5.1 Hausdorff, Regular and Normal Spaces.

In the definition of a T_1 space, isn't it sufficient to simply state that there exists an open set U about x such that $y \notin U$?

Theorem 5.1 A space $(X, \mathfrak{T})$ is T_1 if and only if every point of X is closed.

Proof: First we'll prove the forward direction. Suppose every point $x \in X$ is a closed set. Then $X - \{x\}$ is open, so foe $y \in X$, $y \neq x$, there exists an open set U such that $y \in U$ and $U \cap \{x\} = \emptyset \implies x \notin U$. Analogously, $X - \{y\}$ is open so there exists an open set V containing x such that $x \in V$ but $y \notin V$. Since x, y were arbitrary distinct points of X, we have that X is a T_1 space.

Now suppose that X is a T_1 space. Consider an $x \in X$ and suppose for a contradiction that $y \in X$, $y \neq x$ is a limit point of $\{x\}$. Then every open set of y must contain x. However, this is not possible since x and y are distinct points, X is T_1, and therefore there exists an open set U containing x such that $y \notin U$. Thus y can't be a limit point, which means that no element in $X - \{x\}$ is a limit point of $\{x\}$. Therefore, x must be a closed set, and since x was arbitrary this shows that every point of X must be a closed set.

■

Exercise 5.2 Let X be a space with the finite complement topology. Show that X is T_1.

Solution: Observe that $X - \{x\}$ is an open set in the finite complement topology for all $x \in X$. Then its complement, $X - (X - \{x\}) = \{x\}$ is closed. Therefore every point is a closed set, and by the Theorem 5.1 we have that X is a T_1 space.

□

Exercise 5.3 Show that $\mathbb{R}_{\text{std}}$ is Hausdorff.

Solution: Consider any two distinct points x and y in $\mathbb{R}$. Then observe that we can construct the open sets $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ where $|x - y| < \epsilon$ so that $B(x, \epsilon/2)$ and

$B(y, \epsilon/2)$ are disjoint but $x \in B(x, \epsilon/2)$ and $y \in B(y, \epsilon/2)$. Since x, y were distinct and arbitrary, we have that $\mathbb{R}_{\text{std}}$ is a Hausdorff space.

$\square$

Exercise 5.4 Show that $\mathbb{H}_{\text{bub}}$ is regular.

Solution: We found earlier that all subsets of the x-axis are closed in $\mathbb{H}_{\text{bb}}$. Thus if we have a closed subset A and a point $x \notin U$, we can use exercise 4.10(4) to show that there must exist disjoint open sets U and V such that $x \in U$ and $A \subset V$. Therefore, $\mathbb{H}_{\text{bub}}$ is regular.

$\square$

Exercise 5.5 Show that $\mathbb{R}_{LL}$ is normal.

Solution: Let A, B be two disjoint closed sets. Consider $a \in A$, and observe that $\mathbb{R}_{\text{LL}} - B$ is an open set containing a. Therefore, there exists a basis element $[x, y)$ such that $a \in [x, y) \subset \mathbb{R}_{\text{LL}} - B$. Therefore, $[a, y) \subset [x, y) \subset (\mathbb{R}_{\text{LL}} - B)$. Observe that we can create open sets $[a, y)$ for all $a \in A$. Thus let $U = \bigcup\limits_{a \in A} [a, y)$, which is open as it is the arbitrary union of open sets. Similarly, if we take a $b \in B$ and find a basic open set $[x', y')$ such that $b \in [x', y') \subset \mathbb{R}_{\text{LL}} - A$, then we can define an open set $V = \bigcup\limits_{b \in B} [b, y')$.

Now U and V cannot intersect. Each member $[a, y')$ in the union of U is a subset of $\mathbb{R}_{\text{LL}} - B$, while each member $[b, y')$ in the union of V is a subset of $\mathbb{R}_{\text{LL}} - A$, and if they did intersect then this would require that for some $a \in A$, $b \in B$, $[a, y) \cap [b, y') \neq \emptyset$. However, this is impossible as this would imply that either $b \in [a, y)$ or $a \in [b, y')$, which cannot happen since $[a, y) \subset \mathbb{R}_{\text{LL}} - B$ and $[b, y') \subset \mathbb{R}_{\text{LL}} - A$. Thus we have that $U \cap V = \emptyset$. Since A and B were arbitrary disjoint closed sets in $\mathbb{R}_{\text{LL}}$, we see that X must be normal by the definition of normality.

$\square$

Exercise 5.6 1. Consider $\mathbb{R}^2$ with the standard topology. Let $p \in \mathbb{R}^2$ be a point not in a closed set A. Show that

$$\inf\{d(a, p) | a \in A\} > 0.$$

(Recall that $\inf E$ is the greatest lower bound of a set of real numbers E.)

