Chapter 13
Fundamental Group: Capturing Holes

Theorem 13.2 Given topological spaces X and Y with $S \subset X$, homotopy relative to S is an equivalence relation on the set of all the functions from X to Y. In particular, if $S = \emptyset$, homotopy is an equvialence relation on the set of all continuous functions from X to Y.

Proof: Let f and g be continuous functions from X to Y. Let us denote $f \simeq_S g$ to mean that f and g are homotopic relative to $S \subset X$.

- **Reflexive:** Observe that this relation is reflexive. If $H(x,t) = f(x)$ for all $t \in [0,1]$, then it is trivial that H forms a homotopy relative to S between f and itself.
- **Symmetric:** If $f \simeq_S g$, then there is a continuous function $H : X \times [0,1] \to Y$ such that

$$\begin{aligned} H(x,0) &= f(x) && \text{for all } x \in X \\ H(x,1) &= g(x) && \text{for all } x \in X \\ H(x,t) &= f(x) = g(x) && \text{for all } x \in S, t \in [0,1] \end{aligned}$$

then consider $H(x, 1-t)$ and observe that

$$\begin{aligned} H(x,1) &= f(x) && \text{for all } x \in X \\ H(x,0) &= g(x) && \text{for all } x \in X \\ H(x,1-t) &= f(x) = g(x) && \text{for all } x \in S, t \in [0,1] \end{aligned}$$

is a homotopy that deforms g into f. Thus $g \simeq_S f$, so the relation is symmetric.

- **Transitive:** Now suppose $f \simeq_S g$ and $g \simeq_S h$. Then there exist continuous functions $H : X \times [0,1] \to Y$ and $G : X \times [0,1] \to Y$ such that

$$\begin{aligned} H(x,0) &= f(x) && \text{for all } x \in X \\ H(x,1) &= g(x) && \text{for all } x \in X \\ H(x,t) &= f(x) = g(x) && \text{for all } x \in S, t \in [0,1] \end{aligned}$$

and

$$\begin{aligned} G(x,0) &= g(x) && \text{for all } x \in X \\ G(x,1) &= h(x) && \text{for all } x \in X \\ G(x,t) &= g(x) = h(x) && \text{for all } x \in S, t \in [0,1]. \end{aligned}$$

Now observe that we can construct the function

$$F = \begin{cases} H(x,2t) & 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} < t \le 1 \end{cases}$$

which will be a homotopy relative to S from f to h. Note that continuity here is guaranteed by application of the pasting lemma, since H and G are continuous on the same intervals. In total, we then have that $f \simeq_S g$ and $g \simeq_S h$ imply $f \simeq_S h$, as desired.

■

Theorem 13.3 If α, α', β and β' are paths in a space X such that $\alpha \sim \alpha'$, $\beta \sim \beta'$ and $\alpha(0) = \beta(1)$, then $\alpha \cdot \beta \sim \alpha' \cdot \beta'$.

Proof: Since $\alpha \sim \alpha'$ and $\beta \sim \beta'$, there exists homotopies A and B which connect α to α' and β to β'. Consider the continuous function

$$H(x,t) = \begin{cases} A(2x,t) & 0 \leq x \leq \frac{1}{2} \\ B(2x-1,t) & \frac{1}{2} < x \leq 1 \end{cases}$$

which is a homotopy from $\alpha \cdot \beta$ to $\alpha' \cdot \beta'$ since $H(x,0) = \alpha(x) \cdot \beta(x)$, $H(x,1) = \alpha'(x) \cdot \beta'(x)$, $H(0,t) = \alpha(0) = \alpha'(0)$ and $H(1,t) = \beta(0) = \beta'(0)$.

■

Theorem 13.4 Given paths α, β and γ where the following products are defined, then $(\alpha \cdot \beta) \cdot \gamma \sim (\beta \cdot \gamma) \cdot \alpha$ and $([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma])$

Proof: Consider the homotopy given by

$$H(x,t) = \begin{cases} \alpha\left(\dfrac{4x}{2-t}\right) & 0 \leq x \leq \dfrac{2-t}{4} \\ \beta(4x+t-2) & \dfrac{2-t}{4} \leq x \leq \dfrac{3-t}{4} \\ \\ \gamma\left(\dfrac{4x-3+t}{1+t}\right) & \dfrac{3-t}{4} \leq x \leq 1. \end{cases}$$

Observe that

$$H(x,0) = \begin{cases} \alpha(2x) & 0 \leq x \leq \frac{1}{2} \\ \beta(4x-2) & \frac{1}{2} \leq x \leq \frac{3}{4} \\ \gamma(4x-3) & \frac{3}{4} \leq x \leq 1 \end{cases} = \alpha \cdot (\beta \cdot \gamma)$$

and

$$H(x,1) = \begin{cases} \alpha(2x) & 0 \leq x \leq \frac{1}{4} \\ \beta(4x-2) & \frac{1}{4} \leq x \leq \frac{1}{2} \\ \gamma(4x-3) & \frac{1}{2} \leq x \leq 1 \end{cases} = (\alpha \cdot \beta) \cdot \gamma.$$

Thus we see that H is continuous by the pasting lemma, $H(x,0) = \alpha \cdot (\beta \cdot \gamma)$ and $H(x,1) = (\alpha \cdot \beta) \cdot \gamma$. In addition, we see that $H(0,t) = \alpha \cdot (\beta \cdot \gamma)(0) = (\alpha \cdot \beta) \cdot \gamma(0)$ and $H(1,t) = (\alpha \cdot \beta) \cdot \gamma(1) = \alpha \cdot (\beta \cdot \gamma)(1)$. Thus we have that $\alpha \cdot (\beta \cdot \gamma) \sim (\alpha \cdot \beta) \cdot \gamma$, which implies that

$$[\alpha \cdot (\beta \cdot \gamma)] = [(\alpha \cdot \beta) \cdot \gamma]$$

as desired.

■

Theorem 13.5 Let α be a path with $\alpha(0) = x_0$. Then $\alpha \cdot \alpha^{-1} \sim e_{x_0}$, where e_{x_0} is the constant path at x_0.

