

习题 1.1

1. 证明 $\sqrt{3}$ 为无理数.

证 若 $\sqrt{3}$ 不是无理数, 则 $\sqrt{3} = \frac{p}{q}$, p, q 为互素自然数. $3 = \frac{p^2}{q^2}$, $p^2 = 3q^2$. 3除尽 p^2 ,

必除尽 p , 否则 $p = 3k + 1$ 或 $p = 3k + 2$. $p^2 = 9k^2 + 6k + 1$, $p^2 = 9k^2 + 12k + 4$, 3除 p^2 将余1. 故 $p = 3k$, $9k^2 = 3q^2$, $q^2 = 3k^2$, 类似得3除尽 q . 与 p, q 互素矛盾.

2. 设 p 是正的素数, 证明 \sqrt{p} 是无理数.

证 设 $\sqrt{p} = \frac{a}{b}$, a, b 为互素自然数, 则 $p = \frac{a^2}{b^2}$, $a^2 = pb^2$, 素数 p 除尽 a^2 , 故 p 除尽 a ,

$a = pk$. $p^2 k^2 = pb^2$, $pk^2 = b^2$. 类似得 p 除尽 b . 此与 a, b 为互素自然数矛盾.

3. 解下列不等式:

(1) $|x| + |x-1| < 3$; (2) $|x^2 - 3| < 2$.

解 (1) 若 $x < 0$, 则 $-x + 1 - x < 3$, $2x > -2$, $x > -1$, $(-1, 0)$;

若 $0 < x < 1$, 则 $x + 1 - x < 3$, $1 < 3$, $(0, 1)$;

若 $x > 1$, 则 $x + x - 1 < 3$, $x < 3/2$, $(1, 3/2)$.

$X = (-1, 0) \cup (0, 1) \cup (1, 3/2)$.

(2) $-2 < x^2 - 3 < 2$, $1 < x^2 < 5$, $1 < |x|^2 < 5$, $1 < |x| < \sqrt{5}$, $x = (1, \sqrt{5}) \cup (-\sqrt{5}, -1)$.

4. 设 a, b 为任意实数, (1) 证明 $|a+b| \geq |a| - |b|$; (2) 设 $|a-b| < 1$, 证明 $|a| < |b| + 1$.

证 (1) $|a| = |a+b+(-b)| \leq |a+b| + |-b| = |a+b| + |b|$, $|a+b| \geq |a| - |b|$.

(2) $|a| = |b+(a-b)| \leq |b| + |a-b| < |b| + 1$.

5. 解下列不等式:

(1) $|x+6| > 0.1$; (2) $|x-a| > l$.

解 (1) $x+6 > 0.1$ 或 $x+6 < -0.1$. $x > -5.9$ 或 $x < -6.1$. $X = (-\infty, -6.1) \cup (-5.9, +\infty)$.

(2) 若 $l > 0$, $X = (a+l, +\infty) \cup (-\infty, a-l)$; 若 $l = 0$, $x \neq a$; 若 $l < 0$, $X = (-\infty, +\infty)$.

6. 若 $a > 1$, 证明 $0 < \sqrt[n]{a} - 1 < \frac{a-1}{n}$, 其中 n 为自然数.

证 若 $a > 1$, 显然 $\sqrt[n]{a} = b > 1$. $a-1 = \sqrt[n]{a^n} - 1 = (\sqrt[n]{a} - 1)(b^{n-1} + b^{n-2} + \dots + 1) > n(\sqrt[n]{a} - 1)$.

7. 设 (a, b) 为任意一个开区间, 证明 (a, b) 中必有有理数.

证 取自然数 n 满足 $1/10^n < b-a$. 考虑有理数集合

$A = A_n = \{\frac{m}{10^n} \mid m \in \mathbf{Z}\}$. 若 $A_n \cap (a, b) = \emptyset$, 则 $A = B \cup C$, $B = A \cap \{x \mid x \geq b\}$,

$C = A \cap \{x \mid x \leq a\}$. B 中有最小数 $m_0/10^n$, $(m_0-1)/10^n \in C$,

$b-a \leq m_0/10^n - (m_0-1)/10^n = 1/10^n$, 此与 n 的选取矛盾.

8. 设 (a, b) 为任意一个开区间, 证明 (a, b) 中必有无理数.

证 取自然数 n 满足 $1/10^n < b-a$. 考虑无理数集合 $A_n = \{\sqrt{2} + \frac{m}{10^n} \mid m \in \mathbf{Z}\}$. 以下仿8题.

习题 1.2

1. 求下列函数的定义域:

$$(1) y = \ln(x^2 - 4); (2) y = \ln \sqrt{\frac{1+x}{1-x}}; (3) y = \sqrt{\ln \frac{5x-x^2}{4}}; (4) y = \frac{1}{\sqrt{2x^2+5x-3}}.$$

解 (1) $x^2 - 4 > 0, |x|^2 > 4, |x| > 2, D = (-\infty, -2) \cup (2, +\infty).$

$$(2) \frac{1+x}{1-x} > 0, \begin{cases} 1-x > 0 \\ 1+x > 0 \end{cases} \text{ 或 } \begin{cases} 1-x < 0 \\ 1+x < 0 \end{cases}, -1 < x < 1, D = (-1, 1).$$

$$(3) \frac{5x-x^2}{4} > 1, x^2 - 5x - 4 < 0, x^2 - 5x + 4 = 0, (x-1)(x-4) = 0, x_1 = 1, x_2 = 4.$$

$D = (1, 4).$

$$(4) 2x^2 + 5x - 3 > 0, (2x-1)(x+3) = 0, x_1 = -3, x_2 = 1/2, D = (-\infty, -3) \cup (1/2, +\infty).$$

2. 求下列函数的值域 $f(X)$, 其中 X 为题中指定的定义域.

$$(1) f(x) = x^2 + 1, X = (0, 3), f(X) = (1, 10).$$

$$(2) f(x) = \ln(1 + \sin x), X = (-\pi/2, \pi], f(X) = (-\infty, \ln 2].$$

$$(3) f(x) = \sqrt{3+2x-x^2}, X = [-1, 3], 3+2x-x^2 = 0, x^2 - 2x - 3 = 0, (x+1)(x-3) = 0, x_1 = -1, x_2 = 3, f(X) = [0, f(1)] = [0, 4].$$

$$(4) f(x) = \sin x + \cos x, X = (-\infty, +\infty).$$

$$f(x) = \sqrt{2}(\sin x \cos(\pi/4) + \cos x \sin(\pi/4)) = \sqrt{2} \sin(x + \pi/4), f(X) = [-\sqrt{2}, \sqrt{2}].$$

3. 求函数值:

$$(1) \text{ 设 } f(x) = \frac{\ln x^2}{\ln 10}, \text{ 求 } f(-1), f(-0.001), f(100);$$

$$(2) \text{ 设 } f(x) = \arcsin \frac{x}{1+x^2}, \text{ 求 } f(0), f(1), f(-1);$$

$$(3) \text{ 设 } f(x) = \begin{cases} \ln(1-x), & -\infty < x \leq 0, \\ -x, & 0 < x < +\infty, \end{cases} \text{ 求 } f(-3), f(0), f(5).$$

$$(4) \text{ 设 } f(x) = \begin{cases} \cos x, & 0 \leq x < 1, \\ 1/2, & x = 1, \\ 2^x, & 1 < x \leq 3 \end{cases} \text{ 求 } f(0), f(1), f(3/2), f(2).$$

$$\text{解 (1) } f(x) = \log x^2, f(-1) = \log 1 = 0, f(-0.001) = \log(10^{-6}) = -6, f(100) = \log 10^4 = 4.$$

$$(2) f(0) = 0, f(1) = \arcsin(1/2) = \pi/6, f(-1) = \arcsin(-1/2) = -\pi/6.$$

$$(3) f(-3) = \ln 4, f(0) = 0, f(5) = -5.$$

$$(4) f(0) = \cos 0 = 1, f(1) = 1/2, f(3/2) = 2\sqrt{2}, f(2) = 4.$$

$$4. \text{ 设函数 } f(x) = \frac{2+x}{2-x}, x \neq \pm 2, \text{ 求 } f(-x), f(x+1), f(x)+1, f\left(\frac{1}{x}\right), \frac{1}{f(x)}.$$

$$\text{解 } f(-x) = \frac{2-x}{2+x}, x \neq \pm 2; f(x+1) = \frac{2+x+1}{2-x-1} = \frac{3+x}{1-x}, x \neq 1, x \neq -3,$$

$$f(x)+1=\frac{2+x}{2-x}+1=\frac{4}{2-x}, x \neq \pm 2; f\left(\frac{1}{x}\right)=\frac{2-1/x}{2+1/x}=\frac{2x-1}{2x+1}, x \neq 0, x \neq \pm 1/2,$$

$$\frac{1}{f(x)}=\frac{2+x}{2-x}, x \neq \pm 2.$$

5. 设 $f(x)=x^3$, 求 $\frac{f(x+\Delta x)-f(x)}{\Delta x}$, 其中 Δx 为一个不等于零的量.

$$\text{解 } \frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{(x+\Delta x)^3-x^3}{\Delta x}=\frac{x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3-x^3}{\Delta x}=3x^2+3\Delta x+\Delta x^2.$$

6. 设 $f(x)=\ln x, x>0, g(x)=x^2, -\infty<x<+\infty$, 试求 $f(f(x)), g(g(x)), f(g(x)), g(f(x))$.

$$\text{解 } f(f(x))=f(\ln x)=\ln \ln x, x>1; g(g(x))=g(x^2)=x^4, -\infty<x<+\infty;$$

$$f(g(x))=f(x^2)=\ln x^2, x \neq 0; g(f(x))=g(\ln x)=\ln^2 x, x>0.$$

$$7. \text{ 设 } f(x)=\begin{cases} 0, & x \geq 0, \\ -x, & x < 0; \end{cases} g(x)=\begin{cases} x, & x \geq 0; \\ 1-x, & x < 0, \end{cases} \text{ 求 } f(g(x)), g(f(x)).$$

$$\text{解 } \forall x, g(x) \geq 0, f(g(x))=0.$$

$$g(f(x))=\begin{cases} g(0), & x \geq 0, \\ g(-x), & x < 0. \end{cases}=\begin{cases} 0, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

8. 作下列函数的略图:

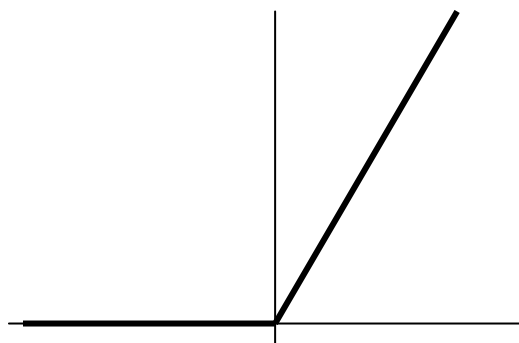
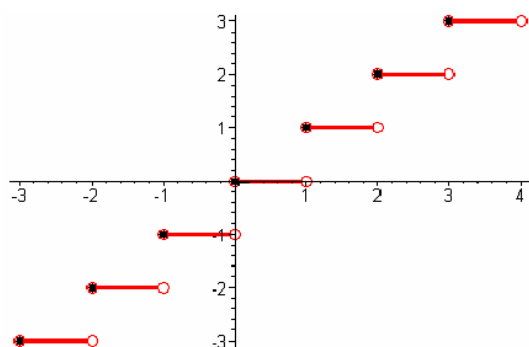
(1) $y=[x]$, 其中 $[x]$ 为不超过 x 的最大整数;

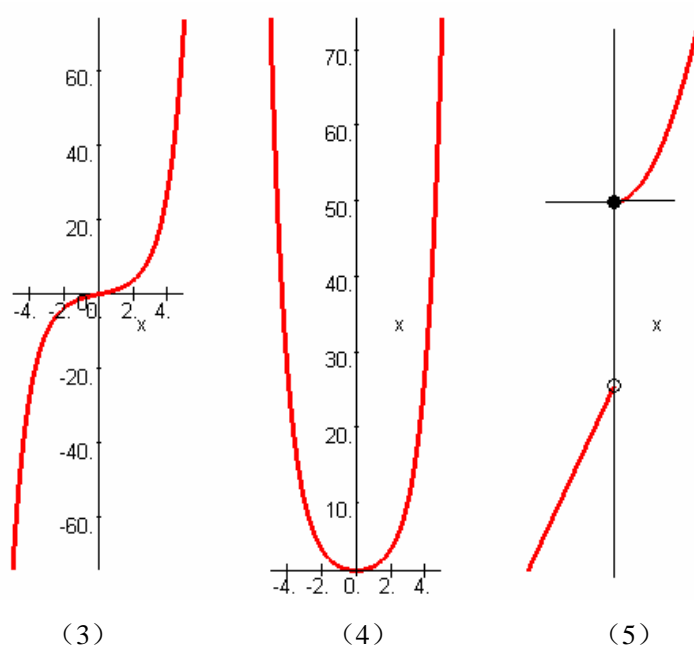
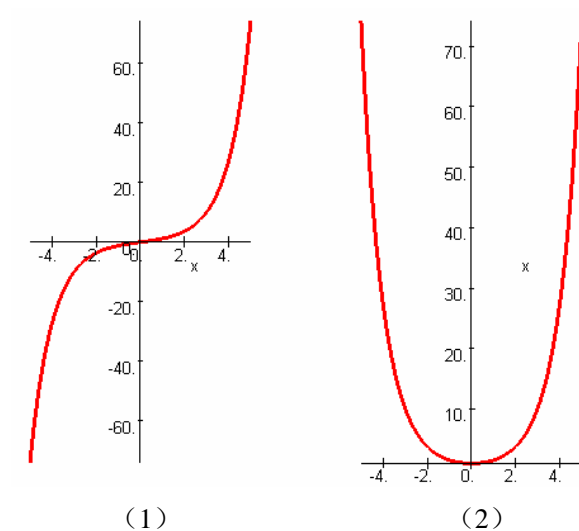
(2) $y=[x]+x$;

(3) $y=\sinh x=\frac{1}{2}(e^x-e^{-x}) (-\infty<x<+\infty)$;

(4) $y=\cosh x=\frac{1}{2}(e^x+e^{-x}) (-\infty<x<+\infty)$;

(5) $y=\begin{cases} x^2, & 0 \leq x < 0, \\ x-1, & -1 \leq x < 0. \end{cases}$





9. 设 $f(x) = \begin{cases} x^2, & x \geq 0, \\ x, & x < 0, \end{cases}$ 求下列函数并且作它们的图形:

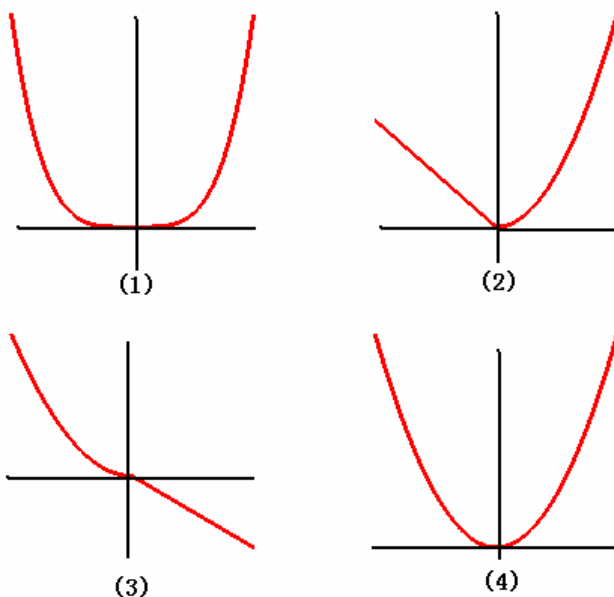
(1) $y = f(x^2)$; (2) $y = |f(x)|$; (3) $y = f(-x)$; (4) $y = f(|x|)$.

解 (1) $y = x^4, -\infty < x < +\infty$.

(2) $y = |f(x)| = \begin{cases} x^2, & x \geq 0, \\ -x, & x < 0. \end{cases}$

(3) $y = f(-x) = \begin{cases} x^2, & -x \geq 0, \\ -x, & -x < 0 \end{cases} = \begin{cases} x^2, & x \leq 0, \\ -x, & x > 0. \end{cases}$

(4) $y = f(|x|) = x^2, -\infty < x < +\infty$.



10.求下列函数的反函数:

(1) $y = \frac{x}{2} - \frac{2}{x} (0 < x < +\infty)$;

(2) $y = \sinh x (-\infty < x < +\infty)$;

(3) $y = \cosh x (0 < x < +\infty)$.

解(1) $\frac{x}{2} - \frac{2}{x} = y, x^2 - 2yx - 4 = 0, x = y + \sqrt{y^2 + 4}, y = x + \sqrt{x^2 + 4} (-\infty < x < +\infty)$.

(2) $\frac{e^x - e^{-x}}{2} = y, z = e^x, z^2 - 2yz - 1 = 0, e^x = z = y + \sqrt{y^2 + 1}, x = \ln(y + \sqrt{y^2 + 1}),$

$y = \ln(x + \sqrt{x^2 + 1}), (-\infty < x < +\infty)$.

(3) $\frac{e^x + e^{-x}}{2} = y, z = e^x, z^2 - 2yz + 1 = 0, e^x = z = y + \sqrt{y^2 - 1}, x = \ln(y + \sqrt{y^2 - 1}),$

$y = \ln(x + \sqrt{x^2 - 1}), (x \geq 1)$.

11.证明 $\cosh^2 x - \sinh^2 x = 1$.

证 $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{(e^{2x} + e^{-2x} + 2) - (e^{2x} + e^{-2x} - 2)}{4} = 1$.

12.下列函数在指定区间内是否是有界函数?

