Chapter 4

Bases, Subspaces, Prodcuts: Creating New Spaces

Theorem 4.1 Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if:
1. $\mathcal{B} \subset \mathcal{T}$
2. for each set U in $\mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subset U$

Proof: First we prove the forward direction. Suppose we have a set $\mathcal{B}$ such that $\mathcal{B} \subset \mathcal{T}$, and for every open set $U \in \mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subset U$. Then let A be an open set of X. For all $a \in A$, there exists an open set $V_a \in \mathcal{B}$ such that $a \in V_a \subset A$. Then observe that

$$\bigcup_{a \in A} V_a = A.$$

Thus we see that every open set in X is the union of elements of $\mathcal{B}$, so $\mathcal{B}$ is a basis for X.

Now we prove the other direction, and suppose $\mathcal{B}$ is a basis for X. First observe that $\mathcal{B} \subset \mathcal{T}$ because this is part of the definition of a basis, so this proves (1). Now let $U \in \mathcal{T}$. Then

$$\bigcup_{B \in \mathcal{B}'} B = U$$

for some subset $\mathcal{B}'$ of $\mathcal{B}$. Thus for any $p \in U$, there must exist at least one $B \in \mathcal{B}'$ such that $p \in B$ and by construction $B \subset U$. Therefore, we have that for each set U in $\mathcal{T}$ and point p in U there is a set B in $\mathcal{B}$ such that $p \in B \subset U$, as desired.

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Ex: the set of length 1 intervals. This does not generate a topology on $\mathbb{R}$ because intersections should be open but sometimes they are intervals of length less than 1.
Another topology is the stick bubble topology. Open sets are balls sitting in the upper half plane or an open ball containing a single point on its circumference which is shared with the boundary of the upper half plane.

Ordered set topology:

Exercise 4.2
1. Let $\mathcal{B}_1 = \{(a,b) \subset \mathbb{R} : a, b \in \mathbb{Q}\}$. Show that $\mathcal{B}_1$ is a basis for the standard topology on $\mathbb{R}$.
2. Let $\mathcal{B}_2 = \{(a,b) \cup (c,d) \subset \mathbb{R} : a < b < c < d$ are distinct irrational numbers.$\}$ Show that $\mathcal{B}_2$ is also a basis for the standard topology on $\mathbb{R}$.

Solution: 1. We can do this using Theorem 4.1. Observe firstly that $\mathcal{B}_1 \subset \mathcal{T}_{\text{std}}$. Now consider an arbitrary open set U in $\mathbb{R}$. By definition, for any $p \in U$ there exists an open ball $B(p, \epsilon(p))$ such that $p \in B(p, \epsilon(p)) \subset U$. Since the rationals are dense in $\mathbb{R}$, for any $p \in U$ there must exist a rationals a/b and c/d such that

$$p - \epsilon(p) < a/b < p < c/d < p + \epsilon(p).$$

Thus $(a/b, c/d) \in \mathcal{B}_1$ and $p \in (a/b, c/d) \subset B(p, \epsilon) \subset U$. This proves (2) of the theorem, so $\mathcal{B}_1$ is a basis for the standard topology on $\mathbb{R}$.

2. We can do this using Theorem 4.1. Observe firstly that $\mathcal{B}_1 \subset \mathcal{T}_{\text{std}}$. Now consider an arbitrary open set U in $\mathbb{R}$. By definition, for any $p \in U$ there exists an open ball $B(p, \epsilon(p))$ such that $p \in B(p, \epsilon(p)) \subset U$. Since the irrationals are dense in $\mathbb{R}$, for any $p \in U$ there must exist a rationals a, b, c and d such that

$$p - \epsilon(p) < a < p < b < c < d < p + \epsilon(p).$$

Thus $(a,b) \cup (c,d) \in \mathcal{B}_2$ and $p \in (a,b) \cup (c,d) \subset B(p, \epsilon) \subset U$. This proves (2) of the theorem, so $\mathcal{B}_2$ is a basis for the standard topology on $\mathbb{R}$.

$\square$

Theorem 4.3 Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if:
1. Each point of X is in some element $\mathcal{B}$
2. if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W of $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.

Proof: First we'll prove the forward direction. Suppose that $\mathcal{B}$ is a basis for some topology $\mathcal{T}$. Since $X \in \mathcal{T}$, we know by Theorem 4.1 that for every $p \in X$ there exists a $B \in \mathcal{B}$ such that

$$p \in \mathcal{B} \subset X.$$

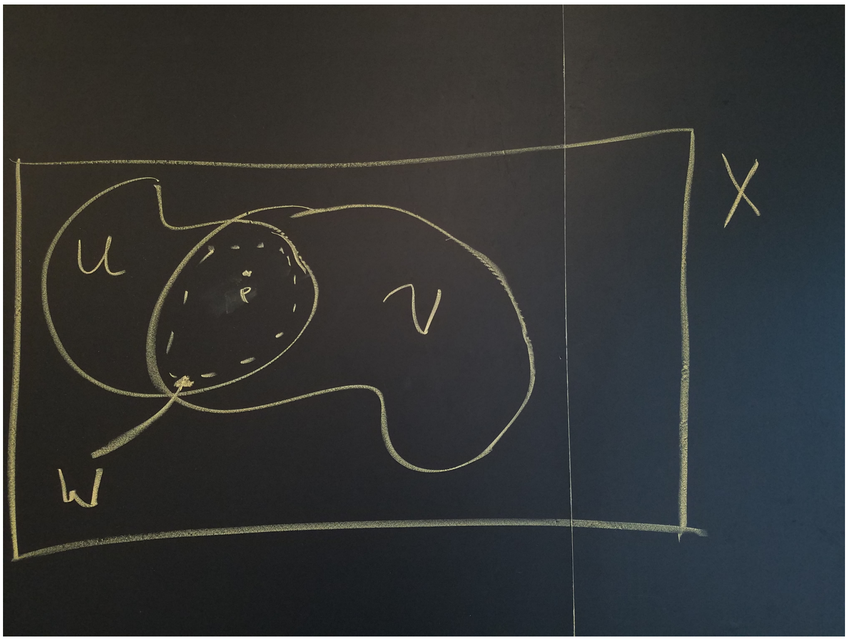

Figure 1: Here we have sketched the conditions (1) and (2), where X is the whole space, $U, V \in \mathcal{B}$ and $W \in (U \cap V)$.

Thus this proves (1). Next observe that if $U, V \in \mathcal{B}$, then $U, V \in \mathcal{T}$. Therefore, $U \cap V \in \mathcal{T}$ is an open set in X. By Theorem 4.1, for any $p \in U \cap V$, there must exist a $W \in \mathcal{B}$ sucht that

$$p \in W \subset U \cap V.$$

This proves (2) which finishes the proof in this direction.

Suppose that (1) and (2) are true. Then consider the set of all possible unions of elements of $\mathcal{B} = \{B_\alpha\}_{\alpha \in \lambda}$, namely the set

$$\mathcal{T} = \{ \bigcup_{\alpha \in \lambda'} B_\alpha : \lambda' \subset \lambda \}.$$

We can now verify the properties that this is a topology.

1. Observe that $\emptyset \in \mathcal{T}$ if we take an empty union of objects.

2. $X \in \mathcal{T}$ by condition (1).

3. Observe that arbitrary unions of elements of $\mathcal{B}$ are within $\mathcal{T}$, since that is by definition how we constructed $\mathcal{T}$.

