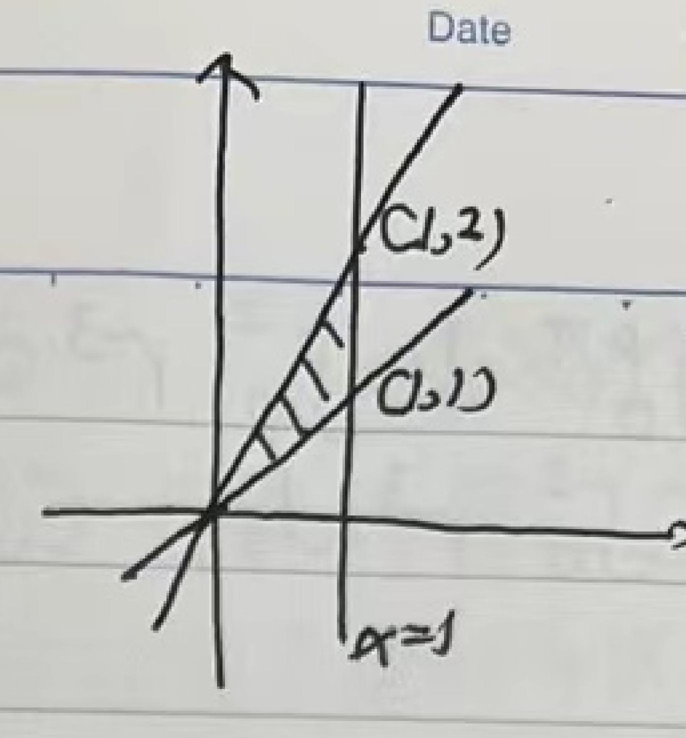
模拟题答案



题1: $I=\int_0^1[\int_x^{2x}(\sqrt{x}+y)dy]dx$

$=\int_0^1(x^{\frac{3}{2}}+\frac{y^2}{2}\Big|_{y=x}^{y=2x})dx$

$=\int_0^1(x^{\frac{3}{2}}+\frac{3}{2}x^2)dx$

$=(\frac{2}{5}x^{\frac{5}{2}}+\frac{1}{2}x^3)\Big|_0^1=\frac{9}{10}$

题2: 球坐标变换

$I=\iiint_{\Omega'}\frac{1}{\rho}(\rho^2\sin\varphi)\,d\rho\, d\varphi\, d\theta$

$\Omega'=\{(\rho,\varphi,\theta):\rho\le 2\sin\varphi\cos\theta,\ \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]\}$

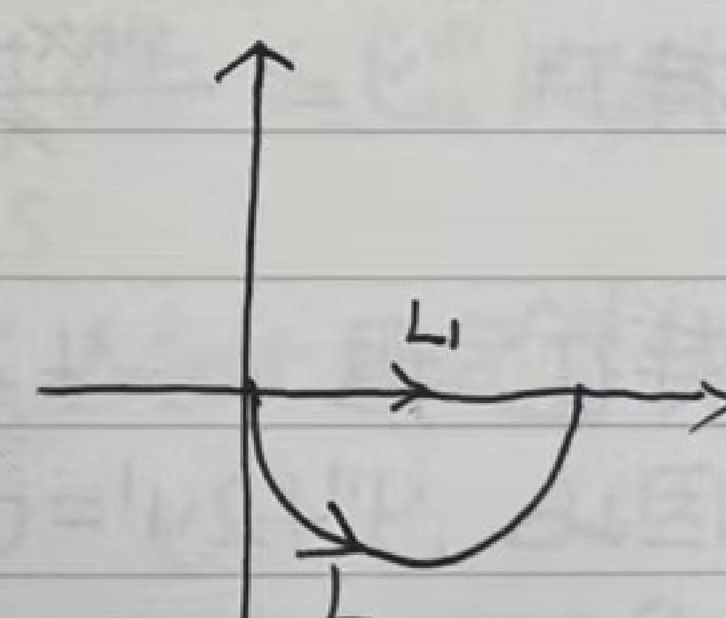
$I=\iiint_{\Omega'}\rho\sin\varphi\, d\rho\, d\varphi\, d\theta$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{\pi}d\varphi\cdot\int_0^{2\sin\varphi\cos\theta}\rho\sin\varphi\, d\rho$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{\pi}2\sin^3\varphi\cos^2\theta\, d\varphi$