Proof: Consider the homotopy

$$H(x,t) = \begin{cases} \alpha(2x) & 0 \le x \le \frac{1-t}{2} \\ \alpha^{-1}(2x-1) & \frac{1-t}{2} \le x \le 1-t \\ e_{x_0} & 1-t \le x \le 1. \end{cases}$$

which traverses α, α^{-1}, and then sits at x_0. Observe that

$$H(x,0) = \begin{cases} \alpha(2x) & 0 \le x \le \frac{1}{2} \\ \alpha^{-1}(2x-1) & \frac{1}{2} \le x \le 1. \end{cases} = \alpha \cdot \alpha^{-1}$$

while

$$H(x,1) = e_{x_0} \quad 0 \le x \le 1.$$

In addition, we have that $\alpha \cdot \alpha^{-1}(x) = e_{x_0}(x)$ for $x = 0, 1$. Also, H is continuous by the pasting lemma. Thus we have that

$$\alpha \cdot \alpha^{-1} \sim e_{x_0}$$

as desired. The proof is nearly identitical to show that $\alpha^{-1} \cdot \alpha \sim e_{x_0}$.

■

Theorem 13.6 The fundamental group $\pi_1(X, x_0)$ is a group. The identity element is the class of homotopolically trivial loops based at x_0.

Proof:

Identity. With a group operation $\cdot$, we see that there is an identity element e_{x_0} such that $[\alpha] \cdot [\alpha^{-1}] = [\alpha^{-1}] \cdot [\alpha] = [e_{x_0}]$

Associativity. We have associativity of products by Theorem 13.4.

Inverse Elements. Inverse elements exist by simply defining $\alpha^{-1}(t) = \alpha(1-t)$. This will still be loop about x_0, and hence will continue to be a member of $\pi_1(X, x_0)$.

Closure. Finally, observe that the product is closed in the group, since any sequence of loops about x_0, their product

$$[\alpha_1] \cdot [\alpha_2] \cdot \ldots \cdot [\alpha_n] = [\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n]$$

will itself be a loop about x_0, and hence by definition., an element which is already in the set. Thus the fundamental group $\pi_1(X, x_0)$ is in fact a group.

■

Theorem 13.7 If X is path connected, then $\pi_1(X,p) \cong \pi_1(X,q)$ where $p, q \in X$.

Proof: We'll do this by constructing a bijective homomorphism. Since X is path connected, there must exist a path γ from p to q. Let α and β be loops centered at p, q respectively. Then observe that the function $\psi : \pi_1(X,p) \to \pi_1(X,q)$ defined as

$$\psi[\alpha] = \gamma^{-1}\alpha\gamma$$

is a homomorphism, since if $\alpha_1, \alpha_2 \in \pi_1(X,p)$,

$$\psi[\alpha_1\alpha_2] = \gamma^{-1}\alpha_1\alpha_2\gamma = \gamma^{-1}\alpha_1\gamma\gamma^{-1}\alpha_2\gamma = \psi[\alpha_1]\psi[\alpha_2].$$

We can similarly construct a homomorphism $\phi : \pi_1(X,q) \to \pi_1(X,p)$ as

$$\phi[\beta] = \gamma\beta\gamma^{-1}.$$

The proof is exactly the same as before: let $\beta_1, \beta_2 \in \pi_1(X,q)$. Then

$$\phi[\beta_1\beta_2] = \gamma\beta_1\beta_2\gamma^{-1} = \gamma\beta_1\gamma^{-1}\gamma\beta_2\gamma^{-1} = \phi[\beta_1]\phi[\beta_2].$$

Now observe that this homomorphism we constructed is in fact the inverse of ψ, since

$$\psi(\phi(\beta)) = \gamma^{-1}\phi(\beta)\gamma = \gamma^{-1}\gamma\beta\gamma^{-1}\gamma = \beta$$
$$\phi(\phi(\alpha)) = \gamma\phi(\alpha)\gamma^{-1} = \gamma\gamma^{-1}\alpha\gamma\gamma^{-1} = \alpha.$$

Therefore, ψ is a bijective homomorphism, which proves that $\pi_1(X,p) \cong \pi_1(X,q)$.

■

Corollay 13.8 Suppose X is a topological space and there is a path between the points p and q in X. Then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.

Proof: Observe that this result is immediate since the proof of Theorem 13.7 relied on the fact that there exists a path between p and q. Thus the proof can be used exactly the same to show that path connectedness between two points is sufficient to guarantee that $\pi_1(X,p) \simeq \pi_1(X,q)$.

■

Exercise 13.9 Let α be a loop into a topological space X. Then $\alpha = \beta \circ \omega|_{[0,1]}$ where ω is the standard wrapping map and β is some continuous function from $\mathbb{S}^1$ into X. This relationship gives a correspondence between loops in X and continuous maps from $\mathbb{S}$ into X.

Solution: Consider the function $\omega^{-1} : \mathbb{S} \to [0,1]$ where $\omega(0) = \omega(1)$. As this is a continuous function, we then see that $\alpha \circ \omega^{-1} : \mathbb{S} \to X$ is a continuous function that maps out the curve X. Define this to be β. Then observe that we can write this as

$$\beta = \alpha \circ \omega^{-1} \implies \alpha = \beta \circ \omega$$

so that α can be written as a continuous from from $\mathbb{S} \to X$ composed with a continuous function from $[0,1] \to \mathbb{S}$, as desired.

□

Theorem 13.10 Let X be a topological space and let p be a point in X. Then a loop $\alpha = \beta \circ \omega|_{[0,1]}$ (where ω is the standard wrapping map and β is a continuous function from $\mathbb{S}^1$ into X) is homotopically trivial if and only if β can be extended to a continuous function from the unit disk $\mathbb{D}^2$ to X.

Proof:

■

Theorem 13.11 Show the following (1 denotes the trivial group):

1. $\pi_1([0,1]) \cong 1$
2. $\pi_1(\mathbb{R}^n) \cong 1$ for $n \geq 1$
3. $\pi_1(X) \cong 1$, if X is a convex set in $\mathbb{R}^n$
4. $\pi_1(X) \cong 1$ if X is a cone.
5. $\pi_1(X) \cong 1$ if X is a star-like space in $\mathbb{R}^n$ (a subset of $\mathbb{R}^n$ is called star-like if there is a fixed point $x_0 \in X$ such that for any $x \in X$, the line segment between x_0 and x lies in X; a five pointed star is an example of a star-like space that is not convex.)