2. Show that $\mathbb{R}^2$ with the standard topology is regular.
3. Find two disjoint closed subsets A and B of $\mathbb{R}^2$ with the standard topology such that

$$\inf\{d(a,b)|a \in A \text{ and } b \in B\} = 0$$

4. Show that $\mathbb{R}^2$ with the standard topology is normal.

Solution:

1. Firstly we know that $\inf\{d(a,p)|a \in A\} \geq 0$ since the distance function is always greater than or equal to zero. Thus we must simply show that it is not zero for any $a \in A$ where $p \notin A$. First, observe that since p is not a limit point of A, there exists an open set $B(p,\epsilon)$ containing p such that $B(p,\epsilon) \cap A = \emptyset$. Therefore, we have that $\inf\{d(a,p)|a \in A\} > \epsilon > 0$, proving the desired result.

2. Let $x \in \mathbb{R}^2$ and suppose A is a closed set not containing x. Since x is not a limit point in A, there exists an open set U containing x such that $U \cap A = \emptyset$. Thus let $B(x,\epsilon) \subset U$. Then if we take each point in $a \in A$ and construct an open ball $B(a,\epsilon/2)$ where $\epsilon = \inf\{d(a,p)|a \in A\}$, then none of these balls intersect U. If we union these set of balls, we'll obtain an open set which contains A but is disjoint with x. Thus by definition, $\mathbb{R}^2$ with the standard topology is regular.

3. Consider the set of points which lies on the line $x = 0$ and $y = \frac{1}{x}$. The function y converges to the y-axis, and while these two are sets are closed and disjoint we see that the inf of their distances between their points converges to 0.

4. If we have two disjoint closed sets A and B, then no point of one set is a limit point of the other. Construct a ball about each point of a given by $B(a, r_a)$ where $r_a = \frac{1}{4}\inf\{d(a,b)|b \in B\}$. By part (a), we know that $r_a > 0$. Similarly, let us constuct balls about each point $b \in B$ of radius $r_b = \frac{1}{4}\inf\{d(a,b)|a \in A\}$ given by $B(b, r_b)$ Now observe that no ball from the set $\{B(a,r_a)|a \in A\}$ intersects with any ball from the set $\{B(b,r_b)|b \in B\}$, and that $A \subset \bigcup_{a\in A} B(a,r_a)$ and $B \subset \bigcup_{b \in B} B(b,r_b)$. Since A and B were arbitrary closed sets, we must have that $\mathbb{R}^2$ is normal.

$\square$

Note: this can be done for all metric spaces, since we didn't necessarily appeal to explicit properties of $\mathbb{R}^2$!

Theorem 5.7 1. A T_2-space (Hausdorff) is a T_1-space.

2. A T_3-space (regular and T_1) is a Hausdorff space, that is a T_2-space.

3. A T_4-space (normal and T_1) is regular and T_1, that is, a T_3-space.

Proof:

1. In a T_2-space, we have that for every x distinct from y of the topological space, there are disjoint open sets U, V such that $x \in U$ and $y \in V$. As an obvious consequence, for each $x \neq y$, there exists open sets U, V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Since this holds for all distinct $x, y \in X$, we can conclude that by defintion X is also a T_1 space.

2. Let x, y be distinct. Since the space is regular, and because $\{x\}$ is a closed set (by T_1), we know that for every y distinct form x, there must exist disjoint open sets U, V such that $\{x\} \subset U$ and $y \in V$. In other words, there exists disjoint open sets such that $x \in U$, $y \in V$. Thus by definition, we have a T_2 space.

3. Observe that since we have a T_1, every point is a closed set. Furthermore, since we have normality, disjoint closed sets may be contained in disjoint open sets. Thus consider a closed set A and a point $x \notin A$. Then since $\{x\}$ and A are disjoint closed sets, we may construct disjoint open sets U, V such that $\{x\} \subset U$ and $A \subset V$. By definition, this shows that X is also a regular space. Since the space is regular, and T_1 by hypothesis, we know that the space must be T_3 as desired.

■

Theorem 5.8 A topological space X is regular if and only if for each point p in X and open set U containing p there exists an open set V such that $p \in V$ and $\overline{V} \subset U$.

Proof: First we prove the forward direction. Suppose that X is regular and consider some $p \in X$.