$=(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\cos^2\theta)\cdot(\int_0^{\pi}\sin^3\varphi)$

$=4\cdot\frac{\pi}{4}\cdot 2\cdot\frac{2}{3}=\frac{4\pi}{3}$

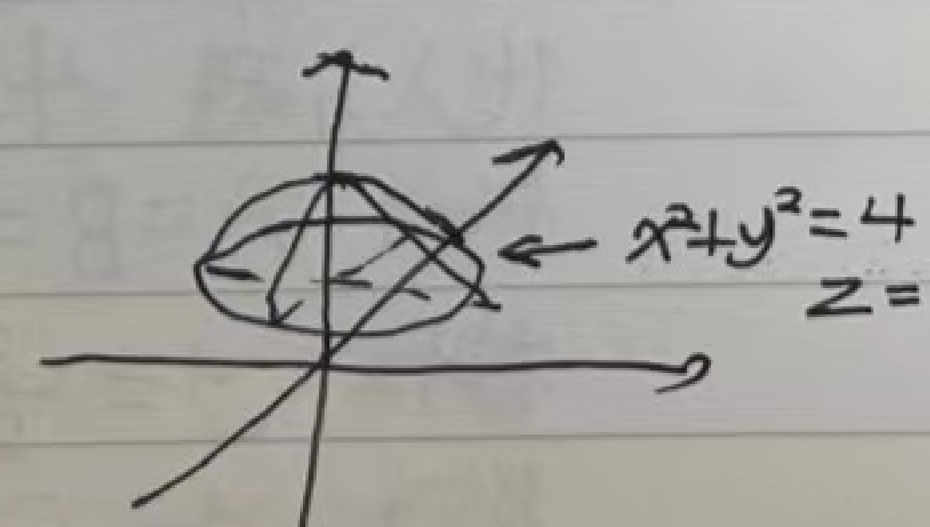


题3: 在$x^2+y^2-2x\le 0$下半圆用路径无关性，注意到

$\partial_y(\frac{-1}{x-y+1})=\partial_x(\frac{1}{x-y+1})$

因此可改为直径L_1上积分

$I=\int_{L_1}\frac{dy-dx}{x-y+1}=\int_0^2\frac{dx}{x+1}=\ln 3$



题4: 圆面$S_1=\{z=1\ \ x^2+y^2=4\}$

$S\cup S_1$围成区域Ω'

用Gauss公式

$\iint_{S,外}+\iint_{S_1下}=\iiint_{\Omega'}(y+x)\,d\sigma \xlongequal{对称性} 0$

$I=-\iint_{S_1下}y(x-z)dydz+x^2dzdx+(y^2+xz)dxdy$

$=\iint_{x^2+y^2\le 4}(y^2+x)dxdy$ （二重积分）

$=\iint_{x^2+y^2\le 4}y^2dxdy$ （x部分有对称性）

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$= \int_0^{2\pi} d\theta \int_0^2 r^3 \cos\theta \, dr$

$= (\int_0^2 r^3 dr)(\int_0^{2\pi} \cos\theta d\theta) = 4\pi$

题5: 一阶ODE $y' + \frac{2}{x} \cdot y = \frac{\sin x}{x}$

常数变异法，对应齐次方程 $y' + \frac{2}{x} y = 0$

有通解 $y = C_0 e^{-2\ln x} = C_0 x^{-2}$

变异解 $y = C_0(x) \frac{1}{x^2}$

代入一阶ODE $C_0'(x) \cdot \frac{1}{x^2} = \frac{\sin x}{x}$

解得 $C_0'(x) = x\sin x$

即 $C_0(x) = -x\cos x + \sin x + C$

由此通解为 $y = \frac{1}{x^2}(\sin x - x\cos x + C)$

代入 $y(\pi) = \frac{1}{\pi}$ 得 $\frac{1}{\pi} = \frac{1}{\pi^2} \cdot (\pi + C) \Rightarrow C = 0$