Proof:

1. Observe that for any loop $\alpha \in [0,1]$, we can write a homotopy between $x_0 = \alpha(0)$ as
$$H(x,t) = tx_0 + (1-t)\alpha(x)$$
2. Again, loop $\alpha(\mathbf{x})$ based at $\mathbf{x_0}$ can be reduced to the trivial loop via the homotopy
$$H(\mathbf{x},t) = t\mathbf{x_0} + (1-t)\alpha(\mathbf{x})$$
3. A convex set in $\mathbb{R}^n$ has the property that every straight line between any two points in the set is entirely contained within the set. Thus we can apply the straight line homotopy to any loop based at x_0 as in (1.) and (2.).
4. Observe that every point on the cone can be connected to the apex via a straight line. Thus we can connect every loop based at the apex to itself via a straight-line homotopy.
5. In a star shaped figure, we can connect every point to one another via straight lines which intersect the fixed point x_0 without leaving the figure. Thus we can apply the straight line homotopy here as well.

■

Exercise 13.12 Show the following:

1. $\pi_1(\mathbb{S}^0, 1) \cong 1$ where $\mathbb{S}^0$ is the zero-dimensional sphere $\{-1, 1\}$, the set of points unit distance from the origin in $\mathbb{R}^1$.
2. $\pi_1(\mathbb{S}^2) \cong 1$.
3. $\pi_1(\mathbb{S}^n) \cong 1$ for $n \geq 3$.

Solution:

1. In $\pi_1(\mathbb{S}^0, 1)$, the only element is the identity itself. Thus this group is literally trivial.
2. Consider a path γ in $\pi_1(\mathbb{S}^2)$, and suppose that γ is not a space filling curve. Then γ will miss at least a single point. Thus we can stereographically project γ on the sphere onto the $\mathbb{R}^2$ plane via a homeomorphism h.

   However, we know that any loop in $\mathbb{R}^2$ is homotopically trivial. Therefore there exists an a homotopy H from $h(\gamma)$ to the trivial loop. Now note that $h^{-1} \circ H$ will be a homeomorphism of γ to the trivial loop on $\mathbb{S}^2$. Thus $\pi_1(\mathbb{S}^2) = 1$.

   Now suppose γ is a space filling curve. Observe that via the Lebesgue number theorem, that this curve must enter and exit a finite number of times. Thus we can shift the curve over a particular point p in the open set, and do this a finite number of times. We can then stereographically project as before to shrink the curve on the surface to a point.

   Thus in either case, we see that any loop on $\mathbb{S}^2$ can be contracted to a single point via stereographic projection and the face that $\pi_1(\mathbb{R}^2) \cong 1$. Therefore we see that $\pi_1(\mathbb{S}^2) \cong 1$ as desired.

□

Exercise 13.13 Show that the cone over the Hawaiian earring is simply connected. Can you generalize your insight?

Solution: First observe that this space is path connected, since each ring of the Hawaiian earing are connected to the single point on the base of the cone and to the apex of the cone.

Now consider the unique point p on the base of the cone for which all rings intersect. Suppose α is a loop based at this point. With Theorem 10.25, we can deduce that α cannot traverse infinitely many rings in the Hawaiian earing. α is continuous and $[0, 1]$ is a compact interval and therefore it cannot be mapped into an infinitely long path, as this image would no longer be compact.

Thus any loop α based at p traverses a finite number of rings. Therefore, we can construct a homotopy H which lifts α over the apex of the cone and towards the point p itself, via a straight line homotopy (which we can do via the definition of the cone). Note that this will always be possible since there will only ever be a finite number of rings to lift over the apex.

Thus we have that $\pi(X, p) \cong 1$, but since this space is path connected we have that $\pi_1(X) \cong 1$. Therefore it is simply connected.

□

Theorem 13.14

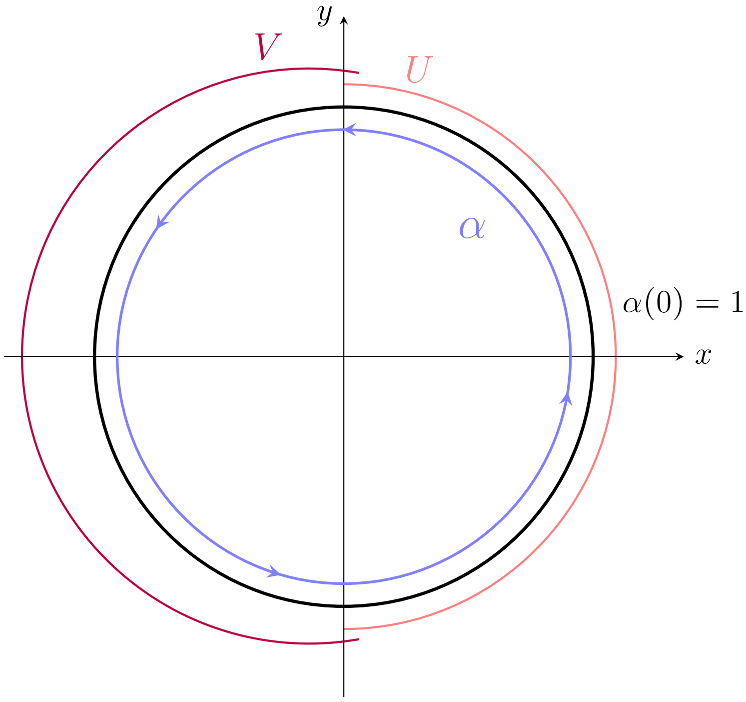
1. Any loop $\alpha : [0, 1] \to \mathbb{S}^1$ with $\alpha(0) = 1$ can be written $\alpha = \omega \circ \tilde{\alpha}$ where $\tilde{\alpha} : [0, 1] \to \mathbb{R}^1$ satisfies $\tilde{\alpha}(0) = 0$ and ω is the standard wrapping map.
2. If $\alpha : [0, 1] \to \mathbb{S}^1$ is a loop, then $\tilde{\alpha}(1)$ is an integer.
3. Loops α_1 and α_2 are equivalent in $\mathbb{S}^1$ if and only if $\tilde{\alpha_1}(1) = \tilde{\alpha_2}(1)$.
4. $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Proof:

1. First observe that we can cover $\mathbb{S}^1$ with two open sets U and V, as demonstrated in the figure below. Since $[0,1]$ is a compact interval, and $\alpha[0,1] \to \mathbb{S}^1$, we know by Theorem 10.25 that we can divide $[0,1]$ into N intervals such that

$$\alpha\left(\left[\frac{i-1}{N}, \frac{i}{N}\right]\right)$$

lies in U or V.



Since U is not all of $\mathbb{S}^1$, we know that $\omega^{-1}(U)$ exists. If we define $\omega(0) = 1$ (i.e., if we specify that our rotation starts at 1) then $\omega^{-1}(U)$ will correspond to a union of open sets around every integer in $\mathbb{R}$. Also, $\omega^{-1}(V)$ will correspond to a union of intervals, each of which do not intersect any member of $\mathbb{Z}$.

