Chapter 7

Compactness: The Next Best Thing To Being Finite

Theorem 7.1 Let X be a finite topological space. Then X is compact.

Proof: Consider an open cover $\mathcal{C}$ of the set X. Since $\mathcal{C}$ covers X, we know that for each $p \in X$ there exists an open set $U_p \in \mathcal{C}$ such that $p \in U_p$. Since X is finite, there are finitely many open sets $U_p \in \mathcal{C}$ such that $p \in U_p$. Therefore, we see that $\{U_p : p \in X\}$ is a finite subcover of $\mathcal{C}$, which shows that X is compact.

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Theorem 7.2 Let C be a compact subset of $\mathbb{R}_{\text{std}}$. Then C has a maximum point, that is, there is a point $m \in C$ such that for every $x \in C$, $x \leq m$.

Proof: Let $\mathcal{C}$ be an open cover of the set. Then it must have some finite subcover $\mathcal{C}'$. However, since the basic open sets of $\mathbb{R}$ are balls, there must be a finite set of basic open sets which cover C. However, every open set is of the form $(x-\epsilon, x+\epsilon)$, where $x \in C$ and $\epsilon > 0$. Take max x which appears in this finite open cover, and observe that for all $c \in C$, $c \leq x$. Thus C must have a maximum, as desired.

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Theorem 7.3 If X is a compact space, then every infinite subset of X has a limit point.

Proof: Consider an infinite subset A of X. Suppose that A has no limit points. Then for every $p \in X$, there exists an open set U_p such that $(U_p - \{p\}) \cap A = \emptyset$. However, this would imply that $\bigcup_{a \in A} U_a \subset X$, so that any open cover would automatically have to be infinite. But this contradicts the fact that X is compact. Therefore, A must have a limit point in X.

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Corollay 7.4 If X is compact and E is a subset of X with no limit point, then E is finite.

Proof: Suppose X is compact and a subset E has no limit point in X. Then for every $p \in X$, we know that there exists an open set U_p which contain p, $(U_p - \{p\}) \cap E = \emptyset$. Then again, $\bigcup_{p \in E} U_p \subset X$. Since every open cover of X must have a finite subcover, we know that there cannot be an infinite number of open sets U_p for $p \in E$, since we could then never cover it with finitely many open sets. Thus $\{U_p : p \in E\}$ has to be restricted to be finite, so that E must be finite, as desired.

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Theorem 7.5 A space X is compact if and only if every collection of closed sets with the finite intersection property has a non-empty intersection.

Proof: Suppose X is compact, and let $\mathcal{C}$ be a collection of closed sets with the finite interscetion property. Suppose that $\bigcap_{C \in \mathcal{C}} C \neq \{p\}$ for some $p \in X$; this is a trivial case of the theorem.
Now let $p_1 \in C_1 \cap C_2$ for $C_1 \neq C_2$ and $C_1, C_2 \in \mathcal{C}$. We can then construct a sequence of points p_i such that

$$p_i \in C_1 \cap C_2 \cap \cdots \cap C_i \cap C_{i+1}$$

where $C_i, C_{i+1} \in \mathcal{C}$ and $C_i \neq C_{i+1}$ for all $i \in \mathbb{N}$. Since $\mathcal{C}$ has the finite intersection property, we know for a fact that we can always find a p_i in the finite intersection.

now if $\mathbb{C}$ has an empty intersection, then this implies that this sequence of points $\{p_i : i \in \mathbb{N}\}$ converges to a point p which is not contained in any $C \in \mathbb{C}$. First of all, we know it will converge to some point in X by Theorem 7.3. Second of all, observe that if p is the limit of this sequence, then for every open set U which contains p, there exists a $N \in \mathbb{N}$ such that for $i > N$, $p_i \in U$. Thus, in other words, if U contains p, then

$$(U - \{p\}) \cap C_i \neq \emptyset$$

for $i \in \mathbb{N}$. Thus p is a limit point for each C_i, and since each C_i is closed, $p \in C_i$ for all $i \in \mathbb{N}$.

