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143 Hw #1

A5. Let A be a 5×5 matrix. Suppose A has distinct eigenvalues $-1, 1, -10, 5, 2$.

(a) What is $\det A$? What is $\operatorname{tr} A$?

(b) If A and B are similar, what is $\det B$? Why?

(c) Do you expect that all eigenvectors of A are mutually orthogonal? Why?

Solution:

(a) Since the product of our eigenvalues (in general, taking into account algebraic multiplicity) result in the determinant, we have that

$$|A| = (-1) \cdot (1) \cdot (-10) \cdot (5) \cdot (2) = 100.$$

On the other hand, the trace is the sum of the eigenvalues, so that

$$\operatorname{tr} A = -1 + 1 - 10 + 5 + 2 = -3.$$

(b) The determinants must be equal. If A and B are similar, then there exists an invertible matrix P such that $A = P^{-1}BP$. Hence we have that

$$|A| = |P^{-1}BP| = |P^{-1}| \cdot |B| \cdot |P| = |B|$$

where we applied the product rule for determinants and used the fact that $|P^{-1}| = |P| = 1$. Therefore $|B| = |A|$.

(c) Not unless A is a real symmetric matrix.

$\square$

A7.

(a) Prove that similar matrices have the same eigenvalues.

(b) Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of a matrix A with associated eigenvectors $x_1, x_2, \dots, x_k$ Prove that $x_1, x_2, \dots, x_k$ are linearly independent.

(c) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation defined by $L(X) = AX$. Let $V_\lambda = \{\zeta \in \mathbb{R}^n \mid L(\zeta) = \lambda\zeta\}$. Prove that V_λ is a subspace of $\mathbb{R}^n$. (This subspace is called the eigenspace associated with λ.)

(d) Let λ be an eigenvalue of A with multiplicity r. Let $\dim V_\lambda = s$. Prove that $s \leq r$. (That is, the dimension of the eigenspace associated with λ is at most the multiplicity of λ.)

Solution:

(a) Suppose that $n \times n$ matrices M and N are similar. Then there exists an invertible matrix P such that $M = P^{-1}NP$. Now observe that the characteristic polynomial of M will be given by

$$|M - \lambda I| = |P^{-1}NP - \lambda I|$$

which we can rewrite as

$$|P^{-1}NP - \lambda P^{-1}IP| = |P^{-1}(N - \lambda I)P| = |P^{-1}| \cdot |N - \lambda I| \cdot |P|.$$

Since P is invertible, we have that $|P| = |P^{-1}| = 1$. Hence we have that

$$|P^{-1}| \cdot |N - \lambda I| \cdot |P| = |N - \lambda I|$$

which implies that $|M - \lambda I| = |N - \lambda I|$. As the characteristic polynomials of the two matrices are the same, we have that their roots must be equal, so that their eigenvalues are the same.

(b) Suppose $\lambda_1, \lambda_2, \dots \lambda_k$ and $x_1, x_2, \dots, x_k$ are distinct eigenvalues and eigenvectors of A, respectively. For the sake of contradiction, suppose that the set of eigenvectors is not linearly independent. Then there exists coefficients c_i, at least one nonzero, such that

$$x_j = \sum_{\substack{i=1 \\ i \neq j}}^{k} c_i x_i.$$

On one hand, we know that $A(x_j) = \lambda_j x_j$. However, the above equation tells us that

$$A(x_j) = \sum_{\substack{i=1 \\ i \neq j}}^{k} A(c_i x_i) = \sum_{\substack{i=1 \\ i \neq j}}^{k} c_i \lambda_i x_i.$$

Therefore, we have that

$$\lambda_j x_k = \sum_{\substack{i=1 \\ i \neq j}}^{k} c_i \lambda_i x_i$$

However, this is a contradiction since our initial hypothesis was that

$$x_j = \sum_{\substack{i=1 \\ i \neq j}}^{k} c_i x_i \implies \lambda_j x_j = \sum_{\substack{i=1 \\ i \neq j}}^{k} c_i \lambda_j x_i.$$

and if we are to have distinct eigenvalues

$$\sum_{\substack{i=1 \\ i \neq j}}^{k} c_i \boldsymbol{\lambda_i} x_i \neq \sum_{\substack{i=1 \\ i \neq j}}^{k} c_i \boldsymbol{\lambda_j} x_i$$

Hence our initial hypothesis is wrong, and our collection of eigenvectors must be linearly independent.