特解 $y = \frac{\sin x - x\cos x}{x^2}$

题6: 特征方程 $\lambda^2 + 2\lambda = 0 \Rightarrow \lambda = 0, -2$

因此 $y'' + 2y' = 0$ 有解子 $y = 1, y = e^{-2x}$

求 $y'' + 2y' = 3 + 4\sin 2x$ 特解. 设为 $y = \frac{3}{2}x + A\sin 2x + B\cos 2x$

$y'' = -4A\sin 2x - 4B\cos 2x$

$y' = \frac{3}{2} + 2A\cos 2x - 2B\sin 2x$

代入方程 $4\sin 2x = (-4A - 4B)\sin 2x + (4A - 4B)\cos 2x$

解得 $A = B = -\frac{1}{2}$

特解 $y = \frac{3}{2}x - \frac{\sin 2x + \cos 2x}{2}$

通解 $y = C_1 + C_2 e^{-2x} + \frac{3}{2}x - \frac{\sin 2x + \cos 2x}{2}$

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题7：设第一卦限部分记作Ω_1，体积为V_1，Ω_1在XoY平面的投影为正方形$[0,a]\times[0,a]$，我们只需计算以$[0,0]$，$[a,0]$，$[a,a]$为顶点的等腰直角三角形为底之部分（记为D_1）

$V=8V_1$

$=16\iint_{D_1}\sqrt{a^2-x^2}\,dxdy$

$=16\int_0^a\left(\int_0^x\sqrt{a^2-x^2}\,dy\right)dx$

$=16\int_0^a x\sqrt{a^2-x^2}\,dx$

$=16\cdot -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}\Big|_{x=0}^{x=a} = \frac{16}{3}a^3$

题8：用Newton-Leibniz公式 $u_x=\frac{\partial u}{\partial x}$ $u_y=\frac{\partial u}{\partial y}$

$u(x,y)=\int_a^x u_x(t,y)\,dt$

$=\int_c^y u_y(x,s)\,ds$

因此 $|u(x,y)|\le\int_a^x|u_x(t,y)|dt\le\int_a^b|u_x(t,y)|dt$

且 $|u(x,y)|\le\int_c^d|u_y(x,s)|ds$

二式相乘 $|u(x,y)|^2\le\left(\int_a^b|u_x(t,y)|dt\right)\left(\int_c^d|u_y(x,s)|ds\right)$

对$(x,y)\in D$做二重积分

$\iint_D|u(x,y)|^2 d\sigma\le\left(\int_c^d dy\cdot\int_a^b|u_x(t,y)|dt\right)\cdot$

$\left(\int_a^b dx\cdot\int_c^d|u_y(x,s)|ds\right)$

注：本题原型为Sobolev嵌入定理

$\le\left(\iint_D|u_x(t,y)|dtdy\right)\left(\iint_D|u_y(x,s)|dxds\right)$

题9：① $\lambda=0$ 时 $u\equiv1$ 为本征函数

② $a_1=-1$ $b_1=1$ $a_2=b_2=0$ $\begin{cases}u''+\lambda u=0\\ u'(0)=u'(L)=0\end{cases}$

解二阶ODE $u''+\lambda u=0$（对某固定λ）

(i) $\lambda=0$ $u=C_1+C_2x$

(ii) $\lambda>0$ $u=C_1\sin\sqrt{\lambda}x+C_2\cos\sqrt{\lambda}x$

(iii) $\lambda<0$ $u=C_1e^{\sqrt{-\lambda}x}+C_2e^{-\sqrt{-\lambda}x}$

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代入两个边界条件 (i) $C_1 = \text{any}$ $C_2 = 0 \Rightarrow \lambda = 0$ 为本征值 $u_\lambda = C$

