Chapter 10
Metric Spaces: Getting some distance

4/8/19 Q: Why does X have to be a metric space?

Lebesgue Number Theorem 10.24 Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of a compact set A in a metric space X. Then there exists a $\delta > 0$ such that for every point $p \in A$, $B(p,\delta) \subset U_\alpha$ for some α. This number δ is called a **Lebesgue number** of the cover.

Proof: Since A is compact, there exists a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$. Now suppose for the sake of contradiction that there does not exists such a δ. Then in order for this to happen, we would need that for every $B(p,\delta)$ containing p there exists a member of $U_{\alpha'} \in \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that $B(p,\delta) \not\subset U_{\alpha'}$. But since we can theoretically propose an infinite number of δ, we must have an infinite number of such U'_αs.

However, we cannot do this in the finite subcover, as it is finite. Therefore the contrary must be true: there exists a δ such that for every $p \in A$, $B(p,\delta) \subset U_\alpha$ for some α. And since this is true for the finite subcover, which is a subset of the open cover, this is definitely true for the open cover.

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Theorem 10.25 Let $\gamma : [0,1] \to X$ be a **path**: a continuous map from $[0,1]$ into the space X. Given an open cover $\{U_\alpha\}$ of X, show that $[0,1]$ can be divided into N intervals of the form $I_i = [\frac{i-1}{N}, \frac{i}{N}]$ such that each $\gamma(I_i)$ lies completely in one set of the cover.

Proof: If $\{U_\alpha\}_{\alpha\in\lambda}$ is an open cover of X, then consider the set $\{\gamma^{-1}(U_\alpha)\}_{\alpha\in\lambda}$. This will be an open cover of γ, since we know γ maps $[0,1]$ into X. However, since γ is compact, we know by Lebesgue Number Theorem that there exists a δ such that $p \in B(p,\delta) \subset \gamma^{-1}(U_\alpha)$ for all $p \in [0,1]$ where $\gamma^{-1}(U_\alpha)$ is some set in the open cover containing p.

Let $\dfrac{1}{N} < \delta$ where N is a positive integer. Then observe that the sequence of intervals

$$\left[\frac{i-1}{N}, \frac{i}{N}\right] \quad 1 \leq i \leq N$$

will each be contained in at least one member of $\gamma^{-1}(U_\alpha)$. Thus

$$\left[\frac{i-1}{N}, \frac{i}{N}\right] \subset \gamma^{-1}(U_\alpha) \implies \gamma\left(\left[\frac{i-1}{N}, \frac{i}{N}\right]\right) \subset U_\alpha.$$

Thus $[0,1]$ can be divided into N intervals of the form $I_i = [\frac{i-1}{N}, \frac{i}{N}]$ such that each $\gamma(I_i)$ lies completely in one set of the cover in X, which is what we set out to show.

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