Differential Geometry

Contents

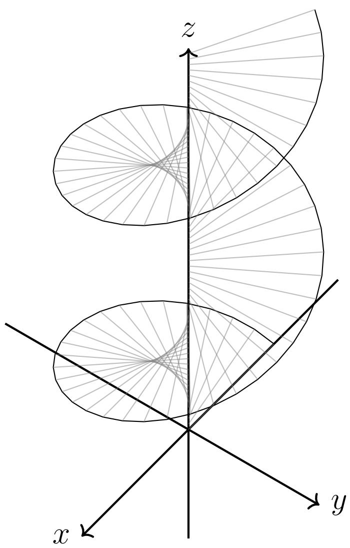
1 Curves.

1.1 Parameterizations and Regular Curves

Differential geometry mainly concerns itself with higher dimension surfaces, or manifolds, although here we'll look at curves. We start with curves as a way to build up to the study of surfaces.

But first, what's the difference between a curve and a surface? Intuitively, the difference is obvious, but from the perspective of an alien from another dimension, it's not so obvious. Saying "a curve looks like a screwed up line, while a surface is like a bent piece of paper" would not suffice for such an alien. What would suffice, is the following.

Definition 1.1. A **curve** is a continuous mapping $f : [0, 1] \to X$, where X is a topological space.



The reason why we use the interval $[0, 1]$ is because we can always rescale our domain to simply be the unit interval. That's what makes curves nice. Thus the difference between a curve and a surface is *where* they come from; a curve is from some interval, while a surface is from a subset of $\mathbb{R}^2$. This is somewhat an oversimplification, and we could offer even nicer defintions for a curve; one could consider it to be a 1-manifold.

The above definition will suffice, since we will mainly be considering when $X = \mathbb{R}^3$; an example of such a curve is on the left, which is simply a helix with parameterization

$$x(t) = \cos(t) \quad y(t) = \sin(t) \quad z(t) = t.$$

Definition 1.2. A **parameterized, differentiable curve** is a differentiable mapping $f : I \to \mathbb{R}^3$, where I is an open subset of $\mathbb{R}$.

The above illustration is such an example, since we can differenitate the curve very nicely:

$$x'(t) = -\sin(t) \quad y'(t) = \cos(t) \quad z'(t) = 1.$$

Note that we cannot work off of the previous definition of a curve since we now need to guarantee that X is a metric space in order to use a concept such as differenitability. The previous definition was simply meant to demonstrate the essence of a curve.

So, why do we now want our curve to be differentiable? The idea of differentiable geometry is to extract information from the geometry of data, and the first step in that direction involves examining the derivative. To do that, it needs to first exist. For example, if we have a curve

$\alpha(t) = (x(t), y(t), z(t))$, knowledge of the derivative $\alpha'(t)$ gives rise to the classic arc length formula $s(t)$ for a curve.

$$s(t) = \int_0^1 \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2)}dt = \int_0^1 |\alpha'(t)|dt.$$

Having the derivative also allows us to **parameterize with respect to arc length**. Such a parameterization is extremely useful. While there are a number of ways to say "at t =blah, where am I?" such a parameterization says "when my *arc length* is, say, 1, where am I?"

Specifically, what we can do is compute $s(t)$ and solve for $t = t(s)$. Then, $\alpha(t(s))$ is our same curve, but this time parameterized with respect to arc length. Then we see that

$$\left|\frac{d(\alpha(t(s)))}{ds}\right| = \left|\frac{d(\alpha(t(s)))}{dt} \cdot \frac{dt}{ds}\right| = \left|\frac{d(\alpha(t(s)))}{dt} \cdot \frac{1}{\alpha'(t)}\right| = 1$$

if $\alpha'(t) \neq 0$. This is really nice! If we imagine the derivative as a tangent vector, then this says such a vector has unit length. This then motivates the following definition.

Definition 1.3. A parameterized, differentiable curve $\alpha : I \to \mathbb{R}^3$ is **regular** if $\alpha'(t) \neq 0$ for $t \in I$.

If you imagine such a curve as a particle moving in three dimensional space, it would have to be one which doesn't stop moving (i.e., its velocity is nonzero at all points).

1.2 Fundamental Theorem of Curves

For this section, we'll continue to consider differentiable curves $\alpha(s) : I \to \mathbb{R}^3$ which are parameterized with respect to arc length. The goal will be to sort of inductively construct a coordinate frame which is accessible due to parameterizations by arc length.

