Groups, Rings, and Modules

Chapter 1

Groups

1.1 Definitions

Definition 1.1.1. A **group** is a set G, equipped with a binary operation $\cdot : G \times G \longrightarrow G$ which satisfies the three following axioms:

(G1) The operation $\cdot$ is **associative**. Specifically, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$

(G2) There exists an element $e \in G$, known as the **identity** of G, such that $a \cdot e = a = e \cdot a$ for all $a \in G$

(G3) For each $a \in G$, there exists an element $a^{-1} \in G$, known as the **inverse** of a, such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

In this case, we say that G **is a group under** $\cdot$, and denote this as $(G, \cdot)$.

Remark 1.1.2.

- Note that in order to have a group, we require a set G, and a binary operation. Hence we cannot say "the set X is a group." This makes no sense, although as we will see, sometimes this is written when the operation is obvious or stated.
- It goes without really explicitly stating that $\cdot$ must also be **closed**; that is, it cannot map elements anywhere outside of G. This is due to our definition that $\cdot : G \times G \longrightarrow G$. That is, the codomain, or range, is always within G.

Definition 1.1.3. Let G be a group. Suppose that, for any two $g, h \in G$ we have

$$g \cdot h = h \cdot g$$

then G is an **abelian** or **commutative** group.

Notation. First observe that we use $\cdot$ in our definition of a group. This is unfortunately the same notation used in modern-day numerical multiplication (i.e., $5 \cdot 3 = 15$). Here, this is not the case; it's just a placeholder for *some* operator. You'll get used to this as you go in group theory.

In group theory, we denote the multiplication of group elements g and h as $g \cdot h$. However, if the operator $\cdot$ is already understood, then we will just write gh. If there is possibility for confusion (i.e., perhaps in a situation in where there are *two* operators in play) we will be more explicit and clear about our operator. Buts for the most part we'll just write gh.

Example. Consider the set $\mathbb{Z}$, and let the operator on the elements of $\mathbb{Z}$ simply be standard addition. This is a group, which we'll demonstrate by showing that this set equipped with the addition operator satisfy the three axioms.

(1) Closed. From elementary mathematics, we know that if $m, n \in \mathbb{Z}$ then $m + n \in \mathbb{Z}$. Thus this set is closed under addition.

(2) Associativity. Observe that for any integers n, m and p,

$$n + (m + p) = (n + m) + p.$$

This is just a basic fact of elementary arithmetic.

(3) Identity. Observe that for any $n \in \mathbb{Z}$,

$$n + 0 = 0 + n = n.$$

Thus 0 is an appropriate choice of an identity.

(4) Inverse. Consider any $n \in \mathbb{N}$. Observe that (1) $-n \in \mathbb{Z}$ and (2)

$$n + (-n) = (-n) + n = 0.$$

Thus every element has an inverse. Note we specified that $-n \in \mathbb{Z}$, as we wanted to emphasize that not only $-n$ exists, but $-n$ is *in* our set $\mathbb{Z}$.

With all three properties satisfied, we have that $\mathbb{Z}$ is a group with addition. More generally, we'd say that $\mathbb{Z}$ is a group under addition, and denote it as $(\mathbb{Z}, +)$.

Note that $\mathbb{Z}$ is not a group under multiplication. Suppose we try to say it is one anyways. Then the most natural step is to first note that 1 should be our identity. After all, for any $n \in \mathbb{Z}$, $1 \cdot n = n \cdot 1 = n$. If we then consider any $m \in \mathbb{Z}$, what is the inverse of m? We'd need a $p \in \mathbb{Z}$ such that

$$m \cdot p = p \cdot m = 1.$$

This has no solution if $m > 1$; for example, there is no integer p such that $5 \cdot p = 1$. In fact, $p = \frac{1}{5}$, can only satisfy this in the set of real numbers, but $\frac{1}{5}$ is not in $\mathbb{Z}$. Thus $\mathbb{Z}$ is not a group under multiplication, but it is a group under addition.

We reiterate again that these two examples highlight the fact that a group requires two things: a set, and a well-defined operator that acts on the set.

It turns out that $\mathbb{Q} \setminus \{0\}$ (not including zero) is a group under multiplication. Also, $\mathbb{R}$ is a group under multiplication and a group under addition. We wont show this (it's not much work) and will instead move onto more interesting examples which capture how versatile the definition of a group really is.

Example. Consider the set of $n \times n$ matrices with determinant 1 and entries in $\mathbb{R}$, where the multiplication is standard matrix multiplication. This is known as the **Special Linear Group** and is denoted $SL_n(\mathbb{R})$. We'll show that this set is a group.

(1) Closed. First we need to check if this operation is closed. That is, for $A, B \in SL_n(\mathbb{R})$, is it true that $AB \in SL_n(\mathbb{R})$?

We know products of $n \times n$ matrices give back $n \times n$ matrices. So the real question is, if two matrices both have determinant 1, will their product necessarily be a matrix whose determinant is also 1? The answer is yes. From linear algebra, we know that

$$\det(AB) = \det(A)\det(B).$$

Now if A, B have determinant 1,

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1.$$

Therefore, $AB \in SL_n(\mathbb{R})$, since AB is $n \times n$ and it has determinant 1.

(2) Associativity. For matricies $A, B, C \in SL_n(\mathbb{R})$, we know from linear algebra that

$$(AB)C = A(BC).$$

That is, matrix multiplication is associative.

(3) Identity. Naturally, the identity matrix I serves as our group identity. This is because for any $A \in SL_n(\mathbb{R})$, $AI = IA = A$.

(4) Inverses. For any $A \in SL_n(\mathbb{R})$, $\det(A) = 1$. Specifically observe that $\det(A) \neq 0$. Therefore by the invertible matrix theorem, A has an inverse element A^{-1} such that $AA^{-1} = A^{-1}A = I$. But the real question is: is it true that $A^{-1} \in SL_n(\mathbb{R})$?

To answer this, observe that $AA^{-1} = I$ and that $\det(I) = 1$. Thus

$$\det(AA^{-1}) = \det(I) = 1.$$

However, since $\det(AA^{-1}) = \det(A)\det(A^{-1})$,

$$\det(AA^{-1}) = 1 \implies \det(A)\det(A^{-1}) = 1.$$

But $A \in SL_n(\mathbb{R})$, so $\det(A) = 1$. Therefore, $\det(A)^{-1} = 1$, so that $A^{-1} \in SL_n(\mathbb{R})$.

Thus $SL_n(\mathbb{R})$ does form a group.

You may be wondeirng: Why did we focus on matrices with determinat 1? Why not consider all matrices in general?

At first, the set of all $n \times n$ matrices with coefficients in $\mathbb{R}$, denoted by $M_{n\times n}(\mathbb{R})$, may seem like a group. But recall from linear algebra that some matrices are not invertible under matrix multiplication. Thus we can't form a group since some matrices would just not have an inverse.

However, consider the following set:

$$G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} - \{0\} \right\}.$$

Clearly, these elements are not invertible matrices under matrix multiplication. However, we can still form a group out of this!

Naturally, we'd want to make the identity element as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, since this is the natural identity when it comes to matrix multiplication. However, this isn't in the above set, which may make you wonder if this really is a group.

(1) Closed. First we show that this is closed. Let

$$A = \begin{pmatrix} a & a \\ a & a \end{pmatrix}, B = \begin{pmatrix} b & b \\ b & b \end{pmatrix}$$

where $a, b \in \mathbb{R} - \{0\}$. Now observe that

$$AB = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix}.$$

Since $2ab \in \mathbb{R} - \{0\}$, we see that $AB \in G$. Hence, the set is closed.

(2) Associativity. Again, from linear algebra, we already know that matrix multiplication is associative.

(3) Identity. What should we make our identity? One can realize that

$$e = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

suffices for an identity. That is, for any $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$, we see that

$$Ae = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = A$$

and

$$eA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = A.$$

Hence $Ae = A = eA$, so that it works correctly as an identity.

(4) Inverses. For any $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$, we can write $A^{-1} = \begin{pmatrix} \frac{1}{2a} & \frac{1}{2a} \\ \frac{1}{2a} & \frac{1}{2a} \end{pmatrix}$. Note that since $a \in \mathbb{R} - \{0\}$ implies that $\frac{1}{2a} \in \mathbb{R} - \{0\}$, so that $A^{-1} \in G$. Now we see that

$$AA^{-1} = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \begin{pmatrix} \frac{1}{2a} & \frac{1}{2a} \\ \frac{1}{2a} & \frac{1}{2a} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$A^{-1}A = \begin{pmatrix} \frac{1}{2a} & \frac{1}{2a} \\ \frac{1}{2a} & \frac{1}{2a} \end{pmatrix} \begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus we see that for every $A \in G$, there exists an element $A^{-1} \in G$ such that $AA^{-1} = e = A^{-1}A$. With all four axioms satisfied, we see that G forms a group.

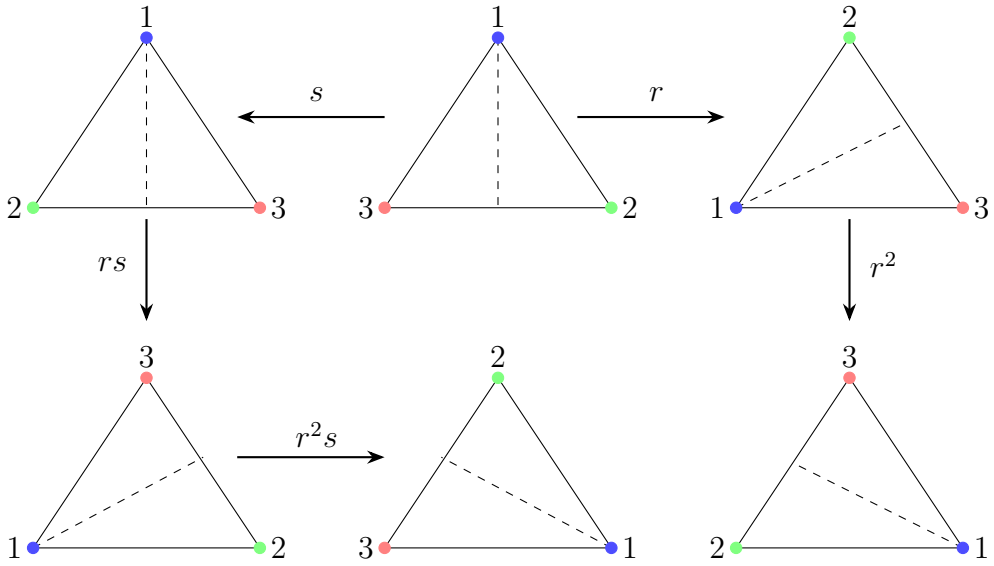
Example. Consider an equilateral triangle. The set of rigid transformations which preserve

symmetry form a group. This is actually just a special case of the more general **Dihedral Group** D_{2n}.

There are many ways we can think of transforming a triangle. But we can always break them down into rotations, denoted by r, of 120°, and reflections across a a diagonal of symmetry, denoted by s. It turns out that the full list of unique rotations we can come up with are

$$\{e, r, r^2, s, rs, r^2 s\}$$

which we can illustrate visually with the triangles below.



Theorem 1.1.4. Let $(G, \cdot)$ be a group. Then the following hold:

1. The identity $e \in G$ is unique
2. The inverse $g^{-1} \in G$ is unique for every $g \in G$.
3. For any $g \in G$, $(g^{-1})^{-1} = g$.
4. Let $g, h \in G$. Then $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$.
5. Let $g_1, g_2, \ldots, g_n \in G$. The product $g_1 \cdot g_2 \cdot \ldots \cdot g_n$ is independent of its bracketing.
6. Let $g, h \in G$. There always exist x, y such that $g \cdot x = h$ and $h \cdot y = g$.

Proof:

1. Suppose there exists another identity element f, different from e, such that $g \cdot f = f \cdot g = g$ for all $g \in G$. Then

$$e = e \cdot f = f$$

so that $e = f$. Therefore, the identity element is unique.

2. Suppose h_1 and h_2 are both inverses of $g \in G$. Then by definition, $h_2 \cdot g = e = g \cdot h_2$ and $h_1 \cdot g = e = g \cdot h_1$. Therefore,

$$h_1 = (h_2 \cdot g) \cdot h_1 = \underbrace{h_2 \cdot (g \cdot h_2)}_{\text{by associativity of G}} = h_2 \cdot e = h_2.$$

Thus $h_1 = h_2$, so that the inverse of g os unique.

3. Observe that for any $g \in G$,

$$g^{-1} \cdot (g^{-1})^{-1} = e$$

by defintion. Multiplying on the left by g on both sides of the equation, we get

$$g \cdot (g^{-1} \cdot (g^{-1})^{-1}) = g \cdot e \implies (g \cdot g^{-1}) \cdot (g^{-1})^{-1} = g$$

by associativity. Since $g \cdot g^{-1} = e$, this then leads to

$$e \cdot (g^{-1})^{-1} = g \implies (g^{-1})^{-1} = g$$

as desired.

4. Note that $(g \cdot h)^{-1} \cdot (g \cdot h) = e$. Therefore,

$$(g \cdot h)^{-1} \cdot (g \cdot h) = e \implies (g \cdot h)^{-1} \cdot g = h^{-1} \implies (g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$$

by first multiplying on the right by g^{-1} and then by h^{-1}, which proves the formula.

5. We can demonstrate this by induction. First write our proposition as

$$P(n) = \begin{cases} \text{For any } g_1, g_2, \ldots, g_n \in G \text{ we have that} \\ g_1 \cdot g_2 \cdots g_n \text{ is independent of its bracketing.} \end{cases}$$

Base Case. For the base case $n = 1$, there is nothing to check.

Inductive Step. Now suppose that $P(n)$ is true for all positive integers $n \leq n_0$. Then let $g_1, g_2, \ldots g_{n+1} \in G$ and consider

$$g_1 \cdot g_2 \cdot \cdots \cdot g_{n+1}.$$

Observe that we clearly have that

$$g_1 \cdot g_2 \cdot \cdots \cdot g_{n+1}. = (g_1 \cdot g_2 \cdot \cdots \cdot g_i) \cdot (g_{i+1} \cdot \cdots \cdot g_{n+1}).$$

for all $1 \leq ilen + 1$. Hence we can apply the inductive hyptohesis to each of the subproducts $(g_1 \cdot g_2 \cdot \cdots g_i)$ and $(g_{i+1} \cdot \cdots g_{n+1})$ generated in each case. Since this exhausts all possible subproducts, and the values do not change by our inductive hypothesis, we see that $P(n+1)$ is true. Hence $P(n)$ holds for all $n \in \mathbb{N}$.

6. Observe that if we have the equation $g \cdot x = h$, then we can multiply both sides on the right by g^{-1} to observe that

$$(g^{-1} \cdot g) \cdot x = g^{-1} \cdot h \implies x = g^{-1} \cdot h.$$

Since x is the product of elements of G (namely, $g^{-1} \cdot h$) and because G is closed under $\cdot$, we have that $x \in G$. Thus a solution exists in G. The proof for the existence of $y \in G$ such that $h \cdot y = g$ is exactly the same.

■

In our study of group theory, many of the groups we'll deal with will actually turn out to be finite. We'll also be interested in breaking down the structures of finite groups (a lot of things can happen). A couple things should be noted about finite groups.

Consider $g \in G$, where G is a **finite group**. Since G must be closed under its product, we note that $g^2 \in G$, $g^3 \in G$, and so on. That is, all powers of g must be in G. But since G is *finite*, there must exist some $m \in \mathbb{N}$ such that $g^m = e$. If not, you could keep raising the power of g, and keep getting new elements. Since you'd never come back to e, you could then generate an infinite set $\{g, g^2, g^3, g^4, \dots\}$ entirely contained in G. But this would imply G is infinite, which it isn't.

Definition 1.1.5. Let G be a group. The **order of an element** $g \in G$ is the smallest integer n such that $g^n = e$.

In addition, if G is a finite group, then we can also talk about the **order of a group** $|G|$, which we define as the number of elements within the group.

The order is denoted as $|g|$; thus we'd say that $|g| = n$ if g has order n. If G is infinite, it may be possible that $|g| = \infty$. On the topic of order, we define that $g^0 = e$.

Subgroups.

Definition 1.1.6. Let G be a group, and consider a subset H of G. We define H to be **subgroup** of G if H is also a group.

The definition is exactly what it sounds like: H is a subgroup if $H \subset G$ and H is a group. You might note that the definition is clear, but determining if a set is a subgroup of G sounds like a lot of work. Fortunately there's the subgroup test.

Theorem 1.1.7. Let H be nonempty and suppose $H \subset G$. Then H is a subgroup if and only if for all $x, y \in H$, $xy^{-1} \in H$.

If H is a subgroup, we usually write $\mathbf{H \leq G}$ if we are trying to be concise.

Proof: ($\implies$) Suppose $H \leq G$. Then since H is a group, for any $x, y \in H$, $xy^{-1} \in H$ since it is closed under multiplication of its elements. This proves the forward direction.

($\impliedby$) Suppose H is nonempty and $H \subset G$ such that for all $x, y \in H$, $xy^{-1} \in H$. We just need to prove H is a group. We already know group multiplication is an associative, binary operation on G, so it is still associative on elements of H. Thus we only need to prove closedness, existence of identity and inverses.

By the definition of H, for all $x, y \in H$, $xy^{-1} \in H$.

Identity. Let $x \in H$. Then clearly $xx^{-1} = e \in H$. Thus H has the identity.

Inverses. Since $x, e \in H$, we see that $ex^{-1} = x^{-1} \in H$. Thus for all $x \in H$, $x^{-1} \in H$.
Closedness. Now let $y \in H$; hence, $y^{-1} \in H$, as just proven. Then $x(y^{-1})^{-1} = xy \in H$, so that H is closed under multiplication of its elements.
Therefore H is (1) a group and (2) a subset of G so that $H \leq G$, as desired. ■

It turns out the intersection of two subgroups is still a subgroup. In fact, the arbitrary intersection of subgroups always produces a subgroup.

Theorem 1.1.8. Let G be a group and $\{H_\alpha\}_{\alpha\in\lambda}$ be a family of subgroups of G. Then the set $H = \bigcap_{\alpha\in\lambda} H_\alpha$ is a subgroup of G.

Proof: First, observe that

$$H = \bigcap_{\alpha\in\lambda} H_\alpha$$

is nonempty. This is because each $H_\alpha \leq G$ and thus the identity of G is contained in each H_α for all $\alpha \in \lambda$. So the identity is in H as well.

Thus let $x, y \in H$. Then $x, y \in H_\alpha$ for all $\alpha \in \lambda$. Since each H_α is a group, y^{-1} exists and is contained in each H_α. Hence, $xy^{-1} \in H_\alpha$ for all $\alpha \in \lambda$, so we have that $xy^{-1} \in H$. Therefore, we see by the subgroup text that $H \leq G$. ■

With the basic properties of a group introduced, we now introduce two more group definitions.

Definition 1.1.9. Let G be a group and $S \subset G$. The **centralizer** of S in G is defined to be the set $C_G(S)$

$$C_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}.$$

In the case where G is abelian, we $C_G(S) = G$ for any nonempty subset S of G. This definition is close to the **center of a group**, which is as follows.

Definition 1.1.10. Let G be a group. Then the **center of a group** G is defined as

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

In this case, we note that $C_G(G) = Z(G)$ and if G is abelian $Z(G) = G$. Finally, we introduce the definition of the normalizer.

Definition 1.1.11. Let G be a group and $S \subset G$. The **normalizer** of S in G is defined as

$$N_G(S) = \{g \in G \mid gS = Sg\}$$

The centralizer and the normalizer are closely related definitions. However, these two concepts highlight the important distinction one must understand between term-by-term equality and set equality. Firstly, we can think of $C_G(S)$ as all $g \in G$ which commutes with each and every single element of S. On the other hand, if $g \in N_G(S)$, it is not necessarily true that $gs = sg$ for all $s \in S$. The only requirement is that gS creates the same set as Sg. Of course, one way to do this is if

$gs = sg$ for all $s \in G$. In that case, $gS = Sg$. But there are other ways to do this, so this definition is more versatile than $C_G(S)$.

One interesting fact is that $C_G(S)$ and $N_G(S)$ are subgroups of G for any $S \subset G$.

Theorem 1.1.12. Let G be a group and $S \subset G$. Then $C_G(S)$ and $N_G(S)$ are both subgroups of G.

Note: if we let $S = G$, we see that $C_G(S) = Z(G)$. Therefore, an immediate corollay of this theorem is that $Z(G)$ is also a subgroup of G!

Proof: Let G be a group and $S \subset G$. To show that $C_G(S) \leq G$, we can use the subgroup test.

Nonempty. First we have to show the set is nonempty. But note that for any S, $e \in C_G(S)$ since $gs = sg$ for any $s \in S$.

Inverses. We now show that if $x, y \in C_G(S)$ then so is xy^{-1}. We know that for all $s \in S$, $xs = sx$ and $ys = sy$. Therefore $s = y^{-1}sy$ and $s = ysy^{-1}$ by solving for s in the last equation. Plugging this into the first equation with x, we get

$$xs = sx \implies x(y^{-1}sy) = (ysy^{-1})x \implies xy^{-1}sy = ysy^{-1}x.$$

Multiplying both sides on the right by y^{-1} leads to

$$xy^{-1}s = ysy^{-1}xy^{-1} \implies xy^{-1}s = syy^{-1}xy^{-1} \implies xy^{-1}s = sxy^{-1}$$

where in the second step we used the fact that $ys = sy$. Thus $xy^{-1} \in C_G(S)$, so by subgroup test we have that $C_G(S) \leq G$.

The proof for $N_G(S)$ is the exact same; simply replace s with S. ■

1.2 Permutation Groups.

One of the most important and well-known types of groups are the permutation groups, which we introduce formally as follows.

Consider a finite set of elements $X = \{1, 2, \ldots, n\}$. We define a **permutation** to be a reordering of the elements of X. More formally, a **permutation** is a bijective mapping $\sigma : X \longrightarrow X$, similar to one that follows.

How can we represent this information? We generally don't use sets to represent permutations, since sets don't care about order. That is, $\{1, 2, 3\} = \{3, 2, 1\}$, etc.

Thus for a set $\{1, 2, \ldots, n\}$, we can represent a permutation σ of the set of elements as follows:

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

where we read this as 1 is assigned to $\sigma(1)$, 2 is assigned to $\sigma(2)$. For example, a permutation that just shifts the elements down the line is

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & 1 \end{pmatrix}.$$

That is, σ sends 1 to 2, 2 to 3 and eventually n to 1. Here we'll denote the set of all permutations of the set $\{1, 2, \ldots, n\}$, or more generally a set of n objects (since we can always enumerate objects with natural numbers) as S_n.

What is interesting about this is that if we define "multiplication" of elements of S_n to be function composition, then the **set of permutations of X form a group** which we show as follows.

Let $X = \{1, 2, \ldots, n\}$.

Closed. For any $\sigma_1, \sigma_2 \in S_n$, we see that $\sigma_2 \circ \sigma_1$ is (1) a composition of bijective functions and therefore is bijective and (2) a permutation of X. One way to think about the composition is that σ_1 rearranges X, and σ_2 rearranges X again. Thus $\sigma_2 \circ \sigma_1 \in S_n$.

Associativity. Associativity is obvious since function composition is associative.

Identity. Observe that the permutation $\sigma_e : X \longrightarrow X$ for which $\sigma_e(i) = i$ is technically a permutation of X. Therefore σ_e acts as the identity element in S_n.

Inverse. Consider a permutation $\sigma \in S_n$. Define σ^{-1} to be the function where if $\sigma(i) = j$, then $\sigma^{-1}(j) = i$. Then by construction, (1) σ^{-1} is a permutation of X and (2) $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = \sigma_e$. Thus S_n contains inverses and composition of the inverses returns σ_e, the identity.

Proposition 1.2.1. For all $n \geq 1$, $|S_n| = n!$

Proof: This is counting the number of ways to rearrange a set of size n, which we know from combinatorics to simply be $n!$ ■

Now that we know that S_n is a group, we'll study the properties of this group.

Recall earlier our notation for representing a permutation $\sigma \in S_n$:

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

This notation sucks, since it includes more information than we actually need to. For instance, the top row is always going to be the same.

A better way to write this is through *cycle decomposition*, which we will soon define.

Definition 1.2.2. Let $\sigma \in S_n$ and suppose $X = \{1, 2, \ldots, n\}$. Suppose that there exists a subset $\{n_1, n_2, \ldots, n_k\}$ of X such that

$$\sigma(n_1) = n_2, \sigma(n_2) = n_3, \ldots, \sigma(n_k) = n_1.$$

Then $\{n_1, n_2, \ldots, n_k\}$ is called a **cycle**, and we denote this cycle as

$$\sigma = \begin{pmatrix} n_1 & n_2 & \cdots & n_k \end{pmatrix}.$$

We then read this as "$n_1 \longrightarrow n_2$, $n_2 \longrightarrow n_3, \ldots, n_k \longrightarrow n_1$".

Why do we care about cycles?
Well, consider an arbitrary cycle $\sigma = \begin{pmatrix} n_1 & n_2 & \cdots & n_k \end{pmatrix}$. Then again, $\sigma(n_1) = n_2, \sigma(n_2) = n_3, \ldots, \sigma(n_k) = n_1$. However, what this is really saying is that

$$\sigma(n_1) = n_2, \sigma^2(n_1) = n_3, \ldots, \sigma^{k-1}(n_1) = n_k, \sigma^k(n_1) = n_1.$$

However, also take a note to observe that

$$\sigma(n_2) = n_3, \sigma^2(n_2) = n_4, \ldots, \sigma^{k-1}(n_2) = n_1, \sigma^k(n_2) = n_2.$$

More generally, we see that the element $\sigma \in S_n$ has order n_k, which is why the cycle length is k.

We care about cycles since, given the fact that S_n is always a finite group, each of its elements will have finite order. Thus, in some way, we can always represent the elements of S_n in this form.

More definitions.
If $\begin{pmatrix} n_1 & n_2 & \cdots & n_k \end{pmatrix}$ and $\begin{pmatrix} n_1' & n_2' & \cdots & n_k' \end{pmatrix}$ share no elements in common, i.e.,

$$\{n_1, n_2, \ldots, n_k\} \cap \{n_1', n_2', \ldots, n_k'\} = \varnothing$$

then the cycles are defined as **disjoint cycles**.

Note that if $\sigma(i) = i$ for some $i \in X$, then this is technically a cycle and we represent the cycle as $\begin{pmatrix} i \end{pmatrix}$. In this case, we say that σ **fixes** i.

For example, suppose we have a permutation $\sigma \in S_5$ where $\sigma(1) = 2, \sigma(2) = 4, \sigma(4) = 2$. Then we have a cycle of length 4 and we denote this as

$$\begin{pmatrix} 1 & 2 & 4 \end{pmatrix}.$$

Since $\sigma \in S_5$, suppose further that $\sigma(3) = 5$ and $\sigma(5) = 3$. Then we see that we have another cycle,

disjoint with the previous cycle, and we write this one as

$$\begin{pmatrix} 3 & 5 \end{pmatrix}.$$

To write the entire permutation, we then can then express σ as

$$\sigma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 5 \end{pmatrix}$$

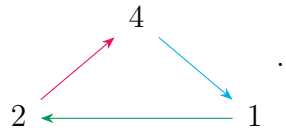
which gives us all the information we need to know on how σ rearranges the elements of X. Such a representation of a permutation is called a **disjoint cycle decomposition**. It will turn out that we can actually express *every* permutation $\sigma \in S_n$ in a product of disjoint cycles.

Remark. In general, 1-cycles are omitted in the representation of a disjoint cycle decomposition. Thus if we have a permutation $\sigma \in S_3$ such that $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(3) = 3$, then we would write this as

$$\sigma = \begin{pmatrix} 1 & 2 \end{pmatrix}.$$

Such a statement leads us to conclude that $\sigma(3) = 3$. And if $\sigma \in S_5$, we would furthermore conclude that not only $\sigma(3) = 3$, but also $\sigma(4) = 4$ and $\sigma(5) = 5$.

Nonuniqueness. One thing to note is that cycles are not unique. For example, we could have written the cycle $\begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$ as $\begin{pmatrix} 2 & 4 & 1 \end{pmatrix}$ or $\begin{pmatrix} 4 & 1 & 2 \end{pmatrix}$, since the other expressions still capture the fact that 1 is sent to 2, 2 is sent to 4, and 4 is sent to 1.



Note that the colors correspond to where the cycle starts. Clearly in the diagram, there are three ways to start the cycle, and hence why there are three nonunique representations for the cycle.
More generally, for any cycle $\begin{pmatrix} i_1 & i_2 & \cdots & i_n \end{pmatrix}$ we have that

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_n \end{pmatrix} = \begin{pmatrix} i_2 & i_3 & \cdots & i_n & i_1 \end{pmatrix} = \cdots = \begin{pmatrix} i_n & i_1 & \cdots & i_{n-1} \end{pmatrix}.$$

1.3 Homomorphism and Isomorphisms.

As with all mathematical objects, now that we have a well defined abstract concept (i.e., a group) we'll now be interested attempting to understand *mappings* between different groups. Mappings of abstract concepts simply helps mathematicians get a better sense of what they're dealing with, and most often provides new insight into understand their objects.

The most important utility of the following definition is that it not only leads one to have a better understanding of groups, but it also helps us understand when two groups are equivalent. For example, D_3 and S_3 equivalent, since one could view D_3 as simply all the permutations of 1, 2, and 3, if we assigned these numbers to the vertices of a triangle.

Definition 1.3.1. Let $(G, \cdot)$ and $(G', *)$ be groups. A **homomorphism** is a mapping $\varphi : G \longrightarrow G'$ such that, for all $a, b \in G$,

$$\varphi(a \cdot b) = \varphi(a) * \varphi(b).$$

Again, here $*$ is the group operation of G'.

Example. Consider the two groups $GL_n(\mathbb{R})$ and $\mathbb{R}\backslash\{0\}$. If we define φ such that, for $A \in GL_n(\mathbb{R})$

$$\varphi(A) = \det(A)$$

then φ defines a homomorphism.

Recall that for for any $n \times n$ matrices A, B that $\det(AB) = \det(A)\det(B)$. Therefore

$$\varphi(AB) = \det(AB) = \det(A)\det(B) = \varphi(A)\varphi(B).$$

Since $\varphi(AB) = \varphi(A)\varphi(B)$, we see that φ satisfies the condition to be a homomorphism.

Proposition 1.3.2. Let $\varphi : G \longrightarrow G'$ be a homomorphism. Then all of the following hold.

1. If e_G is the identity of G and $e_{G'}$ is the identity of G', then $\varphi(e_G) = e_{G'}$.
2. For all $g \in G$, $\varphi(g^{-1}) = \varphi(g)^{-1}$.
3. For $g_1, g_2, \ldots, g_n \in G$, then $\varphi(g_1 \cdot g_2 \cdot \cdots \cdot g_n) = \varphi(g_1)\varphi(g_2)\cdots\varphi(g_n)$. Consequently, if $g = g_1 = g_2 = \cdots = g_n$, then $\varphi(g^n) = \varphi(g)^n$.

Proof: Let $g \in G$, and suppose $\varphi : G \longrightarrow G'$ is a homomorphism.

1. Since $e_G = e_G \cdot e_G$, we have that

$$\varphi(e_G) = \varphi(e_G \cdot e_G) = \varphi(e_G)\varphi(e_G).$$

We also know that $\varphi(e_G) \in G'$, and becuase G' is a group, there exists an inverse $\varphi(e_G)^{-1} \in G$ of $\varphi(e_G)$. Multiplying this on the left (or right) yields

$$e_{G'} = \varphi(e_G)$$

as desired.

2. Since $gg^{-1} = e_G$, and by (1.) we know that $\varphi(e_G) = e_{G'}$. Hence

$$\varphi(e_G) = e_{G'} \implies \varphi(gg^{-1}) = e_{G'} \implies \varphi(g)\varphi(g^{-1}) = e_{G'}.$$

Again, $\varphi(g) \in G'$, and since G' is a group there exist an inverse $\varphi(g)^{-1} \in G$ of $\varphi(g)$. Multiplying on the left by this inverse, we get

$$\varphi(g)\varphi(g^{-1}) = e_{G'} \implies \varphi(g^{-1}) = \varphi(g)^{-1}$$

as desired.

3. This is just repeated application of the homomorphism property. For $g_1, g_2, \ldots g_n \in G$, $g_1 \cdot g_2 \cdot \ldots \cdot g_n = g_1 \cdot (g')$ where $g' = g_2 \cdot g_3 \cdot \ldots \cdot g_n$. Applying the homomorphism property,

$$\varphi(g_1 \cdot g_2 \cdot \ldots \cdot g_n) = \varphi(g_1 \cdot g') = \varphi(g_1)\varphi(g').$$

Repeatedly applying the same idea, starting again with the product $g_2 \cdot g_3 \cdot \ldots \cdot g_n$ yields the result. The fact that $\varphi(g^n) = \varphi(g)^n$ is follows immediately.

■

If φ is a bijective homomorphism (i.e., one-to-one and onto) then we say that φ is an **isomorphism**. Furthermore, if there exists an isomorphism between two spaces G and G', then we say these spaces are **isomorphic** and that $G \cong G'$. As we'll soon see, isomorphisms gives us really nice results (hence the special terminology and notation). In addition, it can sometimes be difficult to tell when two groups G and G' are the same or different. Isomorphisms can help determine when there *isn't* such an equivalence.

As we'll see, the concept of an isomorphism is very powerful. However, proving it may not be that simple, and in ceratin cases the following theorem will be very useful.

Theorem 1.3.3. Let G and H be groups. The homomorphism $\varphi : G \longrightarrow H$ is an isomorphism if and only if there exists a homomorphism $\psi : H \longrightarrow G$ such that $\psi \circ \varphi$ is the identity map on G and $\varphi \circ \psi$ is the identity map on H.

Proof: ($\implies$) Suppose $\varphi : G \longrightarrow H$ is an isomorphism. Since φ is bijective, define the inverse map $\varphi^{-1} : H \longrightarrow G$ such that if $\varphi(g) = g'$ then $\varphi^{-1}(g') = g$.

Note that this is a well defined map due to the surjectivity and injectivity of φ. To show it is a homomorphism, we need to demonstrate that $\varphi^{-1}(h_1 \cdot h_2) = \varphi^{-1}(h_1)\varphi^{-1}(h_2)$. Thus observe that for $h_1, h_2 \in H$ there exist $g_1, g_2 \in G$ such that $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Therefore

$$\varphi(g_1 \cdot g_2) = h_1 \cdot h_2 \implies \varphi^{-1}(h_1 \cdot h_2) = g_1 \cdot g_2 = \varphi^{-1}(h_1) \cdot \varphi^{-1}(h_2).$$

Thus φ^{-1} is a homomorphism.

Now observe that for all $g \in G$ we have that $\varphi^{-1} \circ \varphi(g) = g$ and for all $h \in H$, $\varphi \circ \varphi^{-1}(h) = h$. Thus $\varphi^{-1} \circ \varphi$ is the identity on G while $\varphi \circ \varphi^{-1}$ is the identity on H, which proves this direction.

($\Longleftarrow$) Now suppose $\varphi : G \longrightarrow H$ is a homomorphism and that there exists a homomorphism $\psi : H \longrightarrow G$ such that $\psi \circ \varphi$ is the identity map on G and $\varphi \circ \psi$ is the identity map in H. In other words, ψ and φ are inverses of each other. Thus φ is a bijection function from $G \longrightarrow H$, which implies that φ is an isomorphism. ■

We also introduce the following criteria which is frequently used to evaluate if a homomorphism is one-to-one and/or onto.

Theorem 1.3.4. Let $\varphi : G \longrightarrow G'$ be a homomorphism. Then

1. φ is one-to-one if and only if $\ker(\varphi)$ is trivial. That is, $\ker(\varphi) = \{e_G\}$, where e_G is the identity of G.
2. φ is onto if and only if $\text{im}(\varphi) = G'$.

Therefore, φ is an **isomorphism** if and only if (1) and (2) hold.

Proof:

1. Suppose φ is one-to-one. By proposition 1.1.1, we know that $\varphi(e_G) = e_{G'}$. But since φ is injective we know e_G is the only element in G which is mapped to $e_{G'}$. Therefore $\ker(\varphi) = \{e_G\}$.

   Now suppose $\ker(\varphi) = \{e_G\}$. To show φ is one-to-one, consider $g, h \in G$ such that

   $$\varphi(g) = \varphi(h).$$

   Multiplying both sides by $\varphi(h)^{-1}$ we get

   $$\varphi(g)\varphi(h)^{-1} = e_{G'}.$$

   By proposition 1.1.2, we know that $\varphi(h)^{-1} = \varphi(h^{-1})$. Since φ is a homomorphism, we can then combine the terms to get

   $$\varphi(gh^{-1}) = e_{G'}.$$

   Since $\ker(\varphi) = \{e_G\}$, we see that

   $$gh^{-1} = e_G \implies g = h.$$

   Therefore φ is one to one.

2. Suppose φ is onto. Then $\text{im}(\varphi) = G'$ is just another way of stating this fact.

   Suppose $\text{im}(\varphi) = G'$. Then for every element $g' \in G'$, there exists $g \in G$ such that $\varphi(g) = g'$. That is, φ covers every value in G' so that it is onto.

Thus, we have that a function is isomorphic if and only if it is one to one and onto. Hence, it is isomorphic if and only if (1) and (2) hold. ■

We also make two common definitions for special homomorphisms.

Definition 1.3.5. Let G be a group.

1. If $\varphi : G \longrightarrow G$ is a group homomorphism, then we say that φ is a **endomorphism**.
2. If φ is a bijective endomorphism (an isomophic endomorphism) then we say that φ is an **automorphism**.

Theorem 1.3.6. The set of all automorphisms of a group G, denoted as $\mathrm{Aut}(G)$, forms a group with an operation $\circ$ of function composition.

Proof: We can prove this directly.

Closure. Let φ and ψ be automorphisms. Then $\varphi \circ \psi$ is (1) a homomorphism from $G \longrightarrow G$ and (2) a bijection (as the composition of bijections is a bijetion).

Associativity. In general, function composition is associative.

Identity. Let $i : G \longrightarrow G$ be the identity map. The (1) i is a group homomorphism and (2) a bijection. Therefore $i \in \mathrm{Aut}(G)$ and we can set i as the identity of the group. Note that

$$i \circ \varphi = \varphi = \varphi \circ i$$

for any $\varphi \in \mathrm{Aut}(G)$.

Inverse. Let $\varphi \in \mathrm{Aut}(G)$. Construct the function φ^{-1} as follows. If $\varphi(g) = g'$ for some $g, g' \in G$, then write $\varphi^{-1}(g') = g$. Such an assignment is well-defined since φ is a bijection. Hence we see that

$$\varphi \circ \varphi^{-1} = i = \varphi^{-1} \circ \varphi.$$

Finally, observe that φ^{-1} is (1) a homomorphism and (2) a bijection, so we see that $\varphi^{-1} \in \mathrm{Aut}(G)$. Therefore this forms a group.

■

1.4 Cyclic Groups.

Cyclics groups are a special type of group that are easy to recognize as group structures. In a cyclic group, one can always pinpoint a single element which can "generate" every other element of the group. For example, the subgroup of rotations $\{e, r, r^2, \ldots, r^{n-1}\}$ in dihedral groups is cyclic. In such a subgroup, every element is simply a fininte power of r. We can then think of this subgroup as being generated by a single element, namely r.

It will turn out later that, in every group, there will always be a subset of its elements such that the subset generates the whole group. Here, we're starting small, by just considering groups whose elements can be generated by *one* element.

Definition 1.4.1. A group G is **cyclic** if there exists an element $g \in G$ such that

$$G = \{g^n \mid n \in \mathbb{N}\}.$$

A very trivial example of a cyclic group is the integers under addition. This is because every element in $(\mathbb{Z}, +)$ can be generated by the number 1 (e.g., $3 = 1 + 1 + 1$, $-2 = -1 - 1$). With the definition of cyclic groups at hand, we can introduce theorems about cyclic groups.

Theorem 1.4.2. Let G be a cyclic group, and suppose $G = \{g^n | n \in \mathbb{N}\}$ for some $g \in G$. Then $|G| = |g|$.

Proof: Suppose $|g| = k$ where k is some positive integer. Then since $G = \{g^n | n \in \mathbb{N}\}$,

$$G = \{e, g, g^2, \ldots, g^{k-1}\}.$$

Therefore $|G| = k = |g|$. Now if $|g| = \infty$, then by the same exact reasoning $|G| = \infty$. ■

A consequence of this theorem is that if $|g| = \infty$, then we know that $g^a \neq g^b$ for any $a, b \in \mathbb{N}$ such that $a \neq b$. This would imply that $g^{a-b} = 1$, which otherwise imply the group G to have finite order; a contradiction if $|g| = \infty$ since this implies $|G| = \infty$.

Theorem 1.4.3. Any two cyclic groups of the same order are isomorphic.

Proof: Let G and G' be two cyclic groups and suppose $|G| = |G'|$. Furthermore suppose that $G = \langle g \rangle$ and $G' = \langle g' \rangle$. Construct the homomorphism $\varphi : G \longrightarrow G'$ where

$$\varphi(g^n) = (g')^n$$

for any $n \in \mathbb{N}$. Observe that this is surjective, as the groups are of equal order so for any $(g')^n$ there we can identify the preimage to be g^n. This is also injective, since if $g_1' = g_2'$ are both elements in G', then $g_1' = g_2' = (g')^n$ for some n which corresponds to one and only one element in G; namely, g^n.

As the homomorphism we constructed is surjective and injective, we have that φ is an isomorphism so that the groups are isomorphic.

Note we could have also utilized Theorem 1.3 here, by constructing the homomorphism $\psi : G' \longrightarrow G$ where $\psi((g')^n) = g^n$.

■

We'll now move onto a more useful theorem concerning cyclic groups.

Theorem 1.4.4. Let G be a cyclic group. Then every subgroup of G is cyclic.

Proof: Let H be a subgroup of G. Let m be the smallest integer such that $g^m \in H$. Suppose towards a contradiction that H is not cyclic. That is, there exists an element $h \in H$ such that $h \neq g^{mn}$ for any $m \in \mathbb{N}$.

Since $h \in G$, and G is a cyclic group, it must be *some* integer power of g. Since $h \neq g^m$ for any $m \in \mathbb{N}$, we know that there must exist $q, r \in \mathbb{Z}$ such that $h = g^{qm+r}$ where $0 < r < m$.

Now since H is a group, $h^{-1} = g^{-(qm+r)} \in H$. Furthermore, $g^{(q+1)m} \in H$ since H is closed under products. By the same reasoning,

$$g^{(q+1)m} g^{-(qm+r)} = g^{m-r}$$

is in H. However, this contradicts our assumption that m is the smallest positive integer such that $g^m \in H$. Thus by contradiction H must be cyclic. ■

Remark. This proof utilizes the important idea that, if you *know* G is a group, and g_1, g_2 are *any* two elements of G, then the product $g_1 g_2 \in G$. Furthermore, if $g_1, g_2, \ldots, g_n \in G$, then $g_1^{n_1} g_2^{n_2} \cdots g_k^{n_k} \in G$ for literally any powers $n_1, n_2 \ldots n_k \in \mathbb{Z}$.

We used this idea by (1) observing that $g^m \in H$, so that $g^{(q+1)m} \in H$ for any $q \in \mathbb{Z}$ and (2) reasoning that $g^{(q+1)m} g^{-(qm+r)} \in H$.

In addition, we'll offer a way to think about subgroups of a cyclic group G:

$$G = \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8, g^9, g^10, \ldots\} \tag{1.1}$$
$$= \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8, g^9, g^{10}, \ldots\} \tag{1.2}$$
$$= \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8, g^9, g^{10}, \ldots\} \tag{1.3}$$

In the above representations, the color-highlighted powers of g form subgroups of G. Thus $\{e, g^2, g^4, g^6, \ldots\}$, $\{e, g^3, g^6, g^9, \ldots\}$ and $\{e, g^4, g^8, \ldots\}$ are all subgroups of G. Of course, this process will eventually terminate if G is finite, but we represent the most general case above with the $\ldots$ terms. In addition, if G is finite one of the subgroups may turn out to be just the entire group G.

To find the exact subgroups of a cyclic group we can use the following theorem.

Theorem 1.4.5. Let G be a finite cyclic group of order n. For every positive integer d such that $d|n$, there is exactly one subgroup of order d of G. These are all the subgroups of G.

Proof: Let H be a subgroup of G and suppose $|H| = m \leq n$. We'll first show that $n \mid m$ and then show that for every $d \mid n$, there exists a subgroup of order d.

By the previous theorem, we know that H must be cyclic. Therefore $H = \{h^n \mid n \in \mathbb{N}\}$ for some $h \in G$. However, by Theorem 1.4, we also know that $|H| = |h|$, so that $h^m = e$. Therefore,

$$H = \{e, h, h^2, \ldots, h^{m-1}\}.$$

However, if $G = \{e, g, g^2, \ldots, g^{n-1}\}$ for some $g \in G$, then $h = g^k$ for some positive integer $k < n$. Therefore

$$H = \{g^{jk} \mid j \in \mathbb{N}\}.$$

Since g^k generates H, we can apply Theorem 1.4 to conclude that $|g^k| = m$. In order for this to be true, there must exist some positive integer j such that $g^{jk} = e$.

On one hand, we already know m is the smallest such integer, as we specified $h^m = (g^k)^m = e$. On the other, we also know that $|g| = n$; that is, n is the smallest integer such that $g^n = e$. Therefore, we have that

$$n = mk$$

which proves that $m \mid n$.