Then let $p \in U \subset X$ where U is an open set in X. Observe that U^c is closed and $p \notin U^c$. By regularity, there must exist disjoint open sets V, W such that $p \in V$ and $U^c \subset W$. Now observe that $V \subset W^c$, and since W^c is closed, we know that $\overline{V} \subset W^c$. However, since $U^c \subset W$,

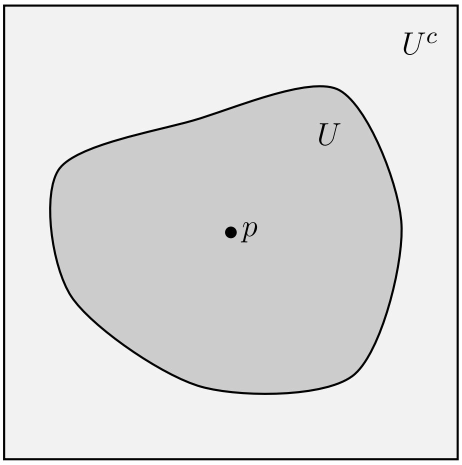
$$U^c \cap W^c = \emptyset \implies U^c \cap \overline{V} = \emptyset \implies \overline{V} \subset U.$$

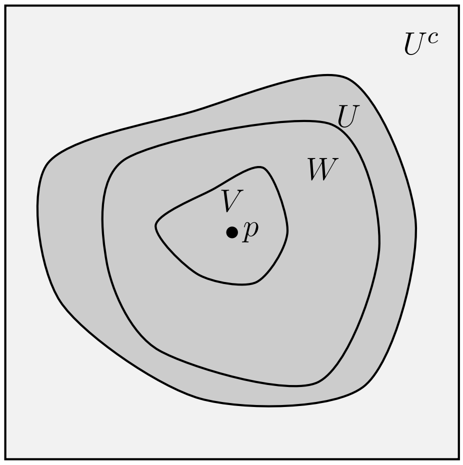
Thus we have found an open set V such that $p \in V$ and $\overline{V} \subset U$, as desired.

Now we prove the reverse direction. Suppose for each $p \in X$ and open set U containing p, there's an open set V such that $p \in V$ and $\overline{V} \subset U$.

Let A be a closed set not containing $x \in X$. As x is not a limit point of x, there exists an open set U such that $x \in U$ and $U \cap A = \emptyset$.

By hypothesis, there must exist an open V such that $p \in V$ and $\overline{V} \subset U$. Then observe





The first picture shows an arbitrary open set U containing p. In the second picture, we see that $U^c \subset W$, so the boundary of W lives inside U. V is disjoint from W, but contains p, so it also lives inside U.

that (1) $x \in V$ and (2) $A \subset (\overline{V})^c$ and $V \cap (\overline{V})^c = \emptyset$. Since A was an arbitrary closed set and x an arbitrary point not in A, and we contained A and x in disjoint, open sets, we have that X must be a regular space by definition.

$\blacksquare$

Theorem 5.9 A topological space X is normal if and only if for each closed set A in X and open set U containing A there exists an open set V such that $A \subset A$ and $\overline{V} \subset U$.

Proof: Suppose that X is normal. Let A be a closed set, and U an open set about A. Since U^c is closed and disjoint from A, there must exist a pair of disjoint open sets V, W such that $U^c \subset V$ and $A \subset W$. Next observe that since V and W are disjoint, we know that $W \subset V^c$. Since V^c is closed, we also know that $\overline{W} \subset V^c$. But $V^c \subset U$; Thus we have that $A \subset W \subset \overline{W} \subset V^c \subset U$. Thus for every closed A and U containing A, there exists an open set W such that $A \subset \overline{W} \subset U$, as desired.

Now suppose that for a closed set A and an open set U containing A there exists an open set V such that $A \subset V$ and $\overline{V} \subset U$. Let B be a closed set which is disjoint from A. Since A and B have no limit points in common, we can see that for each $a \in A$, there exists an open set U_a such that $U_a \cap B = \emptyset$. Thus let $U' = \bigcup\limits_{a \in A} U_a$, which is an open set. Then $U' \cap B = \emptyset$ by construction, and by assumption there must exist an open set W such that $A \subset W$ and $\overline{W} \subset U'$. Next observe that $\overline{U}^c$ is an open set which contains B, so by assumption there exists a W' such that $B \subset W' \subset \overline{W'} \subset \overline{U}^c$. Thus we see that

$A \subset W$ and $B \subset W'$, and $W \cap W' = \emptyset$ since $W \subset U$ but $W' \subset \overline{U}^c$ Since A and B were arbitrary closed sets, and can be contained in disjoint open sets, we have that the space is normal, which proves the assertion.

■

Presented in class 2/20

Theorem 5.10 A topological space X is normal if and only if for each pair of disjoint closed sets A and B there are disjoint open sets U and V such that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$.

Proof: First we prove the forward direction. Suppose that X is a normal space. Then for every pair of disjoint closed sets A and B in X, there exist disjoint open sets such that $A \subset U$ and $B \subset V$. However, by Theorem 5.9, we know that there must exist open sets U' and V' such that $A \subset U' \subset \overline{U'} \subset U$ and $B \subset V' \subset \overline{V'} \subset U$. Since U' and V' are disjoint, this proves the existence of disjoint open sets containing A and B whose intersection of their closures is empty.

Next, suppose that for every pair of disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$. Since A, B are arbitrary disjoint closed sets, $A \subset U$ and $B \subset V$ and $U \cap V = \emptyset$, X satisfies the conditions of a normal space, so X must be normal.