Now consider the derivative of $\alpha'(s)$, which is just $\alpha''(s)$. Given a sufficiently smooth curve which allows a second derivative, what does this quantity represent? Since $|\alpha'(s)| = 1$, observe that

$$|\alpha'(s)| = 1 \implies \alpha'(s) \cdot \alpha'(s) = 1 \implies 2\alpha''(s) \cdot \alpha'(s) = 0 \implies \alpha''(s) \cdot \alpha'(s) = 0.$$

This is a common technique in differential geometry. If you don't know where to start with something, just take the derivative and see what happens!

Hence $\alpha''(s)$ is a vector which is orthogonal to $\alpha'(s)$. Moreover, the fact that $|\alpha'(s)| = 1$ implies that $|\alpha''(s)|$ simply records the rate at which the tangent vector *changes direction.* This motivates the follow definition.

Definition 1.4. If $\alpha(s) : I \to \mathbb{R}^3$ is parameterized by arc length, then we define $k(s) = |\alpha''(s)|$ to be the **curvature** at s. In this case $k(s) : I \to \mathbb{R}_{\geq 0}$.

Again, the function $k(s)$ measures how *quickly* the tangent vector is changing direction. In addition, since $\alpha''(s)$ and $\alpha'(s)$ form a orthogonal pair, we make the next following definition. From here on, we'll denote $\alpha'(s) = \boldsymbol{t}(s)$.

Definition 1.5. if $\alpha(s) : I \to \mathbb{R}^3$ is parameterized by arc length, we define the unit vector $\boldsymbol{n}(s) = \dfrac{\alpha''(s)}{||\alpha''(s)||}$ to be the **normal vector**. Since $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$ therefore form an *orthonormal* pair for each s, we define the plane containing the two vectors to be the **osculating plane**.

Note that for these definitions to hold, we need $k(s) \neq 0$. Thus we'll assume that such is the case going forward. Finally, we finish our construction by considering a vector orthogonal to our osculating plane.

Definition 1.6. Given an osculating plane generated by $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$ on a curve $\alpha(s)$, define the **binormal vector** $\boldsymbol{b}(s)$ as the cross product $\boldsymbol{t}(s) \times \boldsymbol{n}(s)$.

Note that

$$||\boldsymbol{b}(s)|| = ||\boldsymbol{t}(s) \times \boldsymbol{n}(s)|| = ||\boldsymbol{t}(s)|| \cdot ||\boldsymbol{n}(s)|| \sin(\theta) = 1$$

where θ is the angle between $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$, which is of course $\frac{\pi}{2}$. Therefore, the binormal vector is a unit vector. In addition, we have that

$$\boldsymbol{b}'(s) = \boldsymbol{t}'(s) \times \boldsymbol{n}(s) + \boldsymbol{t} \times \boldsymbol{n}'(s) = \boldsymbol{t} \times \boldsymbol{n}'(s).$$

Hence we see that $\boldsymbol{b}'(s)$ and $\boldsymbol{t}$ are orthogonal, so $\boldsymbol{b}'(s)$ is parallel to $\boldsymbol{n}$. Hence we see that

$$\boldsymbol{b}'(s) = \tau(s)\boldsymbol{n}(s)$$

for some function $\tau(s) : I \to \mathbb{R}$. This is generally referred to as the **torsion** of α

At this point, we have the following proposition.

Proposition 1.7. Let $\alpha(s) : I \to \mathbb{R}$ be a regular curve, parameterized with respect to arc length, such that $\alpha''(s) \neq 0$. Then for each $s \in I$, there exists an orthonormal triplet given by $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$, generally called the **Frenet moving frame**. Moreover, we have the so-called **Frenet formulas:**

$$\begin{aligned} \boldsymbol{t}'(s) &= k(s)\boldsymbol{n} \\ \boldsymbol{n}'(s) &= -k(s)\boldsymbol{t}(s) - \tau\boldsymbol{b} \\ \boldsymbol{b}'(s) &= \tau\boldsymbol{n} \end{aligned}$$

However, the converse of this result is true.

Theorem 1.8. Fundamental Theorem of Local Curves. Given the functions $k(s), \tau(s) : I \to \mathbb{R}$, there exists a regular, parameterized curve $\alpha : I \to \mathbb{R}^3$ such that

- s is the arc length
- $k(s)$ is the curvature
- $\tau(s)$ is the torsion.

If $\overline{\alpha}$ is another regular, parameterized curve satisfying the above criterion, then $\overline{\alpha}$ and α differ by a rigid motion.