Now since $\alpha\left(\left[\frac{i-1}{N}, \frac{i}{N}\right]\right) \subset U$ or V, we can map this image to $\mathbb{R}$ via ω^{-1}, starting from the first interval $\left[0, \frac{1}{N}\right]$ which is mapped to a neighborhood of 0 in $\mathbb{R}$. Thus we can define a function $\tilde{\alpha}$ as

$$\tilde{\alpha} = \omega^{-1} \circ \alpha$$

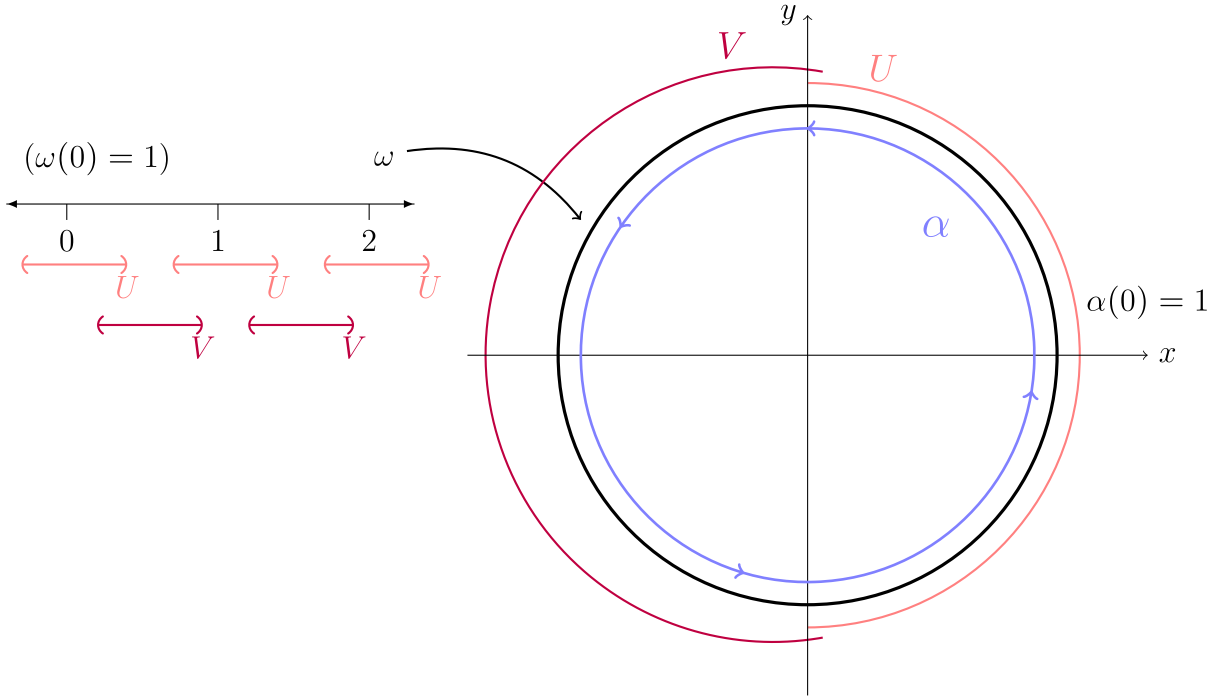
if we specify that $w(0) = 1$ and $\tilde{\alpha} : [0,1] \to \mathbb{R}$ by construction. Therefore, we can write

$$\alpha = \omega \circ \tilde{\alpha}$$

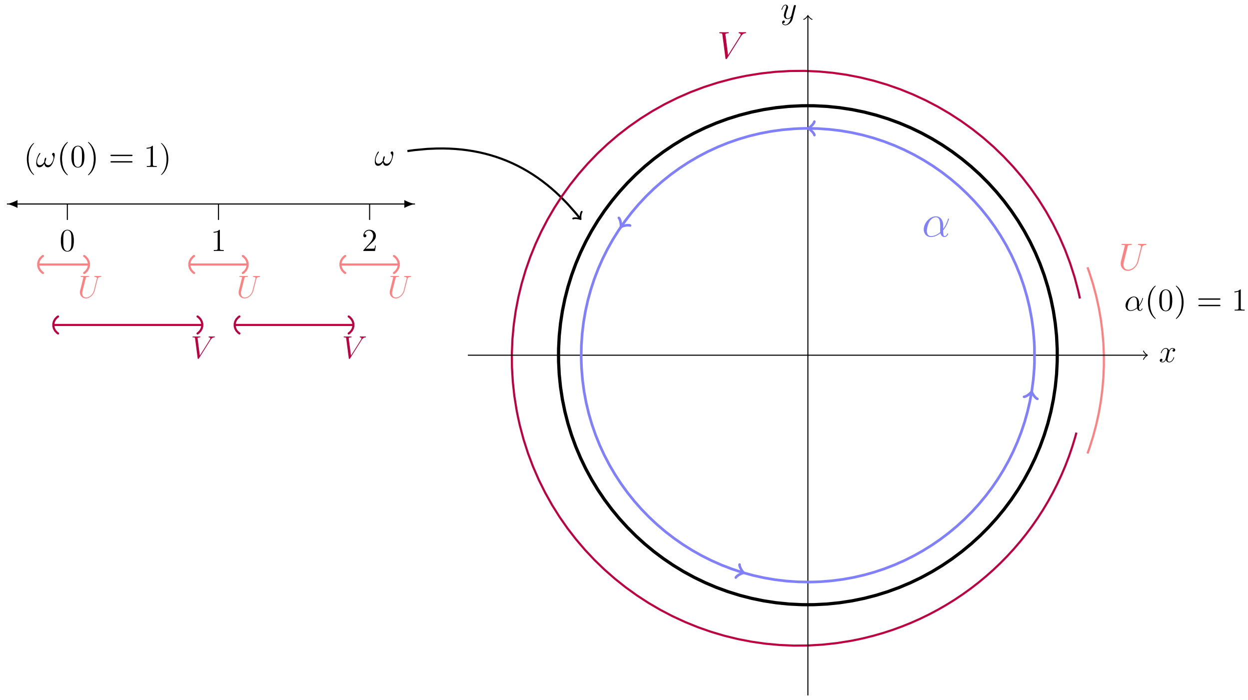
where $\alpha(0) = 0$.

