \xrightarrow{x} \mathfrak{A} \neq 0$, otherwise $\mathfrak{A}^2 = O$. But any nonzero endomorphism on a simple module is an automorphism.

By Theorem 6.14 every R-module is injective and R is isomorphic to a finite product of matrix rings over skew fields.

F. Noetherian ideal theory

Let R be a ring which obeys the ascending chain condition for left ideals. All modules over R will be understood to be left-modules.

Let E be an indecomposable injective and $R \to E$ any nonzero map. If $O \to \mathfrak{A} \to R \to E$ is exact, then $R/\mathfrak{A}$ is embeddable in E and $R/\mathfrak{A}$ is absolutely indecomposable. Equivalently, $\mathfrak{A}$ is not the intersection of two larger ideals, or as classically stated, $\mathfrak{A}$ is indecomposable. Two indecomposable ideals $\mathfrak{A}$, $\mathfrak{B}$ are such that $R/\mathfrak{A}$ and $R/\mathfrak{B}$ have isomorphic injective envelopes iff there exists $x,y \in R$ such that $\{r \in R \mid rx \in \mathfrak{A}\} = \{r \in R \mid ry \in \mathfrak{B}\}$.

Henceforth let R be commutative, that is, a **Noetherian ring.** The last paragraph says that if $R/\mathfrak{A}$ and $R/\mathfrak{B}$ have isomorphic injective envelopes there exists $\mathfrak{C} \subset R$ such that $\mathfrak{A} \subset \mathfrak{C}$, $\mathfrak{B} \subset \mathfrak{C}$, and $R/\mathfrak{C}$ has the same injective envelope. The family of ideals $\mathbf{F}_E$ that appear as kernels of maps $R \to E$ has a unique maximal member $\mathfrak{P}$. Moreover, for any $x \in R$, $\{r \mid rx \in \mathfrak{P}\}$, if not all of R, is a member of $\mathbf{F}_E$. That is $\mathfrak{P}$ is a prime ideal. For any $\mathfrak{A} \in \mathbf{F}_E$ there exists $x \in R$ such that $\{r \mid rx \in \mathfrak{A}\} = \mathfrak{P}$, hence $\mathfrak{P}$ is the only prime in $\mathbf{F}_E$. Every indecomposable injective is the injective envelope of $R/\mathfrak{P}$ for some unique choice of prime ideal $\mathfrak{P}$.

Let $\mathfrak{P}$ and $\mathfrak{P}'$ be prime ideals and E,E' their corresponding injectives. $(E,E') \neq O$ iff $\mathfrak{P} \subset \mathfrak{P}'$.

Let A be a finitely generated module. The injective envelope of $R/\mathfrak{P}$ appears as a summand of the injective envelope of A iff there is $x \in A$ such that $\{r \mid rx = 0\} = \mathfrak{P}$. We shall call such primes the *annihilating* primes of A.

Let $\mathfrak{A}$ be an ideal. The annihilating primes of $R/\mathfrak{A}$ are defined to be the *associated* primes of $\mathfrak{A}$. If $\mathfrak{A}$ has only one associated prime $\mathfrak{P}$, and if $\mathfrak{P}'$ is another prime such that $\mathfrak{A} \subset \mathfrak{P}'$, then there exists a nonzero map from the injective envelope of $R/\mathfrak{P}$ to that of $R/\mathfrak{P}'$ and $\mathfrak{P} \subset \mathfrak{P}'$. That is the intersection of all primes containing $\mathfrak{A}$ is $\mathfrak{P}$.

In any commutative ring R, Noetherian or not, the set $\{x \mid x^n \in \mathfrak{A}$, some $n\}$ (usually called the radical of $\mathfrak{A}$ and written $\sqrt{\mathfrak{A}}$) is the intersection of all primes that contain $\mathfrak{A}$. To prove it note that $\sqrt{\mathfrak{A}}$ is clearly contained in any prime that contains $\mathfrak{A}$. Conversely suppose that $x \notin \sqrt{\mathfrak{A}}$. We wish to find a prime ideal containing $\mathfrak{A}$ but not x. In the formal power series ring $(R/\mathfrak{A})[[X]]$ the inverse of $1 - xX$ is $1 + xX + x^2X^2 + x^3X^3 + \cdots$ and $1 - xX$ is a unit in the polynomial ring $(R/\mathfrak{A})[X]$ iff $x \in \sqrt{\mathfrak{A}}$. Let $\mathfrak{M}$ be a maximal ideal containing $1 - xX$ and $f\colon R \to ((R/\mathfrak{A})[X])/\mathfrak{M}$ the induced ring homomorphism. $f(x) \neq 0$, hence $x \notin Ker(f)$. Since the range of f is a domain, $Ker(f)$ is a prime ideal.

To return to the Noetherian case. If $\mathfrak{A}$ has only one associated prime $\mathfrak{P}$, then $\sqrt{\mathfrak{A}} = \mathfrak{P}$ and for all $x \notin \mathfrak{A}$, $\{r \mid rx \in \mathfrak{A}\} \subset \mathfrak{P} = \sqrt{\mathfrak{A}}$. Thus $\mathfrak{A}$ is a primary ideal with associated prime $\mathfrak{P}$.

The Lasker-Noether ideal theorems are now obtainable by examining the injective envelope E of $R/\mathfrak{A}$. The factorization of E into components, not indecomposable, but each with its own annihilating prime, pulls back to a decomposition of $\mathfrak{A}$ as an intersection of primary ideals. The uniqueness of the primes involved and the primaries corresponding to the minimal primes follows easily.

