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143 Hw #2

Problem 1. Show $\mathbb{R}P^n$ is a differentiable manifold by definition.

Solution: Recall that $\mathbb{R}P^n$ is the set of all lines which pass through the origin of $\mathbb{R}^{n+1}$. Let $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$. Then we can also see that $\mathbb{R}P^n$ can be identified as the quotient sapce $\mathbb{R}^{n+1} - \{0\}/\sim$ with the equivalence relation $(x_1, \ldots, x_{n+1}) \cong (\lambda x_1, \ldots, \lambda x_{n+1})$ where $\lambda \in \mathbb{R} - \{0\}$. In this interpretation, the points of $\mathbb{R}P^n$ will be denoted by the equivalence classes, denoted as $[x_1, x_2, \ldots, x_{n+1}]$. Notice that

$$[x_1, \ldots, x_{n+1}] = \left[\frac{x_1}{x_i}, \ldots, \overbrace{1}^{i\text{−th coordinate}} \ldots, \frac{x_{n+1}}{x_i}\right].$$

for any nonzero x_i in the representation $x_1, \ldots, x_{n+1}$ of the equivalence class. Define subsets $V_i \subset \mathbb{R}P^n$ by

$$V_i = \{[x_1, x_2, \ldots, x_{n+1}] \mid x_i \neq 0\} \quad i = 1, 2, \ldots, n.$$

The claim to prove now is that $\mathbb{R}P^n$ can be covered by the sets $V_1, \ldots, V_n$. This is obtained by the maps $\varphi_i : \mathbb{R}^n \to M$

$$\varphi_i(y_1, \ldots, y_n) = [y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n]$$

where $y_j = \dfrac{x_j}{x_i}$. First, observe that the maps are injective; for if

$$[y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n] = [y_1', \ldots, y_{i-1}', 1, y_{i+1}', \ldots, y_n']$$

then there exists a scalar λ such that

$$(y_1, \ldots, 1, \ldots, y_n) = (\lambda y_1', \ldots, \lambda, \ldots, \lambda y_n') \implies \lambda = 1.$$

Hence the points must be equal. Next, observe that

$$\bigcup_{i=1}^{n} \varphi_i(\mathbb{R}^n) = \{[y_1, \ldots, \overbrace{1,}^{i\text{−th coordinate}} \ldots, y_n] \mid (y_1, \ldots, y_n) \in \mathbb{R}^n, i = 1, 2, \ldots, n\} = \mathbb{R}P^n.$$

Next, consider $\varphi_i^{-1}(V_i \cap V_j)$, and observe that

$$\varphi_i^{-1}(V_i \cap V_j) = \{(y_1, \ldots, y_n) \mid y_j \neq 0\}.$$

Since this is the complement of $\{(y_1,\dots,y_n) \mid y_j = 0\}$, a closed set, we see that it must be open. Now suppose $i > j$; then observe that

$$\begin{aligned}
\varphi_j^{-1}\circ\varphi_i(y_1,\dots,y_n) &= \varphi_j^{-1}[y_1,\dots,y_{i-1},1,y_i,\dots,y_n] \\
&= \varphi_j^{-1}\left[\frac{y_1}{y_j},\dots,y_{j-1},\overbrace{1}^{j-\text{th coordinate}},y_{j+1},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\frac{y_i}{y_j},\dots,\frac{y_n}{y_j}\right] \\
&= \left(\frac{y_1}{y_j},\dots,\frac{y_{j-1}}{y_j},\frac{y_{j+1}}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\frac{y_i}{y_j},\dots,\frac{y_n}{y_j}\right).
\end{aligned}$$

which is differentiable as $y_j \neq 0$. The case is the same if $j > i$; it's simply a matter of notational difference. In this case,

$$\begin{aligned}
\varphi_j^{-1}\circ\varphi_i(y_1,\dots,y_n) &= \varphi_j^{-1}[y_1,\dots,y_{i-1},1,y_i,\dots,y_n] \\
&= \varphi_j^{-1}\left[\frac{y_1}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\frac{y_i}{y_j},\dots,y_{j-1},\overbrace{1}^{j-\text{th coordinate}},y_{j+1},\dots\frac{y_n}{y_j}\right] \\
&= \left(\frac{y_1}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\frac{y_i}{y_j},\dots,\frac{y_{j-1}}{y_j},\frac{y_{j+1}}{y_j},\dots,\frac{y_n}{y_j}\right).
\end{aligned}$$

which is also differentiable since $y_j \neq 0$. In either case, we see that $\varphi_j^{-1}\circ\varphi_i$ is differentiable. In total, we see that $\mathbb{R}P^n$ is a differentiable manifold, as desired.