Second attempt:
First we'll prove the forward direction. Suppose that X is a compact space, and let $\mathcal{C}$ be a collection of closed sets in X with the finite intersection property. For the sake of contradiction, suppose that $\bigcap_{C \in \mathcal{C}} C = \emptyset$. Then observe that $\{C^c : C \in \mathcal{C}\}$ (where ^{c} denotes the complement) is an open cover of X. Since this set is an open cover, it must have a finite subcover, which means that there exist sets $C_1^c, C_2^c \dots C_n^c$ such that

$$\bigcup_{i=1}^{n} C_i^c = X.$$

However, taking the complement of this leads to

$$\bigcap_{i=1}^{n} C_i = \emptyset$$

which contradicts the finite interscetion propety of $\mathcal{C}$. Thus we have a contradiction, which implies that there must exist a $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$ as desired.

Now we prove the other direction. Suppose that for every collection $\mathcal{C}$ of closed sets in X with the finite interesection property, we have that

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset.$$

Now let $\mathcal{U}$ be an open cover of X. Suppose for the sake of contradiction that this does not have a finite subcover. Observe that the set $\{U^c : U \in \mathcal{U}\}$ is a collection of closed subsets in X with the finite intersection property, since each U^c is a closed set which is not disjoint with any other set. Since we know that every collection of closed sets in X with the finite interscetion property has a nonempty interesection, we can conclude that

$$\bigcap_{U \in \mathcal{U}} U^c \neq \emptyset \implies \bigcup_{U \in \mathcal{U}} U \neq X.$$

However, the last equation contradicts the fact that U was an open cover of X. Thus we have that every open cover must have a finite subcover, proving that X is compact as desired, completing the proof.

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Theorem 7.6 A space X is compact if and only if for any open set U in X and any collection of closed sets $\{K_\alpha\}_{\alpha \in \lambda}$ such that $\cap_{\alpha \in \lambda} K_\alpha \subset U$, there exist a finite number of K_α's whose interesection lies in U.

Proof: First we'll prove the forward direction. Suppose X is compact and that $\cap_{\alpha \in \lambda_1} K_\alpha \subset U$ for some index λ_1. Observe that

$$U^c \subset \bigcup_{\alpha \in \lambda} K_\alpha^c$$

so that

$$\bigcup_{\alpha \in \lambda} K^c \cup U$$

is an open cover of X. Observe that this must have a finite subcover, so that λ_1 can at least be finite. Thus there can be a finite number of K_α's, given by $\{K_1, K_2, \dots, K_n\}$ such that

$$\bigcap_{i=1}^{n} K_i \subset U$$

which proves this direction.

Now we'll prove the other direction. Suppose that for every $U \subset X$ and any collection of closed sets $\{K_\alpha\}_{\alpha\in\lambda}$ such that $\bigcap_{\alpha\in\lambda} K_\alpha \subset U$, there exist a finite number of K_α's such that their intersection lies in U.
Now suppose that $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ is an open cover of X. Then observe that the set $\{U_\alpha^c\}_{\alpha\in\lambda}$ is a collection of closed sets such that $\bigcap_{\alpha\in\lambda} U_\alpha^c \subset \emptyset$. By assumption, there must exist a finite number of U_α^c's such that their intersection lies in $\emptyset$. Call these U_α's $U_1^c, U_2^c, \ldots, U_n^c$. Then

$$\bigcap_{i=1}^n U_i^c \subset \emptyset \implies \bigcap_{i=1}^n U_i^c = \emptyset \implies \bigcup_{i=1}^n U_i = X$$

which shows that $\mathcal{U}$ must always have a finite subcover. Therefore, the space is compact as desired.

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Exercise 7.7 If A and B are compact subsets of X, then $A \cup B$ is compact. Suggest and prove a generalization.

Solution: Suppose $\mathcal{W}$ is an open cover of $A \cup B$. Then observe that $\mathcal{W}$ is a cover of both A and B, and since A and B are compact, there exist finite subcovers of $\mathcal{W}$, denoted $\mathcal{W}_A$ and $\mathcal{W}_B$, such that $A \subset \mathcal{W}_A$ and $B \subset \mathcal{W}_B$. Now observe that $\mathcal{W}_A \cup \mathcal{W}_B$ is a finite subcover of $\mathcal{W}$, so that every open cover of $A \cup B$ has a finite subcover. Therefore $A \cup B$ is compact, as desired.