CHAPTER 7

EMBEDDING THEOREMS

We return to the functor category $(\mathscr{A},\mathscr{G})$. In Chapter 5 we observed that $(\mathscr{A},\mathscr{G})$ is a Grothendieck category with a generator, and in Chapter 6 we constructed, under such conditions, injective envelopes.

7.1. FIRST EMBEDDING

Proposition 7.11

If an object $E \in (\mathscr{A},\mathscr{G})$ is injective, then it is a right-exact functor.

Proof:

Let $A' \to A \to A'' \to O$ be any exact sequence in $\mathscr{A}$. Applying the representation functor H we obtain the exact sequence

$$O \to H^{A''} \to H^{A} \to H^{A'} \text{ in } (\mathscr{A},\mathscr{G}).$$

The functor $(-,E)\colon (\mathscr{A},\mathscr{G}) \to \mathscr{G}$ is an exact functor. Hence we obtain the exact sequence

$$(H^{A'},E) \to (H^{A},E) \to (H^{A''},E) \to O \text{ in } \mathscr{G}.$$

By the Yoneda lemma, the above sequence is isomorphic to $E(A') \to E(A) \to E(A'') \to O$ and hence E is right-exact. ∎

A right-exact functor is exact iff it carries monomorphisms into monomorphisms. We introduce the term **mono functor** to describe a functor which preserves monomorphisms. An injective mono functor is, therefore, an exact functor. The next lemma will provide a proof that the injective envelope of a mono functor is an exact functor.

Essential lemma 7.12

Let $M \to E$ be an essential extension in $(\mathscr{A},\mathscr{G})$. If M is a mono functor, then so is E.

Proof:

Suppose E is not a mono functor. There exists, then, a monomorphism $A' \to A$ in $\mathscr{A}$ such that $E(A') \to E(A)$ is not a monomorphism in $\mathscr{G}$. Let $0 \neq x \in E(A')$ be such that

$$[E(A') \to E(A)](x) = 0.$$

We construct the subfunctor $F \subset E$ "generated" by x. Define

$$F(B) = \{y \in E(B) \mid \text{there is } A' \to B \in \mathscr{A} \text{ such that } [E(A') \to E(B)](x) = y\}.$$

It follows that for $B' \to B$

$$[E(B') \to E(B)](F(B')) \subset F(B)$$

and that we may define $F(B' \to B)$ by restriction. F is clearly a set-valued functor. It is seen to be a group-valued functor once it is established that $F(B)$ is a subgroup of $E(B)$, and such is clearly the case. (F is the image of the transformation $H^A \xrightarrow{\eta} E$ such that $\eta(1_A) = x$.)

Since $x \in F(A') \subset E(A')$, we know that $F \neq O$. Since $M \subset E$ is essential, $F \cap M \neq O$. In particular then, there is an object B such that $F(B) \cap M(B) \neq O$. Let $0 \neq y \in F(B) \cap M(B)$. By the construction of F there is a map $A' \to B$ such that $y = [E(A') \to E(B)](x)$. Let

$$\begin{array}{ccc} A' & \to & A \\ \downarrow & & \downarrow \\ B & \to & P \end{array}$$

be a pushout diagram. The pushout theorem asserts that $B \to P$ is a monomorphism. Since M is a mono functor

$$[M(B) \to M(P)](y) \neq 0,$$

and hence

$$\begin{aligned} 0 \neq [E(B) \to E(P)](y) &= [E(B) \to E(P)][E(A') \to E(B)](x) \\ &= [E(A') \to E(P)](x) \\ &= [E(A) \to E(P)][E(A') \to E(A)](x) \\ &= 0, \end{aligned}$$

a contradiction. ▮

Corollary 7.13

A group-valued functor may be embedded in an exact functor iff it is a mono functor. ▮

First embedding theorem, 7.14

Every small abelian category is isomorphic to an exact full subcategory of $\mathscr{G}$. Equivalently, for every small abelian category $\mathscr{A}$ there is an exact embedding functor $\mathscr{A} \to \mathscr{G}$. In the terminology of Chapter 4, every abelian category is very abelian. ▮

Proof:

Consider the group-valued functor $G = \sum_{A \in \mathscr{A}} H^A$. G is a mono functor. Let E be its injective envelope. By 7.13 E is an exact functor. Since G is an embedding functor it follows that any extension of G is an embedding functor. Hence E is an exact embedding functor. ∎

7.2. AN ABSTRACTION

Let $\mathscr{M}(\mathscr{A})$ be the subcategory of $(\mathscr{A},\mathscr{G})$ consisting of all mono functors and all transformations between mono functors. $\mathscr{M}(\mathscr{A})$ is a *full* subcategory of $(\mathscr{A},\mathscr{G})$.

$\mathscr{M}(\mathscr{A})$ is closed under certain operations: any subobject of an object in $\mathscr{M}(\mathscr{A})$ is in $\mathscr{M}(\mathscr{A})$; any product of objects in $\mathscr{M}(\mathscr{A})$ is in $\mathscr{M}(\mathscr{A})$; any essential extension of an object in $\mathscr{M}(\mathscr{A})$ is in $\mathscr{M}(\mathscr{A})$.

Let us abstract the situation. Let $\mathscr{B}$ be a Grothendieck category with injective extensions, and let $\mathscr{M}$ be a full subcategory of $\mathscr{B}$ closed under the three operations of subobject, product, and essential extension. We shall call objects in $\mathscr{M}$ **mono objects.** We have two reasons for this further abstraction: first, the situation occurs in other interesting cases, most noticeably in the category of group-valued presheaves on topological spaces and in the theory of relative homological algebra (see Exercises 7-F and 7-G); second, without abstraction we would be lost in a forest of functors defined on functors.