■

Theorem 5.11 (The Incredible Shrinking Theorem.) A topological space X is normal if and only if for each pair of open sets U, V such that $U \cup V = X$, there exist open sets U', V' such that $\overline{U'} \subset U$ and $\overline{V'} \subset V$ and $U' \cup V' = X$.

Proof: First we prove the forward direction. Suppose X is normal and that U, V are open sets such that $U \cup V = X$. Observe that $U^c \subset V$ and $V^c \subset U$. By Theorem 5.9 there must exist open sets U', V' such that

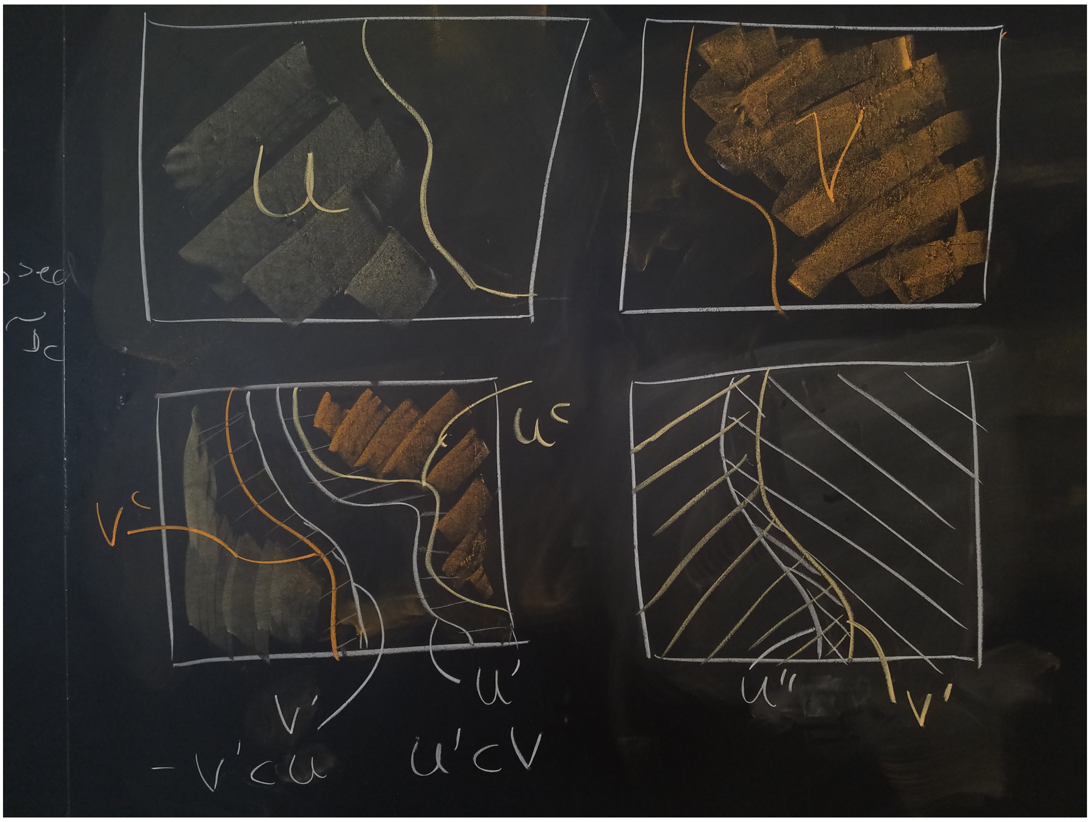
$$U^c \subset U', \overline{U'} \subset V,$$

$$V^c \subset V', \overline{V'} \subset U.$$

Since $(V')^c$ is a closed and $(V')^c \subset V$, we can apply Theorem 5.9 again to conclude that there must exist a set U'' such that

$$(V')^c \subset U'', \overline{U''} \subset V.$$

Now since U'' and V' are open sets such that $\overline{U''} \subset V$, $\overline{V'} \subset U$, and $U'' \cap V' \neq \emptyset$ because $(V')^c \subset U''$, we have that $U'' \cup V' = X$.



Thus this finishes the proof in this direction. Next we prove the other direction. Suppose that for every pair of open sets $U, V \subset X$ such that $U \cup V = X$, there exists open sets U', V' such that $\overline{U'} \subset U$ and $\overline{V'} \subset V$.

Let A and B be disjoint closed sets in X. Observe that A^c, B^c are open sets such that $A^c \cup B^c = X$. Thus there must exist open sets U, V such that

$$\overline{U} \subset A^c \,, \overline{V} \subset B^c, U \cup V = X.$$

Next observe that

$$(A^c)^c \subset (\overline{U})^c \subset U^c \implies A \subset (\overline{U})^c \subset U^c$$
$$(B^c)^c \subset (\overline{V})^c \subset V^c \implies B \subset (\overline{V})^c \subset V^c$$

Since $U \cup V = X$, we have that $U^c \cap V^c = \emptyset$ by DeMorgan's laws. Hence, $(\overline{U})^c$ and $(\overline{V})^c$ are disjoint open sets such that $A \subset (\overline{U})^c$ and $B \subset (\overline{V})^c$. Thus by definition, X is normal.