To show that for each $d \mid n$ there exists a subgroup of order d, simply observe that if $G = \{e, g, g^2, \ldots, g^n\}$ then the set $\{e, g^{n/d}, g^{2n/d}, \ldots, g^{(d-1)n/d}\}$ is a subgroup of order d. This completes the proof. ■

1.5 Left and Right Cosets, Lagrange's Theorem

The result of the previous proof is a special case of a more general theorem we'll come across, know as Lagrange's Theorem. The theorem states that if G is a finite group and H is a subgroup, then $|H| \mid |G|$. That is, the order of H divides G. This is a remarkable and useful result, aiding proofs as we move on from it. But in order to reach Lagrange's Theorem we first discuss the extremely important concept of a **coset** of a group.

Before defining a coset, we first recall the definition of an equivalence relation.

Definition 1.5.1. An equivalence relation on a set G is a binary relation $\sim$ that satisfies the following properties.

Reflexive. For all $a \in G$, $a \sim a$.

Symmetric. If $a \sim b$ then $b \sim a$.

Transitive. If $a \sim b$ and $b \sim c$ then $a \sim c$.

Equivalence classes are a useful concept since they tend to break up a set of objects G into distinct, disjoint sets A_i. More specifically, they partition G. These sets, A_i, are known as **equivalence classes** since their criteria for membership requires that $a \in A_i$ if and only if $a \sim a'$ for all $a' \in A_i$. This is a general strategy in mathematics: to define equivalence classes from *some* equivalence relation to break up a set into disjoint partitions. However, we use the concept of an equivalence class to partition a group G. We use the following relation to do this.

The Relation.

Let G be a group and H be a subgroup of G. If $a, b \in G$, then the relation $\sim$ on G such that $a \sim b$ if and only if $ab^{-1} \in H$ is an equivalence relation.

Proof:

Reflexive. Observe that $a \sim a$, since $aa^{-1} = e \in H$.

Symmetric. First, if $a \sim b$ then $ab^{-1} \in H$. Since H is a group, we know that $(ab^{-1})^{-1} = ba^{-1} \in H$. Thus by our definition we see that $b \sim a$, so that our relation is also symmetric.

Transitive. Now suppose $a \sim b$ and $b \sim c$ for $a, b, c \in G$. Then by definition $ab^{-1} \in H$ and $bc^{-1} \in H$. Since H is a group, and it is closed under products of its elements. Therefore

$$(ab^{-1})(bc^{-1}) = ab^{-1}bc^{-1} = ac^{-1} \in H.$$

Thus we see that $a \sim c$, which proves that our relation is transitive.

■

As our relation is reflexive, symmetric and transitive, we see that it is an equivalence relation. Note however that we could have defined our relation as $a \sim b$ if and only if $a^{-1}b \in H$; such a relation is equivalent to what we just worked with.

First we require a quick definition.

Definition 1.5.2. Consider a subgroup H of G. For any $a \in G$, we define

$$Ha = \{ha \mid \text{ for all } h \in H\}$$

to be the right coset of H. We also define

$$aH = \{ah \mid \text{ for all } h \in H\}.$$

to be the left coset of H. Note that since H is a group, it is closed under products of its elements. Therefore for any $h \in H$

$$hH = Hh = H.$$

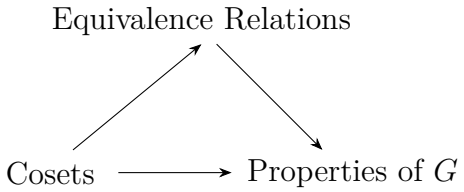
Don't confuse the above equality; take special note that equality above is *set equality*, not term by term equality. Of course we have no idea if $ha = ah$ where $h \in H$ for $a \in G$ unless G is abelian or we have other information.

The Big Idea of Cosets.

Now consider the relation introduced earlier, which we proved is in fact an equivalence relation. Consider an equivalence class of an element $a \in G$, denoted $[a]$. Then we can describe $[a]$ as

$$\begin{aligned}
[a] &= \{g \in G \mid g \sim a\} \\
&= \{g \in G \mid ga^{-1} \in H\} \\
&= \{g \in G \mid ga^{-1} = h \text{ for some } h \in H\} \\
&= \{g \in G \mid g = ha \text{ for some } h \in H\} \\
&= Ha.
\end{aligned}$$

Thus the equivalence classes of the elements of G with respect to some subgroup H are simply just the right cosets of H. (We could have alternatively defined our equivalence relation to be $g \sim a$ if and only if $a^{-1}g \in H$, in which case our above description of $[a]$ would have resulted in being equal to aH. Since both formulations are equivalent, we will simply work with the right cosets of H, namely the sets Ha.)



Once we understand cosets, we can understand a lot about a group, because they're really just equivalence classes!

Since equivalence classes are mathematical objects which partition a set, what we have is the following beautiful idea: We can take a subgroup H of a set G and partition our group G via the right (or left) cosets of H. This is because our cosets are equivalence classes, and as we said before equivalence classes partition sets which they are defined on.

Example

Consider the group $\mathbb{Z}$ and the subgroup $5\mathbb{Z} = \{5n \mid n \in \mathbb{Z}\}$. We can calculate the cosets of this $\mathbb{Z}$

with respect to $5\mathbb{Z}$ as

$$\begin{aligned}
5\mathbb{Z}+1 &= \{5n+1 \mid n \in \mathbb{Z}\} \\
5\mathbb{Z}+2 &= \{5n+2 \mid n \in \mathbb{Z}\} \\
5\mathbb{Z}+3 &= \{5n+3 \mid n \in \mathbb{Z}\} \\
5\mathbb{Z}+4 &= \{5n+4 \mid n \in \mathbb{Z}\} \\
5\mathbb{Z}+5 &= \{5n+5 \mid n \in \mathbb{Z}\} = 5\mathbb{Z}
\end{aligned}$$

Note that we didn't list any other cosets. Well, that's because these are all of the possible distinct cosets of $\mathbb{Z}$ with respect to $5\mathbb{Z}$. For example, the coset $5\mathbb{Z}+37$ is equivalent to $5\mathbb{Z}+2$, since

$$\begin{aligned}
5\mathbb{Z}+37 = \{5n+37 \mid n \in \mathbb{Z}\} &= \{5n+5\cdot 7+2 \mid n \in \mathbb{Z}\} \\
&= \{5(n+7)+2 \mid n \in \mathbb{Z}\} \\
&= 5\mathbb{Z}+2.
\end{aligned}$$

Thus, any other coset we propose is equivalent to one of the five we listed. This is demonstrated in the figure below.

$5\mathbb{Z}+1$	$\cdots$	$-14,$	$-9,$	$-4,$	$1,$	$6,$	$11,$	$\cdots$
$5\mathbb{Z}+2$	$\cdots$	$-13,$	$-8,$	$-3,$	$2,$	$7,$	$12,$	$\cdots$
$5\mathbb{Z}+3$	$\cdots$	$-12,$	$-7,$	$-2,$	$3,$	$8,$	$13,$	$\cdots$
$5\mathbb{Z}+4$	$\cdots$	$-11,$	$-6,$	$-1,$	$4,$	$9,$	$14,$	$\cdots$
$5\mathbb{Z}$	$\cdots$	$-10,$	$-5,$	$0,$	$5,$	$10,$	$15,$	$\cdots$

Note that in this figure we can identify every integer in $\mathbb{Z}$. This assures us that our above list of cosets is in fact complete. In addition, this demonstrates the fact that cosets partition a group. Note that each above coset is disjoint, yet the union of all of the cosets is the entire group G.

As cosets can partition a group, we define $[G:H]$, called the **index**, to be the number of distinct right (or equivalently left) cosets of G. If G is finite, then $[G:H]$ is of course finite. However, G can still be infinite while $[G:H]$ is finite.

Proposition 1.5.3. If G is a group and H is a subgroup, then for $a, b \in G$, $Ha \cap Hb = \emptyset$ or $Ha = Hb$.

This proves the observation we made beforehand in the example with the cosets of $\mathbb{Z}$ with respect to $5\mathbb{Z}$. We saw that the 5 cosets we came up with were distinct and disjoint, which is what this proposition proves is true in general.

Proof: This is simply a consequence of the connection between cosets and equivalence relations of G. Equivalence classes form partitions, so by definition they are disjoint. However, equivalence classes can also be equal to one another (namely, if a, b belong to the same equivalence class A, then $[a] = [b] = A$. This is why equivalence classes, which in our case are cosets, are awesome.) Therefore cosets Ha and Hb are either disjoint or equal to each other.

This, however, can be proven directly. Consider such Ha and Hb. Suppose $Ha \cap Hb \neq \emptyset$. Then by definition of cosets, there exists a h_1 and h_2 such that $h_1 a = h_2 b$. Therefore $a = h_1^{-1} h_2 b$. Since H is a group, and it is closed under products of its elements, there exists a h' such that $h' = h_1^{-1} h_2$. Thus $a = h'b$. Consequently, we see that

$$Ha = \{ha \mid h \in H\} = \{h(h'b) \mid h \in H\} = H(h'b).$$

However, recall earlier that $Hh = H$ for any $h \in H$. Since $h' \in H$, we then have that

$$H(h'b) = (Hh')b = Hb$$

which proves that $Ha = Hb$ as well as the proposition. ■

Proposition 1.5.4. Let G be finite and $a, b \in G$. If H is a subgroup, and Ha and Hb are distinct cosets, then $|Ha| = |Hb|$.

Hence, cosets of G with respect to some subgroup H are always of the same size.

Proof: Construct a bijection $f : Ha \longrightarrow Hb$ given by $f(ha) = hb$. Observe that this is surjective. It is also injective since $hb = hb'$ if and only if $b = b'$, but since we assumed Ha and Hb are distinct, we know by the previous proposition that distinctness implies disjointness. Since we can formulate a bijection the two sets, the sets have the same sizes. ■

The next theorem, credited to Lagrange, demonstrates the usefulness of studying cosets to study finite groups. Our equivalence classes not only parition our group G, but they are also the same size. Therefore, we can always partition a finite group G into equally sized cosets.

Theorem 1.5.5. Let G be a finite group, and suppose H is a subgroup of G. Then $|H|$ divides $|G|$.

Proof: Since G is finite, there are a distinct set of cosets $Ha_1, Ha_2, \ldots, Ha_n$ which partition G. By Proposition 1.3, each set is of equal size; call it k. Therefore, we see that

$$|Ha_1| + |Ha_2| + \cdots + |Ha_n| = |G| \implies kn = |G|.$$

Therefore $|G|$ will always be a multiple of $|H|$. Or, in other words, $|H|$ divides $|G|$. ■

This is the theorem we said was a more general case of Theorem 1.6. The above theorem enables us to understand all the possible subgroups of any finite group G. In fact, the theorem implies more useful consequences of Theorem 1.7.

Corollary 1.5.6. Let G be a finite group and H a subgroup of G. Then we have the following consequences:

1. If G is a finite group and $g \in G$, then $|g|$ divides $|G|$ and $g^{|G|} = e$.
2. Let p be a prime number. If G is a group of order p, then G is a cyclic group.

3. If $\varphi : G \longrightarrow G'$ is a homomorphism between finite groups, then $|\ker \varphi|$ divides G and $|\text{im } \varphi|$ divides G'.
4. $|G| = |H| \cdot [G : H]$ for any subgroup H of G.

Proof:

1. Consider the cyclic subgroup $H = \langle g \rangle$ of G. By Lagrange's theorem, we know that $|H|$ divides $|G|$ isnce H is a subgroup of G. However $|g| = |H|$ since H is cyclic. Therefore $|g|$ divides the order of $|G|$. This implies that $|G| = n|g|$ for some $n \in \mathbb{N}$. Therefore

$$g^{|G|} = g^{n|g|} = (g^{|g|})^n = e^n = e$$

   which is what we set out to show.
2. If $|G| = p$, we know by Lagrange's Theorem we know that there are exactly two subgroups of G, namely the trivial group and the whole group G.
   Thus let $g \in G$, where g is not the identity, and consider the subgroup $H = \langle g \rangle$. Since g is not the identity, H is not the trivial group. But since it is a nontrivial subgroup, and the only nontrivial subgroup of G is itself, we see that our only choice is to let $H = G$. However, H is cyclic, which proves that G is cyclic as well.
3. This result immediately follows from the fact that $\ker \varphi$ is a subgroup of G and $\text{im } \varphi$ is a subgroup of G'. Applying Lagrange's theorem leads to the result.
4. For any subgroup H of G, we know that $[G : H]$ is the number of left or right cosets of G. Since each such set is of size $|H|$, and because they all together partition G, we see that $|G| = |H| \cdot [G : H]$.

■

1.6 Normal subgroups

Normal subgroups are special subgroups which exhibit properties of interest for when we go on to later define the idea of quotient groups, a concept we have touched upon slightly in considering $\mathbb{Z}/2\mathbb{Z}$ and other modulo groups. They are a bit abstract at first, since they have to do with **cosets**. Once you work with normal subgroups for a bit, it will s eventually click and the reasoning behind their definitions becomes clear.

Definition 1.6.1. Let G be a group and suppose H is a subgroup of G. We say that H is **normal** if and only if **for every** $g \in G$, we have that $Hg = gH$. We denote such a relation as $H \trianglelefteq G$.

We make two remarks here.

Commutative Groups. Note that if G is commutative, then H, a subgroup of G, is also commutative. In fact, H commutes with all elements of G. That is, if $H = \{h_1, h_2, \dots\}$ then

$$gH = \{gh_1, gh_2, \dots\} = \{h_1 g, h_2 g, \dots\} = Hg$$

for all $g \in G$. Thus what we're trying to say here is if G is commutative, every subgroup H of G is normal.

Set Equality. If H is normal to G, then $gH = Hg$ all $g \in G$. Be careful with this equation, since what this is not saying is that $gh = hg$ for all $g \in G$ and $h \in H$; that would imply commutativity, and it may be the case that G and H are not commutative groups. That is, the above equation is set equality, not term-by-term equality.

What this does say, however, is if $gH = Hg$, then for each $g \in G$, and for every $h_1 \in H$, there exists an $h_2 \in H$ such that

$$gh_1 = h_2 g.$$

Note here that commutative groups satisfy this because in their case, $h_1 = h_2$ satisfies the equation.

Since our current definition of normality would be exhausting to use directly if we wanted to check if a subgroup is normal, we have the following theorem that helps us check for normality.

Theorem 1.6.2. Let G be a group and H a subgroup of G. The following are equivalent:

1. $H \trianglelefteq G$ for all $g \in G$
2. $gHg^{-1} = H$ for all $g \in G$
3. $gHg^{-1} \subset H$ for all $g \in G$.
4. $(Hg)(Hh) = H(gh)$ for all $g, h \in G$

Proof: We'll prove this by producing a chain of imply statements that can traverse in both directions. Let G be a group and H be a subgroup.

(**1** $\iff$ **2**) If $H \trianglelefteq G$, then $gH = Hg$ for all $g \in G$. Multiplying on the left by g^{-1}, we then see that $gHg^{-1} = H$ for all $g \in G$.

Proving the reverse direction, if $gHg^{-1} = H$ for all $g \in G$ then $gH = Hg$ for all $g \in G$, which means that H is normal by defintion.
(**2** $\iff$ **3**) If $gHg^{-1} = H$ for all $g \in G$ then it is certainly true that $gHg^{-1} \subset H$ for all $g \in G$.

Now we prove the other direction. Suppose $gHg^{-1} \subset H$ for all $g \in G$. Then

$$gHg^{-1} \subset H \implies gH \subset Hg \implies H \subset g^{-1}Hg$$

by multiplying on the right by g and on the left by g^{-1}. However, since we have assumed (3) is true we know that

$$(g^{-1})H(g^{-1})^{-1} \subset H \implies g^{-1}Hg \subset H.$$

By the above equations we then have that $H = g^{-1}Hg$, and multiplying by g^{-1} on the right and g on the left yields that $H = gHg^{-1}$ as desired.
(**2** $\iff$ **4**) Suppose (2). Then observe that $gHg^{-1} = H \implies gH = Hg$ for all $g \in G$. Therefore for $h \in G$,

$$(Hg)(Hh) = H(gH)h = H(Hg)h = H(gh).$$

In the first step we used associativity and in the second step we used the fact that $gH = Hg$.

To prove the other direction, suppose $(Hg)(Hh) = H(gh)$ for all $g, h \in G$. Let $h = e$. Then
■

To show a subgroup H of G is normal, condition (3) of this theorem generally the fastest and easy way to take advtange of. It is usually the least complicated one to show.

Example.
Consider the group $GL_n(\mathbb{R})$ and its subgroup $SL_n(\mathbb{R})$. It turns out that $SL_n(\mathbb{R}) \trianglelefteq GL_n(\mathbb{R})$, which we will show using condition (3).

Let $A \in GL_n(\mathbb{R})$ and suppose $T \in SL_n(\mathbb{R})$. We must show that $ATA^{-1} \in SL_n(\mathbb{R})$ for all $A \in GL_n(\mathbb{R})$ and $T \in SL_n(\mathbb{R})$. Observe that

$$\det(ATA^{-1}) = \det(A)\det(T)\det(A^{-1}) = \det(A)(1)\det(A)^{-1} = 1$$

where we used the basic properties of the determinant for the calculation. Since $\det(ATA^{-1}) = 1$, we have that $ATA^{-1} \in SL_n(\mathbb{R})$ for all A and T in $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$, respectively. Therefore $SL_n(\mathbb{R})$ is normal to $GL_n(\mathbb{R})$.

Example.
One important example is the following: for any group homomorphism φ between two groups G and G', recall that $\ker(\varphi)$ is a subgroup of G. However, we also have that $\ker(\varphi) \trianglelefteq G$, which we'll show as follows.

Proposition 1.6.3. Let G, G' be groups and $\varphi : G \longrightarrow G'$ be a group homomorphism. Then $\ker(\varphi) \trianglelefteq G$.

Proof: We need to show that for all $g \in G$, $h \in \ker(\varphi)$ that $ghg^{-1} \in \ker(\varphi)$. Thus observe that

$$\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g) \cdot 0 \cdot \varphi(g^{-1}) = 0.$$

Since $\varphi(ghg^{-1}) = 0$, we thus see that $ghg^{-1} \in \ker(\varphi)$ for all $g \in G$ and $h \in \ker(\varphi)$, which proves $\ker(\varphi) \trianglelefteq G$. ■

Another important example of normality is the fact that the center of a group $Z(G)$ is normal to G for any group G.

Proposition 1.6.4. Let G be a group. Then $Z(G) \trianglelefteq G$.

Proof: Recall that $Z(G)$ is a subgroup of G, consisting of all the elements of G which commute with every element in G. More precisely,

$$Z(G) = \{z \in G \mid gz = zg \text{ for all } g \in G\}.$$

Now for any $g \in G$ and $z \in Z(G)$, we have that $gzg^{-1} = gg^{-1}z = z$, since z commutes with all elements of G. Therefore $gzg^{-1} \in Z(G) \implies gZ(G)g^{-1} \subset Z(G)$. By the previous theorem, we can conclude that $Z(G) \trianglelefteq G$ as desired. ■

Next, we introduce a small theorem that allows us to quickly and easily identify if a subgroup H of G is normal.

Theorem 1.6.5. If G is a group and H is a subgroup, and $[G : H] = 2$, then $H \trianglelefteq G$.

Proof: Since G has two right (and equivalently two left) cosets, we see that they must be of the form H and Hg where $g \in G \setminus H$ (that is, all of the elements of G which are not in H).

As we said before, there are equivalently two left cosets H and gH where $g \in G \setminus H$. Since the cosets partition G, we see that for any $g \in G \setminus H$ two partitions of G are

$$\{H, Hg\} \text{ and } \{H, gH\}.$$

Since these partition the same set we see that $gH = Hg$ for all $g \in G \setminus H$. Note that we already know that for $g \in H$, $Hg = H$ and $gH = H$ so $gH = Hg$. Therefore, we have all together that $Hg = gH$ for all $g \in G$. ■

In working with normal subgroups, one may form the following questions.

Q: If K is a normal subgroup of H and H is a normal subgroup of G, is K normal to G?

A: Not always. If $H \trianglelefteq K$, then $khk^{-1} \in K$ for all $k \in K$ but there is nothing allowing for us to extend this further and state that $ghg^{-1} \in K$ for all $g \in G$.
However, a special case for when this is true involves $Z(G)$. We know that $Z(G) \trianglelefteq G$. But if $K \trianglelefteq Z(G)$ then it turns out $K \trianglelefteq G$,

1.7 Quotient Groups.

The work done in the previous section on Normal subgroups now leads to the formulation of the **Quotient Group**. Up to this point we've studied groups which have familiar, concerete objects, but now we're going to get a little bit abstract. We're going to look at the useful concept of the quotient group, G/H, which is a **group whose elements are H cosets**. That is, the elements of our group are going to be sets themselves. The operation on the elements of the quotient group can only make sense if the cosets are from a subgroup H which is normal to G.

Theorem 1.7.1. Let G be a group and $H \trianglelefteq G$. Define G/H to be the set consisting of all the possible right (or equivalently left) H cosets. If we equip this set with a product $\cdot$ such that

$$(Ha) \cdot (Hb) = H(ab)$$

then G/H forms a group, called the **Quotient Group**.

Let's review what this is saying. Basically, if we have a normal subgroup H of G, the set of cosets $\{Hg_1, Hg_2, \dots\}$ with the product $Hg_1 \cdot Hg_2 = H(g_1g_2)$ **forms a group**.

Proof:

Identity. To show that this set is a group, we first define the identity element to simply be H. This is a "trivial" coset, and for any Ha, where $a \in G$,

$$(Ha)(H) = Ha$$
$$(H)(Ha) = Ha$$

so H is a natural and apporopriate choice for an identity as it has the property of an identity element.

Associativity. Associativity is derived from the associativity of our group G itself. Observe that for any $a, b, c \in G$ we have

$$(Ha)[(Hb)(Hc)] = Ha[H(bc)] = H(abc)$$
$$[(Ha)(Hb)](Hc) = [H(ab)]Hc = H(abc).$$

Therefore $(Ha)[(Hb)(Hc)] = [(Ha)(Hb)](Hc)$ for all $a, b, c \in G$, so the product relation is associative.

Closedness. The result of our proposed product is always a coset itself ($Ha \cdot Hb = H(ab)$), and since G/H is a set of all H cosets we see that this set is closed under $\cdot$.

Inverses. For any $Ha \in G/H$, where $a \in G$, we see that the inverse element is Ha^{-1}, since

$$(Ha)(Ha^{-1}) = H(aa^{-1}) = H$$
$$(Ha^{-1})(Ha) = H(a^{-1}a) = H$$

and we already defined H to be our identity element. So our proposed inverse makes sense.

Note that $Ha^{-1} \in G/H$ since $a^{-1} \in G$, so an inverse element not only exists but it also exists in G/H

All together, this allows us to observe that we have a group structure, so long as $H \trianglelefteq G$. ■

(Why do we need this the condition that $H \trianglelefteq G$? Well, because the only way we can make damn sure that $(Ha)(Hb) = H(ab)$ is by Theorem 1.10, which requires that $H \trianglelefteq G$.)

Note that there is another way to think about G/H. The elements of the quotient group are cosets, right? However, let us not forget that cosets are simply ***equivalence classes which respect the following equivalence relation*** : if G is a group, H is a subgroup, then for any $a, b \in G$ we say that $a \sim b$ if and only if $ab^{-1} \in H$. Thus we can recast our definition follows:

> Let $H \trianglelefteq G$. Then the set G/H is defined to consist of all of the ~~right (or left) cosets of H in G~~ equivalence classes of the elements of G (under the equivalence relation stated in the previous paragraph).

We thus have two equivalent ways to interpret the meaning of a quotient group. One involves equivalence classes, while the other involves cosets. In our case it seems more complicated to think about equivalence classes. However, in different applications of group theory (such as to algebraic geometry and topology) it will be convenient to interpret quotient groups as equivalence classes. For now, we'll stick with the coset interpretation, since it's the easiest way to understand a quotient group.

Example. Recall that we showed $SL_n(\mathbb{R}) \trianglelefteq GL_n(\mathbb{R})$. Thus the quotient group $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ makes sense by Theorem 1.11, so let's see what this group looks like.

First, the identity element of our group is $SL_n(\mathbb{R})$.

In dealing with quotient groups, you may be wondering the following questions:

Q: If H is a normal subgroup of G, and G is abelian, is G/H abelian? If G/H is abelian, is G abelian?

A: The answer to the first question is yes. Observe that by definition, $G/H = \{aH \mid a \in G\}$. But since H is normal, we know that $gH = Hg$ for all $g \in G$. Thus observe that for $aH, bH \in G/H$, we have that

$$\begin{aligned}(aH)(bH) = (ab)H &= (ba)H \text{ (since } G \text{ is abelian)} \\ &= (bH)(aH).\end{aligned}$$

Thus the set G/H must be abelian.

The answer to the second question is no, not always. If G/H is abelian, we know that

$$(aH)(bH) = (bH)(aH) \implies (ab)H = (ba)H.$$

for all $a, b \in G$. However, this only guarantees **set equality**, not a term-by-term equality (in which

case the group would be abelian). An example of this is D_6 with the subgroup $H = \{1, r, r^2\}$. In this case $H \trianglelefteq D_6$ because all the left cosets are H, sH and therefore $[D_{2n} : H] = 2$ (Hence $H \trianglelefteq G$ by the previous proposition). In addition, $H(sH) = sH = sH(H)$, $sH(sH) = s^2H = (sH)sH$, so G/H is abelian, but the set D_{2n} is itself not an abelian group. Thus, **it is possible for G/H to be ableian while G itself is not abelian**

Another fun example for when the quotient group G/H is abelian, even though the group G is abelian, is the following.

Example. Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R}, a \neq 0 \right\}, \quad H = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbb{R} \right\}.$$

G is subset of $GL_2(\mathbb{R})$ and H is a subgroup of G.

***H* ⊴ *G*.** First we'll show that H is normal to G. Thus let $x \in G$, so that $x = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ for some $a, b \in \mathbb{R}$ where $a \neq 0$. Now let $h \in H$ so that $h = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ for some $c \in \mathbb{R}$. Then observe that

$$\begin{aligned} xhx^{-1} &= \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a & -b/a + c \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & (-b + ca) + b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & ca \\ 0 & 1 \end{pmatrix} \in H. \end{aligned}$$

Therefore, we have that $xhx^{-1} \in H$ for all H, which implies that H is a normal subgroup of G.

***G/H* is abelian.** Now we'll show that G/H is an abelian group. Firstly, what does it mean for a quotient group to abelian? Well, it would mean that for any $x, y \in G$ we have that

$$(Hx) \cdot (Hy) = (Hy) \cdot (Hx).$$

Or, in other words,

$$H(xy) = H(yx).$$

Thus we need some kind of set equality to be happening. Thus consider $h = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, where again $x \in \mathbb{R}$, and suppose $x = \begin{pmatrix} a_x & b_x \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} a_y & b_y \\ 0 & 1 \end{pmatrix}$ where $a_x, a_y, b_x, b_y \in \mathbb{R}$ and $a_y, a_x \neq 0$. Then observe that

$$hxy = \begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_x & b_x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_y & b_y\\ 0 & 1\end{pmatrix} \qquad hyx = \begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_y & b_y\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_x & b_x\\ 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_xa_y & a_xb_y + b_x\\ 0 & 1\end{pmatrix} \qquad = \begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_ya_x & a_yb_x + b_y\\ 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}a_xa_y & a_xb_y + b_x + c\\ 0 & 1\end{pmatrix} \qquad = \begin{pmatrix}a_ya_x & a_yb_x + b_y + c\\ 0 & 1\end{pmatrix}.$$

Note that the (1,1) entry in both matrices are equal; that is, $a_xa_y = a_ya_x$ since they are members of $\mathbb{R}$. Therefore, we see that

$$Hxy = \left\{\begin{pmatrix}a_xa_y & a_xb_y + b_x + c\\ 0 & 1\end{pmatrix} \mid a_x, a_y, b_x, b_y, c \in \mathbb{R}, a_x, a_y \neq 0\right\}$$
$$Hyx = \left\{\begin{pmatrix}a_xa_y & a_yb_x + b_y + c\\ 0 & 1\end{pmatrix}. \mid a_x, a_y, b_x, b_y, c \in \mathbb{R}, a_x, a_y \neq 0\right\}.$$

Since b_x, b_y, c are arbitrary members of $\mathbb{R}$, we can replace their sums with another arbitrary $c', c'' \in \mathbb{R}$. Then we see that

$$Hxy = \left\{\begin{pmatrix}a_xa_y & a_xb_y + c'\\ 0 & 1\end{pmatrix} \mid a_x, a_y, b_y, c' \in \mathbb{R}, a_x, a_y \neq 0\right\}$$
$$Hyx = \left\{\begin{pmatrix}a_xa_y & a_yb_x + c''\\ 0 & 1\end{pmatrix} \mid a_x, a_y, b_x, c'' \in \mathbb{R}, a_x, a_y \neq 0,\right\}.$$

After cleaning up the sets, we can now see they are equal, which wasn't as obvious as it was before. They're equal because their criteria for set memberships are identical; they just have different variables, but that of course does not change their members. Therefore we see that $Hxy = Hyx$ for all $x, y \in G$, which proves that G/H is an abelian group, even though G nor H are abelian.

1.8 Isomorphism Theorems

With our knowledge of homomorphisms, normality and quotient groups, we are now able to develop four important theorems, known as the isomorphism theorems, which are indispensible tools in group theory. The isomorphism theorems give isomorphic relations which we can use to our advntage to understand groups and aid our proofs.

The isomorphism theorems are very deep theorems in abstract algeba. While one may go deeper into algebra, they will come across isomorphism theorems analagous to the ones below again and again.

Theorem 1.8.1. (First Isomorphism Theorem) Let $\varphi : G \longrightarrow G'$ be a homomorphism. Then

$$G/\ker(\varphi) \cong \text{im}(\varphi).$$

This is one of the more useful isomorphism theorems, and says something that matches out intuition. That is, if we quotient out the $\ker(\varphi)$, i.e., the set of all elements which get mapped to 0, then we should obtain something isomorphic to $\text{im}(\varphi)$.

Proof: We'll prove this directly. That is, we'll create a homomorphism between $G/\ker(\varphi)$ and $\text{Im}(\varphi)$, and then show that this homomorphism is one-to-one and onto, and therefore bijective. Thus the groups will be isomorphic.

Let $\varphi : G' \longrightarrow G$ be a homorphism. Write $K = \ker(\varphi)$. Define $\psi : G/K \longrightarrow \text{Im}(\varphi)$ as

$$\psi(gK) = \varphi(g)$$

where $gK \in G/K$ and $g \in G$.

(We'll use left cosets (gK) to talk about elements in G/K to remind the reader that left cosets can be used to characterize a quotient group just as as right cosets can.)

We want this to be a homomorphism. But we pulled this function out of nowhere, so let's check if this is well-defined.

Well-Defined. Suppose $g' \in gK$. Then $gK = g'K$, and our goal will be to show that $\psi(gK) = \psi(g'K)$. Since $g' \in gK$, there exists a $k \in K$ such that $gk = g'$. Then

$$\psi(g'K) = \psi((gk)K) = \psi(gK) = \varphi(g)$$

while

$$\psi(gK) = \varphi(g).$$

Therefore $\psi(g'K) = \psi(gK)$, so the representative g or g' does not matter.

Now that we know this function is not nonsense, we move on to showing it is a homomorphism.

It's a Homomorphism. Let's justify that this is a homorphism. For $gK, g'K \in G/K$,

$$\begin{aligned}\psi(gK \cdot g'K) &= \psi((gg')K) = \varphi(gg') \\ &= \varphi(g)\varphi(g') = \psi(gK)\psi(g'K)\end{aligned}$$

where in the second step we used the fact that φ itself is a homomorphism. Thus we have that ψ is a homomorphism.

We'll now show this is a bijective homomorphism, thereby proving the desired isomorphism.

One-to-One. To show this is one-to-one, we can use Theorem 1.1.3.4. Thus our goal will be to show that $\ker(\psi) = \{e_G\}$, the identity element of G.

Suppose

$$\psi(gK) = e$$

which is the identity in $\text{Im}(\varphi)$ (technically, the identity in G'). Then by construction $\varphi(g) = e$. However, this holds for all $g \in K$ (as this is the kernal of φ). Therefore $\ker(\psi) = \{gK \mid g \in K\} = \{K\}$. But K is the identity in G/K. Thus by Theorem 1.1.3.4, we have that ψ is one-to-one.

Onto. To show this is onto, we'll simply show that for any $h \in \text{Im}(\varphi)$, there exists a $gK \in G/K$ such that $\psi(gK) = h$.

So consider any $h \in \text{Im}(\varphi)$. By definition, $h = \varphi(g)$ for some $g \in G$. Now observe that for the element $gK \in G/K$,

$$\psi(gK) = \varphi(g) = h.$$

Thus ψ is onto.

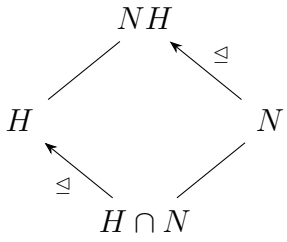
In total, we have showed the following: there exists a bijective homomorphism (i.e., an isomorphism) between $G/K = G/\ker(\varphi)$ and $\text{Im}(\varphi)$. Therefore $G/\ker(\varphi) \cong \text{Im}(\varphi)$ as desired. ■

The second isomorphism theorem summarizes a great deal of useful information concerning groups.

Theorem 1.8.2. (Second Isomorphism Theorem) Let G be a group, H a subgroup of G and N a normal subgroup of G. Then

1. NH is a subgroup of G (HN is also a subgroup)
2. $N \trianglelefteq NH$ (and $N \trianglelefteq HN$)
3. $H \cap N \trianglelefteq H$
4. $H/(H \cap N) \cong NH/N$ (and $H/(H \cap N) \cong HN/N$).

We put parenthesis in some of the statements because while they are true, most people state the second isomorphism theorem by either removing the text in parenthesis or only keeping the text in parenthesis. However, we don't want the reader to get the impression that, for example, $NH \leq G$ but $HN \not\leq G$. We think it is fair to be thorough and precise.



The diagram to the left demonstrates why the Second Isomorphism Theorem is also known as the diamond isomorphism theorem since a relationship between the four main objects in play can be created. The lemma below will clean up the proof of this theorem.

Lemma 1.7.1 Let G be a group. Suppose $N \trianglelefteq G$. Then for any $n \in N$, $h \in G$, there exists $n' \in N$ such that $hn = n'h$ and $nh = hn'$.

Proof: Since N is normal, we know that for any $n \in N$, $hnh^{-1} \in N$ for any $h \in G$. In particular, this means that $hnh^{-1} = n'$ for some $n' \in N$. This implies that $hn = n'h$, which is what we set out to show.

Now we prove the theorem itself. We'll only include proofs for the statments not in paranthesis, because the proofs of the statements in paranthesis are basically identical to the ones we'll offer for those not in paranthesis (e.g., for the proof that $NH \leq G$, only small tweaks are needed to show that $HN \leq G$).

Proof: We'll prove one statement at a time.

1. Consider $NH = \{nh \mid n \in N, h \in N\}$. This is clearly nonempty (N, H are both nonempty), so we can use the Theorem 1.1.1.7, the subgroup test, to prove this.

   Let $n_1h_1, n_2h_2 \in NH$. Our goal is to show that $n_1h_1(n_2h_2)^{-1} \in NH$. Thus observe that

   $$n_1h_1(n_2h_2)^{-1} = n_1h_1h_2^{-1}n_2^{-1} = n_1hn_2^{-1}$$

   where in the last step we know that $h_1h_2^{-1} = h$ for some $h \in H$. Since N is normal, we have by Lemma 1.7.1 that there an $n^* \in N$ such that $hn_2^{-1} = n^*h$. Therefore,

   $$n_1(hn_2^{-1}) = n_1(n^*h) = (n_1n^*)h \in NH$$

   Therefore we have that $n_1h_1(n_2h_2)^{-1} \in NH$, proving that $NH \leq G$.

2. We can prove this directly. Let $nh \in NH$. By Lemma 1.7.1, we know that $nh = hn'$ for some $n' \in N$. Therefore

   $$\begin{aligned}(nh)N(n^{-1}h^{-1}) = (hn')N(n^{-1}h^{-1}) = h(n'Nn^{-1})h^{-1} \\ = hNh^{-1} = N\end{aligned}$$

   where in the last step we used the fact that N is normal and invoked Theorem 1.10.2. By the same theorem, we can then conclude that $N \trianglelefteq NH$.

3. To prove this, first recall that $H \cap N$ is a a subgroup of G since H and N are both subgroups. Now let $a \in H \cap N$ and $h \in H$.

   We can prove normality by Theorem 1.10.3, speficially, that $hah^{-1} \in H$ for all $a \in H \cap N$ and $h \in H$. But since $a \in H$, we already know that $hah^{-1} \in H$. So By Theorem 1.10.3, we thus have that $N \cap H \trianglelefteq H$.

4. To prove this last statement, we first construct a homomorphism $\varphi : H \longrightarrow NH/N$ by defining $\varphi(nh) = Nh$. This is a homomorphism since for $h, h' \in H$,

$$\varphi(nhnh') = Nhnh' = (Nh)(Nnh') = (Nh)(Nh') = \varphi(h)\varphi(h')$$

where in the second step we used the fact that (1) $N \trianglelefteq NH$ and (2) $(Nh)(Nh') = N(hh')$ by Theorem 1.10.4.

Note that φ is onto. For any $Nh \in NH/N$, we note that any $nh \in NH$ maps to this element via φ for any n. Since this is onto, $\text{Im}(\varphi) = NH/N$.

Also, observe that $\ker(\varphi) = H \cap N$, since for any $h \in (H \cap N)$ we have that $\varphi(h) = Nh = N$, which the identity in NH/N.

Now by the First Isomorphism Theorem, we have that

$$H/\ker(\varphi) \cong \text{Im}(\varphi) \implies H/(H \cap N) \cong NH/N$$

as desired.

■

We now move onto the third isomorphism theorem, which matches our intution for when we form a quotient of quotient groups.

Theorem 1.8.3. (Third Isomorphism Theorem) Let K, N be normal subgroups of G, with $N \leq K$. Then $K/N \trianglelefteq G/N$ and

$$(G/N)/(K/N) \cong G/K.$$

Proof: First we'll show that $K/N \trianglelefteq G/N$. Consider any $Nk \in K/N$, where $k \in K$, and any $Ng \in G/N$, where $g \in G$. Our goal will be to show that $(Ng)(Nk)(Ng)^{-1} \in K/N$.

Observe that

$$(Ng)(Nk)(Ng)^{-1} = (Ng)(Nk)(Ng^{-1}) = N(gkg^{-1})$$

where we used the fact that $N \trianglelefteq G$. Since $K \trianglelefteq G$, we know that $gkg^{-1} \in K$. That is, $gkg^{-1} = k'$ for some $k' \in G$. Therefore, $N(gkg^{-1}) \in K/N$ so that $(Ng)(Nk)(Ng^{-1}) \in K/N$. Since g, k were arbitrary, we have by Theorem 1.10.3 that $K/N \trianglelefteq G/N$ as desired.

Next, we'll show that $(G/N)/(K/N) \cong G/K$. We'll do his by constructing an isomorphism between the two groups.

Construct a homomorphism $\varphi : G/N \longrightarrow G/K$ defined as $\varphi(Ng) = Kg$ where $Ng \in G/N$. First, we'll show this is a homomorphism. For any $Ng, Ng' \in G/N$, we have that

$$\begin{aligned}\varphi\Big((Ng)(Ng')\Big) &= \varphi(N(gg')) = Kgg' \\ &= (Kg)(Kg') = \varphi(Ng)\varphi(Ng')\end{aligned}$$

where in the third step we used the fact that $K \trianglelefteq G$. Therefore, this is a homomorphism.
Next, observe that this is onto, since for any $Kg \in G/K$, we know that the element $Ng \in G/N$ maps to Kg via φ. Therefore $\text{Im}(\varphi) = G/K$.
We'll now show that $\ker(\varphi) = K/N$. Observe that

$$\ker(\varphi) = \{Ng : \varphi(Ng) = K\} = \{Ng : Kg = K\} = \{Ng : g \in K\} = K/N.$$

Therefore, $\ker(\varphi) = K/N$.
Finally, we can use the First Isomorphism Theorem to conclude that

$$(G/N)/\ker(\varphi) \cong \text{Im}(\varphi) \implies (G/N)/(K/N) \cong G/K$$

as desired.

■

We now move onto the Fourth Isomorphism Theorem, which is one of the more powerful isomorphism theorems along with the First Isormorphism Theorem.

Theorem 1.8.4. (Fourth Isormorphism Theorem) Let $N \trianglelefteq G$. Then every subgroup of G/N is of the form H/N where $N \leq H \leq G$. Moreover, if H, K are subgroups of G and they contain N, then

1. $H \leq K$ if and only if $H/N \leq K/N$
2. $H \trianglelefteq G$ if and only if $H/N \trianglelefteq G/N$
3. if $H \leq K$ then $[K : H] = [K/N : H/N]$
4. $(H \cap K)/N \cong (H/N) \cap (K/N)$.

$$\begin{aligned} G \geq H_1 &\longrightarrow M_1 \leq G/N \\ G \geq H_2 &\longrightarrow M_2 \leq G/N \\ G \geq H_3 &\longrightarrow M_3 \leq G/N \\ &\vdots \\ G \geq H_n &\longrightarrow M_n \leq G/N \end{aligned}$$

The Fourth Isomorphism Theorem is also commonly known as the correspondence theorem, since what it effectively states is that there is a one-to-one correspondence between subgroups H of G which contain N and the subgroups of G/N.
Thus, if G has n subgroups H_i which contain N, then G/N has n subgroups.

Proof: We first prove the first statement.

Our goal here will be to show that $M \leq G/N \implies M = H/N$ where M is some subgroup of G/N and $N \leq H \leq G$.

Consider a subgroup M of G/N. Let H be the set of all $h \in G$ such that $Nh \in M$. Then observe that $N \subset H$, since the smallest subgroup of G/N is the trivial group, namely $\{N\}$. Therefore $N \subset H \subset G$.

Now we show that $N \leq H \leq G$. To do this, we just need to show that $H \leq G$, which we will do by the subgroup test.

Let $h, h' \in H$. Since $M \leq G/N$, we know that for any $Nh, Nh' \in M$,

$$\underbrace{(Nh')(Nh)^{-1} \in M}_{\text{by the Subgroup Test}} \implies (Nh')(Nh^{-1}) \in M \implies N(h'h^{-1}) \in M.$$

However, in order for $N(h'h^{-1}) \in M$, we have that $h'h^{-1} \in H$. Since h, h' were arbitrary elements of H, we have by the subgroup test we have that $H \leq G$.

But since we have that $N \leq G$, $H \leq G$ and $N \subset H \subset G$, we all together have that $N \leq H \leq G$.

Next, we prove the the statements (1)−(4). To prove (1), we'll show that $H/N \leq K/N \implies H \leq K$ and $H \leq K \implies H/N \leq K/N$ for any subgroups H, K of G which contain N where $N \trianglelefteq G$.

Let H, K be subgroups of G such that $N \subset H$ and $N \subset K$. Furthermore, suppose that $H/N \leq K/N$.

■

The Isomophism Theorems are extremely powerful. The following an application to something which matches our intuiton, but extremely difficult to prove without the isomophism theorems.

Theorem 1.8.5. Let G be a group and H and K be normal subgroups of G. Then

1. HK is a subgroup of G
2. If $\gcd(|H|, |K|) = 1$ then $H \times K \cong HK$.

Proof:

1. Observe that since $H \trianglelefteq G$ and $K \trianglelefteq G$, then obviously $H \leq G$ and hence we can apply the Second Isomorphism Theorem to conclude that $HK \leq G$. Thus we see that for this statement to be true in general we really only need one of the subgroups, either H or K, to be normal to G.

   To prove this, we'll construct an isomorphism between the two groups. In constructing the homomorphism, we'll have to do a bit of work to show our proposed homorphism is in fact a homomorphism, the work which lies in showing elements of H and K commute. Thus we will show this first.

2. The fact that $\gcd(|H|, |K|) = 1$ allows us to concldue that neither $H \not\leq K$ and $K \not\leq H$, since otherwise by Lagrange's theorem the order of one group would divide the other, and obviously we don't have that case here. Thus we know that $H \cap K = \{e\}$, as by our previous argument it would be impossible for them to share any other nontrivial element.

   Since H, K are normal to G we'll have that

$$hkh^{-1} \in K$$
$$kh^{-1}k^{-1} \in H$$

because h and k are both elements in G, and we know for all $a \in G$ that $aha^{-1} \in H$ for $h \in H$ and $aka^{-1} \in K$ for $k \in K$. We can then state that

$$\begin{aligned} \overbrace{(hkh^{-1})}^{\text{A member of } K} k^{-1} &= hkh^{-1}k^{-1} \in K \\ h \underbrace{(kh^{-1}k^{-1})}_{\text{A member of } H} &= hkh^{-1}k^{-1} \in H \end{aligned}$$

by using the fact that H, K are subgroups and are therefore closed under products of their elements. But we showed earlier that $H \cap K = \{e\}$; hence

$$hkh^{-1}k^{-1} \in H \cap K = \{e\} \implies hkh^{-1}k^{-1} = e \implies hk = kh.$$

But h, k were arbitrary elements of H, K, so this shows that products of their elements commute.

Next, consider the function $\varphi : H \times K \to HK$ defined as

$$\varphi((h, k)) = hk.$$

which we will show to be a homomorphism. Observe that if (h_1, k_1) and (h_2, k_2) are in $H \times K$, then

$$\varphi((h_1, k_1) \cdot (h_2, k_2)) = \varphi((h_1h_2, k_1k_2)) = h_1h_2k_1k_2.$$

However, we showed that products of elements between H and K can commute, so that we can rewrite h_2k_1 as k_1h_2 to write

$$\varphi((h_1, k_1) \cdot (h_2, k_2)) = h_1h_2k_1k_2 = h_1k_1h_2k_2 = \varphi((h_1, k_2))\varphi((h_2, k_2)).$$

Thus φ is a homomorphism.