If one considers the collection of all regular, paramterized curves, and then mods out the space by the equivalence relation where two curves are equivalent if they are the same up to some rigid motion (obviously reflexive, symmetric, and transitive by composing matrix transformations) then the above theorem says that every equivalence class is uniquely determined by a triple $(s, k(s), \tau(s))$. As a result, the interpretation of such uniqueness on equivalence classes is that every curve is basically a straight line which is subjected to curvature (bending) and torsion (twisting).

2 Regular Surfaces.

2.1 Manifolds and Regular Surfaces.

In this section, we now move onto surfaces. In the same spirit as the previous section, we first narrow our surfaces of interest since we want to deal with classes of surfaces which are nice enough to say something general. Thus we begin with the concept of a manifold.

Definition 2.1. A ***n*-manifold** is a separable metric space M which is **locally homeomorphic** to $\mathbb{R}^n$. What we mean by locally homeomorphic is that, for each $p \in M$, there is a neighborhood U of p which is homeomorphic to some open set in V in $\mathbb{R}^n$.

Alternatively, we can define a ***n*-manifold** to be a second-countable Hausdorff space M which locally homeomorphic to $\mathbb{R}^n$.

The above defintions are nearly equivalent: a separable metric space is second countable, which is also Hausdorff since a metric space is Hausdorff. On the other hand, a second countable, Hausdorff space is not necessarily a separable metric space, since not every Hausdorff space can form a metric space (lots of weird counterexamples point this out). In some sense, it is better to use the first definition to dodge these weird counterexamples, but lots of people use the second anyways.

As a side note, it turns out that manifolds form a **category**, hilariously called $\mathbf{Man}_p$. The objects are manifolds, and the morphisms are functions which are p-times continuously differentiable.

Definition 2.2. Let $f : M \to N$ be a mapping between two manifolds. If f is a differentiable homeomorphism (that is, its inverse is also differentiable) then x is a **diffeomorphism**.

Definition 2.3. Let M be a n-manifold, and suppose U is a subset of $\mathbb{R}^m$. A diffeomorphism $\boldsymbol{x} : U \to M$ is said to be a **parameterization** of M.

Example. Here's a simple example. The torus is of course a manifold, and a parameterization of such a structure is given by

$$\boldsymbol{x}(\theta, \phi) = ((R\cos(\theta) + r)\cos(\phi), (R\cos(\theta) + r)\sin(\theta), r\sin(\theta))$$

where $\phi, \theta \in [0, 2\pi]$.

Let's now reduce our scope and consider a special type of manifold known, as a regular surface.

Definition 2.4. Let $S \subset \mathbb{R}^3$ and let $p \in S$. If there exists a neighborhood $V \subset \mathbb{R}^3$ of p and an open set $U \subset \mathbb{R}^2$ with a map

$$\boldsymbol{x}(u, v) : U \to V \cap S$$

where

1. $\boldsymbol{x}(u, v)$ is diffeomorphism

3. The differential $d\boldsymbol{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

then S is said to be a **regular surface**. Or, more easily, a **regular surface** is a 2-manifold whose inclusion mapping $S \to \mathbb{R}^3$ is a homeomorphic immersion.

So, what does this mean? This means that to be a regular surface, we need to be able to parameterize open sets of the surface using open sets from $\mathbb{R}^2$. Furthermore, the differential of these parameterizations must behave well (which we'll see later will serve the purpose of allowing us to define a tangent space at each point).

Proposition 2.5. Let $U \subset \mathbb{R}^2$ be open. Suppose $f(u, v) : U \to \mathbb{R}$ is a differentiable function. Then the subset S of $\mathbb{R}^3$

$$S = \Big\{(u, v, f(u, v)) \mid u, v \in U\Big\}$$

is a regular surface.

Proof: To prove this, we examine the map $\boldsymbol{x}(u, v) = (u, v, f(u, v))$. First, note that $\boldsymbol{x}(U) = S$, so that we have the first part of the definition of a regular surface satisfied. We certainly have that $\boldsymbol{x}'(u, v)$ exists and is well defined. In addition, $\boldsymbol{x}$ is clearly a continuous bijection. Note that since $(u, v) \mapsto (u, v, f(u, v))$, we can define

$$\boldsymbol{x}^{-1} = \pi^3_2 : \mathbb{R}^3 \to \mathbb{R}^2 \qquad \pi^3_2(u, v, f(u, v)) = (u, v).$$

Therefore $\boldsymbol{x}^{-1} = \pi^3_2\Big|_S$. Since projection maps are continuous, so is its restriction onto an open set. Hence $\boldsymbol{x}^{-1}$ is continuous.