An example worth keeping in mind is the following: Let R be an integral domain, $\mathscr{B}$ the category of R-modules, and $\mathscr{M}$ the subcategory of torsion-free modules.

Proposition 7.21

Given any $B \in \mathscr{B}$ there is a maximal quotient object lying in $\mathscr{M}$, $B \to M(B)$.

Proof:

Let $\mathscr{F}$ be the family of mono quotients of B, and define $M(B)$ to be a coimage of

$$B \xrightarrow{h} \prod_{B' \in \mathscr{F}} B',$$

where each component of h is the obvious epimorphism. Then $M(B) \in \mathscr{M}$, since $\Pi B' \in \mathscr{M}$ and $M(B)$ is a subobject of $\Pi B'$. Moreover, given any epimorphism $B \to B''$ where $B'' \in \mathscr{M}$ we may find $M(B) \to B''$ such that

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & M(B) \arrow[d] \\ & B'' \end{tikzcd} \quad \text{commutes,}$$

by defining $M(B) \to B''$ as $M(B) \to \Pi B' \xrightarrow{p} B''$. ∎

Proposition 7.22

Let $B \in \mathscr{B}$, $M \in \mathscr{M}$, and $B \to M$ any map. Then there is a unique $M(B) \to M$ such that

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & M(B) \arrow[d] \\ & M \end{tikzcd} \quad \textit{commutes.}$$

In the terminology of Exercise 3-F, *$M(B)$ is the reflection of B in $\mathscr{M}$.*

Proof:

Let $B \to B''$ be the coimage of $B \to M$. Since $\mathscr{M}$ is closed under subobjects, $B'' \in \mathscr{M}$ and the maximality of $M(B)$ among mono quotients insures a map $M(B) \to B''$ such that

$$B \to M(B) \to B'' = M \to B''.$$

Hence, we may define $M(B) \to M$ as $M(B) \to B'' \to M$ where $B'' \to M$ is such that $B \to B'' \to M = B \to M$. Its uniqueness is insured by the fact that $B \to M(B)$ is epimorphic. ∎

Given a map $B' \to B$ we obtain then a unique map $M(B') \to M(B)$ such that

$$\begin{array}{ccc} B' & \to & M(B') \\ \downarrow & & \downarrow \\ B & \to & M(B) \end{array} \qquad \text{commutes.}$$

The uniqueness forces M to be an additive functor. The epimorphisms $B \to M(B)$ produce a natural transformation from the identity functor on $\mathscr{B}$ to M.

Proposition 7.23

The transformation $I \to M$ yields a natural equivalence $(M(A),B) \to (I(A),B)$ for all $A \in \mathscr{B}$, $B \in \mathscr{M}$.

Proof:

The last proposition restated. ∎

We shall say that $T \in \mathscr{B}$ is a **torsion** object if for every $M \in \mathscr{M}$, $(T,M) = O$. Equivalently, T is torsion if $M(T) = O$.

Proposition 7.24

$Ker(B \to M(B))$ is the maximal torsion subobject of B.

Proof:

It is clear that for every torsion object T and map $T \to B$, the image of $T \to B$ lies in $Ker(B \to M(B))$, and hence if $Ker(B \to M(B))$ is torsion it is the maximal such.

Suppose $B'' \in \mathscr{M}$, $K \to B''$ is any map, and $O \to K \to B \to M(B) \to O$ is exact. Let $B'' \to E$ be the injective envelope of B''.

We know that $E \in \mathscr{M}$. Let $B \to E$ be such that

$$\begin{tikzcd} O \arrow[r] & K \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & M(B) \arrow[r] \arrow[dl] & O \\ & B'' \arrow[r] & E & & \end{tikzcd} \qquad \text{commutes,}$$

where $M(B) \to E$ is the map insured by Proposition 7.22. It is clear then that $K \to B'' = 0$ and that K is torsion. ▮

$\mathscr{M}$ is not in general an abelian category. Not every monomorphism in $\mathscr{M}$ appears as a kernel of a map in $\mathscr{M}$.

Borrowing from group theory terminology, let us define a subobject $M' \subset M \in \mathscr{M}$ to be **pure** if the exact sequence $O \to M' \to M \to M/M' \to O$ lies in $\mathscr{M}$, i.e., if M/M' is mono. We shall say that a mono object is **absolutely pure** iff whenever it appears as a subobject of a mono object it is a pure subobject. An everpresent example of such is an injective mono object. Indeed, in the case of torsion-free modules over a domain such are the only examples. In the case of mono functors, however, we find that a mono functor $M \in (\mathscr{A}, \mathscr{G})$ is absolutely pure iff it is left-exact.

First,

Lemma 7.25

If $O \to M_1 \to B \to M_2 \to O$ is exact in $\mathscr{B}$ and $M_1, M_2 \in \mathscr{M}$, then $B \in \mathscr{M}$.

Proof:

Let $M_1 \to E$ be an injective envelope, and $B \to E$ an extension of $M_1 \to E$. Then $B \to E \oplus M_2$ is a monomorphism. ▮

Lemma 7.26

A pure subobject of an absolutely pure subobject is absolutely pure.

Proof:

Let A be absolutely pure, $P \to A$ pure in A, and $P \to M$ any monomorphism into a mono object M.

Let

$$\begin{array}{ccc} P & \to & A \\ \downarrow & & \downarrow \\ M & \to & R \end{array}$$

be a pushout diagram and

$$\begin{array}{ccccccccc} & & O & & O & & O & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ O & \to & P & \to & A & \to & P/A & \to & O \\ & & \downarrow & & \downarrow & & \downarrow & & \\ O & \to & M & \to & R & \to & P/A & \to & O \\ & & \downarrow & & \downarrow & & \downarrow & & \\ O & \to & M/P & \to & R/A & \to & O & & \\ & & \downarrow & & \downarrow & & & & \\ & & O & & O & & & & \end{array}$$

an exact commutative diagram. Since M and P/A are mono, R is mono. Hence R/A is mono and M/P is mono. Thus P is absolutely pure. ∎

Theorem 7.27

A mono functor $M \in (\mathscr{A},\mathscr{G})$ is absolutely pure iff it is left-exact.

