274. Before we can go further into the way choices are made, we need to consider the underlying theory. The basic ideas implemented in *make_choices* are due to John Hobby, who introduced the notion of "mock curvature" at a knot. Angles are chosen so that they preserve mock curvature when a knot is passed, and this has been found to produce excellent results.

It is convenient to introduce some notations that simplify the necessary formulas. Let $d_{k,k+1} = |z_{k+1} - z_k|$ be the (nonzero) distance between knots k and $k+1$; and let

$$\frac{z_{k+1} - z_k}{z_k - z_{k-1}} = \frac{d_{k,k+1}}{d_{k-1,k}} e^{i\psi_k}$$

so that a polygonal line from z_{k-1} to z_k to z_{k+1} turns left through an angle of ψ_k. We assume that $|\psi_k| \le 180^\circ$. The control points for the spline from z_k to z_{k+1} will be denoted by

$$\begin{aligned} z_k^+ &= z_k + \tfrac{1}{3}\rho_k e^{i\theta_k}(z_{k+1} - z_k), \\ z_{k+1}^- &= z_{k+1} - \tfrac{1}{3}\sigma_{k+1} e^{-i\phi_{k+1}}(z_{k+1} - z_k), \end{aligned}$$

where ρ_k and σ_{k+1} are nonnegative "velocity ratios" at the beginning and end of the curve, while θ_k and ϕ_{k+1} are the corresponding "offset angles." These angles satisfy the condition

$$\theta_k + \phi_k + \psi_k = 0, \qquad (*)$$

whenever the curve leaves an intermediate knot k in the direction that it enters.

275. Let α_k and β_{k+1} be the reciprocals of the "tension" of the curve at its beginning and ending points. This means that $\rho_k = \alpha_k f(\theta_k, \phi_{k+1})$ and $\sigma_{k+1} = \beta_{k+1} f(\phi_{k+1}, \theta_k)$, where $f(\theta, \phi)$ is METAFONT's standard velocity function defined in the *velocity* subroutine. The cubic spline $B(z_k, z_k^+, z_{k+1}^-, z_{k+1}; t)$ has curvature

$$\frac{2\sigma_{k+1}\sin(\theta_k + \phi_{k+1}) - 6\sin\theta_k}{\rho_k^2 d_{k,k+1}} \qquad \text{and} \qquad \frac{2\rho_k\sin(\theta_k + \phi_{k+1}) - 6\sin\phi_{k+1}}{\sigma_{k+1}^2 d_{k,k+1}}$$

at $t = 0$ and $t = 1$, respectively. The mock curvature is the linear approximation to this true curvature that arises in the limit for small θ_k and ϕ_{k+1}, if second-order terms are discarded. The standard velocity function satisfies

$$f(\theta, \phi) = 1 + O(\theta^2 + \theta\phi + \phi^2);$$

hence the mock curvatures are respectively

$$\frac{2\beta_{k+1}(\theta_k + \phi_{k+1}) - 6\theta_k}{\alpha_k^2 d_{k,k+1}} \qquad \text{and} \qquad \frac{2\alpha_k(\theta_k + \phi_{k+1}) - 6\phi_{k+1}}{\beta_{k+1}^2 d_{k,k+1}}. \qquad (**)$$

276. The turning angles ψ_k are given, and equation $(*)$ above determines ϕ_k when θ_k is known, so the task of angle selection is essentially to choose appropriate values for each θ_k. When equation $(*)$ is used to eliminate ϕ variables from $(**)$, we obtain a system of linear equations of the form

$$A_k\theta_{k-1} + (B_k + C_k)\theta_k + D_k\theta_{k+1} = -B_k\psi_k - D_k\psi_{k+1},$$

where

$$A_k = \frac{\alpha_{k-1}}{\beta_k^2 d_{k-1,k}}, \qquad B_k = \frac{3-\alpha_{k-1}}{\beta_k^2 d_{k-1,k}}, \qquad C_k = \frac{3-\beta_{k+1}}{\alpha_k^2 d_{k,k+1}}, \qquad D_k = \frac{\beta_{k+1}}{\alpha_k^2 d_{k,k+1}}.$$

The tensions are always $\frac{3}{4}$ or more, hence each α and β will be at most $\frac{4}{3}$. It follows that $B_k \ge \frac{5}{4}A_k$ and $C_k \ge \frac{5}{4}D_k$; hence the equations are diagonally dominant; hence they have a unique solution. Moreover, in most cases the tensions are equal to 1, so that $B_k = 2A_k$ and $C_k = 2D_k$. This makes the solution numerically stable, and there is an exponential damping effect: The data at knot $k \pm j$ affects the angle at knot k by a factor of $O(2^{-j})$.

277. However, we still must consider the angles at the starting and ending knots of a non-cyclic path. These angles might be given explicitly, or they might be specified implicitly in terms of an amount of "curl."

Let's assume that angles need to be determined for a non-cyclic path starting at z_0 and ending at z_n. Then equations of the form

$$A_k\theta_{k-1} + (B_k + C_k)\theta_k + D_k\theta_{k+1} = R_k$$

have been given for $0 < k < n$, and it will be convenient to introduce equations of the same form for $k = 0$ and $k = n$, where

$$A_0 = B_0 = C_n = D_n = 0.$$

If θ_0 is supposed to have a given value E_0, we simply define $C_0 = 1$, $D_0 = 0$, and $R_0 = E_0$. Otherwise a curl parameter, γ_0, has been specified at z_0; this means that the mock curvature at z_0 should be γ_0 times the mock curvature at z_1; i.e.,

$$\frac{2\beta_1(\theta_0 + \phi_1) - 6\theta_0}{\alpha_0^2 d_{01}} = \gamma_0 \frac{2\alpha_0(\theta_0 + \phi_1) - 6\phi_1}{\beta_1^2 d_{01}}.$$

This equation simplifies to

$$(\alpha_0\chi_0 + 3 - \beta_1)\theta_0 + \bigl((3-\alpha_0)\chi_0 + \beta_1\bigr)\theta_1 = -\bigl((3-\alpha_0)\chi_0 + \beta_1\bigr)\psi_1,$$

where $\chi_0 = \alpha_0^2\gamma_0/\beta_1^2$; so we can set $C_0 = \chi_0\alpha_0 + 3 - \beta_1$, $D_0 = (3-\alpha_0)\chi_0 + \beta_1$, $R_0 = -D_0\psi_1$. It can be shown that $C_0 > 0$ and $C_0B_1 - A_1D_0 > 0$ when $\gamma_0 \ge 0$, hence the linear equations remain nonsingular.

Similar considerations apply at the right end, when the final angle ϕ_n may or may not need to be determined. It is convenient to let $\psi_n = 0$, hence $\theta_n = -\phi_n$. We either have an explicit equation $\theta_n = E_n$, or we have

$$\bigl((3-\beta_n)\chi_n + \alpha_{n-1}\bigr)\theta_{n-1} + (\beta_n\chi_n + 3 - \alpha_{n-1})\theta_n = 0, \qquad \chi_n = \frac{\beta_n^2\gamma_n}{\alpha_{n-1}^2}.$$

When *make_choices* chooses angles, it must compute the coefficients of these linear equations, then solve the equations. To compute the coefficients, it is necessary to compute arctangents of the given turning angles ψ_k. When the equations are solved, the chosen directions θ_k are put back into the form of control points by essentially computing sines and cosines.