Observe now that $\ker(\varphi) = \{(e, e)\}$. This is because we know that $H \cap K = \{e\}$, so that if

$$\varphi((h, k)) = hk = e$$

we know it is impossible that this could be because $h = k^{-1}$; otherwise, $H \cap K \neq \{e\}$, which we know is not the case. Hence the only time when $hk = e$ is if both h and k are e, so that $\ker(\varphi) = \{(e, e)\}$.

Observe that $\text{im}(\varphi) = HK$. This is because for any $hk \in HK$, we can simply observe that $h \in H, k \in K$, and therefore there exists a $(h, k) \in H \times K$ such that

$$\varphi(h, k) = hk.$$

Thus every element of HK is covered by our mapping, so φ is injective and hence $\text{im}(\varphi) = HK$.

Finally, what we have shown is that (1) φ is a homomorphism and (2) it is a bijection from $H \times K$ to HK. We can now apply the First Isomorphism Theorem to conclude that

$$H \times K/\ker(\varphi) \cong \text{im}(\varphi) \implies H \times K \cong HK$$

because $\ker(\varphi) = \{(e, e)\}, H \times K/\{(e, e)\} = H \times K$, and $\text{im}(\varphi) = HK$. This completes the proof.

■

The First Isomophism Theorem has a lot of fun applications, one of which we present here.

Theorem 1.8.6. Let G and H be groups such that $|G|$ and $|H|$ are coprime. If $\varphi : G \longrightarrow H$ is a homomorphism, then φ is zero homomorphism.

Proof: By the First Isomorphism Theorem, we see that

$$G/\ker(\varphi) \cong \text{Im}(\varphi).$$

Therefore $|G/\ker(\varphi)| = |\text{Im}(\varphi)|$. However,

$$\begin{aligned} |G/\ker(\varphi)| = |G|/|\ker(\varphi)| &= |\text{Im}(\varphi)| \\ \implies |G| &= |\ker(\varphi)| \cdot |\text{Im}(\varphi)|. \end{aligned}$$

Note that $|\ker(\varphi)| \mid |G|$ and $|\text{Im}(\varphi)| \mid |H|$ by Lagrange's Theorem. However, we said that $|G|$ and $|H|$ are corpime which means that $|\text{Im}(\varphi)| = 1$. Hence we must have that $|\ker(\varphi)| = G$, and since $\ker(\varphi) \leq G$ we have that $\ker(\varphi) = G$. Therefore φ sends every element of G to the identity of H, which is what we set out to show. ■

1.9 Group Actions.

As we shall see, a group action is a special type of mapping one can formulate involving a group G and an arbitrary set of objects X. Specifically, it is a mapping from $G \times X \longrightarrow X$. Thus, a group action is said to make a group G "act" on a set X. It is through this perspective that one can then view group actions as permutations of a set X. This becomes more clear with the formal definition.

Definition 1.9.1. Let G be a group and X an arbitrary set. A **group action** of G on X is a mapping $* : G \times X \longrightarrow X$ that

1. $g_1 * (g_2 * x) = (g_1 \cdot g_2) * x$ for all $g_1, g_2 \in G, x \in X$.
2. $e * x = x$ where $e \in G$ is the identity.

Note that $\cdot$ is the *group multiplication in* G. For notational convenience, we will surpress $\cdot$ in the cases for where it's obvious or implied, as usual.
We also note that we could have defined $* : X \times G \longrightarrow X$. For simplicity, we let G act on the left.

Let's breakdown what this is really saying. For a group action $*$ of G acting on X, we have for all $g \in G$, $x \in X$, the product $g * x$ is mapped to some element $x' \in X$.

Now observe that if we replaced X with G, then we get $* : G \times G \longrightarrow G$. Thus $*$ would just permutate the elements of G. Furthermore, if we let $*$ be the group multiplication $\cdot$ which is already defined in G, then we just get back the definition of a group!

This fits with the intuition that, group multiplication of elements (e.g., $g \cdot g'$ where $g, g' \in G$) simply permutates the elements of a group. That is, if you placed the elements of G in a tuple such as

$$(g_1, g_2, \ldots, g_n)$$

and multiplied this by some $g' \in G$, you would get a tuple

$$(g_1, g_2, \ldots, g_n) \cdot g' = (g_1 \cdot g', g_2 \cdot g', \ldots, g_n \cdot g') = (g_i, g_j, \ldots, g_k)$$

containing all the elements of G, but just in a different order. (In this case we supposed $g_1 \cdot g' = g_i, g_2, \cdot g' = g_j$, and so on.)

This permutation phenonmenon can be found in more general group actions. For a fixed $g \in G$, define $\sigma_g : X \longrightarrow X$ as

$$\sigma_g(x) = g * x.$$

So σ_g maps each x to some other element $x' \in X$. Therefore, a group action can be thought of as a set of maps σ_g, one for every element $g \in G$, each of which can appropriately be composed with one another. That's why it can be thought of as a permutation. The diagram on the right gives an illustration how this plays out for one particular $g \in G$ acting on a set X with five elements.

X		X
x_1	$\xrightarrow{\sigma_g}$	x_2
x_2	$\xrightarrow{\sigma_g}$	x_3
x_3	$\xrightarrow{\sigma_g}$	x_5
x_4	$\xrightarrow{\sigma_g}$	x_1
x_5	$\xrightarrow{\sigma_g}$	x_4

Here's another way to think about a group action. If G acts on X, then the group action $*$ turns

each and every element of $g \in G$ into a *function*, which maps X to X. This agrees with our intuition, since a permutation is exactly a function of X to itself.

Theorem 1.9.2. A finite group of order n is isomorphic to a subgroup of S_n.

This theorem is a powerful theorem that gives us a new way to think about finite groups. It states that every finite group is basically the same as a subgroup of a symmetric group up to an isomorphism.

Proof: To prove this, we'll first construct a group action of G on itself. Then we'll

Consider the group action of G acting on itself, whereby we define $g_1 \cdot g_2 = g_1 g_2$ for $g_1, g_2 \in G$. That is, the group action mapping is simply the multiplication used between the elements of G.

This is a Group Action. (Note: we already pointed out that if we replace X with G in the definition of a group action, and let $\cdot$ be the group multiplication in G, then we just get the definition of a group. Thus a group is a special, but boring, type group action.)

To show this is a group action, let $x \in G$. Then

$$g_1 \cdot (g_2 \cdot x) = g_1 \cdot (g_2 x) = g_1 g_2 x = (g_1 g_2) x = (g_1 g_2) \cdot x.$$

for $g_1, g_2 \in G$. Therefore, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$. The second axiom is satisfied, since if e is the identity of G, then clearly $e \cdot x = ex = x$. We have both axioms satisfied. So this is a group action.

■

Before we lead up to a powerful theorem involving group actions, we must define a few definitions.

Definition 1.9.3. Suppoe G acts on a set X, and let $x \in X$. Then we define the set

$$Gx = \{g * x \mid g \in G\}$$

as the **orbirt** of x.

The orbit basically considers the set of all images one obtains when one grabs a single element of $x \in X$, and multiplies it by every element $g \in G$. Note that since $g \cdot x \in X$ for every $g \in G, x \in X$, we have that $Gx \subset X$.

Orbits are rather interesting since **they partition their acting set** X. That is, if $X = \{x_1, x_2, \ldots, x_n\}$, then $Gx_1 \cup Gx_2 \cup \cdots \cup Gx_n = X$. Note that $x_i \in Gx_i$ for $i = 1, 2, \ldots, n$, so this definitely makes sense.

However, it is possible that $Gx_i = Gx_j$ for some i, j. In such a case we note that for each $g \in G$ there exists a $g' \in G$ such that $gx_i = g'x_j \implies g^{-1}g'x_j = x_i$. Since $g^{-1}g' \in G$, our condition boils down to the following: $Gx_i = Gx_j$ for some i, j if there exists a $g \in G$ such that $gx_j = x_i$. So $Gx_i = Gx_j$ if $x_j \in Gx_i$.

Thus, these things are behaving like cosets (recall that $Gh = Gh'$ if and only if $h' \in Gh$.) and

they partition the acting set X! This understanding will become helpful in the future.

Since orbits form partitions, and it is possible that the set of all orbits will be redundant (i.e., it's possible that $Gx_i = Gx_j$ for some i, j), we offer the following definition.

Definition 1.9.4. Let G be a group, and suppose it acts on a set X. Let $Gx_1, Gx_2, \cdots, Gx_n$ be a distinct set of orbits such that

$$Gx_1 \cup Gx_2 \cup \cdots G_n = X.$$

Then each $x_1, x_2, \ldots, x_n$ are called **representatives of an orbit** of G. We generally denote $R = \{x_1, x_2, \ldots x_n\}$ to be the set of representatives of the orbits.

Thus for some orbit Gx_i, we say that x_i "represents" this orbit. We make this definition since we just showed that it doesn't really matter what representative we pick, since if $x_j \in Gx_j$, $Gx_j = Gx_i$, so x_j could have equally represented this orbit. Thus given this arbitrary-ness, the definition allows us to talk about orbits more easily.

We now offer another definition regarding group actions.

Definition 1.9.5. Suppose G acts on X, and $x \in X$. Then the set

$$G_x = \{g \in G \mid g * x = x\}.$$

is defined to be the **stabilizer** of x.

The stabilizer considers the elements of $g \in G$ which act as an identity to x. Since G_x considers elements of G, we see that $G_x \subset G$. Furthermore, we have the following proposition.

Proposition 1.9.6. Suppose G acts on X, and let $x \in X$. Then $G_x \leq G$.

Proof: Observe first that this is nonempty, since $e * x = x$ for all $x \in X$, where $e \in G$ is the identity. Therefore $e \in G_x$. Next, observe that associativity is inherited from the set G itself. To check for inverses, we note that for any $g \in G$, $g \cdot x = x$, so we can multiply both sides by g^{-1} to get

$$g^{-1} * g * x = g^{-1} * x \implies (g^{-1}g) * x = g^{-1} * x \implies x = g^{-1} * x.$$

Thus $g^{-1} * x = x$ so $g^{-1} \in G$. Finally, observe that the set is closed. Given $g, g' \in G_x$, we see that

$$(gg') * x = g * (g' * x) = g * '(x) = x.$$

Therefore G_x is (1) a subset of G and (2) a group so it is a subgroup of G.

■

We now move onto one of the useful theorems that arises once one realizes the definitions of the orbit and stabilizers.

Theorem 1.9.7. Let G be a finite group, and suppose G acts on a set X. Then for any $x \in X$ we have that

$$|G| = |Gx| \cdot |G_x|.$$

Proof: To show this, we'll construct a bijection between Gx and G/G_x.

Let $g \in G$ so that $gG_x \in G/G_x$. Then construct the map $\psi : G/G_x \longrightarrow Gx$ by

$$\psi(gG_x) = g * x.$$

Note that there is only one element in G/G_x which gets to x; namely, G_x. The calculation is as follows:

$$\psi(G_x) = e * x = x.$$

This map is obviously surjective, since for any $x' \in Gx$, we we know that there exists a $g \in G$ such that $g * x = x'$. Thus $g \notin G_x$, so that gG_x is nontrivial and $\psi(gG_x) = g * x = x'$.

Now to show that this is injective, suppose that $g * x = h * x$. we have that $g^{-1}h * x = x$. Therefore, $gh^{-1} \in G_x$. Furthermoresee that

$$g^{-1}hG_x = G_x \implies hG_x = gG_x.$$

Thus this can only happen if the input is the same. Therefore this is a one-to-one and onto mapping.

Since this is a bijection, we can conclude that

$$|Gx| = |G/G_x| = |G|/|G_x| \implies |G| = |Gx||G_x|$$

as desired. ■

1.10 Conjugation, The Class Equation, and Cauchy's Theorem.

We now touch on a very deep example of a group action, known as conjugation. Let G act on itself "by conjugation", which we define as follows. Let $g, h \in G$. Then

$$g * h = ghg^{-1}$$

is the group action of conjugation. Let's show that this is a group action.

Composition. Let $g_1, g_2, h \in G$. Then observe that

$$\begin{aligned} g_1 * (g_2 * h) = g_1 * (g_2 h g_2^{-1}) &= g_1 g_2 h g_2^{-1} g_1^{-1} \\ &= (g_1 g_2) h (g_1 g_2)^{-1} \\ &= (g_1 g_2) * h \end{aligned}$$

so that the first axiom of a group action is satisfied.

Identity. Observe also that for $e \in G$, the identity of G,

$$e * h = ehe^{-1} = h.$$

Therefore this is a group action.

We'll now show that this group action is very special and important. Conjugation itself is important in math. In Linear Algebra, two matrices which are similar (i.e., A is similar to B if there exists P such that $A = P^{-1}BP$) have **the same rank, determinants, trace, eigenvalues, and much more**. Basically, they represent the same linear transformation, just in different bases. To learn more about conjugation, we make a few definitions with this group action.

Definition 1.10.1. Let G be a group, and let G act on itself by conjugation. For any $h \in G$, **the orbit** of this group action

$$\begin{aligned} Gh &= \{g * h \mid g \in G\} \\ &= \{ghg^{-1} \mid g \in G\} \end{aligned}$$

is known as a **conjugacy class** of G.

Previously we discussed how orbits of a group action partition the set X which is being acted on. Since G acts on itself in this example, we see that **the conjugacy classes form a partition of G!**

Remark. Recall the definition of a centralizer G for a set $A \subset G$:

$$\begin{aligned} C_G(A) &= \{g \in G \mid gs = sg \text{ for all } s \in S\} \\ &= \{g \in G \mid gsg^{-1} = s \text{ for all } s \in S\}. \end{aligned}$$

Therefore for a single point $x \in G$, $C_G(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid g * x = x\}$, where in the last equation we are speaking in terms of group actions. But note that this last set is exactly

the **stabilizer** of G under this group action. **Therefore, $C_G(x) = G_x$ for any $x \in G$ under this group action.**

Furthermore, let $x \in Z(G) = \{z \in G \mid z = gzg^{-1} \text{ for all } g \in G\}$, the center of G. Then we see that $Gx = \{gxg^{-1} \mid g \in G\} = \{x\}$. **So for any $x \in Z(G)$, the orbit is of size one. The sole element it contains is just x.** (We can go even further: the conjugacy classes of an abelian group are all of size one.)

Let's put all of these results together. In general, if G acts on itself via conjugation, then we know its orbits, or conjugacy classes, partition G. Moreover, let $R \subset X$ be a set of orbit representatives (or conjugacy class representatives, if you like). Then

$$|G| = \sum_{x \in R} |Gx|$$

Recall that $|Gx| = 1$ if $x \in Z(G)$. Thus we can write this further as

$$\begin{aligned} |G| = \sum_{x \in Z(G)} |Gx| + \sum_{x \in R \setminus Z(G)} |Gx| &= \sum_{x \in Z(G)} 1 + \sum_{x \in R \setminus Z(G)} |Gx| \\ &= |Z(G)| + \sum_{x \in R \setminus Z(G)} |Gx| \end{aligned}$$

By the Orbit-Stabilizer theorem, we can write $|Gx| = |G|/|G_x|$. Substituting this in, we get

$$|G| = |Z(G)| + \sum_{x \in R \setminus Z(G)} |G|/|G_x|$$

and since $C_G(x) = G_x$,

$$|G| = |Z(G)| + \sum_{x \in R \setminus Z(G)} |G|/|C_G(x)|$$

which is known as the **class equation.** This equation is pretty badass, as it gives us a way to understand the cardinality of a group. This equation is also useful in proofs, as we shall see in the following examples. First, we begin with a lemma.

Lemma 1.10.2. Let G be a group. Then $C_G(x) = G$ if and only if $x \in Z(G)$.

Proof: Suppose $C_G(x) = G$. Then for all $g \in G$, $gx = xg$. However, $Z(G)$ is the set of all G which commutes with every member of G, so $x \in Z(G)$. Now suppose $x \in Z(G)$. Then $gx = xg$ for all $g \in G$. Therefore, $C_G(x) = \{g \in G \mid gx = xg\} = G$. ■

Theorem 1.10.3. Let G be a group such that $|G| = p^n$ for some prime p and $n \in \mathbb{N}$. Then $|Z(G)| > 1$. That is, $|Z(G)| \in \{p, p^2, \ldots, p^n\}$.

Equivalently, this theorem says that $Z(G)$ is nontrivial. Moreover, this implies that **there exists non identity elements of G which commute with every element of G.**

Proof: First observe that $Z(G)$ is a subgroup of G. Therefore, by Lagrange's Theorem, we know that $|Z(G)|$ divides G. Thus $|Z(G)| \in \{1, p, p^2, \ldots, p^n\}$. Our goal is to show that $|Z(G)|$ cannot equal 1.

For the sake of contradicton, suppose $|Z(G)| = 1$. Then by the previous lemma, we see that there is no nontrivial element g of G such $C_G(g) = G$.

Let R be the set of conjugacy class representatives. Then $|G|/|C_G(r)| \in \{p, p^2, \cdots, p^n\}$ for $r \in R \setminus Z(G)$ (since $|Z(G)| = 1$, $R \setminus Z(G)$ simply removes e, the identtiy, from R).

Why can't $|G|/|C_G(r)| = 1$ for any $r \in R \setminus Z(G)$? Well, because for such an r, $r \notin Z(G)$. Therefore $C_G(r) \neq G$, so $|G|/|C_G(r)| \neq 1$.

Now by the class equation, we see that

$$\underbrace{|G|}_{\text{divisible by } p} = |Z(G)| + \overbrace{\sum_{r \in R \setminus Z(G)} |G|/|C_G(r)|}^{\text{divisible by } p}$$

since $|G|/|C_G(r)| \in \{p, p^2, \ldots, p^n\}$ for all $r \in R \setminus Z(G)$. Therefore we see that $|Z(G)|$ must be divisible by p. But this is a contradiction since we said $|Z(G)| = 1$. Therefore, we see that $|Z(G)| \in \{p, p^2, \ldots, p^n\}$.

■

The above theorem can be used to prove the next theorem, whose signifiance demonstrates the power of the class equation. The theorem below is generally proved by proving the above theorem first in the special case for when $|G| = p^2$. But it will be helpful to other proofs later on to consider the more general case as we presented it above.

Theorem 1.10.4. Let G be a group, and suppose $|G| = p^2$ where $p \geq 2$ is prime. Then G is abelian.

Proof: By the previous theorem, we see that $|Z(G)| \in \{p, p^2\}$.) We'll proceed by considering two cases.

$|\mathbf{Z(G)}| = \mathbf{p^2}$. In this case $|G| = |Z(G)|$. Since we also have that $Z(G)$ is a subgroup of G, we can conclude that $G = Z(G)$. Therefore, G is abelian.

$|\mathbf{Z(G)}| = \mathbf{p}$. Recall that $Z(G) \trianglelefteq G$ from Proposition 1.6.4. Therefore, we can speak of the quotient group $G/Z(G)$, which has size $|G|/|Z(G)| = p^2/p = p$. By the corollary to Lagrange's Theorem, this implies that $G/Z(G)$ is cyclic, since it has prime order. Thus there exists a $g \in G$ such that we can represent $G/Z(G)$ as

$$G/Z(G) = \{Z(G), Z(G)g, Z(G)g^2, \ldots, Z(G)g^{p-1}\}.$$

As we already know, cosets partition G. Therefore, let $a, b \in G$, and suppose $a \in Z(G)g^i$ and $b \in Z(G)g^j$. Then there exist $x, y \in Z(G)$ such that $a = xg^i$ and $b = yg^j$. Thus observe that

$$ab = xg^iyg^j = xyg^ig^j = xyg^{i+j} = xyg^jg^i = yg^jxg^i = ba$$

where we used the commutavity of x, y since $x, y \in Z(G)$. Since a, b were arbitrary members of G, this proves that G is abelian.

■

Thus we see that the class equation is useful in proving more general facts about group theory. The class equation can also be used to prove the following important theorem in group theory, known as Cauchy's Theorem.

Theorem 1.10.5. (Cauchy's Theorem) Let G be a finite group and $p \geq 2$ be a prime. If p divides the order of G, then G has an element of order p.

So consider a group G with order n, and suppose

$$n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_n^{i_n}$$

is its prime factorization. Then there exist elements $g_1, g_2, \ldots, g_n$ such that $|g_i| = p_i$ for $i = 1, 2, \ldots, n$.

Another way to visualize this as follows. Consider a group G consisting of 10 elements.

$$\{e,\ g_1,\ g_2,\ g_3,\ g_4,\ g_5,\ g_6,\ g_7,\ g_8,\ g_9\}$$

By Cauchy's theorem, there exists elements of order 2 and 5. So suppose g_1 and g_2 are such elements, i.e., $g_1^2 = 3$ and $g_2^5 = e$. Then we can really rewrite this as

$$\{e,\ g_1,\ g_2,\ g_2^2,\ g_2^3,\ g_2^4,\ g_6,\ g_7,\ g_8,\ g_9\}.$$

However, we know $g_1g_2, g_1g_2^2, g_1g_2^3$ and $g_1g_2^4$ are all in G. Thus we can really write this as

$$\{e,\ g_1,\ g_2,\ g_2^2,\ g_2^3,\ g_2^4,\ g_1g_2,\ g_1g_2^2,\ g_1g_2^3,\ g_1g_2^4\}.$$

Thus we can understand the structure of every single group of order 10. But this can be done for all finite groups!

Proof: In this proof, we'll prove this in a very clevery way by letting a subgroup of a permutation group act on a special set X (both of which we will define). This will then prove the existence of elements of order p.

Let p be a prime which divides $|G|$. Define H to be the cyclic subgroup of S_p generated by $(1\ 2\ \cdots\ p)$.

We can picture H as the group

$$\{(1\ 2\ \cdots\ p), (2\ 3\ \cdots\ p,\ 1), \cdots, (p\ 1\ \cdots\ p-1)\}.$$

Now let H act on the set X defined as

$$X = \{(g_1, g_2, \ldots, g_p) \mid g_1, g_2, \ldots, g_p \in G \text{ and } g_1g_2\cdots g_p = e\}$$

where the $\sigma \in H$ acts on $g \in X$ as

$$\sigma \cdot (g_1, g_2, \ldots, g_p) = (g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(p)}).$$

This H takes a p-tuple in X and permutates the elements. Since H is generated by $(1\ 2\ \cdots\ p)$, it "pushes" the elements g_i in the tuple over to the right, and the elements that are pushed out of the right end of the tuple are pushed back in on the left side.

First we'll show that this is a group action.

This is a Group Action. Let $x \in X$ and $\sigma \in H$. If $x = (g_1, g_2, \ldots, g_p)$, observe that

$$\sigma * x = (g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(p)}).$$

Suppose $h(1) = n$. Then in general $h(i) = (i + n) \bmod p$. Therefore, we see that

$$(g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(p)}) = (g_n, g_{n+1}, \ldots, g_p, g_1, \ldots, g_{n-1}).$$

However, observe that

$$g_1 g_2 \cdots g_p = g_1 g_2 \cdots g_{n-1} g_n g_{n+1} \cdots g_p = e \implies (g_1 g_2 \cdots g_{n-1})(g_n g_{n+1} \cdots g_p) = e.$$

Thus the elements $g_1 g_2 \cdots g_{n-1}$ and $g_n g_{n+1} \cdots g_p$ in G are inverses of each other. But know that if two group elements are inverses, either order of their product returns e. Therefore

$$(g_n g_{n+1} \cdots g_p)(g_1 g_2 \cdots g_{n-1}) = g_n g_{n+1} \cdots g_p g_1 g_2 \cdots g_{n-1} = e.$$

We therefore see that $(g_n, g_{n+1}, \ldots, g_p, g_1, \ldots, g_{n-1}) = \sigma * x \in X$.
Now we verify associativity. For any $\sigma_1, \sigma_2 \in H$, we see that

$$\begin{aligned} \sigma_1 * \sigma_2 * x &= \sigma_1 * (g_{\sigma_2(1)}, g_{\sigma_2(2)}, \ldots, g_{\sigma_2(p)}) \\ &= (g_{\sigma_1(\sigma_2(1))}, g_{\sigma_1(\sigma_2(2))}, \ldots, g_{\sigma_1(\sigma_2(p))}) \\ &= (\sigma_1 \sigma_2) * (g_1, g_2, \ldots, g_p). \end{aligned}$$

Thus $*$ is associative. Finally, if σ is the trivial element,

$$\sigma * x = (g_{\sigma(1)}, g_{\sigma(2)}, \cdots g_{\sigma(p)}) = (g_1, g_2, \ldots, g_p) = x.$$

Therefore this is a group action.

Now that we've shown that this is a group action, we'll argue that the orbits are either of size 1 or p.

The Orbits. For any $x \in X$ such that $x = (g_1, g_2, \ldots, g_p)$, we see that the orbit Hx will simply be all of the permutations of the p-tuple $(g_1, g_2, \ldots, g_p)$. Note however that there are only p many ways to rearrange this tuple, so that any orbit Hx will be of size p.
Of course, the exception to this is if $g_1 = g_2 = \cdots = g_p$. In this case, there are no other ways to reorganize the tuple. Hence the orbit will have size 1.

Finally, we will show that there exists a nontrivial orbit of size 1. This is equivalent to show

that there exists a nontrivial element of G of order p, which we'll eloaborate later.

Orbit of Size 1. First let's count the elements of X. Observe that for any $(g_1, g_2, \ldots, g_p) \in X$, the last element g_p is always determined by the first $p-1$ elements. This is because if we know the first $p-1$ elements, then

$$g_p = (g_1 g_2 \cdots g_{p-1})^{-1}$$

in order for $g_1 g_2 \cdots g_p = e$. Since there are $|G|^{p-1}$ many ways to pick the first $p-1$ elements in any p-tuple of X, we see that $X = |G|^{p-1}$.

Now by hypothesis, p divides $|G|$. Therefore p divides $|X|$ so we may write $|X| = np$ for some integer n.

Since the orbits of X form a partition, the orbits partition a set np elements into orbits of size 1 or size p. We know one orbit of size 1 exists (namely, the trivial orbit $He = \{(e, e, \ldots, e)\}$), so there must exist at least $p-1$ nontrivial other orbits of size 1.

Let Hx' be one of those orbits. Then for some $g \in G$ we have that $Hx = \{(g, g, \ldots, g)\}$. However since $Hx \subset G$, what we have prove is the existence of a nontrivial element $g \in G$ such that $gg \cdots g = g^p = e$, which set out to show.

This completes the proof. ■

Cacuhy's Theorem is an incredibly useful tool one can use in finite group theory. Here's an amazing and useful theorem who's proof is eased via Cauchy's Theorem.

Theorem 1.10.6. Let G be a group $p \geq 2$ a prime. If $|G| = p^n$ for some $n \in \mathbb{N}$, then G has a subgroup of order p^k for all $0 < k < n$.

Note we didn't write $0 \leq k \leq n$. We could have, but we already know that there exists a subgroup of order p^n (namely, G itself) and that there exists a subgrou of order $p^0 = 1$ (namely, the trivial group).

Proof: To prove this, we'll use strong induction on the statement. Specifically, we'll induct on the powers of n.

Let us induct on n in the statement above. Then for $n = 1$, there is no such $k < n$. Hence the statement is vacuously true.

Next suppose that the statement is true up to order p^n, and let G be a group of order p^{n+1}.

By Theorem 1.1.10.3, we already note that $|Z(G)| > 1$ and hence is a multiple of p. By Cauchy's Theorem, we then know that $Z(G)$ contains an element g of order p. Note that (1) $\langle g \rangle$ is a subgroup of $Z(G)$ and (2) $h \langle g \rangle = \langle g \rangle h$ for all $h \in G$ (since, by definition of the center, every element of $Z(G)$ commutes with elements of G). Therefore $\langle g \rangle \trianglelefteq G$.

Let $H = \langle g \rangle$. Since we just showed $H \trianglelefteq G$ we can appropriately discuss the quotient group G/H.

Observe that $|G/H| = |G|/|H| = p^{n+1}/p = p^n$. Thus by hypothesis, G/H has a subgroup of order p^k for all $0 < k < n$. Denote these such subgroups of G/H as

$$\{N_1/H, N_2/H, \cdots N_{n-1}/H\}$$

where $|N_k/H| = p^k$. Since $H \trianglelefteq G$, we know by the Fourth Isomorphism Theorem that every subgroup of G/H is of the form N/H where $H \leq N \leq G$. Thus we see that

$$H \leq N_k \leq G$$

for all $0 < k < n$. But since $|N_k/H| = p^k$, and $|H| = p$, we see that each such N_k will now have order p^{k+1}. Thus what we have shown is that G itself contains subgroups of order k for all $1 < k < n+1$. The subgroup H of order p is the final piece to this puzzle, and allows us to confirm that G has a subgroup of order p^k for all $0 < k < n$. By strong induction this holds for all $\mathbb{N}$, which completes the proof. ■

1.11 Sylow Theorems.

Lagrange's Theorem states that $H \leq G$, then $|H|$ divides $|G|$. However, you may wonder if there is some kind of converse. If k divides $|G|$, is there a subgroup of order k?

By Cauchy's Theorem, we know that if p is a prime which divides then there exists an element of order p. Can we generalize this result further (for example, state *how* many such elements satisfy this)?

The answer to both questions is yes and is achieved through Sylow's Theorem. It's a foundational theorem in finite group theory, as it strengthens our two most power theorems for finite groups: Lagrange's Theorem and Cauchy's Theorem.

Definition 1.11.1. H is a **p-subgroup** of a group G if H is a subgroup of G and $|H| = p^n$ for some $n \geq 1$.

Definition 1.11.2. Let G be a group and let p be a prime such that $p \mid |G|$. Suppose p^k is the largest power such that $p^k \mid |G|$. That is, $|G| = p^k m$ for some integer $m \in G$, $\gcd(p, m) = 1$. Then any subgroup H of G with $|H| = p^k$ is called a **Sylow p-subgroup**.

An equivalent definition is the following: H is a **Sylow-p subgroup** if H is a p-subgroup where $|H| = p^k$.

A Sylow p-subgroup is nothing more than a subgroup H where $|H| = p^k$ and $|G| = p^k m$ where gcd(p, m) = 1.

Definition 1.11.3. Let G be a group and P and Q be subgroups of G. If there exists an element $g \in G$ such that

$$gPg^{-1} = Q$$

then P is **conjugate** to Q.

Recall that if $H \trianglelefteq G$, then for any $g \in G$ we see that $gHg^{-1} = H$. Thus H is conjugate to itself. Also note that if P is a subgroup then so is gPg^{-1}.

Theorem 1.11.4. (Sylow Theorem) Let G be a finite group and p a prime such that $p \mid |G|$. Suppose further that $|G| = p^k m$ where $\gcd(p, m) = 1$. Then

1. There exists a Sylow p-subgroup (equivalently, there exists a subgroup H of G where $H = p^k$) and every p-subgroup of G is contained in some Sylow p-subgroup
2. All Sylow p-subgroups are conjugate to each other, and the conjugate of any Sylow p-subgroup is also a Sylow p-subgroup
3. If n_p is the number of Sylow p-subgroups, then

$$n_p \mid m \quad \text{and} \quad n_p = 1 \bmod m.$$

Proof: We can prove the first part by letting G act on a special set Ω. It will turn out that the stabilizer of our action will be the desired Sylow p-subgroup.

1. Define
$$\Omega = \{X \subset G \mid |X| = p^k\}$$
and let G act on Ω from the left. Observe that for $X \in \Omega$, $g*X = \{gx \mid \text{ for all } x \in X\} = gX$. Since $|gX| = |X| = p^k$, we see that $gX \in \Omega$. Associativity and identity applications are trivial, so we get that this is a group action.

   Now that we have shown that this is a group action, we will consider the orbits of the group aciton.

   Since $|G| = p^k m$, there are $\binom{p^k m}{p^k}$ many ways for us to choose a subset X of G with size p^k. Hence $|\Omega| = \binom{p^k m}{p^k}$. Note that since this is a group action, the orbits form a partition of X. Now from number theory, we know that
$$\binom{p^k m}{p^k} = m \bmod p.$$
   Since the orbits must partition Ω, the above result tells us that we cannot partition G with sets which are divisible by p. In other words, there must exist some orbit $\mathcal{O}$ such that $|\mathcal{O}|$ is not divisible by p.

   Now that we know that there exists an orbit not divisible by p, we will anaylze the corresponding stabilizer of this orbit. This stabilizer will turn out to be our Sylow p-subgroup.

   Let H be the orbit corresponding to $\mathcal{O}$. Then by the Orbit-Stabilizer Theorem
$$|G| = |\mathcal{O}||H| \implies p^k m = |\mathcal{O}||H|.$$
   By the last equation, we see that p^k must divide both sides. However, $|\mathcal{O}|$ is not divisible by p. Hence $|H|$ must be divisible by p^k.

   However, by Lagrange's Theorem, $|H|$ divides $|G| = p^k m$. Therefore $|H| = m$ or $|H| \in \{1, p, p^2, \ldots, p^k\}$. In either case $|H| \leq p^k$ (since $m \leq p$). But we just showed that p^k divides $|H|$, which proves that $|H| = p^k$.

   Since H is a stabilizer, $H \leq G$, so we have effectively proved the existence of a subgroup of order p^k; or, in other words, a Sylow p-subgroup.

2. Suppose H and K are Sylow p-subgroups of G. Then observe that

3.

■

The consequences of this theorem are immediate.

Proposition 1.11.5. Let G be a finite group and suppose $|G| = p^k m$ for some prime p where

$\gcd(p, m) = 1$. Then G has a normal subgroup of order p^k if and only if $n_p = 1$.

Proof: ($\Longrightarrow$) Suppose G has a normal subgroup H of order p^k. By Sylow's Theorem, we know that all other Sylow p-subgroups are conjugate to H. Thus let $g \in G$ and observe that

$$gHg^{-1} = H$$

since H is normal. Therefore, there are no other Sylow p-subgroups so $n_p = 1$.

($\Longleftarrow$) Now suppose that $n_p = 1$. Let H be a sole Sylow p-subgroup of G. Since it is the only Sylow p-subgroup, we see that

$$gHg^{-1} = H$$

for all $g \in G$. However this exactly the definition for H to be a normal subgroup of G. This proves the result. ■

Once you use Sylow's Theorem and study finite groups more, you'll realize that some groups aren't that complicated. For example, consider *any* subgroup of order 4. This can be any wild group you want, but at the end of the day, it turns out one of the following options is true:

$$G \cong \mathbb{Z}/4\mathbb{Z} \quad \text{or} \quad G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The process leading to such a conclusion is known as **classifying groups up to an isomorphism**. That is, you start with a group with a fixed order, and then determine much simpler groups that your group could be isomorphic to. In our example, we say that any group of order 4 can only be two things up to an isomorphism.

The cool thing about Sylow's Theorem is that it is so strong that it allows us to classify groups up to an isomorphism.

In general, when classifying groups up to an isomorphism, it is convenient to do in terms of integer groups $\mathbb{Z}$ or modulo integer groups, as we saw above. This isn't always possible, but when it is, the following theorem comes in handy.

Theorem 1.11.6. Let m, n be positive integers. Then

$$\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

if and only if m and n are coprime.

Example.
Suppose we want to classify all groups of order 1225 up to an isomorphism.
Let G be a group such that $|G| = 1225 = 5^2 7^2$. Then observe $\gcd(5, 7^2) = 1$. By Sylow's theorem, we know that if n_5 is the number of Sylow 5-subgroups of G, then

$$n_5 \Big| 7^2 \quad \text{and} \quad n_5 \equiv 1 \bmod 5.$$

Observe that n_5 can only equal 1. Since $n_5 = 1$, we know by Propsition 1.11.5 that for the unique Sylow 5-subgroup H that $H \trianglelefteq G$. Also note that $|H| = 5^2$.

Now observe that $\gcd(7, 5^2) = 1$. By Sylow's Theorem, we know that if n_7 is the number of Sylow 7-subgroups of G that

$$n_7 \Big| 5^2 \quad \text{and} \quad n_7 \equiv 1 \bmod 7.$$

Note that n_7 must also equal 1. Thus again for the unique Sylow 7-subgroup K, we must have that $K \trianglelefteq G$ and $|K| = 7^2$. Now we can observe that (1) $\gcd(|H|, |K|) = 1$ and (2) $|G| = |H||K|$ so that

$$G \cong H \times K$$

by Theorem 1.1.8.5. Now observe that since $|H| = 5^2$, $H \cong \mathbb{Z}/25\mathbb{Z}$ and $H \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. Since $K = 7^2$, $K \cong \mathbb{Z}/49\mathbb{Z}$ and $H \cong (\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z})$. Therefore, we see that the groups of order 1225 are, up to isomorphism,

(1) $(\mathbb{Z}/25\mathbb{Z}) \times (\mathbb{Z}/49\mathbb{Z})$

(2) $(\mathbb{Z}/25\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z})$

(3) $(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \times \mathbb{Z}/49\mathbb{Z}$

(4) $(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z})$.

We suspect that these are all the groups of order 1225 up to an isomorphism. However, we double check that none of these groups are actually equivalent to each other, i.e., that we have no redundancies.

Observe that (1) is not isomorphic to any of the the other groups, since $(1, 1) \in (\mathbb{Z}/25\mathbb{Z}) \times (\mathbb{Z}/49\mathbb{Z})$, has order 1225 but none of the other groups have an element of order 1225.

In addition, (3) is not isomorphic to (2) or (3) since $(0, 1) \in (\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \times \mathbb{Z}/49\mathbb{Z}$ and has order 49 but no element of either (2) or (3) has an element of either 49.

Finally, we see that (2) is not isomorphic to (4) because $(1, 0) \in (\mathbb{Z}/25\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z})$ is an element of order 25 but there is no element of order 25 in (4). Thus we see that (1) these subgroups are isomorphic to G and (2) none of them are isomorphic to each other. Therefore, this an exhaustive list of all the groups of order 1225 up to isomorphism.

Here's another example in which Sylow's Theorem helps us classify a specific type of group.

Theorem 1.11.7. Let p, q be primes with $p < q$ and suppose p does not divide $q - 1$. If G is a group such that $|G| = pq$, then $G \cong \mathbb{Z}/pq\mathbb{Z}$.

Proof: Let G be a group and $|G| = pq$. Since $\gcd(p, q) = 1$, by the Sylow Theorem, there exists a Sylow p-subgroup and Sylow q-subgroup of G.
Now let n_p and n_q be the number of Sylow p and q-subgroups, respectively. Then observe that

$$n_p \Big| q \qquad n_p \equiv 1 \bmod p$$

so that $n_p = 1$ and

$$n_q \Big| p \qquad n_q \equiv 1 \bmod q.$$

Now observe that $n_p = 1$ or q. However, since p does not divide $q - 1$, we know that

$$q \not\equiv 1 \bmod p.$$

Thus $n_p = 1$. Again, either $n_p = 1$ or p but $p < q$ so

$$n_q \not\equiv 1 \bmod q$$

unless $n_q = 1$. Thus there is one and only one Sylow p-subgroup and Sylow q-subgroup, which we can call H and K respectively. By proposition 1.11.5,

$$H \trianglelefteq G \qquad K \trianglelefteq G.$$

Note that (1) $\gcd(|H|, |K|) = \gcd(p, q) = 1$ and (2) $|G| = |H||K| = pq$. Thus $G \cong H \times K$ by Theorem 1.1.8.5. Now observe that H and K are of prime order, so that $H \cong \mathbb{Z}/p\mathbb{Z}$ and $K \cong \mathbb{Z}/q\mathbb{Z}$. We then see that

$$G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}.$$

From theorem ???, we know that if m, n are positive integers and $\gcd(m, n) = 1$, then

$$\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/mn\mathbb{Z}.$$

Obviously, $\gcd(p, q) = 1$, so that

$$\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/pq\mathbb{Z}.$$

Now isomorphic relations are transitive, so we can finally state that

$$G \cong \mathbb{Z}/pq\mathbb{Z}$$

as desired. ■

1.12 Fundamental Theorem of Finite Abelian Groups.

Due to Sylow's Theorem, it is now an easy task to classify groups of small orders up to an isomorphism by hand. However, abelian groups are even easier to understand. Abelian groups have a simple enough structure that we can actually generalize the structure of *every* ablian group with the following theorems.

First we begin with a lemma.

Lemma 1.12.1. Let G be a finite abelian group. Then G is isomorphic to a direct product of its Sylow p-subgroups.

Proof: Since G is finite, suppose $|G| = p_1^{n_1}p_2^{n_2}\cdots p_n^{n_k}$ where p_i are distinct primes and n_i are positive integers for $i = 1, 2, \ldots, k$.

By Sylow's Theorem, there exist Sylow p_i-subgroups for each $i = 1, 2, \ldots, k$. Denote these subgroups as H_i (and hence $|H_i| = p_i^{n_i}$). Observe that $\gcd(p_i^{n_i}, p_j^{n_j}) = 1$ for any $i \neq j$. Hence, no H_i is a subgroup of any other H_j for $i \neq j$.

(Otherwise, Lagrange would tell us that that's nonsense because the order of subgroup always divides the order of the bigger group; and in this case, $\gcd(p_i^{n_i}, p_j^{n_j}) = 1$.)

We can equivalently state that $H_i \cap H_j = \{e\}$ for $i \neq j$, where e is the identity of G.

Now observe that (1) $H_i \trianglelefteq G$ for all i since G is abelian and (2) $H_i \cap H_j = \{e\}$ and (3)

$$|G| = |H_1| \cdot |H_2| \cdots |H_k|.$$

Therefore, we can repeatedly apply Theorem 1.1.8.5 to conclude that

$$G \cong H_1 \times H_2 \times \cdots \times H_k.$$

So G is a product of its Sylow subgroups. ■

Lemma 1.12.2. Let G be an abelian group and p a prime. Then if $G = p^n$ for some positive integer n, then G is isomorphic to a direct product of its cyclic groups.

Proof: We'll proceed with strong induction. Consider our base case with $n = 1$. Then we see that $G = p$, and by the corollary to Lagrange's Theorem we know that this is cyclic.

For the inductive case, suppose this statement holds up to p^n. Let G be a group such that $|G| = p^{n+1}$. Let g be an nontrivial element of G, and consider the cyclic subgroup $\langle g \rangle$.

Now define H as follows:

$$H = (G \setminus \langle g \rangle) \cup \{e\} = \{h \in G \mid h \neq g^i \text{ for } i = 1, 2, \ldots, m-1\}.$$

We will show that this is a subgroup via the subgroup test. First observe that H is nonempty, since we supposed that $|g| \neq k+1$. Therefore, let $h, h' \in H$. Suppose for the sake of contradiction that $h^{-1} \notin H$. That is,

$$h^{-1} = g^j$$

for some $j = 1, 2, \ldots, m-1$. Then $e = hg^j$. But since the order of g is m, we see that this implies that $h = g^{m-j} \implies h \in H$. This is our contradiction so $h^{-1} \in H$.

Since $h^{-1} \in H$, we see that $h^{-1} \neq g^i$ for any $i = 1, 2, \ldots, m-1$. Since $h' \in H$ we see that

$$h'h^{-1} \neq g^i \text{ for any } i = 1, 2, \ldots, m-1.$$

Thus $h'h^{-1} \in H$, and by the subgroup test we see that H is in fact a subgroup of G.

The result follows immediately after this, but we will elaborate on why.

Note that $|\langle g \rangle| = m \neq 0$ and $H \cup \langle g \rangle = G$. Therefore, we see that $|H| < |G|$. Since H is a subgroup of G, we know by Lagrange's Theorem that $|H|$ divides $|G| = p^{k+1}$. Hence, $|H| = p^j$ for some $j < k+1$.

By construction, we see that (1) $\langle g \rangle \cap H = \{e\}$. Therefore

$$|H \cdot K| =$$

By our inductive hypothesis, we know that H is isomorphic to a direct product of cyclic groups. ■

Theorem 1.12.3. (Fundamental Theorem of Finite Abelian Groups) Let G be a finite group. Then G is a direct product of cyclic groups. (Furthermore, these cyclic groups are Sylow p-groups.)

Proof: The result follows immediately from the previous two lemmas.

Note that any finite ablein group G is isomorphic to a direct product of its Sylow p-groups by Lemma 1.12.1. However, each Sylow p-group is isomorphic to a product of cyclic groups by Lemma 1.12.2. Therefore, we have that G itself is isomorphic to a product of cyclic groups. ■

The Fundamental Theorem of Finite Abelian Groups is analagous to the fundamental theorem of arithmetic (hence the name). While the fundamental theorem of arithmetic allows us to completely factorize integers, the fundamental theorem of finite abelian groups allows us to factorize finite abelian groups.

Example.
Suppose we have an abelian group G of order 16. Then, up to an isomophism, G is isomorphic to one of the following:

$$\mathbb{Z}/16\mathbb{Z}$$
$$\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}2\mathbb{Z}$$
$$\mathbb{Z}4\mathbb{Z} \times \mathbb{Z}4\mathbb{Z}$$
$$\mathbb{Z}4\mathbb{Z} \times \mathbb{Z}2\mathbb{Z} \times \mathbb{Z}2\mathbb{Z}$$
$$\mathbb{Z}2\mathbb{Z} \times \mathbb{Z}2\mathbb{Z} \times \mathbb{Z}2\mathbb{Z} \times \mathbb{Z}2\mathbb{Z}$$

Example.
Observe that $9000 = 9 \cdot 5^3 \cdot 2^3$. We know that all abelian groups of order 9000 are going to be direct

products of cyclic subgroups. In this case, we can represent the isomorphism with $\mathbb{Z}/m\mathbb{Z}$ groups. Now because of the size of G, we know that there are Sylow 9-, 5- and 2-subgroups of G. Thus we can view G as a product of $\mathbb{Z}/n\mathbb{Z}$ groups.