(1) $y = e^{x^2}, x \in (-\infty, +\infty)$; 否

(2) $y = e^{x^2}, x \in (0, 10^{10})$; 是

(3) $y = \ln x, x \in (0, 1)$; 否

(4) $y = \ln x, x \in (r, 1)$, 其中 $r > 0$. 是

(5) $y = \frac{e^{-x^2}}{2 + \sin x} + \cos(2^x), x \in (-\infty, +\infty)$; 是 $|y| \leq \frac{1}{2-1} + 1 = 2$.

(6) $y = x^2 \sin x, x \in (-\infty, +\infty)$; 否.

(7) $y = x^2 \cos x, x \in (-10^{10}, 10^{10})$. 是

13. 证明函数 $y = \sqrt{1+x} - \sqrt{x}$ 在 $(1, +\infty)$ 内是有界函数.

$$\text{证 } y = \sqrt{1+x} - \sqrt{x} = \frac{(\sqrt{1+x} - \sqrt{x})(\sqrt{1+x} + \sqrt{x})}{\sqrt{1+x} + \sqrt{x}} = \frac{1}{\sqrt{1+x} + \sqrt{x}} < \frac{1}{\sqrt{2} + 1} (x > 1).$$

13. 研究函数 $y = \frac{x^6 + x^4 + x^2}{1 + x^6}$ 在 $(-\infty, +\infty)$ 内是否有界.

$$\text{解 } |x| \leq 1 \text{ 时, } \frac{x^6 + x^4 + x^2}{1 + x^6} \leq 3, |x| > 1 \text{ 时, } \frac{x^6 + x^4 + x^2}{1 + x^6} \leq \frac{3x^6}{x^6} = 3,$$

$$|y| = y \leq 3, x \in (-\infty, +\infty).$$

习题 1.3

1. 设 $x_n = \frac{n}{n+2}$ ($n=1, 2, \dots$), 证明 $\lim_{n \rightarrow \infty} x_n = 1$, 即对于任意 $\varepsilon > 0$, 求出正整数 N , 使得当 $n > N$ 时有 $|x_n - 1| < \varepsilon$, 并填下表:

ε	0.1	0.01	0.001	0.0001
N	18	198	1998	19998

证 $\forall \varepsilon > 0$, 不妨设 $\varepsilon < 1$, 要使 $|x_n - 1| = \left| \frac{n}{n+2} - 1 \right| = \frac{2}{n+2} < \varepsilon$, 只需 $n > \frac{2}{\varepsilon} - 2$, 取

$N = \left\lceil \frac{2}{\varepsilon} - 2 \right\rceil$, 则当 $n > N$ 时, 就有 $|x_n - 1| < \varepsilon$.

2. 设 $\lim_{n \rightarrow \infty} a_n = l$, 证明 $\lim_{n \rightarrow \infty} |a_n| = |l|$.

证 $\forall \varepsilon > 0, \exists N$, 使得当 $n > N$ 时, $|a_n - l| < \varepsilon$, 此时 $||a_n| - |l|| \leq |a_n - l| < \varepsilon$, 故 $\lim_{n \rightarrow \infty} |a_n| = |l|$.

3. 设 $\{a_n\}$ 有极限 l , 证明

(1) 存在一个自然数 $N, n < N \mid |a_n| < |l| + 1$;

(2) $\{a_n\}$ 是一个有界数列, 即存在一个常数 M , 使得 $|a_n| \leq M$ ($n=1, 2, \dots$).

证 (1) 对于 $\varepsilon = 1, \exists N$, 使得当 $n > N$ 时, $|a_n - l| < 1$, 此时 $|a_n| = |a_n - l + l| \leq |a_n - l| + |l| < |l| + 1$.

(2) 令 $M = \max\{|l| + 1, |a_1|, \dots, |a_N|\}$, 则 $|a_n| \leq M$ ($n=1, 2, \dots$).

4. 用 $\varepsilon - N$ 说法证明下列各极限式:

$$(1) \lim_{n \rightarrow \infty} \frac{3n+1}{2n-3} = \frac{3}{2}; \quad (2) \lim_{n \rightarrow \infty} \frac{\sqrt[3]{n^2} \sin n}{n+1} = 0;$$

$$(3) \lim_{n \rightarrow \infty} n^2 q^n = 0 \quad (|q| < 1); \quad (4) \lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0;$$

$$(5) \lim_{n \rightarrow \infty} \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{(n-1)n} \right) = 1;$$

$$(6) \lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^{3/2}} + \dots + \frac{1}{(2n)^{3/2}} \right) = 0.$$

证 (1) $\forall \varepsilon > 0$, 不妨设 $\varepsilon < 1$, 要使 $\left| \frac{3n+1}{2n-3} - \frac{3}{2} \right| = \frac{11}{2(2n-3)} < \varepsilon$, 只需 $n > \frac{11}{2\varepsilon} + 3$,

取 $N = \left\lceil \frac{11}{2\varepsilon} + 3 \right\rceil$, 当 $n > N$ 时, $\left| \frac{3n+1}{2n-3} - \frac{3}{2} \right| < \varepsilon$, 故 $\lim_{n \rightarrow \infty} \frac{3n+1}{2n-3} = \frac{3}{2}$.

(2) $\forall \varepsilon > 0$, 要使 $\left| \frac{\sqrt[3]{n^2} \sin n}{n+1} \right| < \varepsilon$, 由于 $\left| \frac{\sqrt[3]{n^2} \sin n}{n+1} \right| \leq \sqrt[3]{n}$, 只需 $\sqrt[3]{n} < \varepsilon, n > \frac{1}{\varepsilon^3}$,

$$\text{取 } N = \left\lceil \frac{1}{\varepsilon^3} \right\rceil, \text{ 当 } n > N \text{ 时 } \left| \frac{\sqrt[3]{n^2} \sin n}{n+1} \right| < \varepsilon.$$

$$(3) |q| = \frac{1}{1+\alpha} (\alpha > 0), n > 4$$

$$|n^2 q^n| = \frac{n^2}{(1+\alpha)^n} = \frac{n^2}{1+n\alpha + \frac{n(n-1)}{2}\alpha^2 + \frac{n(n-1)(n-2)}{6}\alpha^3 + \dots + \alpha^n}$$

$$< \frac{6n}{(n-1)(n-2)\alpha^3} < \frac{24}{n\alpha^3} < \varepsilon, n > \frac{24}{\varepsilon\alpha^3}, N = \max\{4, \left\lceil \frac{24}{\varepsilon\alpha^3} \right\rceil\}.$$

$$(4) \frac{n!}{n^n} \leq \frac{1}{n} < \varepsilon, n > \frac{1}{\varepsilon}, N = \left\lceil \frac{1}{\varepsilon} \right\rceil.$$

$$(5) \left| \left(\frac{1}{1g^2} + \frac{1}{2g^3} + \dots + \frac{1}{(n-1)g^n} \right) - 1 \right|$$

$$= \left| \left(\left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{(n-1)} - \frac{1}{n} \right) \right) - 1 \right| = \frac{1}{n} < \varepsilon, n > \frac{1}{\varepsilon}, N = \left\lceil \frac{1}{\varepsilon} \right\rceil.$$

$$(6) \frac{1}{(n+1)^{3/2}} + \dots + \frac{1}{(2n)^{3/2}} \leq \frac{n}{(n+1)^{3/2}} < \frac{1}{\sqrt{n}} < \varepsilon, n > \frac{1}{\varepsilon^2}, N = \left\lceil \frac{1}{\varepsilon^2} \right\rceil.$$

5. 设 $\lim_{n \rightarrow \infty} a_n = 0$, $\{b_n\}$ 是有界数列, 即存在常数 M , 使得 $|b_n| < M (n=1, 2, \dots)$, 证明 $\lim_{n \rightarrow \infty} a_n b_n = 0$.

$$\text{证 } \forall \varepsilon > 0, \exists \text{ 正整数 } N, \text{ 使得 } |a_n| < \frac{\varepsilon}{M}, |a_n b_n| = |a_n| |b_n| \leq \frac{\varepsilon}{M} M = \varepsilon,$$

$$\text{故 } \lim_{n \rightarrow \infty} a_n b_n = 0.$$

$$6. \text{ 证明 } \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$$

$$\text{证 } \forall \varepsilon > 0, \text{ 要使 } |\sqrt[n]{n} - 1| = \sqrt[n]{n} - 1 < \varepsilon, \text{ 只需 } \frac{n}{(1+\varepsilon)^n} < 1.$$

$$\text{而 } \frac{n}{(1+\varepsilon)^n} = \frac{n}{1+n\varepsilon + \frac{n(n-1)}{2}\varepsilon^2} < \frac{2}{(n-1)\varepsilon^2} < \frac{4}{n\varepsilon^2}, \text{ 只需 } \frac{4}{n\varepsilon^2} < 1, n > \frac{4}{\varepsilon^2}, N = \left\lceil \frac{4}{\varepsilon^2} \right\rceil.$$

7. 求下列各极限的值:

$$(1) \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

$$(2) \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 - 100}{4n^3 - n + 2} = \lim_{n \rightarrow \infty} \frac{1 + 3/n - 100/n^2}{4 - 1/n^2 + 2/n^2} = \frac{1}{4}.$$

$$(3) \lim_{n \rightarrow \infty} \frac{(2n+10)^4}{n^4 + n^3} = \lim_{n \rightarrow \infty} \frac{(2+10/n)^4}{1+1/n} = 16.$$

$$(4) \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{-2n} = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n \right]^{-2} = e^{-2}.$$

$$(5) \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n-1}\right)^{n-1} \left(1 + \frac{1}{n-1}\right)}$$

$$= \frac{1}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n-1}\right)^{n-1} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n-1}\right)} = \frac{1}{e}.$$

$$(6) \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^{n^2} = \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{n}\right)^n \right]^n, \text{ 取 } q \in \left(\frac{1}{e}, 1\right), \exists N, \text{ 当 } n > N \text{ 时, } \left(1 - \frac{1}{n}\right)^n < q$$

$$0 < \left[\left(1 - \frac{1}{n}\right)^n \right]^n < q^n, \lim_{n \rightarrow \infty} q^n = 0, \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{n}\right)^n \right]^n = 0, \text{ 即 } \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^{n^2} = 0.$$

$$(7) \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = e \frac{1}{e} = 1.$$

8. 利用单调有界序列有极限证明下列序列极限的存在性:

$$(1) x_n = \frac{1}{1} + \frac{1}{2^2} + L + \frac{1}{n^2}, x_{n+1} = x_n + \frac{1}{(n+1)^2} > x_n,$$

$$x_n < 1 + \frac{1}{1 \cdot 2} + L + \frac{1}{(n-1)n} = 2 - \frac{1}{n} < 2. x_n \text{ 单调增加有上界, 故有极限.}$$

$$(2) x_n = \frac{1}{2+1} + \frac{1}{2^2+1} + L + \frac{1}{2^n+1}, x_{n+1} = x_n + \frac{1}{2^{n+1}+1} > x_n,$$

$$x_n = \frac{1}{2} + \frac{1}{2^2} + L + \frac{1}{2^n} = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + L + \frac{1}{2^{n-1}} \right) = \frac{1}{2} \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} < 1.$$

x_n 单调增加有上界, 故有极限.

$$(3) x_n = \frac{1}{n+1} + \frac{1}{n+2} + L + \frac{1}{n+n}, x_{n+1} - x_n = \frac{1}{2n+2} - \frac{1}{n+1} = -\frac{1}{2n+2} < 0,$$

$x_{n+1} < x_n, x_n > 0, x_n$ 单调减少有下界, 故有极限.

$$(4) x_n = 1 + 1 + \frac{1}{2!} + L + \frac{1}{n!}, x_{n+1} - x_n = \frac{1}{(n+1)!} > 0,$$

$$x_n \leq 2 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + L + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 3 - \frac{1}{n} < 3.$$

x_n 单调增加有上界, 故有极限.

$$9. \text{ 证明 } e = \lim_{n \rightarrow \infty} \left(1 + 1 + \frac{1}{2!} + L + \frac{1}{n!} \right).$$

$$\begin{aligned}
& \text{证} \left(1 + \frac{1}{n}\right)^n = 1 + n \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + L + \frac{n(n-1)L}{k!} \frac{(n-k+1)}{n^k} + \\
& L + \frac{n(n-1)L}{n!} \frac{(n-n+1)}{n^n} \\
& = 2 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{k!} \left(1 - \frac{1}{n}\right) L \left(1 - \frac{k-1}{n}\right) + \frac{1}{n!} \left(1 - \frac{1}{n}\right) L \left(1 - \frac{n-1}{n}\right) \\
& < 1 + 1 + \frac{1}{2!} + L + \frac{1}{n!} \cdot e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \leq \lim_{n \rightarrow \infty} \left(1 + 1 + \frac{1}{2!} + L + \frac{1}{n!}\right).
\end{aligned}$$

对于固定的正整数 k , 由上式, 当 $n > k$ 时,

$$\left(1 + \frac{1}{n}\right)^n > 2 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{k!} \left(1 - \frac{1}{n}\right) L \left(1 - \frac{k-1}{n}\right),$$

$$\text{令 } n \rightarrow \infty \text{ 得 } e \geq \left(1 + 1 + \frac{1}{2!} + L + \frac{1}{k!}\right),$$

$$e \geq \lim_{k \rightarrow \infty} \left(1 + 1 + \frac{1}{2!} + L + \frac{1}{k!}\right) = \lim_{n \rightarrow \infty} \left(1 + 1 + \frac{1}{2!} + L + \frac{1}{n!}\right).$$

10. 设满足下列条件: $|x_{n+1}| \leq k |x_n|, n = 1, 2, \dots$, 其中是小于1的正数. 证明

$$\lim_{n \rightarrow \infty} x_n = 0.$$

证 由 $|x_{n+1}| \leq k |x_n| \leq k^2 |x_{n-1}| \leq L \cdot k^{n-1} |x_1| \rightarrow 0 (n \rightarrow \infty)$, 得 $\lim_{n \rightarrow \infty} x_n = 0$.

习题 1.4

1. 直接用 ε - δ 说法证明下列各极限等式:

$$(1) \lim_{x \rightarrow a} \sqrt{x} = \sqrt{a} (a > 0); (2) \lim_{x \rightarrow a} x^2 = a^2; (3) \lim_{x \rightarrow a} e^x = e^a; (4) \lim_{x \rightarrow a} \cos x = \cos a.$$

证 (1) $\forall \varepsilon > 0$, 要使 $|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$, 由于 $\frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \frac{|x-a|}{\sqrt{a}}$,

只需 $\frac{|x-a|}{\sqrt{a}} < \varepsilon$, $|x-a| < \sqrt{a}\varepsilon$. 取 $\delta = \sqrt{a}\varepsilon$, 则当 $|x-a| < \delta$ 时, $|\sqrt{x} - \sqrt{a}| < \varepsilon$, 故 $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$.

(2) $\forall \varepsilon > 0$, 不妨设 $|x-a| < 1$. 要使 $|x^2 - a^2| = |x+a||x-a| < \varepsilon$, 由于 $|x+a| \leq |x-a| + |2a| < 1 + |2a|$,

只需 $(1 + |2a|)|x-a| < \varepsilon$, $|x-a| < \frac{\varepsilon}{1 + |2a|}$. 取 $\delta = \min\{\frac{\varepsilon}{1 + |2a|}, 1\}$, 则当 $|x-a| < \delta$ 时,

$|x^2 - a^2| < \varepsilon$, 故 $\lim_{x \rightarrow a} x^2 = a^2$.

(3) $\forall \varepsilon > 0$, 设 $x > a$. 要使 $|e^x - e^a| = e^a(e^{x-a} - 1) < \varepsilon$, 即 $0 < (e^{x-a} - 1) < \frac{\varepsilon}{e^a}$, $1 < e^{x-a} < 1 + \frac{\varepsilon}{e^a}$,

$0 < x-a < \ln\left(1 + \frac{\varepsilon}{e^a}\right)$, 取 $\delta = \min\{\frac{\varepsilon}{1 + |2a|}, 1\}$, 则当 $0 < x-a < \delta$ 时, $|e^x - e^a| < \varepsilon$,

故 $\lim_{x \rightarrow a+} e^x = e^a$. 类似证 $\lim_{x \rightarrow a-} e^x = e^a$. 故 $\lim_{x \rightarrow a} e^x = e^a$.

(4) $\forall \varepsilon > 0$, 要使 $|\cos x - \cos a| = 2 \left| \sin \frac{x+a}{2} \sin \frac{x-a}{2} \right| = 2 \left| \sin \frac{x+a}{2} \right| \left| \sin \frac{x-a}{2} \right| \leq |x-a|$,

取 $\delta = \varepsilon$, 则当 $|x-a| < \delta$ 时, $|\cos x - \cos a| < \varepsilon$, 故 $\lim_{x \rightarrow a} \cos x = \cos a$.

2. 设 $\lim_{x \rightarrow a} f(x) = l$, 证明存在 a 的一个空心邻域 $(a-\delta, a) \cup (a, a+\delta)$, 使得函数 $u = f(x)$ 在该邻域内使有界函数.

证 对于 $\varepsilon = 1$, 存在 $\delta > 0$, 使得当 $0 < |x-a| < \delta$ 时, $|f(x) - l| < 1$, 从而 $|f(x)| = |f(x) - l + l| \leq |f(x) - l| + |l| < 1 + |l| = M$.

3. 求下列极限:

(1) $\lim_{x \rightarrow 0} \frac{(1+x)^2 - 1}{2x} = \lim_{x \rightarrow 0} \frac{2x + x^2}{2x} = \lim_{x \rightarrow 0} \left(1 + \frac{x}{2}\right) = 1$.

(2) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2\left(\frac{x}{2}\right)}{x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$.