4. By condition (2) if $U, V \in \mathcal{B}$ then there exsits a W such that $W \in \mathcal{B}$ and $p \in W \subset (U \cap V)$. Thus observe that

$$U \cap V = \bigcup_{p \in U \cap V} W_p$$

where $W_p \in \mathcal{B}$ such that $p \in W_p \subset U \cap V$. By our definition of our topology, $U \cap V$ must be an open set. Therefore, finite interesections of open sets are open.

Thus we have shown that conditions (1) and (2) generate a topology, which completes the proof.

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Exercise 4.4 Show that the basis proposed above (all sets of the form $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$) for the lower limit topology is in fact a basis.

Solution: We can show that this is a basis by using Theorem 4.3. Observe that for any point $p \in \mathbb{R}$, there exists $a, b \in \mathbb{R}$ such that $a \leq p < b$. Thus every point in $\mathbb{R}$ is in some element of our basis. Next, let $c < d < e < f$ and consider $U = [c, d]$ and $V = [e, f]$. Then $U \cap V = \emptyset \in \mathbb{R}_{\text{LL}}$.
Next, let $c < d$ and $e < d < f$, and consider the sets $U = [c, d)$ and $V = [e, f)$. Then there is a point $p \in (U \cap V)$. Observe that if $\epsilon < p - e$ then the set $p \in [p - \epsilon, f) \subset U \cap V$ and $[p - \epsilon, f)$ is a member of our basis. Thus we have that sets of the form $[a, b)$ form a basis for the lower limit topology.

□

Theorem 4.5 Every open set in $\mathbb{R}_{\text{std}}$ is an open set in $\mathbb{R}_{\text{LL}}$, but not vice versa.

Proof: Consider an open set $B(p, \epsilon)$ in $\mathbb{R}_{\text{std}}$ about a point p of radius $\epsilon > 0$, which is really just an interval $(p - \epsilon, p + \epsilon)$. Then consider the sequence of open sets $\mathbb{R}_{\text{LL}}$:

$$\left\{\left[p - \epsilon\left(1 - \frac{1}{2^n}\right), p + \epsilon\right) : n \in \mathbb{N}\right\}.$$

Recall that an arbitrary union of open sets is open. Then

$$\bigcup_{n \in \mathbb{N}} \left[p - \epsilon\left(1 - \frac{1}{2^n}\right), p + \epsilon\right) = (p - \epsilon, p + \epsilon)$$

is an open set. Thus we have that open sets in $\mathbb{R}_{\text{std}}$ are open in $\mathbb{R}_{\text{LL}}$. However, it is obvious that open sets in $\mathbb{R}_{\text{LL}}$ are not open in $\mathbb{R}_{\text{std}}$, because sets of the form $[a, b)$ are neither open or closed in $\mathbb{R}_{\text{std}}$. Thus this proves the theorem.

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Exercise 4.6 Give an example of two topologies on $\mathbb{R}$ such that neither is finer than the other, that is, the two topologies are not comparable.

Solution: We can define an upper limit topology $\mathbb{R}_{\text{UL}}$ generated by basis sets $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$.

For each $x \in \mathbb{R} \in (x - \epsilon, x + \epsilon] \in \mathbb{R}_{\text{UL}}$. In addition, observe that $(a, b] \cap (c, d] = (b, c]$ if $b < c$ and $(a, d]$ if $b = c$ and $\emptyset$ if $b > c$, all of which are basic open sets in the topology. Thus by Theorem 4.3 this generates a topology.

Now observe that neither of the topologies $\mathbb{R}_{\text{LL}}$ and $\mathbb{R}_{\text{UL}}$ are finer than the other, since neither is a subset of the other. Thus these topologies on $\mathbb{R}$ are not comparable.

□

Exercise 4.7 Check that the collection of sets that we specify as a basis in the double headed snake actually forms a basis for the topology.

Solution: We can verify this using theorem 4.3. Observe first that every point in $\mathbb{R}_{+00}$ is contained within some set in the basis. Next, let U and V be any two sets in the topology. Then if U, V are of the form $(0, b) \cup \{0'\}$, then their intersection will be of the form $(0, a) \cup \{0'\}$ where $a \leq b$ which is a set within our basis. The argument applies again to if U, V are both of the form $(0, b) \cup \{0''\}$ or (a, b).

Next observe that if we intersect U of the form $(0, b) \cup \{0'\}$ with V of the form (a, b) then the intersection is either empty, or of the form (a, b), which is a type of set contained in our absis. If we intersct
Let U be any set in $\mathbb{R}_{+00}$. Let $U = (a, b)$ where $a, b \in \mathbb{R}$ and $a < b$. If U does not contain $\{0'\}$ or $\{0''\}$, then there exits numbers $c, d \in mathbbR$ such that $a < c < d < b$. Then the set (c, d) is in our basis and is a subset of U.

□

Exercise 4.8 In the Double Headed Snake, show that every point is a closed set; however, it is impossible to find disjoint open sets U and V such that $\{0'\} \in U$ and $\{0''\} \in V$.

Solution: Observe that the complement of every point is an interval of the line, for which we can represent as the union of basic open sets and therefore the complement of every point is open. Thus every point must be a closed set.

Next, let U be an open set containing $\{0'\}$ and V an open set containing $\{0''\}$. Then by definition, there must exist basic open sets U_B, V_B such that $0' \in U_B \subset U$ and $0'' \in V_B \subset V$. Since they are basic open sets containing the zeros of the double headed snake, both are either of the form $(0, b) \cup \{0'\}$ or $(0, b) \cup \{0''\}$, so that $U_B \cap V_B \neq \emptyset$. Thus we cannot find disjoint open sets U and V such that $\{0'\} \in U$ and $\{0''\} \in V$.

□

Exercise 4.9 1. In the topological space $\mathbb{R}_{\text{har}}$, what is the closure of the set $H = \{1/n\}_{n\in\mathbb{N}}$?
2. In the topological space $\mathbb{R}_{\text{har}}$, what is the closure of the sets $H^- = \{-1/n\}_{n\in\mathbb{N}}$?
3. Is it possible to find disjoint open sets U, V in $\mathbb{R}_{\text{har}}$ such that $0 \in U$ and $H \subset V$?

Solution:

1. There are no limit points to the set because all sets are of the form (a, b) or $(a, b) - H$. Thus for any neighborhood about a point will always either not contain H or it will exclude H, so by definition no point can be a limit point of H.

2. For H^-, the limit points just consists of the set $\{0\}$. This is any open set which contains 0 must contain points of the sequence $\{\frac{1}{n} | n \in \mathbb{N}\}$. The difference between this question and question (1.) is that in the first question, the points in the sequence were always excluded whenever they interesected any open set containing 0, whereas that's not the case here since we don't care about excluding the negative harmonics.

3. For any open set U containing 0, we must have that there exists an open set U_0 of 0 such that $0 \in V \subset U$. And since the basic open sets are (a, b) or $(a, b) - H$, there must exist points to the right of 0 in the set U; otherwise, 0 would be a limit point. Because any open set containing H must contain points arbitrarily close to 0, it is inevitable for U and V to intersect. This is because for any open set which contains 0, say $(a, 0 + \epsilon)$ for $\epsilon > 0$ and $a < 0$, there exsits an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ and therefore any open set containing H must contain this point and hence intersect with $(a, 0 + \epsilon)$. Of course, any set of the form $(0 - \epsilon, b)$ for $b > 0$ will intersect with

any set containining H by the same argument as before. Thus there does not exist disjoint open sets U, V such that $0 \in U$ and $H \subset V$.

$\square$

Exercise 4.10 1. In $\mathbb{H}_{\text{bub}}$, what is the closure of the set of rational on the x-axis?