For the sake of notation, we'll write that $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$. We can then lists the groups as

$$\mathbb{Z}_9 \times \mathbb{Z}_{5^3} \times \mathbb{Z}_{2^3} \tag{1.1}$$
$$\mathbb{Z}_9 \times \mathbb{Z}_{5^3} \times (\mathbb{Z}_{2^2} \times \mathbb{Z}_2) \tag{1.2}$$
$$\mathbb{Z}_9 \times \mathbb{Z}_{5^3} \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \tag{1.3}$$
$$\mathbb{Z}_9 \times (\mathbb{Z}_{5^2} \times \mathbb{Z}_5) \times \mathbb{Z}_{2^3} \tag{1.4}$$
$$\mathbb{Z}_9 \times (\mathbb{Z}_{5^2} \times \mathbb{Z}_5) \times (\mathbb{Z}_{2^2} \times \mathbb{Z}_2) \tag{1.5}$$
$$\mathbb{Z}_9 \times (\mathbb{Z}_{5^2} \times \mathbb{Z}_5) \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \tag{1.6}$$
$$\mathbb{Z}_9 \times (\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5) \times \mathbb{Z}_{2^3} \tag{1.7}$$
$$\mathbb{Z}_9 \times (\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5) \times (\mathbb{Z}_{2^2} \times \mathbb{Z}_2) \tag{1.8}$$
$$\mathbb{Z}_9 \times (\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5) \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \tag{1.9}$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_{5^3} \times \mathbb{Z}_{2^3} \tag{1.10}$$

$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_{5^3} \times (\mathbb{Z}_{2^2} \times \mathbb{Z}_2) \tag{1.11}$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_{5^3} \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \tag{1.12}$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_{5^2} \times \mathbb{Z}_5) \times \mathbb{Z}_{2^3} \tag{1.13}$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_{5^2} \times \mathbb{Z}_5) \times (\mathbb{Z}_{2^2} \times \mathbb{Z}_2) \tag{1.14}$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_{5^2} \times \mathbb{Z}_5) \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \tag{1.15}$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5) \times \mathbb{Z}_{2^3} \tag{1.16}$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5) \times (\mathbb{Z}_{2^2} \times \mathbb{Z}_2) \tag{1.17}$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5) \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \tag{1.18}$$

(It's a christmas tree!) Recall the fact that $\mathbb{Z}/mn\mathbb{N} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ iff $\gcd(m, n) = 1$. Thus we see that

1. $\mathbb{Z}_9 \ncong \mathbb{Z}_3 \times \mathbb{Z}_3$
2. $\mathbb{Z}_{5^3} \ncong \mathbb{Z}_5 \times \mathbb{Z}_{5^2}$ and $\ncong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$
3. $\mathbb{Z}_{2^3} \ncong \mathbb{Z}_2 \times \mathbb{Z}_{2^2}$ and $\ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Therefore, we see that none of the groups (1) - (18) are isomorphic to each other, so this exhaustive list of abelian groups of order 9000 up to isomorphism is complete.

It turns out that our fundamental theorem for finite abelian groups can actually be strengthened. This strengthened version isn't that useful, since it is sufficiently useful to know that every finite abelian group is a product of cyclic groups. Nevertheless its proof is fun.

Theorem 1.12.4. Let G be a finite abelian group. Then there exist integers $a_1, a_2, \ldots, a_k$ such that

$$G \cong \mathbb{Z}/a_1\mathbb{Z} \times \mathbb{Z}/a_2\mathbb{Z} \times \ldots \mathbb{Z}/a_k\mathbb{Z}$$

where $a_i \mid a_{i+1}$.

Proof: Let G be a finite abelian group and suppose $|G| = p_1^{k_1}p_2^{k_2}\cdots p_n^{k_n}$. Since G is abelian, we know by Lemma 1.12.1 that it is isomorphic to a product of Sylow subgroups. Therefore, we see that

$$G \cong H_1 \times H_2 \times \cdots \times H_n$$

where for some $H_1, H_2, \ldots H_n$ Sylow subgroups, and $|H_i| = p_i^{k_i}$. However, observe that for each $i \leq n$,

$$H_i \cong \underbrace{(\mathbb{Z}/p_i\mathbb{Z}) \times (\mathbb{Z}/p_i\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_i\mathbb{Z})}_{k_i\text{-many times}}.$$

Substituting for each H_i, we then have that

$$G \cong \overbrace{(\mathbb{Z}/p_1\mathbb{Z}) \times (\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_1\mathbb{Z})}^{k_1\text{-many times}} \times \overbrace{(\mathbb{Z}/p_2\mathbb{Z}) \times (\mathbb{Z}/p_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_2\mathbb{Z})}^{k_2\text{-many times}} \times \cdots \times \underbrace{(\mathbb{Z}/p_n\mathbb{Z}) \times (\mathbb{Z}/p_n\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_n\mathbb{Z})}_{k_n\text{-many times}}.$$

Therefore, we can rewrite G as

$$\begin{aligned} G \cong (\mathbb{Z}/p_1\mathbb{Z}) \times (\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times \Big((\mathbb{Z}/p_1\mathbb{Z}) \times (\mathbb{Z}/p_2\mathbb{Z}) \times (\mathbb{Z}/p_2\mathbb{Z}) \times \cdots\Big) \\ \times \Big((\mathbb{Z}/p_2\mathbb{Z}) \times (\mathbb{Z}/p_3\mathbb{Z}) \times (\mathbb{Z}/p_3\mathbb{Z}) \times \cdots\Big) \\ \times \cdots \Big((\mathbb{Z}/p_{n-2}\mathbb{Z}) \times (\mathbb{Z}/p_{n-1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_{n-1}\mathbb{Z})\Big) \\ \times \Big((\mathbb{Z}/p_{n-1}\mathbb{Z}) \times (\mathbb{Z}/p_n\mathbb{Z}) \times (\mathbb{Z}/p_n\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_n\mathbb{Z})\Big). \end{aligned}$$

That is, we can factor it into a product where the i-th factor includes one $\mathbb{Z}/p_i\mathbb{Z}$ factor and $k_{i+1}-1$ many factors of $\mathbb{Z}/p_{i+1}\mathbb{Z}$.

Let us make the following observation. By Theorem 1.1.11.6 we know that

$$\mathbb{Z}/hm\mathbb{Z} \cong \mathbb{Z}/h\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \tag{1.1}$$

since $\gcd(h, m) = 1$. Thus we can collapse the products (in the last equation of G) back together to observe that

$$G \cong (\mathbb{Z}/p_1^{k_1-1}\mathbb{Z}) \times (\mathbb{Z}/p_1p_2^{k_2-1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_{n-1}p_n^{k_n}\mathbb{Z})$$

since by repeated application of equation (1),

$$\mathbb{Z}/p_i p_{i+1}^{k_{i+1}-1} \cong \mathbb{Z}/p_i\mathbb{Z} \times \overbrace{\mathbb{Z}/p_{i+1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_{i+1}\mathbb{Z}}^{(k_{i+1}-1)-\text{many times}}$$

for $1 < i < n-2$, and

$$\mathbb{Z}/p_{n-1}p_n^{k_n}\mathbb{Z} \cong (\mathbb{Z}/p_{n-1}\mathbb{Z}) \times \overbrace{(\mathbb{Z}/p_n\mathbb{Z}) \times (\mathbb{Z}/p_n\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_n\mathbb{Z})}^{k_n-\text{many times}}.$$

Since we have that

$$G \cong (\mathbb{Z}/p_1^{k_1-1}\mathbb{Z}) \times (\mathbb{Z}/p_1 p_2^{k_2-1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_{n-1}p_n^{k_n}\mathbb{Z})$$

if we let $a_1 = p_1^{k_1-1}$ and $a_i = p_{i-1}p_i^{k_i-1}$ for $1 < i < n$ and $a_k = p_{n-1}p_n^{k_n}$, then we see that

$$G \cong \mathbb{Z}/a_1\mathbb{Z} \times \mathbb{Z}/a_2\mathbb{Z} \times \cdots \mathbb{Z}/a_k\mathbb{Z}$$

where $a_i \Big| a_{i+1}$ for $0 < i < n$, as desired. ■

Chapter 2

Rings

2.1 Definitions.

While many mathematical objects come in the form of groups, we also know that there are objects and spaces which require more than one operation. Can we generalize them?

For example, we know that the integers $\mathbb{Z}$ form a group under addition. But don't we also know that multiplication of elements of $\mathbb{Z}$ also yield elements of $\mathbb{Z}$? Isn't this another type of group-like structure we would like to generalize?

We could do this on $\mathbb{R}$ too. It's a group under addition, but we know it's closed under multiplication and has some identity element.

This is where rings come into play, which we define as follows.

Definition 2.1.1. Let R be a set. We define $(R, +, \cdot)$ to be a **ring** if there exist binary operations $+ : R \times R \longrightarrow R$ and $\cdot : R \times R \longrightarrow R$ (referred to as addition and multiplication) such that

(**R1**) **Group addition.** $(R, +)$ is an **abelian group**, with 0 denoted as the identity. (In this group, the additive inverse of an element a is always denoted $-a$.)

(**R2**) **Closure.** For all a, $b \in R$, we have that $a \cdot b \in R$.

(**R3**) **Associativity.** For all a, b, $c \in R$, we have that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(**R4**) **Distributivity.** Similarly, we have that $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.

(**R4**) There exists an element $1 \neq 0$ in R such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$. This is the **unit of the ring**.

Remark 2.1.2.

- As usual, if the multiplication operation $\cdot$ is specified and well-understood, then we will drop $\cdot$ and write multiplication of ring elements as gh instead of $g \cdot h$.
- Axioms (**R5**) is technically optional. However, we don't really care about rings without unity, so we just add it to ou defintion.

Proposition 2.1.3. Suppose R is a ring with identity $1 \neq 0$. Then

1. $0 \cdot a = a \cdot 0$ for all $a \in R$

2. $-(a \cdot b) = (-a) \cdot b = a \cdot (-b)$ for all $a, b \in R$
3. $-a = a \cdot (-1) = (-1) \cdot a$
4. $(-a) \cdot (-b) = a \cdot b$
5. The multiplicative identity is unique.

This is just the stuff you would expect from a ring R based on the fact that many domains you've seen are in fact rings, and some of these facts are obvious in those domains.

Proof:

1. Observe that

$$\begin{aligned}(0 \cdot a) + (0 \cdot a) &= (0 + 0) \cdot a \text{ (by R4)} \\ &= (0 \cdot a) + 0 \text{ (since } 0 + 0 = 0)\end{aligned}$$

where we added 0 to the righthand side (which of course does not change the value of the equation.) Subtracting $(0 \cdot a)$ from both sides, we get that

$$0 \cdot a = 0.$$

Similarly, observe that

$$\begin{aligned}(a \cdot 0) + (a \cdot 0) &= a \cdot (0 + 0) \text{ (by R4)} \\ &= (a \cdot 0) + 0 \text{ (since } 0 + 0 = 0)\end{aligned}$$

where again, we added 0 to both sides. Subtracting $-(a \cdot 0)$ from both sides, we get

$$a \cdot 0 = 0$$

as desired.

2. First we'll show that $-(a \cdot b) = (-a) \cdot b$. To prove this, observe that

$$\begin{aligned}(a \cdot b) - [(a \cdot b)] &= 0 \\ &= a \cdot 0 \text{ (which we just proved)} \\ &= a \cdot [b + (-b)] \\ &= a \cdot b + a \cdot (-b) \text{ (by R4)}\end{aligned}$$

and adding $-(a \cdot b)$ to both sides yields

$$-(a \cdot b) = a \cdot (-b)$$

as desired. Now we'll show that $-(a \cdot b) = (-a) \cdot b$. Observe that

$$\begin{aligned}(a \cdot b) - [(a \cdot b)] &= 0 \\ &= 0 \cdot b \text{ (which we just proved)} \\ &= [a + (-a)] \cdot b \\ &= a \cdot b + (-a) \cdot b \text{ (by R4)}\end{aligned}$$

and adding $-(a \cdot b)$ gives that

$$-(a \cdot b) = (-a) \cdot b$$

which proves the asserition.

3. Simply let $b = 1$ in the previous statements.
4. To prove that $(-a) \cdot (-b) = a \cdot b$, first observe that for any $c \in R$ we already proved that

$$(-a) \cdot c = a \cdot (-c).$$

Thus let $c = -b$. Then observe that

$$(-a) \cdot (-b) = a \cdot [-(-b)]$$

and from group theory, we know that $-(-b) = b$. Therefore, we see that

$$(-a) \cdot (-b) = a \cdot b$$

as desired.

5. To prove that uniqueness of the multiplicative identity, first suppose that it is not unique. That is, there exists elements 1_1 and 1_2 such that

$$1_1 \cdot a = a \cdot 1_1 = a \qquad 1_2 \cdot a = a \cdot 1_2 = a.$$

for all $a \in R$. Then observe that

$$1_1 = 1_1 \cdot 1_2 = 1_2$$

so that the uniqueness must hold.

■

An example of a ring is of course $\mathbb{Z}$, but that's boring. Is $(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$, where n is a positive integer, a ring? Let's check if it is.

Abelian. Since addition is commutative, we already know that $\mathbb{Z}/n\mathbb{Z}$ is abelian (in fact, it is cyclic.)
Associativity. Let a, b and $c \in \mathbb{Z}/n\mathbb{Z}$. Now obviously, $a(bc) = (ab)c$ under *standard* or "normal"

multiplication of integers. Therefore we see that

$$\begin{aligned} a \cdot (b \cdot c) &= a(bc) \bmod n \\ &= (ab)c \bmod n \\ &= (a \cdot b) \cdot c. \end{aligned}$$

Distributivity. Let a, b and c be defined as before. Again, we know that $a(b+c) = ab + ab$ in $\mathbb{Z}$. Therefore

$$\begin{aligned} a \cdot (b + c) &= a(bc) \bmod n \\ &= (ab + ac) \bmod n \\ &= ab \bmod n + ac \bmod n \\ &= a \cdot b + a \cdot c. \end{aligned}$$

The argument is exactly the same to prove left distributivity. Altogether, we see that $\mathbb{Z}/n\mathbb{Z}$ satisfies the axioms of a ring when endowed with modulo addition for $+$ and modulo multiplication for $\cdot$.

Multiplication yielding zeros.
For our ring $\mathbb{Z}$, we know that the only way to ever obtain 0 by multiplication is to just take 0 itself and multiply it by an integer. Thus in this ring, if n, m are nonzero then we always know that $n \cdot m$ is nonzero.

However, note that in $\mathbb{Z}/n\mathbb{Z}$, we have that $a \cdot b = 0$ if and only if $a \cdot b$ is a multiple of n.

If n is prime, then there are no elements in $\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}$ whose product will be a multiple of n. This is just because nothing divides n.

But if n is composite, then there exist integers pq such that $n = pq$, and since $p < n$ and $q < n$, you can be certain that $p, q \in \mathbb{Z}/n\mathbb{Z}$. Then we'd see that $pq = 0$ in $\mathbb{Z}/n\mathbb{Z}$. If p or q are also composite, then there are even more combinations of integers in $\mathbb{Z}/n\mathbb{Z}$ whose product yields 0 in $\mathbb{Z}/n\mathbb{Z}$.

So in the ring $\mathbb{Z}$, multiplication of nonzero elements will be nonzero. But in the ring $\mathbb{Z}/n\mathbb{Z}$ there are many ways one one can multiply elements to get zero (if n is not prime). Obviously these are both rings, but they're behaving differently! Hence we introduce the following definitions.

Definition 2.1.4. Let $(R, +, \cdot)$ be a ring and suppose $a \neq 0$ and $b \neq 0$ are elements of R, while

$$a \cdot b = 0.$$

Then a and b are *both* called **zero divisors** of the ring R. Note that 0 is not a zero divisor. Meanwhile, if R has an identity, and for some $a \in R$ there exists a $b \in R$ such that

$$ab = 1 = ba$$

then we call *both* a and b **units** in R. It turns out the set of units of a ring R form an abelian group, which we denote as R^*.

Note that $\mathbb{Z}$ has no zero divisors, and its unit group R^* is just $\{1, -1\}$. We can see that since if

$ab = 1$ for $a, b \in \mathbb{Z}$, then we know that $a = b = 1$ or -1.

On the other hand, $\mathbb{Z}/n\mathbb{Z}$ can have a more interesting unit group. Observe that if there exists integers $p, q \in \mathbb{Z}/n\mathbb{Z}$ such that

$$pq = n + 1$$

then we see that $p \cdot q = pq \bmod n = n + 1 \bmod n = 1$ in $\mathbb{Z}/n\mathbb{Z}$. If either p or q are composite, then R^* becomes even more interesting.

As a more specific example, observe that the ring $\mathbb{Z}/10\mathbb{Z}$ has units $\{1, 3, 7, 9\}$ and zero divisors $\{2, 4, 6, 8\}$.

Lemma 2.1.5. A zero divisor can never be a unit.

Proof: Let R be a ring and suppose $a \in R$ is a zero divisor. Then there exists an element $b \in R$ where $b \neq 0$ and $ab = 0$. Now suppose that a is also a unit, so that there exists a $c \in R$ such tha $ac = ca = 1$. Then observe that

$$\begin{aligned} 1 = ca \implies b &= (ca)(b) \\ &= c(ab) \\ &= c(0) \\ &= 0 \end{aligned}$$

which is a contradiction since we said $b \neq 0$. Hence a cannot be a unit. ■

We'll next prove another useful lemma which is commonly known as the cancellation law.

Lemma 2.1.6. Let R be a ring, and $a \in R$ such that $a \neq 0$. If a is not a zero divisor, then for any $b, c \in R$ such that $ab = ac$ we have that $b = c$. In addition, if $ba = ca$ then $b = c$.

Proof: Suppose $ac = ab$ for some elements $a, b, c \in R$ where a is not a zero divisor. Then observe that

$$ab = ac \implies ac - ab = 0 \implies a(b - c) = 0.$$

Since a is not a zero divisor, the only way for the above equation to hold is if $b - c = 0 \implies b = c$. Proving the analagous statement is identical to this proof. ■

Now that we have identified terms and can describe the specific elements of a ring R based on their properties, we again return to our observation that $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ behaved differnetly. This is not uncommon in ring theory, so we can divide rings into specific classes as follows.

Definition 2.1.7. Let R be a ring.

1. If R is commutative ring with identity and has no zero divisors, then R is said to be an **integral domain.**
2. The ring R is a said to be a **division ring** if every element of R has a multiplicative inverse. An equivalent condition is if $R^* = R \setminus \{0\}$.
3. If R is a commutative division ring, then R is said to be a **field**.

You've probably read textbooks that called $\mathbb{R}$ or $\mathbb{C}$ fields. This is what they're talking about.

Proposition 2.1.8.

1. If $(R, +, \cdot)$ is an integral domain, then the cancellation law holds for all elements of R.
2. $(R, +, \cdot)$ is an integral domain if and only if for $a, b \in R$, the equation $a \cdot b = 0$ implies either $a = 0$ or $b = 0$.
3. $(R, +, \cdot)$ is a division ring if and only if $ax = b$ and $ya = b$ are solvable in R for every $a, b \in R$ where $a \neq 0$.

Consider again the ring $\mathbb{Z}/p\mathbb{Z}$ where p is a positive integer. We noted that if p is prime then there are no zero divisors. Thus we could state that this an integral domain. However, we can strengthen this even further and state that this is a field, as follows.

Let $a \in \mathbb{Z}/p\mathbb{Z}$ be nonzero. We know that there exists an inverse a^{-1} such that

$$aa^{-1} = 1 \bmod p$$

if a is coprime with p, which is of course true. Since every element has a multiplicative invesrse we see that $\mathbb{Z}/p\mathbb{Z}$ is a division ring. Since this is a commutative division ring, we have that $\mathbb{Z}/p\mathbb{Z}$ is a field. Observe that we could have more easily proved thiss tatemen with the following theorem.

Theorem 2.1.9. Any finite integral domain is a field.

Proof: Let R be a finite integral domain and let $a \in R$ be nonzero. Construct a function $\varphi_a : R \longrightarrow R$ by $\varphi_a(b) = ab$ for $b \in R$.

Suppose $\varphi_a(b) = \varphi_a(c)$ for $b, c \in R$. Then $ab = ac \implies b = c$ since R is an integral domain. Therefore φ_a is injective, and it is clearly surjective so that $\varphi_a(R) = R$.

Since φ_a is bijective for each $a \in R$, we know there always exists a $b \in R$ such that $\varphi_a(b) = 1 \implies ab = 1$. In other words, each $a \in R$ has an inverse, proving R is a division ring. Since R is an integral domain and thus a commutative we have that it is a commutative divison ring, and hence a field. ■

As we said before $\mathbb{Z}/p\mathbb{Z}$ is an integral domain. Since it's also finite, this allows us to conclude it is a field, which we proved before we proved the above theorem.

The following is apparently too difficult for any introductory algebra book to prove in terms of elementary language.

Theorem 2.1.10. Any finite division ring is a field.

With the integral domain, division ring and field introduced, we have a solid footing in the fundamentals of ring theory. We move forward by introducing the concept of a subring.

Subrings.

Definition 2.1.11. Let R be a ring and S be a nonempty subset of R. Then S is a **subring** of R if S is a ring under the addition and multiplication equipped on R.

Specifically, S is a subring if S is an abelain group under addition and is closed under multiplication.

Examples.
We already have an example from our previous work. We know that $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$, where n is a positive integer, are both rings. Since $\mathbb{Z}/n\mathbb{Z} \subset \mathbb{Z}$, we see that $\mathbb{Z}/n\mathbb{Z}$ is a subring of $\mathbb{Z}$.

Define the set $\mathbb{Z}[i] = \{m + ni : m, n \in \mathbb{Z}\}$. Then this is a ring. This is clearly an abelian group under addition (0 is the identity, associativity is obvious, closedness is clear, inverse of any given element is the same element with coefficients of opposite sign). Multiplicative associativity and left and right distributions are clear. However, since $\mathbb{Z}[i] \subset \mathbb{C}$, we see that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$.

Let R be a ring. Then the set of $n \times n$ matrices with entries R, denoted $M_n(R)$, forms a ring. Addition on this set forms an abelian group. And we know from linear algebra that matrix multiplication is associative and left and right distributive. It turns out this ring has many interesting subrings, which we'll list here

Diagonals.
$$D_n(R) = \{A \in \mathbb{R} : a_{ij} = 0 \text{ if } i \neq j\}$$

Upper Triangulars.
$$T^n(R) = \{A \in M_n(R) : a_{ij} = 0 \text{ if } i > j\}$$

Lower Triangulars.
$$T_n(R) = \{A \in M_n(R) : a_{ij} = 0 \text{ if } i < j\}.$$

These are all subrings of $M_n(R)$.

Let G be abelian. Then End(G), the set of endomorphism (homomorphisms from G to itself), forms a ring under addition as function addition and multiplication as function composition.

Abelian Group. First observe that this is a commutative structure since G is abelian. We just have to show that this is a group.

Identity. Let 0_G be the identity element of G. Construct the identity element for End(G) to be the zero map 0 defined as $0 : G \longrightarrow G$ such that $0(g) = 0_G$ for all $g \in G$.

Associativity. Since G is associative, and the images of elements in End(G) are in G, associativity is inherited.

Closedness. Let $f, g \in \text{End}(G)$ and define $h = f + g$. Then $h : G \longrightarrow G$, and is obviously a homomorphism, so that $h \in \text{End}(G)$.

Inverses. Let $f \in \text{End}(G)$. Then construct the function $f^{-1} : G \longrightarrow G$ such that $f^{-1}(g) = -h$ whenever $f(g) = h$. (Note that $-h$ is the inverse of h.) Then we see that $f^{-1}(g) + f(g) = 0$ for all $g \in G$, and that $f^{-1}(g) \in \text{End}(G)$, so that f^{-1} is an inverse of f.

Multiplicatively Closed. Observe that if $h : f \circ g$, then $h : G \longrightarrow G$, and it is a homomorphism. Hence $h \in \text{End}(G)$. Therefore our multiplcative operator is closed.

Multiplicative Associativity. This holds in our case since function composition is in general associative for homomorphisms.

Distributivity. Let $f, g, h \in \text{End}(G)$. Then observe that

$$f(g+h) = f \circ (g+h) = f \circ g + f \circ h = fg + fh$$

and

$$(g+h)f = (g+h) \circ f = g \circ f + h \circ f = gf + hf$$

by linearity of f, g and h (since homomorphisms in general are linear functions).

Therefore, we have that $\text{End}(G)$ forms a ring under function additon and composition.

Polynomial Rings. Polynomials are an interesting example of a ring, which we construct as follows.

Let $R[x]$ be the set of all functions $f : \mathbb{Z}^+ \longrightarrow R$ such that $f(n) = 0$ for all but finitely many n. These functions will be the coefficients to our polynomials, and we want them to be finite, so we request that only finitely many of our cofficients are nonzero. That is, $f(n)$ represents the n-th coefficient.

Define addition and multiplication for two $f, g \in R[x]$ as

$$(f+g)(n) = f(n) + g(n) \quad \text{and} \quad (f \cdot g)(n) = \sum_{i=0}^{n} f(i)g(n-i).$$

This last formula is the formula for the n-th coefficient from the product of two polynomials. We'll show this is a ring.

Abelian. First we'll show this is an abelian group under addition.

Identity. Let $0_R \in R$ be the 0 element of R. If we define 0 to be the map $0(n) : \mathbb{Z} \longrightarrow R$ such that $0(n) = 0_R$ for all $n \in \mathbb{Z}$, then clearly $0 \in R[x]$ and $0 + f = f + 0 = f$ for any $f \in R[x]$. It is our additivite identity.

Associativity. Associativity is derived from the fact that R is associative under addition.

Closedness. To show this is closed we, show that $f + g$ is nonzero for at most finitely many elements for any $f, g \in R[x]$. Simply observe if f is nonzero for k-many elements and g is nonzero for l-many elements then $(f + g)$ is nonzero for at most $(l + k)$-many elements. Therefore $(f + g) \in R[x]$.

Inverses. For any $f \in R[x]$, define f^{-1} to be $f^{-1}(n) = -f(n)$ for all $n \in \mathbb{Z}^+$. Obviously f^{-1} is nonzero for at most finitely many elements if f is, so $f^{-1} \in R[x]$, and $f^{-1}(n) + f(n) = 0$ for any $n \in \mathbb{Z}^+$. Therefore $R[x]$ contains inverses.

Multiplicatively Closed. Observe now that this is closed under multiplication. For any $f, g \in R[x]$, we can simply observe that since f, g are nonzero for at most finitely many values of $n \in ZZ^+$, we note that

$$fg(n) = \sum_{i=1}^{n} f(i)g(n-i)$$

is a function which is nonzero for at most finitely many values, since it is always a finite sum of f and g.

Multiplicative Associativity. Let $f, g, h \in R[x]$. Then observe that

$$\begin{aligned}(fg)h(n) = \sum_{i=0}^{n}(fg)(i)h(n-i) &= \sum_{i=0}^{n}\left(\sum_{j=0}^{i} f(j)g(i-j)\right)h(n-i)\\ &= f(0)g(0)h(n) + \Big(f(0)g(1) + f(1)g(0)\Big)h(n-1)\\ &+ \Big(f(0)g(2) + f(1)g(1) + f(2)g(0)\Big)h(n-2) + \cdots\\ &= \sum_{i=0}^{n} f(n)\sum_{j=0}^{n-i} g(j)h(n-j-i)\\ &= \sum_{i=0}^{n} f(n)(gh)(n-i)\\ &= f(gh)(n).\end{aligned}$$

Therefore multiplicative associativity is satisfied.

Distributivity. Since the image of our functions are elements in R, distributivity is inherited from the ring R, which must be left and right distributed.

Therefore we see that $R[x]$ forms a rings. We'll now realize that this is the set of polynomials by describing the function a stupidly simple function:

$$x^n(m) = \begin{cases}1 & \text{if } n = m\\ 0 & \text{otherwise}\end{cases}.$$

Then observe that for any $f \in R[x]$, we may uniquely associate with it the following object:

$$f = \sum_{n=0}^{\infty} f(n)x^n.$$

The ∞ in the upper limit is there to allow us to define any polynomial of an arbitrary degree. We know it will always a finite polynomial since we said that $f(n) \neq 0$ for at most finitely many n.

Thus what we've shown is that the space $R[x]$, constructed by focusing on the coefficients, defining their rules for polynomial multiplication, and realizing the polynomial structure we wanted, is in fact a ring!

Note that if we don't assume that $f(n)$ is nonzero for finitely many n, then we'll end up constructing a different ring, know as the **formal power series** ring denoted $R[[x]]$. This has the same rules of addition and multiplication, so the ring structure doesn't change. The only thing that changes is that $R[[x]]$ includes infinitely long polynomials.

Thus, we see that $R[x] \subset R[[x]]$ and therefore $R[x]$ is a subring of $R[[x]]$.

In group theory there was a Subgroup Test which simplified the task of determine whether or not a subspace form a group or not. Fortunately, such a tool is available in ring theory.

Theorem 2.1.12.Subring Test. Let R be a ring and $S \subset R$. Then S is a **subring** of R if and only if, for all $x, y \in S$ we have that $x - y \in S$ and $xy \in S$.

Proof: ($\Longrightarrow$) Suppose $S \subset R$ is a subring. Then certain $x - y \in S$ and $xy \in S$.

($\Longleftarrow$) Suppose now that $x - y \in S$ and $xy \in S$ for all $x, y \in S$. The first condition immediately $(S, +)$ is a subgroup of $(R, +)$, since $x - y \in S$ for all $x, y \in S$ is just the subgroup test. Now observe that $xy \in S$ for all $x, y \in S$. Since R is an abliean group under addition and is closed under multiplication of its elements, we have that S is a subring of R as desired. ■

It turns out that arbitrary intersections of subrings produce a subring, an important result we include here.

Theorem 2.1.13. Let R be a ring and $\{S_\alpha\}_{\alpha \in \lambda}$ be a family of subrings of R. Then $S = \bigcap_{\alpha \in \lambda} S_\alpha$ is a subring of R.

Proof: From group theory, we know that the arbitrary intersection of subgroups is again a group. So $S = \bigcap_{\alpha \in \lambda} S_\alpha$ is an abelian subgroup of R. Therefore, we just need to check that S is closed under multiplication.

From group theory, we know that the arbitrary intersection of a family of subgroups is a group. Thus S is an abelian group, and we just need to check that it is closed under multiplication.

For any $s, s' \in S$ we know that $s, s' \in S_\alpha$ for all $\alpha \in \lambda$. Since each subring is obviously closed under multiplication we see that $ss' \in S_\alpha$ for all $\alpha \in \lambda$. Hence, $ss' \in S$ as desired. ■

2.2 Ring homomorphisms.

After one understand the fundamentals of group theory, they go on to construct maps between different groups. This is the same strategy we'll follow here, since we can definitely define **ring homomorphisms** between rings.

The ring homomorphisms are also useful since they can help us deduce when two rings R and S are the "same," a concept which evolves into the concept of isomorphisms.

Definition 2.2.1. Let R and S be rings, and $f : R \longrightarrow S$. We define f to be a **ring homomorphism** if it preserves addition and multiplication; that is, if

$$f(a+b) = f(a) + f(b) \quad \text{and} \quad f(ab) = f(a)f(b)$$

for all $a, b \in R$. If f is a bijection, then we say that f is a **ring isomorphism**.

Note that a ring homomorphism is simply a group homomorphism, with the extra condition of which preserves multiplication of the ring elements. Therefore the following proposition, which hold for group homomorphisms, holds for ring homomorphisms too.

Proposition 2.2.2. Let R and S be rings and $f : R \longrightarrow S$ a ring homomorphism. Then

1. if $0_R \in R$ and $0_S \in S$ are zero elements, then $f(0_R) = 0_S$.
2. if $f(-a) = -f(a)$ for all $a \in R$
3. $f(a_1 a_2 \cdots a_n) = f(a_1)f(a_2)\cdots f(a_n)$ for all $a_1, a_2, \ldots a_n \in R$
4. $f(a_1 + a_2 + \cdots + a_n) = f(a_1) + f(a_2) + \cdots + f(a_n)$ for all $a_1, a_2, \ldots a_n \in R$.

Proof: Observe that

$$\begin{aligned} \varphi(r) + \varphi(-r) &= \varphi(r + (-r)) \\ &= \varphi(r) \\ &= 0 \\ &= \varphi(r) + [-\varphi(r)]. \end{aligned}$$

Since $(R, +)$ is a group, subtract $\varphi(r)$. ■

Note that it is not necessarily true that $f(1_R) = 1_S$. In group theory, it was always guaranteed that we could map the identity element from one group to another. In our case, that's still true: $f(0_R) = 0_S$. Group identity of $(R, +)$ is still mapped to the identity of $(S, +)$. But this is mapping the *additive* identity of R to the *additive* identity of S.

What we're saying is that *multiplicative* identities may not always be mapped to each other.

Now since we can't always guarantee that $f(1_R) = 1_S$, we also can't guarantee that $f(a^{-1}) = f(a)^{-1}$ for some invertible $a \in R$. However, there is a clear cut case for when these things do happen.

Proposition 2.2.3. Let R and S be rings and $\varphi : R \longrightarrow S$ a nonzero ring homomorphism. Denote $1_R \in R$ and $1_S \in S$ to be the respective multiplicative identities. Then

1. If $\varphi(1_R) \neq 1_S$ then $\varphi(1_R)$ is a zero divisors of S.
2. If S is an integral domain then $\varphi(1_R) = 1_S$.
3. If $\varphi(1_R) = 1_S$ and $u \in R$ is a unit then $\varphi(u)$ is a unit in S. In other words, $\varphi(R^*) \subset S^*$
4. If $\varphi(1_R) = 1_S$ and if $u \in R$ has an inverse $u^{-1} \in R$ then $\varphi(u^{-1}) = \varphi(u)^{-1}$.

An immediately corollary is this: $\varphi : R \longrightarrow S$ is a not nonzero ring homomorphism if and only if $\varphi(1_R) \neq 0_S$. Furthermore, If S is an integral domain then $\varphi(R^*) \subset S^*$ for any homomorphism $\varphi : R \longrightarrow S$.

Proof:

1. Suppose $\varphi(1_R) \neq 1_S$. Since $1_R 1_R = 1_R$, we know that

$$\begin{aligned} \varphi(1_R 1_R) - \varphi(1_R) = 0_S &\implies \varphi(1_R)\varphi(1_R) - \varphi(1_R) = 0_S \\ &\implies \Big(\varphi(1_R) - 1_S\Big)\varphi(1_R) = 0_S. \end{aligned}$$

   Since $\varphi(1_R) \neq 1_S$, either $\varphi(1_S) = 0$ or it is a zero divisor of S.

   Suppose $\varphi(1_R) = 0_S$ and let $a \in R$. Then

$$\begin{aligned} \varphi(a) = \varphi(1_R a) &= \varphi(1_R)\varphi(a) \\ &= 0_S \varphi(a) \\ &= 0_S. \end{aligned}$$

   Thus we see that φ send every element of R to 0_S. However, this cannot be the case since we supposed that φ is a nonzero homomorphism. Therefore $\varphi(1_R) \neq 0$, leaving us with no choice but to conclude that $\varphi(1_R)$ is a zero divisor in S as desired.

2. Suppose S is an integral domain, and that $\varphi(1_R) \neq 1_S$ for the sake of contradiction. Then observe for any $a \in R$

$$\varphi(1_R a) - \varphi(a) = 0_S \implies \varphi(1_R)\varphi(a) - \varphi(a) = 0_S \implies (\varphi(1_R) - 1_S)\varphi(a) = 0_S.$$

   Since $\varphi(1_R) \neq 1_S$, and φ is a nonzero homomorphism, this implies that $\varphi(a)$ and $\varphi(1_R) - 1_S$ are zero divisors in S for at least one $a \in R$. However, this is a contradiction since S is an integral domain and hence has no zero divisors. Thus by contradiction $\varphi(1_R) = 1_S$.

3. Suppose $\varphi(1_R) = 1_S$ and let u be a unit in R. Then $uv = 1_R$ for some $v \in R$. So

$$\varphi(uv) = \varphi(1_R) = 1_S \implies \varphi(u)\varphi(v) = 1_S.$$

   Therefore, $\varphi(u)$ is a unit in S. Next, since $uu^{-1} = 1_R$,

$$\varphi(1_R) = 1_S \implies \varphi(uu^{-1}) = 1_S \implies \varphi(u)\varphi(u^{-1}) = 1_S \implies \varphi(u)^{-1} = \varphi(u^{-1})$$

   as desired.

4. Suppose $\varphi(1_R) = 1_S$ and that $u \in R$ has some inverse $u^{-1} \in R$. Since $uu^{-1} = 1_R$,

$$\varphi(1_R) = 1_S \implies \varphi(uu^{-1}) = 1_S \implies \varphi(u)\varphi(u^{-1}) = 1_S \implies \varphi(u)^{-1} = \varphi(u^{-1})$$

as desired.

■

Examples.
Let $n \in \mathbb{Z}$, and define the function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ as

$$f(m) = nm.$$

Then this is a homomorphism if and only if $n = 0$ or 1. Suppose otherwise. Then observe that the second condition of the definition of a ring homomorphism specifies that

$$\begin{aligned} f(ab) = f(a)f(b) \implies nab &= nanb \\ &= n^2ab. \end{aligned}$$

This is only true if $n = 0$ or 1, which is our contradiction.

Instead, we can construct the following function to form a homomorphism between $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$, where n is a positive integer. Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ such that

$$f(m) = [m]$$

where $[m] = \{k \in \mathbb{Z} \mid k = m \bmod n\}$.

Suppose we construct a homomorphism $\varphi : \mathbb{R}[x] \longrightarrow S$. (Recall that $\mathbb{R}[x]$ is the set of finite polynomials with coefficients in $\mathbb{R}$). Define φ as

$$\varphi(p(x)) = p(i).$$

First, observe that this is surjective, since for any $a + bi \in \mathbb{C}$ we can send $a + bx \in \mathbb{R}$ to this element via φ. Therefore $\mathrm{Im}(\varphi) = \mathbb{C}$.

Let us now describe $\ker(\varphi)$. First suppose that $p(i) = 0$ for some $p(x) \in \mathbb{R}[x]$. At this point, we know that $p(x)$ must be at least a second degree or greater polynomial. Therefore we can express $p(x)$ as

$$p(x) = q(x)(x^2 + 1) + bx + a$$

for some $q(x) \in \mathbb{R}[x]$. Then

$$\begin{aligned} p(i) &= q(i)(i^2 + 1) + bi + a \\ &= a + bi \end{aligned}$$

but this implies that $a + bi = 0 \implies a = b = 0$. Therefore, $p(i) = 0$ if and only if $p(x) = q(x)(x^2 + 1)$

some $q(x) \in \mathbb{R}[x]$. In other words,

$$\ker(\varphi) = \{p(x) \in \mathbb{R}[x] \mid (x^2+1)\Big|p(x)\}.$$

As in group theory, we have the following theorem regarding isomorphisms. We won't prove this again.

Theorem 2.2.4. Let R and S be rings. A ring homomorphism $f : R \longrightarrow S$ is an isomorphism if and only if there exists a h omomorphism $g : S \longrightarrow R$ such that $f \circ g$ is the identity map on R and $g \circ f$ is the identity map on S.

With the ring homomorphism defined, we again have $\ker(f)$ and $\mathrm{Im}(f)$ as valid and important concepts.

Definition 2.2.5. Let R and S be rings and $f : R \longrightarrow S$ a ring homomorphism. Then we define

$$\ker(f) = \{a \in R \mid f(a) = 0\}$$

and

$$\mathrm{Im}(f) = \{f(a) \mid a \in R\}.$$

Proposition 2.2.6. Suppose $f : R \longrightarrow S$ is a ring homomorphism. Then

1. The kernal $\ker(f)$ is a subring of R.
2. The image $\mathrm{Im}(f)$ is a subring of S.

Caveat: Recall that "subrings" are rings that might not possibly contain 1, the multiplcative identity.

Proof:

1. We can show this using the Subring Criterion. As we stated before, $f(0) = 0$. Hence $0 \in \ker(f)$ so that $\ker(f)$ is nonempty.

   To prove this, observe that

   $$\begin{aligned} f(0) + f(0) &= f(0+0) \\ &= (0) \\ &= f(0) + 0. \end{aligned}$$

   Since $(R, +)$ is a group, we can subtract $f(0)$ from both sides to get $f(0) = 0$.

   Next, we want to show that $r_1, r_2 \in \ker(f) \implies r_1 - r_2 \in \ker(f)$. Since we showed that

$f(-r) = -f(r)$ for all $r \in R$, we know that

$$\begin{aligned} f(r_1 - r_2) &= f(r_1) + f(-r_2) \\ &= f(r_1) - f(r_2) \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

Hence, we see that $r_1 - r_2 \in \ker(f)$.

Now again suppose $r_1 r_2 \in \ker(f)$. Then

$$\begin{aligned} f(r_1 r_2) &= f(r_1) f(r_2) \\ &= 0 \end{aligned}$$

so that $r_1 r_2 \in \ker(f)$. By the subring test, we see that $\ker(f)$ is a subring of R.

2. We can similarly prove this via the Subring Test. First, observe that $f(0) = 0$, so that $0 \in \text{Im}(f)$. Hence, $\text{Im}(f)$ is nonempty.

   Next, suppose $s_1, s_2 \in \text{Im}(f)$. Then we want to show that $s_1 - s_2 \in \text{Im}(f)$. Now

$$\begin{aligned} s_1 - s_2 &= f(r_1) - f(r_2) \\ &= f(r_1 - r_2). \end{aligned}$$

   This shows that $s_1 - s_2 \in \text{Im}(f)$. Finally, we'll show that $s_1 s_2 \in \text{Im}(f)$. Observe that

$$s_1 \times s_2 = f(r_1) f(r_2) = f(r_1 r_2).$$

   Hence we see that $s_1 \times s_2 \in \text{Im}(f)$. Thus $\text{Im}(f)$ is a subring of R.

■

Finally, we end this section by noting that two important and useful mathematical identites continue to hold in the context of rings. We won't offer their proofs though since they are a bit tedious.

Proposition 2.2.7. Let R be a ring and let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be elements of R. Then

$$(a_1 + a_2 + \cdots + a_m)(b_1 + b_2 + \cdots + b_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j$$

Proposition 2.2.8 (Binomial Theorem)**.** Let R be a ring (with identity) and let $a, b \in R$ with $ab = ba$. Then for any $n \in \mathbb{N}$

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b_{n-k}.$$

2.3 Ideals and Quotient Rings.

Consider a ring homomorphism $f : R \longrightarrow S$. Let $a \in R$ and suppose $b \in \ker(f)$. Then

$$f(ab) = f(a)f(b) = 0f(b) = 0.$$

Therefore, if $a \in \ker(R)$, then $ab \in \ker(R)$ for all $b \in R$. Many subrings behave this way and are particularly interesting, so we give them a special name!
First, we'll introduce the concept of a coset.

Definition 2.3.1. Let $(R, +, \cdot)$ be a ring with identity $1 \neq 0$. Suppose I is a subring. Then we define the set

$$\overline{a} = a + I = \{a + i \in R \mid i \in I\}$$

to be a **coset** I in R. Since R is an abelian group under addition, we see that

$$a + I = I + a$$

for all $a \in R$. Hence, left and right cosets are the concept here. Finally, we define the **collection of cosets** by

$$R/I = \{\overline{a} \mid a \in R\}.$$

We are now ready to introduce the concept of an ideal.

Definition 2.3.2. Let R be a ring and suppose $I \subset R$. Then we define I to be an **ideal** of R if and only if

1. I is an additive subgroup of R
2. $rI \subset I$ for all $r \in R$
3. $Ir \subset I$ for all $r \in R$

An ideal is simply an interesting subring R' of a ring R which sort of "sucks in" elements of R and sends them into R'. That is, $rr' \in R'$ for every $r \in R$ and $r' \in R'$.

We've already seend many examples of this, although we don't usually think of them that way. For instance, it's a well known fact that for any integer times an even number is again an even number. Algebraically, for $n \in \mathbb{Z}$ and $k \in 2\mathbb{Z}$ we have that $nk \in 2\mathbb{Z}$ and $kn \in 2\mathbb{Z}$.

Thus $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$. In fact, if k is any even integer then $k\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

The set of odd integers is not an ideal of $\mathbb{Z}$, since we could always take an even number $n \in \mathbb{Z}$ and any odd k, and multiply them to obtain an even number nk which is obviously not in the set of odd integers.

If $I \subset R$ satisfies (2) then I is said to be a **left ideal**. On the other hand if $I \subset R$ satisfies (3) then it is said to be a **right ideal**.

Thus any ideal I is both a left and right ideal. In addition, the concept of a left ideal is identical to a right ideal in a commutative ring.

Theorem 2.3.3. Suppose $I \subset R$ is a proper subring. Then the following are equivalent:

1. $I = \ker(f)$ for some $f : R \longrightarrow S$
2. $r \cdot x = x \cdot r \in I$ for any $r \in R$, $x \in I$
3. R/I is a ring with $\overline{1} \neq \overline{0}$.
4. I is an ideal.

Proof:

1. We'll show $(i) \implies (ii)$. Assume $I = \ker(f)$. Given $r \in R$ and $i \in I$,

$$\varphi(r \cdot i) = \varphi(r) \cdot \varphi(i) = \varphi(r) \cdot 0 = 0$$
$$\varphi(i \cdot r) = \varphi(i) \cdot \varphi(r) = 0 \cdot \varphi(r) = 0$$

   This shows that $r \cdot i, i \cdot r \in \ker(f)$.

ii. We'll show that $(ii) \implies (iii)$. Assume $ri, ir \in I$ for all $r \in R, i \in I$. We'll show that this is a ring with $\overline{1} \neq \overline{0}$.