(3) $\lim_{x \rightarrow 0} \frac{\sqrt{x+a} - \sqrt{a}}{x} = \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+a} + \sqrt{a})} = \frac{1}{2\sqrt{a}} (a > 0)$.

(4) $\lim_{x \rightarrow 1} \frac{x^2 - x - 2}{2x^2 - 2x - 3} = \frac{-2}{-3}$.

(5) $\lim_{x \rightarrow 0} \frac{x^2 - x - 2}{2x^2 - 2x - 3} = \frac{-2}{-3}$.

$$(6) \lim_{x \rightarrow \infty} \frac{(2x-3)^{20}(2x+2)^{10}}{(2x+1)^{30}} = \frac{2^{30}}{2^{30}} = 1.$$

$$(7) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \lim_{x \rightarrow 0} \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})} = 1.$$

$$(8) \lim_{x \rightarrow -1} \left(\frac{1}{x+1} - \frac{3}{x^3+1} \right) = \lim_{x \rightarrow -1} \frac{x^2 - x + 1 - 3}{(x+1)(x^2 - x + 1)} = \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x+1)(x^2 - x + 1)}$$

$$= \lim_{x \rightarrow -1} \frac{(x+1)(x-2)}{(x+1)(x^2 - x + 1)} = \lim_{x \rightarrow -1} \frac{(x-2)}{(x^2 - x + 1)} = \frac{-3}{3} = -1.$$

$$(9) \lim_{x \rightarrow 4} \frac{\sqrt{1+2x} - 3}{\sqrt{x} - 2} = \lim_{x \rightarrow 4} \frac{(\sqrt{1+2x} - 3)(\sqrt{x} + 2)(\sqrt{1+2x} + 3)}{(\sqrt{x} - 2)(\sqrt{x} + 2)(\sqrt{1+2x} + 3)}$$

$$= \lim_{x \rightarrow 4} \frac{(2x-8)(\sqrt{x} + 2)}{(x-4)(\sqrt{1+2x} + 3)} = \frac{2 \cdot 4}{6} = \frac{4}{3}.$$

$$(10) \lim_{x \rightarrow 1} \frac{x^n - 1}{x - 1} = \lim_{y \rightarrow 0} \frac{(1+y)^n - 1}{y} = \lim_{y \rightarrow 0} \frac{ny + \frac{n(n-1)}{2}y^2 + L + y^n}{y} = n.$$

$$(11) \lim_{x \rightarrow \infty} \left(\sqrt{x^2+1} - \sqrt{x^2-1} \right) = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x^2+1} + \sqrt{x^2-1}} = 0.$$

$$(12) \lim_{x \rightarrow 0} \frac{a_0 x^m + a_1 x^{m-1} + L + a_m}{b_0 x^n + b_1 x^{n-1} + L + b_n} (b_n \neq 0) = \frac{a_m}{b_n}.$$

$$(13) \lim_{x \rightarrow \infty} \frac{a_0 x^m + a_1 x^{m-1} + L + a_m}{b_0 x^n + b_1 x^{n-1} + L + b_n} (a_0 \neq 0) = \begin{cases} a_0 / b_0, & m = n \\ 0, & n > m \\ \infty, & m > n. \end{cases}$$

$$(14) \lim_{x \rightarrow \infty} \frac{\sqrt{x^4+8}}{x^2+1} = \lim_{x \rightarrow \infty} \frac{\sqrt{1+8/x^4}}{1+1/x^2} = 1.$$

$$(15) \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+3x} - \sqrt[3]{1-2x}}{x+x^2}$$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt[3]{1+3x} - \sqrt[3]{1-2x})(\sqrt[3]{1+3x}^2 + \sqrt[3]{1+3x}\sqrt[3]{1-2x} + \sqrt[3]{1-2x}^2)}{(x+x^2)(\sqrt[3]{1+3x}^2 + \sqrt[3]{1+3x}\sqrt[3]{1-2x} + \sqrt[3]{1-2x}^2)}$$

$$= \lim_{x \rightarrow 0} \frac{5x}{x(1+x)(\sqrt[3]{1+3x}^2 + \sqrt[3]{1+3x}\sqrt[3]{1-2x} + \sqrt[3]{1-2x}^2)}$$

$$= \lim_{x \rightarrow 0} \frac{5}{(1+x)(\sqrt[3]{1+3x}^2 + \sqrt[3]{1+3x}\sqrt[3]{1-2x} + \sqrt[3]{1-2x}^2)} = \frac{5}{3}.$$

$$(16) a > 0, \lim_{x \rightarrow a+0} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}} = \lim_{x \rightarrow a+0} \left(\frac{\sqrt{x} - \sqrt{a}}{\sqrt{x^2 - a^2}} + \frac{1}{\sqrt{x+a}} \right)$$

$$= \lim_{x \rightarrow a+0} \left(\frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x+a}\sqrt{x-a}(\sqrt{x} + \sqrt{a})} + \frac{1}{\sqrt{x+a}} \right)$$

$$\begin{aligned}
&= \lim_{x \rightarrow a+0} \left(\frac{(x-a)}{\sqrt{x+a}\sqrt{x-a}(\sqrt{x}+\sqrt{a})} + \frac{1}{\sqrt{x+a}} \right) \\
&= \lim_{x \rightarrow a+0} \left(\frac{\sqrt{x-a}}{\sqrt{x+a}(\sqrt{x}+\sqrt{a})} + \frac{1}{\sqrt{x+a}} \right) = \frac{1}{\sqrt{2a}}.
\end{aligned}$$

4. 利用 $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ 及 $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ 求下列极限:

$$(1) \lim_{x \rightarrow 0} \frac{\sin \alpha x}{\tan \beta x} = \lim_{x \rightarrow 0} \frac{\sin \alpha x}{\sin \beta x} \lim_{x \rightarrow 0} \cos \beta x = \frac{\alpha}{\beta}.$$

$$(2) \lim_{x \rightarrow 0} \frac{\sin(2x^2)}{3x} = \lim_{x \rightarrow 0} \frac{\sin(2x^2)}{2x^2} \lim_{x \rightarrow 0} \frac{2x^2}{3x} = 1 \cdot 0 = 0$$

$$(3) \lim_{x \rightarrow 0} \frac{\tan 3x - \sin 2x}{\sin 5x} = \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 5x} - \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 5x} = \frac{3}{5} - \frac{2}{5} = \frac{1}{5}.$$

$$(4) \lim_{x \rightarrow 0+} \frac{x}{\sqrt{1-\cos x}} = \lim_{x \rightarrow 0+} \frac{x}{\sqrt{2} \sin \frac{x}{2}} = \sqrt{2}.$$

$$(5) \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a} = \lim_{x \rightarrow a} \frac{\cos \frac{x+a}{2} \sin \frac{x-a}{2}}{\frac{x-a}{2}} = \cos a.$$

$$(6) \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^{-x} = \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^{\frac{x}{k}(-k)} = \left[\lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^{\frac{x}{k}} \right]^{-k} = e^{-k}.$$

$$(7) \lim_{y \rightarrow 0} (1-5y)^{1/y} = \left[\lim_{y \rightarrow 0} (1-5y)^{1/(5y)} \right]^{-5} = e^{-5}.$$

$$(8) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{x+100} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \left[\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) \right]^{100} = e.$$

5. 给出 $\lim_{x \rightarrow a} f(x) = +\infty$ 及 $\lim_{x \rightarrow -\infty} f(x) = -\infty$ 的严格定义.

$\lim_{x \rightarrow a} f(x) = +\infty$: 对于任意给定的 $A > 0$, 存在 $\delta > 0$, 使得当 $0 < |x - a| < \delta$ 时 $f(x) > A$.

$\lim_{x \rightarrow -\infty} f(x) = -\infty$: 对于任意给定的 $A > 0$, 存在 $\Delta > 0$, 使得当 $x < -\Delta$ 时 $f(x) < -A$.

习题 1.5

1. 试用 $\varepsilon - \delta$ 说法证明

(1) $\sqrt{1+x^2}$ 在 $x=0$ 连续

(2) $\sin 5x$ 在任意一点 $x=a$ 连续.

证(1) $\forall \varepsilon > 0$, 要使 $|\sqrt{1+x^2} - \sqrt{1+0^2}| = \frac{x^2}{\sqrt{1+x^2} + 1} < \varepsilon$. 由于 $\frac{x^2}{\sqrt{1+x^2} + 1} \leq x^2$, 只需

$x^2 < \varepsilon$, $|x| < \sqrt{\varepsilon}$, 取 $\delta = \sqrt{\varepsilon}$, 则当 $|x| < \delta$ 时有 $|\sqrt{1+x^2} - \sqrt{1+0^2}| < \varepsilon$, 故 $\sqrt{1+x^2}$ 在 $x=0$ 连续.

(2)(1) $\forall \varepsilon > 0$, 要使 $|\sin 5x - \sin 5a| = 2 \left| \cos \frac{5x+5a}{2} \right| \left| \sin \frac{5(x-a)}{2} \right| < \varepsilon$.

由于 $2 \left| \cos \frac{5x+5a}{2} \right| \left| \sin \frac{5(x-a)}{2} \right| \leq 5|x-a|$, 只需 $5|x-a| < \varepsilon$, $|x-a| < \frac{\varepsilon}{5}$,

取 $\delta = \frac{\varepsilon}{5}$, 则当 $|x-a| < \delta$ 时有 $|\sin 5x - \sin 5a| < \varepsilon$, 故 $\sin 5x$ 在任意一点 $x=a$ 连续.

2. 设 $y = f(x)$ 在 x_0 处连续且 $f(x_0) > 0$, 证明存在 $\delta > 0$ 使得当 $|x - x_0| < \delta$ 时 $f(x) > 0$.

证 由于 $f(x)$ 在 x_0 处连续, 对于 $\varepsilon = f(x_0)/2$, 存在 $\delta > 0$ 使得当 $|x - x_0| < \delta$ 时

$|f(x) - f(x_0)| < f(x_0)/2$, 于是 $f(x) > f(x_0) - f(x_0)/2 = f(x_0)/2 > 0$.

3. 设 $f(x)$ 在 (a, b) 上连续, 证明 $|f(x)|$ 在 (a, b) 上也连续, 并且问其逆命题是否成立?

证 任取 $x_0 \in (a, b)$, f 在 x_0 连续. 任给 $\varepsilon > 0$, 存在 $\delta > 0$ 使得当 $|x - x_0| < \delta$ 时

$|f(x) - f(x_0)| < \varepsilon$, 此时 $||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)| < \varepsilon$, 故 $|f|$ 在 x_0 连续. 其逆命题

不真, 例如 $f(x) = \begin{cases} 1, & x \text{ 是有理数} \\ -1, & x \text{ 是无理数} \end{cases}$ 处处不连续, 但是 $|f(x)| \equiv 1$ 处处连续.

4. 适当地选取 a , 使下列函数处处连续:

$$(1) f(x) = \begin{cases} \sqrt{1+x^2}, & x < 0, \\ a+x & x \geq 0; \end{cases} \quad (2) f(x) = \begin{cases} \ln(1+x), & x \geq 1, \\ a \arccos \pi x, & x < 1. \end{cases}$$

解 (1) $\lim_{x \rightarrow 0-} f(x) = \lim_{x \rightarrow 0-} \sqrt{1+x^2} = 1 = f(0)$, $\lim_{x \rightarrow 0+} f(x) = f(0) = a = 1$.

(2) $\lim_{x \rightarrow 1+} f(x) = \lim_{x \rightarrow 1+} \ln(1+x) = \ln 2 = f(1)$, $\lim_{x \rightarrow 1-} f(x) = \lim_{x \rightarrow 1-} a \arccos \pi x = -a = f(1) = \ln 2$,
 $a = -\ln 2$.

5. 利用初等函数的连续性及定理3求下列极限:

$$(1) \lim_{x \rightarrow +\infty} \cos \frac{\sqrt{1+x} - \sqrt{x}}{x} = \cos \lim_{x \rightarrow +\infty} \frac{\sqrt{1+x} - \sqrt{x}}{x} = \cos 0 = 1.$$

$$(2) \lim_{x \rightarrow 2} x^{\sqrt{x}} = 2^{\sqrt{2}}.$$

$$(3) \lim_{x \rightarrow 0} e^{\frac{\sin 2x}{\sin 3x}} = e^{\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x}} = e^{\frac{2}{3}}.$$

$$(4) \lim_{x \rightarrow \infty} \arctan \frac{\sqrt{x^4+8}}{x^2+1} = \arctan \lim_{x \rightarrow \infty} \frac{\sqrt{x^4+8}}{x^2+1} = \arctan 1 = \frac{\pi}{4}.$$

$$\begin{aligned}
 (5) \lim_{x \rightarrow \infty} \sqrt{(\sqrt{x^2+1} - \sqrt{x^2-2})|x|} &= \sqrt{\lim_{x \rightarrow \infty} [(\sqrt{x^2+1} - \sqrt{x^2-2})|x|]} \\
 &= \sqrt{\lim_{x \rightarrow \infty} \left[\frac{3|x|}{\sqrt{x^2+1} + \sqrt{x^2-2}} \right]} = \sqrt{\lim_{x \rightarrow \infty} \left[\frac{3}{\sqrt{1+1/x^2} + \sqrt{1-2/x^2}} \right]} = \sqrt{\frac{3}{2}}.
 \end{aligned}$$

6. 设 $\lim_{x \rightarrow x_0} f(x) = a > 0$, $\lim_{x \rightarrow x_0} g(x) = b$, 证明 $\lim_{x \rightarrow x_0} f(x)^{g(x)} = a^b$.

$$\text{证 } \lim_{x \rightarrow x_0} f(x)^{g(x)} = \lim_{x \rightarrow x_0} e^{(\ln f(x))g(x)} = e^{\lim_{x \rightarrow x_0} [(\ln f(x))g(x)]} = e^{b \ln a} = a^b.$$

7. 指出下列函数的间断点及其类型, 若是可去间断点, 请修改函数在该点的函数值, 使之称为连续函数:

(1) $f(x) = \cos \pi(x - [x])$, 间断点 $n \in \mathbf{Z}$, 第一类间断点.

(2) $f(x) = \operatorname{sgn}(\sin x)$, 间断点 $n\pi, n \in \mathbf{Z}$, 第一类间断点.

(3) $f(x) = \begin{cases} x^2, & x \neq 1, \\ 1/2, & x = 1. \end{cases}$ 间断点 $x = 1$, 第一类间断点.

(4) $f(x) = \begin{cases} x^2 + 1, & 0 \leq x \leq 1 \\ \sin \frac{\pi}{x-1}, & 1 < x \leq 2, \end{cases}$ 间断点 $x = 1$, 第二类间断点.

(5) $f(x) = \begin{cases} \frac{1}{2-x}, & 0 \leq x \leq 1, \\ x, & 1 < x \leq 2, \\ \frac{1}{1-x}, & 2 < x \leq 3. \end{cases}$ 间断点 $x = 2$, 第一类间断点.

8. 设 $y = f(x)$ 在 \mathbf{R} 上是连续函数, 而 $y = g(x)$ 在 \mathbf{R} 上有定义, 但在一点 x_0 处间断.

问函数 $h(x) = f(x) + g(x)$ 及 $\varphi(x) = f(x)g(x)$ 在 x_0 点是否一定间断?

解 $h(x) = f(x) + g(x)$ 在 x_0 点一定间断. 因为如果它在 x_0 点连续,

$g(x) = (f(x) + g(x)) - f(x)$ 将在 x_0 点连续, 矛盾. 而 $\varphi(x) = f(x)g(x)$ 在 x_0 点未必间断. 例如 $f(x) \equiv 0, g(x) = D(x)$.

习题 1.6

1. 证明: 任一奇数次实系数多项式至少有一实根.

证设 $P(x)$ 是一奇数次实系数多项式, 不妨设首项系数是正数, 则 $\lim_{x \rightarrow +\infty} P(x) = +\infty$, $\lim_{x \rightarrow -\infty} P(x) = -\infty$, 存在 $A, B, A < B, P(A) < 0, P(B) > 0$, P 在 $[A, B]$ 连续, 根据连续函数的中间值定理, 存在 $x_0 \in (A, B)$, 使得 $P(x_0) = 0$.

2. 设 $0 < \varepsilon < 1$, 证明对于任意一个 $y_0 \in \mathbf{R}$, 方程 $y_0 = x - \varepsilon \sin x$ 有解, 且解是唯一的.

证令 $f(x) = x - \varepsilon \sin x, f(-|y_0| - 1) = -|y_0| - 1 + \varepsilon < -|y_0| \leq y_0$,

$f(|y_0| + 1) \geq |y_0| + 1 - \varepsilon > |y_0| \geq y_0, f$ 在 $[-|y_0| - 1, |y_0| + 1]$ 连续, 由中间值定理, 存在 $x_0 \in [-|y_0| - 1, |y_0| + 1], f(x_0) = y_0$. 设 $x_2 > x_1$,

$f(x_2) - f(x_1) = x_2 - x_1 - \varepsilon(\sin x_2 - \sin x_1) \geq x_2 - x_1 - \varepsilon |x_2 - x_1| > 0$, 故解唯一.

3. 设 $f(x)$ 在 (a, b) 连续, 又设 $x_1, x_2 \in (a, b), m_1 > 0, m_2 > 0$, 证明存在 $\xi \in (a, b)$ 使得

$$f(\xi) = \frac{m_1 f(x_1) + m_2 f(x_2)}{m_1 + m_2}.$$

证如果 $f(x_1) = f(x_2)$, 取 $\xi = x_1$ 即可. 设 $f(x_1) < f(x_2)$, 则

$$f(x_1) = \frac{m_1 f(x_1) + m_2 f(x_1)}{m_1 + m_2} \leq \frac{m_1 f(x_1) + m_2 f(x_2)}{m_1 + m_2} \leq \frac{m_1 f(x_2) + m_2 f(x_2)}{m_1 + m_2} = f(x_2),$$

在 $[x_1, x_2]$ 上利用连续函数的中间值定理即可.