■

Exercise 5.12

1. Describe an example of a topological space that is T_1 but not T_2.

2. Describe an example of a topological space that is T_2 but not T_3.
3. Describe an example of a topological space that is T_3 but not T_4.

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1. Finite complement topology is an example. First, every set of the form $X - \{p\}$ where $p \in X$ is open, so the complement $X - (X - \{p\}) = \{p\}$ is closed. Hence by Theorem 5.1, X is T_1.
   Now suppose X is T_2, Then for all $p, q \in X$ and $p \neq q$, there exists disjoint, open sets U, V such that $p \in U$, $q \in V$. However, $U \cap V \implies V \subset U^c$. But this is a contradiction since U^c is finite, by construction, and V must at least be infinite (since we know that V^c is finite.) Thus X is not T_2.

   Another example would be the countable complement topology, and the proof is almost exactly as the one presented for the finite complement topology.

2. The harmonic set is T_2 but not T_3. This is because for any two points $p, q \in \mathbb{R}$ we can contain them in disjoint open sets (a, b) and (c, d) where $a < p < b < c < q < d$ or $c < q < d < a < p < b$. If either p or q are in H, then it is vacuously true that we can contain them in an open set disjoint from any open set containing another point because there are no open sets which contain elements of H.

   The harmonic set is not T_3 since (1) H is a closed set (as it has no limit points) and (2) no open set can contain H. Therefore, it cannot be regular, and hence not T_3.

3. In Exercise 5.4, we showed that $\mathbb{H}_{\text{bub}}$ is regular. Observe that by Exercise 4.10.3 every point on the x-axis can be contained in disjoint open sets, and it is trivial that two distinct points in $\{(x, y) : y > 0\}$ can be contained in disjoint open sets. Thus $\mathbb{H}_{\text{bub}}$ is T_3. However, by Exercise 4.10.5 the rationals and irrationals on the x-axis cannot be contained in disjoint open sets, and the rationals and irrationals are closed sets. Hence $\mathbb{H}_{\text{bub}}$ is not normal. Therefore, $\mathbb{H}_{\text{bub}}$ is T_3 but not T_4.

Exercise 5.14 Show that $\mathbb{H}_{\text{bub}}$ is not normal.

In the previous chapter, we saw that there does not exist disjoint open sets U and V in $\mathbb{H}_{\text{bub}}$ such that $\mathbb{Q} \in U$ and $\mathbb{R} - \mathbb{Q} \subset V$. However, observe that $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are closed sets. With this example we see that $\mathbb{H}_{\text{bub}}$ cannot be a normal set.

Theorem 5.15 Order topologies are T_1, Hausdorff, regular and normal.

Proof: ($\mathbf{T_1}$.) Suppose X has the order topology. Let $a \in X$. Observe that $\{x \in X | a < x\} \cup \{x \in X\ x > a\}$ is the union of two open sets, so it is open, and hence its complement, which is $\{a\}$, is closed. Thus every singleton set is a closed set, so X is T_1.

(**Hausdorff.**) Consider two distinct points $a, b \in X$, and suppose without loss of generatlity that $a < b$. If there exists an element $c \in X$ such that $a < c < b$ then

$$\{x \in X | x < c\} \text{ and } \{x \in X | c < x\}$$

are two disjoint open sets which contain a, b respectively.
If there is no $c \in X$ such that $a < c < b$, then

$$\{x \in X | x < b\} \text{ and } \{x \in X | a < x\}$$

are two disjoint open sets which contain a, b respectively. Thus X is also a Hausdorff space.

(**Regular.**) Now let A be a closed set and suppose $x \notin A$. Since x is not a limit point of A, we see that there must exist an open set U which contains x and $U \cap A = \emptyset$. Then a open set which is disjoint from U and contains A is $A^c - U$, which shows that the space is regular.

■

Theorem 5.16 Let X and Y be Hausdorff. Then $X \times Y$ is Hausdorff.

Proof: If X and Y are both Hausdorff, then for two distinct $p = (p_x, p_y)$ and $q = (q_x, q_y)$ both in $X \times Y$, there exists disjoint open sets $U_{p_x}, U_{q_x} \in \mathcal{T}_X$ such that $p_x \in U_{p_x}$, $q_x \in U_{q_x}$ and another pair of disjoint sets $V_{p_y}, V_{q_y} \in \mathcal{T}_Y$ such that $p_y \in U_{p_y}$ and $q_y \in U_{q_y}$. Now observe that $p \in U_{p_x} \times U_{p_y}$ and $q \in V_{q_x} \times V_{q_y}$ while $U_{p_x} \times U_{p_y}$ is disjoint with $V_{q_x} \times V_{q_y}$. Since p, q were arbitrary distinct points in $X \times Y$, we have that $X \times Y$ is a Hausdorff space.