   First, we define that

$$\overline{a} + \overline{b} = \overline{a + b}$$
$$\overline{a}\overline{b} = \overline{ab}.$$

   We first need to show that these definitions are well-defined. Suppose $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$. Then $a_1 = a_2 + x$ and $b_1 = b_2 + y$ for some $x, y \in I$. Then

$$a_1 + b_1 = (a_2 + b_2) + (x + y).$$

   Since $I \subset R$ is a subring, $x + y \in I$. So,

$$\overline{a_1 + b_1} = \overline{a_1 a_2}.$$

   Simiarly, $\cdot$ is well defined on R/I. Again, suppose $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$. Then $a_1 = a_2 + x$ and $b_1 = b_2 + y$ for some $x, y \in I$. Then

$$\begin{aligned} a_1 \cdot b_1 &= (a_2 + x) \cdot (b_2 + y) \\ &= (a_2 \cdot b_2) + [(a_2 \cdot y) + (x \cdot b_2) + (x \cdot y)]. \end{aligned}$$

   I is a subring, so $x \cdot y \in I$. Now (ii) is true, so $a_2 \cdot y \in I$ and $x \cdot b_2 \in I$. Therefore, $\overline{a_1 \cdot b_1} = \overline{a_2 \cdot b_2}$.

   Finally, we'll show that $(R/I, +, \cdot)$ is a ring.

   (R1: Addition) Observe that $\overline{0} \in R/I$ is the identity and $\overline{-a}$ are inverses of $\overline{a} \in R/I$.

(R2: Closure) The set is closed by construction on $\cdot$.
(R3: Assoc), (R5: Distributivity) hold for R/I because they hold for R.
(R4: Identity) The identitty holds for $\overline{1} \in R/I$. One can check that $\overline{1} \neq \overline{0}$.

iii Now we can show that $(iii) \implies (i)$. Assume $S = R/I$ is a ring. Define

$$\varphi : R \longrightarrow S \quad a \mapsto \overline{a} = a + I$$

One checks that $\ker(\varphi) = I$.

iv. Our work in the previous section has allowed us to prove $(i) \implies (iv)$. Now observe that we can prove $(iv) \implies (i)$ by simply considering the map in (iii).

■

Theorem 2.3.4. (Properties of Ideals) Let R be a ring and I, J ideals of R. Then

1. $I + J$ is an ideal of R. (Note we may extend this to larger, finit sums)
2. $IJ = \left\{ \sum_{k=1}^{n} i_k j_k \mid \text{for all } n \in \mathbb{N}, i_k \in I, j_k \in J \right\}$ is an ideal of R. (Note we can extend this to larger, finite products.)
3. $I \cap J$ is an ideal of R. Morever, if $\{I_\alpha\}_{\alpha \in \lambda}$ is a family of ideals of R, then $\bigcap\limits_{\alpha \in \lambda} I_\alpha$ is an ideal of R.

Proof:

1. By the Second Isomorphism Theorem we know that $I + J$ is a subring of R. Thus, we just need it to be closed under multiplication for it to be an ideal.

   Let $i + j \in I + J$ and let $r \in R$. then $r(i + j) = ri + rj \in I + J$, since $ri \in I$ and $rj \in J$. Similarly, $(i + j)r \in I + J$, so that $I + J$ is an ideal of R.

2. In words, IJ is the set of all finite sums of elements of the form ij where $i \in I$ and $j \in J$. Thus is clearly an abelian group. To show it is closed under multiplication, let $r \in R$. Then observe that $r(\sum_{k=1}^{n} i_k j_k) = \sum_{k=1} r i_k j_k$. Now $r i_k \in I$ for all k since I is an ideal. Therefore $r(\sum_{k=1}^{n} i_k j_k) \in IJ$.

   For similar reasons $(\sum_{k=1}^{n} i_k j_k) r \in I$, so that IJ is an ideal.

3. By our knowledge of group theory we know that intersections of subgroups form a group, so that this is an abelian subgroup. To see it is an ideal we just need to check it is closed under scalar multiplication.

   Let $i \in I \cap J$. Then $i \in I$ and $i \in J$. Hence, $ir \in I$ and $ri \in J$, and $ri \in I$ and $rj \in J$ as I and J are ideals. Hence $ir \in I \cap J$ and $ri \in I \cap J$, so that $I \cap J$ is an ideal.

   The more general statement has the same proof structure.

■

Lemma 2.3.5. If S is a nonempty partially ordered set in which every chain $I_1 \subset I_2 \subset \cdots$ has an upper bound I, then S has a maximal element M.

Theorem 2.3.6. (Properties of Ideals) Let $(R, +, \cdot)$ be a ring with identity $1 \neq 0$. Consider a chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots \subseteq R$ of proper ideals of R.

1. $I = \bigcup_{n \geq 1} I_n$ is a proper ideal of R.
2. Each proper ideal I of R is contained in a maximal ideal M of R.

Proof:

1. ***I is nonempty.***

   Observe that I is nonempty if at least one I_k is nonempty.

   $a, b \in I \implies a - b \in I$.

   Pick $a, b \in I$. Then $a \in I_n$ and $b \in I_m$ for some n, m. Without loss of generality assume $n \leq m$. Then $I_n \subseteq I_m$. Thus $a \in I_m$ as well, and since I_m is an ideal, we see that $a - b \in I_m$. Hence $a - b \in I$.

   $ra \in I$ if $r \in R, a \in I$.

   If $a \in I$ then $a \in I_k$ for some k. Since I_k is an ideal, we have that $ra \in I_k$. Hence $ra \in I$.

   $I \neq R$.

   Suppose on the contrary that $I = R$. Then for every $r \in R$ there exists an integer k such that $r \in I_k$. In particular, for some $u \in R^\times$ (the unit group), there is a k such that $u \in I_k$. Since $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k$, we see that all ideal $I_1, I_2, \ldots, I_k$ are not proper (as they contain a unit.)

   However, this is a contradition, since each I_n must be proper. Thus I cannot be all of R.

2. Consider any proper ideal I_1 of R. If I_1 is not maximal, then there exists an ideal I_2 such that $I_1 \subset I_2$. If I_2 is not maximal, then there exists an ideal I_3 such that $I_2 \subset I_3$. Now construct the set

   $$S = \{I_n \text{ is proper } \mid I_n \subset I_{n+1}\}.$$

   where $I_n \in S_j$ whenever there exists a proper ideal I_{n+1} where $I_n \subset I_{n+1}$.

   If this set is finite, then we take the maximal element (relative to partial ordering on subset inclusion) M as the maximal ideal.

   Suppose on the other hand that this set is infinite. By part (a), we see that every $I_n \in S$ is a subset of the proper ideal $\bigcup_{n \geq 1} I_n$, so that this is an upper bound on the set of elements S_J (in terms of set inclusion). Hence by Zorn's lemma, we see that there must exist a maximal

element $M \in S$. As all members of S are proper ideals, we see that M is by definition a maximal ideal where $M \neq R$. As I_1 was arbitrary, we see that all ideals are contained in some maximal ideal M, as we set out to show.

■

The following is a useful example of an ideal known as the nilradical:

Proposition 2.3.7. Let $(R, +, \cdot)$ be a commutative ring with $1 \neq 0$, and let $I \subset R$ be a proper ideal. The *radical* of I is the set

$$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

1. $\sqrt{I}$ is an ideal containing I.
2. $\sqrt{I}$ is the intersection of all prime ideals P which contain I.

Proof:

1. First observe that $I \subset \sqrt{I}$. Since for any $r \in I$, we see that $r^1 = r \in I$. Hence $r \in \sqrt{I}$. Now we'll show that $\sqrt{I}$ is an ideal.

   $\underline{\boldsymbol{\sqrt{I} \neq \varnothing}}$.

   Since $I \subset \sqrt{I}$, we see that $\sqrt{I}$ is nonempty.

   $\underline{\boldsymbol{a, b \in \sqrt{I} \implies a - b \in \sqrt{I}}}$.

   Let $a, b \in \sqrt{I}$. Then there exist positive integers m, n such that $a^m \in I$ and $b^n \in I$. Now observe that

   $$(a-b)^{n+m} = \sum_{k=0}^{m+n} \binom{m+n}{k} a^{n+m-k}(-b)^k.$$

   by the binomial theorem. Observe that when $k \leq n$,

   $$\begin{aligned} k \leq n &\implies n - k \geq 0 \\ &\implies n + m - k \geq m. \end{aligned}$$

   Hence we see that $a^{n+m-k} = a^{n-k}a^m \in I$ because $a^m \in I$. Since I is an ideal, we see that

   $$\sum_{k=0}^{n} \binom{m+n}{k} a^{n+m-k}(-b)^k$$

   is a sum of terms in I, so therefore it is in I.
   Now suppose $k > n$. Then we get that

   $$n < k \implies k = n + j \text{ for some } j \in \mathbb{Z}^+.$$

   Therefore we see that $b^k = b^j b^n \in I$. Since I is an ideal, the sum

   $$\sum_{k=n+1}^{n} \binom{m+n}{k} a^{n+m-k}(-b)^k$$

is a sum of terms in I. Hence the total sum is in I. Now we see that

$$\sum_{k=0}^{m+n}\binom{m+n}{k}a^{n+m-k}(-b)^k=\sum_{k=0}^{n}\binom{m+n}{k}a^{n+m-k}(-b)^k+\sum_{k=n+1}^{n}\binom{m+n}{k}a^{n+m-k}(-b)^k$$

so that $(a-b)^{m+n}=\sum_{k=0}^{m+n}\binom{m+n}{k}a^{n+m-k}(-b)^k$ is a sum of two terms in I, and hence is in I. Thus we have that $a,b\in\sqrt{I}\implies a-b\in\sqrt{I}$.

ra* ∈ *I* if *r* ∈ *R*, *a* ∈ *I.

Suppose that $a\in\sqrt{I}$. Then $a^n\in I$ for some positive integer n. Since R is a commutative ring, we see that $(ra)^n=r^na^n\in I$ since $a^n\in I$ and I is an ideal. Thus $ra\in I$ for any $r\in R$, $a\in I$.

$\sqrt{I}\neq R$.

Suppose that $\sqrt{I}=R$. Then for every $r\in R$, there exists a positive integer n such that $r^n\in I$.
Then in particular for some unit $u\in R^\times$ we have that $u^m\in I$ for some integer m. However, since $R^\times$ is a group under multiplication, we know that $u^m\in R^\times$. Hence u^m is a unit. Since $u^m\in I$, this implies that I contains a unit, which ultimately implies that $I=R$.

(Note: It is a fact from class that if an ideal I of R contains a unit, it is all of R. I am utilizing this fact. Please don't dock off points for this literal fact from class.)

However, this is a contradition since we assumed that I was proper. Hence $\sqrt{I}\neq R$, which proves that it is a proper ideal.

2. First we prove the hint.

Following the hint, suppose $x\notin\sqrt{I}$. If we let $D=\{1,x,x^2,\dots\}$, pick a maximal ideal M in the ring $S=D^{-1}R/D^{-1}I$.
Let $\varphi:R\longrightarrow S$ where $\varphi(r)=\dfrac{r}{1}+D^{-1}I$. Let P be the pull-back of M under φ. We'll now prove the hint by showing P is prime, $x\notin\sqrt{I}\implies x\notin P$ and that $I\subset P$.

***P* is prime**.

First observe we need to make sure that the pullback is well defined, in the sense that if M is maximal then P is prime. First observe that since M is maximal, it is prime by our previous lemma. Thus we know from Hw 2 that we need to show two things.

1. **$\varphi(1)=1$**. Observe that
$$\varphi(1)=\frac{1}{1}+D^{-1}I$$
which is the identity element in $D^{-1}R/D^{-1}I$. Hence, $\varphi(1)=1$. $\varphi^{-1}(P)$ is a prime ideal. From problem 2, we know that this allows us to conclude the pull-back is well defined.
2. ***P* = φ^{-1}(*M*) is prime**. (It may help the reader for me to refer to P explicitly as

$\varphi^{-1}(M)$, in terms of clarity of the solution, so I'll follow that convention.)

$\varphi^{-1}(M)$ is nonempty. Observe that $\varphi(0) = \dfrac{0}{1} + D^{-1}I \in M$, as M is an ideal of $D^{-1}R/D^{-1}I$ and hence contains the zero element. Therefore $0 \in \varphi^{-1}(M) = P$ and so P is nonempty.

$a, b \in \varphi^{-1}(M) \implies a - b \in \varphi^{-1}(M)$. Let $a, b \in \varphi^{-1}(M)$. Then $\varphi(a), \varphi(b) \in M$. Hence, we see that

$$\begin{aligned} \varphi(a), \varphi(b) \in M &\implies \varphi(a) - \varphi(b) \in M \text{ (since } M \text{ is a prime ideal)} \\ &\implies \varphi(a - b) \in M \text{ (by homomorphism properties)} \\ &\implies a - b \in \varphi^{-1}(M). \end{aligned}$$

Therefore, we see that $a - b \in \varphi^{-1}(M)$ if $a, b \in \varphi^{-1}(M)$.

$ra \in \varphi^{-1}(M)$ if $r \in R, p \in \varphi^{-1}(M)$. We'll show that $r \cdot a \in \varphi^{-1}(M)$ for all $r \in R$. Observe that

$$\varphi(r \cdot a) = \varphi(r)\varphi(a).$$

Since $\varphi(a) \in M$, and M is a prime ideal, $s\varphi(a) \in M$ for all $s \in D^{-1}R/D^{-1}I$. In particular, since $\varphi(r) \in D^{-1}R/D^{-1}I$, we see that $\varphi(r)\varphi(a) \in M$. Therefore, $\varphi(r \cdot a) \in M$ so that $r \cdot a \in \varphi^{-1}(M)$.

$ab \in \varphi^{-1}(M) \implies a \in \varphi^{-1}(M)$ or $b \in \varphi^{-1}(M)$ Suppose $ab \in \varphi^{-1}(M)$. Then we see that

$$\varphi(ab) \in M \implies \varphi(a)\varphi(b) \in M.$$

Since M is a maximal, and hence a prime ideal (as proven earlier), we see that either $\varphi(a) \in M$ or $\varphi(b) \in M$. In either case, we see that either $a \in \varphi^{-1}(M)$ or $b \in \varphi^{-1}(M)$, which is what we set out to show.

$\varphi^{-1}(M)$ is proper. Finally, we show that $\varphi^{-1}(M)$ is proper. Suppose that $\varphi^{-1}(M) = R$. Then

$$\varphi^{-1}(M) = R \implies \varphi(R) = M.$$

Thus we see that $\varphi(r) \in M$ for all $r \in R$. Let $r = 1$.

$$\varphi(r) \in M \implies \varphi(1) \in M \implies 1 \in M$$

since we have that $\varphi(1) = 1$. However, $1 \notin M$ since M is maximal and hence proper. As we've reached a contradiction, we see that the pullback P must always be proper.

Thus we see that the pullback is well-defined (i.e., if M is prime, so is its pullback P) in this case and that P is prime.

(Note: it was technically unnecessary to do all of this work. Even in terms of clarity, I could have just referenced Hw 2, problem 3, and argued that the work carries over via the $\varphi(1) = 1$ argument, since that was the only reason we need R and S to be integral domains

there, and then used the fact that maximal ideals are prime. However, I included the full work to be explicitly clear.)

Next, we continue and prove the hint.
$x \notin \sqrt{I} \implies x \notin P$.

Recall we supposed $x \notin \sqrt{I}$. Now if M is an ideal of $D^{-1}R/D^{-1}I$, then by the Fourth Isomorphism theorem we have that M corresponds to some ideal M' of $D^{-1}R$ where $D^{-1}I \subset M'$. Hence we can write

$$M = M' + D^{-1}I.$$

(Note: before you dock off points, the above choice of notation was introduced by Professor Goins himself. I think it's a bit unorthodox, which you may also think as well, but again, Goins used this notation so I will as well.)

Now suppose for a contradiction that $x \in P$. Then we have that $\frac{x}{1} + D^{-1}I \in M$. For this to be the case, we need that $\frac{x}{1} \in M'$. Since M' is an ideal of $D^{-1}R$, we know that $r\frac{x}{1} \in M$ for all $r \in D^{-1}R$. In particular, we see that

$$\frac{1}{x} \cdot \frac{x}{1} \in M \implies \frac{1}{1} \in M'.$$

As $\frac{1}{1}$ is a unit, this implies that $M' = D^{-1}R$ (Note: It is a fact from class that if an ideal I of R contains a unit, it is all of R. I am utilizing this fact. Please don't dock off points for this literal fact from class.)
However, by the Fourth Isomorphism Theorem, this implies that $M = D^{-1}R/D^{-1}I$; a contradiction to the assumption that M is a maximal ideal. Thus we see that $x \notin P$.

$I \subset P$.

Now since M is an ideal, we see that it contains the zero element $D^{-1}I$. Now observe that for any $i \in I$,

$$\varphi(i) = \frac{i}{1} + D^{-1}I = D^{-1}I \in M.$$

Therefore we see that $I \subset \varphi^{-1}(M) = P$.

As this point we have shown that if P is the pullback of M under the given homomorphism, then (1) the pull back is well-defined (2) P is prime (3) if $x \notin \sqrt{I}$ then $x \notin P$ and (4) $I \subset P$.
Now consider the fact that $x \notin \sqrt{I} \implies x \notin P$. Let $\bigcap_{I \subset P', \text{ prime}} P'$ denote the intersection of all prime ideals containing I. Since

$$\bigcap_{I \subset P', \text{ prime}} P' \subset P$$

because P is a prime ideal contaning I, we see that if $x \notin P$ then $x \notin \bigcap_{I \subset P', \text{ prime}} P'$. As we proved that if $x \notin \sqrt{I}$, then $x \notin P$, we see that

$$x \notin \sqrt{I} \implies x \notin \bigcap_{I \subset P', \text{ prime}} P'.$$

Taking the contrapositive of the statement, we can then conclude that

$$x \in \bigcap_{I \subset P', \text{ prime}} P' \implies x \in \sqrt{I}$$

which ulimately implies that $\bigcap_{I \subset P', \text{prime}} P' \subset \sqrt{I}$.

$x \in \sqrt{I} \implies x \in P$

To show the reverse inclusion, suppose $x \in \sqrt{I}$, and let P be a prime ideal such that $I \subset P$. Then $x^n \in I$ for some positive integer n.
Suppose for the sake of contradiction that $x \notin P$ Let N be the smallest positive integer such that $x^N \in I$. Since $x^N \in I \subset P$, we see that $x^N \in P$. Note that

$$x^N = x \cdot x^{N-1} \in P.$$

Since P is a prime ideal, either $x \in P$ or $x^{N-1} \in P$. However, by assumption $x \notin P$. Thus we must have that $x^{N-1} \in P$. But since $I \subset P$, this implies that $x^{N-1} \in I$. This contradicts our choice of N as the smallest positive integer as $x^N \in I$. We have our contradiction, so we must have that $x \in P$.
Since $x \in \sqrt{I} \implies x \in P$ for every prime ideal P such that $I \subset P$, we see that

$$\sqrt{I} \subset \bigcap_{I \subset P, \text{ prime}} P.$$

Since we already showed that $\bigcap_{I \subset P, \text{ prime}} P \subset \sqrt{I}$, both set inclusions imply that

$$\sqrt{I} = \bigcap_{I \subset P, \text{ prime}} P$$

as desired.

■

Proposition 2.3.8. Let R be a ring and I, J be ideals of R such that $I \subset J \subset R$. Then I is an ideal of J.

Proof: To prove this, simply observe that for any $j \in J$ and $i \in I$ we have that $ij \in I$ and $ji \in I$.
■

A primary example of an ideal is any kernal of a homomorphism.

Lemma 2.3.9. Let $\varphi : R \longrightarrow S$ be ring homomorphism. Then $\ker(\varphi)$ is an ideal of R.

We already partially showed this earlier, and the full proof is not difficult.

Lemma 2.3.10. If R is a division ring then the only ideals of R are $\{0\}$ and R itself.

Proof: Of course, $\{0\}$ is an ideal for any ring. Therefore let I be a nonzero ideal. Then

$$ir \in I$$

for any $i \in I$ and $r \in R$. Since R is a division ring, every element has a multiplicative inverse (except 0). Hence for any nonzero i we can choose $r = i^{-1}$ to conclude that $ii^{-1} = 1_R \implies 1_R \in I$.

Since $1_R \in I$, we can set $r \in R$ to be any element to conclude that $1_R r = r \implies r \in I$. Therefore $I = R$. So every ideal is either R or $\{0\}$. ■

Proposition 2.3.11. Let R be an integral domain. Any ring homomorphism φ from R to an arbitrary ring S is injective or the zero map.

Proof: Since $\ker(\varphi)$ is an ideal of R, it is either $\{0\}$, in which case φ in injective, or R, in which case φ is the zero map. ■

Next we can introduce the concept of a quotient ring, which involves quotienting out an ideal. Note that for a ring R and an ideal I, the concept of R/I makes sense since R is an abelian group, while I is a subgroup and is therefore a normal group to R. Thus we make the following definition.

Definition 2.3.12. Let R be a ring and I an ideal of R. Then R/I, the set of all elements $r + I$ where $r \in R$, is defined to be a **quotient ring** whose operations are specified as follows.

Addition. For any $r + I, s + I \in R/I$ we have that

$$(r + I) + (s + I) = (r + s) + I.$$

Multiplication. For $r + I, s + I \in R/I$ we have that

$$(r + I) \cdot (s + I) = rs + I.$$

First, let's check that this is even sensical. Again, we know from our group theory intuition that R/I definitely makes sense when looked at as an additive group. The identity is I, inverses exist, it is closed and of course associative. Nothing has changed from our group theory perspective.

We want R/I to not only be an abelian group, but *also* a ring, we defined multiplication of elements as $(r + I) \cdot (s + I) = rs + I$. Thus we'll check the validity such multiplication.

The issue at hand is that, for any $r + I \in R/I$, there are many ways we can represent the element. For instance, for any $r' \in R$ such that $r = r' + i$ for some $i \in I$, we have that $r + I = r' + I$. That is, the way we decide to represent our elements is not unique. Thus we just need to check that the way we defined multiplication doesn't depend on the chosen representative of an element $r + I \in R/I$.

To do this suppose that $r+I=r'+I$ and $s+I=s'+I$ are elements of R/I. Then $r=r'+i$ and $s=s'+j$ for some $i,j\in I$. Therefore, $(r'+I)(s'+I)=r's'+I$. On the other hand

$$\begin{aligned}(r+I)\cdot(s+I) &= rs+I\\ &= (r'+i)(s'+j)+I\\ &= r's'+\underbrace{r'j+is'+ij}_{\text{all are in } I}+I\\ &= r's'+I.\end{aligned}$$

where in the last step we used the fact that since I is an ideal, $r'j\in I$ and $is'\in I$. Obviously $ij\in I$ as well. Therefore $(r+I)(s+I)=(r'+I)(s'+I)$, so our definition for multiplication is clear and well-defined.

You may be wondering the following: In a quotient ring R/I, why does I have to be an ideal of R? To answer this, note in the second to last step above, we used the fact that I was an ideal of R to conclude that $r'j, is'\in I$. If I hadn't been an ideal, we wouldn't have been able to absorb these elements into I. Hence, we wouldn't have been able to make sure that our desired multiplication is well-defined. So this is why a quotient ring must always quotient out an ideal, and why we can't just quotient out any subring of R.

Definition 2.3.13. Consider the following map $\pi : R\longrightarrow R/I$, known as the **projection map**, defined as

$$\pi(r)=r+I.$$

Note that this is a stupidly simple map. It's so stupid it almost doesn't even deserve a name. But it will be *convenient* to be able to refer back to the concept of associating an element $r\in R$ with a coset $r+I\in R/I$ as a **projection**. It's so convenient that if you go on in algebra you won't stop this "coset" mapping, yet everytime you see it you'll probably think it's dumb.

Also notice that in this case $\ker(\pi)=I$, and that $\text{Im}(\pi)=R/I$.

2.4 Isomorphism Theorems.

With the concept of a quotient ring defined, we can formulate analagous Isomorphism Theorems as we had in group theory. As we move forward, recall that the main ingredients of the isomorphism theorems in group theory were **normal subgroups** and **quotient groups**. For our ring isomorphism theorems, the "normal groups" will be **ideals** while the "quotient groups" will be the **quotient rings**.

The reasons for having such analogous theorems available to us for ring theory is that groups are a special case of rings. The only thing that makes a group different from a ring is that we've just added a few extra axioms. But it turns out that, even after adding these extra axioms, the Isomorphism Theorems still hold.

If you go on in algebra you'll see the Isomorphism Theorems again, proved for algebraic objects called **modules**. In fact, the Isomorphism Theorems were first proved by Emmy Noether in terms of modules; not groups, or rings, but the theorems hold for groups and rings since groups and rings are special cases of modules.

Theorem 2.4.1.(First Isomorphism Theorem.) If R and S are rings, and $\varphi : R \longrightarrow S$ is a homomorphism, then

$$R/\ker(f) \cong \mathrm{Im}(f).$$

Proof: The proof of this is analogous to the proof in group theory. We construct a homomorphism $\varphi : G/\ker(f) \longrightarrow \mathrm{Im}(f)$ by defining

$$\varphi(r + \ker(f)) = f(r).$$

Observe that for any nonzero $s \in \mathrm{Im}(f)$, there exists a $r \in R$ such that $f(r) = k$. Since s is nonzero, $r \notin \ker(f)$. However, observe that $f(r + \ker(f)) = k$. Therefore φ is surjective.

Now observe that φ is one to one. Suppose that

$$\varphi(r + \ker(f)) = \varphi(r' + \ker(f))$$

for some elements $r + \ker(f), r' + \ker(f) \in R/\ker(f)$. Then $f(r) = f(r')$. But this implies that $f(r) - f(r') = 0$ or that $f(r - r') = 0 \implies r - r' \in \ker(f)$. Therefore $r - r' = s$ for some $s \in \ker(f)$ so that

$$r + \ker(f) = r' + s + \ker(f) = r' + \ker(f).$$

Thus we have that $r + \ker(f) = r' + \ker(f)$, proving that φ is injective. Altogether we have constructed an isomorphism from $R./\ker(f)$ to $\mathrm{Im}(f)$, which proves the theorem. ■

As an application of this, we can revisit one of the examples we computed. Earlier we found that for a homomorphism $\varphi : \mathbb{R}[x] \longrightarrow \mathbb{C}$ defined as

$$\varphi(p(x)) = p(i)$$

that $\text{Im}(f) = \mathbb{C}$ and $\ker(f) = \{p(x) \in \mathbb{R}[x] \mid (x^2+1) \Big| p(x)\}$. Now that we can equivalently dtescribe the kernal as $K = \{p(x) \in \mathbb{R}[x] \mid p(x) = q(x)(x^2+1) \text{ for some } q(x) \in \mathbb{R}[x]\}$. Therefore, by the First Isomorphism Theorem,

$$\mathbb{R}[x]/K \cong \mathbb{C}.$$

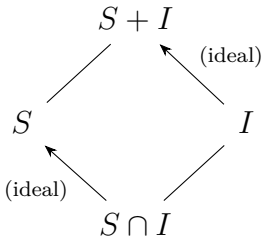
That is, the set of complex numbers is isomorphic to $\mathbb{R}[x]/K$. Well, what is this set? This set is all the elements of the form

$$q(x) + K$$

where $q(x) \in \mathbb{R}[x]$ is an element which does not have x^2+1 as a factor. Thus, the complex numbers are isomorphic to the equivalence class of polynomials which are not divisble by x^2+1.

Theorem 2.4.2. (Second Isomorphism Theorem.) Let R be a ring, I an ideal of R, and S a subring of R. Then

1. $S + I$ is a subring of R
2. I is an ideal of $S + I$
3. $S \cap I$ is an ideal of S
4. $(S + I)/I \cong S/(S \cap I)$.



The diagram on the left is analogous to the one used in the second isomorphism theorem for groups. Hence, this is again known as the diamond theorem.
Although it is important to have this diagram in mind, it is also important to remmeber that $(S + I)/I \cong S/(S \cap I)$ (given the appropriate hypotheses).

Proof:

1. To prove the first statement we first make the following connection. From the Second Isomorphism Theorem for groups, we know that $S + I$ is an abelian group. We just need to show it is closed under multiplication. Thus let $(s+i), (s'+i') \in S + I$. Then

$$(s+i)(s'+i') = \underbrace{ss'}_{\text{in } S} + \overbrace{si' + is' + ii'}^{\text{in } I}.$$

   Therefore, we see that $(s+i)(s'+i') \in I$, so that $S + I$ is closed under multiplication. Therefore it is a subring of R.

2. Let $s + i \in S + I$, and let $j \in I$. Then observe that

$$(s+i)j = sj + ij \quad \text{and} \quad j(s+i) = js + ji.$$

However, since I is an ideal, $sj, js \in I$, and clearly $ij, ji \in I$. Therefore, $(s+i)I \subset I$ and $I(s+j) \subset I$ for any $(s+j) \in S+I$, which shows that I is an ideal of this set.

3. From our study of groups, we know that $S \cap I$ is an abelain group. We just need to check that it is closed under multiplication. Thus for any $i \in S \cap I$ and $s \in S$, we see that $is \in I$ since I is an ideal.

   But $i \in S \cap I \implies i \in S$. Therfore is is also a product of two elements in S.

   Since $is \in I$ and $is \in S$, we see that $is \in S \cap I$, proving that it is an ideal of S.

4. Consider the projection map $\pi : R \longrightarrow R/I$ restircted to S, which we'll define as $\pi|_S : S \longrightarrow R/I$. (What we mean by "restricted" is that, we let π do its job, but we only let it act on elements in $S \subset R$.)

   Note that $\ker(\pi|_S) = S \cap I$, while $\text{Im}(\pi|_S) = (S+I)/I$ (namely, all the elements of the form $s+I$ where $s \notin I$.) Thus by the First Isomorphism Theorem, we have that

$$S/\ker(\pi|_S) \cong \text{Im}(\pi|_S) \implies S/(S \cap I) \cong (S+I)/I$$

   as desired.

■

Theorem 2.4.3. (Third Isomorphism Theorem) Let R be a ring and I and J ideals of R such that $I \subset J$. Then

1. J/I is an ideal of R/J
2. $R/J \cong (R/I)/(J/I)$.

Proof: For this theorem, we offer a two-in-one proof. Construct the ring homomorphism $\varphi : R/I \longrightarrow R/J$ as follows:

$$f(r+I) = r + J.$$

We first demonstrate that this is well-defined. Suppose $r+I = r'+I$; that is, there exists a $i \in I$ such that $r - r' = i$. Then observe that

$$f(r+I) = r + J = r' + i + I = r' + I = f(r'+I).$$

Thus this homomorphism is well defined. Now observe that

$$\begin{aligned} \ker(f) &= \{r+I \in R/I \mid r+J = J\} \\ &= \{r+I \in R/J \mid r \in J\} = J/I. \end{aligned}$$

Now the first result comes by recalling that the kernal is an ideal of the domain ring; that is, J/I is an ideal of R/J. The second result comes from realizing that $\text{Im}(f) = R/J$, and by applying the

First Isomorphism Theorem to that

$$(R/I)/(J/I) = R/J.$$

■

Theorem 2.4.4.(Fourth Isomorphism Theorem.) Let R be a ring, S a subring of R and I an ideal of R. Then every subring of R/I is of the form S/I where $I \subset S \subset R$. Moreover, ideals J of R containing I correspond to ideals of R/I.

2.5 Principal, Maximal and Prime Ideals.

We'll now move more deeper into ring theory. The results prior were already things we've been familiar with, since they were true for groups. It is here that we'll move onto new, deeper concepts regarding the ideal of a ring.

Let $X \subset R$. Then we can talk about the **subring generated by** X as the smallest subring containing X, or equivalently, the intersection of all the subrings containing X. More explicitly, we can define it to be the set of all finite sums of elements of X.

Similarly, we can define the **ideal generated by** X, which again is the smallest ideal containing X or equivthalently the intersection of all ideals contianing X. More explicitly, if R is a ring with identity, then the ideal generated by X is

$$\langle X \rangle = \left\{ \sum_{i=1}^{n} r_i x_i s_i \mid r_i, s_i \in R, x_i \in X \text{ and } n \in \mathbb{N} \right\}$$

while if R is commutative (and again, has an identity) this becomes

$$\langle X \rangle = \left\{ \sum_{i=1}^{n} r_i x_i \mid r_i \in R, x_i \in X \text{ and } n \in \mathbb{N} \right\}.$$

Note: this is not valid for rings without identity.

Why are these formulas correct? We'll show this for the more general case for when R may not be commutative. Specifically, we'll show that these formulas are not only ideals, but that they are the smallest ideals containing X as we have claimed.

If $r \in R$ then

$$r\left(\sum_{i=1}^{n} r_i x_i s_i\right) = \sum_{i=1}^{n} (r r_i) x_i s_i \in \langle X \rangle$$

since $r r_i \in R$ for each $i \in \{1, 2, \ldots, n\}$. Similarly,

$$\left(\sum_{i=1}^{n} r_i x_i s_i\right) r = \sum_{i=1}^{n} r_i x_i (s_i r) \in \langle X \rangle$$

since again, $s_i r \in R$ for each $i \in \{1, 2, \ldots, n\}$. Thus this is an ideal. Now let X' be an ideal which contains X. Pick an arbitrary element $\sum_{i=1}^{n} r_i x_i s_i \in \langle X \rangle$. Observe that since X' is an ideal containing X, we know that $r_i x_i s_i \in X'$ for each $i \in \{1, 2, \ldots, n\}$, and hence the sum itself, $\sum_{i=1}^{n} r_i x_i s_i$, must be in X'.

Thus for any ideal X' containing X, we see that $\langle X \rangle \subset X'$. Hence, $\langle X \rangle$ is the smallest ideal containing X. The proof is similar, and easier, for the case of $\langle X \rangle$ when R is commutative.

Definition 2.5.1. Let R be commutative and $X = \{a\}$, where $a \in R$. Then the ideal generated by X given by

$$\langle X \rangle = \{ra \mid r \in R\}$$

is said to be a **principal ideal generated by** a. Note that since R is commutative, we could have also written $\langle X \rangle = \{ar \mid r \in R\}$.

One may also view a principal ideal generated by a as the set Ra (or again, equivalently as aR).

What's an example of this? Consider the ring $\mathbb{Z}$. Then the subring $2\mathbb{Z}$ is a principle ideal generated by the element 2. That is

$$2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$$

which is pretty basic fact that we already know. But note that every ideal I of $\mathbb{Z}$ is of this form.

To see this, consider an ideal I of $\mathbb{Z}$ and suppose i is the smallest positive element of I. First observe that there is no $j \in I$ such that $i < j < 2i$. Suppose there was. Then $j = i + k$ for some $0 < k < i$. Since I is closed, we know that

$$j - i = i + k - i = k$$

is a member of I. But this contradicts our assumption that i was the smallest positive element of I. Therefore, there is no $j \in I$ such that $i < j < 2i$, so that I is of the form

$$I = \{\dots, -2i, -i, 0, i, 2i, \dots\}.$$

Hence, I is generated by i, a single element, so that I is **principal**. Thus every ideal of $\mathbb{Z}$ is principal. Rings who exhibit this type of behavior get a special name.

Definition 2.5.2. Let R be an integral domain. Then R is a **principal ideal domain (PID)** if every ideal of R is principal.

As we just showed, $\mathbb{Z}$ is a principal ideal domain.

Definition 2.5.3. Let R be a ring. Then an ideal $M \neq R$ is called **maximal** if, for any other ideal I such that $M \subset I \subset R$ we have that $M = I$ or $I = R$.

An example of this is the ring $\mathbb{Z}$ with the ideal $p\mathbb{Z}$ where p is prime. To show this, suppose I is an ideal such

$$p\mathbb{Z} \subset I \subset \mathbb{Z}$$

and further that there exists an $i \in I$ such that $i \notin p\mathbb{Z}$.

Since i is not divisible by p, we know by Fermat's Little Theorem that

$$i^{p-1} = 1 \bmod p.$$

Thus for some $n \in \mathbb{Z}$, $pn - i^{p-1} = 1$. Since $p\mathbb{Z} \subset I$ and $i \in I \implies i^{p-1} \in I$, we see that $pn - i^{p-1} = 1 \in I$. Since $1 \in I$, we can repeatedly add and subtract 1 to generated all of $\mathbb{Z}$, and since this must all be contained in I, we have that $I = \mathbb{Z}$. Thus $p\mathbb{Z}$ is maximal.

Theorem 2.5.4. Let R be a ring and I a proper ideal (i.e., $I \neq R$.) Then there is a maximal ideal of R containing I. Furthermore, if R is a ring with identity, then there are always maximal ideals.

How do we know when a given ideal is maximal or not? That is, how do we know there aren't "bigger" proper ideals which contain the one we are interested in?

Theorem 2.5.5. Let R be a commutative ring with identity. Then an ideal M of R is maximal if and only if R/M is a field. In other words,

$$M \text{ is maximal} \iff R/M \text{ is a field.}$$

Proof: ($\Longrightarrow$) Suppose M is a maximal ideal. To show that R/M is a field, we first realize that it is commutative since R is commutative. Thus we just need to show it is a divison ring.

Let $a + M \in R/M$ where $a \notin M$. Since aR and M are both ideals, we have that $aR + M$ is an ideal. Note that $M \subset aR + M$, which implies that $aR + M = R$. Therefore, there exists an element $r \in R$ and $m \in M$ such that $ar + m = 1$.

Since $ar = 1 - m$, consider $r + R \in R/M$, and observe that

$$(a + M)(r + M) = ar + M = 1 - m + M = 1 + M$$

so that $r + M$ is the desired inverse of $a + M$. Since $a \notin M$, this shows that R/M is a division ring. Since it is commutative, it is a field.

($\Longleftarrow$) Now suppose that R/M is a field. It is a commutative division ring, so that by Lemma 2.3.10 we know that its only ideals are either 0 or R/M. But by the Fouth Isomorphism Theorem, we know that these ideals of R/M correspond to M and R. Thus there are no other ideals of R containing M other than M and R, proving that M is maximal. ■

Definition 2.5.6. Let R be a commutative ring. Then an ideal P is said to be **prime** if $P \neq R$ and if $ab \in P$, then either $a \in P$ or $b \in P$. Furthermore, if there exists a $p \in R$ such that $Rp = \langle p \rangle$ is a prime ideal, then p is said to be prime.

You may wonder why we would make this definition, since ideals tend to suck in elements of R (i.e., $ri \in R$ for all $i \in I$, $r \in R$). However, just because $ab \in I$ does not mean $a \in I$ or $b \in I$. Consider for instance the ring $4\mathbb{Z}$. Obviously, $4 \in 4\mathbb{Z}$, but $2 \cdot 2 = 4$ and yet $2 \notin 4\mathbb{Z}$.

Prime ideals have a similar theorem that maximal ideals have.

Theorem 2.5.7. Let R be a commutative ring with identity. Then an ideal P of R is prime if and only if R/P is an integeral domain. That is,

$$P \text{ is prime} \iff R/P \text{ is an integral domain.}$$

Proof: ($\Longrightarrow$) Suppose P is a prime ideal. To show that R/P is an integral domain, we must show that it has no zero divisor (as we already know it is commutative). Thus let $r + P \in R/P$ where $r \notin P$. Suppose for the sake of contradiction that

$$(r + P)(r' + P) = P$$

for some $r' + P \in R/P$ where $r' \notin P$ (i.e., that there are zero divisors). Then this implies that $rr' \in P$. Since P is prime, we have that either $r \in P$ for $r' \in P$, which is a contradiction since our hypothesis was that $r, r' \notin p$. Therefore R/P is commutative and has no zero divisors, so it is an integral domain.

($\Longleftarrow$) Now suppose that R/P is an integral domain. Then

$$(r + P)(r' + P) \neq P$$

for any r and $r' \notin P$. In other words, if $r, r' \notin P$ then $rr' \notin P$. Taking the contrapositive of this statement, we have equivalently that if $rr' \in P$ then r or $r' \in P$ (notice how that "and" changed to an "or" upon negation) which proves that P is prime. ■

Finally, we can combine all of these theorems into one useful criterion for primeness.

Theorem 2.5.8. Let R be a commutative ring with identity. Let I be an ideal. If I is maximal then I is prime. In other words,

$$I \text{ is maximal} \implies I \text{ is prime.}$$

Proof: By Theorem 1.2.5.5, if I is maximal then R/I is a field. But since R/M is a field, it is an integral domain. Hence by Theorem 1.2.5.7, we have that I is a prime ideal, which proves the theorem. ■

The corollaries of these theorems are immediate.

Corollary 2.5.9. Let R and S be commutative rings with identity, and suppose $\varphi : R \longrightarrow S$ be a surjective ring homomorphism. Then

1. If S is a field, then $\ker(\varphi)$ is a maximal ideal of R.
2. If S is an integral domain then $\ker(\varphi)$ is a prime ideal of R.

Proof:

1. If φ is surjective, then by the First Isomorphism Theorem $R/\ker(\varphi) = \text{Im}(\varphi) = S$. Therefore $R/\ker(\varphi)$ is a field, so by Theorem 1.2.5.5 we have that $\ker(\varphi)$ is maximal.
2. In this case, we again have that $R/\ker(\varphi) = S$, so that $R/\ker(\varphi)$ is an integral domain. Therefore $\ker(\varphi)$ is a prime ideal of R by Theorem 1.2.5.7.

■

2.6 Ring of Fractions.

As rings were modeled based on the behavior of the integers, we can hypothesize that it is possible to generalize the construction of $\mathbb{Q}$ from $\mathbb{Z}$. Such a construction is possible, and studying it will utilize all of the concepts introduced up to this point. However, there is a very subtle issue that when we try to do this. The main issue is in defining $\frac{a}{b}$ when $b \neq 0$ doesn't have an inverse.

Definition 2.6.1. Say $(R, +, \cdot)$ is a commutative ring with $1 \neq 0$. We say that $D \subset R$ is a **multiplicative set** if

1. $1 \in D$
2. $x \cdot y \in D$ for all $x, y \in D$.

Condition 1. is not really crucial. We just need to make sure that D is nonempty.

An example of a multiplicative set includes $D = \{1\}$. A less boring example is $D = \{2n+1 \mid n \in \mathbb{Z}\}$, since the product of two odd integers is odd. And the maximal example we can come up with for any ring is to set $D = R$.

Now we introduce a proposition regarding a relation on elements.

Proposition 2.6.2. Define a relation $\sim$ on $R \times D$ by saying $(a, b) \sim (c, d)$ whenever we can find $x \in D$ such that

$$x \cdot (a \cdot d - b \cdot c) = 0.$$

Then $\sim$ is an equivalence relation on $R \times D$.

Proof: We must show the three properties of an equivalence relation.

Reflexivity. $(a, b) \sim (a, b)$. Choosing $x = 1$, then we see that

$$x \cdot (a \cdot b - a \cdot b) = 0.$$

Symmetry. $(a, b) \sim (c, d)$ if and only if $(c, d) \sim (a, b)$.

Say $(a, b) \sim (c, d)$. We can find an $x \in D$ such that

$$x \cdot (a \cdot d - b \cdot c) = 0.$$

Now consider

$$x \cdot (c \cdot b - d \cdot a) = (-1) \cdot [x \cdot (a \cdot d - b \cdot c)] = 0$$

This shows that $(c, d) \sim (a, b)$

Transitivity. If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $(a, b) \sim (e, f)$.

We can find $x, y \in D$ such that

$$x \cdot (a \cdot d - b \cdot c) = y \cdot (c \cdot f - d \cdot e) = 0.$$

Denote $z = d \cdot x \cdot y$. This is an element of D since it is a multiplicative set. Now

$$\begin{aligned} z \cdot (a \cdot f - b \cdot e) &= yfxad - yfxbc - xbycf - xbyde \\ &= y \cdot f[x \cdot (a \cdot d - b \cdot c)] + x \cdot b[y \cdot (c \cdot f - d \cdot e)] \\ &= 0. \end{aligned}$$

■

We'll define the collection of all equivalence classes by the set

$$D^{-1}R = \left\{ \frac{a}{b} \mid a \in R, b \in D \right\}$$

where

$$\frac{a}{b} = \Big\{ (c,d) \in R \times D \mid (a,b) \sim (c,d) \Big\}.$$

Why are we doing this? Let's say that $R = \mathbb{Z}$, and $D = \mathbb{Z} \setminus \{0\}$. Then D is a multiplicative set and $(a,b) \sim (c,d)$ if and only if $ab - bc = 0$. As an example, $(1,2) \sim (2,4)$ implies that in our set, $\frac{1}{2} = \frac{2}{4}$. In other words, we're basically saying we don't care whether or not the fraction is in reduced terms.

Proposition 2.6.3. Let R be a ring with identity $1 \neq 0$.

1. $D^{-1}R$ is a commutative ring
2. $D^{-1}R = \left\{ \frac{0}{1} \right\}$ is the trivial ring if and only if $0 \in D$.
3. The units $(D^{-1}R)^\times$ contains
$$D^{-1}D = \left\{ \frac{a}{b} \mid a, b \in D \right\}.$$

Proof:

1. Define addition and multiplication.

   Lemma 2.6.4. The following is well-defined.

$$\begin{aligned} + : D^{-1}R \times D^{-1}R &\longrightarrow D^{-1}R \\ \frac{a}{b}, \frac{c}{d} &\longmapsto \frac{ad - bc}{bd} \end{aligned}$$

   Suppose that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$. We can find $x, y \in D$ such that

$$x \cdot (a_1 b_2 - a_2 b_1) = y \cdot [c_1 d_2 - c_2 d_1] = 0.$$

   Denote $z = xy \in D$. Then

$$\begin{aligned} & z[(a_1 d_1 + c_1 b_1)(b_2 d_2) - (a_2 d_2 + c_2 b_2)(b_1 d_1)] \\ = {} & y d_1 d_2 [x(a_1 b_2 - a_2 b_1)] + x b_2 b_1 [y(c_1 d_2 - c_2 d_1)] = 0. \end{aligned}$$

Hence $(a_1d_1 + c_1b_1, b_1d_1) \sim (a_2d_2 + c_2d_2, b_2d_2)$.