2. In $\mathbb{H}_{\text{bub}}$, which subsets of the x-axis are closed?

3. In $\mathbb{H}_{\text{bub}}$, let A be a countable set on the x-axis and z a point on the x-axis not in A. Then there exist open sets U and V such that $A \subset U$ and $z \in V$. (Do you need the countability hypothesis on A?)

4. In $\mathbb{H}_{\text{bub}}$, let A and B be countable sets on the x-axis such that A and B are disjoint. Then there exists open sets U and V such that $A \subset U$ and $B \subset V$.

5. In $\mathbb{H}_{\text{bub}}$, let A be the rational numbers and let B be the irrational numbers. Do there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$?

Solution:

1. Suppose that a limit point was no a rational on the x-axis. Obviously such a point cannot be one which isn't on the x-axis since we could easily find an open set which contained that point but didn't intersect the x-axis. Thus our candidates for limit points for our set are reduced to irrationals on the x-axis. But observe that any open set which contains a rational doesn't necessarily contain an irrational. For example, any basic open set about a rational on the x-axis given by $(p, 0)$ does not contain an irrational. Thus the closure of the rationals is simply the rationals.

2. Consider the set $\{(x, y) \in \mathbb{R}^2 | y > 0\}$; that is, everything but the x-axis. Observe that this set is simply the uncountable union of all possible basic open sets of the form $B((x, y), r)$ where $0 < r \leq y$, so therefore this set must be open. Thererfore, its complement, the x-axis, must be closed. Furthermore, if we included in our uncountable union an arbitrary number of sticky bubbles of the form $B((x, y), r) \cup \{(x, 0\}$, which would form an open set, the complement would be a subset of the x-axis and it would be closed since the complement of open sets are closed. Thus all subsets of the x axis are open.

3. We argue that the countability hypothesis is not necessary. For any set U containing A, U must contain a set of sticky bubbles B_x of radius r_x containing each point $x \in A$. Then if we can construct a sticky bubble of radius $r' < \max\{\sqrt{(z-x)^2 + r_x^2} -$

$r_x^2\}$ about z, we see that U and V do not intersect. Thus there does exists disjoint open sets U and V such that $A \subset U$ and $z \in V$.

4. Let $a \in A, b \in B$, and arbitrarily assign sticky balls $\{B_a\}$ each of radius $r_a > 0$ for each $a \in A$. Then for each $b = (x_b, y_b)$, assign b a sticky ball B_b of radius r such that $r < r'$ for all

   $r' \in \{\sqrt{(x_a - x_b)^2 + r_a^2} - r_a^2 | r_a$ is the radius of the sticky ball of point $(x_a, y_a) \in A\}$.

   If we do this for each point $b \in B$, and union the sticky balls $\{B_b\}$, we'll obtain an open set V such that $B \subset V$. If we union all the sticky balls B_a, we'll again obtain an open set U such that $A \subset U$, which proves the exercise.

5.

$\square$

Exercise 4.11 Check that the arithmetic progressions form a basis of a topology on $\mathbb{Z}$.

Solution: We can use Theorem 4.3 for this. Let the set of arithmetic progression be $\mathcal{B}$ and let $q \in \mathbb{Z}$. Then observe that $q \in \{p \cdot n : n \in \mathbb{Z}\} \in \mathcal{B}$. Thus condition (1) of Theorem 4.3 is satisfied.

Now let $U, V \in \mathcal{B}$, and suppose $U = \{a_1 n + b_1 : n \in \mathbb{Z}\}$ and $V = \{a_2 n + b_2 : n \in \mathbb{Z}\}$. Suppose $U \cap V$ is nonempty. Then $q \in U \cap V$ for some $q \in \mathbb{Z}$. However, in order for q to be in the intersection, we must have tht a_1 and a_2 are coprime. If they are coprime, then by Bezout's theorem that there exist integers m_1, m_2 such that $m_1 n_1 + m_2 n_2 = 1$. We then know by the Chinese Remainder Theorem that $q = b_1 + (b_2 - b_1) m_1 a_1$. Therefore, we see that

$$q \in \{b_1 + (b_2 - b_1) n a_1 : n \in \mathbb{Z}\} \subset U \cap V.$$

Since this is an arithmetic progression, this set lies in $\mathcal{B}$. Therefore, we see that condition (3) of Theorem 4.3 is satisfied, so that the arithmetic progressions do form a basis for a topology on $\mathbb{Z}$.

$\square$

Theorem 4.12 There are infinitely many primes.

Proof: Let p be prime and consider the set $p\mathbb{Z}$. Observe that this is a closed set since it is the complement of the union of sets $p\mathbb{Z}+1, \dots, p+(p-1)$ which are of the forms of basic open sets.

Now observe that nonempty open sets are always open. This is because every open set must contain a basic open set, which are by definition infinite sets.

Thus suppose that there are infinitely many primes $p_1, p_2, \dots, p_n$. Then

$$\bigcup_{i=1}^{n} p_i\mathbb{Z}$$

is a closed set as it is the finite union of closed sets. However, note that

$$\left(\bigcup_{i=1}^{n} p_i\mathbb{Z}\right)^c = \{-1, 1\}.$$

This should be an open set, since $\bigcup_{i=1}^{n} p_i\mathbb{Z}$ is closed. But this is a contradiction since $\{-1, 1\}$ is a finite set and hence cannot be open. Thus there must be an infinite number of primes.

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Exercise 4.18 Let X be totally ordered by $<$. Let $\mathcal{S}$ be the collection of sets of the following forms

$$\{x \in X | x < a\} \quad \text{or} \quad \{x \in X | x > a\}$$

for $a \in X$. Then $\mathcal{S}$ forms a subbasis for the order topology on X.

Solution: We can prove this using Theorem 4.14. Observe that the first condition is satisfied because $\mathcal{S} \subset \mathcal{T}$. Next, let $p \in U \in \mathcal{T}$, and suppose that U is of the form $\{x \in X | x < a\}$ or $\{x \in X | a < x\}$. Then observe that $U \in \mathcal{S}$ so that $\bigcap_{n=1}^{1} U \subset U$. Finally, suppose that U is of the form $\{x \in X | a < x < b\}$. Then we can simply intersect the sets $S_1, S_2 \in \mathcal{S}$ where $S_1 = \{x \in X | a < x\}$ and $S_2 = \{x \in X | x < b\}$ to get that $\bigcap_{n=1}^{2} S_n = \{x \in X | a < x < b\} \subset U$. Thus by condition (2) of Theorem 4.14, we have that $\mathcal{S}$ must be a subbasis for the order topology.

□

Exercise 4.19 Verify that the order topology on $\mathbb{R}$ with the usual $<$ order is the standard topology on $\mathbb{R}$.

Solution: Every set of the order topology is of the form $\{x \in \mathbb{R} | x < a\}$ or $\{x \in \mathbb{R} | a < x\}$ or $\{x \in \mathbb{R} | a < x < b\}$ where $a, b \in \mathbb{R}$.

Consider a point p in a set U of the form of $\{x \in \mathbb{R} | x < a\}$ or $\{x \in \mathbb{R} | a < x\}$. Then observe that $p \in B(p, |a - p|)$ is a ball containing p inside U. By definition, these are therefore open sets in the standard topology on $\mathbb{R}$.

If instead U is of the form $\{x \in \mathbb{R} | a < x < b\}$, then for any $p \in U$ observe that $p \in B(p, \min\{p - a, b - p\})$, so that U must also be open in the standard topology on $\mathbb{R}$.