This can be extended to finitely many unions of compact sets. Suppose that $A_1, A_2, \ldots, A_n$ are compact. Then $A_1 \cup A_2 \cup \ldots A_n$ is compact. This is because any open cover $\mathcal{W}$ of $A_1 \cup A_2 \cup \ldots A_n$ is also an open cover for each $A_1, \ldots, A_n$, so there are finite subcovers $\mathcal{W}_{A_i}$ such that $\mathcal{W}_{A_i}$ covers A_i for $i = 1, 2, \ldots, n$. Therefore, $\mathcal{W}_{A_1} \cup \cdots \cup \mathcal{W}_{A_n}$ is a finite subcover of $\mathcal{W}$ containing $A_1 \cup \cdots \cup A_n$, so that $A_1 \cup \cdots \cup A_n$ is compact. However, this cannot be extended to infinitely many unions of compact sets since unioning infinitely many finite subcovers will not yield a finite subcover.

□

Theorem 7.8 Let A be a closed subspace of a compact space. Then A is compact.

Proof: Let X be compact and A a closed subspace of X. Then any closed set in A can be expressed as $D \cap A$, where D is closed in X. Since X is comapct, by Theorem 7.5, any collection of closed sets in X with the finite intersection property has a nonempty intersection. But closed sets in A are closed sets in X, so that any collection of closed sets in A with the finite intersection property have a nonempty interesection, which proves that A is a compact set. Therefore, A is compact.

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Theorem 7.9 Let A be a compact subspace of a Hausdorff space X. Then A is closed.

Proof: Let $q \in X - A$. Since X is Hausdorff, for any $p \in A$, there exist disjoint open sets U_p and V_p such that $p \in U_p$ and $q \in V_q$. Now observe that the set $\{U_p | p \in A\}$ is an open cover of A, where each member corresponds to a disjoint open set V_p of the point q. Since A is a compact set, we know that the set must have a finite subcover; denote it as $\{U_{p_1}, U_{p_2}, \ldots, U_{p_n}\}$. Then the set $\bigcap_{i=1}^n V_{p_i}$ is a an open set containing q, (open because the interesection is finite) which is disjoint from A. Since this must hold for all $q \in X - A$, this shows that $X - A$ is an open set. Therefore, A is closed, as desired.

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Exercise 7.10 Construct an example of a compact subset of a topological space that is not closed.

Solution: On the discrete topology, an finite set is an open set, although as we saw from Theorem 7.1 any finite set is also a compact set.

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Exercise 7.11 Must the intersection of two compact sets be compact? Add hypothesis, if necessary. Extend any theorems you discover, if possible.

Theorem 7.12 Every compact, Hausdorff space is normal.

Proof: First we can show that X is regular. Suppose A is closed and consider any $p \notin A$. Then observe that, since X is Hausdorff, for each $a \in A$, there are disjoint open sets U_a and V_a such that $a \in U_a$ and $p \in U_a$. Then

$$U = \{U_a : a \in A\}$$

is an open cover of A, and since A is closed Theorem 7.8 guarantees that A is compact, and therefore there is a finite subcover

$$U' = \{U_a : a \in F\}$$

where F is a finite subset of A. Therefore, the set $V = \bigcap_{a \in F} V_a$ is an open set containing p but is entirely disjoint from all sets in U' by construction. Since A and $p \notin A$ were arbitrary, and we contained them in disjoint open sets, then we have that X is regular.

Now let A be closed and U be an open set containing A. Then note that for each $a \in A$ that $a \in B_a \subset U$ where B_a is some basic open set. Thus $\{B_a : a \in A\}$ is an open cover of A. By compactness of A, there must exist a finite subcover, given by $\{B_a : a \in F\}$ where F is a finite subset of A.