Lemma 2.6.5. The following is well-defined:

$$\begin{aligned} \cdot : D^{-1}R \times D^{-1}R &\longrightarrow D^{-1}R \\ \frac{a}{b}, \frac{c}{d} &\longmapsto \frac{ac}{bd} \end{aligned}$$

Again, say that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$. Denote $z = xy \in D$. Then

$$z[(a_1c_1) \cdot (b_2d_2) - (a_2c_2)(b_1d_1)] = yc_1d_2[x(a_1b_2 - a_2b_1)] + xa_2b_1[y(c_1d_2 - c_2d_1)] = 0$$

Hence $(a_1c_1, b_1d_1) \sim (a_2c_2, b_2d_2)$. Showing that $D^{-1}R$ is a commutative ring with these properties is straightforward.

2. We show that $D^{-1}R$ is trivial if and only if $0 \in D$.

   Say $D^{-1}R = \left\{\frac{0}{1}\right\}$. Since $1 \in R$, we see that $\frac{1}{1} \in D^{-1}R$ so $\frac{1}{1} = \frac{0}{1}$. By definition, we can find an $x \in D$ such that

   $$x \cdot (1 \cdot 1 - 0 \cdot 1) = 0$$

   so $x = 0$. Hence $0 \in D$.

   Now suppose $0 \in D$. Pick any $\frac{a}{b} \in D^{-1}R$. For $x = 0$, we have that $x \cdot (a \cdot 1 - b \cdot 0) = 0$. Hence we see that $\frac{a}{b} = \frac{0}{1}$, so that $D^{-1}R = \left\{\frac{0}{1}\right\}$.

3. If $a, b \in D$, then $\frac{a}{b} \in D^{-1}R$ so that $\frac{a}{b} \cdot \frac{b}{a} = \frac{1}{1}$. Hence $\frac{a}{b} \in (D^{-1}R)^*$. That is, $DD^{-1} \subset (D^{-1}R)^*$.

■

$D^{-1}R$ is the ring of fractions of R by D. We also remark that if $D \subset R \setminus \{0\}$, then $D^{-1}R$ is a commutative ring with $\frac{1}{1} \neq \frac{0}{1}$. Finally, we also note that the elements of D are invertible in $D^{-1}R$. We want to construct $D^{-1}R$ as the "smallest" ring such that D is invetible.

Next we introduce a theorem.

Theorem 2.6.6. Let $(R, +, \cdot)$ be a commutative ring with $1 \neq 0$, and $D \subset R$ be a multiplicative set. Denote $S = D^{-1}R = \{a/b \mid a \in R, b \in D\}$ as the *ring of fractions of R by D*. Then

1. Let I be an ideal of R. Then $D^{-1}I = \{x/b \mid x \in I, b \in D\}$ is an ideal of S. (This is called the *extension* of I to S.)

2. Let J be an ideal of S. Define the ring homomorphism $\pi : R \longrightarrow S$ which sends $r \mapsto r/1$. Then the preimage $\pi^{-1}(J)$ is an ideal of R. (This is called the *contraction* of J to R.)

3. For any ideal J of S we have that $D^{-1}[\pi^{-1}(J)] = J$. (That is, "extension is the left inverse of contraction.")

Proof:

1. To show this we proceed as follows.

$D^{-1}I$ is nonempty.

Observe that since D is a multiplicative set we have that $1 \in D$. Hence if I is nonempty, then $\frac{i}{1} \in D^{-1}I$ for all $i \in I$. Hence it is nonempty.

$a - b \in D^{-1}I$ if $a, b \in D^{-1}I$.

Let $a, b \in D^{-1}I$. Then $a = \frac{i_1}{d_1}$ and $b = \frac{i_2}{d_2}$. Then observe that

$$a - b = \frac{i_1 d_2 - i_2 d_1}{d_1 d_2}.$$

Since I is an ideal, $i_1 d_2 . i_2 d_1 \in I$. Therefore their difference $i_1 d_2 - i_2 d_1 \in I$. Since D is a multiplicative set we have that $d_1 d_2 \in D$. Hence we see that $\frac{i_1 d_2 - i_2 d_1}{d_1 d_2} \in D^{-1}I$, so that $a - b \in D^{-1}I$ whenever $a, b \in D^{-1}I$.

$s \cdot a \in D^{-1}I$ if $s \in S, a \in D^{-1}I$

Let $s \in S$ and $a \in D^{-1}I$. Then $s = \frac{r}{d}$ and $a = \frac{i}{d'}$ for $r \in R, i \in I, d, d' \in D$. Hence observe that

$$s \cdot a = \frac{r}{d} \cdot \frac{i}{d'} = \frac{ri}{dd'}.$$

Since I is an ideal, we see that $ri \in I$. Since D is a multiplicative set, we also see that $dd' \in D$. Hence, we see that $\frac{ri}{dd'} \in D^{-1}I$, so that $s \cdot a \in D^{-1}I$ if $s \in S$ and $a \in D^{-1}I$.

Thus in total we have that $D^{-1}I$ is an ideal of S.

2. To show this is an ideal, we proceed as follows.

$\pi^{-1}(J)$ is nonempty.

Observe that if J is nonempty, then $\frac{0}{1} \in J$. This is because if J is an ideal, then $sj \in J$ for all $s \in S$ and $j \in J$. Hence let $s = \frac{0}{1}$, and $j = \frac{r}{d} \in J$. Then

$$sj \in J \implies \frac{0}{1} \cdot \frac{r}{d} \in J \implies \frac{0}{d} \in J.$$

Now observe that $(0, 1) \sim (0, d)$ for any $d \in D$. This is because

$$x \cdot (0 \cdot d - 0 \cdot 1) = x \cdot (0) = 0$$

is automatically satisfied by any $x \in D$. Hence, $(0, 1) \sim (0, d)$. Now we argue that

$$\pi^{-1}(J) = \{r \in R \mid \pi(r) \in J\}.$$

is nonempty. Observe that $\pi\left(\frac{0}{1}\right) = \frac{0}{1}$, which we know is true because π, as a ring homomorphism, is also a group homomorphism between the abelian groups R and S (i.e., the abelian groups we get when we remove the multiplicative ring stucture on them). We know from group theory that group homomorphisms map additive zero elements from one group to the additive zero element of the other group, so that $\pi\left(\frac{0}{1}\right) = \frac{0}{1}$.

As we just showed, $\frac{0}{1} \in J$. Since $\pi\left(\frac{0}{1}\right) = \frac{0}{1}$, we see that $\frac{0}{1} \in \pi^{-1}(J)$, so it is nonempty.

$a - b \in \pi^{-1}(J)$ if $a, b \in \pi^{-1}(J)$.

Suppose $a, b \in \pi^{-1}(J)$. Then $\pi(a) = \frac{a}{1}$ and $\pi(b) = \frac{b}{1}$ are both members of J. Since J is an ideal, we know that on one hand,

$$\frac{a}{1}, \frac{b}{1} \in J \implies \frac{a}{1} - \frac{b}{1} \in J \implies \frac{a-b}{1} \in J.$$

On the other hand, observe that $\pi(a-b) = \frac{a-b}{1}$, which we just showed is inside J. Therefore $a - b \in \pi^{-1}(J)$ when $a, b \in \pi^{-1}(J)$, which is what we set out to show.

$r \cdot a \in \pi^{-1}(J)$ if $r \in R, a \in \pi^{-1}(J)$.

Now suppose that $r \in R$ and $a \in \pi^{-1}(J)$. Then $\pi(a) = \frac{a}{1} \in J$ by definition. Hence, observe that

$$\pi(r \cdot a) = \frac{ra}{1} = \frac{r}{1} \cdot \frac{a}{1}.$$

Observe that since J is an ideal of S, $\frac{r}{1} \cdot \frac{a}{1} \in R$. Hence we see that $\pi(r \cdot a) \in J$, so that $r \cdot a \in \pi^{-1}(J)$ whenever $r \in R$, $a \in \pi^{-1}(J)$, as desired.

3. We can prove equality between the sets by demonstrating mutual subset properties.

$D^{-1}[\pi^{-1}(J)] \subset J$

Consider any $\frac{a}{b} \in D^{-1}[\pi^{-1}(J)]$ where by definition $a \in \pi^{-1}(J)$ and $b \in D$.

Since $a \in \pi^{-1}(J)$ we know that $\pi(a) = \frac{a}{1} \in J$. Consider the element $\frac{1}{b} \in S$. Since J is an ideal of S, we know that $sj \in J$ for all $s \in S, j \in J$. Set $j = \frac{a}{1}$ and $s = \frac{1}{b}$, and observe that

$$sj \in J \implies \frac{1}{b} \cdot \frac{a}{1} = \frac{a}{b} \in J.$$

Hence we have that $D^{-1}[\pi^{-1}(J)] \subset J$.

$J \subset D^{-1}[\pi^{-1}(J)]$

Now consider any $j = \frac{a}{b} \in J$. To prove this direction, we just need to show that $a \in \pi^{-1}(J)$ since b is already a member of D. And to prove that, we just need to show that $\frac{a}{1} \in J$. Hence we formalize our claim:

Claim: $\frac{a}{b} \in J \implies \frac{a}{1} \in J$.

To show this, observe that $\frac{b}{1} \in S$ and since J is an ideal,

$$\frac{b}{1} \cdot \frac{a}{b} \in J \implies \frac{ab}{b} \in J.$$

Now observe that $(ab, b) \sim (a, 1)$, since

$$x \cdot (ab \cdot 1 - b \cdot a) = x \cdot (ab - ab) = 0$$

is satisfied by any choice of $x \in D$. Therefore, $\frac{ab}{b} = \frac{a}{1}$, and since $\frac{ab}{b} \in J$ we have that $\frac{a}{1} \in J$. Finally, since $\frac{a}{1} \in J$, we see that $a \in \pi^{-1}(J)$. Therefore, $\frac{a}{b} \in D^{-1}[\pi^{-1}(J)]$, so that $J \subset D^{-1}[\pi^{-1}(J)]$ as desired.
With both directions, we can then conclude that $D^{-1}[\pi^{-1}(J)] = J$, which is what we set out to show.

■

Theorem 2.6.7. Let $(R, +, \cdot)$ be a commutative ring with $1 \neq 0$, and $D \subset R$ be a multiplicative set. Let P be an ideal of R with the property that if $d \cdot x \in P$ for $d \in D$ then $x \in P$. Then the following are equivalent.

i. P is a prime ideal of R with $P \cap D = \varnothing$

ii. $D^{-1}P$ is a prime ideal of the ring of fractions $D^{-1}R$.

Proof:

i $\implies$ ii. Suppose P is disjoint with D. Then we show that $D^{-1}P$ is a prime ideal of $D^{-1}R$.

$D^{-1}P$ is nonempty.

If P is nonempty, then since $1 \in D$, we know that $\frac{p}{1} \in D^{-1}P$ for each $p \in P$. Hence, it is nonempty.

$a, b \in D^{-1}P \implies a - b \in D^{-1}P$.

Suppose $a, b \in D^{-1}P$. Write $a = \frac{p_1}{d_1}$ and $b = \frac{p_2}{d_2}$. Then we see that

$$a - b = \frac{p_1 d_2 - p_2 d_1}{d_1 d_2}.$$

Observe that $p_1 d_2, p_1 d_2 \in P$ since P is an ideal of R. Hence, $p_1 d_2 - p_2 d_1 \in P$, and since $d_1 d_2 \in D$, we have that $a - b \in D^{-1}P$ whenever $a, b \in D^{-1}P$.

$r \cdot a \in D^{-1}P$ for $r \in D^{-1}R, a \in D^{-1}P$.

Consider an element $a = \frac{p}{d} \in D^{-1}P$ and $r = \frac{r'}{d'} \in D^{-1}R$. Then

$$r \cdot a = \frac{r'}{d'} \cdot \frac{p}{d} = \frac{r'p}{d'd}.$$

Since P is a prime ideal, we see that $r'p \in P$ and $d'd \in D$ as it is a multiplicative set. Therefore it is an ideal.

$a \cdot b \in D^{-1}P \implies a \in D^{-1}P$ or $b \in D^{-1}P$

Let $a = \frac{r_1}{d_1}$ and $b = \frac{r_2}{d_2}$ where $r_1, r_2 \in R$ and $d_1, d_2 \in D$. Now

$$a \cdot b \in D^{-1}P \implies \frac{r_1 r_2}{d_1 d_2} \in D^{-1}P.$$

Then we see that $r_1 r_2 \in P$. Since P is a prime ideal disjoint with D, we see that $r_1 \in P$ or $r_2 \in P$ and $r_1, r_2 \notin D$ by assumption. Therefore, we see that $a \in D^{-1}P$ or $b \in D^{-1}P$, so that $D^{-1}P$ is a prime ideal.

***ii* $\implies$ *i*.** Suppose $D^{-1}P$ is a prime ideal of $D^{-1}R$.

P is nonempty.

If $D^{-1}P$ is nonempty, then since D at least contains 1, there exists at least one $\frac{a}{1} \in D^{-1}P$ where $a \in P$. Hence we see that P is nonempty.

$a, b \in P \implies a - b \in P$.

Suppose $a, b \in P$. Then we see that $\frac{a}{1}, \frac{b}{1} \in D^{-1}P$. Hence,

$$\frac{a}{1} - \frac{b}{1} \in D^{-1}P \implies \frac{a-b}{1} \in P.$$

Since $\frac{a-b}{1} \in D^{-1}P$, we see that $a - b \in P$. Hence $a, b \in P \implies a - b \in P$.

$rp \in P$ if $r \in R, p \in P$.

Consider $\frac{p}{1} \in D^{-1}P$ and $\frac{r}{1} \in D^{-1}R$ for any $r \in R$. Then since $D^{-1}P$ is an ideal,

$$\frac{r}{1} \cdot \frac{p}{1} \in D^{-1}P \implies \frac{rp}{1} \in D^{-1}P.$$

Thus we see that $rp \in P$. Therefore $r \in R, p \in P \implies rp \in P$.

$D \cap P = \varnothing$.

Suppose that $D \cap P \neq \varnothing$. Then there exists a $p \in P$ where $p \in D$. Hence, observe that $\frac{p}{p} = \frac{1}{1} \in D^{-1}P$. Then since $D^{-1}P$ is an ideal of $D^{-1}R$, we see that $pr \in D^{-1}P$ for any

$p \in D^{-1}P$ and $r \in D^{-1}R$. Thus see that for any $\frac{a}{b} \in D^{-1}R$,

$$\frac{1}{1} \cdot \frac{a}{b} \in D^{-1}P \implies \frac{a}{b} \in D^{-1}P$$

which shows that $D^{-1}P = D^{-1}R$. Hence, if we want $D^{-1}P$ to be a proper ideal, we need that $D \cap P = \varnothing$.

ab* ∈ *P* ⟹ *a* ∈ *P* or *b* ∈ *P

Suppose that $a = \frac{p_1}{d_1}$ and $b = \frac{p_2}{d_2}$ such that

$$\frac{p_1}{d_1} \cdot \frac{p_2}{d_2} \in D^{-1}P \implies \frac{p_1}{d_1} \text{ or } \frac{p_2}{d_2} \in D^{-1}P.$$

Since $\frac{p_1}{d_1} \cdot \frac{p_2}{d_2} \in D^{-1}P$, we have that $\frac{p_1 p_2}{d_1 d_2} \in D^{-1}P$, which implies that $p_1 p_2 \in P$.

The fact that $\frac{p_1}{d_1} \in D^{-1}P$ or $\frac{p_2}{d_2} \in D^{-1}P$ implies that $p_1 \in P$ or $p_2 \in P$. Hence we see that $p_1 p_2 \in P \implies p_1 \in P$ or $p_2 \in P$, which proves that P is a prime ideal.

With both directions proven, we can conclude that the two given statments are in fact equivalent. ■

Localization. The construction we have been implementing relates to a concept as localization, which we define as follows.

Definition 2.6.8. Let $(R, +, \cdot)$ be a commutative ring with identity $1 \neq 0$. Let $S \subset R$ and define $D = R - S$. We define the **localization** of R at S as the ring of fractions

$$R_P = D^{-1}R = \left\{ \frac{r}{d} \mid r \in R, d \in D \right\}.$$

It turns out that if we localize at a prime ideal, nice things happen. Specifically, the localization contains a unique maximal ideally.

Generally, rings do not have unique maximal ideals, although the definition of a maximal ideal can often confuse people. For example, consider the ring $\mathbb{Z}$. Then for any prime p, we see that $p\mathbb{Z}$ is a maximal ideal; given the infinitude of the primes, we have infinitely many maximal primes.

For rings that do have a unique, maximal ideal, we give them a special name.

Definition 2.6.9. Let $(R, +, \cdot)$ be a ring. If R has a unique, maximal ideal M, then we say that R is a **local ring**.

Theorem 2.6.10. Let $(R, +, \cdot)$ be an integral domain, and P be a prime ideal of R.

1. The set $D = R - P$ is a *multiplicative set*.
2. The *localization* of R at P, $R_P = D^{-1}R$, is an integral domain.
3. The ring R_P is a *local* ring i.e., R_P has a unique maximal ideal M_P.

Proof:

1. First we show that $1 \in D$. Suppose P is a prime ideal such that $R \neq P$. Then observe that $1 \notin P$. For if $1 \in P$, then for any $r \in R$, we'd see that

$$1 \cdot r = r \in P.$$

Since r is arbitrary, we'd have that $R = P$, a contradiction. Therefore, we see that $1 \in R - P = D$, which proves the first property.

Now we show $x, y \in D \implies xy \in D$. Since P is a prime ideal, we know that $p_1 p_2 \in P \implies p_1 \in P$ or $p_2 \in P$. Hence the reverse negative of the statement is true: if $p_1 \notin P$ and $p_2 \notin P$ then $p_1 p_2 \notin P$.

Therefore for any $x, y \in D = R - P$, we see that $xy \notin P$. Hence $xy \in D$, which proves the second property.

2. Since we already know that R_P is a commutative ring, it suffices to show that there are no zero divisors. Suppose on the contrary that $\frac{a}{b}, \frac{c}{d} \in R_P$ are zero divisors of each other. Hence, $a \neq 0$ and $c \neq 0$. Then we see that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{0}{1}.$$

In this case, we see that there exists a $x \in D$ such that

$$x \cdot (ac \cdot 1 - bd \cdot 0) \implies x \cdot (ac).$$

Since $a, c \neq 0$, and R is an integral domain, we see that $ac \neq 0$. But x is also nonzero, while $x \cdot (ac) = 0$. This cannot happen since R is an integral domain, so we have a contradiction. Therefore there are no zero divisors, and since R_P is a commutative, this makes it an integral domain.

3. Let $M_P = D^{-1}P$. We'll show that this is our unique, maximal ideal.

First observe that in the previous theorem, if P is a prime ideal and $P \cap D = \varnothing$, then $D^{-1}P$ is a prime ideal of $D^{-1}R$.

In our case, $D = R - P$. Hence $P \cap D = \varnothing$, so we may conclude that M_P is a prime ideal.

Now we show M_p is maximal. Let I be a proper ideal, i.e. $I \neq D^{-1}R$, and suppose $I \not\subset M_P$. That is, there exist an element $\frac{a}{b} \in I$ such that $\frac{a}{b} \notin M_P$.

Since $\frac{a}{b} \notin M_P$, we see that $a \notin P$. Hence, $a \in R - P = D$, and of course $b \in D$ as well. Now consider the element $\frac{b}{a}$. Observe that $\frac{b}{a} \notin M_P$, since $b, a \in D$, as shown earlier. Since $\frac{a}{b} \in I$, we have that

$$\frac{a}{b} \cdot \frac{b}{a} \in I \implies = \frac{1}{1} \in I.$$

Since I contains $\frac{1}{1}$, we have that $I = D^{-1}R$. This is because we see that for any $\frac{c}{d} \in D^{-1}R$,

$$\frac{c}{d} = \frac{c}{d} \cdot \frac{1}{1} \in I \implies \frac{c}{d} \in I.$$

Hence, $I = D^{-1}R$. But we assumed I was a proper ideal; thus we have a contradiction, so

we see that $M_P = D^{-1}P$ is in fact maxmial. Since we assumed I was *any* ideal of $D^{-1}R$, this also proves that M_P is a unique maximal ideal, since what we showed is that any other ideal is automatically contained in M_P.

■

As an example, consider the prime ideal $\{0\}$ of the ring $\mathbb{Z}$. Then the localization of $\mathbb{Z}$ at $\{0\}$ is given by

$$\left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} - \{0\}\right\}$$

which is just the rational numbers.

Theorem 2.6.11. Let $(R, +, \cdot)$ be an integral domain, and P be a prime ideal of R. Let R_P be the localization of R at P, and M_P be its unique maximal ideal. Consider the map $\varphi : R/P \longrightarrow R_P/M_P$ which sends $r + P \mapsto r/1 + M_P$.
1. φ is a well-defined, injective ring homomorphism.
2. φ is an isomorphism if P is a maximal ideal.

Proof:

1. **Well-defined.**

First observe that this function is well-defined. Suppose that $r+P = r'+P$; that is, $r = r'+p$ for some $p \in P$. Observe that

$$\begin{aligned}\varphi(r+P) = \frac{r}{1} + M_P &= \frac{r'+p}{1} + M_P \\ &= \frac{r'}{1} + \frac{p}{1} + M_P \\ &= \frac{r'}{1} + M_P \\ &= \varphi(r'+P)\end{aligned}$$

where in the fourth step we used that fact that $\dfrac{p}{1} \in M_P$ since $p \in P$, $1 \in D$. Since $\varphi(r+P) = \varphi(r'+P)$, we see that this function is well-defined.

Ring homomorphism.

We demonstrate that $\varphi : R/P \longrightarrow R_P/M_P$ is a ring homomorphism.
$\boldsymbol{\varphi(a+b) = \varphi(a) + \varphi(b).}$ Let $a = r + P$ and $b = r' + P$ be elements or R/P. Then

$$\begin{aligned}\varphi(a+b) = \varphi\Big((r+r') + P\Big) &= \frac{r+r'}{1} + M_P \\ &= \frac{r}{1} + \frac{r'}{1} + M_P \\ &= \left(\frac{r}{1} + M_P\right) + \left(\frac{r'}{1} + M_P\right) \\ &= \varphi(a) + \varphi(b)\end{aligned}$$

which is what we set out to show.

$\varphi(ab) = \varphi(a)\varphi(b)$. Again, suppose $a = r + P$ and $b = r' + P$. Then observe that

$$\begin{aligned}\varphi(ab) = \varphi\Big((r+P)(r'+P)\Big) &= \varphi(rr' + P) \\ &= \frac{rr'}{1} + M_P \\ &= \left(\frac{r}{1} + M_P\right)\left(\frac{r'}{1} + M_P\right) \\ &= \varphi(a)\varphi(b)\end{aligned}$$

which is what we set out to show.

With these two properties, we have that $\varphi : R/P \longrightarrow R$ is a ring homomorphism.

Injectivity.

Next we show that this is an injective function. Suppose $r + P, r' + P \in D^{-1}R$ such that

$$\varphi(r + P) = \varphi(r' + P).$$

Then we have that $\dfrac{r}{1} + M_P = \dfrac{r'}{1} + M_P$. In other words, we see that

$$\frac{r}{1} = \frac{r'}{1} + \frac{a}{b}$$

for some $\dfrac{a}{b} \in M_P$. This further implies that $\dfrac{r}{1} = \dfrac{r'b + a}{b}$. Hence, $(r, 1) \sim (r'b + a, b)$. For this equivalence to occur, there must exist an element $x \in D$ such that

$$x \cdot (rb - (r'b + a)) = 0.$$

Since R is an integral domain, and $x \neq 0$, we require that $rb = r'b + a$. Rearranging, we see that this implies that

$$rb - r'b = a \implies (r - r')b = a.$$

The above equality implies that $(r - r')b \in P$; recall that $\dfrac{a}{b} \in M_P = D^{-1}P$, so that $a \in P$ and $b \in D = R - P$.

Since P is a prime ideal, we thus see that either $r - r' \in P$ or $b \in P$. Since we just said that $b \notin P$, we must have that $r - r' \in P$. In other words,

$$r - r' = p \implies r = r' + p$$

for some $p \in P$. Hence,

$$r + P = r' + p + P = r' + P.$$

Therefore we see that $\varphi(r + P) = \varphi(r' + P) \implies r + P = r' + P$, so that φ is injective.

2. Suppose P is a maximal ideal (in addition to being prime). To demonstrate that $\varphi : R/P \longrightarrow R_P/M_P$ is an isomorphism, it suffices to show that φ is surjective, since in the previous step

we already showed that it was injective.
Since P is a maximal ideal, we see that R/P is a field. Therefore inverse elements exist, so for the element $b+P$, there exists an element $b'+P$ where $b' \notin P$ such that

$$(b+P)(b'+P) = 1+P.$$

That is, $bb' = 1+p$ for some $p \in P$.
Let $r = ab'$, and consider the elements $\frac{a}{b}+M_P$ and $\frac{r}{1}+M_P$. Since we want $\frac{a}{b}+M_P$ to be nontrivial, we let $a, b \in D$ (that is, $a \notin P$ or else in that case $\frac{a}{b} \in M_P$).
Observe that $\frac{r}{1} \notin M_P$ since $a \in D, b' \in D$, and because D is a multiplicative set, $r = ab \in D$. Therefore $r \notin P$, so that $\frac{r}{1} \notin M_P$. Hence, $\frac{r}{1}+M_P \neq \frac{0}{1}+M_P$.
Now observe that

$$\begin{aligned}
\left(\frac{a}{b}+M_p\right) - \left(\frac{r}{1}+M_P\right) &= \left(\frac{a}{b}-\frac{r}{1}\right)+M_P \\
&= \frac{a-br}{b}+M_P \\
&= \frac{a-b(ab')}{b}+M_P \\
&= \frac{a-abb'}{b}+M_P \text{ (by commutativity)} \\
&= \frac{a-a(1+p)}{b}+M_P \\
&= \frac{-ap}{b}+M_P.
\end{aligned}$$

Since P is an ideal, we see that $-ap \in P$. Therefore, $\frac{-ap}{b} \in M_P$, so that $\frac{-ap}{b}+M_P = \frac{0}{1}+M_P$. Thus what we've shown is that

$$\left(\frac{a}{b}+M_P\right) - \left(\frac{r}{1}+M_P\right) = \frac{0}{1}+M_P.$$

which implies that

$$\frac{a}{b}+M_P = \frac{r}{1}+M_P.$$

Thus we see that $\varphi(r+P) = \frac{r}{1}+M_P = \frac{a}{b}+M_P$. However, $\frac{a}{b}+M_P$ was an arbitrary element of R_P/M_P. Hence, we've shown that for any $\frac{a}{b}+M_P$, there exists an $r+P \in R/P$ such that $\varphi(r+P) = \frac{a}{b}+M_P$. In particular, $\varphi(0+P) = \frac{0}{1}+M$. Therefore φ is surjective, and as we already showed it is injective, this makes it an isomorphism.

■

2.7 PIDs and Euclidean Domains.

Rings were invented in order to generalize mathematical domains which are parralel to the properties of integers and polynomials, since there exist many mathematical structures which share such properties. Two major properites of interest include the **fundamental theorem of arithmetic** and **factorization** via Euclid's algorithm. These concepts generalize to rings, specifically to Principal Ideal Domains, which we will demonstrate in this section.

First we begin with defintions.

Definition 2.7.1. Let R be a commutative ring, and suppose $a, b \in R$ are nonzero. Then

1. We say a **divides** b if $b = ac$ for some $c \in R$. This is denoted as $a \mid b$.
2. Let a not be a unit. Then a is **irreducible** if $a = bc$ implies b or c is a unit.
3. If a is not a unit, then a is **prime** if $a|bc$ implies $a|b$ or $a|c$.

As a consquence of these definitions, we have the following proposition.

Proposition 2.7.2. Let R be an integral domain and suppose $a, b \in R$ are nonzero. Then

1. If $\langle a \rangle$ is the principal ideal generated by a, then $a \mid b$ if and only if $\langle b \rangle \subset \langle a \rangle$.
2. The element a is a prime element of R if and only if $\langle a \rangle$ is a prime ideal.
3. If $a \mid b$ then $au \mid bv$ for any units $u, v \in R$
4. If $a \mid b$ and a is not a unit then b is not a unit.
5. If p is prime and $p \mid a_1 a_2 \cdots a_n$ then $p \mid a_i$ for some $i \in \{1, 2, \ldots, n\}$.

Proof:

1. ($\Longrightarrow$) Suppose $a \mid b$. Then $b = ac$ for some $c \in R$. Now observe that

$$\langle b \rangle = \{rb \mid r \in R\} = \{r(ac) \mid r \in R\} = \langle a \rangle c.$$

   Since $\langle a \rangle c \subset \langle a \rangle$, we have that $\langle b \rangle \subset \langle a \rangle$, as desired.

   ($\Longleftarrow$) Now suppose that $\langle b \rangle \subset \langle a \rangle$. Then $b \in \langle a \rangle$, so that $b = ac$ for some $c \in R$. Hence, $a \mid b$, which proves the result.

2. This is just the definition of an element being prime in R.
3. Suppose $a \mid b$ and let u, v be units. Since $b = ac$ for some $c \in R$, we see that $bu = acv = avc$. Therefore $bu \mid av$. (Since u, v were units, they are not zero divisors, so they did not change the value of the equation.)
4. Suppose a is not a unit and $a \mid b$ for some $b \in R$. Suppose for the sake of contradiction that b is a unit. Then $b = ac$ for some $c \in R$, and furthermore there exists a $d \in R$ such that $br = 1$.

   Therefore, $bd = acd \implies 1 = acd$. Hence a is a unit since $a(bd) = (bd)a = 1$. But this is a contradiction since we said a was not a unit, which completes the proof.

5. Let p be a prime element and suppose $p \mid a_1 a_2 \cdots a_n$ for elements $a_1, a_2, \ldots, a_n \in R$. Then $a_1 a_2 \cdots a_n = pb$ for some $b \in R$. Hence we see that $a_1 a_2 \cdots a_n \in Rp$. Since p is prime, Rp is a prime ideal and hence one element a_i where $i \in \{1, 2, \ldots, a_n\}$ must be in P. In other words, $p \mid a_i$ for some $i \in \{1, 2, \ldots, n\}$, as desired.

■

The above propsition generalizes rules that we hold to be familiar in $\mathbb{Z}$. For example, we know the units of $\mathbb{Z}$ are $\{1, -1\}$, and it is obvious to us that if $a \mid b$ for some integers a, b then $-1 \cdot a \mid -1 \cdot b$ and $1 \cdot a \mid 1 \cdot b$. We also know that if p is prime integer and $p \mid a_1 a_2 \cdots a_n$ for some integers $a_1, a_2 \ldots, a_n$ then $p \mid a_i$ for some $i \in \{1, 2, \ldots, n\}$. The proposition just tells us that our intuition on $\mathbb{Z}$ does in fact generalize to integral domains, and what we've seen in $\mathbb{Z}$ is just a tiny snap shot of algebra at work.

Proposition 2.7.3. Let R be an integral domain. If $p \in R$ is prime then p is also irreducible.

Proof: Let $p \in R$ be prime and suppose $p = qm$ for some $q, m \in R$. Then by definition we know that $p \mid q$ or $p \mid m$. Without loss of generality suppose $p \mid q$. Then $q = pc$ for some $c \in R$. Then we have that

$$p = qm \implies p = pcm \implies 1 = cm$$

where we used the cancellation law, valid on integral domains. Then m is a unit, and hence by definition this implies that p is irreducible. ■

Keep in mind that the converse of the above statement is not true. It will, however, turn out to be true for PIDs.

We now see that greatest common divisors can be generalized to integral domains.

Definition 2.7.4. Let R be an integral domain and let $A \subset R$ be nontrivial. Then we define

Say $(R, +, \cdot)$ is an integral domain. That is, a commutative rwith $1 \neq 0$ having no zero divisors. We say it is a Euclidean Domain if

1. We have a map $N : R \longrightarrow mathbbZ_{\geq 0}$ with $N(0) = 0$.
2. Given $a, b \in R$, we can find $q, r \in R$ with $a = bq + r$ and either $r = 0$ or $N(r) < N(b)$.

Proposition 2.7.5. Every ideal in a Euclidean Domain is principal. That is,

$$\text{(Euclidean Domain)} \subseteq \text{(Principal Ideal Domains)} \subseteq \text{(Integral Domain)}$$

Question: Are there PIDs that are not Euclidean Domains?

Unique Factorization Domain

Definition 2.7.6. Say $(R, +, \cdot)$ is an integral domain. Pick $r \in R$ that is neither zero nor a unit, i.e., $r \notin R^\times \cup \{0\}$.

We say that R is **irreducible** if when $r = ab$ either $a \in R^\times$ or $b \in R^\times$. Otherwise, we say that r is **reducible**.

Example. Say $R = \mathbb{Z}$. Then

$$r = 5 \text{ is irreducible.}$$
$$r = 6 = 2 \cdot 3 \text{ is irreducible.}$$

Definition 2.7.7. We say R is a **Unique Factorization Domain** (UFD) if the following holds for all $r \in R^\times \cup \{0\}$:

1. $r = p_1 p_2 \cdots p_m$ is a finite product of irreducibles $p_i \in R$.
2. If $r = q_1 q_2 \cdots q_m$ is another factorization into irreducibles, then $n = m$ and $q_i = q_i \cdot p_i$ for some $u_i \in R^\times$.

Again, let $R = \mathbb{Z}$ and observe that $r = 6 = 2 \cdot 3 = (-2) \cdot (-3)$. Really, we see that $2 = (-1) \cdot 2$ and $3 = (-1) \cdot 3$.

Proposition 2.7.8. Say R is a UFD. Given $r \notin R^\times \cup \{0\}$, the following are equivalent:

1. If $I = (r)$ is a nonzero prime ideal of R.
2. r is irreducible.

Remark: We avoid saying "r is prime", we say "$I = (r)$ is prime".

Proof:

i $\implies$ ***ii*** Assume $I = (r)$ is a prime Write $r = a \cdot b$. We want to show either $a \in R^\times$ and $b \in R^\times$. We have $a \cdot b = r \in I$, so by definition either $a \in I$ or $b \in I$. Either $a = v \cdot r$ where $b = u \cdot r$ for some $u, v \in R$.
Hence either $r = a \cdot b = (v \cdot r) \cdot b$ or $r = (v \cdot a) \cdot r$. Since R is an integral domain, the cancellation law holds. Hence either

$$r = 0 \quad \text{or} \quad v \cdot b = 1 \quad \text{if} \quad a \in I$$
$$r = 0 \quad \text{or} \quad v \cdot a = 1 \quad \text{if} \quad b \in I$$

Since $r \notin R^\times \cup \{0\}$, either $v \cdot b = 1$ or $u \cdot a = 1$. Hence either $b \in R^\times$ or $a \in R^\times$ is a unit.

ii $\implies$ ***i.*** Assume $r \in R$. We'll show that I is a **proper** ideal. Since $r \notin R^\times$, we know that $I \neq R$.
Now we show that I is prime. Say $a, b \in R$ satisfying $a \cdot b \in I$. We must show that either $a \in I$ or $b \in I$. Since $a \cdot b \in I$, we can write $a \cdot b = r \cdot c$ for some $c \in R$.

Case #1. Either $a, b = 0$. Then either $a, b \in I$.

Case #2. Either $a, b \in R^\times$. Without loss of generality, say $a \cdot u = 1$. Then $b = u \cdot r \cdot c \in I$.

Case #3. $a, b \notin R^\times \cup \{0\}$. Since R is a UFD, factor a and b.

$$a = p_1 p_2 \cdots p_n$$
$$b = q_1 q_2 \cdots q_m.$$

Hence $a \cdot b = p_1 \cdots p_n q_1 \cdots q_m$. But r is an irreducible that divides $a \cdot b = c \cdot r$. Thus either $q_i = u \cdot r$ or $p_j = v \cdot r$ for some i, j. But then either

$$a = (v \cdot r) p_2 \cdots p_n \in I = (r)$$
$$b = (v \cdot r) q_2 \cdots q_m \in I = (r)$$

if for example $i, j = 1$.

■

Definition 2.7.9. Say $(R, +, \cdot)$ is an integral domain. We say R is a **principal ideal domain** (PID) if every ideal $I = (r)$ is principal.

Proposition 2.7.10. 1. If R is a Euclidean Domain, then R is a PID.

2. If R is a PID, then R is a UFD.

Here's a diagram of what we have so far.

$$(\text{Euclidean Domains}) \subseteq (\text{Principal Ideal Domains}) \subseteq (\text{UFDs}) \subseteq (\text{Integral Domains})$$

Proof: We showed (1) before. Now we show (2). Assume R is a PID. We'll show that every $r \in R$ that is nonzero has unique factorization.

$$r = u \cdot p_1 p_2 \cdots p_n$$

where $u \in R^\times$ and p_i are irreducibles. We will in two parts: (i) existence (of at least one factorzation) and (ii) uniqueness (of at most one factorization).
Existence. Consider

$$S = \{I = (r) \mid r \in R - \{0\} \text{ does not have a factorization}\}.$$

We want to show that $S = \varnothing$. Thus assume otherwise. Pick some $I_1 = (r_1)$ in S. Then $r_1 \notin R^\times \cup \{0\}$ and r_1 is not irreducible. This means that r_1 is reducible, so write

$$r_1 = r_2 \cdot r_2' \quad r_2, r_2' \in R^\times \cup \{0\}.$$

Consider $I_2 = (r_2)$ and $I_2' = (r_2')$. If both r_2, r_2' have factorizations into irreducibles, then so would $r_1 = r_2 \cdot r_2'$. Thus without loss of generality, r_2 does not have a factorization into irreducibles. Thus the ideal $I_2 \in S$. Observe $I_1 \subsetneqq I_2$. We then have a chain of proper ideals:

$$I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_n \subsetneqq \cdots \subsetneqq R.$$

Let $I = \bigcup_{n\geq 1} I_n = (r_0)$ as the maximal ideal. ask: is $I \in S$? If $I \in S$, then we see that I is not maximal. If $I \notin S$, then r_0 can be written as a product of irreducibles.

Uniqueness. We show uniqueness. Consider a proof by induction via the following statement:

$$P(n) = \text{"If } r \in R - \{0\} \text{ has some factorization}$$
$$r = u \cdot p_1 \cdots p_n \text{ into } n \text{ irreducibles, then such a factorization is unique."}$$

We will show $P(n)$ is true for $n = 0, 1, 2$.

Base Case. If $r \neq 0$ has a factorization into $n = 0$ irreducibles, then $r = u$ is a unit. Say it has a second factorization

$$r = u \cdot q_1 \cdots q_n \text{ for } m > 0.$$

Then

$$1 = (u^{-1} \cdot v \cdot q_1 \cdots q_m) \cdot q_m$$

So $q_m \in R^\times$ is a unit. This is a contradiction

Induction. Suppose that $P(n_0)$ is true for some $n_0 \geq 0$. Then we'll show that $P(n_0 + 1)$ is true. Consider $r \neq 0$ that is the product of n irreducibles:

$$\begin{aligned} r &= u \cdot p_1 \cdots p_n \\ &= (u \cdot p_1 \cdots p_{n_0}) \cdot p_n \end{aligned}$$

Say we have a second factorization:

$$r = v \cdot q_1 \cdots q_m.$$

We then have the following facts:
$P = (p_n)$ is a prime ideal of R. (Recall $I = (r)$ is prime if and only if r is irreducible.)
$\overline{Q_i} = q_i \cdot P$ is a coset in $\overline{R} = R/P$. Not all $\overline{q_i} \neq \overline{0}$. Why? Because

$$\overline{u} \cdot \overline{q_1} \cdots \overline{q_m p_m} = \overline{u \cdot q_1 \cdots q_m} \cdot \overline{p_m} = 0.$$

Since $\overline{R}$ is an integral domain, without of generality, suppose $\overline{q_m} = \overline{0}$. Hence $q_m = q_m \cdot p_n$. Now consider

$$(u \cdot p_1 \cdots p_{n_0}) \cdot p_n = r = (u_m \cdot v \cdot q_1 \cdots q_{m-1}) \cdot p_n.$$

Since we're in an integral domain, we can use the cancellation law. Hence we have that

$$u \cdot p_1 \cdots p_{n_0} = (u_m \cdot v) \cdot q_1 \cdots q_{m-1}.$$

Now we can invoke the inductive hypothesis. We see that $m - 1 = n_0$ and $q_i = u_i \cdot p_i$ for some unit $u_i \in R^\times$. However, this is equivalent to saying that $m = n$. and as we showed $q_m = u_m \cdot p_n$, we have that the factorizations are the same. Hence $P(n)$ is true for all positive integers.

■

Primes, irreducibles and Maximal Ideals.

Proposition 2.7.11. Say $(R, +, \cdot)$ is a PID. Then the following are equivalent for $r \in R$ with $r \notin R^\times \cup \{0\}$:

i. r is irreducible.

ii $I = (r)$ is a prime ideal.

iii. $I = (r)$ is a maximal ideal.

Proof: First observe that $i \iff ii$ since they are true in a UFD, and we know that a PID is a UFD. Hence we just need to show that $ii \iff iii$.

We show that $(ii \iff iii)$. By way of contradiction, sy $I = (r)$ is a prime that is not maximal. Then we can find an ideal $M = (a)$ such that $I \subsetneqq M \subsetneqq R$. This means that $a|r$. That is, one can find $a, b \in R$ such that $r = a \cdot b$.

Since I is prime and $a \cdot b \in I$, either $a \in I$ or $b \in I$. If $a \in I$, then $M = (a) \subset I$, which would imply that $M = I$. Thus we have a contradiction.

Thus we need that $b \in I$. Since $b \in I = (r)$, write $b = u \cdot r$. Then $r = a \cdot b = (u \cdot a) \cdot r$. Since $r \neq 0$, we find $u \cdot a = 1$. This means that $a \in R^\times$ so $M = (a) = R$. But $M \subsetneqq R$, so again we find a contradiction.

Thus we see that I must be a maximal ideal. The other direction, that every maximal ideal is a prime ideal, is trivial. ■

Dedekind-Hasse Norms.

Motivating questions.

1. Are there examples of PIDs that are not Euclidean Domains?
2. Are there examples of PIDs that are not $\mathbb{Z}$?

Say $(R, +, \cdot)$ is an integral domain. A **norm** is a map $\mathbb{N} : R \longrightarrow \mathbb{Z}_{\geq 0}$. such that $\mathbb{N}(0) = 0$.

On the other hand, a **Dedekind-Hasse** Norm is a norm $\mathbb{N}$ satisfying

DH1. $N(0) = 0$

DH2. $N(a) > 0$ for all $a > 0$

DH3. For all nonzero $a, b \in R$, either

1. b divides a, i.e., $a = q \cdot b$ or
2. there exits $s, t \in R$ such that

$$x = s \cdot a - t \cdot b \in (a, b)$$

satisfies $0 < N(x) < N(b)$.

Remark: If in (DH3) we can **always** choose $s = 1$, then we say that R is a Euclidean Domain.

Examples.
Let $R = \mathbb{Z}$. This a Euclidean Domain. The statement (DH3) is the Euclidean algorithm for $N(a) = |a|$.

Let R be any integral domain. Then $S = R[x]$ is also an integral domain. We define a norm on S be saying $N(g(x)) = \deg(g)$ if $g \neq 0$. If $g(x) \equiv 0$ is the zero polynomial, then $\deg(g) = -\infty$. So define $N(0) = 0$. This is not Dedekind-Hasse. As an example, consider $R = \mathbb{Z}$. Then let $I = (2, x)$ in S. This is not a principal ideal.

Now consider F a field. Let $S' = F[x]$ which is a PID. Define $N(g(x = 2^{\deg(g)}$. And $N(0) = 2^{-\infty} = 0$.

2.8 Polynomial Rings (for Galois Theory).

Definition 2.8.1. Say $(R, +, \cdot)$ is a ring with identity $1 \neq 0$. We will always assume R is of this form.

1. If x is indeterminate (i.e. a variable) a **polynomial** in x is a formal sum

$$a_d x^d + \cdots + a_1 x + a_0.$$

   with $a_k \in R$.

2. We say the **degree** $\deg(p) = d$ if a_d. Otherwise, set $\deg(p) = -\infty$ if $a_d = \cdots = a_0 = 0$.

3. Let $R[x]$ be the collectionn of all such $p(x)$. This is the **polynomial ring** over R.

4. More generally, say $\{x_1, x_2, \ldots, x_n\}$ is a collection of n indeterminates. Inductively define $R[x_1, x_2, \ldots, x_n] = S[x_n]$ where $S = R[x_1, x_2, \ldots, x_{n-1}]$. A typical element is in the form

$$p(x_1, \cdots, x_n) = \sum_{i_1=1}^{d_1} \cdots \sum_{i_n=1}^{d_n} a(i_1, \cdots, i_n) x_1^{i_1} \cdots x_n^{i_n}.$$

   We then say that this $p(x_1, \ldots, x_n)$ is a polynomial in n variables.

5. Assuming that the highest coefficient $a(d_1, \ldots, d_n) \neq 0$, we then say the degree is

$$\deg(p) = \max\{i_1 + \cdots + i_n\} \text{ (fix here).}$$

   Otherwise, if all terms are zero, i.e., if all $a(i_1, \ldots, i_n) = 0$, then we set $\deg(p) = -\infty$.

Example. Consider the polynomial ring $R[x, y]$ as the polynomial ring in $n = 2$ variables. Let $p(x, y) = 1 + x^3 + y^3$. Then $\deg(p) = 3$.

Proposition 2.8.2. $(R[x_1, \cdots, x_n], +, \cdot)$ is a ring with $1 \neq 0$.

Proof: We'll showt this by induction:

$$P(n) = \text{"}(R[x_1, \ldots, x_n]) \text{ is a ring with } 1 \neq 0\text{"}.$$

We have already shown the base case $P(1)$ is true. Assume that $P(n)$ is true for some n_0. We show $P(n)$ is true for $n = n_0 + 1$.