Finally, note that if $q \in U$,

$$d\boldsymbol{x}_q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{pmatrix}.$$

As the columns are linearly independent, we see that $d\boldsymbol{x}_q$ is injective. Therefore, S is a regular surface.

$\blacksquare$

This proposition guarantees that a real valued functions in second variables, which are differenitable, can reliably form regular surfaces. An example of this would be something like a topographical map; after all, a topographical map is a three dimensional surface projected down into two dimensions, whose values allow us to undestand how steep a certain hike is. In addition to this, this proposition allows us to establish a functor between the category of differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}$ and the category of regular surfaces.

Definition 2.6. Let $f : U \to \mathbb{R}^m$ be a differentiable map where $U \subset \mathbb{R}^n$, and let $p \in U$. If $df_p : \mathbb{R}^n \to \mathbb{R}^m$ is not injective, then we say $p \in U$ is a **critical point**, and that $f(p)$ is a **critical value**. If $q \in \mathbb{R}^m$ is not a critical point then it is a **regular value**.

Proposition 2.7. Let $f : U \to \mathbb{R}$ be a differentiable function, where $U \subset \mathbb{R}^3$, and suppose $a \in f(U)$ is a regular value. Then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Proof: Let $p \in f^{-1}(a)$. Then since a is a regular value, we see that $df_p = (f_x, f_y, f_z)$ is an injective mapping. For this to be the case, we the partials f_x, f_y, f_z cannot simultaneously be zero when evaluated at p. Without loss of generality, suppose f_z is nonzero.

Now consider the mapping $F : U \to \mathbb{R}^3$ where $F(x, y, z) = (x, y, f(x, y, z))$. Then we have that

$$dF_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ f_x & f_y & f_z \end{pmatrix}.$$

Since $\det(dF_p) = f_z \neq 0$, linear algebra tells us that this mapping is invertible. By the inverse function theorem, there exists open subsets V of p and W of $F(p)$ such that $F : V \to W$ is one to one, and $F^{-1} : W \to V$ is differentiable. Let $F^{-1} = (u, v, g(u, v, t))$ where $(u, v, t) \in W$. Then $g(u, v, t) : \mathbb{R}^3 \to \mathbb{R}$ is differentiable, and so is $h(u, v) = g(u, v, a) : \mathbb{R}^2 \to \mathbb{R}$. However, observe that

$$F(f^{-1}(a) \cap V) = W \cap \{(u, v, a) \mid (u, v) \in \mathbb{R}^2\}.$$

Therefore, the graph generated by $(u, v, h(u, v))$ is $f^{-1}(a) \cap V$. Thus we see that the properties of F allow $f^{-1}(a)$ to be a regular surface.

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Proposition 2.8. Let $S \subset \mathbb{R}^3$ be a regular surface. Then for each $p \in S$, there exists a neighborhood V of p such that V is the graph of a differentiable function.

What we mean by being a graph of a function is a that one of the three must be true.

$$V = \{(x, y, f(x, y)) \mid x, y \in \mathbb{R}^2\}$$
$$V = \{(x, g(x, z), z) \mid x, z \in \mathbb{R}^2\}$$
$$V = \{(h(y, z), y, z) \mid y, z \in \mathbb{R}^2\}$$

where f, g and h are differentiable functions.

Proof: Let $\boldsymbol{x} : U \to S$ be a parameterization of S. Then by definition, $\boldsymbol{x}$ is a diffeomorphic immersion $\boldsymbol{x} : U \to V \cap S$. Because $\boldsymbol{x}$ is an immersion, we know that $d\boldsymbol{x}_q$ is injective where $q = f^{-1}(p)$. If we write $\boldsymbol{x} = (x(u, v), y(u, v), z(u, v))$ then

$$d\boldsymbol{x}_q = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

but injectivity of $d\boldsymbol{x}_q$ implies that the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)} \quad \frac{\partial(x,z)}{\partial(u,v)} \quad \frac{\partial(y,z)}{\partial(u,v)}$$

cannot simultaneously be zero at q. Without loss of generality, suppose that $\dfrac{\partial(x,y)}{\partial(u,v)}(q)$ is nonzero. If we let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map where $(x,y,z) \mapsto (x,y)$, then we see that $\pi \circ \boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^2$; specifically, $\pi \circ \boldsymbol{x} : U \to \mathbb{R}^2$. The punchline is we can apply the inverse function theorem to *this* function so that there exists open subsets of $\mathbb{R}^2$, W_1 of q and W_2 of $\pi(p)$, such that $(\pi \circ \boldsymbol{x})^{-1} : W_2 \to W_1$ is differentiable. Now let $V = \boldsymbol{x}(V_1)$, so that $\pi(V) = V_2$. Then construct the map $g(u,v) = z(u,v)$. Then we see that

$$V = \{(x, y, g(x,y)) \mid x, y \in \mathbb{R}^2\}$$

so that V is the graph of a differentiable function, as desired.