2. Since $\alpha(1) = \alpha(0)$, we see that α must return to U at some point. And since there we can subdivide $[0,1]$ into finite intervals to keep track of the mapping, there will always be at most a finite number of rotations made around $\mathbb{S}$.

Now let us shrink U containing 1 in $\mathbb{S}$. As we shrink around 1, $\omega^{-1}(U)$ will still be a union of nieghborhoods of integers in $\mathbb{R}$. And since we are to free to shrink U, we see that the value of $al\tilde{p}ha$ must be an integer.



If $\tilde{\alpha}(1)$ is not an integer, then we can shrink U in $\mathbb{R}$ past this non-integer value. However, this implies that α is not a closed curve in $\mathbb{S}^1$ since $\alpha(1) \neq \alpha(0)$. Hence, $\omega^{-1} \circ \alpha = \tilde{\alpha}$ takes on integer values.



3. Suppose $\tilde{\alpha}_1(1) = \tilde{\alpha}_2(1)$. Note that $\tilde{\alpha_1}(0) = \alpha_2\tilde{(}0)$, and as these are paths in $\mathbb{R}$ we can construct a stright line homotopy between the two paths relative to $\{0, 1\}$. Thus $\tilde{\alpha_1} \sim \tilde{\alpha_2}$, and there exists a homotopy H from $\tilde{\alpha_1}$ to $\tilde{\alpha_2}$.

   Since ω is continuous, $\omega \circ H$ is a continuous and a homotopy between α_1 and α_2, since (1) $(\omega \circ H)(0, t) = \omega \circ \tilde{\alpha_1} = \alpha_1$ and (2) $(\omega \circ H)(1, t) = \omega \circ \tilde{\alpha_2} = \alpha_2$, and the homotopy retains the endpoints.

4. Now consider the function $\gamma : \mathbb{S} \to \mathbb{Z}$ given by

$$\phi(\gamma) = \tilde{\gamma}(1).$$

Let $\gamma_1, \gamma_2 \in \pi_1(\mathbb{S}^1)$. Then observe that $\gamma_1 \cdot \gamma_2$ will be a path which comples $\tilde{\gamma}_1$ rotations, followed by $\tilde{\gamma}(2)$ rotations. Therefore

$$\phi(\gamma_1 \cdot \gamma_2) = \tilde{\gamma_1}(1) + \tilde{\gamma_2}(1) = \phi(\gamma_1) + \phi(\gamma_2).$$

Thus ϕ is a homomorphism. Now observe that part (3) of this problem proves injectivity, while surjectivity comes from the fact that for any $n \in \mathbb{Z}$, we can create a loop α such that α completes n rotations in $\mathbb{S}$, giving that $\tilde{\alpha}(1) = n$. Therefore ϕ is bijective and hence an isomorphism, so that $\pi_1(\mathbb{S}) \cong \pi_1(\mathbb{Z})$.

■

Theorem 13.15 Let $(X, x_0), (Y, y_0)$ be path connected spaces. Then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

via the canonical map that takes a loop γ in $X \times Y$ to $(p \circ \gamma, q \circ \gamma)$ where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are projection maps.

Proof: Observe that the map $(p \circ \gamma, q \circ \gamma)$ as defined above is a homomorphism. To show this, let $\gamma \in \pi_(X \times Y, (x_0, y_0))$. Then

$$\phi(\gamma) = (p \circ \gamma, q \circ \gamma) = (\gamma_x, \gamma_y) = (\gamma_x, e_{y_0}) \cdot (e_{x_0}, \gamma_y) = \phi(\gamma_x) \cdot \phi(\gamma_y)$$

where γ_x is a loop in X based at x_0, γ_y a loop in Y based at y_0. Observe that this is bijective since every loop in $\pi_1(X \times Y, (x_0, y_0))$ is mapped to a loop in $\pi_1(X, x_0) \times \pi_1(Y, y_0)$, and every loop in $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ can be written as a loop in $\pi_1(X \times Y, (x_0, y_0))$. Therefore this is a isomorphism and thus $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

■

Exercise 13.16 Find:

1. $\pi_1(X)$ where X is a solid torus.
2. $\pi_1(\mathbb{S}^2 \times \mathbb{S})$
3. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2)$
4. $\pi_1(X)$, where X is a direct product of k_n copies of $\mathbb{S}^n$, with $k_n = 0$ for n sufficiently large.

Exercise 13.18 Check that for a continuous function $f : X \to Y$, the induced homomorphism f_* is well-defined (that is, the image of an equivalence class is independent of the chosen representative.) Show that it is indeed a group homomorphism.

Solution: Observe that since $\alpha \sim \beta$, there exists a homotopy H between the two paths that fixes the endpoints. Then observe that $f \circ H$ is (1) a continuous function and (2) a homotopy from $f(\alpha)$ to $f(\beta)$. Thus $f(\alpha) \sim f(\beta)$. Therefore, if $\alpha \sim \beta$ then $f(\alpha) \sim f(\beta) \implies [f \circ \alpha] = [f \circ \beta] \implies f_*([\alpha]) = f_*([\beta])$ so that our definition is well defined.

To show it is a group homomorphism, observe that

$$f_*([\alpha \cdot \beta]) = [f \circ (\alpha \cdot \beta)] = [f \circ \alpha \cdot f \circ \beta] = [f \circ \alpha] \cdot [f \circ \beta] = f_*([\alpha]) \cdot f_*([\beta]).$$

Thus we see that this forms a group homomorphism.

□

Theorem 13.19 The following are true:

1. If $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ are continuous maps, then $(g \circ f)_* = g_* f_*$.
2. If $\mathrm{id} : (X, x_0) \to (Y, y_0)$ is the identity map, then $\mathrm{id}_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity homomorphism.

Proof:

1. Let $[\alpha] \in \pi_1(X, x_0)$. Then

   $$(g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha] = [g \circ (f \circ \alpha)] = g_*([f_*([\alpha])]) = g_* \circ f_*([\alpha])$$

   so that $(g \circ f)_* = g_* f_*$.

2. Let $[\alpha] \in \pi_1(X, x_0)$. Then

   $$id_*([\alpha]) = [id \circ \alpha] = [\alpha].$$

   Since this is a homomorphism on the group which sends every group element to itself, we have that this is an identity homomorphism.

■

Theorem 13.20 If $h : X \to Y$ is a homeomorphism then

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is a group isomorphism. Thus homeomorphic, path-connected spaces have isomorphic fundamental groups.