4. 设 $y = f(x)$ 在 $[0, 1]$ 上连续且 $0 \leq f(x) \leq 1, \forall x \in [0, 1]$. 证明在存在一点 $t \in [0, 1]$ 使得 $f(t) = t$.

证 $g(t) = f(t) - t, g(0) = f(0) \geq 0, g(1) = f(1) - 1 \leq 0$. 如果有一个等号成立, 取 t 为 0 或 1. 如果等号都不成立, 则由连续函数的中间值定理, 存在 $t \in (0, 1)$, 使得 $g(t) = 0$, 即 $f(t) = t$.

5. 设 $y = f(x)$ 在 $[0, 2]$ 上连续, 且 $f(0) = f(2)$. 证明在 $[0, 2]$ 存在两点 x_1 与 x_2 , 使得 $|x_1 - x_2| = 1$, 且 $f(x_1) = f(x_2)$.

证令 $g(x) = f(x+1) - f(x), x \in [0, 1]$.

$g(0) = f(1) - f(0), g(1) = f(2) - f(1) = f(0) - f(1) = -g(0)$. 如果 $g(0) = 0$, 则 $f(1) = f(0)$, 取 $x_1 = 0, x_2 = 1$. 如果 $g(0) \neq 0$, 则 $g(0), g(1)$ 异号, 由连续函数的中间值定理, 存在 $\xi \in (0, 1)$ 使得 $g(\xi) = f(\xi+1) - f(\xi) = 0$, 取 $x_1 = \xi, x_2 = \xi + 1$.

第一章总练习题

1. 求解下列不等式:

$$(1) \left| \frac{5x-8}{3} \right| \geq 2.$$

$$\text{解 } \frac{|5x-8|}{3} \geq 2, |5x-8| \geq 6, 5x-8 \geq 6 \text{ 或 } 5x-8 \leq -6, x \geq \frac{14}{5} \text{ 或 } x \leq \frac{2}{5}.$$

$$(2) \left| \frac{2}{5}x - 3 \right| \leq 3,$$

$$\text{解 } -3 \leq \frac{2}{5}x - 3 \leq 3, 0 \leq x \leq 15.$$

$$(3) |x+1| \geq |x-2|$$

$$\text{解 } (x+1)^2 \geq (x-2)^2, 2x+1 \geq -4x+4, x \geq \frac{1}{2}.$$

2. 设 $y = 2x + |2-x|$, 试将 x 表示成 y 的函数.

$$\text{解 当 } x \leq 2 \text{ 时, } y = x+2, y \leq 4, x = y-2; \text{ 当 } x > 2 \text{ 时, } y = 3x-2, y > 4, x = \frac{1}{3}(y-2).$$

$$x = \begin{cases} y-2, & y \leq 4 \\ \frac{1}{3}(y-2), & y > 4. \end{cases}$$

3. 求出满足不等式 $\sqrt{1+x} < 1 + \frac{1}{2}x$ 的全部 x .

$$\text{解 } x \geq -1. \quad 2\sqrt{1+x} < x+2, 4(1+x) < x^2 + 4x + 4, x^2 > 0. x \geq -1, x \neq 0.$$

4. 用数学归纳法证明下列等式:

$$(1) \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$$

证 当 $n=1$ 时, $2 - \frac{1+2}{2^1} = \frac{1}{2}$, 等式成立. 设等式对于 n 成立, 则

$$\begin{aligned} \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n+1}{2^{n+1}} &= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}} \\ &= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} = 2 - \frac{2n+4-(n+1)}{2^{n+1}} = 2 - \frac{(n+1)+3}{2^{n+1}}, \end{aligned}$$

即等式对于 $n+1$ 也成立. 故等式对于任意正整数皆成立.

$$(2) 1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2} \quad (x \neq 1).$$

$$\text{证 当 } n=1 \text{ 时, } \frac{1 - (1+1)x^1 + 1x^{1+1}}{(1-x)^2} = \frac{(1-x)^2}{(1-x)^2} = 1, \text{ 等式成立.}$$

设等式对于 n 成立, 则

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} + (n+1)x^n = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2} + (n+1)x^n$$

$$\begin{aligned}
&= \frac{1 - (n+1)x^n + nx^{n+1} + (1-x)^2(n+1)x^n}{(1-x)^2} \\
&= \frac{1 - (n+1)x^n + nx^{n+1} + (1-2x+x^2)(n+1)x^n}{(1-x)^2} \\
&= \frac{1 - (n+1)x^n + nx^{n+1} + (x^n - 2x^{n+1} + x^{n+2})(n+1)}{(1-x)^2} \\
&= \frac{1 - (n+1)x^n + nx^{n+1} + (x^n - 2x^{n+1} + x^{n+2})(n+1)}{(1-x)^2} \\
&= \frac{1 - (n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2},
\end{aligned}$$

即等式对于 $n+1$ 成立.由归纳原理,等式对于所有正整数都成立.

5. 设 $f(x) = \frac{|2+x| - |x| - 2}{x}$

(1) 求 $f(-4), f(-1), f(-2), f(2)$ 的值;

(2) 将 $f(x)$ 表成分段函数;

(3) 当 $x \rightarrow 0$ 时 $f(x)$ 是否有极限;

(4) 当 $x \rightarrow -2$ 时是否有极限?

解 (1) $f(-4) = \frac{2-4-2}{-4} = -1, f(-1) = \frac{1-1-2}{-1} = 2, f(-2) = \frac{-2-2}{-2} = 2, f(2) = \frac{4-2-2}{2} = 0.$

(2) $f(x) = \begin{cases} -4/x, & x \leq -2; \\ 2, & -2 < x \leq 0; \\ 0, & x > 0. \end{cases}$

(3) 无. 因为 $\lim_{x \rightarrow 0-} f(x) = 2, \lim_{x \rightarrow 0+} f(x) = 0 \neq \lim_{x \rightarrow 0} f(x).$

(4) 有. $\lim_{x \rightarrow -2-} f(x) = \lim_{x \rightarrow -2-} (-4/x) = 2, \lim_{x \rightarrow -2+} f(x) = \lim_{x \rightarrow -2+} 2 = 2 = \lim_{x \rightarrow -2} f(x), \lim_{x \rightarrow -2} f(x) = 2.$

6. 设 $f(x) = [x^2 - 14]$, 即 $f(x)$ 是不超过 $x^2 - 14$ 的最大整数.

(1) 求 $f(0), f\left(\frac{3}{2}\right), f(\sqrt{2})$ 的值;

(2) $f(x)$ 在 $x=0$ 处是否连续?

(3) $f(x)$ 在 $x=\sqrt{2}$ 处是否连续?

解 (1) $f(0) = [-14] = -14, f\left(\frac{3}{2}\right) = \left[\frac{9}{4} - 14\right] = \left[-6 + \frac{1}{4}\right] = -7, f(\sqrt{2}) = [-12] = -12.$

(2) 连续. 因为 $\lim_{x \rightarrow 0} f(x) = \lim_{y \rightarrow 0+} [y - 14] = -14 = f(0).$

(3) 不连续. 因为 $\lim_{x \rightarrow \sqrt{2}+} f(x) = -12, \lim_{x \rightarrow \sqrt{2}-} f(x) = -11.$

7. 设两常数 a, b 满足 $0 \leq a < b$, 对一切自然数 n , 证明:

(1) $\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)b^n; (2) (n+1)a^n < \frac{b^{n+1} - a^{n+1}}{b - a}.$

$$\text{证 } \frac{b^{n+1} - a^{n+1}}{b - a} = \frac{(b-a)(b^n + b^{n-1}a + \dots + a^n)}{b-a} < b^n + b^{n-1}b + \dots + b^n = (n+1)b^n,$$

$$\text{类似有 } \frac{b^{n+1} - a^{n+1}}{b-a} > (n+1)a^n.$$

$$8. \text{对 } n=1, 2, 3, \dots, \text{ 令 } a_n = \left(1 + \frac{1}{n}\right)^n, b_n = \left(1 + \frac{1}{n}\right)^{n+1}.$$

证明: 序列 $\{a_n\}$ 单调上升, 而序列 $\{b_n\}$ 单调下降, 并且 $a_n < b_n$.

证令 $a = 1 + \frac{1}{n+1}, b = 1 + \frac{1}{n}$, 则由7题中的不等式,

$$\frac{\left(1 + \frac{1}{n}\right)^{n+1} - \left(1 + \frac{1}{n+1}\right)^{n+1}}{\frac{1}{n} - \frac{1}{n+1}} < (n+1) \left(1 + \frac{1}{n}\right)^n,$$

$$\left(1 + \frac{1}{n}\right)^{n+1} - \left(1 + \frac{1}{n+1}\right)^{n+1} < (n+1) \left(1 + \frac{1}{n}\right)^n \frac{1}{n(n+1)}$$

$$\left(1 + \frac{1}{n}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n \frac{1}{n} < \left(1 + \frac{1}{n+1}\right)^{n+1},$$

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

$$(n+1) \left(1 + \frac{1}{n+1}\right)^n < \frac{\left(1 + \frac{1}{n}\right)^{n+1} - \left(1 + \frac{1}{n+1}\right)^{n+1}}{\frac{1}{n} - \frac{1}{n+1}}$$

$$(n+1) \left(1 + \frac{1}{n+1}\right)^n \frac{1}{n(n+1)} < \left(1 + \frac{1}{n}\right)^{n+1} - \left(1 + \frac{1}{n+1}\right)^{n+1}$$

$$\left(1 + \frac{1}{n+1}\right)^n \frac{1}{n} < \left(1 + \frac{1}{n}\right)^{n+1} - \left(1 + \frac{1}{n+1}\right)^{n+1}$$

$$\left(1 + \frac{1}{n+1}\right)^n \left(\frac{1}{n} + 1 + \frac{1}{n+1}\right) < \left(1 + \frac{1}{n}\right)^{n+1}.$$

$$\text{我们证明 } \frac{1}{n} + 1 + \frac{1}{n+1} > \left(1 + \frac{1}{n+1}\right)^2.$$

$$\Leftrightarrow \frac{1}{n} + 1 + \frac{1}{n+1} > 1 + \frac{2}{n+1} + \frac{1}{(n+1)^2}$$

$$\Leftrightarrow \frac{1}{n(n+1)} > \frac{1}{(n+1)^2}. \text{最后不等式显然成立.}$$

$$\text{当 } n \rightarrow \infty \text{ 时, } \left(1 + \frac{1}{n}\right)^n \rightarrow e, \left(1 + \frac{1}{n}\right)^{n+1} \rightarrow e, \text{ 故 } \left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$$

9. 求极限

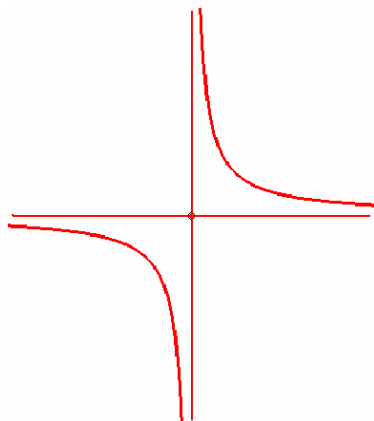
$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{n^2}\right)$$

$$\text{解} \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{n^2}\right)$$

$$= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdots \frac{n}{n} \cdot \frac{n+1}{n} = \frac{1}{n} \cdot \frac{n+1}{2} \rightarrow \frac{1}{2} (n \rightarrow \infty).$$

10. 作函数 $f(x) = \lim_{n \rightarrow \infty} \frac{nx}{nx^2 + a}$ ($a \neq 0$) 的图形.

$$\text{解} f(x) = \lim_{n \rightarrow \infty} \frac{nx}{nx^2 + a} = \begin{cases} 0, & x = 0; \\ 1/x, & x \neq 0. \end{cases}$$



11. 在关于有界函数的定义下, 证明函数 $f(x)$ 在区间 $[a, b]$ 上为有界函数的充要条件为存在一个正的常数 M 使得 $|f(x)| < M, \forall x \in [a, b]$.

证 设存在常数 M_1, N 使得 $M_1 \leq f(x) \leq N, \forall x \in [a, b]$, 取 $M = \max\{|M_1|, |N|\} + 1$, 则有 $|f(x)| < M, \forall x \in [a, b]$.

反之, 若存在一个正的常数 M 使得 $|f(x)| < M, \forall x \in [a, b]$, 则 $-M < f(x) < M, \forall x \in [a, b]$.

12. 证明: 若函数 $y = f(x)$ 及 $y = g(x)$ 在 $[a, b]$ 上均为有界函数, 则 $f(x) + g(x)$ 及 $f(x)g(x)$ 也都是 $[a, b]$ 上的有界函数.

证 存在 M_1, M_2 , $|f(x)| < M_1, |g(x)| < M_2, \forall x \in [a, b]$. $|f(x) + g(x)| \leq |f(x)| + |g(x)| < M_1 + M_2$, $|f(x)g(x)| = |f(x)| |g(x)| < M_1 M_2, \forall x \in [a, b]$.

13. 证明: $f(x) = \frac{1}{x} \cos \frac{\pi}{x}$ 在 $x = 0$ 的任一邻域内都是无界的, 但当 $x \rightarrow 0$ 时 $f(x)$ 不是无穷大量.

证 任取一个邻域 $(-\delta, \delta), \delta > 0$ 和 $M > 0$, 取正整数 n , 满足 $\frac{1}{n} < \delta$ 和 $n > M$, 则 $\left|f\left(\frac{1}{n}\right)\right| = n > M$,

故 $f(x)$ 在 $(-\delta, \delta)$ 无界. 但是 $x_n = \frac{1}{2n+1/2} \rightarrow 0, f(x_n) = (2n+1/2) \cos(2n+1/2)\pi = 0 \nrightarrow \infty$,

故当 $x \rightarrow 0$ 时 $f(x)$ 不是无穷大量.

14. 证明 $\lim_{n \rightarrow \infty} n(x^n - 1) = \ln x (x > 0)$.

证 令 $x^{\frac{1}{n}} - 1 = y_n$, 则 $\frac{1}{n} \ln x = \ln(1 + y_n)$, $n = \frac{\ln x}{\ln(1 + y_n)}$. $\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} x^{\frac{1}{n}} - 1 = 0$.

注意到 $\lim_{y \rightarrow 0} \frac{\ln(1 + y)}{y} = \lim_{y \rightarrow 0} \ln(1 + y)^{\frac{1}{y}} = \ln \lim_{y \rightarrow 0} (1 + y)^{\frac{1}{y}} = \ln e = 1$,

我们有 $n(x^n - 1) = \frac{y_n \ln x}{\ln(1 + y_n)} \rightarrow \ln x (n \rightarrow \infty)$.

15. 设 $f(x)$ 及 $g(x)$ 在实轴上有定义且连续. 证明: 若 $f(x)$ 与 $g(x)$ 在有理数集处处相等, 则它们在整个实轴上处处相等.

证 任取一个无理数 x_0 , 取有理数序列 $x_n \rightarrow x_0$, $f(x_0) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) = g(x_0)$.

16. 证明 $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.

证 $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = \lim_{y \rightarrow 0} \frac{2 \sin^2 y}{4y^2} = \frac{1}{2} \left(\lim_{y \rightarrow 0} \frac{\sin y}{y} \right)^2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$.

17. 证明: (1) $\lim_{y \rightarrow 0} \frac{\ln(1 + y)}{y} = 1$; (2) $\lim_{x \rightarrow 0} \frac{e^{x+a} - e^x}{x} = e^a$.

证 (1) $\lim_{y \rightarrow 0} \frac{\ln(1 + y)}{y} = \lim_{y \rightarrow 0} \ln(1 + y)^{\frac{1}{y}} = \ln \lim_{y \rightarrow 0} (1 + y)^{\frac{1}{y}} = \ln e = 1$.

(2) $\lim_{x \rightarrow 0} \frac{e^{x+a} - e^a}{x} = \lim_{x \rightarrow 0} \frac{e^a(e^x - 1)}{x} = e^a \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = e^a \lim_{y \rightarrow 0} \frac{y}{\ln(1 + y)} = e^a \frac{1}{\lim_{y \rightarrow 0} \frac{\ln(1 + y)}{y}} = e^a \frac{1}{1} = e^a$.

18. 设 $y = f(x)$ 在 a 点附近有定义且有极限 $\lim_{x \rightarrow a} f(x) = 0$, 又设 $y = g(x)$ 在 a 点附近有定义, 且是有界函数. 证明 $\lim_{x \rightarrow a} f(x)g(x) = 0$.

证 设 $|g(x)| < M, 0 < |x - a| < \delta_0$. 对于任意 $\varepsilon > 0$, 存在 $\delta_1 > 0$, 使得当 $0 < |x - a| < \delta_1$ 时 $|f(x)| < \varepsilon / M$. 令 $\delta = \min\{\delta_1, \delta_0\}$, 则 $0 < |x - a| < \delta$ 时,

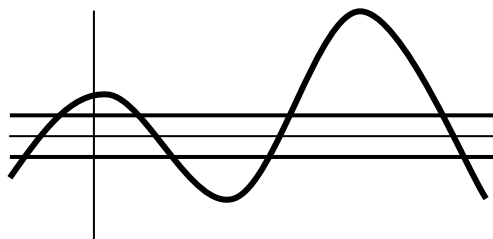
$|f(x)g(x)| = |f(x)| |g(x)| < \frac{\varepsilon}{M} gM = \varepsilon$, 故 $\lim_{x \rightarrow a} f(x)g(x) = 0$.