Finally observe that every open set in the standard topology on $\mathbb{R}$ is of the form of a set in the order topology. This is because every open set in $\mathbb{R}_{\text{std}}$ can be bounded from either one or both ends, both possibilities which are captured by elements of the standard topology on $\mathbb{R}$. Thus we can conclude that the order topology on $\mathbb{R}$ with the usual $<$ order is the standard topology on $\mathbb{R}$.

$\square$

Exercise 4.20
Draw pictures of various open sets in the lexigraphically ordered square.

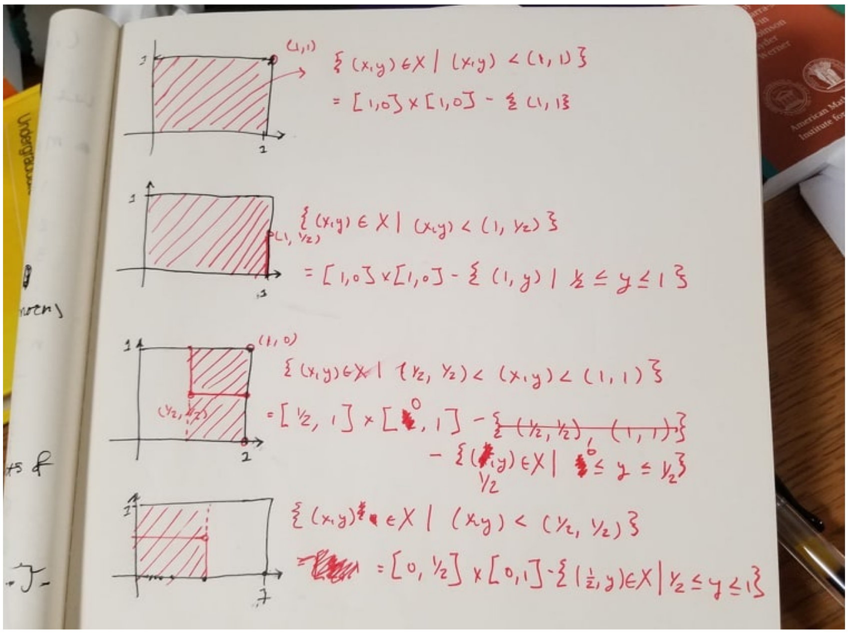

Figure 2: Here we consider three open sets $\{(x,y) \in X|(x,y) < 1\}$, $\{(x,y) \in X|(x,y) < (1,1/2)\}$, $\{(x,y) \in X|(1/2,1/2) < (x,y) < (1,1)\}$ and $\{(x,y) \in X|(x,y) < (1/2,1/2)\}$ and sketch their drawings. Since these are technically basis elements, we can also imagine unioning these sets around to obtain new open sets to imagine what the topology looks like.

4.21 In the lexigraphically ordered square find the closures of the following subsets:

$$A = \left\{\left(\frac{1}{n}, 0\right) | n \in \mathbb{N}\right\}$$
$$B = \left\{\left(1 - \frac{1}{n}, \frac{1}{2}\right) | n \in \mathbb{N}\right\}$$
$$C = \{(x,0) \, | 0 < x < 1\}$$
$$D = \left\{\left(x, \frac{1}{2}\right) | 0 < x < 1\right\}$$
$$E = \left\{\left(\frac{1}{2}, y\right) | 0 < y < 1\right\}.$$

Solution: For set A, we argue that the set of limit points or the set A is simply the point $(0,1)$. This is because for any open interval containing $(0,1)$, we must have the set wrap back around and include a set of points $(x,0)$ where $x > 0$ and is very small. Since the sequence $\{\frac{1}{n}\}$ converges to 0, we see that an open set $(0,1)$ must include points of the sequence. Therefore (0,1) is a limit point.

For set B, observe that there are no limit points of B. The only possible limit point would be $(1, \frac{1}{2})$, but we can find an open set containing this point but no point of B. For example, the set $\{(x, y) \in X | (1, 0) < (x, y)\}$ contains $(1, \frac{1}{2})$ but not any point of the set B.

For the set C, we can see that the set of limit points will simply be the set $\{(x, 1) \in X | 0 \leq x < 1\} \cup \{(1, 0)\}$. Basically, the the closure is the top and bottom lines of the unit square, minus the two points $(0, 0)$ and $(1, 1)$.

The set D has no limit points. This is because for any point $p = (a, b)$, we can simply construct an open neighborhood about p given by

$$\{(a, y) \in X | |y - b| < \min\{|b - \frac{1}{2}|, b - 1\}\}$$

which does not intersect the set if $b \neq \frac{1}{2}$. In the case where $b = \frac{1}{2}$, any neighborhood of the form

$$U = \{(x, y) \in X | (a, 0) < (x, y) < (a, 1)\}$$

contains p, but $(U - \{p\}) \cap D = \emptyset$. Therefore, there are no limit points to the set so $\overline{D} = D$.

For the set E, we see that the set of limit points is simply $\{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}$. This is because any open set containing either of these points must definitely contain points of the set E. Thus the closure is $E \cup \{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}$

$\square$

Theorem 4.25 Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. Then the collection of sets $\mathcal{T}_Y$ is in fact a topology on Y.

Proof: We can show that $\mathcal{T}_Y$ is a topology by showing that it satisfies the four criteria for the definition of a topology.

1. Since $\mathcal{T}_Y = \{U | U = V \cap Y \text{ where } V \in \mathcal{T}\}$, we can take $V = \emptyset$ to observe that $\emptyset \in \mathcal{T}_Y$.

2. Next, since $X \in \mathcal{T}$, we can take $V = X$ to observe that $X \cap Y = Y \in \mathcal{T}_Y$.

3. Suppose $U, V \in \mathcal{T}_Y$ and consider $U \cap V$. Since $U = U' \cap Y$ and $V = V' \cap Y$ for some $U', V' \in \mathcal{T}$, we have that $U \cap V = (U' \cap Y) \cap (V' \cap Y) = (U' \cap V') \cap Y$. Since $(U' \cap V') \in \mathcal{T}$, then we know that $(U' \cap V') \cap Y = (U \cap V) \in \mathcal{T}_Y$. Thus $\mathcal{T}_Y$ is closed under finite intersections.

4. Now suppose $U_\alpha \in \mathcal{T}_Y$ for all $\alpha \in \lambda$, where λ is an arbitrary index. Then for each α there must exist a $V_\alpha \in \mathcal{T}$ such that $U_\alpha = V_\alpha \cap Y$. Thus $\bigcup_{\alpha\in\lambda} U_\alpha = \bigcup_{\alpha\in\lambda}(V_\alpha \cap Y) = \bigcup_{\alpha\in\lambda}(V_\alpha) \cap Y$. Since $\bigcup_{\alpha\in\lambda} V_\alpha \in \mathcal{T}$, we know that $\bigcup_{\alpha\in\lambda} V_\alpha \cap Y \in \mathcal{T}_Y \implies \bigcup_{\alpha\in\lambda} U_\alpha \in \mathcal{T}_Y$. Thus we have that arbitrary unions of open sets are contained in $\mathcal{T}_Y$.

Thus, we have that $\mathcal{T}_Y$ is a topology on Y.

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Exercise 4.26 Consider $Y = [0, 1)$ as a subspace for $\mathbb{R}_{\text{std}}$ In Y, is the set $[1/2, 1)$ open, closed, neither or both?

Solution: The set $[1/2, 1)$ is not open but is closed under this topology. Firstly, there does not exist an element $V \in \mathbb{R}_{\text{std}}$ such that $[1/2, 1) = V \cap Y$. Thus $[1/2, 1)$ is not open in $\mathcal{T}_Y$. However, it is a closed set since it contains its only limit point $1/2$. This is because every open set which contains $1/2$ must intersect with $[1/2, 1)$. Thus $[1/2, 1)$ is closed in the Y subspace topology.