By definition, $R[x_1, \ldots, x_n] = S[x_n]$ where $S = R[x_1, \ldots, x_{n_0}]$. By our inductive hypothesis, we have that $(S, +, \cdot)$ is a ring with $1 \neq 0$. Hence by $P(1)$ we have that $(S[x_n], \cdot, +)$ is also a ring with $1 \neq 0$. Hence $P(n_0 + 1)$ is true. ■

Proposition 2.8.3. Assume R is an integral domain.

1. Suppose $p, q \in R[x_1, \cdots x_n]$. Then
$$\deg(p \cdot q) = \deg(p) + \deg(q).$$
2. $R[x_1, \ldots, x_n]$ is also an integral domain.
3. The units of $R[x_1, \ldots, x_n]$ is $R^\times$.

Proof:

1. If either $p = 0$ or $q = 0$, then $p \cdot q = 0$. Then observe that
$$\deg(p \cdot q) = \deg(p) + \deg(q)$$
doesn't make sense unless we choose to set $\deg(0) = -\infty$. Hence we see that
$$\deg(p \cdot q) = -\infty.$$
We could make it positive infinity, but we assume that it is negative for subtle reasons later on.

Now assume that $p, q \neq 0$. We write our polynomials
$$p = \sum_{i_1, \ldots, i_n}^{d_1, d_2, \cdots, d_n} a(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n} \qquad q = \sum_{j_1, \ldots, j_n}^{e_1, e_2, \cdots, e_n} a(j_1, \ldots, j_n) x_1^{j_1} \cdots x_n^{j_n}$$
where $\deg(p) = d_1 + d_2 + \cdots + d_n$ and $\deg(q) = e_1 + e_2 + \cdots + e_n$. Then we see that
$$p \cdot q = \sum_{k_1, \ldots, k_n}^{d_1 + e_1, d_2 + e_2, \cdots, d_n + e_n} c(i_1, \ldots, i_n) x_1^{k_1} \cdots x_n^{k_n}$$
where
$$c(k_1, \ldots, k_n) = \sum_{i_1 + j_1 = k_1, \ldots, i_n + j_n = k_n.} a(i_1, \ldots i_n) b(j_1, \ldots, j_n).$$
The leading term is $c(d_1 + e_1, \ldots, d_n + e_n) = a(d_1, \ldots, d_n) b(e_1, \ldots, e_n)$. Hence, the leading term is also nonzero, so the degree must be
$$d_1 + e_1 + \cdots + d_n + e_n = \deg(p) + \deg(q).$$
There are actually many ways to define the degree of a polynomial in $R[x_1, \ldots, x_n]$ when$n \geq 2$.

2., 3. We can show both (2) and (3) at the same time via induction. Let

$$P(n) = \text{"}R[x_1, \ldots, x_n] \text{ is an integral domain and } R[x_1, \ldots, x_n]^\times = R^\times\text{."}$$

We've done this in the one-variable case, so that $P(1)$ is true. We can invoke our inductive hypothesis to suppose that $P(n_0)$ is true for some n_0. We next show that $n = n_0 + 1$ is true.

By definition, our ring $R[x_1, \ldots, x_{n_0}] = S[x_n]$ for

$$S = R[x_1, \ldots, x_{n_0}].$$

By our inductive hypothesis, we know that (1) S is an integral domain. Since $P(1)$ is true, we know that $S[x_{n_0}]$ is an integral domain. By (2), we know that $S^\times = R^\times$. By $P(1)$, we see that $S[x_n] = R^\times$. This show that $P(n)$ is true for all n.

■

Proposition 2.8.4. Say $(R, +, \cdots)$ is a ring with $1 \neq 0$ and $I \subsetneq R$ is a ideal. Denote $S = R[x_1, \ldots, x_n]$.

1. $J = I[x_1, \ldots, x_n]$ is a proper ideal of S
2. $S/J = (R/J)[x_1, \ldots, x_n]$
3. Moreover, say R is commutative. If $I \subset R$ is a prime ideal of R, then J is a prime ideal of S.

Proof:

1., 2. Denote $\overline{R} = R/I$ as a ring with identity $1 \neq 0$, which holds since we are working with a proper ideal. We know both S and $\overline{R}[x_1, \ldots, x_n]$ is also a ring with $1 \neq 0$.

Consider the following "reduction mod I" map:

$$\varphi : R[x_1, \ldots, x_n] \longrightarrow \overline{R}[x_1, \ldots, x_n]$$

where if $p = \sum_{i_1, \ldots, i_n}^{d_1, d_2, \cdots, d_n} a(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n}$ then

$$\overline{p} = \sum_{i_1, \ldots, i_n}^{d_1, d_2, \cdots, d_n} \overline{a(i_1, \ldots, i_n)} x_1^{i_1} \cdots x_n^{i_n}$$

where $\overline{a} = a + I$ in R/I.
Claim: This is a ring homomorphism. In fact, this map is surjective.
Neither of these are difficult to show.

Observe that

$$\ker(\varphi) = \left\{ p = \sum_{i_1, \ldots, i_n}^{d_1, d_2, \cdots, d_n} a(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n} mid a(i_1, \ldots, i_n) \in I \right\} = I[x_1, \ldots, x_n].$$

The First Isomorphism Theorem for Rings states that

$$\begin{aligned} S/J &\cong R[x_1, \ldots, x_n]/\ker(\varphi) \\ &\cong \mathrm{Im}(\varphi) \\ &= (R/I)[x_1, \ldots, x_n]. \end{aligned}$$

At this point, we've shown (2). Now observe that $J \subsetneqq R$ since $(R/I)[x_1, \ldots, x_n]$ is a ring with identity $1 \neq 0$. This proves (1). To show (3), say R is a commutative ring and $I \subset R$ is a prime ideal. Since I is prime, we see that (R/I) is an integral domain. Hence we see that $(R/I)[x_1, \ldots, x_n]$ is an integral domain. Since $S/J \cong (R/I)[x_1, \ldots, x_n]$, and we know that $R[x_1, \ldots, x_n]$ is also a commutative ring, we see that $J \subset S$ must be a prime ideal as well.

■

Chapter 3

Modules

3.1 Definitions.

In group theory, we started with a set G equipped with a bilinear operation $\cdot : G \times G \longrightarrow G$ which mapped G to itself. The operation was required to be associative, and there needed to be inverses and an identity element.

In ring theory, we went further to assume R was not only an abelian group, we placed the group operation with $+ : R \times R \longrightarrow R$ and then defined a *multiplication* $\cdot : R \times R \longrightarrow R$ which is was associative and left- and right- distributive.

Finally, we reach module theory, which considers again an abelian group M with operation $+ : M \times M \longrightarrow M$ but lets a ring R act on M, whose addition $+R \times R \longrightarrow R$ agrees with the one which acts on M but whose multiplication $\cdot : R \times M \longrightarrow M$ acts on R and M.

Note how abelian groups and rings are special cases of modules. This will be more clear once we introduce the axioms.

Definition 3.1.1. Let R be a ring with identity, and M an abelian group equipped with $+ : M \times M \longrightarrow M$. Then M is an **left R-module** if we equip $R \times M$ with multiplication $\cdot : R \times M \longrightarrow M$ and for all $m \in M$ and $a, b \in R$

1. $a(m_1 + m_2) = am_1 + am_2$
2. $(a + b)m = am + bm$
3. $(ab)m = a(bm)$
4. $1_R m = m$ where 1_R is the identity of R.

Alternatively, an abelian group M is a **right R-module** if we equip $M \times R \longrightarrow M$ with multiplication $\cdot : M \times R \longrightarrow M$ and for all $m \in M$ and $a, b \in R$

1. $(m + n)a = ma + na$
2. $m(a + b) = ma + mb$
3. $m(ab) = (ma)b$
4. $m1_R = m$ where 1_R is the identity of R.

Notice that we can think of these products as a group action, or sort of a "ring action" acting on M. That is, an R-module M is just an abelian group that a ring R can act on. If you have an abelian group N that R simply cannot act on and satisfy the above axioms, then N is not an R-module.
For convenience, we will develop the theory of R-modules by solely working with left R-modules,

since all proofs and statements will be equivalent up to a swap of variables for right R-modules.
Examples.

1. Note that if R is commutative, then a left R-module coincides with a right R-module. To see this, let M be a left R-module. Then construct the right R-module by defining the multiplication as
$$m \cdot r = rm.$$
Then we see that for all $m \in M$, $a, b \in R$,
   1. $(m_1 + m_2) \cdot a = a(m_1 + m_2) = am_1 + am_2 = m_1 \cdot a + m_2 \cdot a$ ✓
   2. $m(a + b) = (a + b)m = am + bm = m \cdot a + m \cdot b$ ✓
   3. $m \cdot (ab) = (ab)m = (ba)m = b(am) = b(m \cdot a) = (m \cdot a) \cdot b$ ✓
   4. $m \cdot 1_R = 1_R m = m$. ✓

   Note that in part (c) is where we used the fact that R is commutative. So whenever R is commutative, the existence of a left R-module automatically implies that existence of a right R-module, and vice versa.
2. Let R be a ring. Then if we substitute $M = R$ in the above definition, and let the multiplication $\cdot$ be the multiplication on R then R is a left and a right R-module. This is because R is an abelian group which is associative and left- and right-distributive. Hence, it satisfies all of the above axioms.

   So keep in mind that a ring R is just a left- and right-R module that acts on R.

Here's another example which shows that abelian groups are simply $\mathbb{Z}$ modules.

Proposition 3.1.2. Let G be an abelian group. Then G is a left and right $\mathbb{Z}$-module.

Proof: Let $\mathbb{Z}$ act on G as follows. Define
$$ng = \begin{cases} g + g + \cdots + g \ (n \text{ times}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ (-g) + (-g) \cdots (-g) \ (n \text{ times}) & \text{if } n < 0 \end{cases}$$
and
$$gn = \begin{cases} g + g + \cdots + g \ (n \text{ times}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ (-g) + (-g) \cdots (-g) \ (n \text{ times}) & \text{if } n < 0. \end{cases}$$
Then with this definition of multiplication, it is easy to show that the axioms (a)-(d) are satisfied. ■

3. If R is a ring and I is a left (right) ideal of R, then I is a left (right) R-module.
4. Let V be a vector space defined over a field F. Then V is an F-module. (Now it is clear why there are a million axioms in the definition of a vector space!)

With R-modules introduced and understood, we can jump right into homomorphisms.

Definition 3.1.3. Let R be a ring and M and N be R-modules. We define $f : M \longrightarrow N$ to be an **R-module homomorphism** if

1. $f(m_1 + m_2) = f(m_1) + f(m_2)$ for any $m_1, m_2 \in M$
2. $f(am) = af(m)$ for all $a \in R$ and $m \in M$.

If f is a bijective R-module homomorphism, then we say that f is an **isomorphism** and that $M \cong N$.

Thus we see that R-module homomorphisms must not only be linear over the elements of M, but they must also pull out scalar multiplication by elements of R.

Recall earlier that we said a vector space V over a field F is an F-module. Now if W is another vector space and $T : V \longrightarrow W$ is a linear transformation, then we see that T is also an F-module homomorphism!

In the language of linear algebra, a **linear transformation** is usually defined as a function $T : V \longrightarrow W$ such that for any $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v} \in \mathbf{V}$ and $\alpha \in F$ we have that

1. $T(\mathbf{v_1} + \mathbf{v_2}) = \mathbf{T}(\mathbf{v_1}) + \mathbf{T}(\mathbf{v_2})$
2. $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$.

As we will see, linear algebra is basically a special case of module theory.

Definition 3.1.4. Let R be a ring and M and N a pair of R-modules. Then $\hom_R(M, N)$ is the set of all R-module homomorphisms from M to N.

It turns out we can turn $\hom_R(M, N)$ into an abelian group, and under special circumstances it can actually be an R-module itself. It will be the case that $\hom_R$ will actually be an important functor, but that is for later.

To turn this into an abelian group, we define addition of the elements to be

$$(f + g)(m) = f(m) + g(m)$$

for all $f, g \in \hom_R(M, N)$. We let the identity be the zero map, and realize associativity and closedness are a given to conclude that this is in fact an abelian group.

Suppose we want to make R, our ring, act on $\hom_R(M, N)$ in order for it to be an R-module. Then we define scalar multiplication to be $(af)(m) = a(f(m))$; a pretty reasonable definition for scalar multiplication.

This issue with this is that $\hom_R(M, N)$ will not be closed under scalar multiplication of elements of R unless R is a commutative ring.

We'll demonstrate this as follows. Let $b \in R$ and $f \in \hom_R(M, N)$. Then the second property of an R-module homomorphism tells us that $f(bm) = bf(m)$ for all $m \in M$. Now suppose we try to use our definition of scalar multiplication, and consider af where $a \in R$. Then if we try to see if af will pass the second criterion for being an R-module homomorphism, we see that

$$(af)(bm) = a(f(bm)) = a(bf(m)) = abf(m).$$

That is, we see that af isn't an R-module homomorphism because $(af)(bm) \neq b(af)(m)$ (which is required for an R-module homomorphism); rather, $(af)(bm) = abf(m)$. Now if R is a commutative

ring, then

$$abf(m) = baf(m)$$

so we can then say that $(af)(bm) = b(af)(m)$, in which case af passes the test for being an R-module homomorphism.

This proves the following propsition, which will be useful for reference for later.

Proposition 3.1.5. Let M and N be R-modules. Then $\hom_R(M, N)$ is an abelian group. Furthermore, it is an R-module if and only if R is a commutative ring.

Next, we make the following definitions for completeness.

Definition 3.1.6. Let R be a ring and M and N be R-modules. If $f : M \longrightarrow N$ is an R-module homomorphism, then

1. The set $\ker(f) = \{m \in M \mid f(m) = 0\}$ is the **kernal** of f
2. The set $\text{Im}(f) = \{f(m) \mid m \in M\}$ is the **image** of f.

3.2 Submodules, Quotient Modules and Isomorphism Theorems.

Definition 3.2.1. Let M be a R-module. Then a set $N \subset M$ is said to be a **submodule** of M if N is also a R-module.

What do we need for $N \subset M$ to be a submodule? For N to be a submodule,

- N needs to be an nonnempty abelian group
- Axioms (a) - (d) in Definition 1.1 must be satisfied
- N needs to be closed under multiplication of R. That is, $\cdot : R \times M|_N \longrightarrow N$, where $M|_N$ is M restricted to N (namely, just N).

However, since $N \subset M$, axioms (a) - (d) are already satisfied for N. In addition, if N is a nonempty subgroup of M then it is automatically abelian. Thus N is a **R-submodule** of M if N is a subgroup of M and N is closed under multiplication of elements from R. This leads us to the following submodule test.

Theorem 3.2.2. (Submodule Test.) Let M be an R-module and $N \subset M$ be nonempty. Then N is an R-submodule of M if and only if $an_1 + bn_2 \in N$ for all $n_1, n_2 \in N$ and $a, b \in R$.

Proof: ($\Longrightarrow$) If N is an R-submodule of N then obviously $an_1 + bn_2 \in N$ for all $n_1, n_2 \in N$ and $a, b \in R$.

($\Longleftarrow$) Suppose $an_1 + bn_2 \in N$ for all $n_1, n_2 \in N$ and $a, b \in R$. First observe that N is nonempty. Now setting $a = 1$ and $b = -1$ we see that $n_1 - n_2 \in N$ for all $n_1, n_2 \in N$, and thus by the subgroup test we see that N is a subgroup of N.

Since $an_1 + bn_2 \in N$ for all $a, b \in R$ we see that N is closed under multiplication of elements of R.

Since N is an abelian subgroup of M and is closed under multiplication of elements of R, we see that N is an R-submodule as desired. ■

Example.
An immediate example we can create from our previous discussions the fact that if $f : M \longrightarrow N$ is an R-module homomorphism then

1. $\ker(f)$ is an R-submodule of M.
2. $\mathrm{Im}(f)$ is an R-submodule of N.

As we saw in group and ring theory, arbitrary intersections of subgroups or subrings resulted in subgroups and subrings. Thus the following theorem should be of no surprise.

Theorem 3.2.3. Let R be a ring and M an R-module. If $\{N_\alpha\}_{\alpha \in \lambda}$ be a set of R-submodules of M, then $N = \bigcap_{\alpha \in \lambda} N_\alpha$ is a submodule of M.

Proof: First observe that $N = \bigcap_{\alpha\in\lambda} N_\alpha$ is nonempty, since $0 \in N_\alpha$ (the identity) for all $\alpha \in \lambda$. Thus for any $n_1, n_2 \in N$ we know that $n_1, n_2 \in N_\alpha$ for all $\alpha \in \lambda$. Since each such N_α is an R-submodule, we know that $an_1 + bn_2 \in N_\alpha$ for all $\alpha \in \lambda$ for any $a, b \in R$. Hence, $an_1 + bn_2 \in N$ for all $a, b \in R$, proving that N is an R-submodule as desired. ■

Note that what ring R is under discussion, we will just state a R-submodule as simply a submodule.

Quotient Modules.

As we discovered quotient groups in group theory and quotient rings in ring theory, it should again be no surprise that we can formalize the concept of quotient modules.

In group theory, a quotient group G/H only made sense if the group H being quotiened out was **normal** to G. This guaranteed that our desired group operation in the quotient group worked and made sense as desired. In ring theory, a quotient ring R/I only made sense if the ring I being quotiened out was an **ideal** of R. Since we wanted R/I to be a ring, we needed not only addition but multiplication to be well-defined, but well-definedness only worked when I was an ideal.

In both cases, we couldn't quotient out just any subgroup or a subring to get a quotient group or quotient ring. They had to be special subsets (e.g. normal groups, ideals). However, in module theory, it does happen to be the case that we can just quotient out a submodule to get a quotient module.

To define a quotient module, we first consider an R-module M and a submodule N of M. To turn R/N into an R-module, we first turn this into an abelian group, which we can perfectly do since N is a subgroup of M, an abelian group, so M/N makes sense. A result from group theory tells us that if M is abelian then M/N is abelain.

Next, to turn this into an R-module we define scalar multiplication as

$$r(m + N) = rm + N$$

where $r \in R$, and multiplication of elements as

$$(m + N)(m' + N) = mm' + N.$$

As always, when defining a quotient object we're worried about the ability of our multiplication to preserve equivalence of elements. This is usually where we run into trouble in group theory or ring theory, in which case we modify the set N which we're quotienting out. In group theory, we'd turn N into normal group, and in ring theory we'd turn N into an ideal. Here we'll leave N alone, since it works out in the end.

Thus suppose that

$$m + N = m' + N$$

that is, $m = m' + n$ for some $n \in N$. Then to check if our multiplicaton is well-defined, we observe that for $a \in R$

$$am + N = a(m' + n) + N = am' + an + N$$

and since N is a submodule, it is closed under scalar multiplication of elements of R. Hence, $an \in N$, so that

$$am' + an + N = am' + N.$$

Thus we see that $am + N = am' + N$, so that our scalar multiplication is well-defined. This leads to the following definition.

Definition 3.2.4. Let R be a ring and M an R-module. If N is a submodule of M, then we defined M/N to be the **quotient R-module** of M with respect to N. As we showed earlier, this is in fact an R-module.

As before, it should be no surprise that the Noether Isomorphism Theorems apply to modules as well. In fact, the Noether Isomorphism Theorems were first introduced by Emmy Noether for modules; not through groups or for rings. The Isomorphism Theorems hold for groups and rings since abelian groups and rings are special cases of modules.

First, we introduce two homomorphisms which seem as if they are so stupidly simple that they don't even deserve a definition; yet, they do.

Definition 3.2.5. Let R be a ring and M and N be R-module homomorphisms. Then we define the following R-module homomorphisms.

1. The map $\pi : M \longrightarrow M/N$ given by
$$\pi(m) = m + N$$
is said to be the **projection map**. Note that π is **surjective**, and that $\ker(\pi) = N$ (since $m + N = N$ if and only if $m \in N$.)
2. The map $i : M/N \longrightarrow M$ given by
$$i(m + N) = m$$
is known as the **inclusion map**. More generally, if $M' \subset M$, the **inclusion map** can also be defined as $i : M' \longrightarrow M$ where
$$i(m') = m'$$
for all $m' \in M'$. Note that i is **injective**, and in the first case $\text{Im}(i) = M/N \cup \{0\}$ and in the second case $\text{Im}(i) = M'$.

Theorem 3.2.6. (First Isomorphism Theorem) Let R be a ring and M and N be R-modules. If $f : M \longrightarrow N$ is an R-module homomorphism, then

$$M/\ker(f) \cong \text{Im}(f).$$

Proof: The proof is the same as before. Define the map $\varphi : M/\ker(f) \longrightarrow N$ as

$$\varphi(m + \ker(f)) = f(m).$$

We quickly show that this is well-defined. If $m + \ker(f) = m' + \ker(f)$ for some $m, m' \in M$, then $m = m' + k$ for some $k \in K$. Therefore,

$$\varphi(m + \ker(f)) = f(m) = f(m' + k) = f(m') = \varphi(m' + \ker(f)).$$

Next, we show this is in fact an R-module homomorphism. Linearity is obvious, so we check the second criterion. Now for any $a \in R$ we see that

$$\varphi(a(m + \ker(f))) = \varphi(am + \ker(f)) = f(am) = af(m) = a(\varphi(m + \ker(f)))$$

where we pulled the a outside from $f(am)$ to make $af(m)$ from the fact that f is an R-module homomorphism.

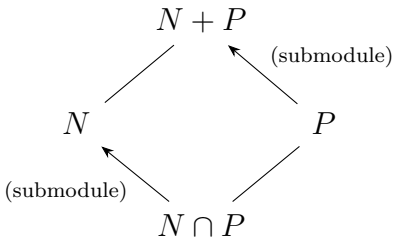
Now we make two observations. First, we see that there is a one-to-one correspondence between $M/\ker(f) \longrightarrow \text{Im}(f)$. Second, this implies that φ is an isomorphism between the two modules, so that

$$M/\ker(f) \cong \text{Im}(f)$$

as desired. ■

Theorem 3.2.7.(Second Isomorphism Theorem.) Let R be a ring and M and N and P be submodules of M. Then

$$(N + P)/P \cong N/(N \cap P).$$



The diagram on the left is the same one we used in group theory and ring theory. That is, the second isomorphism theorem can still be described using the diamond diagram.

Proof: Construct the projection map $\pi : M \longrightarrow M/P$ and let π' be the restriction of π to N. Then we see that $\ker(\pi') = N \cap P$, while

$$\text{Im}(\pi') = \{\pi'(n) \mid n \in N\} = \{n + P \mid n \in N\} = (N + P)/P.$$

Thus by the First Isomorphism Theorem we have that

$$N/\ker(\pi') \cong \text{Im}(\pi') \implies (N + P)/P \cong N/(N \cap P)$$

as desired. ■

Theorem 3.2.8.(Third Isomorphism Theorem) Let R be a ring and M an R-module. Suppose N and P submodules such that $P \subset N$. Then

$$M/N \cong (M/P)/(N/P).$$

Proof: Construct the map $f : M/P \longrightarrow M/N$ by defining $f(m+P) = m+N$ where $m+P \in M/P$ and $m+N \in M/N$. First observe that this is a surjective mapping since $P \subset M$, so the correspondence $m+P \longrightarrow m+N$ will cover all of M/N.

Now observe that

$$\ker(f) = \{m+p \mid m \in N\} = N/P.$$

Therefore, by the First Isomorphism Theorem

$$(M/P)/\ker(f) \cong M/N \implies (M/P)/(N/P) \cong M/N$$

as desired. ■

Theorem 3.2.9.(Fourth Isomorphism Theorem) Let R be a ring and M an R-module. Suppose N is a submodule of M. Then every submodule of M/N is of the form P/N where $N \subset P \subset M$.

Another way to understand this statement is to realize there is a one to one correspondence between the submodules of M containing N and the submodules of M/N.

Proof: ■

3.3 Generating Modules, Torsions, Annihilators.

The concepts we have introduced so far are not new. In fact, this is the third time you've probably seen all of these concepts. However, module theory is very deep, and here is where we will start seeing new concepts.

Generating Modules.
Let M be an R-module and suppose $S \subset M$ where S is nonempty. Denote the smallest submodule of M containing S as $\langle S \rangle$, and observe that

$$\langle S \rangle = \bigcap_{\alpha \in \lambda} S_\alpha$$

where $\{S_\alpha\}_{\alpha \in \lambda}$ is a family of submodules containing S.

Note that this is in fact a submodule, since arbitrary intersections of submodules yield a submodule.

Now consider the set of all finite linear combinations of elements of S with coefficients in R. That is,

$$S' = \left\{ \sum_{i=1}^{n} a_i s_i \mid a_i \in R, s_i \in S \text{ for all } i \in N \right\}.$$

This is of course also an R-module. We claim that $\langle S \rangle = S'$.

$\langle \mathbf{S} \rangle \subset \mathbf{S'}$. To see this, consider any $s \in \langle S \rangle$. Then $s \in S_\alpha$ for all $\alpha \in \lambda$. Furthermore, since $S \subset S'$ we see that S' is one of the members of the families of all submodules containing S. Therefore we must have that $s \in S'$, and thus $s = \sum_{i=1}^{n} a_i s_i$ for some $a_i \in R$ and $s_i \in S$. Hence, $\langle S \rangle \subset S'$.

$\mathbf{S'} \subset \langle \mathbf{S} \rangle$. Simply observe that since $\langle S \rangle$ contains S, and because $\langle S \rangle$ is a submodule, it must be that $\langle S \rangle$ contains all linear combinations of elements of S with coefficients in R. That is, $\sum_{i=1}^{n} a_i s_i \in \langle S \rangle$ for any $a_i \in R, s_i \in S$.

Thus what we have shown is the following theorem.

Theorem 3.3.1. Let M be an R-module and suppose $S \subset M$. Then if $\langle S \rangle$ is the smallest submodule of M containing S then

$$\langle S \rangle = \bigcap_{\alpha \in \lambda} S_\alpha$$

where $\{S_\alpha\}_{\alpha \in \lambda}$ is the family of submodules containing S. More explicilty, we have that

$$\langle S \rangle = \left\{ \sum_{i=1}^{n} a_i s_i \mid a_i \in R, s_i \in S \text{ for all } i \in N \right\}.$$

The above theorem then leads a very useful definition that we will work with frequently.

Definition 3.3.2. Let M be an R-module and $S \subset M$.

1. The R-submodule $\langle S \rangle$ of M is called the **submodule of M generated by S.**

2. If $M = \langle S \rangle$ for some $S \subset M$, then we say that M **is generated by** S. Hence, the elements of S are referred to as the **generators** of M.
3. If M is generated by S and S is finite, then we say M **is finitely generated by** S. In this case we refer to $|S|$, denoted as $\mu(M)$, as the **rank** of M. Furthermore, if $S = \{x_1, x_2, \ldots, x_n\}$ it is convenient to write $M = \langle x_1, x_2, \ldots, x_n \rangle$.

In the case of a cyclic group G, we see that G has rank one and one generator. Thus we see that this concept is generalized in module theory if the generating set for some module is of cardinality one.

Note we can also union modules together to get another module.

Definition 3.3.3. Let M be an R-module and $\{S_\alpha\}$ a family of submodules of R. Then the **submodule generated by** $\{N_\alpha\}_{\{}\alpha \in \lambda\}$ is

$$\left\langle \bigcup_{\alpha \in \lambda} S_\alpha \right\rangle$$

which we often denote as $\sum_{\alpha \in \lambda} S_\alpha$.

Next we move onto the concept of an annihilator, which we define as follows.

Definition 3.3.4. Let M be an R-module, and suppose $X \subset M$. Then we define the set

$$\text{Ann}(X) = \{a \in R \mid ax = 0 \text{ for all } x \in X\}$$

to be the **annihilator of** X.

Note that $\text{Ann}(X) \subset R$ and that $0 \in \text{Ann}(X)$ for any $X \subset M$. If $\text{Ann}(X) = \{0\}$, we of course say that it is **trivial**.

The annihilator captures all of the coefficients of R which annihilate every element in X. Thus one can imagine that the size of $\text{Ann}(X)$ increases as the size of $X \subset M$ increases (of course, if $\text{Ann}(X)$ is not trivial for all X.)

Proposition 3.3.5. Let M be an R-module and let $X \subset M$ be nonempty. Then

1. $\text{Ann}(X)$ is a left ideal of R
2. If N is a submodule of M, then $\text{Ann}(N)$ is an ideal of R.
3. If R is commutative and N is a cyclic submodule of M generated by $x \in N$ then $\text{Ann}(N) = \{a \in R \mid ax = 0\}$.

Proof:

1. Let $X \subset M$. In order for $\text{Ann}(X)$ to be a left ideal, it must be a subring of R which absorbs left multiplication.

   It's a Subring. We can apply the subring test to prove this. Recall earlier we said that $0 \in \text{Ann}(X)$ for any $X \subset M$, so Ann is nonempty.
   Now let $a, b \in \text{Ann}(X)$, so that $ax = bx = 0$ for all $x \in X$. Then clearly $abx = 0$ and $(a - b)x = ax - bx = 0$ so that $ab \in \text{Ann}(X)$ and $a - b \in \text{Ann}(X)$. Thus it is a subring of R.

It's an ideal. For any $r \in R$ and $a \in \text{Ann}(X)$ we have that $rax = r(ax) = 0$. Therefore $ra \in I$ for all $r \in R$, which proves it is a left ideal.
Why isn't it also a right ideal? Well, observe that we would need $arx = 0$ whenever $a \in \text{Ann}(X)$ and $r \in R$. This would require either that $ar = 0$, which we can't always guarantee, or that $rx \in X$. But we don't know if $rx \in X$; we could only guarantee that if X was an R-module, which we'll see in the next proof.

2. Let N be a submodule of M. Since we showed in (1.) that $\text{Ann}(X)$ is a left ideal for any $X \subset M$, we must simply show that $\text{Ann}(N)$ also absorbs right multiplication of R as well in order to show it is an ideal.

   Thus let $r \in R$ and $a \in \text{Ann}(N)$. Since N is a submodule of N we know that $rx \in N$. Hence $a(rx) = 0$ as $an = 0$ for all $n \in N$. Therefore $ar \in \text{Ann}(N)$ whenever $r \in R$ and $a \in \text{Ann}(N)$, proving that $\text{Ann}(X)$ absorbs right multiplication and is therefore an ideal.

3. Consider $\text{Ann}(N)$ where N is cyclic and generated by x and let $|N| = k$. Suppose we have an $a \in R$ such that $ax = 0$. Then we see that $ax^2 = (ax)x = 0$, $ax^3 = (ax)x^2 = 0$, and that in general $ax^j = (ax)x^{j-1} = 0$ for any $j \in \{1, 2, \ldots, k\}$. Since every $n \in N$ is of the form x^j for some $j \in \{1, 2, \ldots, k\}$ we have $an = 0$ for all $n \in N$. Hence,

$$\text{Ann}(X) = \{a \in R \mid an = 0 \text{ for all } n \in N\} = \{a \in R \mid ax = 0\}.$$

Where does commutativity come into play?

■

Note that in general we denoted $\text{Ann}(x)$ to be $\text{Ann}(N)$ where N is cyclic and generated by x. This only really makes sense if R is commutative.

For an R-module M and a family of submodules $\{N_\alpha\}_{\alpha \in \lambda}$, we have been able to define the arbitrary intersection and addition of submodules. Next we define the product of R-modules. We now define a product of R-modules.

If M is an R-module and I is an ideal, then we can define

$$IM = \left\{ \sum_{i=1}^{n} a_i m_i \mid a_i \in R, m_i \in M \text{ for } n \in N \right\}$$

which is a submodule of R. Thus our main properties of submodules are intersection, addition, and products with ideals.

Definition 3.3.6. Let R be an integral domain and M an R-module. Suppose $x \in M$. Then

1. If $\text{Ann}(x) \neq \{0\}$ then we define x to be **torsion element**. We define the set of torsion elements of M to be the **torsion submodule** and denote this as M_τ.
2. If $M_\tau = \{0\}$ then we say M is **torsion free**, while if $M_\tau = M$ we say that M is **torsion module**.

Proposition 3.3.7. Let R be an integral domain and M an R-module. Then

1. M_τ is a submodule of M.

2. M/M_τ is torsion free.

Proof:

1. We can use the submodule test to show that this is a submodule of M. First, recall that M_τ is nonempty as it always contains 0. Let $a, b \in R$ and suppose $m_1, m_2 \in M_\tau$. Then observe that $(am_1 + bm_2)x = am_1x + bm_2x = 0$ since $m_1x = 0$ and $m_2x = 0$. Hence, this is a submodule.
2. Suppose that $(m + M_\tau)x = M_\tau$ where $m \in M$ for some $x \in M$. Then this implies that $mx \in M_\tau$. Therefore there exists a $n \in R$ such that $n(mx) = (nm)x = 0$. Since R is an integral domain, $nm \neq 0$ so that we must have $x = 0$. Thus the torsion module is trivial.

■

Proposition 3.3.8. Let R be an integral domain and suppose $M = \langle x_1, x_2, \ldots, x_n \rangle$ (that is, M is finitely generated). Then

$$\mathrm{Ann}(M) = \mathrm{Ann}(x_1) \cap \mathrm{Ann}(x_2) \cap \cdots \cap \mathrm{Ann}(x_n).$$

Note that $0 \in \mathrm{Ann}(x_i)$ for all $i \in \{1, 2, \ldots, n\}$. Hence, the above intersection is never empty.

Proof:

$\mathrm{Ann}(\mathbf{M}) \subset \mathrm{Ann}(\mathbf{x_1}) \cap \mathrm{Ann}(\mathbf{x_2}) \cap \cdots \cap \mathrm{Ann}(\mathbf{x_n})$. Observe that for any $a \in \mathrm{Ann}(M)$, we see that $ax = 0$ for all $x \in M$. In particular, $ax_i = 0$ for $i \in \{1, 2, \ldots, n\}$. Hence we see that $a \in \mathrm{Ann}(x_i)$ for each x_i, so that $a \in \mathrm{Ann}(x_1) \cap \mathrm{Ann}(x_2) \cap \cdots \cap \mathrm{Ann}(x_n)$.

$\mathrm{Ann}(\mathbf{x_1}) \cap \mathrm{Ann}(\mathbf{x_2}) \cap \cdots \cap \mathrm{Ann}(\mathbf{x_n}) \subset \mathrm{Ann}(\mathbf{M})$. Observe that for $a \in \mathrm{Ann}(x_1) \cap \mathrm{Ann}(x_2) \cap \cdots \cap \mathrm{Ann}(x_n)$ we see that $ax_i = 0$ for each $i \in \{1, 2, \ldots, n\}$.

By 1.3.3.1 we know that for any $m \in M$ we have that $m = \sum_{i=1}^{n} a_i x_i$ for some $a_i \in R$ and $x_i \in M$. But note that

$$am = a\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a a_i x_i = \sum_{i=1}^{n} a_i (a x_i) = 0$$

where in the last step we used the commutativity of R. Therefore $a \in \mathrm{Ann}(X)$, proving that $\mathrm{Ann}(x_1) \cap \mathrm{Ann}(x_2) \cap \cdots \cap \mathrm{Ann}(x_n) \subset \mathrm{Ann}(M)$.

With both directions of the proof complete, we can conclude that $\mathrm{Ann}(M) = \mathrm{Ann}(x_1) \cap \mathrm{Ann}(x_2) \cap \cdots \cap \mathrm{Ann}(x_n)$ as desired. ■

3.4 Cartesian Products and Direct Sums.

In group and ring theories, we can make sense of the idea of a cartesian product of groups or rings. Thus it is again no surprise that we can construct cartesian products of modules.

However, we shall see a theme that is common in most areas of mathematics: infinite products behave differently than finite products. In fact, it usually turns out that our intuitive definiton for the products of our objects (in our case, modules) is usually wrong and cumbersome, even though it feels intuitive. That is, we generally want to define products using the cartesian notion, but this usually just gives us more problems.

The alternative is to come up with a definition of multiplication that *is* cartesian for finite products, but is not exactly cartesian for infinite products. This will make sense once we are more specific by what we mean.

Definition 3.4.1. Let $M_1, M_2, \dots, M_n$ be a set of R-modules. We define

$$\prod_{i=1}^{n} M_i = M_1 \times M_2 \times \cdots \times M_n$$

as the **cartesian product** of these R-modules whose elements are of the form $(x_1, x_2, \dots, x_n)$ where $x_i \in M_i$ for $i \in \{1, 2, \dots, n\}$.

More generally, if $\{M_\alpha\}_{\alpha\in\lambda}$ is an arbitrary family of R-modules then $\prod_{\alpha\in\lambda} M_\alpha$ is the **arbitrary cartesian product**.

You may now ask how we notate, or even describe these elements. We can't put them in a tuple, since they're not finite. We could put them in a tuple like "$(x_1, x_2, \dots)$", where the ellipsis implies an infinite list of elements, but that would only take care of at most countable families of R-modules.

Instead, we use the following idea as elements of $\prod_{\alpha\in\lambda} M_\alpha$ being "functions." This is an abstract, yet quite useful strategy used in different areas of mathematics to deal with arbitrary products.

Let us first literally describe our elements. An element $a \in \prod_{\alpha\in\lambda} M_\alpha$ is uniquely determined by selecting one element $m_\alpha \in M_\alpha$ for each $\alpha \in \lambda$. This is how a tuple works. For example, in $\mathbb{R}^3$, we separately pick 3 elements out of 3 separate copies of $\mathbb{R}$ to form a tuple $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Thus for each $a \in \prod_{\alpha\in\lambda}$ we may associate a with a function $f_a : \lambda \longrightarrow \prod_{\alpha\in\lambda} M_\alpha$ which iterates through all $\alpha \in \lambda$ and picks out an element M_α. For example, if we know that, for $i \in \lambda$, the i-th coordinate of a is x, then $f_a(i) = x$.

$$\begin{array}{lccccc} \lambda : \{\dots, & \alpha, & \beta, & \gamma, & \dots\} \\ & \downarrow f_a & \downarrow f_a & \downarrow f_a & \\ a = (\dots, & x_\alpha, & x_\beta & x_\gamma, & \dots) \end{array}$$

The above diagram illustrates our descriptions so far, where in the case above we have that the α-th

element of a is x_α, the β-th element of a is x_β, and so on. With that said, we can now restate that

$$\prod_{\alpha\in\lambda} M_\alpha = \{\text{All functions } f \mid f(\alpha) \in M_\alpha \text{ where } \alpha \in \lambda\}.$$

and move onto understanding why we want to adjust our definition for multiplication of R-modules.

It turns out that we can make the arbitrary cartesian product into an R-module.

Proposition 3.4.2. If $\{M_\alpha\}_{\alpha\in\lambda}$ is a family of R-modules, then $\prod_{\alpha\in\lambda} M_\alpha$ is an R-module.

Proof:

Abelian Group. First observe that $\prod_{\alpha\in\lambda} M_\alpha$ is an abelian group if we realize the identity is the zero map f (i.e., the "tuple" of all zeros) and endow an operation of addition as follows. For $f_1, f_2 \in \prod_{\alpha\in\lambda} M_\alpha$ we have that

$$(f_1 + f_2)(\alpha) = f_1(\alpha) + f_2(\alpha)$$

for all $\alpha \in \lambda$. Note that this makes sense since $f_1(\alpha), f_2(\alpha) \in M_\alpha$. Hence the sum will be an element in M_α. Also, if $f \in \prod_{\alpha\in\lambda} M_\alpha$, we define the inverse to be f^{-1} where $f^{-1}(\alpha) = -f(\alpha)$. Commutativity is inherited from commutativity of all M_α, and so we have an abelian group.

Ring Multiplication. Let $a \in R$. Then define

$$(af)(\alpha) = a(f(\alpha))$$

for all $\alpha \in \lambda$. Observe that, since each M_α is an R-module, we have that $f(\alpha) \in M_\alpha \implies af(\alpha) \in M_\alpha$ for all $\alpha \in \lambda$. Thus our multiplcation is well-defined. It is then a simple exercise to check that the axioms of an R-module are satisfied via our operations.

■

Since our above argument was a bit abstract, we reintroduce it in the language of finite products. Again, we can turn a finite cartesian product of R-modules into an R-module with the following operations.

1. Let $(m_1, m_2, \ldots, m_n), (p_1, p_2, \ldots, p_n) \in M_1 \times M_2 \times \cdots \times M_n$. Then let us define addition of elements as

$$(m_1, m_2, \ldots, m_n) + (p_1, p_2, \ldots, p_n) = (m_1 + p_1, m_2 + p_2, \ldots, m_n + p_n).$$

2. For any $a \in R$ and $(m_1, m_2, \ldots, m_n) \in M_1 \times M_2 \times \cdots \times M_n$ we define scalar multiplication as

$$a(m_1, m_2, \ldots, m_n) = (am_1, am_2, \ldots, am_n).$$

Again, it is then simple to check that this satisfies the axioms for an R-module.

When we think of multiplying sets together, cartesian products usually come to mind. They are the most natural to us since it has been ingrained in us to think this way since primary school. How-

ever, it turns out in many areas of mathematics that the cartesian approach to defining multiplication of objects leads to undersirable properties, and objects often misbehave under a cartesian definition.

As we said earlier, the problems arise when the products get infinite. Hence the solution involves defining a new kind of multiplication which is the same as a cartesian product for *finite* products, but is different for infinite products.

This leads to the concept of direct sums, which we will use instead of cartesian products (we will soon see why).

Definition 3.4.3. Let $\{M_\alpha\}_{\alpha\in\lambda}$ be a family of R-modules. Then we define the **direct sum** of $\{M_\alpha\}_{\alpha\in\lambda}$ as

$$\bigoplus_{\alpha\in\lambda} M_\alpha = \{\text{All functions } f \mid f(\alpha)\in M_\alpha \textbf{ and } f(\alpha)=0 \text{ except for finitely many } \alpha\in\lambda\}.$$

The only difference between the direct sum and the cartesian product is that, for any point $a\in\bigoplus_{\alpha\in\lambda} M_\alpha$, all indices of a are zero except for finitely many indices. So only finitely many indices are nonzero for a direct sum, while in a cartesian product there may be finite, countable or uncountably many nonzero indices.

Thus, note that for a finite product, the direct sum and the cartesian product are the exact same thing. There is no difference when the product is finite. In other words,

$$M_1\times M_2\times\cdots\times M_n = M_1\oplus M_2\oplus\cdots\oplus M_n.$$

Proposition 3.4.4. The direct sum of a family $\{M_\alpha\}_{\alpha\in\lambda}$ of R-modules is an R-module. In fact, $\bigoplus_{\alpha\in\lambda} M_\alpha$ is an R-submodule of $\prod_{\alpha\in\lambda} M_\alpha$.

Proof: Note that $\bigoplus_{\alpha\in\lambda} M_\alpha \subset \prod_{\alpha\in\lambda} M_\alpha$. Thus we can use the submodule test to check if is in fact an R-module. Observe that for any $a,b\in R$ and $f_1,f_2\in\bigoplus_{\alpha\in\lambda} M_\alpha$, we have that

$$a(f_1)(\alpha)+b(f_2)(\alpha)\in\bigoplus_{\alpha\in\lambda} M_\alpha$$

since the function $a(f_1)(\alpha)+b(f_2)(\alpha)$ will be nonzero for only finitely many values. (In fact, if f_1 is nonzero for k-many values and f_2 is nozero for l many values, then $a(f_1)(\alpha)+b(f_2)(\alpha)$ is nonzero for at most $k+l$-many values). Hence this passes the submodule test. ■

Why do we prefer direct sums over cartesian products?

The answer lies in the following observation. Suppose $\{M_\alpha\}_{\alpha\in\lambda}$ is a family of R-modules and that for each $\alpha\in\lambda$ there exists a homomorphism $\varphi_\alpha : M_\alpha\longrightarrow N$. Let $a\in\prod_{\alpha\in\lambda} M_\alpha$ and represent a with the map $f_a:\lambda\longrightarrow\prod_{\alpha\in\lambda} M_\alpha$. Thus $f_a(\alpha)\in M_\alpha$ is the α-th coordinate of our point a.

If we try to define a homomorphism $\varphi : \prod_{\alpha\in\lambda} M_\alpha \longrightarrow N$ in a natural, linear way such as

$$\varphi(a) = \sum_{\alpha\in\lambda} \varphi_\alpha(f_a(\alpha))$$

where $a \in \prod_{\alpha\in\lambda} M_\alpha$, then observe that the above sum is nonsense. What the hell is an infinite sum of module elements of N supposed to represent? Also, there's no way to make sure this is even well-defined!

However, if we instead consider $\bigoplus_{\alpha\in\lambda} M_\alpha$, then creating a natural homomorphism $\varphi : \bigoplus_{\alpha\in\lambda} M_\alpha \longrightarrow N$ where again

$$\varphi(a) = \sum_{\alpha\in\lambda} \varphi_\alpha(f_a(\alpha))$$

works out fine. We see that φ is valid because $f_a(\alpha) = 0$ for all but finitely many $\alpha \in \lambda$. Hence, the above sum will only ever consist of a sum of finite elements.

The next important two theorems demonstrate the importance of the direct sum.

Theorem 3.4.5. Let M be an R-module and suppose $M_1, M_2, \ldots, M_n$ are submodules such that

1. $M = M_1 + M_2 + \cdots + M_n$
2. $M_j \cap (M_1 + M_2 + \cdots + M_{j-1} + M_{j+1} + \cdots + M_n) = \{0\}$ for all $j \in \{1, 2, \ldots, n\}$.