Proof: If $h : X \to Y$ is a homeomorphism, then h is continuous and bijective. Therefore, if $\alpha \in \pi_1(X, x_0)$, then $h_* = [h \circ \alpha]$. Since h^{-1} exists and is continuous, we then now if $\beta \in \pi_1(Y, y_0)$ then $h_*^{-1} = [h^{-1} \circ \alpha]$ is also a homomorphism. Now observe that

$$h_*[h_*^{-1}[\beta]] = [h \circ [h^{-1} \circ \beta]] = [\beta]$$
$$h_*^{-1}[h_*[\alpha]] = [h^{-1} \circ [h \circ \alpha]] = [\alpha].$$

Therefore, we see that h_*^{-1} is the inverse homomorphism of h_*, which implies that the homomorphism is bijective. Therefore, the two groups are isomorphic.

■

Theorem 13.22 If $f, g : (X, x_0) \to (Y, y_0)$ are continuous functions and f is homotopic to g relative to x_0, then $f_* = g_*$.

Proof: Since $f \simeq g$, there exists a homotopy H such that $H(x, 0) = f(x), H(x, 1) = g(x)$ and $H(x_0, t) = f(x_0) = g(x_0)$ for all $t \in [0, 1]$. Let $\alpha \in \pi_1(X, x_0)$. Then observe that $H(\alpha(x), t)$ is (1) continuous and (2) a homotopy from $f \circ \alpha$ to $g \circ \alpha$. Therefore

$$f \circ \alpha \simeq g \circ \alpha \implies [f \circ \alpha] = [g \circ \alpha] \implies f_* = g_*.$$

■

Lemma 13.23 Homotopy equivalence of spaces is an equivalence relation.

Proof: We can show that this satisfies the axioms for an equivalence relation.

Reflexive. Observe that if we let $f = g = id_X$, then $g \circ f = id_X$ and $f \circ g = id_X$. Thus a topological space X is homotopy equivalent to itself.

Symmetric. The defintion of homotopy equivalence makes this obvious. Suppose X is homotopy equivalent to Y. By definition, there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f \simeq id_X \qquad f \circ g \simeq id_Y.$$

Thus Y is homotopy equvalent to X since there exists continuous functions $g : Y \to X$ and $f : X \to Y$ such that

$$f \circ g \simeq id_Y \qquad g \circ f \simeq id_X.$$

Transitive. Suppose X is homotopy equivalent to Y which is homotopy equivalent to Z. By definition, there exist continuous functions $f_1 : X \to Y$, $f_2 : Y \to X$, $g_1 : Y \to Z$, $g_2 : Z \to Y$ such that

$$f_1 \circ f_2 \simeq id_X \qquad f_2 \circ f_1 \simeq id_Y \text{ and } g_1 \circ g_2 \simeq id_Y \qquad g_2 \circ g_1 \simeq id_Z$$

■

Theorem 13.24 If $f : X \to Y$ is a homotopy equivalence and $y_0 = f(x_0)$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. In particular, if $X \sim Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof: Since f is continuous from X to Y, we already have a homomorphism from $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, given by f_*. Now consider the induced homomorphism g_* where

$$g \circ f \simeq id_X \quad \text{and} \quad f \circ g \simeq id_Y.$$

Then observe that if $\alpha \in \pi_1(X, x_0)$ then

$$g_* \circ f_*[\alpha] = (g_* \circ f_*)[\alpha] = [g \circ f \circ \alpha].$$

Now since $g \circ f \simeq id_X$, we know that there exists a homotopy H such that $H(x, 0) = g \circ f$ and $H(x, 1) = id_X$ while $H(x_0, t) = g \circ f(x_0) = id_X(x_0) = x_0$. Then observe that $H(\alpha(x), t)$ is a homotopy from $g \circ f \circ \alpha$ to α. As these two paths are homotopic, their equivalence classes should be the same. Therefore, we see that

$$[g \circ f \circ \alpha] = [\alpha].$$

Now if $\beta \in \pi_1(Y, y_0)$

$$f_* \circ g_*[\beta] = (f_* \circ g_*)[\beta] = [f \circ g \circ \beta] = [\beta]$$

By the same argument. Therefore, g_* is an inverse homomorphism of f_*, so that f_* is ultimately an isomorphism between the two groups. Thus $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

■

Exercise 13.25 Show that for $n \geq 0$, $\mathbb{R}^{n+1} - \{0\}$ can be strong deformation retracted onto $\mathbb{S}^n$

Solution: Consider the homotopy $R : \mathbb{R}^{n+1} \to \mathbb{S}^n$ such that

$$R(\mathbf{x}, 0) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^{n+1}$$
$$R(x, 1) = r(x) \quad \text{for all } \mathbf{x} \in \mathbb{R}$$
$$R(a, t) = a \quad \text{for all } a \in \mathbb{S}^n$$

where

$$r(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||}.$$

Thus we see that $\mathbb{R}^{n+1} - \{0\}$ can be strong deformation retracted onto $\mathbb{S}^n$.

□

Lemma 13.26 If A is strong deformation retract of X, then A and X are homotopy equivalent.

Proof: Since A is a strong deformation retract of X, we know that there exists a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Consider also the inclusion map $i : A \to X$. Observe that

$$i \circ r : X \to X \quad r \circ i : A \to A$$

and

$$r \circ i = id_A \quad i \circ r \simeq id_X.$$

Thus by defintion we see tha $A \sim X$.

■

Theorem 13.27 $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, for any $n \neq 2$.

Proof: Suppose for contradiction that $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^n$. Then if we poke a hole in $\mathbb{R}^2$, we can embed a circle in the space such that its interior contains the hole. We can then strong deformation retract $\mathbb{R}^2$ onto the boundary of the disk. We then see the fundamental group is $\mathbb{Z}$ by Theorem 13.24.

However, we know that $\mathbb{R}^n$, $n \geq 2$ with one hole missing is still a trivial group. In $\mathbb{R}^n$, we can move around the circle we embdedded in $\mathbb{R}^2$ to still compute a trivial fundamental group. But this is a contradiction, since these two spaces are said to be homeomorphic but their fundamental groups are inconsistent under change. Therefore, $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n \geq 2$.

Also observe that $\mathbb{R}^2$ is not homemorphic to $\mathbb{R}$. This is because $\pi_1(\mathbb{R}^2) \cong \mathbb{R} \times \mathbb{R}$. Also, $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^0$, as this is a single point. If we delete this point the fundamental group is empty, while the fundamental group of $\mathbb{R}^2$ would become equivalent to $\mathbb{Z}$, and hence these two spaces are not homemorphic. Therefore, we see that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n \neq 2$.