19. 设 $y = f(x)$ 在 $(-\infty, +\infty)$ 中连续, 又设 c 为正的常数, 定义 $g(x)$ 如下

$$g(x) = \begin{cases} f(x) & \text{当 } |f(x)| \leq c \\ c & \text{当 } f(x) > c \\ -c & \text{当 } f(x) < -c \end{cases}$$

试画出 $g(x)$ 的略图, 并证明

$g(x)$ 在 $(-\infty, +\infty)$ 上连续.



证(一)若 $|f(x_0)| < c$, 则存在 $\delta_0 > 0$, 当 $|x - x_0| < \delta_0$ 时 $|f(x)| < c$, $g(x) = f(x)$,

$$\lim_{x \rightarrow x_0} g(x) = \lim_{x \rightarrow x_0} f(x) = f(x_0) = g(x_0).$$

若 $f(x_0) > c$, 则存在 $\delta_0 > 0$, 当 $|x - x_0| < \delta_0$ 时 $f(x) > c$, $g(x) = c$,

$$\lim_{x \rightarrow x_0} g(x) = \lim_{x \rightarrow x_0} c = c = g(x_0).$$

若 $f(x_0) = c$, 则 $g(x_0) = c$. 对于任意 $\varepsilon > 0$, 不妨设 $\varepsilon < c$, 存在 $\delta > 0$, 使得当 $|x - x_0| < \delta$ 时

$$|f(x) - c| < \varepsilon. \text{ 设 } |x - x_0| < \delta. \text{ 若 } f(x) \leq c, \text{ 则 } g(x) = f(x), |g(x) - g(x_0)| = |f(x) - c| < \varepsilon,$$

$$\text{若 } f(x) > c, \text{ 则 } g(x) = c, |g(x) - g(x_0)| = 0 < \varepsilon.$$

证(二)利用 $g(x) = \min\{f(x), c\} + \max\{f(x), -c\} - f(x)$.

$$\max\{f_1(x), f_2(x)\} = (|f_1(x) - f_2(x)| + f_1(x) + f_2(x)) / 2.$$

$$\min\{f_1(x), f_2(x)\} = (-|f_1(x) - f_2(x)| + (f_1(x) + f_2(x))) / 2.$$

$$20. \text{ 设 } f(x) \text{ 在 } [a, b] \text{ 上连续, 又设 } \eta = \frac{1}{3}[f(x_1) + f(x_2) + f(x_3)],$$

其中 $x_1, x_2, x_3 \in [a, b]$. 证明存在一点 $c \in [a, b]$, 使得 $f(c) = \eta$.

证若 $f(x_1) = f(x_2) = f(x_3)$, 则 $\eta = f(x_1)$, 取 $c = x_1$ 即可.

否则设 $f(x_1) = \min\{f(x_1), f(x_2), f(x_3)\}$, $f(x_3) = \max\{f(x_1), f(x_2), f(x_3)\}$,

$f(x_1) < \eta < f(x_3)$, f 在 $[x_1, x_3]$ 连续, 根据连续函数的中间值定理, 存在一点 $c \in [a, b]$,

使得 $f(c) = \eta$.

21. 设 $y = f(x)$ 在点 x_0 连续而 $g(x)$ 在点 x_0 附近有定义, 但在 x_0 不连续问 $kf(x) + lg(x)$

是否在 x_0 连续, 其中 k, l 为常数.

解如果 $l = 0$, $kf(x) + lg(x)$ 在 x_0 连续; 如果 $l \neq 0$, $kf(x) + lg(x)$ 在 x_0 不连续, 因否则

$g(x) = [(kf(x) + lg(x)) - kf(x)] / l$ 将在 x_0 连续.

22. 证明 Dirichlet 函数处处不连续.

证任意取 x_0 . 取有理数列 $x_n \rightarrow x_0$, 则 $D(x_n) \rightarrow 1$; 取无理数列 $x'_n \rightarrow x_0$, 则 $D(x'_n) \rightarrow 0$;

故 $\lim_{x \rightarrow x_0} D(x)$ 不存在, $D(x)$ 在 x_0 不连续.

23. 求下列极限:

$$(1) \lim_{x \rightarrow \infty} \left(\frac{1+x}{1+2x} \right)^{|x|} = 0; (2) \lim_{x \rightarrow +\infty} (\arctan x) \sin \frac{1}{x} = \frac{\pi}{2} \cdot 0 = 0;$$

$$(3) \lim_{x \rightarrow 0} \frac{\tan 5x}{\ln(1+x^2) + \sin x} = \lim_{x \rightarrow 0} \frac{\tan 5x / x}{x[\ln(1+x^2)] / x^2 + \sin x / x} = \frac{5}{1} = 5.$$

$$(4) \lim_{x \rightarrow 1} (\sqrt{x})^{\frac{1}{\sqrt{x}-1}} = \lim_{y \rightarrow 0} (1+y)^{1/y} = e.$$

24. 设函数 $y = f(x)$ 在 $[0, +\infty)$ 内连续, 且满足 $0 \leq f(x) \leq x$. 设 $a_1 \geq 0$ 是一任意数, 并假定 $a_2 = f(a_1), a_3 = f(a_2), \dots$, 一般地 $a_{n+1} = f(a_n)$. 试证明 $\{a_n\}$ 单调递减, 且极限 $\lim_{n \rightarrow \infty} a_n$ 存在.

若 $l = \lim_{n \rightarrow \infty} a_n$, 则 l 是方程 $f(x) = x$ 的根, 即 $f(l) = l$.

证 $a_{n+1} = f(a_n) \leq a_n, \{a_n\}$ 单调递减. 又 $a_{n+1} = f(a_n) \geq 0 (n = 1, 2, \dots), \{a_n\}$ 单调递减有下界,

故 a_n 有极限. 设 $l = \lim_{n \rightarrow \infty} a_n$, 则 $l = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n) = f(l)$.

25. 设函数 $y = E(x)$ 在 $(-\infty, +\infty)$ 内有定义且处处连续, 并且满足下列条件:

$$E(0) = 1, E(1) = e, E(x+y) = E(x)gE(y).$$

证明 $E(x) = e^x (\forall x \in (-\infty, +\infty))$.

证用数学归纳法易得 $E(x_1 + L + x_n) = E(x_1)gE(x_n)$. 于是 $E(nx) = E(x)^n$.

设 n 是正整数, 则 $E(n) = E(1 + L + 1) = E(1)^n = e^n$.

$1 = E(0) = E(n + (-n)) = E(n)gE(-n) = e^n gE(-n)$, $E(-n) = e^{-n}$. 于对于任意整数 $E(n) = e^n$.

对于任意整数 n , $E(1) = E(n \frac{1}{n}) = E(n)gE(\frac{1}{n}) = e^n gE(\frac{1}{n})$, $gE(\frac{1}{n}) = e^{\frac{1}{n}}$.

$$E(\frac{m}{n}) = E(m \frac{1}{n}) = \left(E(\frac{1}{n}) \right)^m = \left(e^{\frac{1}{n}} \right)^m = e^{\frac{m}{n}}. \text{ 即对于所有有理数 } r, E(r) = e^r.$$

对于无理数 x , 取有理数列 $x_n \rightarrow x$, 由 $E(x)$ 的连续性,

$$E(x) = \lim_{n \rightarrow \infty} E(x_n) = \lim_{n \rightarrow \infty} e^{x_n} = e^{\lim_{n \rightarrow \infty} x_n} (e^x \text{ 的连续性}) = e^x.$$

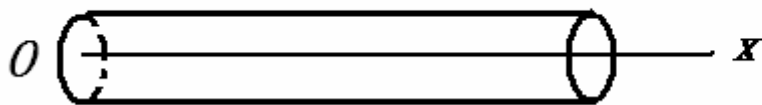
习题 2.1

1. 设一物质细杆的长为 l , 其质量在横截面的分布上可以看作均匀的. 现取杆的左端点为坐标原点 O , 杆所在直线为 x 轴. 设从左端点到细杆上任一点 x 之间那一段的质量为 $m(x) = 2x^2$ ($0 \leq x \leq l$)

(1) 给自变量 x 一个增量 Δx , 求的相应增量 Δm ;

(2) 求比值 $\frac{\Delta m}{\Delta x}$, 问它的物理意义是什么?

(3) 求极限 $\lim_{\Delta x \rightarrow 0} \frac{\Delta m}{\Delta x}$, 问它的物理意义是什么?



解(1) $\Delta m = 2(x + \Delta x)^2 - 2x^2 = 2(x^2 + 2x\Delta x + \Delta x^2) - 2x^2 = 2(2x\Delta x + \Delta x^2)$.

(2) $\frac{\Delta m}{\Delta x} = \frac{2(2x\Delta x + \Delta x^2)}{\Delta x} = 2(2x + \Delta x)$. $\frac{\Delta m}{\Delta x}$ 是 x 到 $x + \Delta x$ 那段细杆的平均线密度.

(3) $\lim_{\Delta x \rightarrow 0} \frac{\Delta m}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2(2x + \Delta x) = 4x$. $\lim_{\Delta x \rightarrow 0} \frac{\Delta m}{\Delta x}$ 是细杆在点 x 的线密度.

2. 根据定义, 求下列函数的导函数:

(1) $y = ax^3$; (2) $y = \sqrt{2px}$, $p > 0$; (3) $y = \sin 5x$.

解(1) $y' = \lim_{\Delta x \rightarrow 0} \frac{a(x + \Delta x)^3 - ax^3}{\Delta x}$
 $= a \lim_{\Delta x \rightarrow 0} \frac{(x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) - x^3}{\Delta x} = a \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + \Delta x^2) = 3ax^2$.

(2) $y' = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{2p(x + \Delta x)} - \sqrt{2px}}{\Delta x} = \sqrt{2p} \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$
 $= \sqrt{2p} \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \Delta x} + \sqrt{x})}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \sqrt{2p} \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$
 $= \sqrt{2p} \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{\sqrt{2p}}{2\sqrt{x}}$.

(3) $y' = \lim_{\Delta x \rightarrow 0} \frac{\sin 5(x + \Delta x) - \sin 5x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2 \cos \frac{5(2x + \Delta x)}{2} \sin \frac{5\Delta x}{2}}{\Delta x}$
 $= \lim_{\Delta x \rightarrow 0} \frac{2 \cos \frac{5(2x + \Delta x)}{2} \sin \frac{5\Delta x}{2}}{\frac{5\Delta x}{2}} = 5 \lim_{\Delta x \rightarrow 0} \cos \frac{5(2x + \Delta x)}{2} \lim_{\Delta x \rightarrow 0} \frac{\sin \frac{5\Delta x}{2}}{\frac{5\Delta x}{2}} = 5 \cos 5x$.

3.求下列曲线 $y = f(x)$ 在指定点 $M(x_0, f(x_0))$ 处的切线方程:

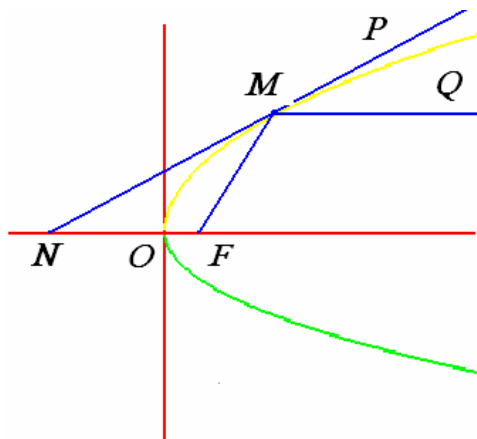
(1) $y = 2^x, M(0, 1)$; (2) $y = x^2 + 2, B(3, 11)$.

解(1) $y' = 2^x \ln 2, y'(0) = \ln 2$, 切线方程 $y - 1 = \ln 2(x - 0), y = (\ln 2)x + 1$.

(2) $y' = 2x, y'(3) = 6$, 切线方程: $y - 11 = 6(x - 3)$.

4.试求抛物线 $y^2 = 2px (p > 0)$ 上任一点 $M(x, y) (x > 0, y > 0)$ 处的切线斜率,

并证明:从抛物线的焦点 $F\left(\frac{p}{2}, 0\right)$ 发射光线时,其反射线一定平行于 x 轴.



证 $y = \sqrt{2px}, y' = \frac{2p}{2\sqrt{2px}} = \frac{p}{y}$, 过点 M 的切线 PMN 方程: $Y - y = \frac{p}{y}(X - x)$.

切线与 x 轴交点 $N(X_0, 0), -y = \frac{p}{y}(X_0 - x), X_0 = x - \frac{y^2}{p} = -x$.

$$FN = \frac{p}{2} + x, FM = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = \sqrt{\left(x - \frac{p}{2}\right)^2 + 2px}$$

$$= \sqrt{x^2 + px + \left(\frac{p}{2}\right)^2} = \sqrt{\left(x + \frac{p}{2}\right)^2} = x + \frac{p}{2} = FN, \text{ 故 } \angle FNM = \angle FMN.$$

过 M 作 PQ 平行于 x 轴, 则 $\angle PMQ = \angle FNM = \angle FMN$.

5.曲线 $y = x^2 + 2x + 3$ 上哪一点的切线与直线 $y = 4x - 1$ 平行, 并求曲线在该点的切线和法线方程.

解 $y' = 2x + 2 = 4, x_0 = 1, y_0 = 6, k = 4$

切线方程: $y - 6 = 4(x - 1), y = 4x + 2$. 法线方程: $y - 6 = \left(-\frac{1}{4}\right)(x - 1), y = -\frac{1}{4}x + \frac{25}{4}$.

6. 离地球中心 r 处的重力加速度 g 是 r 的函数,其表达式为

$$g(r) = \begin{cases} \frac{GMr}{R^3}, & r < R; \\ \frac{GM}{r^2}, & r \geq R \end{cases} \quad \text{其中 } R \text{ 是地球的半径, } M \text{ 是地球的质量, } G \text{ 是引力常数.}$$

(1) 问 $g(r)$ 是否为 r 的连续函数:

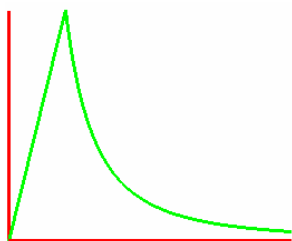
(2) 作 $g(r)$ 的草图;

(3) $g(r)$ 是否是 r 的可导函数.

解 明显地, $r \neq R$ 时 $g(r)$ 连续. $\lim_{r \rightarrow R-} g(r) = \lim_{r \rightarrow R-} \frac{GMr}{R^3} = \frac{GM}{R^2},$

$\lim_{r \rightarrow R+} g(r) = \lim_{r \rightarrow R+} \frac{GM}{r^2} = \frac{GM}{R^2} = \lim_{r \rightarrow R-} g(r), g(r)$ 在 $r = R$ 连续.

(2)



(3) $r \neq R$ 时 $g(r)$ 可导.

$g'_-(R) = \frac{GM}{R^3}, g'_+(R) = -\frac{2GM}{R^3} \neq g'_-(R), g(r)$ 在 $r = R$ 不可导.

7. 求二次函数 $P(x)$, 已知: 点 $(1, 3)$ 在曲线 $y = P(x)$ 上, 且 $P'(0) = 3, P'(2) = 1$.

$$\text{解 } P(x) = ax^2 + bx + c, P'(x) = 2ax + b. \begin{cases} a + b + c = 3 \\ b = 3 \\ 4a + b = 1 \end{cases}$$

$$b = 3, a = -\frac{1}{2}, c = 3 - (a + b) = \frac{1}{2}, P(x) = -\frac{1}{2}x^2 + 3x + \frac{1}{2}.$$

8. 求下列函数的导函数:

$$(1) y = 8x^3 + x + 7, y' = 24x^2 + 1.$$

$$(2) y = (5x + 3)(6x^2 - 2), y' = 5(6x^2 - 2) + 12x(5x + 3) = 90x^2 + 36x - 10.$$

$$(3) y = (x + 1)(x - 1) \tan x = (x^2 - 1) \tan x, y' = (2x) \tan x + (x^2 - 1) \sec^2 x.$$

$$(4) y = \frac{9x + x^2}{5x + 6}, y' = \frac{(9 + 2x)(5x + 6) - 5(9x + x^2)}{(5x + 6)^2} = \frac{5x^2 + 12x + 54}{(5x + 6)^2}.$$

$$(5) y = \frac{1 + x}{1 - x} = -1 + \frac{2}{1 - x} (x \neq 1), y' = \frac{2}{(1 - x)^2}.$$

$$(6) y = \frac{2}{x^3 - 1} (x \neq 1), y' = \frac{-6x^2}{(x^3 - 1)^2}.$$

$$(7) y = \frac{x^2 + x + 1}{e^x}, y' = \frac{(2x + 1)e^x - e^x(x^2 + x + 1)}{e^{2x}} = \frac{-x^2 + x - 1}{e^x}.$$

$$(8) y = x \lg 10^x, y' = 10^x + x \lg 10^x \ln 10 = 10^x(1 + x \ln 10).$$

$$(9) y = x \cos x + \frac{\sin x}{x}, y' = \cos x - x \sin x + \frac{x \cos x - \sin x}{x^2}.$$

$$(10) y = e^x \sin x, y' = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x).$$

9. 定义: 若多项式 $P(x)$ 可表为 $P(x) = (x - x_0)^m g(x)$, $g(x_0) \neq 0$

则称 x_0 是 $P(x)$ 的 m 重根. 今若已知 x_0 是 $P(x)$ 的 k 重根, 证明 x_0 是 $P'(x)$ 的 $(k-1)$ 重根 ($k > 2$).

$$\text{证 } P(x) = (x - x_0)^k g(x), g(x_0) \neq 0$$

$$P'(x) = k(x - x_0)^{k-1} g(x) + (x - x_0)^k g'(x)$$

$$= (x - x_0)^{k-1} (kg(x) + (x - x_0)g'(x)) = (x - x_0)^{k-1} h(x),$$

$h(x_0) = kg(x_0) \neq 0$, 由定义 x_0 是 $P'(x)$ 的 $(k-1)$ 重根.

