A “Coherent” Proof of Mac Lane’s Coherence Theorem

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Definition of a Category

A **category** $\mathcal{C}$ is mathematical dataset.

It consists of:

- a collection of "things" which we call **objects** and denote them as $A, B, C, \ldots$
- a collection of "arrows" *between* objects, which we call **morphisms** and denote them as $f : A \to B$
- a **composition** operator $\circ$ acting on morphisms.

such that a few rules hold. Details are not important, and we demonstrate a few examples.

Some Familiar Categories

Sets form a category called **Set**.

Objects. All Sets

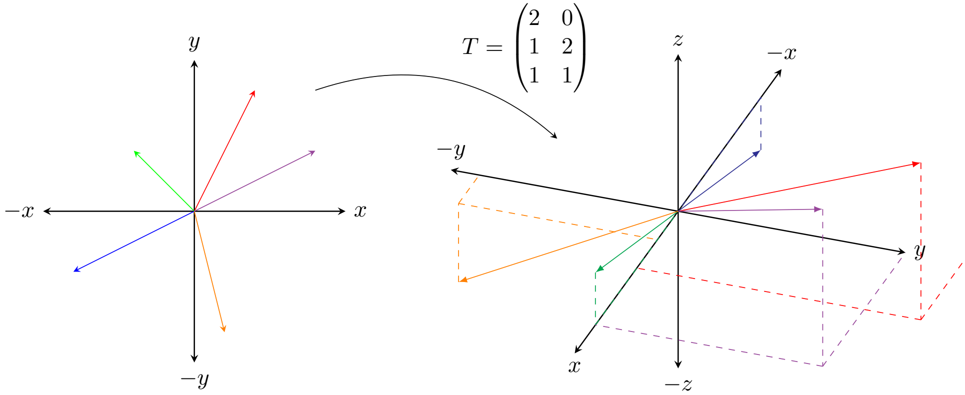
Morphisms. Functions $f : X \to Y$ between sets.

Some Familiar Categories

Vector spaces over a field k form a category called $\mathbf{Vect}_k$.

Objects. All vector spaces V over k.

Morphisms. Linear transformations $T : V \to W$.

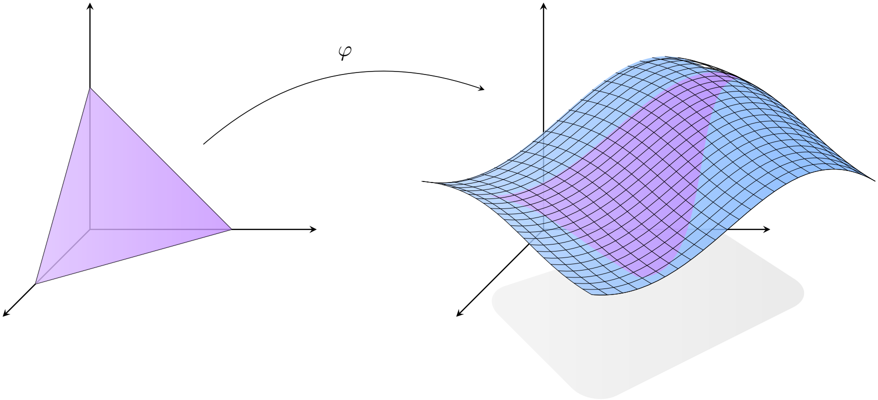


Some Familiar Categories

Topological spaces form a category called **Top**.

Objects. All topological spaces X.

Morphisms. Continuous functions $\varphi : X \to Y$.



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- Algebraic Geometry, Complex Analysis use **sheaves, stalks, and germs** which are now seen to be categorical concepts
- Algebraic Topology and Differential Geometry use **co/homology theories** and **abelian categories**
- Knot Theory, areas of theoretical physics use **monoidal categories**

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- These categories are naturally equipped with a way to
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  - combine morphisms to create new morphisms
- We now look at two examples: **Set** and $\mathbf{Vect}_k$

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$$\lambda_A : \{\bullet\} \times A \xrightarrow{\sim} A \qquad (\bullet, a) \mapsto a$$
$$\rho_A : A \times \{\bullet\} \xrightarrow{\sim} A \qquad (a, \bullet) \mapsto a$$

$\mathbf{Vect}_k$: Another "Category with Multiplication"

- $\mathbf{Vect}_k$ has $\otimes$, the tensor product: For vector spaces U, V with bases $\{e_i\}_{i\in I}, \{f_j\}_{j\in J}$, the vector space $U \otimes V$ has basis $\{e_i \otimes f_j\}$