■

Exercise 13.28 Let x and y be two points in $\mathbb{R}^2$. Show that $\mathbb{R}^2 - \{x, y\}$ strong deformation retracts onto the figure eight. In addition, show that $\mathbb{R}^2$. Show that $\mathbb{R}^2 - \{x, y\}$ strong deformation retracts onto a theta space.

Solution: Observe that the figure eight and the theta space both have two holes inside of them. If we configure both of these holds to individually contain x and y, then we can retract $\mathbb{R}$ ont the boundaries of the figure eight and theta space.

Note we would not be able to do this without first poking two holes in $\mathbb{R}$, since we would otherwise not be able to retract the interior of the each hole in the figure eight or theta space to its boundaries (for the same reason we can't retract $\mathbb{D}^2$ to its boundary).

□

Theorem 13.29 If $r : X \to A$ is a strong deformation retraction and $a \in A$, then $\pi_1(X, a) \cong \pi_1(A, a)$.

Proof: Suppose $r : X \to A$ is a strong deformation retraction. Then by Lemma 13.26, we know that A and X are homotopy equivalent. Moreover, by Theorem 13.24, we have that $\pi_1(X, a) \cong \pi_1(A, a)$ for $a \in A$.

■

Exercise 13.30 Calculate the fundamental group of the following spaces.

1. An annulus.
2. A cylinder.
3. The Möbius Band.
4. An open 3-ball with a diameter removed.

Solution:

1. Suppose that we embed an annulus at the origin of the complex plane. Then it is given by $\{z : R_1 < |z| < R_2\}$ where $R_1 < R_2$. It should be fairly obvious that we can construct a strong deformation retract to the set of points $\{z : |z| = R_1\}$; that is, to the inner circle of the annulus. As the fundamental group of the circle is $\mathbb{Z}$, we can thus conclude that the fundamental group of the annulus is also $\mathbb{Z}$ by Theorem 13.29.

2. For a cylinder, we can embedd such a structure in $\mathbb{R}^3$, which in this space we can construct a strong deformation retract between the set of points on the cylinder to one of the two disks which define the top and the bottom of the cylinder. Since the fundamental group of a disk is trivial, we see that the fundamental group of the cylinder must also be trivial by Theorem 13.29.

3. Observe that if we take a Möbius band and strong deformation retract the set of points to one of its boundaries, we'll get a closed curve, which we can then form a strong deformation retraction to a circle. Since the fundamental group of a circle is $\mathbb{Z}$, we see that the fundamental group of the Möbius band is also $\mathbb{Z}$.

   However, if the cylinder does not have a filled top (i.e., the cylinder is just a piece of paper folded on its ends) then the group is trivial

4. Observe that we can strong deformation retract an open 3-ball with a diameter removed to a circle with a hole removed from its center. Since this has a fundamental group of $\mathbb{Z}$, we see that by Theorem 13.27 that the open 3-ball with diameter removed also has a fundamental group of $\mathbb{Z}$.

□

Theorem 13.32 Let A be a retract of X via the inclusion $i : A \hookrightarrow X$ and retraction $r : X \hookrightarrow A$. Then for $a \in A$, $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective and $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.

Proof: First note that for any $\alpha \in \pi_1(A, a)$ we have that

$$r_* \circ i_*([\alpha]) = [r \circ i \circ \alpha] = [\alpha].$$

Therefore, we see that $r_* \circ i_*([\alpha]) = id_*$ is the identity homomorphism on $\pi_1(A, a)$. Now suppose that i_* is not injective. Then we'll have that $r_* \circ i_* \neq id_*$, which is a contradiction. Furthermore, if r_* is not surjective then $r_* \circ i_* \neq id_*$. Thus we see that i_* is injective and r_* is surjective.

■

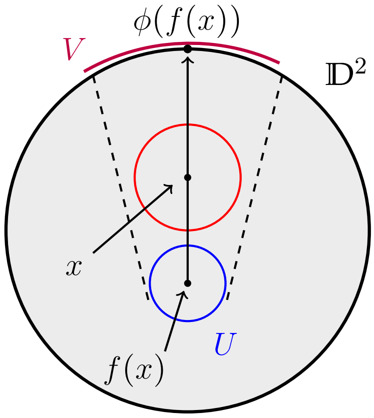
Theorem 13.33 (No retraction theorem for $\mathbb{D}^2$.) There is no retraction from $\mathbb{D}^2$ to its boundary.

Proof: Suppose for a contradiction that there exists a retraction $r : \mathbb{D}^2 \to \mathbb{S}^1$. Then the inclusion map $i_* : \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{D}^2)$ should be injective, this is impossible since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ while $\pi_1(\mathbb{D}^2)$ is trivial. Thus there is no such r.

■

Theorem 13.34 (Brouwer Fixed Point Theorem for $\mathbb{D}^2$.) Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be a continuous map. Then there is some $x \in \mathbb{D}^2$ for which $f(x) = x$.

Proof: Suppose for a contradiction that there exists a continuous function $f : \mathbb{D}^2 \to \mathbb{D}^2$ such that $f(x) \neq x$ for all $x \in \mathbb{D}^2$. Consider the retraction $\phi(x) = x$ if $x \in \mathbb{S}^1$ and $\phi(x) = f(x)|_{\text{proj}}$ where $f(x)_{\text{proj}}$ is the projection of $f(x)$ to its boundary through the straightline between $f(x)$ and x.



Observe that this function is continuous, since for any open set containing $V \subset \mathbb{S}^1$ containing $\phi(f(x))$ there exists an open set $U \in \mathbb{D}^2$ containing $f(x)$ such that $\phi(U) \subset V$. See the figure.

Note that what we have is a continuous retraction of $\mathbb{D}^2$ to its boundary, $\mathbb{R}$. However, we know from Theorem 13.33 that this is a contradiction. Thus there cannot be any such f, so that for a continuous $f : \mathbb{D}^2 \to \mathbb{D}^2$ there must exist an $x \in \mathbb{D}^2$ such that $f(x) = x$, as desired.

■

Theorem 13.40 Let $X = U \cup V$, where U and V are open and path connected and $U \cap V$ is path-connected, simply connected and nonempty. Then $\pi_1(X) \cong \pi_1(U) \star \pi_1(V)$.

Proof: Let $x_0 \in U \cap V$ and let Γ be a loop based at x_0. Then observe that $\Gamma : [0, 1] \to X$ is a path from a compact interval, and U and V form an open cover of X. By Theorem 10.25 we may divide the interval into finite subintervals such that the image of each interval lies in U or V.