$\square$

Problem 2. Show why the set of tangent vectors which tangent to all the curves starting from a point on a manifold M form a linear space (called a tangent space at p of M, denoted T_pM).

Solution: Let $\alpha : (-\varepsilon,\varepsilon) \to M$ be a curve on M where $\alpha(0) = p$. Recall that if $\mathcal{D}$ denotes the set of differentiable functions f on M defined at p, then the tangent vectors at p are given by the function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ where

$$\alpha'(0)f = \left.\frac{d(f\circ\alpha)}{dt}\right|_{t=0}.$$

Let $x : U \subset \mathbb{R}^n \to M$ be a parameterization where $p \in x(U)$. Then $x^{-1}\circ\alpha : (-\varepsilon,\varepsilon) \to \mathbb{R}^n$ is differentiable. That is, $x^{-1}\circ\alpha = (x_1(t),\dots,x_n(t))$ for some differentiable $x_i(t)$. So we see that

$$\begin{aligned}
\left.\frac{d(f\circ\alpha)}{dt}\right|_{t=0} &= \left.\frac{d}{dt} f\circ x(x_1(t),\dots,x_n(t))\right|_{t=0} \\
&= \left.\frac{d}{dt} f(x_1(t),\dots,x_n(t))\right|_{t=0} \\
&= \sum_{i=1}^n x_i'(0)\left.\frac{\partial f}{\partial x_i}\right|_{t=0}
\end{aligned}$$

Thus we see that $\alpha'(0)(f) = \sum_{i=1}^{n} x_i'(0)\frac{\partial f}{\partial x_i}\Big|_{t=0}$; hence we may express the operator $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ in the basis $\left(\frac{\partial}{\partial x_i}\right)_0$, where the zero denotes the evaluation at zero. This basis is orthogonal, since each $\left(\frac{\partial}{\partial x_i}\right)_0$ demonstrates the tangent vector of the map $x(0, \dots, x_i, \dots, 0)$, where the x_i appears in the i-th coordinate. Moreover, this is the dual basis of $\{dx_1, \dots, dx_n\}$. Therefore we see that T_pM can be endowed with an n-orthogonal vector, which shows that it is an n-dimensional vector space, as desired.

$\square$

Problem 3. Show we can put a differentiable structure on a tangent bundle of a differentiable manifold.

Solution: Let M be a differentiable n-manifold. Denote the tangent bundle as the set

$$TM = \{(p, v) \mid p \in M, v \in T_pM\}.$$

We'll show that TM itself, known as the **tangent bundle**, is itself a manifold.

Since M is differentiable, there exists a (maximal) differentiable structure $\{(U_\alpha), \varphi_\alpha\}$ with $\alpha \in \lambda$, an indexing set, with $\varphi_\alpha : U_\alpha \to M$ which satisfy the three properities required of a differentiable manifold.

Denote the coordinates of U_α by $(x_1^\alpha, \dots, x_n^\alpha)$, and suppose we denote $\left\{\frac{\partial}{\partial x_1^\alpha}, \dots, \frac{\partial}{\partial x_n^\alpha}\right\}$ as the basis for the tangent space induced by $\varphi_\alpha(U_\alpha)$. Now define the functions $\psi_\alpha : U_\alpha \times \mathbb{R}^n \to TM$ as

$$\psi_\alpha((x_1^\alpha, \dots, x_n^\alpha), (u_1, \dots, u_n)) = \left(\varphi_\alpha(x_1^\alpha, \dots, x_n^\alpha), \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i^\alpha}\right)$$

Observe that this map makes sense since $\varphi_\alpha(x_1^\alpha, \dots, x_n^\alpha)$ is of course a point p on M and $\sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i^\alpha}$ is a vector in T_pM. Hence the above tuple is in TM.