□

Exercise 4.27 Consider a subspace Y of a topological space X. Is every subset $U \subset Y$ that is open in Y also open in X?

Solution: Consider an open set $U \in \mathcal{T}_Y$. Then there exists a $V \in \mathcal{T}_X$ such that $U = V \cap Y$. While the set U is then technically open in $\mathcal{T}_Y$, it is possible that $V \cap Y$ is not an open set in $\mathcal{T}_X$, which would only happen in $Y \notin \mathcal{T}_X$. Thus we must have that Y be open in order for the above statement to be true.

□

Theorem 4.28 Let $(Y, \mathcal{T}_Y)$ be a subsapce of $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if there is a set $D \subset X$, closed in $(X, \mathcal{T})$, such that $C = D \cap Y$.

Proof: Let us first prove the forward direction. Suppose $D \subset X$ is closed in $(X, \mathcal{T})$ and let $C = D \cap Y$. Since D is closed in $(X, \mathcal{T})$, we have that $X - D$ is open in the same topology. Now since $X - D \in \mathcal{T}$, observe that $(X - D) \cap Y \in (Y, \mathcal{T}_Y)$ by definition. Since $(X - D) \cap Y$ is open in $(Y, \mathcal{T}_Y)$, the complement $Y - ((X - D) \cap Y)$ is closed. However, observe that $Y - ((X - D) \cap Y) = Y - (Y - D) = D \cap Y = C$, so that we have concluded that C is a closed set.

Now we prove the other direction. Suppose that C is closed in $(Y, \mathcal{T}_Y)$. Since C is closed, $Y - C$ is open in $(Y, \mathcal{T}_Y)$. Thus by definition, there exists a set $A \in \mathcal{T}_X$ such that $A \cap Y = Y - C$. Since A is open, we know that $X - A$ is closed in $(X, \mathcal{T})$. Call this set D. Now observe that

$$D \cap Y = (X - A) \cap Y = (Y - A) \cap Y = C$$

because $A \cap Y = Y - C$. Thus we have found a set D which closed in $(X, \mathcal{T})$ such that $C = D \cap Y$. Having proved both direction, this proves the theorem.

■

Corollary 4.29 Let $(Y, \mathcal{T}_Y)$ be a subspace $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if $\text{Cl}_X(C) \cap Y = C$

Proof: First we'll prove the forward direction. Suppose that $\text{Cl}_X(C) \cap Y = C$. Then let p be a limit point of C in the topological space $(Y, \mathcal{T}_Y)$. Then for every set U open in $(Y, \mathcal{T}_Y)$ which contains p we have that $(U - \{p\}) \cap C \neq \emptyset$. Since U is an open set, there exists a set $V \in \mathcal{T}_X$ such that $U = V \cap Y$. Thus $(V - \{p\}) \cap C \neq \emptyset$ for all open sets $V \in \mathcal{T}_X$ which contain p. This implies that $p \in \text{Cl}_X(C)$. But $p \in \text{Cl}_X(C) \cap Y = C$, so that C contains all of its limit points in $(Y, \mathcal{T}_Y)$. Thus C is closed in $(Y, \mathcal{T}_Y)$.

Now let us prove from the other direction. Suppose that C is closed in $(Y, \mathcal{T}_Y)$. Then C contains all of its limit points in this topological space. We previously showed that all of the limit points in C in $(Y, \mathcal{T}_Y)$ must be in $\text{Cl}_X(C)$. And since $C \subset Y$, we must have that $\text{Cl}_X(C) \cap Y = C$, which is what we set out to show in this direction. Having proven both directions, this proves the corollary.

■

Exercise 4.31 Consider the following subspaces of the lexicographically ordered square.
1. $D = \{(x, \frac{1}{2}) | 0 < x < 1\}$
2. $E = \{(\frac{1}{2}, y) | 0 < y < 1\}$
3. $F = \{(x, 1) | 0 < x < 1\}$.

As sets they are all lines. Describe their relative topologies, especially noting any connections to topologies you have seen already.

Solution:

1. For this set, the relative topology establishes that open sets are simply subsets of the line D. This is because we can imagine creating open sets in the lexicographically ordered squared and intersecting them with out line segment to contain either a point or a subset of D. Thus all subsets of D are open sets, which is similar to the discrete topology we encountered earlier.

2. The relative topology established by this set only contains the empty set and the set E itself, which mirrors the indiscrete topology. We arrive at this conclusion by the fact that intersecting this set with an open set simply yields either the empty set or E itself.

3. By the definition of the relative topology, we see that

$$\mathcal{T}_F = \{U : U = F \cap V, V \in \mathcal{T}_{\text{sq}}\}$$

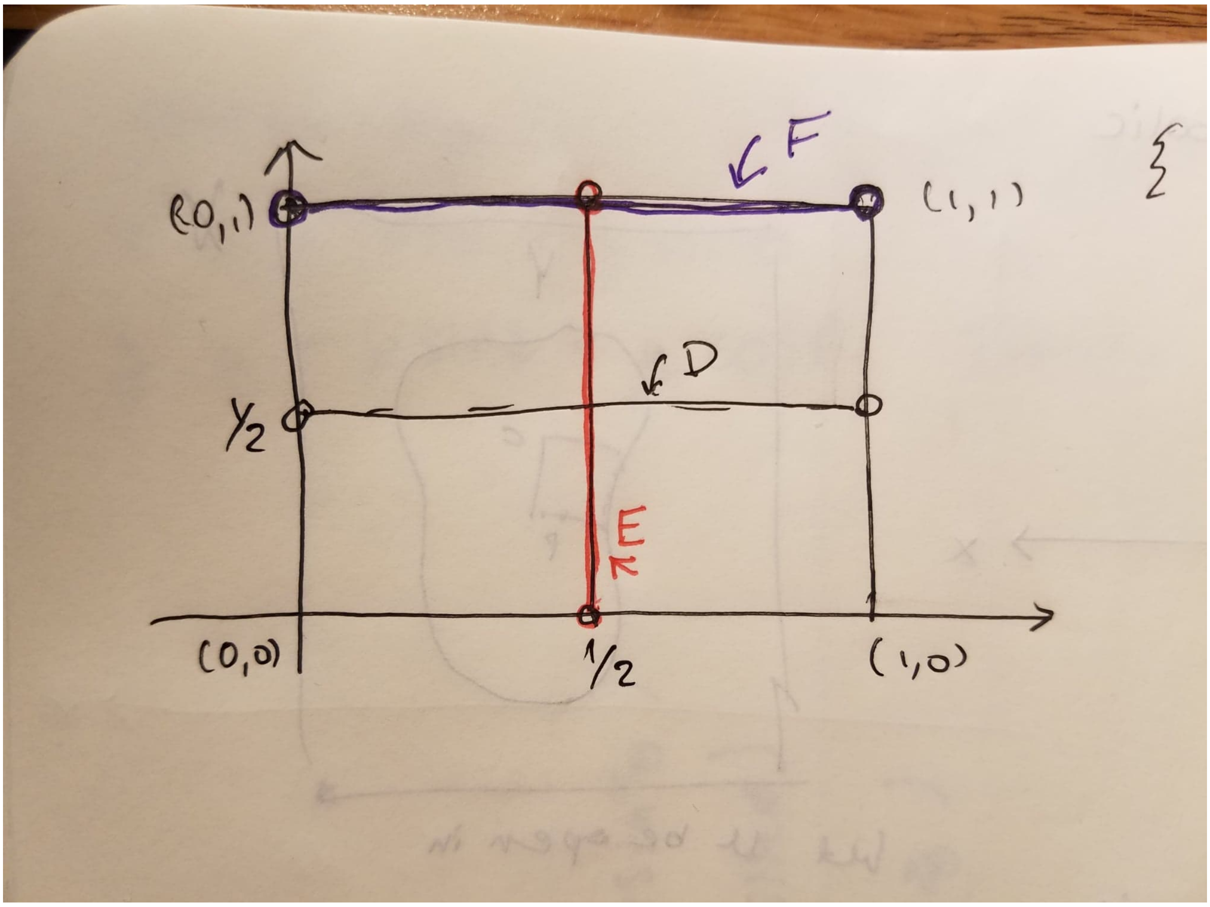
where $\mathcal{T}_{\text{sq}}$ is the topology of the lexicographically ordered square. If $V \in \mathcal{T}_{\text{sq}}$ and $V \subset F$, then obviously $V \in \mathcal{T}_F$. Thus we suspect that subsets of F will be in $\mathcal{T}_F$. In our case, the subsets of F include the empty set, intervals of the form $a < x < b$, $a \leq x < b$, $a \leq x \leq b$, and $a \leq x \leq b$ where $a, b \in (0, 1)$. However, the last two forms are not allowed in the topology, since the rightmost endpoints are not included. Because of this, we see that $\mathcal{T}_F$ has a connection with $\mathbb{R}_{LL}$, whose topology consists of intervals where the left hand point is inclusive but the righthand point is not inclusive.

