Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.]

2.7. (a) $\dim \mathbf{P}^n = n$.
(b) If $Y \subseteq \mathbf{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \bar{Y}$. [*Hint*: Use (Ex. 2.6) to reduce to (1.10).]

2.8. A projective variety $Y \subseteq \mathbf{P}^n$ has dimension $n - 1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbf{P}^n$.

2.9. *Projective Closure of an Affine Variety.* If $Y \subseteq \mathbf{A}^n$ is an affine variety, we identify $\mathbf{A}^n$ with an open set $U_0 \subseteq \mathbf{P}^n$ by the homeomorphism φ_0. Then we can speak of $\bar{Y}$, the closure of Y in $\mathbf{P}^n$, which is called the *projective closure* of Y.
(a) Show that $I(\bar{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).
(b) Let $Y \subseteq \mathbf{A}^3$ be the twisted cubic of (Ex. 1.2). Its projective closure $\bar{Y} \subseteq \mathbf{P}^3$ is called the *twisted cubic curve* in $\mathbf{P}^3$. Find generators for $I(Y)$ and $I(\bar{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\bar{Y})$.

2.10. *The Cone Over a Projective Variety* (Fig. 1). Let $Y \subseteq \mathbf{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbf{A}^{n+1} - \{(0, \ldots, 0)\} \rightarrow \mathbf{P}^n$ be the map which sends the point with affine coordinates $(a_0, \ldots, a_n)$ to the point with homogeneous coordinates $(a_0, \ldots, a_n)$. We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \ldots, 0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbf{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \ldots, x_n]$.
(b) $C(Y)$ is irreducible if and only if Y is.
(c) $\dim C(Y) = \dim Y + 1$.
Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbf{P}^{n+1}$. This is called the *projective cone* over Y.

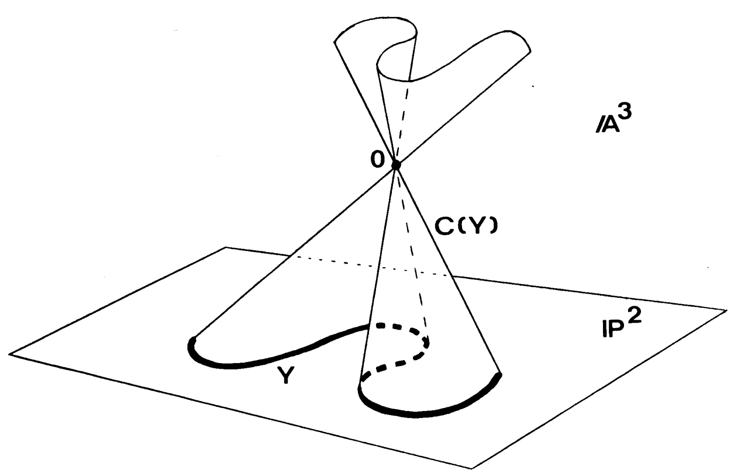

Figure 1. The cone over a curve in $\mathbf{P}^2$.