10. 若 $f(x)$ 在 $(-a, a)$ 中有定义, 且满足 $f(-x) = f(x)$, 则称 $f(x)$ 为偶函数. 设 $f(x)$

是偶函数, 且 $f'(0)$ 存在, 试证明 $f'(0) = 0$.

$$\text{证 } f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{f(-x) - f(0)}{x} = -\lim_{x \rightarrow 0} \frac{f(-x) - f(0)}{-x} = -f'(0), f'(0) = 0.$$

$$11. \text{ 设 } f(x) \text{ 在 } x_0 \text{ 处可导, 证明 } \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} = 2f'(x_0).$$

$$\begin{aligned} \text{证 } \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} &= \frac{1}{2} \lim_{\Delta x \rightarrow 0} \left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - \frac{f(x_0 - \Delta x) - f(x_0)}{\Delta x} \right] \\ &= \frac{1}{2} \lim_{\Delta x \rightarrow 0} \left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} + \frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} \right] \\ &= \frac{1}{2} \left[\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} \right] = \frac{1}{2} [f'(x_0) + f'(x_0)] = f'(x_0). \end{aligned}$$

12. 一质点沿曲线 $y = x^2$ 运动, 且已知时刻 t ($0 < t < \pi/2$) 时质点所在位置

$P(t) = (x(t), y(t))$ 满足: 直线 \overline{OP} 与 x 轴的夹角恰为 t . 求时刻 t 时质点的位置速度及加速度.

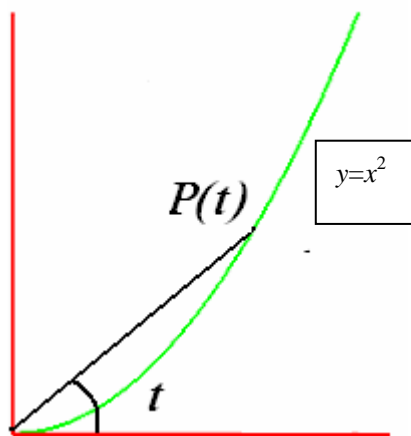
$$\text{解 } \frac{y(t)}{x(t)} = \frac{x^2(t)}{x(t)} = x(t) = \tan t, y(t) = \tan^2 t,$$

位置 $(\tan t, \tan^2 t)$,

$$v'(t) = (\sec^2 t, 2 \tan t \sec^2 t),$$

$$v''(t) = (2 \sec^2 t \tan t, 2 \sec^4 t + 4 \tan^2 t \sec^2 t)$$

$$= 2 \sec^2 t (\sec^2 t, 2 \tan^2 t).$$



13.求函数

$$f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

在 $x=0$ 的左右导数.

$$\text{解 } f'_-(0) = \lim_{x \rightarrow 0^-} \frac{\frac{x}{1+e^{1/x}}}{x} = \lim_{x \rightarrow 0^-} \frac{1}{1+e^{1/x}} = 1, f'_+(0) = \lim_{x \rightarrow 0^+} \frac{\frac{x}{1+e^{1/x}}}{x} = \lim_{x \rightarrow 0^+} \frac{1}{1+e^{1/x}} = 0.$$

14.设 $f(x) = |x-a|\varphi(x)$, 其中 $\varphi(x)$ 在 $x=a$ 处连续且 $\varphi(a) \neq 0$. 证明 $f(x)$ 在 $x=a$ 不可导.

$$\text{证 } f'_-(a) = \lim_{x \rightarrow a^-} \frac{(a-x)\varphi(x)}{x-a} = -\varphi(a), f'_+(a) = \lim_{x \rightarrow a^+} \frac{(x-a)\varphi(x)}{x-a} = \varphi(a) \neq f'_-(a).$$

习题 2.2

1. 下列各题的计算是否正确, 指出错误并加以改正:

$$(1) (\cos \sqrt{x})' = -\sin \sqrt{x}, \text{ 错. } (\cos \sqrt{x})' = -\sin \sqrt{x} \sqrt{x}' = -\frac{\sin \sqrt{x}}{2\sqrt{x}}.$$

$$(2) [\ln(1-x)]' = \frac{1}{1-x}, \text{ 错. } [\ln(1-x)]' = \frac{1}{1-x} (1-x)' = \frac{1}{x-1}.$$

$$(3) \left[x^2 \sqrt{1+x^2} \right]' = (x^2)' (\sqrt{1+x^2})' = 2x \frac{x}{\sqrt{1+x^2}}, \text{ 错.}$$

$$\begin{aligned} \left[x^2 \sqrt{1+x^2} \right]' &= (x^2)' (\sqrt{1+x^2}) + (x^2) (\sqrt{1+x^2})' = 2x\sqrt{1+x^2} + x^2 \frac{x}{\sqrt{1+x^2}} \\ &= 2x\sqrt{1+x^2} + \frac{x^3}{\sqrt{1+x^2}} = \frac{2x+3x^3}{\sqrt{1+x^2}}. \end{aligned}$$

$$(4) \left[\ln |x+2\sin^2 x| \right]' = \frac{1}{x+2\sin^2 x} (1+4\sin x) \cos x, \text{ 错.}$$

$$\left[\ln |x+2\sin^2 x| \right]' = \frac{1}{x+2\sin^2 x} (1+4\sin x \cos x).$$

2. 记 $f'(g(x)) = f'(u)|_{u=g(x)}$. 现设 $f(x) = x^2 + 1$.

(1) 求 $f'(x)$, $f'(0)$, $f'(x^2)$, $f'(\sin x)$;

(2) 求 $\frac{d}{dx} f(x^2)$, $\frac{d}{dx} f(\sin x)$;

(3) $f'(g(x))$ 与 $[f(g(x))]'$ 是否相同? 指出两者的关系.

解 (1) $f'(x) = 2x$, $f'(0) = 0$, $f'(x^2) = 2x^2$, $f'(\sin x) = 2\sin x$.

$$(2) \frac{d}{dx} f(x^2) = f'(x^2) (x^2)' = 2x^2 \cdot 2x = 4x^3.$$

$$\frac{d}{dx} f(\sin x) = f'(\sin x) (\sin x)' = 2\sin x \cos x = \sin 2x.$$

(3) $f'(g(x))$ 与 $[f(g(x))]'$ 不同, $[f(g(x))]' = f'(g(x))g'(x)$.

3. 求下列函数的导函数:

$$(1) y = \frac{2}{x^3-1}, y' = -\frac{2 \cdot 3x^2}{(x^3-1)^2} = -\frac{6x^2}{(x^3-1)^2}.$$

$$(2) y = \sec x, y' = ((\cos x)^{-1})' = -(\cos x)^{-2} (\cos x)' = -(\cos x)^{-2} (-\sin x) = \tan x \sec x.$$

$$(3) y = \sin 3x + \cos 5x, y' = 3\cos 3x - 5\sin 5x.$$

$$\begin{aligned} (4) y &= \sin^3 x \cos 3x, y' = 3\sin^2 x \cos x \cos 3x - 3\sin^3 x \sin 3x \\ &= 3\sin^2 x (\cos x \cos 3x - \sin x \sin 3x) = 3\sin^2 x \cos 4x. \end{aligned}$$

$$(5) y = \frac{1 + \sin^2 x}{\cos x^2}, y' = \frac{2 \sin x \cos x \cos x^2 - (1 + \sin^2 x)(-\sin x^2)2x}{\cos^2 x^2}$$

$$= \frac{\sin 2x \cos x^2 + 2x(1 + \sin^2 x)(\sin x^2)}{\cos^2 x^2}.$$

$$(6) y = \frac{1}{3} \tan^3 x - \tan x + x, y' = \tan^2 x \sec^2 x - \sec^2 x + 1$$

$$= \tan^2 x \sec^2 x - \tan^2 x = \tan^2 x (\sec^2 x - 1) = \tan^4 x.$$

$$(7) y = e^{ax} \sin bx, y' = ae^{ax} \sin bx + be^{ax} \cos bx = e^{ax} (a \sin bx + b \cos bx).$$

$$(8) y = \cos^5 \sqrt{1+x^2}, y' = 5 \cos^4 \sqrt{1+x^2} (-\sin \sqrt{1+x^2}) \frac{x}{\sqrt{1+x^2}}$$

$$= -\frac{5x \cos^4 \sqrt{1+x^2} \sin \sqrt{1+x^2}}{\sqrt{1+x^2}}.$$

$$(9) y = \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|, y' = \frac{1}{2} \frac{1}{\tan \left(\frac{x}{2} + \frac{\pi}{4} \right)} \sec^2 \left(\frac{x}{2} + \frac{\pi}{4} \right)$$

$$= \frac{1}{2} \frac{1}{\tan \left(\frac{x}{2} + \frac{\pi}{4} \right)} \frac{1}{\cos^2 \left(\frac{x}{2} + \frac{\pi}{4} \right)} = \frac{1}{2 \sin \left(\frac{x}{2} + \frac{\pi}{4} \right) \cos \left(\frac{x}{2} + \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sin \left(x + \frac{\pi}{2} \right)} = \frac{1}{\cos x} = \sec x.$$

$$(10) y = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| (a > 0, x \neq \pm a), y' = \frac{1}{2a} \frac{x+a}{x-a} \frac{(x+a) - (x-a)}{(x+a)^2} = \frac{1}{x^2 - a^2}.$$

4. 求下列函数的导函数:

$$(1) y = \arcsin \frac{x}{a} (a > 0), y' = \frac{1}{\sqrt{1 - \left(\frac{x}{a} \right)^2}} \frac{1}{a} = \frac{1}{\sqrt{a^2 - x^2}}.$$

$$(2) y = \frac{1}{a} \arctan \frac{x}{a} (a > 0), y' = \frac{1}{a} \frac{1}{1 + \left(\frac{x}{a} \right)^2} \frac{1}{a} = \frac{1}{a^2 + x^2}.$$

$$(3) y = x^2 \arccos x (|x| < 1), y' = 2x \arccos x - \frac{x^2}{\sqrt{1-x^2}}.$$

$$(4) y = \arctan \frac{1}{x}, y' = \frac{1}{1 + \frac{1}{x^2}} \frac{-1}{x^2} = -\frac{1}{1+x^2}.$$

$$(5) y = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} (a > 0),$$

$$y' = \frac{1}{2}\sqrt{a^2 - x^2} + \frac{x}{2} \frac{-2x}{\sqrt{a^2 - x^2}} + \frac{a^2}{2} \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \frac{1}{a}$$

$$= \frac{1}{2}\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}} + \frac{a^2}{\sqrt{a^2 - x^2}} = \sqrt{a^2 - x^2}.$$

$$(6) y = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \frac{x + \sqrt{x^2 + a^2}}{a} \quad (a > 0)$$

$$y' = \frac{1}{2}\sqrt{x^2 + a^2} + \frac{x}{2} \frac{x}{\sqrt{x^2 + a^2}} + \frac{a^2}{2} \frac{1}{x + \sqrt{x^2 + a^2}} \left(1 + \frac{x}{\sqrt{x^2 + a^2}}\right)$$

$$= \frac{1}{2}\sqrt{x^2 + a^2} + \frac{x^2}{2\sqrt{x^2 + a^2}} + \frac{a^2}{2\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2}.$$

$$(7) y = \arcsin \frac{2x}{x^2 + 1}, x \neq \pm 1.$$

$$y' = \frac{1}{\sqrt{1 - \frac{4x^2}{(x^2 + 1)^2}}} \frac{2(x^2 + 1) - 2x \cdot 2x}{(x^2 + 1)^2} = 2 \frac{1}{|x^2 - 1|} \frac{1 - x^2}{x^2 + 1} = \frac{2 \operatorname{sgn}(1 - x^2)}{x^2 + 1}.$$

$$(8) y = \frac{2}{\sqrt{a^2 - b^2}} \arctan \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right) \quad (a > b \geq 0).$$

$$y' = \frac{2}{\sqrt{a^2 - b^2}} \frac{1}{1 + \frac{a-b}{a+b} \tan^2 \frac{x}{2}} \sqrt{\frac{a-b}{a+b}} \sec^2 \frac{x}{2} \left(\frac{1}{2} \right)$$

$$= \frac{1}{a+b + (a-b) \tan^2 \frac{x}{2}} \sec^2 \frac{x}{2} = \frac{1}{(a+b) \cos^2 \frac{x}{2} + (a-b) \sin^2 \frac{x}{2}}$$

$$= \frac{1}{a+b \cos x}.$$

$$(9) y = (1 + \sqrt{x})(1 + \sqrt{2x})(1 + \sqrt{3x}), \ln y = \ln(1 + \sqrt{x}) + \ln(1 + \sqrt{2x}) + \ln(1 + \sqrt{3x})$$

$$y'/y = \frac{1}{2(1 + \sqrt{x})\sqrt{x}} + \frac{2}{2(1 + \sqrt{2x})\sqrt{2x}} + \frac{3}{2(1 + \sqrt{3x})\sqrt{3x}},$$

$$y' = y \left[\frac{1}{2(1 + \sqrt{x})\sqrt{x}} + \frac{2}{2(1 + \sqrt{2x})\sqrt{2x}} + \frac{3}{2(1 + \sqrt{3x})\sqrt{3x}} \right].$$

$$(10) y = \sqrt{1 + x + 2x^2}, y' = \frac{1 + 4x}{2\sqrt{1 + x + 2x^2}}.$$

$$(11) y = \sqrt{x^2 + a^2}, y' = \frac{x}{\sqrt{x^2 + a^2}}.$$

$$(12) y = \sqrt{a^2 - x^2}, y' = \frac{-x}{\sqrt{a^2 - x^2}}.$$

$$(13) y = \ln(x + \sqrt{x^2 + a^2}), y' = \frac{1}{x + \sqrt{x^2 + a^2}} \left(1 + \frac{x}{\sqrt{x^2 + a^2}} \right) = \frac{1}{\sqrt{x^2 + a^2}}.$$

$$(14) y = (x-1)\sqrt[3]{(3x+1)^2(2-x)}. \ln y = \ln(x-1) + \frac{2}{3}\ln(3x+1) + \frac{1}{3}\ln(2-x),$$

$$\frac{y'}{y} = \frac{1}{x-1} + \frac{2}{3x+1} + \frac{1}{3} \frac{-1}{2-x}$$

$$y' = y \left[\frac{1}{x-1} + \frac{2}{3x+1} + \frac{1}{3} \frac{-1}{2-x} \right].$$

$$(15) y = e^x + e^{e^x}, y' = e^x + e^{e^x} \mathfrak{E}^x = e^x(1 + e^{e^x}).$$

$$(16) y = x^{a^a} + a^{x^a} + a^{a^x} (a > 0).$$

$$y' = a^a x^{a^a-1} + a^{x^a} \ln a (ax^{a-1}) + a^{a^x} \ln a a^x \ln a$$

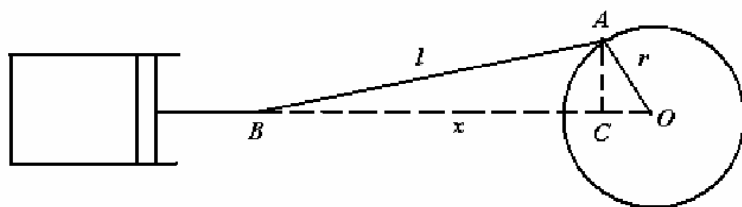
$$= a^a x^{a^a-1} + a \ln a a^{x^a} x^{a-1} + a^{a^x} a^x \ln^2 a.$$

5.一雷达的探测器瞄准着一枚安装在发射台上的火箭,它与发射台之间的距离是400m. 设 $t=0$ 时向上垂直地发射火箭,初速度为0,火箭以的匀加速度 8m/s^2 垂直地向上运动;若雷达探测器始终瞄准着火箭. 问:自火箭发射后10秒钟时,探测器的仰角 $\theta(t)$ 的变化速率是多少?

$$\text{解 } x(t) = \frac{1}{2} g t^2 = 4t^2, \tan \theta(t) = \frac{x(t)}{400} = \frac{t^2}{100},$$

$$\theta(t) = \arctan \frac{t^2}{100}, \theta'(t) = \frac{1}{1 + \left(\frac{t^2}{100} \right)^2} \mathfrak{E} \frac{t}{50}, \theta'(10) = \frac{1}{1 + \left(\frac{10^2}{100} \right)^2} \mathfrak{E} \frac{10}{50} = 0.1 (\text{弧度}/\text{s}).$$

6.在图示的装置中,飞轮的半径为2m且以每秒旋转4圈的匀角速度按顺时针方向旋转. 问:当飞轮的旋转角为 $\theta = \frac{\pi}{2}$ 时,活塞向右移动的速率是多少?



解 $x(t) = 2 \cos 8\pi t + \sqrt{36 - 4 \sin^2 8\pi t},$

$$x'(t) = -16\pi \sin 8\pi t + \frac{-8 \sin 8\pi t \cos 8\pi t (8\pi)}{2\sqrt{36 - 4 \sin^2 8\pi t}},$$

$$\alpha(t) = 8\pi t = \frac{\pi}{2},, t_0 = \frac{1}{16}, x'(\frac{1}{16}) = -16\pi.$$

活塞向右移动的速率是 $16\pi \text{m/s}$.

习题 2.3

1. 当 $x \rightarrow 0$ 时, 下列各函数是 x 的几阶无穷小量?

(1) $y = x + 10x^2 + 100x^3$. 1阶.

