Chapter 9
Connectedness: When Things Don't Fall Into Pieces

Theorem 9.1 The following are equivalent:

1. X is connected
2. there is no continuous function $f : X \to \mathbb{R}_{\text{std}}$ such that $f(X) = \{0, 1\}$
3. X is not the union of two disjoint nonempty separated sets
4. X is not the union of two disjoint nonempty closed sets
5. the only subsets of X that are both closed and open in X are both the empty set and X itself
6. for every pair of points p and q and every open cover $\{U_\alpha\}_{\alpha\in\lambda}$ of X there exists a finite number of U_α's, $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that $p \in U_{\alpha_1}$, $q \in U_{\alpha_n}$ for each $i < n$, $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$.

Proof:

(**1** $\implies$ **2**) Suppose X is connected, and for contradiction that there is a continuous function $f : X \to \mathbb{R}_{\text{std}}$ such that $f(X) = \{0, 1\}$. However, this would imply that $f^{-1}(1)$ and $f^{-1}(0)$ are (1) disjoint open sets in X such that (2) their union is X. However, that contradicts the fact that X is connected by definition. Therefore, there is no continuous function $f : X \to \mathbb{R}_{\text{std}}$ such that $f(X) = \{0, 1\}$.

(**2** $\implies$ **1**) Now if there is no continuous function $f : X \to \mathbb{R}_{\text{std}}$ such that $f(X) = \{0, 1\}$, then that means X cannot be split into two disjoint open sets whos union is X, which implies that X is connected.

(**1** $\implies$ **3**) Since X is connected, it is not the union of two nonempty disjoint open subsets of X. However, suppose A, B are two separated sets such that $A \cup B = X$.

(**3** $\implies$ **1**) Suppose now that X is not the union of two disjoint nonempty separated sets. Then X is not union of two disjoint open sets, so that X is connected.

(**1** $\implies$ **4**) Suppose X is connected, and for contradiction that $X = A \cup B$ where A and B are disjoint nonempty closed sets. Then we can construct a continuous function from $f : X \to \{0, 1\}$, where $f^{-1}(0) = A$ and $f^{-1}(1) = B$. However, this contradictions the fact that X is connected, so that X is no the union of two disjoint nonempty closed sets.

(**4** $\implies$ **1**) Suppose X is not the union of two disjoint nonempty closed sets. Then there is no continuous function $f : X \to \{0, 1\}$ since $f^{-1}(0)$ and $f^{-1}(1)$ cannot be open or closed. Thus X must be connected.

(**1** $\implies$ **5**) Suppose X is connected. Suppose there is a set such that $A \neq X$

and $A \neq \emptyset$ is open and closed. Then $A^c \cup A = X$. However, that would mean X is the union of two disjoint non empty open sets, which is a contradiction. Thus the only open and closed sets are X and $\emptyset$.

(**5** $\implies$ **1**) Suppose the only open and closed sets in X are X and $\emptyset$. Suppose for contradiction that X is not connected, so that $X = A \cup B$ for two disjoint nonempty open sets. Then $X^c = (A \cup B)^c = A^c \cap B^c = \emptyset$. However, this is a contradiction since their intersection must be nonempty. Therefore, X is connected.

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Exercise 9.2 **Exercise 9.2** Which of the following spaces are connected?

1. $\mathbb{R}$ with the discrete topology?
2. $\mathbb{R}$ with the indiscrete topology?
3. $\mathbb{R}$ with the finite complement topology?
4. $\mathbb{R}_{\text{LL}}$?
5. $\mathbb{Q}$ as a subspace of $\mathbb{R}_{\text{std}}$?
6. $\mathbb{R} - \mathbb{Q}$ as a subspace of $\mathbb{R}_{\text{std}}$?