Proof:

Since M may be embedded in a functor that is both absolutely pure and left-exact, namely its injective envelope, it suffices to prove that a pure subfunctor of a left-exact functor is left-exact.

Let $O \to M \to E \to F \to O$ be exact in $(\mathscr{A},\mathscr{G})$, E left-exact, F mono. Let $O \to A' \to A \to A''$ be exact in $\mathscr{A}$. Consider the commutative diagram

$$\begin{array}{ccccccc} & & O & & O & & O \\ & & \downarrow & & \downarrow & & \downarrow \\ O & \to & M(A') & \to & M(A) & \to & M(A'') \\ & & \downarrow & & \downarrow & & \downarrow \\ O & \to & E(A') & \to & E(A) & \to & E(A'') \\ & & \downarrow & & \downarrow & & \\ O & \to & F(A') & \to & F(A) & & \\ & & \downarrow & & \downarrow & & \\ & & O & & O & & \end{array}$$

The hypothesis of 2.64 is satisfied: F is mono iff M is left-exact. ∎

We return to the abstract situation: a Grothendieck category $\mathscr{B}$ and a full subcategory $\mathscr{M}$ closed with respect to subobjects, products, and essential extensions. We define $\mathscr{L}$ to be the full subcategory of absolutely pure objects.

Given $M \in \mathscr{M}$ we say that $M \to R$, $R \in \mathscr{L}$, is a **reflection** of M in $\mathscr{L}$ if for every map $M \to L$, $L \in \mathscr{L}$, there is a unique map $R \to L$ such that

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & R \arrow[d] \\ & L \end{tikzcd} \quad \text{commutes.}$$

Recognition theorem 7.28

If the sequence $O \to M \to R \to T \to O$ is exact in $\mathscr{B}$, M mono, R absolutely pure, T torsion, then $M \to R$ is a reflection of M in $\mathscr{L}$.

Proof:

Consider any $M \to L$, $L \in \mathscr{L}$. Let $L \to E$ be an injective envelope and $E \to F$ a cokernel of $L \to E$. Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} O & \to & M & \to & R & \to & T & \to & O \\ & & \downarrow & & \downarrow & & \downarrow & & \\ O & \to & L & \to & E & \to & F & \to & O \end{array}$$

where $R \to E$ is any commutative map insured by the injectiveness of E, $T \to F$ the commutative map arising from the exactness of rows.

E is mono by the essential theorem, F is mono since L is absolutely pure. Hence $T \to F = 0$ and $Im(R \to E) \subset L$. Thus is obtained a map $R \to L$ such that

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & R \arrow[d] \\ & L \end{tikzcd}$$

commutes.

The uniqueness is seen easily by considering two extensions of $M \to L$. Their difference $R \xrightarrow{\delta} L$ is such that $M \to R \xrightarrow{\delta} L = 0$, hence $R \xrightarrow{\delta} L$ factors through $R \longrightarrow T$. But T is torsion, L is mono, and $\delta = 0$. ▮

Construction theorem 7.29

For every mono object $M \in \mathcal{M}$ there is a monomorphism $M \to R$ which is a reflection of M in $\mathcal{L}$.

Proof:

Embed M into any absolutely pure object E (an injective envelope will do).

Construct the exact commutative diagram

$$\begin{tikzcd} & O \arrow[d] & O \arrow[d] & O \arrow[d] & \\ O \arrow[r] & M \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & O \\ O \arrow[r] & M \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & O \\ & O \arrow[r] & M(F) \arrow[r] \arrow[d] & M(F) \arrow[r] \arrow[d] & O \\ & & O & O & \end{tikzcd}$$

by starting with the middle row, then the right-hand column, then the bottom row, then the top row (nine lemma, 2.65).

T is torsion, R is a pure subobject of an absolutely pure object, and hence absolutely pure. The top row fits the last theorem. ∎

Choosing $M \to R(M)$ a reflection in $\mathscr{L}$ for each $M \in \mathscr{M}$, we obtain an additive functor $\mathscr{M} \xrightarrow{R} \mathscr{L}$ and a natural transformation from the identity functor on $\mathscr{M}$, $I \to R$ that induces an isomorphism $(I(M),L) \to (R(M),L)$ for every $M \in \mathscr{M}, L \in \mathscr{L}$.

7.3. THE ABELIANNESS OF THE CATEGORIES OF ABSOLUTELY PURE OBJECTS AND LEFT-EXACT FUNCTORS

Theorem 7.31

$\mathscr{L}$ is abelian and every object has an injective envelope.

Proof:

Axiom 0. The zero object is obvious.

Axiom 1, 1*. For $M \in \mathscr{M}$ it is the case that $M \in \mathscr{L}$ iff $M \to R(M)$ is an isomorphism. R is an additive functor. Hence $\mathscr{L}$ is closed under the formation of products and sums.

Axiom 2. Lemma 7.26 asserts that the $\mathscr{B}$-kernel of $(L_1 \to L_2) \in \mathscr{L}$ is in $\mathscr{L}$ and hence $\mathscr{L}$ has kernels. Moreover, a map in $\mathscr{L}$ is an $\mathscr{L}$-monomorphism iff it is a $\mathscr{B}$-monomorphism.