We must now show that our set of maps, $(U_\alpha \times \mathbb{R}^n, \psi_\alpha)$, establish that TM is a differentiable manifold. First observe that these maps are injective. Injectivity in the first coordinate is inherited from the injectivity of each φ_α. It is easy to see that injectivity is established in the second coordinates since

$$\sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i^\alpha} = \sum_{i=1}^{n} u_i' \frac{\partial}{\partial x_i^\alpha} \implies u_i = u_i', i = 1, 2, \dots, n.$$

Hence each ψ_α must be injective.

Now observe that

$$\bigcup_{\alpha \in \lambda} \psi_\alpha \left(U_\alpha \times \mathbb{R}^n\right) = \bigcup_{\alpha \in \lambda} \{(p, v) \mid p \in \varphi_\alpha(U_\alpha), v \in T_pM\} = TM.$$

This is because firstly $\bigcup_{\alpha\in\lambda}\varphi_\alpha(U_\alpha) = M$ and secondly since $(u_1,\dots,u_n)\in\mathbb{R}^n$ is allowed to vary, we see that

$$\begin{aligned}
\psi_\alpha(U_\alpha\times\mathbb{R}^n) &= \bigcup_{(x_1^\alpha,\dots,x_n^\alpha)\in U_\alpha} \psi_\alpha(\{(x_1^\alpha,\dots,x_n^\alpha)\}\times\mathbb{R}^n)\\
&= \bigcup_{(x_1^\alpha,\dots,x_n^\alpha)\in U_\alpha} \left\{\left(\varphi_\alpha(x_1^\alpha,\dots,x_n^\alpha), \sum_{i=1}^n u_i\frac{\partial}{\partial x_i^\alpha}\right) \mid (u_1,\dots,u_n)\in\mathbb{R}^n\right\}\\
&= \left\{(\varphi_\alpha(x_1^\alpha,\dots,x_n^\alpha), v) \mid (x_1^\alpha,\dots,x_n^\alpha)\in U_\alpha, v\in \operatorname{span}\left\{\frac{\partial}{\partial x_1^\alpha},\dots,\frac{\partial}{\partial x_n^\alpha}\right\}\right\}\\
&= \{(p,v)\mid p\in\varphi_\alpha(U_\alpha), v\in T_pM\}
\end{aligned}$$

since $\operatorname{span}\left\{\frac{\partial}{\partial x_1^\alpha},\dots,\frac{\partial}{\partial x_n^\alpha}\right\} = T_{\varphi_\alpha(x_1^\alpha,\dots,x_n^\alpha)}M$. Hence we are able to appropriately cover TM with our set of maps.

Now suppose $\psi_\alpha(U_\alpha\times\mathbb{R}^n)\cap\psi_\beta(U_\beta\times\mathbb{R}^n) = W$ is nonempty. Then $U_\alpha\cap U_\beta$ is nonempty, and therefore is an open set, so that $\psi_\alpha^{-1}(W) = U_\alpha\cap U_\beta\times\mathbb{R}^n$ is an open set.

Finally, consider $(p,v)\in W$. Then we have that

$$\begin{aligned}
(p,v) &= (\varphi_\alpha(x_1^\alpha,\dots,x_n^\alpha), d\varphi_\alpha(v_\alpha))\\
&= (\varphi_\beta(x_1^\beta,\dots,x_n^\beta), d\varphi_\beta(v_\beta))
\end{aligned}$$

for some $(x_1^\alpha,\dots,x_n^\alpha)\in U_\alpha, (x_1^\beta,\dots,x_n^\beta)\in U_\beta$, and $v_\alpha, v_\beta\in\mathbb{R}^n$. Now observe that

$$\begin{aligned}
\psi_\beta^{-1}\circ\psi_\alpha((x_1^\alpha,\dots,x_n^\alpha),v_\alpha) &= \psi_\beta^{-1}(\varphi_\alpha(x_1^\alpha,\dots,x_n^\alpha), d\varphi_\alpha(v_\alpha))\\
&= (\varphi_\beta^{-1}\circ\varphi_\alpha(x_1^\alpha,\dots,x_n^\alpha), d(\varphi_\beta^{-1}\circ\varphi_\alpha)(v_\alpha)).
\end{aligned}$$