$\blacksquare$

2.2 Change of Parameters and Differentiability

In the definition of a regular surface, we basically need to collect a series of parameterizations $\boldsymbol{x}_i : U_i \to S$ where U_i are open subsets of $\mathbb{R}^2$. However, sometimes we can't find these parameterizations, or we may wonder if there are other parameterizations of the surface. That is, is there are "right" way to parameterize a regular surface?

The answer is no.

Theorem 2.9.Change of Parameters. Let S be a regular surface and p a point on S. Suppose we have two parameterizations of the regular surface:

$$\boldsymbol{x} : U \to S \qquad \boldsymbol{y} : V \to S$$

where U, V are subsets of $\mathbb{R}^2$ and such that $p \in \boldsymbol{x}(U) \cap \boldsymbol{y}(V) = W$. Then the change of parameters

$$h = \boldsymbol{x}^{-1} \circ \boldsymbol{y} : \boldsymbol{y}^{-1}(W) \to \boldsymbol{x}^{-1}(W)$$

is a diffeomorphism.

The above theorem says that parameterizations of a regular surface are in some sense the same thing. One can pass diffeomorphically between them!

Proof: Clearly h is a homeomorphism, so we just need to show h and h^{-1} are differenitable. To do this, consider $q \in \boldsymbol{y}^{-1}(W)$ so that $h(q) \in \boldsymbol{x}^{-1}(W)$. Since $\boldsymbol{x}$ is a parameterization, suppose it has the form $\boldsymbol{x} = (x(u,v), y(u,v), z(u,v))$ and that without loss of generality

$$\frac{\partial(x,y)}{\partial(u,v)}(h(q)) \neq 0.$$

To apply the inverse function theorem, we construct the map $F : U \times \mathbb{R} \to \mathbb{R}^3$ where

$$F(u,v,t) = (x(u,v), y(u,v), z(u,v) + t).$$

Therefore, $F(u,v,0) = \boldsymbol{x}$. Note that

$$\det(dF_{h(q)}) = \begin{vmatrix} x_u & x_v & 0 \\ y_u & y_v & 0 \\ z_u & z_v & 1 \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}(h(q)) \neq 0.$$

Therefore, we can apply the inverse function theorem to conclude that there exists a neighborhood N of $\boldsymbol{x}(h(q))$ such that F^{-1} exists and is differentiable. Since N is open, continuity implies that there exists an open set $U' \subset \mathbb{R}^2$ of q such that $\boldsymbol{y}(U') \subset M$. Now observe that

$$h|_U = F^{-1} \circ \boldsymbol{y}|_U.$$

Since F^{-1} is differenitable on $\boldsymbol{y}(U)$ and $\boldsymbol{y}$ is itself differentiable, we can conclude that h must be differenitable on U. But since q is an arbitrary point of V, we see that h is differentiable. The argument here can be dualized to show that h^{-1} is differentiable, so that h, the change of coordinates, is a diffeomorphism.

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Since the specific parameterization of regular surfaces are all diffeomorphic, the following definition of differentiability is well-defined.

Definition 2.10. Let S be a regular surface and let $V \subset S$ be open. A function

$$f : V \to \mathbb{R}$$

is said to be **differentiable** at $p \in V$ if for any parameterization $\boldsymbol{x} : U \to S$, with $U \subset \mathbb{R}^2$ open, we have $f \circ \boldsymbol{x} : U \to \mathbb{R}$ is differentiable at $\boldsymbol{x}^{-1}(p)$. Hence, f is **differentiable** on all of S if it is differentiable at all points.

We can also define differentiability between surfaces. Let S_1 and S_2 be surfaces, and suppose that $V_1 \subset S_1$. Then a function

$$\phi : V_1 \to S_2$$

is differentiable if for any parameterizations

$$\boldsymbol{x}_1 : U_1 \to S_1 \qquad \boldsymbol{x}_2 : U_2 \to S_2$$

with $U_1, U_2 \subset \mathbb{R}^2$, the map

$$\boldsymbol{x}_{\boldsymbol{2}}^{-1} \circ \phi \circ \boldsymbol{x}_{\boldsymbol{1}} : U_1 \to U_2$$

is differentiable at $q = \boldsymbol{x}_1^{-1}(p)$.