$\square$

Exercise 4.32 Verify that the collection of basic open sets above satisfies the conditions of Theorem 4.3, thus confirming that this collection is a basis for a topology.

Solution: First observe that the first condition of Theorem 4.3 is satisfied, since for any $(p, q) \in X \times Y$ there exist open sets $U \in X$ and $V \in Y$ such that $p \in U$ and $q \in V$. Thus there exists a basic open set $U \times V$ such that $(p, q) \in U \times V$, which shows that each point of $X \times Y$ is in some basic open set.

Now suppose U, V are basic open sets. Then $U = A \times B$ and $V = C \times D$ for some open sets $A, C \in \mathcal{T}_X$ and $B, D \in \mathcal{T}_Y$. Let $p \in U \cap V = (A \cap C) \times (B \cap D)$. Then observe that $(A \cap C) \in \mathcal{T}_X$ and $(B \cap D) \in \mathcal{T}_Y$. Since the basis consists of the product of all open sets in X and all open sets in Y, we see that $(A \cap C) \times (B \cap D)$ must be a basic open set. Thus we have a basis element $W = (A \cap C) \times (B \cap D)$ such that $p \in W \subset U \cap V$,



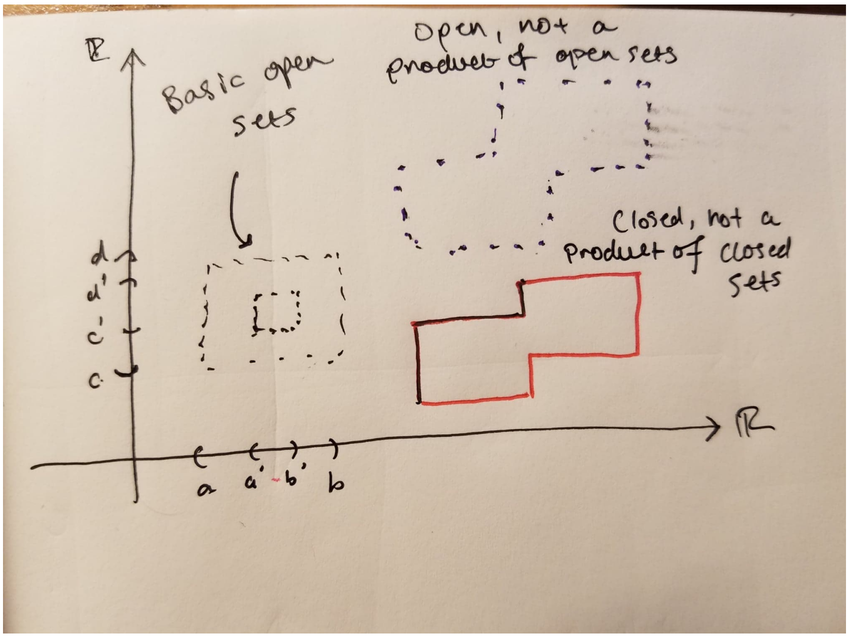
which satisfies the second part of Theorem 4.3. Thus the proposed collection is in fact a basis for the topology.

□

Exercise 4.33 Draw examples of basic and arbitrary open sets in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ using the standard topology on $\mathbb{R}$. Find (i) an open set in $\mathbb{R} \times \mathbb{R}$ that is not the product of open sets, and (ii) a closed set in $\mathbb{R} \times \mathbb{R}$ that is not the product of closed sets.

Exercise 4.34 Is the product of closed sets closed?

Solution: Yes. Let $p \in \overline{U}$ and $q \in \overline{V}$. Then for every open set U_p containing p and V_q containing q, we'll have that $U_p \cap U \neq \emptyset$ and $V_q \cap V \neq \emptyset$. Therefore we see that $(U_p \times U_q) \cap (U \times V) \neq \emptyset$, meaning that $(p, q) \in \overline{U \times V}$. Thus we have that $\overline{U} \times \overline{V} \subset \overline{U \times V}$.



Now suppose $(p,q) \in \overline{U \times V}$. Then every basic open set of the form $U' \times V'$ containing (p,q) intersects $U \times V$. In other words, for every open set $U' \in \mathcal{T}_X$ containing p, then $U' \cap U \neq \emptyset$. Similarly, for every open $V' \in \mathcal{T}_Y$ containing q, $V' \cap V \neq \emptyset$ Thus we must have that $p \in \overline{U}$ and $q \in \overline{V}$, so that $(p,q) \in \overline{U} \times \overline{V}$, which implies that $\overline{U \times V} \subset \overline{U} \times \overline{V}$.

Since $\overline{U \times V} \subset \overline{U} \times \overline{V}$ and $\overline{U} \times \overline{V} \subset \overline{U \times V}$, we have that $\overline{U \times V} = \overline{U} \times \overline{V}$. Thus if U, V are closed, $\overline{U} = U$ and $\overline{V} = V$, so $U \times V = \overline{U \times V}$. So the product of closed sets is in fact closed.

□

Exercise 4.35 Show that the product topology $X \times Y$ is the same as the topology generated by the subbasis of inverse images of open sets under the projection functions, that is the subbasis is $\{\pi_X^{-1}(U) | U \text{ is open in } X\} \cup \{\pi_Y^{-1}(V) | V \text{ is open in } Y\}$.