(c) For a given eigenvalue λ of A, we first observe that V_λ is nonempty, since it at least contains the eigenvector x_λ associated with λ. Now suppose $x, y \in V_\lambda$. Then observe that

$$A(x+y) = A(x) + A(y) = \lambda x + \lambda y = \lambda(x+y).$$

Hence $x + y \in V_\lambda$. Finally, for any scalar multiple c and $x \in V_\lambda$,

$$A(cx) = cA(x) = c\lambda x = \lambda(cx).$$

So $cx \in V_\lambda$. As V_λ is nonempty, closed under vector addition and scalar multiplication, we see that it is a subspace of $\mathbb{R}^n$ as desired.

(d) Observe that if $\dim V_\lambda = s$, then there exists a linearly independent set of vectors $x_1, x_2, \ldots, x_s$ where

$$A(x_i) = \lambda x_i$$

for $i = 1, 2, \ldots, s$. If A is $n \times n$, consider $n - s$ vectors $y_1, \ldots, y_{n-s}$ such that $\{x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_{n-s}\}$ is a linearly independent set of n vectors. Then let M be the change of basis matrix on A from the standard basis to the basis $\{x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_{n-s}\}$. We can then represent this matrix as

$$V = MAM^{-1}.$$

Then V and A are similar. Since they are similar, V and A must have the same eigenvalues. But since V is in the basis $\{x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_{n-s}\}$, the first $s \times n$ entires of the matrix must be zero, except along that diagonal, with a value of λ.

Since these matrices are similar, their characteristic polynomials are the same. Therefore, we have that

$$|A - \zeta I| = |V - \zeta I| = (x - \lambda)^s p(x)$$

where $p(x)$ is some $(n-s)$-degree complex polynomial. However, we know that the algebraic multiplicity of λ is r, so that $s \leq r$, as desired.

□

A12. A linear transformation $L : V \to V$, where V is an n-dimensional Euclidean space, is called *orthogonal* if $\langle Lu, Lw\rangle = \langle v, w\rangle$.

(a) Let A be an $n \times n$ matrix. Show that A is orthogonal if and only if the columns (and rows) of A form an orthonormal basis for $\mathbb{R}^n$.

(b) Let S be an orthonormal basis for V and let the matrix A represent the orthogonal linear transformation L with respect to S. Prove that A is an orthogonal matrix.

(c) Prove that for any vectors $u, v \in \mathbb{R}^n$, $\langle Lu, Lv\rangle = \langle u, v\rangle$ if and only if for any $u \in \mathbb{R}^n$, $||Lu|| = ||u||$.

(d) Let $L : V \to V$ be an orthogonal linear transformation. Show that if λ is an eigenvalue of L, then $|\lambda| = 1$.

Solution:

(a) Suppose A is orthogonal. Recall that for an orthogonal matrix we have that $A^{-1} = A^T$. Hence it is invertible. That is, $AA^T = AA^{-1} = I$. Now suppose A is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}$$

where x_i are the column vectors. Note that since A is invertible, the columns vectors are linearly independent. Now

$$AA^T = I \iff \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{pmatrix} = I$$

$$\iff \begin{pmatrix} \langle x_1, x_1\rangle & \langle x_2, x_1\rangle & \cdots & \langle x_n, x_1\rangle \\ \langle x_1, x_2\rangle & \langle x_2, x_2\rangle & \cdots & \langle x_n, x_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, x_n\rangle & \langle x_2, x_n\rangle & \cdots & \langle x_n, x_n\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and the above holds $\langle x_i, x_j\rangle = 1$ if and only if $i = j$ and is zero otherwise. Since $\langle x_i, x_i\rangle = ||x_i|| = 1$, this shows that each vector x_i is normalized. Since $\langle x_i, x_j\rangle = 0$ for $i \neq j$, this shows that each vector is mutually orthogonal to one another. Finally, since the vectors are linearly independent, we see that the columns of A, $x_1, x_2, \dots, x_n$, form an orthonormal basis of $\mathbb{R}^n$.

