# Chapter 6

# Countable Features of Spaces: Size Restrictions

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**Exercise 6.1** Show that $A$ is dense in $X$ if and only if every non-empty open set of $X$ contains a point of $A$.

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*Solution:* First we prove the forward direction. Suppose that $A$ is a dense subset in $X$. Then by definition, $\overline{A} = X$. Thus every point of $X$ is a limit point of $A$, which means that for every point $p \in X$ and every open set $U$ which contains $p$ we see that

$$(U - \{p\}) \cap A \neq \emptyset.$$

Since this holds for all $p \in X$, we see that every open set in $X$ must contain points in $A$, which proves this direction.

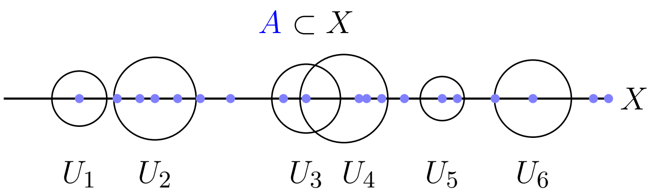


Figure 1:

Here we see the set $A$ is a dense subset in $X$. The sets $U_1, \ldots U_6$ denote arbitrary open sets in $X$.

Now suppose that every nonempty open set of $X$ contains a point of $A$. Then this means that for any $p \in X$, any open set $U$ containing $p$ must also contain a point in $A$. By definition, this is a limit point. Since $p$ was an arbitrary point of $x$, we must have that every element of $X$ is a limit point of $A$. Therefore, we must have that $\overline{A} = X$, which finishes the proof in this direction.

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**Exercise 6.2** Show that $\mathbb{R}_{\text{std}}$ is separable. With which of the topologies on $\mathbb{R}$ that you have studied is $\mathbb{R}$ not separable?

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*Solution:* Observe that a countable dense subset in $\mathbb{R}_{\text{std}}$ is the set of rationals. This is because every nonempty open set of $\mathbb{R}$ on the standard topology contains points of $\mathbb{Q}$. By our previous exercise, this allows us to conclude that $\mathbb{Q}$ is dense in $\mathbb{R}$. Since the rationals

are countable, this in total allows us to conclude that $\mathbb{R}_{\text{std}}$ has a countable dense subset, and is therefore separable by definition.
However, this wouldn't hold for the discrete topology on $\mathbb{R}$, since it does not have a countable dense subset with this topology. The countable complement is also not separable, since every open set in the topology must be uncountable and hence finding a countable but dense subset of $X$ is impossible.

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**Exercise 6.4** Find a separable space that contains a subspace that is not separable in the subspace topology.

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*Solution: Lemma.* An uncountable set with the discrete topology is not separable.

*Proof.* For the sake of contradiction suppose that $X$ is uncountable and is separable under the discrete topology. Then there exists a countable dense set $A$ such that $\overline{A} = X$. However, since $X$ has the discrete topology, we know that $A = \overline{A} = X$; a contradiction since $A$ is countable while $X$ is uncountable. Thus $X$ is not separable under the discrete topology.

Now consider a topology on an uncountable set $X$ given by

$$\mathcal{T} = \{\emptyset\} \cup \{U \subset X\}_{p \in X}$$

where $p \in X$. Observe that $\{p\}$ is dense in this set since every open set in $\mathcal{T}$ contains $\{p\}$ by construction. Since $\{p\}$ is countable and dense, $X$ is separable on this topology.

Consider the subspace $X - \{p\}$. For any $U \subset (X - \{p\})$, we see that $U \cup \{p\} \subset \mathcal{T}$, so that $U$ is open in $X - \{p\}$. Thus every subset of $X - \{p\}$ is open, which implies that this is an uncountable discrete space. However, we know that an uncountable discrete space is not separable, so that $X - \{p\}$ is not separable.

□

**Presented in Class ?**

**Theorem 6.5** If $X$ and $Y$ are separable spaces, then $X \times Y$ is separable.

***Proof:*** Suppose $X$ or $Y$ are separable spaces. Then there exist countable sets $A$ and $B$ such that $\overline{A} = X$ and $\overline{B} = Y$. Using the fact that $\overline{A} \times \overline{B} = \overline{A \times B}$, we see that

$$\overline{A \times B} = \overline{A} \times \overline{B} = X \times Y.$$

Thus $A \times B$ is dense in $X \times Y$. But also observe that $A \times B$ is countable, since we can form a bijection between $A \times B$ and $A$ or $B$ (namely the projection function). Thus $X \times Y$ must be separable because it contains a countable dense subset, which is what we set out to show.