Solution: Let U be open. Then for $p \in U$, there exists a basic open set $W \in \mathcal{T}_{\text{prod}}$, the product topology, such that

$$p \in W \subset U.$$

Observe that $W = W_x \times W_y$ where $W_x \in \mathcal{T}_x$ and $W_y \in \mathcal{T}_y$. Note also that

$$W = W_x \times W_y = (W_x \times Y) \cap (X \times W_y).$$

Let $\mathcal{S}$ be the set of inverse images of open sets under the projection functions. Then we also know that $\pi_X^{-1}(W_x) = W_x \times Y$, while $\pi_X^{-1}(W_y) = X \times W_y$, which are in $\mathcal{S}$. We can then state that

$$W = \pi_X^{-1}(W_x) \cap \pi_Y^{-1}(W_y).$$

Therefore, we see that

$$p \in \pi_X^{-1}(W_x) \cap \pi_Y^{-1}(W_y) \subset W.$$

Since for each $V \in \mathcal{S}$ we have that $V \in \mathcal{T}_{\text{prod}}$, (1) $\mathcal{S} \subset \mathcal{T}_{\text{prod}}$ and (2) for any open set U and point $p \in W$ there exists elements $\pi_X^{-1}(W_x), \pi_X^{-1}(W_y) \in \mathcal{S}$ such that $p \in \pi_X^{-1}(W_x) \cap \pi_Y^{-1}(W_y) \subset U$, we have by Theorem 4.14 that $\mathcal{S}$ is a subbasis of the product topology, as desired.

□

Exercise 4.36 Using the standard topology on $\mathbb{R}$, is the product topology $\mathbb{R} \times \mathbb{R}$ the same as the standard topology on $\mathbb{R}^2$?

Solution: Consider $B(p, R)$, a disk of radius R centered at $p = (p_x, p_y)$, which is an open set in $\mathbb{R}_{\text{std}}$.

Observe that for each $q = (q_x, q_y) \in B(p, R)$, we can construct a set $U_q = (q_x - \epsilon, q_x + \epsilon)$ containing q_x if

$$p_x - R < q_x - \epsilon \quad q_x + \epsilon < p_x + R$$

and similarly we can for the set $V_q = (q_y - \delta, q_y + \delta)$ containing q_y if

$$p_y - R < q_y - \delta \quad q_y + \epsilon < p_y + R.$$

Therefore, $q \in U \times V \subset B(p, R)$. Since for each $q \in B(p, R)$ we can find an open $W_p = U_q \times V_q$ such that $p \in W_q \subset B(p, R)$, we see that

$$\bigcup_{q \in B(p,R)} W_q = B(p, R).$$

Thus we see that the product topology is a subset of the standard topology on $\mathbb{R}$. Now consider a basic open set $W = U \times V$ in the product topology on $\mathbb{R} \times \mathbb{R}$. Thus $U = (a, b)$ and (c, d), where a, b, c, d may or may not be finite.

Observe that for any $p = (p_x, p_y) \in U \times V$, we can contain it in a ball $B(p, \epsilon)$ where

$$\epsilon < \min\{\min\{b - p_x, p_x - a\}, \min\{c - p_y, p_y - c\}\}.$$

Therefore, we see that for any $p \in U \times V$ there exists an open ball $B(p, \epsilon)$ such that $p \in B(p, \epsilon) \subset U \times V$. Hence

$$\bigcup_{p \in U \times V} B(p, \epsilon_p) = U \times V$$

so that the standard topology is a subset of the product topology. Since we show the converse, we must have that the standard topology is equivalent to the product topology on $\mathbb{R}$.

$\square$

Exercise 4.37 A basis for the product topology on $\prod_{\alpha \in \lambda} X_\alpha$ is the collection of all sets of the form $\prod_{\alpha \in \lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all but finitely many α.

Solution: Consider an open set U in the product topology $\mathcal{T}_{\text{prod}}$. Then for each $p \in U$, there exists a subbasic open set in $\mathcal{S}$ such that

$$p \in \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \subset U$$

Now consider the family of sets described in the problem, and call this set $\mathcal{T}'_{\text{prod}}$. Observe that we can write

$$\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) = \cdots \times U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \cdots = \prod_{\alpha \in \lambda} U_\alpha$$

where $U_\alpha = X_\alpha$ for all $\alpha \in \lambda \backslash \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $U_{\alpha_1}, U_{\alpha_2}, \ldots U_{\alpha_n}$ are all restricted open sets in the spaces $X_{\alpha_1}, X_{\alpha_2}, \ldots X_{\alpha_n}$, respectively. Thus $\mathcal{S} \subset \mathcal{T}'_{\text{prod}}$.

$$p \in \prod_{\alpha \in \lambda} U_\alpha \subset U. \tag{1}$$

Now observe that for any $V \in \mathcal{T}'_{\text{prod}}$, $V = \prod_{\alpha \in \lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all but finitely many α,

$$V = \prod_{\alpha \in \lambda} U_\alpha = \cdots \times U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \cdots = \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$$

Thus we see that $\mathcal{T}'_{\text{prod}} \subset \mathcal{S}$, so that these two collections of sets generate the same topology: namely, the product topology. More specifically, we see that (1) $\mathcal{T}'_{\text{prod}} \subset \mathcal{T}_{\text{prod}}$ and (2) equation (1) satisfies Theorem 4.3, which proves that $\mathcal{T}_{\text{prod}}$ forms a basis for the product topology, as desired.

$\square$

Exercise 4.38 Let $\mathcal{T}$ be the topology on 2^X with basis generated by the subbasis $\mathcal{S}$.

1. Every basic open set in 2^X is both open and closed.

2. Show that if a collection of subbasic open sets of 2^X has the property that every point of 2^X lies in at least one of those subbasic open sets, then there are two subbasic open sets in that collection such that every point of 2^X lies in one of those two subbasic open sets.

3. Show that if a collection of basic open sets of 2^X has the property that every point of 2^X lies in at least one of those basic open sets, then there are a finite number of basic open sets in that collection such that every point of 2^X lies in one of those basic sets.

Solution:

1. Consider an arbitrary basic open set U in the product topology of 2^X. Then observe that U is of the form

$$U = \{f \in 2^X : f(a_1) = \delta_1, f(a_2) = \delta_2, \ldots, f(a_n) = \delta_n\}$$

where $a_1, \ldots, a_n \in A$ and $\delta_1, \delta_2, \ldots, \delta_n \in \{0, 1\}$. Then observe that

$$U^c = \{f \in 2^X : f(a_1) = |\delta_1 - 1|, f(a_2) = |\delta_2 - 1|, \ldots, f(a_n) = |\delta_n - 1|\}$$

Since U is open, U^c is closed. However, U^c is still of the form of basic open set, which means that U is closed. Therefore, every basic open set in 2^X is open and closed.

2. Let our subbasic open cover be $\{U_\alpha\}_{\alpha \in \lambda}$ where $U_\alpha \in \mathcal{T}$ for all $\alpha \in \lambda$. Now suppose there aren't two subbasic open sets such that every point of 2^X lies in one or the other. Then observe that this is not a cover of 2^X since, if a point of 2^X lies in one set U, then it does not lie in U^c. Thus in this case it would not even be a cover.

3. Let $\{U_\alpha\}_{\alpha \in \lambda}$ be our cover as previously defined. Fix $p \in 2^X$, and observe it lives in some subbasic set

$$U = \{f \in 2^X : f(a_1) = \delta_1, f(a_2) = \delta_2, \ldots, f(a_n) = \delta_n\}.$$

Observe that every point of x either lies in this set, or its coordinate values $f(a_1), f(a_2), \ldots, f(a_n)$ differ in at least one coordinate from the restriction offered by U. Since every coordinate can have at most 2 different values, we see that there are 2^n different ways that the coordinate values $f(a_1), f(a_2), \dots, f(a_n)$ could differ from the restriction offered in U. Thus we can have at most $2^n + 1$ basic open sets which contain all the points in X, which means that there are at most a finite number of basic open sets that cover every point of 2^X.