(2) $y = (\sqrt{x+2} - \sqrt{2}) \sin x = \frac{x \sin x}{\sqrt{x+2} + \sqrt{2}}$, 2阶.

(3) $y = x(1 - \cos x) = x \cdot 2 \sin^2 \frac{x}{2}$, 2阶.

2. 已知: 当 $x \rightarrow 0$ 时, $\alpha(x) = o(x^2)$. 试证明 $\alpha(x) = o(x)$.

证 $\frac{\alpha(x)}{x} = \frac{\alpha(x)}{x^2} \cdot x = o(1) \cdot x = o(1)$.

3. 设 $\alpha(x) = o(x)$ ($x \rightarrow 0$), $\beta(x) = o(x)$ ($x \rightarrow 0$). 试证明: $\alpha(x) + \beta(x) = o(x)$ ($x \rightarrow 0$).

证 $\frac{\alpha(x) + \beta(x)}{x} = \frac{\alpha(x)}{x} + \frac{\beta(x)}{x} = o(1) + o(1) = o(1)$.

上述结果有时可以写成 $o(x) + o(x) = o(x)$.

4. 计算下列函数在指定点 x_0 处的微分:

(1) $y = x \sin x$, $x_0 = \pi/4$. $y' = \sin x + x \cos x$, $y'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \left(1 + \frac{\pi}{4}\right)$, $dy = \frac{1}{\sqrt{2}} \left(1 + \frac{\pi}{4}\right) dx$.

(2) $y = (1+x)^\alpha$ ($\alpha > 0$ 是常数).

$y' = \alpha(1+x)^{\alpha-1}$, $y'(0) = \alpha$, $dy = \alpha dx$.

5. 求下列各函数的微分:

(1) $y = \frac{1-x}{1+x} = -1 + \frac{2}{1+x}$, $y' = -\frac{2}{(1+x)^2}$, $dy = -\frac{2dx}{(1+x)^2}$.

(2) $y = xe^x$, $y' = e^x + xe^x = e^x(1+x)$. $dy = e^x(1+x)dx$.

6. 设 $y = \frac{2}{x-1}$ ($x \neq 1$), 计算当 x 由 3 变到 3.001 时, 函数的增量和向相应的微分.

解 $y' = -\frac{2}{(x-1)^2}$, $y'(3) = -\frac{1}{2}$.

$\Delta y = \frac{2}{2.001} - 1 = -\frac{0.001}{2.001}$, $dy = -\frac{0.001}{2}$.

7. 试计算 $\sqrt[5]{32.16}$ 的近似值.

解 $\sqrt[5]{32.16} = 2\sqrt[5]{1 + .16/32} = 2\left(1 + \frac{1}{5} \cdot \frac{.16}{32}\right) = 2.002$.

8. 求下列方程所确定的隐函数的导函数:

(1) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ ($a > 0$). $\frac{1}{3}x^{-\frac{1}{3}} + \frac{1}{3}y^{-\frac{1}{3}}y' = 0$, $y' = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$.

(2) $(x-a)^2 + (y-b)^2 = c^2$ (a, b, c 为常数).

$2(x-a) + 2(y-b)y' = 0$, $y' = -\frac{x-a}{y-b}$.

$$(3) \arctan \frac{y}{x} = \ln \sqrt{x^2 + y^2}.$$

$$\frac{-\frac{y}{x^2} + \frac{y'}{x}}{1 + \left(\frac{y}{x}\right)^2} = \frac{x + yy'}{x^2 + y^2}, \frac{xy' - y}{x^2 + y^2} = \frac{x + yy'}{x^2 + y^2}, xy' - y = x + yy', y' = \frac{x + y}{x - y}.$$

$$(4) y \sin x - \cos(x - y) = 0$$

$$y' \sin x + y \cos x + \sin(x - y)(1 - y') = 0,$$

$$y' = \frac{y \cos x + \sin(x - y)}{\sin(x - y) - \sin x}.$$

9. 求下列隐函数在指定的点 M 的导数:

$$(1) y^2 - 2xy - x^2 + 2x - 4 = 0, M(3, 7)$$

$$2yy' - 2y - 2xy' - 2x + 2 = 0, y' = \frac{y + x - 1}{y - x}, y'(3) = \frac{7 + 3 - 1}{7 - 3} = \frac{9}{4}.$$

$$(2) e^{xy} - 5x^2y = 0, M\left(\frac{e^2}{10}, \frac{20}{e^2}\right).$$

$$e^{xy}(y + xy') - 10xy - 5x^2y' = 0, y' = \frac{10xy - ye^{xy}}{xe^{xy} - 5x^2}, y'\left(\frac{e^2}{10}\right) = \frac{20 - \frac{20}{e^2}e^2}{\frac{e^2}{10}e^2 - 5\frac{e^4}{100}} = 0.$$

10. 设 $y = f(x)$ 由下列参数方程给出, 求 $y' = \frac{dy}{dx}$:

$$(1) \begin{cases} x = 2t - t^2 \\ y = 3t - t^3 \end{cases}$$

$$\frac{dy}{dx} = \frac{3 - 3t^2}{2 - 2t} = \frac{3}{2}(1 + t), (t \neq 1).$$

$$(2) \begin{cases} x = t \ln t \\ y = e^t \end{cases} \frac{dy}{dx} = \frac{e^t}{\ln t + 1}, t \neq 1/e.$$

$$(3) \begin{cases} x = \arccos \frac{1}{\sqrt{1+t^2}} \\ y = \arcsin \frac{t}{\sqrt{1+t^2}} \end{cases}$$

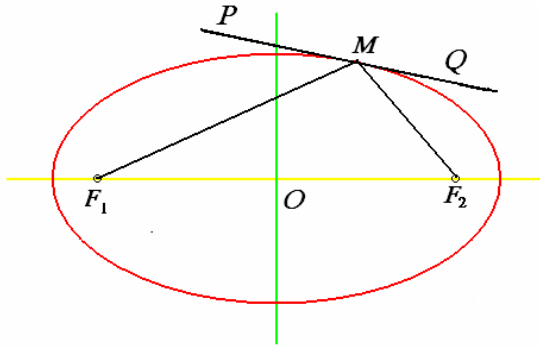
$$\frac{dy}{dx} = \frac{-\frac{1}{\sqrt{1-\frac{1}{1+t^2}}} \left(-\frac{1}{2}\right) \frac{2t}{(1+t^2)^{3/2}}}{\frac{1}{\sqrt{1-\left(\frac{t^2}{1+t^2}\right)}} \frac{t}{\sqrt{1+t^2}}} = \operatorname{sgn}(t), t \neq 0.$$

11. 试求椭圆周 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 上一点 $M_0(x_0, y_0)$ 处的切线方程与法线方程. 并证明: 从椭圆的一个焦点向椭圆周上任一点 M 发射的光线, 其反射线必通过椭圆的另一个焦点.

$$\frac{2x}{a^2} + \frac{2yy'}{b^2}, y' = -\frac{b^2x}{a^2y}.$$

$$\text{切线方程: } y - y_0 = \left(-\frac{b^2x_0}{a^2y_0} \right) (x - x_0), \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1.$$

$$\text{法线方程: } y - y_0 = \left(\frac{a^2y_0}{b^2x_0} \right) (x - x_0), a^2y_0x - b^2x_0y = (a^2 - b^2)x_0y_0$$



焦点 $F_1(-c, 0), F_2(c, 0), c^2 = a^2 - b^2 (a > b)$. 设 $y_0 \neq 0$. 切线斜率 $k = -\frac{b^2x_0}{a^2y_0}$.

$$MF_1 \text{ 的斜率 } k_1 = \frac{y_0}{x_0 + c}, MF_2 \text{ 的斜率 } k_2 = \frac{y_0}{x_0 - c}.$$

$$\begin{aligned} \tan \angle F_2MQ &= \frac{k - k_2}{1 + kk_2} = \frac{-\frac{b^2x_0}{a^2y_0} - \frac{y_0}{x_0 - c}}{1 - \frac{b^2x_0}{a^2y_0} \frac{y_0}{x_0 - c}} = -\frac{b^2x_0(x_0 - c) + a^2y_0^2}{a^2y_0(x_0 - c) - b^2x_0y_0} \\ &= -\frac{a^2b^2 - b^2cx_0}{(a^2 - b^2)x_0y_0 - a^2cy_0} = -\frac{b^2(a^2 - cx_0)}{c^2x_0y_0 - a^2cy_0} = \frac{b^2(a^2 - cx_0)}{cy_0(a^2 - cx_0)} = \frac{b^2}{cy_0}; \end{aligned}$$

$$\begin{aligned} \tan \angle PMF_1 &= \frac{k_1 - k}{1 + kk_1} = \frac{\frac{y_0}{x_0 + c} - \frac{b^2x_0}{a^2y_0}}{1 - \frac{b^2x_0}{a^2y_0} \frac{y_0}{x_0 + c}} = -\frac{b^2x_0(x_0 + c) + a^2y_0^2}{a^2y_0(x_0 + c) - b^2x_0y_0} \\ &= \frac{a^2b^2 + b^2cx_0}{(a^2 - b^2)x_0y_0 + a^2cy_0} = \frac{b^2(a^2 + cx_0)}{c^2x_0y_0 + a^2cy_0} = \frac{b^2(a^2 + cx_0)}{cy_0(a^2 + cx_0)} = \frac{b^2}{cy_0} = \tan \angle F_2MQ. \end{aligned}$$

$\angle PMF_1$ 和 $\angle F_2MQ$ 都在区间 $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$, 故 $\angle PMF_1 = \angle F_2MQ$.

习题 2.4

n

(1) $y = x^n, y^{(n)} = n!$.

(2) $y = e^x, y^{(n)} = e^n$.

(3) $y = \frac{1}{1+x} = (1+x)^{-1} (x \neq -1), y^{(n)} = (-1)(-1-1)L \dots (-1-n+1)(1+x)^{-1-n} = \frac{(-1)^n n!}{(1+x)^{n+1}}$.

(4) $y = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}, y^{(n)} = (-1)^n n! \left(\frac{1}{x^{n+1}} - \frac{1}{(1+x)^{n+1}} \right)$.

2. 设 $y(x) = e^x \cos x$, 证明 $y'' - 2y' + 2y = 0$.

证 $y' = e^x \cos x - e^x \sin x = e^x (\cos x - \sin x)$,

$y'' = e^x (\cos x - \sin x) + e^x (-\sin x - \cos x) = e^x (-2 \sin x)$,

$y'' - 2y' + 2y = e^x (-2 \sin x) - 2e^x (\cos x - \sin x) + 2e^x \cos x = 0$,

3. 设 $y = \frac{x-3}{x+4} (x \neq -4)$, 证明 $2y'^2 = (y-1)y''$.

证 $y = \frac{x-3}{x+4} = 1 - \frac{7}{x+4}, y' = \frac{7}{(x+4)^2}, y'' = -\frac{14}{(x+4)^3}$.

$2y'^2 = \frac{98}{(x+4)^4}, (y-1)y'' = \left(-\frac{7}{x+4} \right) \left(-\frac{14}{(x+4)^3} \right) = \frac{98}{(x+4)^4} = 2y'^2$.

4. 设 $y = (1-x)(2x+1)^2(3x-1)^3$, 求 $y^{(6)}, y^{(7)}$.

解 $y^{(6)} = 6! \cdot 6! = 108, y^{(7)} = 0$.

5. 要使 $y = e^{\lambda x}$ 满足方程 $y'' + py' + qy = 0$ (其中 p, q 为常数), λ 该取哪些值?

解 $y' = \lambda e^{\lambda x}, y'' = \lambda^2 e^{\lambda x}, y'' + py' + qy = (\lambda^2 + p\lambda + q)e^{\lambda x} = 0, e^{\lambda x} \neq 0$,

λ 该取方程 $\lambda^2 + p\lambda + q = 0$ 的根.

6. 飞轮绕一定轴转动, 转过的角度 θ 与时间 t 的关系为 $\theta = t^3 - 2t^2 + 3t - 1$, 求飞轮转动的角速度与角加速度.

解 角速度 $\theta' = 3t^2 - 4t + 3$, 角加速度 $\theta'' = 6t - 4$.

7. 设 $f(x) = \frac{1}{(1-x)^n}$, 其中 n 为一个正整数, 求 $f^{(k)}(x), k$ 为一个正整数.

解 $f(x) = \frac{1}{(1-x)^n} = (1-x)^{-n}, f^{(k)}(x) = (-n)(-n-1)L \dots (-n-k+1)(1-x)^{-n-k} (-1)^k$

$= \frac{n(n+1)L \dots (n+k-1)}{(1-x)^{n+k}}, f^{(k)}(0) = n(n+1)L \dots (n+k-1)$.

8. 设 $y = x^2 \ln(1+x)$, 求 $y^{(50)}$.

解 由 Leibniz 公式,

$$\begin{aligned}
y^{(50)} &= x^2 (\ln(1+x))^{(50)} + 50g(2x)g(\ln(1+x))^{(49)} + \frac{50g49}{2}g^2g(\ln(1+x))^{(48)} \\
&= x^2 ((1+x)^{-1})^{(49)} + 50g(2x)g((1+x)^{-1})^{(48)} + \frac{50g49}{2}g^2g((1+x)^{-1})^{(47)} \\
&= x^2 (-1)(-2)L \quad (-1-49+1)(1+x)^{-50} + 100x(-1)(-2)L \quad (-1-48+1)(1+x)^{-49} + \\
&\quad 2450(-1)(-2)L \quad (-1-47+1)(1+x)^{-48} \\
&= -x^2 49!(1+x)^{-50} + 100g48!(1+x)^{-49} - 2450g47!(1+x)^{-48} = \frac{-2g47!}{(1+x)^{50}}(x^2 + 50x + 1225).
\end{aligned}$$

9.验证函数 $y = C_1 e^{ax} + C_2 e^{bx}$ (其中 C_1 与 C_2 为任意常数)是微分方程 $y'' - (a+b)y' + aby = 0$ 的解.

$$\begin{aligned}
\text{证 } y' &= (C_1 e^{ax} + C_2 e^{bx})' = C_1 a e^{ax} + C_2 b e^{bx}, y'' = (C_1 a e^{ax} + C_2 b e^{bx})' = C_1 a^2 e^{ax} + C_2 b^2 e^{bx}, \\
y'' - (a+b)y' + aby &= C_1 a^2 e^{ax} + C_2 b^2 e^{bx} - (a+b)(C_1 a e^{ax} + C_2 b e^{bx}) + ab(C_1 e^{ax} + C_2 e^{bx}) = 0.
\end{aligned}$$

10.验证函数 $y = (C_1 x + C_2) e^{ax}$ (其中 C_1 与 C_2 为任意常数)是微分方程 $y'' - 2ay' + a^2 y = 0$ 的解.

$$\begin{aligned}
\text{证 } y' &= ((C_1 x + C_2) e^{ax})' = C_1 e^{ax} + a(C_1 x + C_2) e^{ax} = e^{ax} (aC_1 x + C_1 + aC_2), \\
y'' &= e^{ax} a(aC_1 x + C_1 + aC_2) + e^{ax} (aC_1) = e^{ax} (a^2 C_1 x + a^2 C_2 + 2aC_1), \\
y'' - 2ay' + a^2 y &= e^{ax} (a^2 C_1 x + a^2 C_2 + 2aC_1) - 2ae^{ax} (aC_1 x + C_1 + aC_2) + a^2 (C_1 x + C_2) e^{ax} = 0.
\end{aligned}$$

验证函数 $y = C_1 \cos \omega t + C_2 \sin \omega t$ (其中 C_1 与 C_2 为任意常数)是微分方程 $y'' + \omega^2 y = 0$ 的解.

$$\begin{aligned}
\text{证 } y' &= -C_1 \omega \sin \omega t + C_2 \omega \cos \omega t, y'' \\
&= -C_1 \omega^2 \cos \omega t - C_2 \omega^2 \sin \omega t = -\omega^2 (C_1 \cos \omega t + C_2 \sin \omega t) = -\omega^2 y.
\end{aligned}$$

习题 2.5

求下列不定积分：

$$1. \int \left(\frac{a}{\sqrt{x}} - \frac{b}{x^2} + 3C\sqrt[3]{x^2} \right) dx = 2a\sqrt{x} + \frac{b}{x} + \frac{9C}{5}x^{5/3} + C.$$

$$2. \int (1 + \sqrt{x})^2 dx = \int (1 + 2\sqrt{x} + x) dx = x + \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 + C.$$

$$3. \int a \sec^2 x dx = a \tan x + C.$$

$$4. \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C.$$

$$5. \int \cot^2 \varphi d\varphi = \int (\csc^2 \varphi - 1) d\varphi = -\cot^2 \varphi - 1 + C.$$

$$6. \int \frac{x^2 + 3}{1 + x^2} dx = \int \left(1 + \frac{2}{1 + x^2} \right) dx = x + 2 \arctan x + C.$$

$$7. \int \left(\frac{3}{\sqrt{x}} + \frac{4}{\sqrt{1-x^2}} \right) dx = 6\sqrt{x} + 4 \arcsin x + C.$$

$$8. \int (1 + \cos^2 x) \sec^2 x dx = \int (\sec^2 x + 1) dx = \tan x + x + C.$$

$$9. \int \frac{1-x}{1-\sqrt[4]{x}} dx = \int (1 + \sqrt[4]{x} + \sqrt[4]{x^2} + \sqrt[4]{x^3}) dx = x + \frac{4}{5}x^{5/4} + \frac{2}{3}x^{3/2} + \frac{4}{7}x^{7/4} + C$$

$$10. \int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3} \right) dx = \ln |x| - \frac{2}{x} - \frac{6}{x^2} + C.$$

$$11. \int \frac{(1-x)^2}{x\sqrt[3]{x}} dx = \int \frac{1-2x+x^2}{x^{4/3}} dx = \int (x^{-4/3} - 2x^{-1/3} + x^{2/3}) dx \\ = -3x^{-1/3} - 3x^{2/3} + \frac{3}{5}x^{5/3} + C.$$

$$12. \int (2 \cosh x - \sinh x) dx = 2 \sinh x - \cosh x + C.$$

$$13. \int \left(\frac{3x^2-1}{x^2} + \frac{(x+1)^2}{\sqrt{x}} \right) dx = \int \left(3 - \frac{1}{x^2} + x^{3/2} + 2x^{1/2} + x^{-1/2} \right) dx \\ = 3x + \frac{1}{x} + \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 2\sqrt{x} + C.$$

$$14. \int \frac{1}{\sin^2 x \cos^2 x} dx = \int \left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \right) dx = -\cot x + \tan x + C.$$

$$15. \int \frac{2^{x+1} + 3^{x-2}}{6^x} dx = \int \left(2 \left(\frac{1}{3} \right)^x + \frac{1}{9} \left(\frac{1}{2} \right)^x \right) dx \\ = -2 \left(\frac{1}{3} \right)^x / \ln 3 - \frac{1}{9} \left(\frac{1}{2} \right)^x / \ln 2 + C.$$

$$16. \int \frac{1}{x^2(1+x^2)} dx = \int \left(\frac{1}{x^2} - \frac{1}{1+x^2} \right) dx = -\frac{1}{x} - \arctan x + C.$$

17. 求解微分方程 $y''(x) = a + be^{-x}$ (a, b 为常数).