By regularity, we know that for each $a \in B_a$ there exists an open set V_a such that $a \in V_a$ and $\overline{V_a} \subset B_a$. Therefore, we see that $V = \bigcap\limits_{a \in F} V_a$ is an open set containing A and

$$\overline{V} \subset \bigcap_{a \in F} \overline{V_a} \subset U.$$

Thus we have contained A in an open set V such that $A \subset V$ and $\overline{V} \subset A$. By Theorem 5.9, we can conclude that X is normal, as desired.

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Theorem 7.13 Let $\mathcal{B}$ be a basis for a space X. Then X is compact if and only if every cover of X by basic open sets in $\mathcal{B}$ has a finite subcover.

Proof: Suppose that X is compact and has a basis $\mathcal{B}$. Suppose that we cover X by basic open sets $B_{\alpha \in \lambda}$ such that $B_\alpha \in \mathcal{B}$ for all $\alpha \in \lambda$. Then because X is compact, there exsits a finite subcover, which we can express as $\{B_\alpha : \alpha \in \lambda'\}$ where λ' is a countable index. Thus we see that every cover of X by basic open sets in B has a finite subcover.

Now we prove the other direction. Suppose that every cover of X by basic open sets in $\mathcal{B}$ has a finite subcover. First observe that for any open cover $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$, each

U_α can be expressed as the union of basis elements $\{B_{\gamma(\alpha)} : \gamma \in \lambda'\}$. If $\mathcal{U}$ covers X, then the set of basic elements $\{B_\gamma : \gamma \in \lambda\}$ will still contain X. But since every cover of X by basic open sets in the basis have a finite subcover, there exists a finite set which covers X which we can denote as $\{B_\alpha : \alpha \in \lambda''\}$, where λ'' is a finite index. Hence $\mathcal{U}$ has a finite subcover, which implies that X is a compact space.

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Theorem 7.18 (The tube lemma) Let $X \times Y$ be a product space with Y compact. If U is an open set of $X \times Y$ containing the set $x_0 \times Y$, then there is some open set W in X containing x_0 such that U contains $W \times Y$ (called a "tube" around $x_0 \times Y$).

Proof: Let U be an open set in $X \times Y$ containing $x_0 \times Y$. Suppose for each $y \in Y$ we contain y in a set U_y and consider the product $U_x(y) \times U_y$, where $U_x(y) \times U_y \subset U$. Then since Y is compact, there exists a finite subcover of $\{U_y | y \in Y\}$. Suppose this is given by $\{U_{y_1}, \dots, U_{y_n}\}$. Then observe that

$$x_0 \subset \left(\bigcap_{i=1}^n U_{x(y_i)}\right) \times \left(\bigcup_{i=1}^n U_{y_n}\right) = \left(\bigcap_{i=1}^n U_{x(y_i)}\right) \times Y \subset U \times Y$$

so that $W = \bigcap_{i=1}^n U_{x(y_i)}$ is an open set in X such that $W \times Y \subset U$, as desired.

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Theorem 7.19 Let X and Y be compact spaces. Then $X \times Y$ is compact.

Proof:

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Heine-Borel Theorem 7.20 Let A be a subset of $\mathbb{R}^n$ with the standard topology. Then A is compact if and only if A is closed and bounded.

Proof: Let $A \subset \mathbb{R}$ and suppose A is compact. Since $A \subset \mathbb{R}^n$, we now that it must be the product of compact sets $A_i \in \mathbb{R}$, $i = 1, 2, \dots n$. By Theorem 7.15, each such A_i must be closed and bounded. Hence their product, A, must also be closed and bounded, which proves this direction.

Now suppose A is closed and bounded. Then A must be a product of closed, bounded sets $[a_i, b_i]$ where $a_i \leq b_i$ and $i = 1, 2, \dots, n$. However, by Theorem 7.14, each such set is compact, and by Theorem 7.19 their product must also be compact. Hence, A is compact, which proves the theorem.

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Alexander Subbasis Theorem 7.21 Let $\mathcal{S}$ be a subbasis for a space X. Then X is cmpact if and only if every subbasic open cover has a finite subcover.

Proof:

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Tychonoff's Theorem 7.22 Any product of compacts sets is compact.

Proof:

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