Then

$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

Proof: Construct the map $f : M_1 \oplus M_2 \oplus \cdots \oplus M_n \longrightarrow M$ as

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n.$$

It is simple to check that this is an R-module homomorphism. Observe that by (1) $\mathrm{Im}(f) = M$. Now suppose $(x_1, x_2, \ldots, x_n) \in \ker(f)$. Then we see that

$$x_1 + x_2 + \cdots + x_n = 0 \implies x_i = -(x_1 + x_2 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n)$$

for all $i \in \{1, 2, \ldots, n\}$. But by (2), we know that no such x_i can exist. Therefore $x_1 = x_2 = \cdots = x_n = 0$. Hence, f is an isomorphism, which yields the desired result. ■

The above result can be generalized to arbitrary direct sums. However, if we were dealing with cartesian products, we would not be able to generalize the above theorem to arbitrary direct sums.

Theorem 3.4.6. Let M be an R-module and suppose $\{M_\alpha\}_{\alpha\in\lambda}$ is a family of R-modules such that

1. $M = \sum_{\alpha\in\lambda} M_\alpha$
2. $M_\beta \bigcap \sum_{\alpha\in\lambda\setminus\{\beta\}} M_\alpha = \{0\}$ for all $\beta \in \lambda$

then

$$M \cong \bigoplus_{\alpha\in\lambda} M_\alpha$$

The proof is the exact same as before, although the notation is annoying.

3.5 Exact Sequences and the Hom Functor.

This section will be the first encounter with the extremely important algebraic concept of an *exact sequence*, which is something you may have already seen before without even knowing it.

Definition 3.5.1. Let R be a ring. We define a **sequence** of R-modules to be a chain of homomorphisms between R-modules, generally denoted as

$$\cdots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_{i+2}} \cdots$$

we say that that the above sequence is **exact** at M_i if $\text{Im}(f_i) = \ker(f_{i+1})$. Hence, an exact sequence is a sequence which is exact at every M_i.

Short Exact Sequences.
Looking at "short" exact sequences aids out analysis of longer or infinite exact sequences.

Proposition 3.5.2. Let M_1, M_2 and M be R-modules. Then

1. The sequence $0 \longrightarrow M_1 \xrightarrow{f} M$ is exact if and only if f is injective.
2. The sequence $M \xrightarrow{g} M_2 \longrightarrow 0$ is exact if and only if g is surjective.
3. The sequence $0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$ is exact if and only if f is injective and g is injective.

Proof:

1. ($\Longrightarrow$) Suppose the sequence $0 \longrightarrow M_1 \xrightarrow{f} M$ is exact. Then we have that $\text{Im}(0) = \ker(f) \implies \ker(f) = \{0\}$. Therefore we see that f is injective.

   ($\Longleftarrow$)Now suppose f is injective. Then $\ker(f) = 0$. Since $\text{Im}(0) = \{0\}$ we see $\text{Im}(0) = \ker(f)$, so that the sequence $0 \longrightarrow M_1 \xrightarrow{f} M$ is exact.

2. ($\Longrightarrow$) Suppose the sequence $M \xrightarrow{g} M_2 \longrightarrow 0$ is exact. Then we see that $\text{Im}(g) = \ker(0) = M_2$, since the zero map simply takes all of M_2 and sends it to 0. Hence we see that g is surjective.

   ($\Longleftarrow$) Now suppose g is surjective. Then $\text{Im}(g) = M_2$ and we also have that $\ker(0) = M_2$. Therefore $\text{Im}(g) = \ker(0)$ so that we have an exact sequence.

3. By applying (1.) and (2.), the result follows.

■

The above proposition offers the following definitions.

Definition 3.5.3. Let M_1, M_2 and M be R-modules. If the sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

is exact then we say it forms an **short exact sequence**. Furthermore, if there exists an R-module N such that $M = N \oplus \text{Im}(f) = N \oplus \ker(g)$ (since $\text{Im}(f) = \ker(g)$) then we say the above sequence is **split exact**.

In this case, we say N or $\text{Im}(f)$ is a **direct summand** of M.

We can offer a few short exact sequences with some familiar objects.

Examples.

1. Let M be an R-module with a submodule N. If $i : N \longrightarrow M$ is the inclusion map and $\pi : M \longrightarrow M/N$ is the projection map, then the sequence

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{\pi} M/N \longrightarrow 0$$

is exact.
$\mathbf{Im(i) \subset ker\,(\pi)}$. Observe that if $n \in N$ then

$$\pi(i(n)) = \pi(n) = n + N = N$$

so that $\mathrm{Im}(i) \subset \ker(g)$.
$\mathbf{ker\,(\pi) \subset Im(i)}$. Suppose $m \in \ker(\pi)$. Then we see that $\pi(m) = m + N = N$, so that $m \in N$. Since $m \in N$, we know that $i(m) = m$. Therefore m is the image of some element in M mapped by i (namely, just m itself). Hence $\ker(\pi) \subset \mathrm{Im}(i)$.

With both directions, we can conclude that $\mathrm{Im}(i) = \ker(\pi)$ so so that the sequence is exact.

2. Let N and P be R-modules. If we define $i' : N \longrightarrow N \oplus P$ where $i'(n) = (n, 0)$ and $\pi' : N \oplus P \longrightarrow P$ where $\pi'(n, p) = p$, we see that the sequence

$$0 \longrightarrow N \xrightarrow{i'} N \oplus M \xrightarrow{\pi'} M \longrightarrow 0$$

is exact. We can realize this by simply observing that $\ker(\pi')$ is the set of all elements $(n, 0) \in N \oplus P$, which is exactly the image of i'. Therefore $\mathrm{Im}(i') = \ker(\pi')$, so that the sequence is exact.

3. The sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{f} \mathbb{Z}_{pq} \xrightarrow{g} \mathbb{Z}_q \longrightarrow 0$$

where $f : \mathbb{Z}_p \longrightarrow \mathbb{Z}_{pq}$ is given by $f(n) = qn$ and $g : \mathbb{Z}_{pq} \longrightarrow \mathbb{Z}_q$ is given by $g(n) = n \bmod q$, then this sequence is exact. In fact, it is a split exact sequence. From group theory, we know that

$$\mathbb{Z}_{mn} \cong \mathbb{Z}_m \oplus \mathbb{Z}_n$$

if and only if m and n are coprime. In our case, p and q are distint primes and hence are coprime so that $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$. We'll later show that this will be sufficient to conclude that this is a split sequence.

4. If instead we have the sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{f} \mathbb{Z}_{p^2} \xrightarrow{g} \mathbb{Z}_p \longrightarrow 0$$

where $f : \mathbb{Z}_p \longrightarrow \mathbb{Z}_{p^2}$ is given by $f(n) = pn$ and $g : \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_p$ is given by $g(n) = n \bmod p$, then this becomes an exact sequence. However, this is not split exact as p is obviously not comprime with itself, and hence

$$\mathbb{Z}_{p^2} \ncong \mathbb{Z}_p \oplus \mathbb{Z}_p$$

which is why this is not a split exact sequence.

The last two examples can be generalized into a theorem, which include other criterion for when a short exact sequence is split exact.

Theorem 3.5.4. Let M_1, M_2 and M be R-modules such that

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

is exact. Then the following are equivalent:

1. There exists a homomorphism $\alpha : M \longrightarrow M_1$ such that $\alpha \circ f = 1_{M_1}$
2. There exists a homomorphism $\beta : M_2 \longrightarrow M$ such that $g \circ \beta = 1_{M_2}$
3. The above sequence is split exact.

Furthermore, we see that

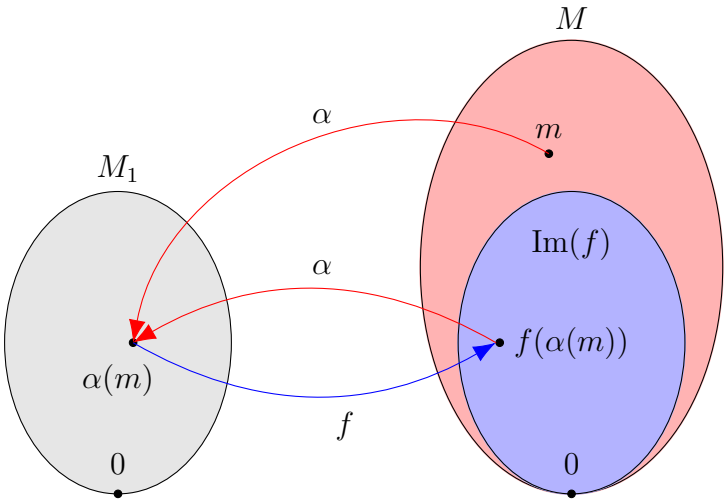
$$\begin{aligned} M &\cong \operatorname{Im}(f) \oplus \ker(\alpha) \\ &\cong \ker(g) \oplus \operatorname{Im}(\beta) \\ &\cong M_1 \oplus M_2. \end{aligned}$$

Proof:

(1 $\implies$ 3). Suppose there exists an $\alpha : M \longrightarrow M_1$ such that $\alpha \circ f = 1_{M_1}$. Let $m \in M_1$. Then observe that

$$\begin{aligned} \alpha(m - f(\alpha(m))) &= \alpha(m) - \alpha(f(\alpha(m))) \\ &= \alpha(m) - (\alpha \circ f)(\alpha(m)) \\ &= \alpha(m) - \alpha(m) \\ &= 0 \end{aligned}$$

where in the third step we used the fact that $\alpha \circ f = 1_{M_1}$, and hence $\alpha(f(m)) = m$ for all $m \in M_1$. Hence, $m - f(\alpha(m)) \in \ker(\alpha)$.



$\alpha(m)$ and $\alpha(f(\alpha(m)))$ are mapped to the same element. Therefore, their difference is zero, so that $m - f(\alpha(m)) \in \ker(\alpha)$.

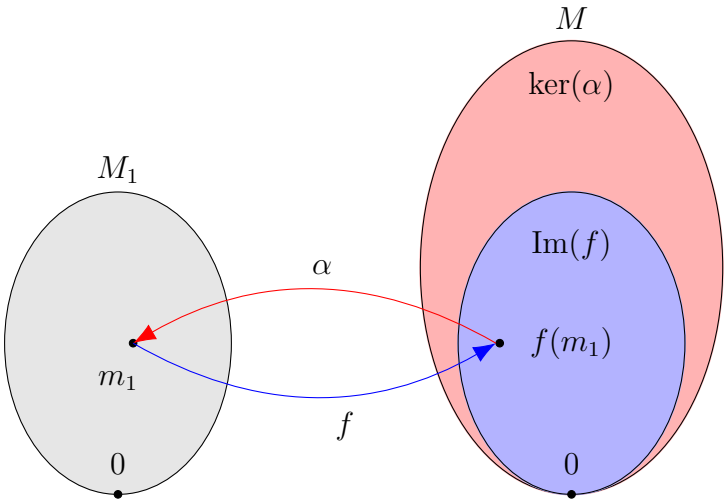
Since $f : M_1 \longrightarrow M$ is injective, we see that $\alpha : M \longrightarrow M_1$ is surjective. To see this, let $m' \in M_1$. Then there exists an $m'' \in M$ such that $\alpha(m'') = m'$; namely, $m'' = f(m')$ works. Since α is surjective, we see that

$$\{f(\alpha(m)) \mid m \in M\} = \{f(m_1) \mid m_1 \in M_1\} = \text{Im}(f).$$

That is, $f(\alpha(M)) = f(M_1) = \text{Im}(f)$. And because $m - f(\alpha(m)) \in \ker(\alpha)$ for all $m \in M$, we see that $m \in \text{Im}(f) + \ker(\alpha)$ for all $m \in M$. Hence, $M \subset \text{Im}(f) + \ker(\alpha)$. But both $\text{Im}(f)$ and $\ker(\alpha)$ are subsets of M. Therefore, $M = \text{Im}(f) + \ker(\alpha)$.
Now let $x \in \ker(\alpha) \cap \text{Im}(f)$. Then $f(y) = x$ for some $y \in M_1$, and $\alpha(x) = 0$ as well. Hence,

$$\alpha(f(y)) = \alpha(x) = 0.$$

But $\alpha \circ f = 0$, which implies that $y = 0$. Therefore $\ker(\alpha) \cap \text{Im}(f) = \{0\}$.



By Theorem 1.3.4.5, we see that this implies that

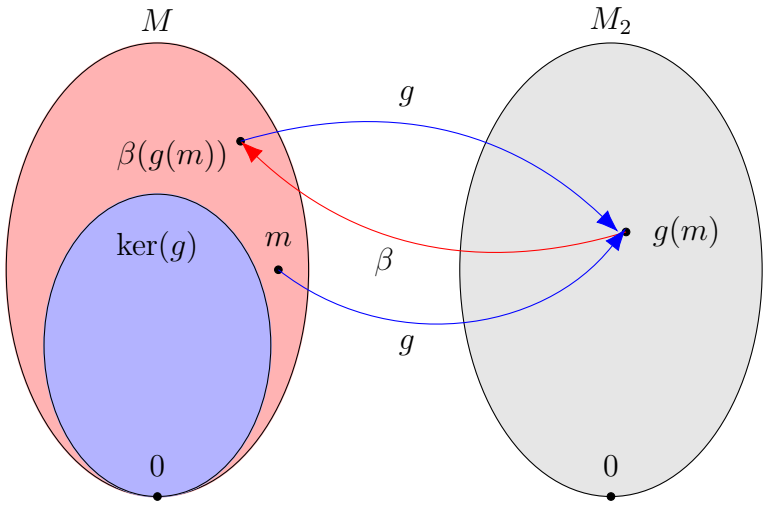
$$M \cong \text{Im}(f) \oplus \ker(\alpha).$$

Hence, M is split exact as $\mathrm{Im}(f)$ is a direct summand of M.

(2 $\implies$ 3). Suppose (2) holds. We'll show that $m - \beta(g(m)) \in \ker(g)$ for all $m \in M$.
To show this, observe that

$$\begin{aligned} g[m - \beta(g(m))] &= g(m) - g \circ \beta(g(m)) \\ &= g(m) - g(m) \\ &= 0 \end{aligned}$$

where in the second step we used the fact that $g \circ \beta = 1_{M_2}$. Therefore, $m - \beta(g(m)) \in \ker(g)$.



$g(m)$ and $g(\beta(g(m)))$ are mapped to the same element in M_2, so their difference is zero. Therefore, $m - \beta(g(m)) \in \ker(g)$.

Now note that

$$\{\beta(g(m)) \mid m \in M\} = \{\beta(m_2) \mid m_2 \in M_2\} = \mathrm{Im}(\beta)$$

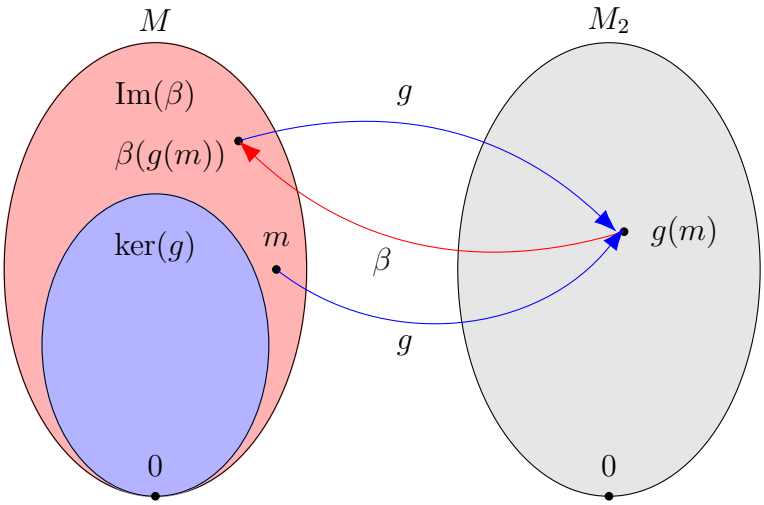
where in the second step we used the fact that g is surjective. That is, $\beta(g(M)) = \beta(M_2) = \mathrm{Im}(\beta)$. Therefore we see that

$$m \in \mathrm{Im}(\beta) + \ker(g)$$

for all $m \in M$ which implies that $M \subset \mathrm{Im}(\beta) + \ker(g)$. But since $\mathrm{Im}(\beta)$ and $\ker(g)$ are both subsets of M, we see that $M = \mathrm{Im}(\beta) + \ker(g)$.
Now let $m' \in \mathrm{Im}(\beta) \cap \ker(g)$. Then there exists an $m_2 \in M_2$ such that $\beta(m_2) = m'$. Furthermore, since $m' \in \ker(g)$,

$$0 = g(m') = g(\beta(m_2)) = m_2$$

since $g \circ \beta = 1_{M_2}$. Hence, $m_2 = 0$, so that $\beta(m_2) = 0 = m$. Therefore $m = 0$, so that $\mathrm{Im}(\beta) \cap \ker(g) = \{0\}$.



By Theorem 1.3.4.5, we have that

$$M \cong \mathrm{Im}(\beta) \oplus \ker(g)$$

so that M is split exact, as one of its direct summands is $\ker(g)$.

(1 $\implies$ 2). Suppose (1) holds. Construct a function $\beta : M_2 \longrightarrow M$ defined by

$$\beta(u) = v - f(\alpha(v))$$

where $g(v) = u$. Since G is surjective, we know that such a v exists, although we don't know if it is the only $v \in M$ which maps to u, and if that could cause us problems. Thus we'll show that this definition is well defined (i.e. independent of the choice of v).

Well-defined. Suppose $g(v') = u$ for some other $v' \in M$. Then

$$\begin{aligned} g(v') - g(v) &= v - f(\alpha(v)) - (v' - f(\alpha(v'))) \\ &= (v - v') - f(\alpha(v)) + f(\alpha(v')) \\ &= (v - v') - f(\alpha(v) - \alpha(v')) \\ &= (v - v') - f(\alpha(v - v')) \\ &= 0. \end{aligned}$$

We will prove the conclusion made in red, i.e., $(v - v') - f(\alpha(v - v')) = 0$.

To see this, first note that, as we proved earlier, $x - f(\alpha(x)) \in \ker(\alpha)$ for all $x \in M$. Hence, $(v - v') - f(\alpha(v - v')) \in \ker(\alpha)$.

Furthermore, since $g(v) = g(v')$, we see that $g(v - v') = 0 \implies v - v' \in \ker(g)$. But $\ker(g) = \mathrm{Im}(f)$, so that $v - v' \in \mathrm{Im}(f)$. Obviously $f(\alpha(v - v')) \in \mathrm{Im}(f)$ for any $v \in M$, so that $(v - v') - f(\alpha(v - v')) \in \mathrm{Im}(f)$.

Thus we have that $(v - v') - f(\alpha(v - v')) \in \mathrm{Im}(f) \cap \ker(\alpha) = \{0\}$, so that $g(v) - g(v') = 0$.

Next observe that for any $u \in M_2$ we have that

$$\begin{aligned} g \circ \beta(u) &= g(v - f(\alpha(v))) \\ &= g(v) - (g \circ f)(\alpha(v)) \\ &= g(v) \end{aligned}$$

where in the second step we used the fact that $(g \circ f) = 0$ as $\ker(g) = \mathrm{Im}(f)$. Thus we have that $g \circ \beta = 1_{M_2}$, so that such a desired $\beta : M_2 \longrightarrow M$ exists.

(2 $\implies$ 1). Suppose (2) holds. Construct a function $\alpha : M \longrightarrow M_1$ defined by

$$\alpha(m) = f^{-1}(m - \beta(g(m))).$$

Note that we must be careful since we're dealing with an inverse. To even make such a statement, we first recall that f is injective, so an inverse from $f^{-1} : \mathrm{Im}(f) \longrightarrow M$ certainly exists. But it only exists if its domain is at most $\mathrm{Im}(f)$. Thus we check that $m - \beta(g(m)) \in \mathrm{Im}(f)$ for all $m \in M$.

Earlier we proved that $m - \beta(g(m)) \in \ker(g)$, and we know that $\ker(g) = \mathrm{Im}(f)$ as the sequence is exact. Therefore, we already know that $m - \beta(g(m)) \in \mathrm{Im}(f)$.

Hence, α makes sense since f^{-1} exists and $m - \beta(g(m)) \in \mathrm{Im}(f)$ for all $m \in M$.

Now observe that for any $m_1 \in M_1$,

$$\begin{aligned} \alpha \circ f(m_1) &= f^{-1}(f(m_1) - \beta(g(f(m_1)))) \\ &= f^{-1}f(m_1) - f^{-1}(0) \\ &= m_1 \end{aligned}$$

since $g(f(m_1)) = 0$ for all $m_1 \in M$. Thus such a desired α exists.

(3 $\implies$ 1 & 2). Suppose that

$$M \cong M' \oplus M''$$

where $M' = \mathrm{Im}(f) = \ker(g)$, and M'' is some other summand of M. Define a projection map $\pi : M \longrightarrow M'$ as

$$\pi(m) = \begin{cases} m & \text{if } m \in M' \\ 0 & \text{otherwise} \end{cases}$$

and similarly the injective map $i : M'' \longrightarrow M$ as $i(m'') = m''$ for all $m'' \in M$.

Consider $\pi \circ f : M \longrightarrow M'$. Since $M' = \mathrm{Im}(f)$, this is clearly an isomorphism. Now define $\alpha = (\pi \circ f)^{-1} \circ \pi_1$ and observe that $\alpha \longrightarrow M \longrightarrow M_1$ and

$$\alpha \circ f = (\pi \circ f)^{-1} \circ \pi_1 \circ f = 1_{M_1}.$$

Hence, (3) $\implies$ (1).

Similarly, observe that $g \circ i : M'' \longrightarrow M_2$ is also an isomorphism. To see this, first observe that $M' = \ker(g)$, and since $M \cong M' \oplus M''$ we know that $M' \cap M'' = \{0\}$. Therefore, if $m \in M''$ is nonzero, then $m \notin \ker(g)$. Hence $g(i(m)) \neq 0$ if and only if $m = 0$, so that $g \circ i$ is one to one. Now surjectivity is clear, as g itself is a surjective function.

Now define $\beta = i \circ (g \circ i)^{-1}$, and observe that $\beta : M_2 \longrightarrow M$ and

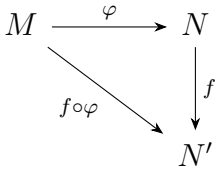
$$g \circ \beta = g \circ i \circ (g \circ i)^{-1} = 1_{M_2}.$$

Therefore $(3 \implies 2)$, which completes the entire proof. ■

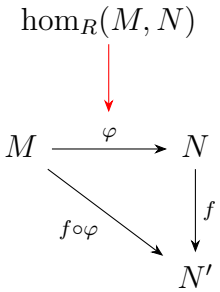
That was a long ass proof, but the theorem is very powerful and worthwhile. Next, we'll reintroduce the concept of hom().

Inducing Homomorphisms.

Let M, N and N' be R-modules, and let $\varphi : M \longrightarrow N$ and $f : N \longrightarrow N'$ be R-modules homomorphisms. Then we see that the diagram to the right commutes.



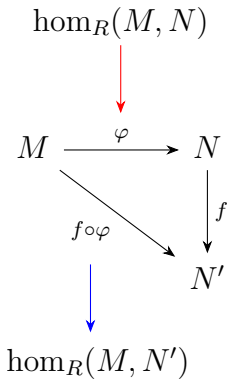
However, suppose we feed the above diagram with arbitrary $\varphi : M \longrightarrow N$. That is, we keep $f : N \longrightarrow N'$ fixed, but let $\varphi : M \longrightarrow N$ vary over all possible φ. This is equivalent to grabbing elements from the abelian group $\hom_R(M, N)$. We can denote this with a red arrow, to remind the reader that this arrow "picks" φ.



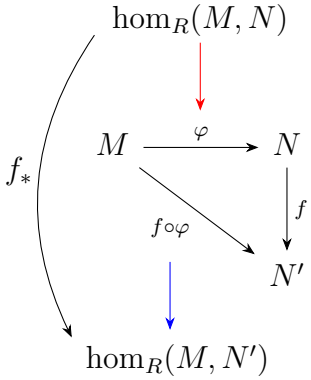
Note that we've described a well-defined system for assigning for each $\varphi \in \hom_R(M, N)$ a function

$$f \circ \varphi.$$

Also, $f \circ \varphi : M \longrightarrow N'$, so that $f \circ \varphi \in \hom_R(M, N')$. We can denote this with a blue arrow, to communicate that $\hom_R(M, N')$ "accepts" $f \circ \varphi$ (after all, it is an element of the set).



What we've just described is an *induced* function, which we denote as f_*. That is, if we fix f, then we can create a homomorphism f_* between the abelian groups $\hom_R(M, N)$ and $\hom_R(M, N')$, where for each element $\varphi \in \hom_R(M, N)$ we assign it the function $f \circ \varphi \in \hom_R(M, N')$.



$\hom_R(-, M)$.

The we restate our results. If N, N' are R-module homomorphisms and $f : N \longrightarrow N'$ is an R-module homomorphism, then for any R-module M we can create an induced homomorphism

$$f_* : \hom_R(M, N) \longrightarrow \hom_R(M, N')$$

defined as

$$f_*(\varphi) = f \circ \varphi.$$

$\hom_R(M, -)$.
Similarly, if N, N' are again R-modules and $g : N' \longrightarrow N$ is a an R-module homomorphism, then for any R-module M, there is an induced homomorphism

$$g^* : \hom_R(N, M) \longrightarrow \hom_R(N', M)$$

defined as

$$g^*(\psi) = \psi \circ g.$$

It turns out in category theory that the behavior of these functions fit the definition of a **functor**. $\hom_R(-, M)$ is known as a covariant functor, while $\hom_R(M, -)$ is known as a contravariant functor. We won't delve too much into this.

Since the $\hom_R$ groups are abelian, we see that f_* and g_* are in fact group homomorphisms. If R is commutative, then we know that $\hom_R$ forms an R-module in which case f_* and g_* become R-module homomorphisms.

Now suppose a family of R-modules $\{M_i \mid i \in \mathbb{N}\}$ associated with a set of homomorphisms $\{f_i \mid f_i : M_{i-1} \longrightarrow M_i, i \in \mathbb{N}\}$ for a long sequence, not necessarily exact.

$$\cdots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_{i+2}} \cdots$$

Then if we apply the $\hom_R(M, -)$ functor, then we see that the above sequence implies a sequence between the hom groups:

$$\cdots \xrightarrow{(f_{i-1})_*} \hom_R(M, M_{i-1}) \xrightarrow{(f_i)_*} \hom_R(M, M_i) \xrightarrow{(f_{i+1})_*} \hom_R(M, M_{i+1}) \xrightarrow{(f_{i+2})_*} \cdots$$

and applying the $\hom_R(-, M)$ functor we get

$$\cdots \xleftarrow{(f_{i-1})^*} \hom_R(M, M_{i-1}) \xleftarrow{(f_i)^*} \hom_R(M, M_i) \xleftarrow{(f_{i+1})^*} \hom_R(M, M_{i+1}) \xleftarrow{(f_{i+2})^*} \cdots$$

Thus the long sequence of R-modules implies the existence of two other long sequences of abelian groups. The interesting thing is that the two sequences are similar but differ in the direction of the arrows (this is why we denote the functions separately with an asterik either in the subscript or superscript). Furthermore, the direction of the arrows in the first seuqence of M_i R-modules determines the direction of the arrows in the other two sequences.

Theorem 3.5.5. Let M_1, M and M_2 be R-modules, and suppose $f : M_1 \longrightarrow M$ and $g : M \longrightarrow M_2$ are R-modules. Then the sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \tag{3.1}$$

is exact if and only if the sequence

$$0 \longrightarrow \hom_R(N, M_1) \xrightarrow{f_*} \hom_R(N, M) \xrightarrow{g_*} \hom_R(N, M_2) \tag{3.2}$$

is an exact sequence of abelian groups. Furthermore, the sequence

$$M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0 \tag{3.3}$$

is exact if and only if

$$\hom_R(M_1, N) \xleftarrow{f^*} \hom_R(M, N) \xleftarrow{g^*} \hom_R(M_2, N) \longleftarrow 0 \tag{3.4}$$

is an exact sequence of abelain groups.

Proof: To show that the sequence between the hom abelian groups is exact, we need to check that (1) f_* is injective and (2) $\text{Im}(f_*) = \ker(g_*)$.

f_* is injective. Suppose that $f_*(\psi) = 0$ for some $\varphi \in \hom_R(N, M_1)$. Then

$$f_*(\psi) = 0 \implies f(\psi(n)) = 0$$

for all $n \in M$. However, f is injective, so that $\ker(f) = \{0\}$. Therefore $\psi(n) \in \ker(f) = \{0\}$ for all n, which means that ψ is the zero function. Therefore $\ker(f_*) = \{0\}$ (where the zero here stands for the zero function between N and M_1) so that f_* is injective.

$\text{Im}(f_*) \subset \ker(g_*)$. Let $\varphi \in \hom_R(N, M_1)$. Then observe that

$$g_*(f_*(\varphi)) = g_*(f \circ \varphi) = g \circ f \circ \varphi = 0$$

since $g \circ f = 0$ as $\text{Im}(f) = \ker(g)$. Therefore we see that $\text{Im}(f_*) \subset \ker(g_*)$.

$\ker(g_*) \subset \text{Im}(f_*)$. Let $\psi \in \hom_R(N, M)$ and suppose that $g_*(\psi) = 0$. Note that

$$g_*(\psi) = 0 \implies g(\psi(n)) = 0$$

for all $n \in N$. Since $\ker(g) = \text{Im}(f)$, we know for all $n \in N$ that $\psi(n) \in \text{Im}(f)$. Therefore, there exist a set of $y \in M_1$ such that $f(y) = \psi(n)$, and since f is one to one this correspondence is uniquely determined.

Thus construct a function $\tau : N \longrightarrow M_1$ such that

$$\tau(n) = f^{-1}(\psi(n)).$$

As we discussed, this function is well defined since f is one-to-one, and therefore there is always a unique value of $f^{-1}(\psi(n))$ for each n. Now note that this function is an R-module homomorphism since, for any $n_1, n_2 \in N$ and $a \in R$

$$\begin{aligned}\tau(n_1+n_2) &= f^{-1}(\psi(n_1+n_2)) \\ &= f^{-1}(\psi(n_1)+\psi(n_2)) \\ &= f^{-1}(\psi(n_1)) + f^{-1}(\psi(n_2)) \\ &= \tau(n_1)+\tau(n_2)\end{aligned}$$

and

$$\begin{aligned}\tau(an_1) &= f^{-1}(\psi(an_1)) \\ &= f^{-1}(a\psi(n_1)) \\ &= af^{-1}(\psi(n)) \\ &= a\tau(n_1).\end{aligned}$$

Therefore we see that $\tau \in \hom_R(N_1, M)$ and that

$$f_*(\tau) = f_*(f^{-1}(\psi)) = f(f^{-1}(\psi)) = \psi.$$

Hence, $\psi \in \mathrm{Im}(f_*)$. Hence $\ker(g_*) \subset \mathrm{Im}(f_*)$, which proves that $\ker(g_*) = \mathrm{Im}(f_*)$.

To prove the reverse direction, we will assume the exactness of the second sequence and show that (1) f is injective and (2) $\mathrm{Im}(f) = \ker(g)$.

***f* is injective.** Suppose that sequence 3.2 is exact for all R-modules N. Then let $N = \ker(f)$, and since $N \subset M_1$ consider the inclusion map $i : N \longrightarrow M_1$. Note however that for any $n \in N$ we see that

$$f_*(i(n)) = f(i(n)) = 0$$

since $\mathrm{Im}(i) = \ker(f)$. Hence, $i \in \ker(f_*)$. However, since $f_* : \hom_R(N, M_1) \longrightarrow \hom_R(N, M)$ is injective, we know that $\ker(f_*) = 0$. Therefore we have that $i = 0$, (i.e. it is a zero map). But since we defined this to be the *inclusion* map, we have that $N = \{0\}$. Hence, $\ker(f) = N = \{0\}$, so that f is one to one.

$\mathrm{Im}(f) \subset \ker(g)$. Let $N = M_1$, and let $1_{M_1} : M_1 \longrightarrow M_1$ be the identity. Then we see that

$$0 = g_*(f_*(1_{M_1})) = g \circ f$$

by exactness of sequence 3.2. Therefore we see that $\mathrm{Im}(f) \subset \ker(g)$.

$\ker(g) \subset \mathrm{Im}(f)$.

■

Theorem 3.5.6. Let N be an R-module. If

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

is a split exact sequence of R-modules, then

$$0 \longrightarrow \hom_R(N, M_1) \xrightarrow{f_*} \hom_R(N, M) \xrightarrow{g_*} \hom_R(N, M_2) \longrightarrow 0$$

and

$$0 \longleftarrow \hom_R(M_1, N) \xleftarrow{f^*} \hom_R(M, N) \xleftarrow{g^*} \hom_R(M_2, N) \longleftarrow 0$$

are split exact sequences of abelian groups (R-modules if R is commutative).

Proof: By the previous theorem, we only need to show that g_* and f^* are surjective and that the two sequences split. Since the first sequence splits, let $\beta : M_2 \longrightarrow M$ be the function which splits the first sequence. Consider the function $\beta_* : \hom_R(N, M_2) \longrightarrow \hom_R(N, M)$. Then observe that for any $\psi \in \hom_R(N, M_2)$ that

$$g_* \circ \beta_*(\psi) = g_*(\beta(\psi)) = g \circ \beta(\psi) = \psi.$$

Therefore, we see that $g_* \circ \beta_* = 1_{\hom_R(M_2,N)}$. Hence by Theorem 1.3.5.4, we see that β_* splits the second sequence. However, note also that $g_* \circ \beta_* = 1_{\hom_R(M_2,N)}$ implies that g_* is surjective. Therefore the second sequence is split exact.

As for the third sequence, consider the function $\alpha^* : \hom_R(M_1, N) \longrightarrow \hom_R(M, N)$. Note that for any $\varphi \in \hom_R(M, N)$, we have that

$$\alpha^* \circ f^*(\varphi) = \alpha^*(f(\varphi)) = \alpha \circ f(\varphi) = \varphi.$$

Hence we see that $\alpha^* \circ f^*$ splits the third sequence. Furthermore, the fact that $\alpha^* \circ f^* = 1_{\hom_R(M,N)}$ implies that f^* is surjective. Thus in total we have that the third sequence is in fact a split exact sequence. ■

The next theorem is a nice result that shows that $\hom_R$ is somewhat of a "linear" operator.

Theorem 3.5.7. Let M_1, M_2 and M be R-modules. Then

$$\hom_R(M, M_1 \oplus M_2) \cong \hom_R(M, M_1) \oplus \hom_R(M, M_2)$$

and

$$\hom_R(M_1 \oplus M_2, M) \cong \hom_R(M_1, M) \oplus \hom_R(M_2, M).$$

These are in general isomorphisms of abelian groups, but can be isomorphisms of R-modules if R is commutative.

Proof: Consider one of our earlier examples of a split exact sequences:

$$0 \longrightarrow M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \longrightarrow 0$$

where i defined as $i(m_1) = (m_1, 0)$ is the inclusion map and π defined by $\pi(m_1, m_2) = m_2$ is the projection map. As this is split exact, we can apply the previous theorem to gaurantee the existence of sequences

$$0 \longrightarrow \hom_R(M, M_1) \xrightarrow{i_*} \hom_R(M, M_1 \oplus M_2) \xrightarrow{\pi_*} \hom_R(M, M_2) \longrightarrow 0$$

and

$$0 \longleftarrow \hom_R(M_1, M) \xleftarrow{i^*} \hom_R(M_1 \oplus M_2, M) \xleftarrow{\pi^*} \hom_R(M_2, M) \longleftarrow 0$$

which are both split exact. Then by applying Theorem 1.3.5.4 we have that

$$\hom_R(M, M_1 \oplus M_2) \cong \hom_R(M, M_1) \oplus \hom_R(M, M_2)$$

and

$$\hom_R(M_1 \oplus M_2, M) \cong \hom_R(M_1, M) \oplus \hom_R(M_2, M).$$

■

3.6 Free R-modules.

Free modules are the type of modules that you are probably already familiar with. Basically, they're modules who have some kind of generating set, which can create all other elements. As we can think of modules as vector spaces, we know that vectors spaces always have some kind of basis set, at least when they can be thought of as existing in $\mathbb{R}^n$. It turns out that having a basis leads to many desirable properties.

First, we make a definition on linear independence, a concept required for discussing bases, and then formally define a free module.

Definition 3.6.1. Let R be a ring and M an R-module. Then the set $S = \{x_1, x_2, \ldots, x_n\}$ with $S \subset M$ is said to be **linearly independent** if and only if the only solution to the equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

is $a_1 = a_2 = \cdots = a_n = 0$ (where $a_1, a_2, \ldots, a_n \in R$).

If S is the smallest linear independent subset of M, then we say that S is a **basis** for M, in which case M is said to be a **free** R-module.

Hence, an R-module is a module with a basis.

This is the exact same definition of linear independence we've seen in linear algebra. Nothing is new here. It is a classic exercise in linear algebra to check the following statement, which we offer here.

Proposition 3.6.2. S is a basis for some R-module M if and only if every $x \in M$ can be written uniquely as

$$x = a_1x_1 + a_2x_2 + \ldots a_nx_n$$

where $a_i \in R$ and $x_i \in S$ for $i = 1, 2, \ldots, n$.

Examples

1. Consider the R-module $M_{m,n}(R)$. Observe that a basis for this module consists of

$$\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

2. Consider an abelian group G. Then as we showed before, G is technically a $\mathbb{Z}$-module. However, if G is finite, then it is not a free $\mathbb{Z}$-module.

   Suppose to the contrary that it is, and that $S = \{x_1, x_2, \ldots, x_n\}$ is a linearly independent set which forms a basis of G. Then if $\{o_1, o_2, \ldots, o_n\}$ is a set such that $o_i = \text{order}(x_1)$ (which exists, by finiteness of G) then

$$o_1x_1 + o_2x_2 + \cdots + o_nx_n = 0.$$

   Hence, the set $\{x_1, x_2, \ldots, x_n\}$ is not linearly independent, so G is not a free $\mathbb{Z}$-module.

3. Consider the set $R[X]$. Observe that a suitable generating basis is

$$\{x^n \mid n \in \mathbb{N}\}$$

which is probably something you already knew.

4. Suppose M_1 and M_2 are free modules with bases S_1, S_2. Then the set $M_1 \oplus M_2$ is a free module, since it has a basis
$$\{(x,0) \mid x \in S_1\} \cup \{(0,y) \mid y \in S_2\}.$$
More generally, if $\{M_\alpha\}_{\alpha\in\lambda}$ is a of free modules where S_α is the basis of M_α, then we see that
$$\oplus_{\alpha\in\lambda} M_\alpha$$
is also a free module with basis
$$\bigcup_{\alpha\in\lambda} \{(\delta_{jk} s_{j\alpha}) \mid s_{j\alpha} \in S_j\}.$$
where δ_{jk} is the Kronecker delta function.

Proposition 3.6.3. Let M be a free R-module. Suppose the basis of the set is S. Let N be an R-module and $h : S \longrightarrow N$ a function. Then there exists a function $f \in \hom_R(M,N)$ such that $f\mid_S = h$.

Theorem 3.6.4. Let R be commmutative and M and N free modules with bases. Then $\hom_R(M,N)$ is a finitely generated free module.

Proof: Suppose the basis for M is $S = \{x_1, x_2, \ldots, x_n\}$, and the basis for N is $T = \{y_1, y_2, \ldots, y_m\}$. Define a set of functions for $1 \le i \le m$ and $1 \le j \le n$ such that
$$f_{ij}(x_k) = \begin{cases} y_j & \text{if } k = i \\ 0 & \text{if } k \neq j \end{cases}.$$
By the previous proposition, we know that each f_{ij} is a element in $\hom_R(M,N)$. Now let $f \in \hom_R(M,N)$ be arbitrary. Since T is a basis for N, we know that for each $v_k \in S$ there exists coefficients $a_{k1}, a_{k2}, \ldots, a_{kn}$ such that
$$f(v_k) = a_{k1}y_1 + \cdots + a_{kn}y_n.$$
However, observe that
$$\begin{aligned} f(v_k) &= a_{i1}y_1 + \cdots + a_{in}y_n \\ &= a_{k1}f_{k1}(x_k) + a_{k2}f_{k2}(x_k) + \cdots + a_{kn}f_{kn}(x_k). \end{aligned}$$

Therefore, we see that for any $b \in M$,

$$\begin{aligned} f(b) &= f(a_{b1}x_1 + \cdots + a_{bm}x_m) \\ &= a_{b1}f(x_1) + \cdots + a_{bm}f(x_m) \\ &= a_{b1}[a_{11}f_{11}(x_1) + a_{12}f_{12}(x_1) + \cdots + a_{1n}f_{1n}(x_1)] \\ &\quad + a_{b2}[a_{21}f_{21}(x_2) + a_{22}f_{22}(x_2) + \cdots + a_{2n}f_{2n}(x_2)] \\ &\quad + \cdots \\ &\quad + a_{bm}[a_{m1}f_{m1}(x_m) + a_{m2}f_{m2}(x_2) + \cdots + a_{mn}f_{mn}(x_m)]. \end{aligned}$$

Therefore we see that $\{f_{ij}\}$ generates $\hom_R(M,N)$, so that $\hom_R(M,N)$ is finitely generated. ■

The previous theorem doesn't hold if M and N are not finitely generated, since there are many counter examples to such a claim. Let $R = \mathbb{Z}$ and $M = \oplus_{i=1}^{\infty}\mathbb{Z}$. Then observe that

$$\hom_R(M,\mathbb{Z}) \cong \prod_{i=1}^{\infty}\mathbb{Z}.$$

by Theorem 1.13. However, we see that while $\mathbb{Z}$ is finitely generated and M is finitely generated, but $\prod_{i=1}^{\infty}\mathbb{Z}$ is not. (The proof is nontrivial.)

Proposition 3.6.5. Let M be a free R-module with basis $S = \{x_j\}_{j\in J}$ and suppose I is an ideal of R. Let $\pi : M \longrightarrow M/I$. Then M/IM is a R/I-module and is free with basis $\pi(S) = \{\pi(x_j)\}_{j\in J}$.

Proof:

***M/IM* is an *R/I*-module.** First recall that IM is a submodule of M. Therefore it makes sense to consider the quotient M/IM. Then we can define a mapping $\cdot : R/I \times M/IM \longrightarrow M/IM$ as follows. Let $r + I \in R/I$ and $m + IM \in M/IM$. Then define the mapping as

$$\begin{aligned} (r+I)\cdot(m+IM) &= r(m+IM) \\ &= rm + rIM \\ &= rm + IM. \end{aligned}$$

Since M is an R-module, $rm \in M$ so that $rm + IM$ is in fact in M/IM. The other module properties may be easily verified without difficulty by using this mapping.

***M/IM* is free.** Suppose $m + IM$ is an element of M/IM. Since $\pi : M \longrightarrow M/I$ is a surjective mapping, we see that there exists at least one $m \in M$ such that $\pi(m) = m + IM$. Now since m is free, there exists a unique representation of m of its basis elements, i.e., there exists $\{a_j\}_{j\in J}$, a subset of R, such that

$$m = \sum_{j\in J} a_j x_j \implies \pi(m) = \pi\left(\sum_{j\in J} a_j x_j\right) = \sum_{j\in J} a_j \pi(x_j) + IM$$

Hence $m + IM = \sum_{j\in J} a_j\pi(x_j) + IM$. To finish showing that $\{\pi(x_j)\}_{j\in J}$ is a basis for M/IM,

we only have to show that it is a linearly independent set. So consider the equation

$$\sum_{j\in J} a_j\pi(x_j) = 0 + IM$$

for some constants $\{a_j\}_{j\in J}$ in $\mathbb{R}$. Suppose additionally for contradiction that not all of the constants are nonzero. Then we that $\sum_{j\in J} a_j\pi(x_j)$ is an element of IM. However, this is a contradiction since none of the elements of $\{\pi(x_j)\}_{j\in J}$ is allowed to be in IM. Hence this set generates M/IM and is linearly independent, so it is a basis.

■

We can introduce an even more useful proposition regarding free modules, and more generally all modules.

Proposition 3.6.6. Let M be an R-module. Then

$$M \cong F/K$$

for a free module F and some submodule K of F. That is, M is the quotient of some free module F. Furthermore, if M is finitely generated, then such an F is finitely generated and $\mu(F) = \mu(M)$.

Proof: Suppose $S = \{x_j\}_{j\in J}$ is a set of elements which generate M. Note that, even in the worst case scenario, such an S exists since we can at most take $S = M$. Now suppose $F = \oplus j \in JR$, which is a free module. Construct the module homomorphism $\psi : F \longrightarrow M$ as

$$\psi((a_j)_{j\in J}) = \sum_{j\in J} a_j x_j.$$

Observe that since S generates M, such a homomorphism is surjective onto M. Hence, we see that M is the quotient of some free module F.

Now suppose that F is finitely generated. Then S is a finite set, so that F is also finitely generated (since in this case it is the direct sum of at most a finite number of copies of R).

Now if M is finitely generated, and is a quotient of F, then clearly $\mu(M) \leq \mu(F)$. However, we also know that $\mu(F) \leq |J| \leq \mu(M)$. Therefore, we see that $\mu(M) = \mu(F)$. ■

Definition 3.6.7. Let M be an R-module and F a free R-module. Then the short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

is called a **free presentation** of M. Note by the previous proposition that every R-module has a free presentation.

Presentations are particularly useful since they make free modules convenient to work with.

Proposition 3.6.8. Suppose F is a free R-module. Then every short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow F \longrightarrow 0$$

is a split exact sequence.

Proof: Let $S = \{x_j\}_{j \in J}$ be a basis for F. Now suppose $f : M \longrightarrow F$ is the surjective function in the above exact sequence. Now construct a function $\psi : F \longrightarrow M$ as follows: $\psi(x_j) = m_j$ if and only if $f(m_j) = x_j$. Since f is surjective, note that this will always be possible. Such a function may not be unique, but we don't care; we just want to know it exists.

By proposition 3.6.3, we know that there exists a unique function $h : F \longrightarrow M$ such that $h|_S = \psi$. Therefore we see that $f \circ h = 1_F$, so that by theorem 3.5.4, we see that the sequence is in fact split exact. ■