$\square$

Exercise 4.39 In the product space $2^{\mathbb{R}}$, what is the closure of the set Z consisting of all elements of $2^{\mathbb{R}}$ that are 0 on every rational coordinate, but may be 0 or 1 on any irrational coordinate? Equivalently, thinking of $2^{\mathbb{R}}$ as subsets of $\mathbb{R}$, what is the closure of the set Z consisting of all subsets of $\mathbb{R}$ that do not contain any rational?

Solution: Observe that

$$Z^c = \bigcup_{n\in\mathbb{N}}^{\infty} \{f : 2^{\mathbb{R}} | f(a_1) = f(a_2) = \cdots = f(a_n) = 0, a_1, a_2, \dots, a_n \in \mathbb{Q}\}$$

where $a_1, a_2, \dots, a_n$ are distinct but arbitrarily chosen of $\mathbb{Q}$. Note that this is an uncountable union of open sets, since we are only making a finite number of restrictions on the coordinates. Therefore Z must be closed, so $\overline{Z} = Z$.

$\square$

Exercise 4.40 Find a subset A of $2^{\mathbb{R}}$ and a limit point x of A such that no sequence in A converges to x. For an ever greater challenge, determine whether you can find such an example if A is countable.

Solution: Observe that the point $p \in 2^{\mathbb{R}}$ such that

$$p(a) = 1 \quad \forall a \in \{\pi + q : q \in \mathbb{Q}\}$$
$$p(a) = 0 \quad \forall a \in \mathbb{R} - \{\pi + q : q \in \mathbb{Q}\}$$

is a limit point of Z, the set consisting of all elements of $2^{\mathbb{R}}$ that are 0 on every rational coordinate and 0 or 1 on all others. (Note: π was chosen randomly. We could have done it with any other irrational. All we want is a point such that it's x-th coordinate is 1 for a countable number of irrational x. We know $\mathbb{Q}$ is countable, so adding π to every element of this set generates a countable set of irrationals, which is how we want to design our point.)

Any open set containing p must be of the form

$$\{f \in 2^{\mathbb{R}} | f(\pi + q_1) = f(\pi + q_2) = \cdots = f(\pi + q_n) = 1, q_1, q_2, \ \ldots, q_n \in \mathbb{Q}\}$$

and hence will intersect Z.

By Theorem 3.30, if there exists a sequence of elements of Z which converge to p then $p \in \overline{Z}$. But in 4.39 we saw $\overline{Z} = Z$, and clearly $p \notin Z$. Hence $p \notin \overline{Z}$, so there is no sequence of elements of Z which converge to p.

$\square$

Exercise 4.41 Let $\mathbb{R}^\omega$ be the countable product of copies of $\mathbb{R}$. So every point in $\mathbb{R}^\omega$ is a sequence $(x_1, x_2, x_3, \ldots)$. Let $A \subset \mathbb{R}^\omega$ be the set consisting of all points with only positive coordinates. Show that in the product topology, $\mathbf{0} = (0, 0, 0, \ldots)$ is a limit point of the set A, and there is a sequence of points in A converging to $\mathbf{0}$. Then show that in the box topology, $\mathbf{0} = (0, 0, 0, \ldots)$ is a limit point of the set A, but there is no sequence of points in A converging to $\mathbf{0}$.

Solution: Let U be an open set containing $\mathbf{0}$. Suppose $\mathbf{0}$ is in the basic open set B of the product topology so that $B = \Pi_{\alpha \in \omega} U_\alpha$ where U_α is open in $\mathbb{R}$, and $U_\alpha = \mathbb{R}$ for all but finitely many α. Then for each $\alpha \in \omega$ corresponding to $U_\alpha \neq \mathbb{R}$, the open set must contain $\mathbf{0}$, and hence it must contain positive points of $\mathbb{R}$. Since the rest of the U_α's such that $U_\alpha = \mathbb{R}$ obviously contain positive coordinates of $\mathbb{R}$, we see that the basic open set, and hence the set $U - \{\mathbf{0}\}$, must have a nonempty intersection with A. Therefore $\mathbf{0}$ is a limit point of A in the product topology.

I claim that the sequence $(\frac{1}{n}, \frac{1}{n}, \ldots)$ is a sequence which converges to $\mathbf{0}$ in the product topology. Observe that we can contain $\mathbf{0}$ in a basic open set B, where again $B = \Pi_{\alpha \in \omega} U_\alpha$ and U_α is open in $\mathbb{R}$ while $U_\alpha = \mathbb{R}$ for all but finitely many α. Thus for each $U_\alpha \neq \mathbb{R}$, let $n_\alpha \in \mathbb{N}$ be such that $\frac{1}{n_\alpha} \in U_\alpha$. Now let

$$n = \min \left\{ n_\alpha \middle| \frac{1}{n_\alpha} \in U_\alpha \right\}.$$

Then for $i > n$, we see that $(\frac{1}{i}, \frac{1}{i}, \ldots) \in B$. Thus every open set about $\mathbf{0}$ will contain points of the sequence, which shows that this sequence converges to $\mathbf{0}$ in the product topology.

Now we'll show that there is no sequence which converges to $\mathbf{0}$ in the box topology. Suppose for the sake of contradiction that there is a sequence of points

$$\begin{matrix} (a_{11} & a_{12} & a_{13} & \dots) \\ (a_{21} & a_{22} & a_{23} & \dots) \\ (a_{31} & a_{32} & a_{33} & \dots) \end{matrix}$$

which converge to $\mathbf{0}$. Then observe that we can construct an open set about $\mathbf{0}$ in the box topology as follows. Let (a_1, a_2) contain 0 but exclude (a_{11}). Let (a_3, a_4) contain $\mathbf{0}$ but exclude (a_{22}). If we continue in this fashion, we'll construct an open set in the box topology

$$(a_1, a_2) \times (a_3, a_4) \times \dots$$

which all contain $\mathbf{0}$ but exclude every point of the proposed sequence. The fact that we can create this open set containing $\mathbf{0}$ but no element of the sequence contradicts our claim, which shows that no sequence in the box topology can converge to $\mathbf{0}$.

□

Exercise 4.42 Show that the set $2^{\mathbb{N}}$ in the box topology is a discrete space, whereas the set $2^{\mathbb{N}}$ in the product topology has no isolated points.

Solution: Observe that we can think of an open set here under the box topology as a a set of points where we are allowed to make an infinite number of restrictions on each coordinate. With this perspective, it is then clear that every point is a basic open set, since every point p has a restriction on every single coordinate. Since every point is open, we have that all subsets of $2^{\mathbb{N}}$ are open, which implies that the set is a discrete space under the box topology.

Consider any basic open set in $2^{\mathbb{N}}$ under the product topology:

$$U = \{f \in 2^{\mathbb{N}} : f(a_1) = \delta_1, \dots, f(a_n) = \delta_n\} \quad \delta_1, \dots, \delta_n \in \{0, 1\}$$

Observe that the set contains the point

$$p = (\dots, \overbrace{\delta_1}^{f(a_1)}, \dots, \overbrace{\delta_2}^{f(a_2)}, \dots, \overbrace{\delta_n}^{f(a_n)}, \dots, \overbrace{\delta_{n+1}}^{\text{unrestricted by } U}, \dots)$$

where $\delta_{n+1} \in \{0, 1\}$. But U also contains another point p' such that

$$p' = (\dots, \overbrace{\delta_1}^{f(a_1)}, \dots, \overbrace{\delta_2}^{f(a_2)}, \dots, \overbrace{\delta_n}^{f(a_n)}, \dots, \overbrace{1 - \delta_{n+1}}^{\text{unrestricted by } U}, \dots).$$

Thus no basic open set in $2^{\mathbb{N}}$ in the product topology contains a single element. Hence, there are no isolated points.

□