Axiom 3. Given a monomorphism $L_1 \to L_2 \in \mathscr{L}$ let $O \to L_1 \to L_2 \to M \to O$ be exact in $\mathscr{B}$. The absolute purity of L_1 asserts that $M \in \mathscr{M}$. Then $L_1 \to L_2 = Ker(L_2 \to M \to R(M))$.

Axiom 2*. Let $L_1 \to L_2 \in \mathscr{L}$ and $L_1 \to L_2 \to F \to O$ be exact in $\mathscr{B}$. Then $L_2 \to F \to M(F) \to R(M(F)) = Cok(L_1 \to L_2)$.

Axiom 3*. The above construction shows that a map $L_1 \to L_2 \in \mathscr{L}$ is an $\mathscr{L}$-epimorphism iff the $\mathscr{B}$-cokernel of $L_1 \to L_2$ is torsion. Let $L_1 \to L_2$ be an $\mathscr{L}$-epimorphism, and $M \to L_2$ the $\mathscr{B}$-image of $L_1 \to L_2$, $O \to M \to L_2 \to T \to O$ exact in $\mathscr{B}$. T is torsion and the recognition theorem asserts that $L_2 = R(M)$. Hence if $L_0 \to L_1 = Ker(L_1 \to M)$, then $Cok(L_0 \to L_1) = L_1 \to M \to R(M)$ and every $\mathscr{L}$-epimorphism is an $\mathscr{L}$-cokernel.

Since monomorphisms are the same in $\mathscr{B}$ and $\mathscr{L}$, if E is a $\mathscr{B}$-injective envelope of an $\mathscr{L}$-object, it is injective in $\mathscr{L}$. ▌

Returning to $(\mathscr{A},\mathscr{G})$ we define $\mathscr{L}(\mathscr{A}) \subset (\mathscr{A},\mathscr{G})$ to be the full subcategory of left-exact functors. The last theorem asserts that $\mathscr{L}(\mathscr{A})$ is an abelian category with injective envelopes. The representation functor $H\colon \mathscr{A} \to (\mathscr{A},\mathscr{G})$ factors through $\mathscr{L}(\mathscr{A})$.

Theorem 7.32

$\mathscr{L}(\mathscr{A})$ is complete and has an injective cogenerator.

Proof:

The construction of products in $\mathscr{L}(\mathscr{A})$ is straightforward. Surprisingly, the construction of sums in $\mathscr{L}(\mathscr{A})$ is also straightforward. Given a family of left-exact functors $\{F_i\}$ their sum as defined in $(\mathscr{A},\mathscr{G})$ is already left-exact and is the sum defined in $\mathscr{L}(\mathscr{A})$.

The product of all the functors $\{H^A\}_{A\in\mathscr{A}}$ is also left-exact and a generator for $\mathscr{L}(\mathscr{A})$. Proposition 3.37 now implies that $\mathscr{L}(\mathscr{A})$ has an injective cogenerator.

Theorem 7.33

$H\colon \mathscr{A} \to \mathscr{L}(\mathscr{A})$ is an exact *full embedding.*

Proof:

We know that H is a full embedding (5.36). Let $O \to A' \to A \to A'' \to O$ be exact in $\mathscr{A}$. We wish to show that $O \to H^{A''} \to H^A \to H^{A'} \to O$ is exact in $\mathscr{L}(\mathscr{A})$. Such is the case iff the sequence $O \to (H^{A'},E) \to (H^A,E) \to (H^{A''},E) \to O$ is exact for

E an injective cogenerator in $\mathscr{L}(\mathscr{A})$. This last sequence is isomorphic by the Yoneda theorem, 5.34, to

$$G \to E(A') \to E(A) \to E(A'') \to O$$

and this sequence is always exact iff E is an exact functor. The exactness of E was proved in the essential lemma 7.12. ∎

Theorem 7.34 (Mitchell)

Every abelian category is fully *abelian.*

Proof:

The last theorem shows that for every small abelian category there is an exact full contravariant embedding into a complete abelian category with an injective cogenerator. By taking the dual of the range category, we obtain for every small abelian category an exact full embedding (covariant) into a complete abelian category with a projective generator. Theorem 4.44 implies therefore that for every small abelian category there is an exact full embedding into a category of modules. ∎

EXERCISES

A. Effaceable and torsion functors

Let $F \in (\mathscr{A}, \mathscr{G})$, $A \in \mathscr{A}$, $x \in F(A)$. x is an *effaceable element* if there is a monomorphism $A \to B$ such that $[F(A) \to F(B)](x) = 0$. F is an *effaceable functor* if all elements in F are effaceable.

1. Subfunctors and quotient functors of effaceable functors are effaceable.
2. The only effaceable mono functors are trivial.
3. Effaceable functors are torsion functors.
4. Define $T(A) = \{x \in F(A) \mid x \text{ is effaceable}\}$. T is a subfunctor of F. (Use the pushout theorem.)

5. F/T is mono.

6. T is the maximal torsion subfunctor of F and torsion functors are effaceable.

B. Effaceable functors and injective objects

If $\mathscr{A}$ has injective extensions then $F \in (\mathscr{A},\mathscr{G})$ is effaceable iff $F(Q) = O$ for all injective $Q \in \mathscr{A}$.

C. 0th right-derived functors

Define $R_0\colon (\mathscr{A},\mathscr{G}) \to \mathscr{L}(\mathscr{A}) = (\mathscr{A},\mathscr{G}) \xrightarrow{M} \mathscr{M}(\mathscr{A}) \xrightarrow{R} \mathscr{L}(\mathscr{A})$ and $F \to R_0(F) = F \to M(F) \to R(M(F))$. $F \to R_0(F)$ is the *0th right-derived functor of F.*

1. For any $F \to L$, $L \in \mathscr{L}(\mathscr{A})$ there is a unique factorization $R_0(F) \to L$ such that $F \to L = F \to R_0(F) \to L$.