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If $\{g_k\}_{k \in K}$ is the basis of W, the isomorphism is

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- For any vector space V over k, we have $k \otimes V \cong V \cong V \otimes k$. The isomorphisms are

$$\lambda_V : k \otimes V \xrightarrow{\sim} V \qquad 1 \otimes e_i \mapsto e_i$$
$$\rho_V : V \otimes k \xrightarrow{\sim} V \qquad e_i \otimes 1 \mapsto e_i$$

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  - For all A, B, C, an isomorphism $\alpha : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$

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  - Some product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which can create new objects
  - For all A, B, C, an isomorphism $\alpha : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$
  - Some special object I such that $A \otimes I \cong I \otimes A \cong A$ given by two "throw-away" isomorphisms
$$\lambda_A : I \otimes A \xrightarrow{\sim} A$$
$$\rho_A : A \otimes I \xrightarrow{\sim} A$$

Monoidal Categories

These are all what motivate the following definition.

A **monoidal category** $\mathcal{M} = (\mathcal{M}, \otimes, I)$ is a category $\mathcal{M}$ equipped with a "product" $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, an identity object I, and three natural isomorphisms

$$\begin{aligned}
\alpha_{A,B,C} &: A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C && \textbf{(Associator)}\\
\lambda_A &: I \otimes A \xrightarrow{\sim} A && \textbf{(Left Unit)}\\
\rho_A &: A \otimes I \xrightarrow{\sim} A && \textbf{(Right Unit)}
\end{aligned}$$

subject to some **coherence conditions**.

Monoidal Category Axioms

Coherence Condition #1. For all objects A, B in $\mathcal{M}$, the diagram commutes.

$$\begin{tikzcd} A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\ & A \otimes B & \end{tikzcd}$$

Monoidal Category Axioms

Coherence Condition #2. For all objects A, B, C, D in $\mathcal{M}$, the diagram commutes.

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A \otimes B, C, D}"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "\alpha_{A,B,C \otimes D}"] \arrow[d, "1_A \otimes \alpha_{B,C,D}"'] & & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A, B \otimes C, D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes 1_D"']
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- Using the monoidal product $\otimes$, we may use 3 objects A, B, C to generate 2 new objects

$$A \otimes (B \otimes C) \quad (A \otimes B) \otimes C.$$

- Using 4 objects, we can generate 5 new objects

$$A \otimes (B \otimes (C \otimes D)), \quad A \otimes ((B \otimes C) \otimes D), \quad ((A \otimes B) \otimes C) \otimes D$$
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- Using 5 objects, we can generate 14 new objects.

Coherence for Three Objects

- **Q:** Are the two objects $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ isomorphic?

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- **A:** They better be! And they are, using the isomorphism:

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

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- **A:** They better be! And they are, using the isomorphism:

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- We can therefore unambiguously write $A \otimes B \otimes C$.

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- Your favorite way is $(A \otimes (B \otimes C)) \otimes D$
- **Q:** Are these isomorphic?
- **A:** Yes! We have the isomorphism below.

$$
\begin{tikzcd}
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha^{-1}"'] & & \\
A \otimes (B \otimes (C \otimes D)) \arrow[r, "1\otimes\alpha"] & A \otimes ((B \otimes C) \otimes D) \arrow[r, "\alpha"] & (A \otimes (B \otimes C)) \otimes D
\end{tikzcd}
$$

Coherence for Four Objects

Suppose we want to multiply together objects A, B, C, D.

- My favorite way to do it is $(A \otimes B) \otimes (C \otimes D)$
- Your favorite way is $(A \otimes (B \otimes C)) \otimes D$
- However, I also have the isomorphism

$$(A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha} ((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha^{-1} \otimes 1} (A \otimes (B \otimes C)) \otimes D$$

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- This is troubling. We have potentially *different* isomorphisms!

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- Your favorite way is $(A \otimes (B \otimes C)) \otimes D$
- This is troubling. We have potentially *different* isomorphisms! To be coherent, we need the diagram below to commute.

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[d, "1 \otimes \alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[l, "\alpha^{-1}"'] \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \arrow[d, "\alpha^{-1} \otimes 1"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D
\end{tikzcd}$$

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- **Q:** We could ask for this diagram to commute, but what about the other ways of multiplying A, B, C, D?