■

Theorem 5.17 Let X and Y be regular. Then $X \times Y$ is regular.

Proof: Suppose X and Y are regular, and let $p = (p_x, p_y) \in X \times Y$. Suppose p is contained in an open set W. Then there exists a basic open set of the form $U \times V$ which contains p and where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$. Now since X and Y are regular, we can use Theorem 5.8 to conclude the existence of open sets U' and V' such that $p_x \in U'$ and $\overline{U'} \subset U$ and $p_y \in V'$ while $\overline{V'} \subset V$. Therefore, $\overline{U'} \times \overline{V'} \subset U \times V$.

Now observe $p \in U' \times V'$ and that, by Exercise 4.34, $\overline{U' \times V'} = \overline{U'} \times \overline{V'} \subset U \times V$ and hence is entirely contained in W. Since p and W were arbitrary, this shows that $X \times Y$ satisfies Theorem 5.8, allowing us to conclude that $X \times Y$ is a regular space.

■

Presented in class 2/25/19

Theorem 5.19 Every Hausdorff is hereditarily Hausdorff.

Proof: Let Y be a subset of X, and consider the relative topology of Y given by

$$\mathcal{T}_Y = \{V | V = Y \cap U, U \in \mathcal{T}_X\}.$$

Since X is Hausdorff, we know that for any distinct pair of points $p, q \in Y$, which are obviously also points in X, there exist disjoint open sets $U', V' \in \mathcal{T}_X$ such that $p \in U'$ and $q \in V'$. Next observe that $U'' = Y \cap U'$ and $V'' = Y \cap V'$ are two disjoint open sets in $\mathcal{T}_Y$ such that $p \in U''$ and $q \in V''$. Since p, q were arbitrary points of Y, we have that Y must also be a Hausdorff space.

■

Theorem 5.20 Every regular space is hereditarily regular.

Proof: Let Y be a subset of X endowed with the relative topology on X. Then consider C closed in $(Y, \mathcal{T}_Y^{\text{rel}})$, and a point $x \in Y$ such that $x \notin C$. From Theorem 4.28, we know that C is closed if and only if there exists a set D closed in $(X, \mathcal{T}_X)$ such that $C = D \cap Y$.

Since X is regular, and we know that $x \notin D$, then for the set D closed in $(X, \mathcal{T}_X)$ there exist disjoint sets $U, V \in (X, \mathcal{T}_X)$ such that $D \subset U$ and $x \in V$. Now observe that $U' = U \cap Y$ and $V' = V \cap Y$ are disjoint sets open in $\mathcal{T}_Y^{\text{rel}}$ such that $C \subset U'$ and $x \in V'$. Since C was an arbitrary closed set in $(Y, \mathcal{T}_Y^{\text{rel}})$ and x was a arbitrary point of Y but not of C, the topological space $(Y, \mathcal{T}_Y^{\text{rel}})$ satisfies the properties of being regular so Y is a regular space.

■

Theorem 5.23 Let A be a closed subset of a normal space X. Then A is normal when given the relative topology.

Proof: Let X be a normal space, and consider the relative topology on A:

$$\mathcal{T}_A^{\text{rel}} = \{U | U = A \cap V \text{ where } V \in \mathcal{T}_X\}$$

Now consider a pair of disjoint closed sets D, C closed in $(A, \mathcal{T}_A^{\text{rel}})$. Then by Theorem

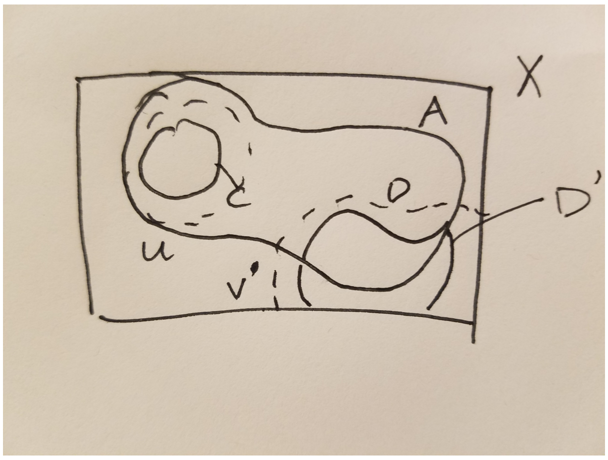

Figure 1: In this figure, we drew a closed set C completely contained in A and a closed set D which shares limit points with A in $(X, \mathcal{T}_X)$.

4.28, we know that there must exist sets D' and C' closed in $(X, \mathcal{T})$ such that $C = A \cap C'$ and $D = A \cap D'$. Now observe that since C and D are result of intersecting two sets which are closed in $(X, \mathcal{T}_X)$, we must have that C and D are also sets closed in $(X, \mathcal{T}_X)$.