2. If $O \to T_1 \to F \to R \to T_2 \to O$ is exact in $(\mathscr{A},\mathscr{G})$, T_1, T_2 torsion and R left-exact, then $R = R_0(F)$.

3. Given $F \to R \in (\mathscr{A},\mathscr{G})$, $R \in \mathscr{L}(\mathscr{A})$, where $\mathscr{A}$ has injective extensions; $F \to R$ is the 0th right-derived functor iff $O \to F(Q) \to R(Q) \to O$ is exact for all injective $Q \in \mathscr{A}$.

4. Let $O \to A \to Q \to A'' \to O$ be exact in $\mathscr{A}$, Q injective. Then $F(A) \to Ker(F(Q) \to F(A'')) = F(A) \to R_0F(A)$.

D. Absolutely pure objects

In the abstract situation define

$$R_0\colon \mathscr{B} \to \mathscr{L} = \mathscr{B} \xrightarrow{M} \mathscr{M} \xrightarrow{R} \mathscr{L}.$$

1. R_0 is an exact functor. (Use an injective cogenerator on $\mathscr{L}$.)

2. R_0: $\mathscr{B} \to \mathscr{L}$ preserves right roots, as do all reflectors, and we may construct right roots for $\mathscr{L}$ by constructing them in $\mathscr{B}$ and then reflecting in $\mathscr{L}$. Since R_0: $\mathscr{B} \to \mathscr{L}$ is also left-exact we obtain a proof via Exercise 5-E that $\mathscr{L}$ is a Grothendieck category.

E. Computations of 0th right-derived functors

Let $F \in (\mathscr{A},\mathscr{G})$. For each $A \in \mathscr{A}$ consider the set of pairs $S(A) = \{(A \to B, y) \mid A \to B$ is a monomorphism, $y \in F(B)\}$. Given two elements in $S(A)$ define $(A \to B_1, y_1) \equiv (A \to B_2, y_2)$ iff there exist monomorphisms $B_1 \to B$, $B_2 \to B$ such that $[F(B_1) \to F(B)](y_1) = [F(B_2) \to F(B)](y_2)$.

1. There is a functor $R \in (\mathscr{A},\mathscr{G})$ such that $R(A)$ is the set of equivalence classes in $S(A)$, and the functions $F(A) \xrightarrow{\eta_A} R(A)$, $\eta_A(x) = [A \xrightarrow{1} A, x]$ yield a natural transformation.
2. The kernel and cokernel of η are effaceable.
3. R is left-exact.
4. $F \to R$ is the 0th right-derived functor of F. (Use 7-G-2.)

F. Sheaf theory

Let X be a topological space, $\mathscr{T}$ the category of open sets and "restriction" maps (the dual of the category of open sets and inclusion maps). $(\mathscr{T},\mathscr{G})$ is called the category of *group-valued presheaves on X*. Given an open set $U \subset X$ let $H^U \in (\mathscr{T},\mathscr{G})$ be defined by

$$H^U(V) = \begin{cases} Z & \text{if } V \subset U \\ 0 & \text{otherwise.} \end{cases}$$

$$H^U(V_1 \to V_2) = \begin{cases} 1 & \text{if } V_1 \subset U \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{U_i\}$ be a family of open sets, $U = \bigcup U_i$, $U_{ij} = U_i \cap U_j$. Define the sequence $\sum_{ij} H^{U_{ij}} \xrightarrow{(g_1 - g_2)} \Sigma H^{U_i} \xrightarrow{f} H^U$ by

$$H^{U_{kl}} \to \Sigma H^{U_{ij}} \xrightarrow{g_1} \Sigma H^{U_i} = H^{U_{kl}} \to H^{U_k} \to \Sigma H^{U_i}$$

$$H^{U_{kl}} \to \Sigma H^{U_{ij}} \xrightarrow{g_2} \Sigma H^{U_i} = H^{U_{kl}} \to H^{U_l} \to \Sigma H^{U_i}$$

$$H^{U_k} \to \Sigma H^{U_i} \xrightarrow{f} H^U = H^{U_k} \to H^U.$$

We shall call all such sequences the family of fundamental sequences in $(\mathscr{T},\mathscr{G})$.

1. All fundamental sequences are exact.
2. For $F \in (\mathscr{T},\mathscr{G})$ we say that F is *substantial* if $O \to (A,F) \to (B,F)$ is exact for all fundamental $C \to B \to A$ in $(\mathscr{T},\mathscr{G})$. An essential extension of a substantial presheaf is substantial.

3. For $F \in (\mathscr{T},\mathscr{G})$ we say that F is a *sheaf* if $O \to (A,F) \to (B,F) \to (C,F)$ is exact for all fundamental $C \to B \to A$ in $(\mathscr{T},\mathscr{G})$. An injective substantial presheaf is a sheaf.

We may apply the abstract situation of this chapter to prove that the full subcategory of sheaves $\mathscr{S}(X)$ is an abelian category with injective envelopes and that there is an exact functor $(\mathscr{T},\mathscr{G}) \xrightarrow{S} \mathscr{S}(X) \subset (\mathscr{T},\mathscr{G})$ and a transformation from the identity functor $I \to S$ such that for every $F \to T$, $T \in \mathscr{S}(X)$ there is a unique map $S(F) \to T$ such that

$$\begin{array}{ccc} I(F) & \to & S(F) \\ & \searrow \quad \swarrow & \\ & T & \end{array} \qquad \text{commutes.}$$

$\mathscr{S}(X)$ is a Grothendieck category (Exercise 7-D), but the inclusion functor $\mathscr{S}(X) \to (\mathscr{T},\mathscr{G})$, unlike $\mathscr{L}(\mathscr{A}) \to (\mathscr{A},\mathscr{G})$, is not directly continuous.