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- **Q:** We could ask for this diagram to commute, but what about the other ways of multiplying A, B, C, D?
- **A:** To satisfy all cases, it suffices to require the diagram below to commute.

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha_{A,B,C\otimes D}"] \arrow[d, "1_A \otimes \alpha_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A\otimes B,C,D}"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes 1_D"']
\end{tikzcd}$$

Hence, the pentagon axiom.

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- Commutativity of the pentagon allows me, up to isomorphism, to unambiguously write “$A \otimes B \otimes C \otimes D$”.

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- **Q:** Okay, awesome! How about the 14 objects generated by A, B, C, D, E? Are they all isomorphic? Can we also unambiguously write "$A \otimes B \otimes C \otimes D \otimes E$"?

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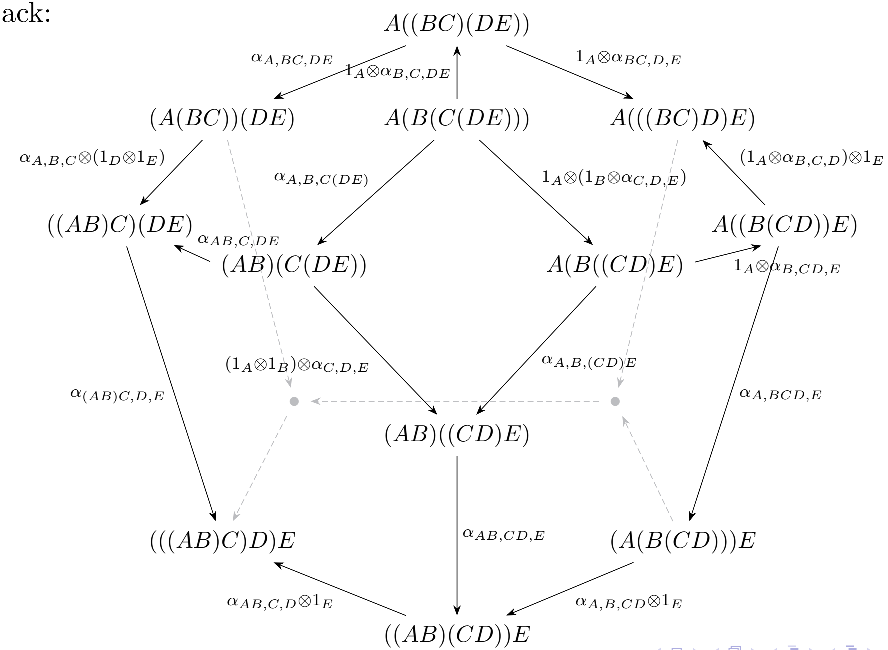
- **Q:** Okay, awesome! How about the 14 objects generated by A, B, C, D, E? Are they all isomorphic? Can we also unambiguously write "$A \otimes B \otimes C \otimes D \otimes E$"?
- **A:** Yes they are all isomorphic. They assemble into the following diagram.

Front:

$$\begin{tikzcd}[column sep=small]
 & & & A((BC)(DE)) \arrow[dll, "1_A\otimes\alpha_{BC,D,E}"'] \arrow[drr, "\alpha_{A,BC,DE}"] & & & \\
 & A(((BC)D)E) \arrow[ddr, "\alpha_{A,B(CD),E}"] & & & & (A(BC))(DE) \arrow[ddl, "\alpha_{A(BC),D,E}"'] \arrow[dr, "\alpha_{A,B,C}\otimes(1_D\otimes 1_E)"] & \\
A((B(CD))E) \arrow[ur, "(1_A\otimes\alpha_{B,C,D})\otimes 1_E"] \arrow[ddr, "\alpha_{A,BCD,E}"'] & & & & & & ((AB)C)(DE) \arrow[ddl, "\alpha_{(AB)C,D,E}"] \\
 & & (A((BC)D))E \arrow[rr, "\alpha_{A,BC,D}\otimes 1_E"] & & ((A(BC))D)E \arrow[dr, "(\alpha_{A,B,C}\otimes 1_D)\otimes 1_E"'] & & \\
 & (A(B(CD)))E \arrow[ur, "(1_A\otimes\alpha_{B,C,D})\otimes 1_E"'] \arrow[drr, "\alpha_{A,B,CD}\otimes 1_E"'] & & & & (((AB)C)D)E & \\
 & & & ((AB)(CD))E \arrow[urr, "\alpha_{AB,C,D}\otimes 1_E"'] & & &
\end{tikzcd}$$

Back: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- It is *not* an axiom of monoidal categories for the previous diagram to commute
- Any two ways of multiplying 5 objects are connected via many potentially different paths
- What about the objects generated by multiplying 6 objects, or 7 objects, and so on?