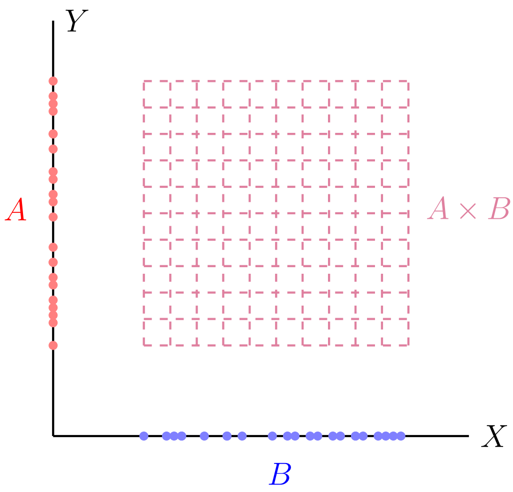


Figure 2: Here in this picture, we see that if $A$ and $B$ are countable dense subsets, then their product forms a countable dense subset.

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**Theorem 6.6** The space $2^{\mathbb{R}}$ is separable.

***Proof:*** Consider

$$A = \left\{ f \in 2^{\mathbb{R}} : \bigcup_{i=1}^{n} [p_i, q_i] | p_i, q_i \in \mathbb{Q} \middle| \forall x \in [p,q], f(x) = 1, x \notin [p_i, q_i], f(x) = 0 \right\}$$
$$\bigcup \left\{ f \in 2^{\mathbb{R}} : \bigcup_{i=1}^{n} [p_i, q_i] | p_i, q_i \in \mathbb{Q} \middle| \forall x \in [p,q], f(x) = 0, x \notin [p_i, q_i], f(x) = 1 \right\}$$

that is, we only consider intervals $[p, q]$, which have rational endpoints, and finitely union them. To construct the points $f$ in our set, we assign either a 1 or a 0 to $f(x)$ when $x$ lies in any of the finite intervals $[p_i, q_i]$. (This actually doesn't have to be done with $\mathbb{Q}$, but rather any set dense in $\mathbb{R}$.)

Theorem 2.14 guarantees that this is an at most countable set. Observe that our set is really a subset of all finite subsets of $\mathbb{Q}$, which itself is a countable set.

Observe that this set is dense in $2^{\mathbb{R}}$. Consider an open set

$$U = \{f \in 2^{\mathbb{R}} : f(a_1) = \delta_1, \ldots, f(a_n) = \delta_n\}$$

where $\delta_1, \ldots, \delta_n \in \{0, 1\}$. Then the point $f \in 2^{\mathbb{R}}$ such that $f(a_i) = \delta_i, f(x) = 0$ otherwise, is a point in $U$. Call this point $y$.

Since $\mathbb{R}$ is normal, there exist disjoint closed neighborhoods $[p_i, q_i]$ such that $a_i \in [p_i, q_i]$ Then observe that the set

$$\bigcup_{i=1}^{n} \{f \in 2^{\mathbb{R}} : f(x) = \delta_i \text{ if } x \in [p_i, q_i], f(x) = 0 \text{ otherwise}\}$$

is (1) a subset of $A$ and (2) contains $y$. Therefore, $A$ and $U$ have a nonempty intersetion, and since $U$ was an arbitrary open set of $2^{\mathbb{R}}$, we see that $A$ is dense in $2^{\mathbb{R}}$. Since it is also countable, we have that $2^{\mathbb{R}}$ is separable, as desired.

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**Theorem 6.9** Let $X$ be a $2^{\text{nd}}$ countable space. Then $X$ is separable.

***Proof:*** Let $p_i$ be some point of $B_i$, $i \in \mathbb{N}$, where $B_i$ is a basic open set from our countable basis. Then for any open set $V$ of $X$, we know that $V$ will intersect $\{p_i\}_{i \in \mathbb{N}}$ since by definition $V$ must contain some basic open set $B_i$ for which $p_i \in B_i$. Thus by Exercise 6.1, $\{p_i\}$ is dense, and since it's countable we have that $X$ is separable.

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**Exercise 6.10**

1. The space $\mathbb{R}_{\text{std}}$ is $2^{\text{nd}}$ countable (and hence separable).
2. The space $\mathbb{R}_{\text{LL}}$ is separable but not $2^{\text{nd}}$ countable.
3. The space $\mathbb{H}_{\text{bub}}$ is seprarble but not $2^{\text{nd}}$ countable.

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*Solution:*

1. Consider the open set $(a, b)$. Then observe that

$$\left(\bigcup_{p\in\mathbb{Q}, a\leq p}(p,\infty)\right)\cap\left(\bigcup_{q\in\mathbb{Q}, q\leq b}(\infty,q)\right)=(a,b).$$

   Therefore, we can generate $(a, b)$ by open sets with rational endpoints, which shows that $\mathbb{R}$ has a countable basis. Specifically, the family of open sets $\{(p, q) : p, q \in \mathbb{Q}\}$ forms a countable basis for $\mathbb{R}_{\text{std}}$, so that by definition $\mathbb{R}_{\text{std}}$ is $2^{\text{nd}}$ countable.