G. Relative homological algebra

Let $\mathscr{A}$ be a small additive category and M a family of monomorphisms which appear as kernels in $\mathscr{A}$ and such that

(0) For every $A \in \mathscr{A}$, $1_A \in M$.

(1) M is closed under composition.

(2) If $A \to B \to C \in M$ then $A \to B \in M$.

(3) If $A \to B \in M$ and $A \to C \in \mathscr{A}$ then there exist maps $C \to D \in M$ and $B \to D \in \mathscr{A}$ such that

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array} \qquad \text{commutes.}$$

We give some examples of such families:

1. The family of all monomorphisms in an abelian category.

2. The family of all splitting monomorphisms in an additive category.

3. Let $\mathscr{A}$ be an additive category with pushouts and a cogenerator C.
Define M to be those maps $A \to B$ such that $(B,C) \to (A,C)$ is epimorphic.

4. As in the last example except that instead of using a cogenerator use a covariant embedding functor $\mathscr{A} \to \mathscr{G}$ which preserves pushouts.

Define $\mathscr{M}(\mathscr{A})$ to be the full subcategory of those functors in $(\mathscr{A},\mathscr{G})$ which carry maps in M into monomorphisms in $\mathscr{G}$. $\mathscr{M}(\mathscr{A})$ is closed under essential extensions and $\mathscr{L}(\mathscr{A})$, the subcategory of absolutely pure functors in $\mathscr{M}(\mathscr{A})$, is abelian. If $\mathscr{A}$ has kernels, the functors in $\mathscr{L}(\mathscr{A})$ may be identified as those which are "M-left-exact."

Suppose that $\mathscr{A}$ has cokernels. We may define $E \subset \mathscr{A}$ to be the family of epimorphisms which appear as cokernels of maps in M. We assume that E satisfies the dual of the properties listed above. (If $\mathscr{A}$ is abelian this turns out to be a gratuitous assumption.) Define $A' \to A \to A''$ to be *relatively exact* in $\mathscr{A}$ if $A' \to A = A' \to K \to A$, $A \to A'' = A \to F \to A''$, $A' \to K \in E$, $K \to A \in M$, $A \to F \in E$, $F \to A'' \in M$, and $K \to A = Ker(A \to F)$. An exact functor $Q\colon \mathscr{A} \to \mathscr{G}$ is one that carries relatively exact sequences into exact sequences.

By the weak embedding theorem there exists an exact functor $Q\colon \mathscr{A} \to \mathscr{G}$ which is *faithfully left-exact*, that is, $Q(A') \to Q(A)$ is mono iff $A' \to A \in M$. Through dualization, we may obtain an exact functor which is *faithfully right-exact*.

Let $\bar{M} \subset \mathscr{L}(\mathscr{A})$ be the family of monomorphisms such that $T' \to T \in M$ iff $(T,Q) \to (T',Q) \to O$ is exact for all exact $Q \in \mathscr{L}(\mathscr{A})$. By the last paragraph, $H^{A''} \to H^A \in \bar{M}$ iff $A \to A'' \in E$, and $H^{A''} \to H^A \to H^{A'}$ is exact relative to $\bar{M}$ iff $A' \to A \to A''$ is exact relative to M. Let $\mathscr{L}_1$ be a small exact subcategory of $\mathscr{L}(\mathscr{A})$ which contains the representable functors and embed $\mathscr{L}_1$ into $\mathscr{L}(\mathscr{L}_1^*)$ in a manner dual to that described above.

The composed full embedding $\mathscr{A} \to \mathscr{L}(\mathscr{L}_1^*)$ is exact and faithfully so, that is, only relatively exact sequences are carried into exact sequences.

The full metatheorem holds for the relative case.

APPENDIX

In writing and preparing this book I repeatedly told myself that I would give everyone his credit in the appendix. Now the book is written, the proofs are read, the publisher is waiting, and I realize I don't know who is to be credited for what. There are some who learn by reading, I am told. The material in this book I have learned either by discovery or by conversation.

The origin of concepts, even for a scholar, is very difficult to trace. For a nonscholar such as me, it is easier. But less accurate. Nonetheless, I have a few stories to tell. I shall tell them. I shall read all the letters that refute them. I shall hope for enough book buyers to pay for a revision.

To start at the beginning, MacLane tells me that there is an intellectual ancestry for the words "category" and "functor" in Kant's *Critique of Pure Reason.* As I said in the Introduction, he should know, for he and Eilenberg defined them.

The definitions in Chapter 1 are also the work of Eilenberg and MacLane. That statement requires a definition of "work." In 1940 algebraic entities were defined by the remnants of generators and relations. MacLane's definition of "product" [20] as the solution of a universal mapping problem was revolutionary. So revolutionary that it was not immediately absorbed even by the most category minded people. It was common to define finite direct sums as suggested in Theorem 2.41, which definition can only apply to additive categories and allows, even there, no generalization to the infinite case.

The axioms for abelian categories in Chapter 2 are new. The first set of equivalent axioms appears in Buchsbaum's dissertation [2], where they are said to describe an "exact" category. The word "abelian" has stuck, partly to honor MacLane who suggested the whole idea [20], partly because Grothendieck writes in French and "abelian" seems to mean "very nice structure" in French [10]. (There are two words: "Abelian" and "abelian.")

The word "pullback" and the ubiquity of the concept I learned from Lang, who also pointed out the pullback theorem and its importance. I plead guilty to "pushout" and "difference kernel."