But we already know that $\varphi_\beta^{-1}\circ\varphi_\alpha$ is differentiable. Hence, $\psi_\beta^{-1}\circ\psi_\alpha$ must also be differentiable. Thus we see that the tangent bundle of a differentiable manifold is also a differentiable manifold, as $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in\lambda}$ provides a differentiable stucture on TM.

$\square$

Problem 4. If M is a manifold and G is a group that acts discontinuously on M, Show M/G is a manifold (see page 23).

Solution: For this to work, we actually need M to be a *differentiable* manifold. This will show up later.

To show this, let p be a point of M and consider the neigborhood U of p such that $U \cap \varphi_g(U) = \varnothing$ for all nontrivial $g \in G$; a neighborhood which is guaranteed to exist since we suppose G acts discontinuously on M.

Since M is a differentiable manifold, pick a parameterization $x : V \to M$ such that $x(V) \subset U$. Then we see that $\pi \circ x : V \to M/G$ is an injective mapping. For if $\pi(p_1) = \pi(p_2)$ for some distinct $p_1, p_2 \in x(V)$, then $p_1 = g'p_2$ for some $g' \in G$. In this case, G then no longer acts discontinuously (as then $U \cap \varphi_{g'}(U) \neq \varnothing$). Hence this mapping $y = \pi \circ x : V \to M/G$ is injective.

As $y : V \to M/G$ covers M/G, we see must show that for any for any two analagous mappings $y_1 = \pi \circ x_1 : V_1 \to M/G$ and $y_2 = \pi \circ x_2 : V_2 \to M/G$, we have that $y_1(V_1) \cap y_2(V_2) \neq 0$ implies that $y_1^{-1} \circ y_2$ is differentiable.

Define

$$\begin{aligned} \pi_1 &= \pi\Big|_{x_1(V_1)} : x_1(V_1) \to M/G \\ \pi_2 &= \pi\Big|_{x_2(V_2)} : x_2(V_2) \to M/G. \end{aligned}$$

Suppose $y_1(V_1) \cap y_2(V_2) \neq \varnothing$. Then for $q \in y_1(V_1) \cap y_2(V_2)$, let $r = (\pi_2 \circ x_2)^{-1}(p)$. Let W be a neighborhood of r such that $(\pi_2 \circ x_2)(W) \subset y_1(V_1) \cap y_2(V_2)$. Then

$$y_1^{-1} \circ y_2\Big|_W = x_1^{-1} \circ \pi_1^{-1} \circ \pi_2 \circ x_2.$$

Note that x_1^{-1} and x_2 are necessarily differentiable; hence for $y_1^{-1} \circ y_2$ to be differentiable, we need $\pi_1^{-1} \circ \pi_2$ to be differentiable on the restriction of $x_2(W)$. To show this, suppose $p_2 = \pi_1^{-1} \circ \pi_2(p_1)$. Then $\pi_1(p_2) = \pi_2(p_1)$, so that p_1 and p_2 are in the same equivalence class in M/G. Therefore, $p_1 = gp_2$ for some element $g \in G$. However, we know that the only function which achieves this is the *unique* diffeomorphism $\varphi_g : M \to M$. Hence we see that

$$\pi_1^{-1} \circ \pi_2 = \varphi_g\Big|_{x_2(W)} = \varphi_g\Big|_{x_2(W)}$$

so that $\pi_1^{-1} \circ \pi_2$ is differentiable on the restriction of $x_2(W)$. Hence we see that

$$y_1^{-1} \circ y_2\Big|_W = x_1^{-1} \circ \pi_1^{-1} \circ \pi_2 \circ x_2$$

is differentiable, and that the mapping (V, y) provides a differenitable structure on M/G, as desired.

□