Now since X is normal and C and D are disjoint and closed in $(X, \mathcal{T}_X)$, there must exist disjoint, open sets U and V in $(X, \mathcal{T}_X)$ such that $C \subset U$ and $D \subset V$. Next, let $U' = A \cap U$ and $V' = A \cap V$ and observe that (1) U' and V' are disjoint open sets in $(A, \mathcal{T}_A^{\text{rel}})$ and (2) $C \subset U'$ and $D \subset V'$. This is because $C \subset A$, so if $C \subset U$ then we are certain that $C \subset U \cap A = U'$; an identical argument applies to D. Now since C and D were arbitrary closed sets of A, we have that A is a normal space when given the relative topology.

■

Exercise 5.25 Let Y be a subspace of a topological space X, and let A and B be two disjoint closed subsets of Y in the subspace topology. Show that both $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, where the closures are taken in X.

Solution: If A and B are two disjoint closed sets in the subspace topology, then observe that no point of one set is a limit point of the other. Thus for every point of $a \in A$, we can construct a set $U_a \in \mathcal{T}_Y$ such that $U_a \cap B = \emptyset$. Similarly, for every point $b \in B$ we can construct a set $U_b \in \mathcal{T}_Y$ such that $U_b \cap A = \emptyset$.

Let a be a limit point of A in $(X, \mathcal{T})$, and consider an open set $U \in \mathcal{T}_X$ containing a. Then let $U' = U \cap Y$, and observe that $U' \cap B = \emptyset$ because A and B are disjoint closed sets in $(Y, \mathcal{T}_{\mathcal{Y}})$. Thus we see that a cannot be a limit point of B, so that $\overline{A} \cap B = \emptyset$, where the closure is taken in X. We can repeat the same argument for B since the argument is symmetric, and conclude that $A \cap \overline{B} = \emptyset$ as well, giving the desired result.

□

Theorem 5.26 The space X is a completely normal space if and only if X is hereditarily normal.

Proof: Suppose X is completely normal. Let Y be a subset and consider any two disjoint, closed sets C and D in the subspace topology

$$\mathcal{T}_Y = \{U | U = V \cap Y : V \in \mathcal{T}_X\}.$$

By Exercise 5.25, these sets are separated in X.

Since C and D are separated in X, there exist two disjoint open sets A, B such that $C \subset A$, $D \subset B$. Therefore, $A \cap Y$ and $B \cap Y$ are two open sets in $\mathcal{T}_Y$ such that $C \subset A \cap Y$ and $B \cap Y$, which shows that Y is normal. Since Y was an arbitrary subset we have that X is hereditarily normal.

Suppose now that X is heredetiarily normal, and consider two separated subsets A and B of X. Denote the subspace $A \cup B$ as Y. Then observe that

$$Y \cap \overline{B} = (A \cup B) \cap \overline{B} = (A \cap \overline{B}) \cup (B \cap \overline{B}) = B.$$

Thus B is closed in Y, since $\overline{B}$ is closed in X, and by Theorem 4.28 this implies that $Y \cap \overline{B} = B$ is a closed set in the subspace Y. By analogous reasoning, we also have that A is closed in Y.

Since X is heredetiarily normal, Y is a subspace, and A, B are disjoint closed sets in Y, we can contain A and B in disjoint open sets U and V in Y. However, we also know $U = U' cap Y$ and $V = V' \cap Y$ for $U', V' \in \mathcal{T}_X$.

■

Theorem 5.29 (The Normality Lemma). Let A and B be subsets of a topological space X and let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be two collections of open sets such that

1. $A \subset \bigcup_{i\in\mathbb{N}} U_i$
2. $B \subset \bigcup_{i\in\mathbb{N}} V_i$
3. for each i in $\mathbb{N}$, $\overline{U_i} \cap B = \emptyset$ and $\overline{V_i} \cap A = \emptyset$.

Then there exist open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Proof: Suppose that (1) and (2) hold, and let

$$U = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \left(U_n - \bigcup_{i=1}^{n} \overline{V_i} \right)$$

and

$$V = \bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} \left(V_n - \bigcup_{i=1}^{n} \overline{U_i} \right).$$

Note that U and V are open because they each are the countable union of open sets. This is because each $\bigcup_{i=1}^{n} \overline{U_i}$ and $\bigcup_{i=1}^{n} \overline{V_i}$ finite unions of closed sets, and hence are closed. Thus each $U_n - \bigcup_{i=1}^{n} \overline{V_i}$ and $V_n - \bigcup_{i=1}^{n} \overline{U_i}$ are open sets by Theorem 3.15.