We now claim that we can write any loop Γ as product of loops in $\pi_1(U)$ and $\pi_1(V)$. Since $U \cap V$ is path connected, we know that for every every time our path intersects $U \cap V$ there exists a point $p \in U \cap V$ such that we can glue a new path γ from p to x_0. This path will either lie entirely in U or in V, thus becoming a member of $\pi_1(U)$ and $\pi_1(V)$. And by Theorem 10.25, this can be done a finite number of times.

Now consider the function
$$\phi(\Gamma) = \alpha_1\beta_1 \cdots \alpha_n\beta_n$$
where $\alpha_1\beta_1 \cdots \alpha_n\beta_n$ is the decomposition of Γ and $\alpha_i \in \pi_1(U)$ while $\beta_i \in \pi_i(V)$ for $i = 1, 2, \ldots, n$. We'll now show this is a homomorphism. For any two paths Γ_1, Γ_2, each have some decomposition $\alpha_1^{(1)}\beta_1^{(1)} \cdots \alpha_n^{(1)}\beta_n^{(1)}$ and $\Gamma_2 = \alpha_1^{(2)}\beta_1^{(2)} \cdots \alpha_n^{(2)}\beta_n^{(2)}$. Therefore, we see that
$$\Gamma_1 \cdot \Gamma_2 = \alpha_1^{(1)}\beta_1^{(1)} \cdots \alpha_n^{(1)}\beta_n^{(1)}\alpha_1^{(2)}\beta_1^{(2)} \cdots \alpha_n^{(2)}\beta_n^{(2)}$$

so that

$$\phi(\Gamma_1 \cdot \Gamma_2) = \alpha_1^{(1)}\beta_1^{(1)} \cdots \alpha_n^{(1)}\beta_n^{(1)}\alpha_1^{(2)}\beta_1^{(2)} \cdots \alpha_n^{(2)}\beta_n^{(2)} = \phi(\Gamma_1)\phi(\Gamma_2).$$

Thus observe that for any $\alpha_1\beta_1 \cdots \alpha_n\beta_n \in \pi_1(X)$, this corresponds to some unique path $\Gamma \in pi_1(X)$ such that

$$\phi(\Gamma) = \alpha_1\beta_1 \cdots \alpha_n\beta_n$$

Thus we get surjectivity for free, since any member of $\pi_1(U) \star \pi_1(V)$ corresponds to some path $\Gamma \in \pi_1(X)$; therefore, the image of Γ under ϕ is then the element of $\pi_1(U) \star \pi_1(V)$ we began with. However, this also lends uniqueness, since every member of $\pi_1(U) \star \pi_1(V)$ corresponds uniquely to some path of $\Gamma \in \pi_1(X)$. Therefore we see that this is a isomorphism, so that $\pi_1(X) \cong \pi_1(U) * \pi_1(V)$.

■

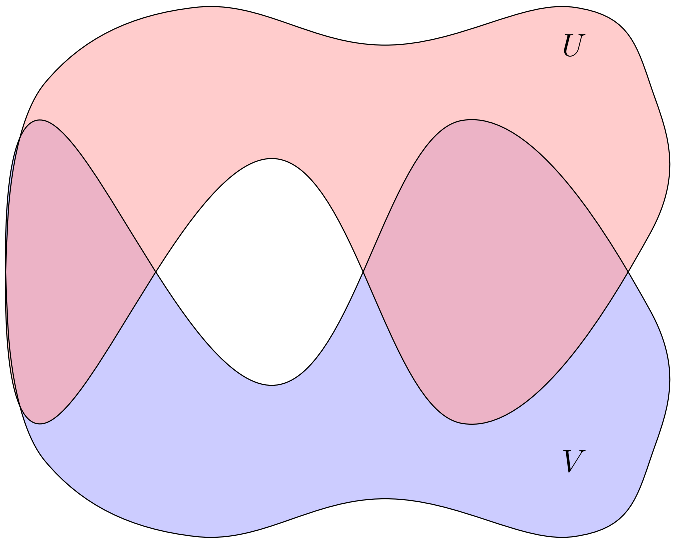
Exercise 13.41 Let X be the bouqeuet of n circles. What is $\pi_1(X)$?

Solution: The bouquet of n circles simply identifies a point on a set of n circles to the same point. Thus we see by repeated application of Theorem 13.30, $\pi_1(X) \cong \pi_1(\mathbb{S}) * \pi_1(\mathbb{S}) * \cdots * \pi_1(\mathbb{S}) = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$. That is, the free product of n groups of $\mathbb{Z}$.

□

Exercise 13.32 Find a path-connected space X with open, path connected subsets U and V of X such that $X = U \cup V$ where U and V are both simply connected, but X is not simply connected. Conclude that the hypothesis that $U \cap V$ is path connected is necessary.

Solution: Consider the sets



where we see that $U \cap V$ is not path connected. In this example we see that the consequence of this is that the union of the sets $U \cup V$ is no longer simply connected, and hence it has a nontrivial fundamental group. However, if we were to ignore the condition that $U \cap V$ be path connected, then Van Kampen's theorem in this case would otherwise guarantee that its fundamental group should be the free product of two trivial groups, and hence be a trivial group itself. Thus path connectedness of $U \cap V$ is a necessary condition for Van Kampen's theorem to be true.

□

Theorem 13.44 Let X be a wedge of two cones over two Hawaiian earrings where they are identified at the points of tangency of the circles of each Hawaiian earring, as in Figure 13.9. Then $\pi_1(X) \ncong 1$.

Proof: Suppose α_n is a loop on the n-th ring on the left Hawaiian earring, while β_n is a loop on the n-th ring on the right Hawaiian earing. Thus consider the path

$$\gamma = \alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_k\beta_k$$

where $k \in \mathbb{N}$. Observe that $\gamma \in \pi_1(X)$ (more specifically, its equivalence class is a member). However, if we attempt to lift this path towards the apex of the cone, which we can individually do without any issue for $\alpha_1\alpha_2\cdots\alpha_k$ and $\beta_1\beta_2\cdots\beta_k$, we run into an issue as $k \to \infty$. This is because $[0,1]$ is a compact interval and cannot be mapped into an infinite path, so that if we attempted to form any homotopy it would automatically fail to be compact and hence continuous if we try to lift the path γ up simultaneously. Thus this path is not homotopic to a point, so that $\pi_1(X) \neq 1$.

■