Solution:

1. Every subset of $\mathbb{R}$ is open and closed. This violates Theorem 9.1(5) so that $\mathbb{R}$ is not connected under the discrete topology.
2. The only sets which are open and closed are $\mathbb{R}$ and $\emptyset$. Thus by Theorem 9.1(5) $\mathbb{R}$ is connected under the indiscrete topology.
3. For contradiction suppose there is a set $U \subset \mathbb{R}$ which is open and closed and not $\mathbb{R}$ or the emptyset.

   Since U is open, U^c is finite. However $(U^c)^c = U$ is infinite and hence U^c is not an open set. But this contradicts the assumption that U was open and closed. Thus $\mathbb{R}$ is connected on the finite complement topology.
4. Consider a basic open set $[a, b)$. Observe that

   $$[a, b)^c = (-\infty, a) \cup [b, \infty)$$

   which is the union of two open sets, and hence is open. Thus $[a, b)$ is open and closed. By Theorem 9.1.5, we have that $\mathbb{R}_{\text{LL}}$ is not connected.
5. Observe that $(\mathbb{Q} \cap (-\infty, \pi))$ and $(\mathbb{Q} \cap (\pi, \infty))$ are disjoint, separated sets in the subspace $\mathbb{Q}$ and

   $$(\mathbb{Q} \cap (-\infty, \pi)) \cap (\mathbb{Q} \cap (\pi, \infty)) = \mathbb{Q}.$$

   Thus $\mathbb{Q}$ is not open as a subspace of $\mathbb{R}_{\text{std}}$.

6. Observe that $(\mathbb{R}-\mathbb{Q})\cap(-\infty,0)$ and $(\mathbb{R}-\mathbb{Q})\cap(0,\infty)$ are disjoint separated sets and
$$((\mathbb{R}-\mathbb{Q})\cap(-\infty,0))\cup((\mathbb{R}-\mathbb{Q})\cap(0,\infty))=\mathbb{R}-\mathbb{Q}.$$
Thus $\mathbb{R}-\mathbb{Q}$ is not connected.

□

Theorem 9.3 The space $\mathbb{R}_{\text{std}}$ is connected.

Proof: The only closed and open sets in $\mathbb{R}_{\text{std}}$ are the emptyset and $\mathbb{R}$ itself, so that by Theorem 9.1(5) we can conclude that $\mathbb{R}_{\text{std}}$ is connected.

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Theorem 9.4 Let A and B be separated subsets of a space X. If C is a connected subset of $A\cup B$, then either $C\subset A$ or $C\subset B$.

Proof: Observe that if C is a connected subset of $A\cup B$, where A and B are separated in X, then C is not the union of two disjoint open sets in the $A\cup B$ subspace topology.

Suppose for the sake of contradiction that $C\subset A$ and $C\subset B$. Then observe that
$$C\subset A\cap B=\emptyset$$
which is a contradiction since C is nonempty. Thus it must be that $C\subset A$ or $C\subset B$.

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Theorem 9.5 Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of connected subsets of X and E another connected subset of X that for each $\alpha\in\lambda$, $E\cap C_\alpha\neq\emptyset$. Then $E\cup(\bigcup\limits_{\alpha\in\lambda}C_\alpha)$ is connected.

Proof: Suppose for the sake of contradicition that $E\cup(\bigcup\limits_{\alpha\in\lambda}C_\alpha)$ is not connected. Then $E\cup(\bigcup\limits_{\alpha\in\lambda}C_\alpha)=A\cup B$ where A and B are some separated sets in X. Observe that since E is a connected subset of X, we have by Theorem 9.4 that $E\subset A$ or $E\subset B$. Without loss of generality suppose $E\subset A$. Then since each C_α is a connected subset of $A\cup B$, Theorem 9.4 implies that $C_\alpha\subset B$ for at least one $\alpha\in\lambda$. However, this is a contradiction since $E\cap C_\alpha\neq\emptyset$ for all $\alpha\in\lambda$, while $A\cap B=\emptyset$. Therefore, we must have that $E\cup(\bigcup\limits_{\alpha\in\lambda})$ is connected.