Now observe that $A \subset U$, $B \subset V$. We'll show this is true for A, since the argument that this is true for B will be identical. Thus let $a \in A$. Then $a \in U_n$ for some $n \in \mathbb{N}$. However, $a \notin \overline{V_i}$ for any $i \in \mathbb{N}$, so that $a \in U_n - \bigcup_{i=1}^{n} \overline{V_i}$. Therefore, $a \in \bigcup_{n\in\mathbb{N}} \left(U_n - \bigcup_{i=1}^{n} \overline{V_i} \right) = U$, so that $A \subset U$.

Finally, observe that $U \cap V = \emptyset$. If not, then there exists an $x \in U \cap V$. meaning that for some $m, n \in \mathbb{N}$,

$$x \in U_n - \bigcup_{i=1}^{n} \overline{V_i}$$
$$x \in V_m - \bigcup_{i=1}^{m} \overline{U_i}.$$

Without loss of generality, suppose $n \leq m$. Then by the second equation, we see that $x \notin U_i$ for $i \leq m$. However, this implies that $x \notin U_n$ since $n \leq m$, which contradicts the first above equation. Thus there cannot be such an x, and $U \cap V = \emptyset$.

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Presented sketch 2/27/18

Theorem 5.30 If X is normal and $C = \cup_{i \in \mathbb{N}} K_i$ is the union of closed sets K_i in X, then the subspace C is normal.

Proof:

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Theorem 5.31 Suppose a space X is regular and countable. Then X is normal.

Proof: Consider two sets A and B. Since X is regular, we know by an application of the definition that for all $a \in A$, there exist open sets $\{U_a\}$ such that $a \in U_a$ which are each disjoint with $\overline{B}$. Similarly, there must exist open sets $\{U_b\}$ such that $b \in U_b$ which are each disjoint with $\overline{A}$.

Now by Theorem 5.8, we know that for each open set U_a containing a, there exists an open set V_a such that $a \in V_a$ and $\overline{V_a} \subset U_a$. Similarly, for each open set U_b containing b, there exists an open set V_b such that $b \in V_b$ and $\overline{V_b} \subset U_b$

Observe that $A \subset \bigcup\limits_{a \in A} V_a$, $B \subset \bigcup\limits_{b \in B} V_b$, and that $\overline{V_b} \cap A = \overline{V_a} \cap B = \emptyset$ for all $a \in A$ and $b \in B$. Since A, B are at most countable, the sets $\{U_a\}_{a \in A}$ and $\{U_b\}_{b \in B}$ are at most countable.
Thus by the Normality Lemma, we can then conclude there exist open sets U and V such that $A \subset U$ and $B \subset V$ while $U \cap V = \emptyset$. Therefore, we can conclude that X is normal, which is what we set out to show.

■

Presented 2/27/18

Theorem 5.32 Suppose a space X is regular and has a countable basis. Then X is normal.

Proof: Consider two disjoint subsets A and B of X. Since they are disjoint, we know that for each $a \in A$, there exists an open set U_a such that $U_a \cap B = \emptyset$ for all $a \in A$. Similarly for each $b \in B$, we know that there exists an open set $U_b \cap A = \emptyset$ for all $b \in B$.

Observe that the sets $\{U_a\}$ and $\{U_b\}$ may or may not be countable. However, since we have a countable basis $\mathcal{B} = \{B_1, B_2, \dots\}$, we know by Theorem 4.1 that each $a \in A$ is contained in some basis element B_i such that $a \in B_i \subset U_a$, where $i \in \mathbb{N}$. Thus let

$\mathcal{B}_A$ be the set of basis elements such that $A \subset \bigcup_{B_A \in \mathcal{B}_A} B_A$. Similarly, every $b \in B$ is contained in a basis element B_j such that $b \in B_j \subset U_b$, where $j \in \mathbb{N}$. Now let $\mathcal{B}_B$ by the set of basis elements such that $B \subset \bigcup_{B_B \in \mathcal{B}_B} B_B$.

Now by Theorem 5.8, for each $a \in A$, there exists an open set $V_{j(a)}$ such that $a \in V_{j(a)}$ and $\overline{V_j(a)} \subset B_{i(a)}$ where $j(a) \in \mathbb{N}$ and $i(a) \in \mathbb{N}$ is the index which corresponds to the set in $\{B_1, B_2, \dots\}$ such that $a \in B_{i(a)}$. Similarly for B, we know that for each $b \in B$ there exists an open set $W_{j(b)}$ such that $b \in W_{j(b)}$ where $j \in \mathbb{N}$ and $\overline{W_j(b)} \subset B_{i(b)}$ where $i(b)$ is defined analogously for how we defined $i(a)$.

Finally, observe that $A \subset \bigcup_{j \in \mathbb{N}} V_j$, $B \subset \bigcup_{j \in \mathbb{N}} W_j$, and that $V_j \cap B = W_j \cap A = \emptyset$ for each $j \in \mathbb{N}$. Since these conditions satisfy that the normality lemma, we have that X is normal, which is what we set out to show.

$\blacksquare$