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Theorem (Mac Lane, 1970)

Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. For each object A, there exists a strict monoidal functor $\Psi_A : \mathcal{W} \to \mathcal{M}$, where $\mathcal{W}$ is the free monoidal category on a single object.

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Theorem (Mac Lane, 1970)

Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. For each object A, there exists a strict monoidal functor $\Psi_A : \mathcal{W} \to \mathcal{M}$, where $\mathcal{W}$ is the free monoidal category on a single object.

Rough Translation:

All diagrams built in a canonical way from α, λ, ρ commute.

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- nLab has an abandoned, incomplete attempt to explain the proof

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- No vague language is used, and readers understand precisely what the theorem statement is saying
- Each nontrivial step is confirmed
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- Save future researchers weeks of study in understanding the theorem
- Prevent confusion and misinformation about the theorem (more of an issue outside professional mathematics, e.g., Math SE, Wikipedia)
- Allow researchers to understand and reuse Mac Lane's proof technique, as it is extremely clever and has been used in other settings

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- Also produced clean proofs of G.M. Kelly's work in simplifying Mac Lane's original monoidal category axioms
- Also got to study abelian categories and persistence homology through Vin de Silva!

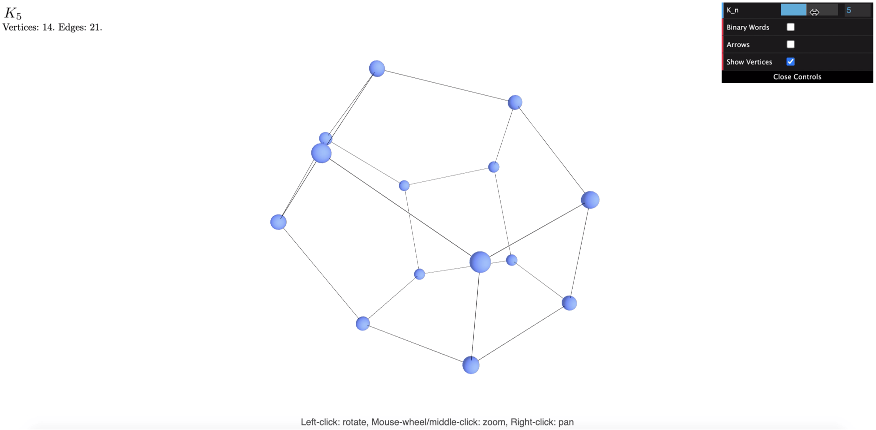
Outcome of Thesis: Encoded a monoidal category

- Encoded a monoidal category on one object in Python (an important category which appears extensively in Mac Lane's proof) to ease computations

(requries Adobe Acorbat)

Outcome of Thesis: Associahedron App

- Encoded the associahedra diagrams in a `three.js` web application for interactive viewing



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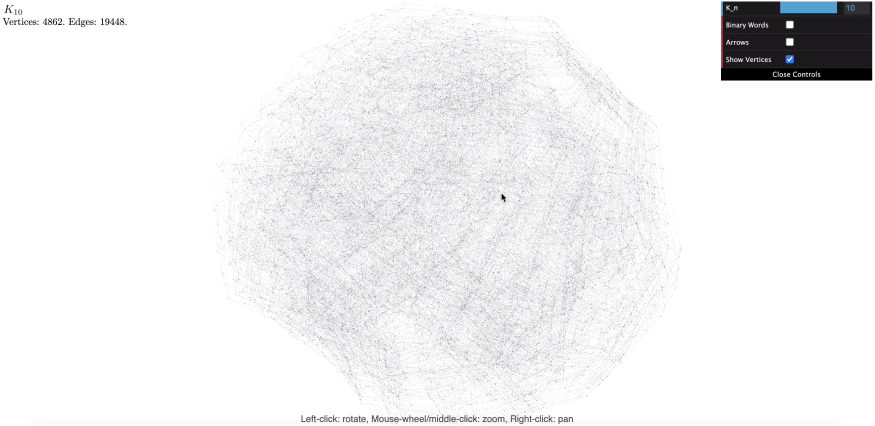
Outcome of Thesis: Associahedron App

- For fun, here's K_{10} with 4862 vertices and 19448 edges:



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Thank you for listening!