(ii) $\begin{cases} \sqrt{\lambda} C_1 = 0 \\ \sqrt{\lambda} C_1 \cos\sqrt{\lambda}L - \sqrt{\lambda} C_2 \sin\sqrt{\lambda}L = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 0 \\ C_2 \text{若非}0 \text{ 必 } \sqrt{\lambda}L = k\pi \quad k\in\mathbb{N}^* \end{cases}$

本征值 $\lambda = (\frac{k\pi}{L})^2$ 本征函数 $u_\lambda = C_2 \cos(\frac{k\pi x}{L})$

(iii) $\begin{cases} \cancel{C_1 + C_2 = 0}\ \sqrt{-\lambda} C_1 - \sqrt{-\lambda} C_2 = 0 \\ \sqrt{-\lambda} C_1 e^{\sqrt{-\lambda}L} - \sqrt{-\lambda} C_2 e^{-\sqrt{-\lambda}L} = 0 \end{cases} \Rightarrow C_1 = C_2 = 0 \Rightarrow \lambda < 0$ 无本征值

综上：$\lambda = (\frac{k\pi}{L})^2$ $k\in\mathbb{N}^*$ 为本征值

对应本征函数 $u = C\cos(\frac{k\pi x}{L})$

$a_1 = b_1 = 0$ $a_2 = b_2 = 1$ 时同理分类讨论

得本征值 $\lambda = (\frac{k\pi}{L})^2$ $k\in\mathbb{N}$ 对应本征函数 $u = C\sin(\frac{k\pi x}{L})$

③ 在方程左右同乘 $u(x)$ 在 $[0, L]$ 积分

$\int_0^L u''(x)u(x)\,dx + \lambda\int_0^L u^2(x)\,dx = 0$

而 $\int_0^L u''(x)u(x)\,dx = u(x)u'(x)\Big|_{x=0}^{x=L} - \int_0^L (u'(x))^2\,dx$

$= u(L)u'(L) - u(0)u'(0) - \int_0^L (u'(x))^2\,dx$

$u(L)u'(L) - u(0)u'(0) = (-\frac{b_2}{b_1}) u(L)^2 - (\frac{a_2}{a_1}) u(0)^2$

代入得 $\underbrace{-\frac{b_2}{b_1}u(L)^2 - \frac{a_2}{a_1}u(0)^2 - \int_0^L (u'(x))^2 dx}_{\le 0} + \lambda\underbrace{\int_0^L (u^2(x))\,dx}_{>0} = 0$

上式说明 $\lambda \ge 0$

如 $\lambda = 0 \Rightarrow u(L) = u(0) = \int_0^L (u'(x))^2 dx = 0$

$\Rightarrow u(L) = u(0) = 0$ $u' \equiv 0 \Rightarrow u \equiv 0$ 与 u 是本征函数矛盾

综上 $\lambda > 0$

④ $u_\lambda''(x) + \lambda u_\lambda(x) = 0$ $u_\mu''(x) + \mu u_\mu(x) = 0$

分别乘以 u_μ 和 u_λ 做积分 $\begin{cases} \int_0^L u_\lambda'' u_\mu\,dx + \lambda\int_0^L u_\lambda u_\mu\,dx = 0 \\ \int_0^L u_\lambda u_\mu''\,dx + \mu\int_0^L u_\lambda u_\mu\,dx = 0 \end{cases}$ (*)

而 $\int_0^L u_\lambda'' u_\mu\,dx = u_\lambda' u_\mu\Big|_{x=0}^{x=L} - \int_0^L u_\lambda' u_\mu'\,dx$

$= (u_\lambda' u_\mu - u_\lambda u_\mu')\Big|_{x=0}^{x=L} + \int_0^L u_\lambda u_\mu''\,dx$

↳ 代入验证知此项为0

Campus 得 $\int_0^L u_\lambda'' u_\mu\,dx = \int_0^L u_\lambda u_\mu''\,dx = 0$ 与方程组(*)中λ≠μ矛盾