Now observe that it doesn't matter if we started with A or A^T, since in the end $(A^T)^T = A$. Therefore the entire process could be repeated by first starting with A^T. This then shows that the rows of A also form an orthonormal basis of $\mathbb{R}^n$.

(b) Since A represents L with respect to the orthonormal basis B, write $S = \{x_1, x_2, \dots, x_n\}$, so that

$$A = \begin{pmatrix} [L(x_1)]_S & [L(x_2)]_S & \cdots & [L(x_n)]_S \end{pmatrix}.$$

Observe however that

$$\begin{aligned}
AA^T &= \begin{pmatrix} [L(x_1)]_S & [L(x_2)]_S & \cdots & [L(x_n)]_S \end{pmatrix} \begin{pmatrix} [L(x_1)]_S \\ [L(x_2)]_S \\ \cdots \\ [L(x_n)]_S \end{pmatrix} \\
&= \begin{pmatrix} \langle [L(x_1)]_S, [L(x_1)]_S \rangle & \langle [L(x_2)]_S, [L(x_1)]_S \rangle & \cdots \langle [L(x_n)]_S, [L(x_1)]_S \rangle & \\ \langle [L(x_1)]_S, [L(x_2)]_S \rangle & \langle [L(x_2)]_S, [L(x_2)]_S \rangle & \cdots \langle [L(x_n)]_S, [L(x_2)]_S \rangle & \\ \vdots & \vdots & \ddots & \vdots \\ \langle [L(x_1)]_S, [L(x_n)]_S \rangle & \langle [L(x_2)]_S, [L(x_n)]_S \rangle & \cdots \langle [L(x_n)]_S, [L(x_n)]_S \rangle & \end{pmatrix} \\
&= \begin{pmatrix} \langle [x_1]_S, [x_1]_S \rangle & \langle [x_2]_S, [x_1]_S \rangle & \cdots & \langle [x_n]_S, [x_1]_S \rangle \\ \langle [x_1]_S, [x_2]_S \rangle & \langle [x_2]_S, [x_2]_S \rangle & \cdots & \langle [x_n]_S, [x_2]_S \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle [x_1]_S, [x_n]_S \rangle & \langle [x_2]_S, [x_n]_S \rangle & \cdots & \langle [x_n]_S, [x_n]_S \rangle \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}
\end{aligned}$$

where in the last step we used the fact that $x_1, x_2, \dots, x_n$ is an orthonormal basis. Hence we see that $AA^T = I$, so that A is an orthonormal matrix.

(c) Suppose that $\langle L(u), L(v) \rangle = \langle u, v \rangle$ for every $u, v \in \mathbb{R}^n$. Then in particular $\langle L(u), L(u) \rangle = \langle u, u \rangle \implies ||L(u)|| = ||u||$ by definition.

On the other hand, suppose $||L(u)|| = ||u||$ for all $u \in \mathbb{R}^n$. Then for any vectors $u, v \in \mathbb{R}^n$, we have that

$$\begin{aligned}
||L(u+v)|| = ||u+v|| &\implies \langle L(u+v), L(u+v) \rangle = \langle u+v, u+v \rangle \\
&\implies \langle L(u) + L(v), L(u) + L(v) \rangle = \langle u, v \rangle + \langle u, v \rangle \\
&\implies \langle L(u), L(v) \rangle + \langle L(u), L(v) \rangle = \langle u, v \rangle + \langle u, v \rangle \\
&\implies 2 \langle L(u), L(v) \rangle = 2 \langle u, v \rangle \\
&\implies \langle L(u), L(v) \rangle = \langle u, v \rangle
\end{aligned}$$

where we repeatedly used the linearity in the first and second coordinates of the dot product.

With both directions proved, we see that $\langle Lu, Lv \rangle = \langle u, v \rangle$ if and only if for any $u \in \mathbb{R}^n$, $||Lu|| = ||u||$ for any vectors $u, v \in \mathbb{R}^n$ as desired.

(d) Suppose λ is an eigenvalue of A with associated eigenvector x. Then $Ax = \lambda x$. Thus

$$Ax = \lambda x \implies ||Ax|| = ||\lambda x|| \implies ||x|| = |\lambda| \cdot ||x|| \implies |\lambda| = 1$$

where we used the fact that since A is orthogonal, $||Ax|| = ||x||$. Hence the maginitude of every eigenvalue of A is one.

$\square$