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Theorem 9.6 Let C be a connected subset of the topological space X. If D is a subset of X such that $C\subset D\subset\overline{C}$, then D is connected.

Proof: Suppose that C is a connected subset of X and for the sake of contradiction that D such that $C \subset D \subset \overline{C}$ is not connected. Then there exists disjoint open sets A and B such that $A \cup B = D$. Since C is connected, we know by Theorem 9.5 that $C \cap A = \emptyset$ or $C \cap B = \emptyset$. Without loss of generality, suppose that $C \cap A = \emptyset$. Then this is a contradiction since $A \subset D \subset \overline{C}$. Therefore, we must have that D is connected.

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Theorem 9.8 Let X be a topological space, C a connected subset of X, and $X - C = A \big| B$. Then $A \cup C$ and $B \cup C$ are each connected

Proof: Suppose that $X - C = A \cup B$ where A and B are separated. Now suppose that $A \cup C$ is not connected, so that $A \cap C = U \cup V$ where U, V are open. Now suppose that $U \cap C \neq \emptyset$ and $V \cap C \neq \emptyset$. Then $(U \cap C) \cup (V \cap C) = A \cap C$

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Theorem 9.12 Let $f : X \to Y$ be a continuous, surjective function. If X is connected, then Y is connected.

Proof: Suppose $f : X \to Y$ is a continuous, surjective function. We can do proof by contradiction. Suppose X is connected but Y is not connected. By Theorem 9.1 part 5, there exists a set $V \subset Y$, $V \neq \emptyset$ $V \neq Y$, such that V is open and closed in Y. By continuity, $f^{-1}(V)$ is both open and closed in X, and by surjectivity, $f^{-1}(V)$ is a proper subset of X. Thus X has an open and closed set, one which is not $\emptyset$ or X, which contradicts the fact that X is not connected by Theorem 9.1 part 5. Thus if X is connected, Y is connected, as desired.

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Theorem 9.13 (Intermediate Value Theorem!) Let $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ be a continuous map. If $a, b \in \mathbb{R}$ and r is a point of $\mathbb{R}$ such that $f(a) < r < f(b)$ then there exists a point c in (a, b) such that $f(c) = r$

Proof: Observe that $\mathbb{R}_{\text{std}}$ is connected. Since $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$, connected should be preserved.

Suppose there does not exist a point $c \in (a, b)$ such that $f(c) = r$. Then $f(x) < r$ or $r < f(x)$ for all $x \in (a, b)$. However since $f(\mathbb{R}) = \mathbb{R}$, this implies that $\mathbb{R}_{\text{std}}$ is not connected, which contradicts the fact that $\mathbb{R}_{\text{std}}$ is connected. Therefore such a c must exist.

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Theorem 9.18 Each component of X is connected, closed, and not contained in any strictly larger connected subset of X.

Proof: Consider a component $C = \bigcup_{\alpha \in \lambda} C_\alpha$ of p in X, where each C_α is connected and $p \in C_\alpha$ for all $\alpha \in \lambda$. Observe that we can apply Theorem 9.5 to conclude that C is connected, since (1) no member of the union of C is disjoint from any other member (as they all contain p) and (2) each member is connected.

Suppose that C is not closed. Then there is a point $q \notin C$ and an open set U containing q such that $(U - \{q\}) \cap C \neq \emptyset$.

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Theorem 9.35 A path connected space is connected.

Proof: Suppose X is path connected but not connected. Then there exist two disjoint open subsets A, B such that $A \cup B = X$. Observe that any point in A cannot be joined together with any point B by a path, a contradiction to the path connectivity of X. Thus X must be connected.

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Theorem 9.36 The flea and comb space is connected but not pathwise connected. (The flea and comb space is the union of the topologist's comb and the point (0, 1).)

Proof: Let A be the set of the comb space. This is obviously path connected, and so it is connected by Theorem 9.35. Observe now that

$$A \subset A \cup flea \subset \overline{A}$$

so that $A \cup flea$, the flea and comb space, must be connected.

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