2. By Exercise 6.2, we found that $\mathbb{R}_{\text{std}}$ is separable since the rationals form a countable, dense subset in $\mathbb{R}_{\text{std}}$. Thus every set $(a, b)$ contains a rational. However, $(a, b) \subset [a, b)$, which which means that every set $[a, b)$ must also intersect the rationals. By Exercise 6.1, we can then conclude that the rationals are dense in $\mathbb{R}_{\text{LL}}$, and since they are countable this implies that $\mathbb{R}_{\text{LL}}$ is separable.

3. Observe that the positive rationals $\mathbb{Q}^+$ form a dense set of $\mathbb{R}^+ \cup 0$. That is, any set with $(a, b)$ with $a, b \geq 0$ must contain a rational. By Theorem 6.5, we can then conclude that $(\mathbb{Q}^+)^2$ is dense in $(\mathbb{R}^+)^2$, and $\mathbb{Q}^+)^2$ is clearly countable. Thus by definition $(\mathbb{R}^+)^2$ is separable.

   However, observe that this is not $2^{nd}$ countable. Observe that if we are to cover this space, we need to cover the $x$-axis. But every point on the $x$ axis needs an individual sticky bubble to cover it, and since there are an uncountable number of such $x$, it would be impossible to cover all of them with a countable number of sticky bubbles. Therefore this space is not $2^{nd}$ countable.

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**Theorem 6.11** Every uncountable set in a $2^{\text{nd}}$ countable space has a limit point.

***Proof:*** Suppose we have an uncountable set $A$ in $X$, and for the sake of contradiction suppose that $U$ has no limit points. Then every point of $A$ is an isolated point, which means that there exists an open set $U$ such that $U \cap A = \{p\}$ for all $p \in A$. Note that for every such $U$ there exists a $B$ basic open set such that $B \subset U$. Thus $p \in B \subset U$. However, there are only countably many basic open sets, while an uncountable number of $p \in A$, which is a contradiction since we cannot contain an uncountable number of points with a countable number of basic open sets. Thus $A$ must have a limit point, which is what we set out to show.

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**Theorem 6.14** Let $X$ be $2^{\text{nd}}$ countable. Then $X$ is $1^{\text{st}}$ countable.

***Proof:*** Let $X$ be a $2^{\text{nd}}$ countable space and $x \in X$. Then $X$ has a countable basis $\mathcal{B}$. Consider the set $\mathcal{B}_x$ of all $B \in \mathcal{B}$ such that $x \in B$. Since $\mathcal{B}$ is a basis, we know that for any open set $U$ containing $x$ there exists a $B_x \in \mathcal{B}$ such that

$$p \in B_x \subset U.$$

But $p \in B_x$ so $B_x \in \mathcal{B}_x$. Therefore, $\mathcal{B}_x$ is a neighborhood basis of $x$. However, $\mathcal{B}_x \subset \mathcal{B}$ so $\mathcal{B}_x$ is countable. Therefore, every point of $x$ as countable neighborhood basis so it is a $1^{\text{st}}$ countable space.

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**Theorem 6.15** If $X$ is a topological space, $p \in X$, and $p$ has a countable neighborhood basis, then $p$ has a nested countable neighrborhood basis.

***Proof:*** Let $\mathcal{B}$ be the countable neighborhood basis for $p$. Observe that for $B \in \mathcal{B}$, (1) $p \in B$ and (2) $B$ is an open set, so by Theorem 3.3 we must be able to contain $p$ in some neighborhood $U$ such that $p \in U \subset B$. By the defintion of a neighborhood basis, there must exist another $B' \in \mathcal{B}$ such that $p \in B' \subset U$. Hence, for every $B \in \mathcal{B}$, there exists an element $B' \in \mathcal{B}$ such that

$$p \in B' \subset B.$$

Since $\mathcal{B}$ is countable, we can construct an at most countable set of nested open sets which form a neighborhood basis of $p$. Thus $p$ has a nested countable neighborhood basis as desired.

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**Theorem 6.18** Suppose $x$ is a limit point of the set $A$ in a $1^{\text{st}}$ countable space $X$. Then there is a sequence of points $\{a_i\}_{i\in\mathbb{N}}$ in $A$ that converges to $x$.

***Proof:*** Since $x$ is a limit point of $A$, for every open set $U$ such that $x \in U$ we have that $(U - \{x\}) \cap A \neq \emptyset$. Since $x$ is also a point in a first countable space, it has a countable neighborhood basis. By Theorem 6.15, $x$ must therefore also have a nested countable neighborhood basis $\mathcal{B}$.

Since $\mathcal{B}$ is countable, we can write $\mathcal{B} = \{B_1, B_2, \ldots\}$. Let $i \in \mathbb{N}$. Now since each $B_i \in \mathcal{B}$ must contain some point $a_i \in A$, $a_i \neq x$, any open set of $x$ will contain some $a_i$ such that $a_i \in B_i \subset U$. Thus we must have that $\{a_i\}_{i\in\mathbb{N}}$ to be a sequence of points of $A$ which converges to $x$.

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