Since this note is already so personal (it certainly isn't objective) let me relate my awakening as a graduate student to the newness of my own language. I was brought up, as an undergraduate at Brown, by Massey and Buchsbaum to think in exact sequences. The notion of exactness seemed as fundamental as the notion of continuity must seem to an analyst. And then one day at Princeton my advisor, Norman Steenrod, calmly told me how he and Eilenberg—just a few years before—had chosen the word "exact."

By now I have heard the story from both Eilenberg and Steenrod, the combined version being somewhat as follows: in writing *Foundations of Algebraic Topology* [7] they so recognized the importance of the choice that they used the word "blank" throughout most of the manuscript. After entertaining an unrecorded number of possibilities they settled on "exact." It was initially suggested by history: the exact sequence in DeRham's theorem is about exact differentials. It was chosen because it is descriptive, it is short, it translates easily, and it inflects well ("exactly," "exactness").

The notion of projective objects is implicit in much early work. MacLane called them "free" objects [20] (and in a footnote used the word "fascist" for the dual). The words "projective" and "injective" appear in Cartan and Eilenberg [4]. MacLane's "integral" objects [20] are the first generators. To be precise, an integral object is a generator which does not contain any generators as direct summands and which has no nontrivial idempotents. He observed that the only integral object in the category of groups is the group of integers, thus anticipating all the Chapter 1 exercises. The word "generator" appears in Grothendieck [10].

I might have been the first to observe that the additive structure of an abelian category is implied by the other axioms. On the other hand, MacLane knew [20] that the additive structure could be recovered from the way in which maps compose. The specific proof of the associativity, commutativity, and identity of the two operations is probably from Eckmann and Hilton, who seem to be responsible for the concept of groups in categories. I learned the proof from Eilenberg who also devised the neat construction of additive inverses.

The "classical" lemmas that close Chapter 2 have their origins in algebraic topology (and hence, so does the entire subject). I believe that Eilenberg, Hurewicz, MacLane, and Steenrod were the prime movers. To Buchsbaum [2] goes the credit for demonstrating that the lemmas are categorically provable. He had been anticipated by MacLane's proof [20] that any map between extensions of the same objects was an isomorphism. The repeated use of pullbacks and pushouts that I use, I trace to Lang.

I believe that the term "skeleton" applied to categories is Isbell's, who also knew the facts in Exercise 3-A. The concept of direct limit first appears in Steenrod's dissertation. Allow me to go back a bit. Emma Noether is credited with selling the idea that the homology of a space is a group, not a set of numerical invariants. The "mother of modern algebra" is more than that. She seems to be the mother of modern mathematics. (There were some fathers too.) Again, I point out that groups used to be generators and relations. After Emma Noether they were things. Now, when Steenrod wrote his dissertation, Cech cohomology was still a set of numerical invariants. In order to define it in a way such that he could prove the universal coefficient theorem he needed direct limits. So he invented them.

Adjoint functors were defined by Kan [16], who borrowed their name from functional analysis and who exposed their properties as outlined in Exercises 3-G and 3-I. Except for Watts' theorem in 3-N [22], the adjoint functor theorems that are developed in the rest of the Chapter 3 exercises appeared in my dissertation [8]. I never published them before now. In a new subject it is often very difficult to decide what is trivial, what is obvious, what is hard, what is worth bragging about. A man learns to think categorically, he works out a few definitions, perhaps a theorem, more likely a lemma, and then he publishes it. Very often his exercise, though unpublished, has been in the folklore from the beginning. Very often it has been published faithfully every year. I think the notion of "generator" has appeared regularly, each time under a new name, since MacLane defined his integral objects in 1950.

It was not until my unpublished dissertation began to be rather frequently cited for its adjoint functor theorems that I entertained their publication. I tried to write them as a separate chapter but the chapter grew longer than the rest of the book. I did validate the exercises as exercises during the 1963 NSF Summer Institute in Algebra and the participating students should be blessed for their service.

Mitchell's theorem of Chapter 4 appeared in his dissertation [21].

The possible importance of functor categories was pointed out to me by Watts, along with the niceness of the representation functor. The nature of the Yoneda transformation was first worked out by Yoneda [23].

Baer discovered and proved the existence of enough injective modules [1], using as a start his theorem herein known as 6.14. Injective envelopes were discovered by Eckmann and Schopf [5], who constructed them by first taking any injective extension and then minimizing. Grothendieck showed that the Baer construction of injectives worked in Grothendieck categories with generators [10]. Yes, Grothendieck discovered, but did not name, Grothendieck categories. Mitchell [21] was the first to construct injective envelopes in one sweep as maximal essential extensions.

The weak embedding theorem was obtained independently by Heron [13], Lubkin [18], and myself [8]. Our proofs were entirely different. I do not think that it was coincidence that I had just read Hurewicz and Wallman's *Dimension Theory* [15], which embeds topological spaces into Euclidean space via a theorem about function spaces.

For some time now there has been a flow of ideas between Gabriel and myself. We have never met, or even corresponded. At first we didn't even know each other's name. (I was known as "a student of Xxxx" [9]. But I was not a student of Xxxx.) Anyway, Gabriel first noticed the nice nature of the category of left-exact functors. The proofs using injectives seem to be mine. And to repeat, Mitchell put things together for the full embedding theorem.

The term "effaceable" is Grothendieck's. Relative homological algebra has its roots, as does just about all of homological algebra,in Hochshild. Moreover, he made it explicit in [14], as did Buchsbaum [2] and Heller [12].

Finally, let it be understood that this is not meant to be a history of categories and functors. Much work has been done on many aspects of the subject not even hinted at in this work.

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