解 $y' = \int (a + be^{-x}) dx = ax - be^{-x} + C_1, y$

$$= \int (ax - be^{-x} + C_1) dx = \frac{1}{2} ax^2 + be^{-x} + C_1 x + C_2.$$

18. 设 $f(x)$ 满足方程 $xf'(x) + f(x) = x^2 + 1$, 求 $f(x)$.

解 $(xf(x))' = x^2 + 1, xf(x) = \int x^3 + 1 dx = \frac{1}{4} x^4 + x + C,$

$$f(x) = \frac{1}{4} x^3 + 1 + \frac{C}{x}.$$

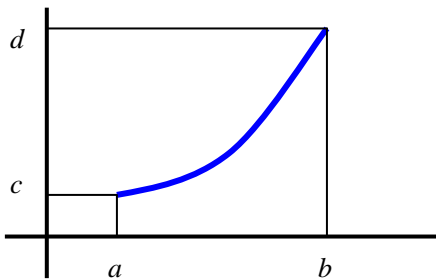
习题 2.6

1. 根据定积分的定义直接求下列积分:

$$(1) \int_a^b k dx = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n k \Delta x_i = \lim_{\lambda \rightarrow 0} k(b-a) = k(b-a).$$

$$\begin{aligned} (2) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(a + \frac{i(b-a)}{n}\right) \frac{b-a}{n} &= \lim_{n \rightarrow \infty} \left(a(b-a) + \frac{(b-a)^2}{n^2} \sum_{i=1}^n i\right) \\ &= a(b-a) + (b-a)^2 \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \sum_{i=1}^n i\right) = a(b-a) + (b-a)^2 \lim_{n \rightarrow \infty} \frac{1}{n^2} \frac{n(n+1)}{2} \\ &= a(b-a) + (b-a)^2 \lim_{n \rightarrow \infty} \frac{(1+1/n)}{2} = a(b-a) + \frac{(b-a)^2}{2} = \frac{b^2 - a^2}{2}. \end{aligned}$$

2. 设函数 $x = \varphi(y)$ 在 $[c, d]$ 上连续且 $\varphi(y) > 0$. 试用定积分表示曲线 $x = \varphi(y)$, $y = c$, $y = d$ 及 y 轴所围的图形的面积; 又设 $c \geq 0$, 函数 $x = \varphi(y)$ 在 $[c, d]$ 上严格递增, 试求积分和 $\int_c^d \varphi(y) dy + \int_a^b \psi(x) dx$, 其中 $y = \psi(x)$ 是 $x = \varphi(y)$ 的反函数, $a = \varphi(c)$, $b = \varphi(d)$.



$$\text{解 } \int_c^d \varphi(y) dy + \int_a^b \psi(x) dx = bd - ac.$$

3. 写出函数 $y = x^2$ 在区间 $[0, 1]$ 上的 Riemann 和, 其中分割为 n 等分, 中间点 ξ_i 为分割小区间的左端点. 求出当 $n \rightarrow \infty$ 时 Riemann 和的极限.]

$$\text{解 } s_n = \sum_{i=0}^{n-1} \frac{i^2}{n^2} \frac{1}{n} = \frac{1}{6n^3} (n-1)n(2n-1) = \frac{1}{6} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right) \rightarrow \frac{1}{3} (n \rightarrow \infty).$$

4. 求定积分 $\int_0^1 \sqrt{x} dx$.

解 $y = \sqrt{x}$ 的反函数 $x = y^2$, 当 $x = 0$ 时, $y = 0$, 当 $x = 1$ 时, $y = 1$. 由 2, 3 题

$$\int_0^1 \sqrt{x} dx = 1 - \int_0^1 y^2 dy = 1 - \frac{1}{3} = \frac{2}{3}.$$

5. 证明下列不等式

$$(1) \frac{\pi}{2} < \int_0^{\pi/2} (1 + \sin x) dx < \pi.$$

$$\text{证 } \int_0^{\pi/2} (1 + \sin x) dx > \int_0^{\pi/2} (1) dx = \frac{\pi}{2}. \int_0^{\pi/2} (1 + \sin x) dx < \int_0^{\pi/2} (2) dx = \pi.$$

$$(2) \sqrt{2} < \int_0^1 \sqrt{2+x-x^2} dx < \frac{3}{2}.$$

证 $2+x-x^2 = (1+x)(2-x) = 0$, $x_1 = -1$, $x_2 = 2$. 当 $x \in (-\infty, 1/2)$ 时, $\sqrt{2+x-x^2}$ 递增,

当 $x \in (1/2, +\infty)$ 时, $\sqrt{2+x-x^2}$ 递减, 故

$$\sqrt{2} = \int_0^1 \sqrt{2} dx < \int_0^1 \sqrt{2+x-x^2} dx < \int_0^1 \sqrt{2+1/2-1/4} dx = \frac{3}{2}.$$

6. 判断下列各题中两个积分值之大小:

$$(1) \int_0^1 e^x dx > \int_0^1 e^{x^2} dx.$$

$$(2) \int_0^{\pi/2} x^2 dx > \int_0^{\pi/2} (\sin x)^2 dx.$$

$$(3) \int_0^1 x dx < \int_0^1 \sqrt{1+x^2} dx.$$

7. 设函数 $y = f(x)$ 在 $[a, b]$ 上有定义, 并且假定 $y = f(x)$ 在任何闭子区间上有最大值和最小值. 对于任意一个分割: $T: x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ 记 m_i 为 $f(x)$ 在 $[x_{i-1}, x_i]$ 中的最小值, M_i 为 $f(x)$ 在 $[x_{i-1}, x_i]$ 中的最大值. 证明

$y = f(x)$ 在 $[a, b]$ 上可积的充要条件是极限 $\lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n m_i \Delta x_i$ 与 $\lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n M_i \Delta x_i$ 存在并且相等.

证 设 $y = f(x)$ 在 $[a, b]$ 上可积, 则 $\lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n m_i \Delta x_i = \lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = \int_a^b f(x) dx,$

$$\lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n M_i \Delta x_i = \lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n f(\eta_i) \Delta x_i = \int_a^b f(x) dx.$$

设 $\lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n m_i \Delta x_i = \lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n M_i \Delta x_i = I$, 则

$$\sum_{i=1}^n m_i \Delta x_i \leq \sum_{i=1}^n f(\xi_i) \Delta x_i \leq \sum_{i=1}^n M_i \Delta x_i,$$

由夹挤定理, $\lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = I.$

习题 2.7

1. 求下列变上(下)限积分所定义的函数的导函数:

$$(1) F(x) = \int_1^{x^2} \frac{dt}{1+t^2}, F'(x) = \frac{1}{1+x^2}.$$

$$(2) G(x) = \int_0^{1+x^2} \sin t^2 dt, G'(x) = 2x \sin(1+x^2)^2.$$

$$(3) H(x) = \int_x^1 t^2 \cos t dt, H'(x) = -x^2 \cos x.$$

$$(4) L(x) = \int_x^{x^2} e^{-t^2} dt, L'(x) = 2xe^{-x^2} - e^{-x^2}.$$

2. 设 $y = f(x)$ 在 $[a, b]$ 上连续. 证明 $F_0(x) = \int_a^x f(t)dt$ 在 a 处有右导数, 且 $F'_+(a) = f(a)$.

$$\text{证 } \frac{F_0(a+\Delta x) - F_0(a)}{\Delta x} = \frac{1}{\Delta x} \int_a^{a+\Delta x} f(t)dt = \frac{1}{\Delta x} f(\xi)\Delta x \quad (a \leq \xi \leq a+\Delta x)$$

$$= f(\xi) \rightarrow f(a) \quad (\Delta x \rightarrow 0+) \text{ 故 } F'_+(a) = f(a).$$

3. 设 $f(x)$ 在 $[a, b]$ 上连续. 假定 $f(x)$ 有一个原函数 $F(x)$ 且 $F(a) = 0$. 证明

$$\text{当 } a \leq x \leq b \text{ 时 } F(x) = \int_a^x f(t)dt.$$

$$\text{证 } G(x) = \int_a^x f(t)dt. G(a) = 0, \text{ 由变上限积分求导定理, } G'(x) = f(x). F'(x) = f(x),$$

$$F(a) = 0.$$

$$(G(x) - F(x))' = G'(x) - F'(x) = f(x) - f(x) = 0, G(x) - F(x) = C, x \in [a, b].$$

$$C = F(a) - G(a) = 0, F(x) = G(x) = \int_a^x f(t)dt, x \in [a, b].$$

$$4. \text{证明: 当 } x \in (0, +\infty) \text{ 时, } \ln x = \int_1^x \frac{dt}{t}.$$

$$\text{证 由于 } (\ln x)' = \frac{1}{x}, \left(\int_1^x \frac{dt}{t} \right)' = \frac{1}{x}, x \in (0, +\infty), \ln 1 = \int_1^1 \frac{dt}{t} = 0, \text{ 故 } \ln x = \int_1^x \frac{dt}{t}.$$

5. 设 $y = f(x)$ 在 $[a, b]$ 上可积, 且 $|f(x)| \leq L, (\forall x \in [a, b])$, 其中 L 为常数. 证明

变上限积分 $F(x) = \int_a^x f(t)dt$ 在 $[a, b]$ 上满足 Lipschitz 条件:

$$|F(x_1) - F(x_2)| \leq L |x_1 - x_2|, (x_1, x_2 \in [a, b]).$$

证不妨设 $x_1 < x_2$,

$$|F(x_1) - F(x_2)| = \left| \int_a^{x_2} f(t)dt - \int_a^{x_1} f(t)dt \right| = \left| \int_{x_1}^{x_2} f(t)dt \right| \leq \int_{x_1}^{x_2} |f(t)|dt \leq \int_{x_1}^{x_2} Ldt = x_2 - x_1.$$

6. 求函数 $G(x) = \int_0^x e^t \int_0^t \sin z dz dt$ 的二阶导数.

$$\text{解 } G'(x) = e^x \int_0^x \sin z dz, G''(x) = e^x \int_0^x \sin z dz + e^x \sin x = e^x (1 - \cos x) + e^x \sin x.$$

习题 2.8

4. 将下列积分改成若干个区间上定积分之和, 然后分别使用Newton-Leibniz公式求其值:

(1)

1. 用Newton-Leibniz公式计算下列定积分:

$$(1) \int_0^1 x^3 dx = \frac{x^4}{4} \Big|_0^1 = \frac{1}{4}.$$

$$(2) \int_a^b e^x dx = e^x \Big|_a^b = e^b - e^a.$$

$$(3) \int_0^{3\pi} \sin x dx = -\cos x \Big|_0^{3\pi} = 2.$$

$$(4) \int_1^2 \frac{dx}{x} = \ln x \Big|_1^2 = \ln 2.$$

$$(5) \int_0^\pi (2 \sin x + x^3) dx = \left[-2 \cos x + \frac{x^4}{4} \right]_0^\pi = 4 + \frac{\pi^4}{4}.$$

$$(6) \int_0^1 (x^5 + \frac{1}{3}x^3 + \frac{1}{2}x + 1) dx = \left[\frac{x^6}{6} + \frac{x^4}{12} + \frac{x^2}{4} + x \right]_0^1 = \frac{3}{2}.$$

2. 验证 $\frac{1}{2}x^2 - \frac{1}{x}$ 是 $x + \frac{1}{x^2}$ 的一个原函数并计算定积分 $\int_2^4 \left(x + \frac{1}{x^2} \right) dx$. 试问下式

$$\int_{-1}^1 \left(x + \frac{1}{x^2} \right) dx = \left(\frac{1}{2}x^2 - \frac{1}{x} \right) \Big|_{-1}^1 \quad \text{是否成立: 为什么?}$$

解 $\left(\frac{1}{2}x^2 - \frac{1}{x} \right)' = \left(\frac{1}{2}x^2 \right)' - (x^{-1})' = x + x^{-2} = x + \frac{1}{x^2}$, 故 $\frac{1}{2}x^2 - \frac{1}{x}$ 是 $x + \frac{1}{x^2}$ 的一个原函数.

$$\int_2^4 \left(x + \frac{1}{x^2} \right) dx = \left(\frac{1}{2}x^2 - \frac{1}{x} \right) \Big|_2^4 = \frac{25}{4}.$$

$$\int_{-1}^1 \left(x + \frac{1}{x^2} \right) dx = \left(\frac{1}{2}x^2 - \frac{1}{x} \right) \Big|_{-1}^1 \quad \text{不成立. 因为 } x + \frac{1}{x^2} \text{ 在 } [-1, 1] \text{ 不可积.}$$

3. 将下列极限中的和式视作适当函数的Riemann和, 然后使用Newton-Leibniz公式求出其值:

$$(1) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \sin \frac{k}{n} = \int_0^1 \sin x dx = -\cos x \Big|_0^1 = 1 - \cos 1.$$

$$(2) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^3}{n^4} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \left(\frac{k}{n} \right)^3 = \int_0^1 x^3 dx = \frac{x^4}{4} \Big|_0^1 = \frac{1}{4}.$$

$$(3) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \frac{1}{1+k/n} = \int_0^1 \frac{dx}{1+x} = \ln(1+x) \Big|_0^1 = \ln 2.$$

4.将下列积分改成若干个区间上定积分之和,然后分别使用Newton-Leibniz公式求其值:

$$(1) \int_{-1}^1 |x| dx = \int_0^1 x dx - \int_{-1}^0 x dx = \frac{x^2}{2} \Big|_0^1 - \frac{x^2}{2} \Big|_{-1}^0 = 1.$$

$$(2) \int_{-1}^1 \operatorname{sgn} x dx = \int_0^1 1 dx + \int_{-1}^0 (-1) dx = 1 - 1 = 0.$$

$$(3) \int_0^1 x \left| \frac{1}{2} - x \right| dx = \int_0^{1/2} x \left(\frac{1}{2} - x \right) dx + \int_{1/2}^1 x \left(x - \frac{1}{2} \right) dx$$

$$= \left(\frac{x^2}{4} - \frac{x^3}{3} \right) \Big|_0^{1/2} + \left(\frac{x^3}{3} - \frac{x^2}{4} \right) \Big|_{1/2}^1 = \frac{1}{16} - \frac{1}{24} + \frac{1}{3} - \frac{1}{4} - \frac{1}{24} + \frac{1}{16} = \frac{1}{8}.$$

$$(4) \int_0^{2\pi} |\sin x| dx = \int_0^{\pi} \sin x dx - \int_{\pi}^{2\pi} \sin x dx = -\cos x \Big|_0^{\pi} + \cos x \Big|_{\pi}^{2\pi} = 2 + 2 = 4.$$

$$(5) \int_0^2 (x - [x]) dx = \int_0^1 x dx + \int_1^2 (x - 1) dx = \frac{x^2}{2} \Big|_0^1 + \left(\frac{x^2}{2} - x \right) \Big|_1^2$$

$$= \frac{1}{2} - \left(-\frac{1}{2} \right) = 1.$$

5.设 $F(x)$ 在 $[a, b]$ 上有连续的导函数 $F'(x)$.试证明:存在一点 $c \in [a, b]$,使得 $F(b) - F(a) = F'(c)(b - a)$.

$$\text{证 } F(b) - F(a) = \int_a^b F'(x) dx (\text{Newton-Leibniz公式})$$

$$= F'(c)(b - a) (\text{定积分中指中值公式}).$$

第二章总练习题

1. 讨论函数 $f(x) = \begin{cases} |x-3| & x \geq 1 \text{ 时} \\ \frac{x^2}{4} - \frac{3}{2}x + \frac{13}{4}, & x < 1 \text{ 时} \end{cases}$ 的连续性和可导性.

解 $x \neq 1$ 时 $f(x)$ 可导. $f(1-0) = \lim_{x \rightarrow 1} \left(\frac{x^2}{4} - \frac{3}{2}x + \frac{13}{4} \right) = 2$;

$f(1+0) = \lim_{x \rightarrow 1} |x-3| = 2 = f(1-0) = f(1)$, f 在 $x=1$ 连续.

$$f'_+(1) = (3-x)'|_{x=1} = -1, f'_-(1) = \left(\frac{x^2}{4} - \frac{3}{2}x + \frac{13}{4} \right)' \Big|_{x=1} = \left(\frac{x}{2} - \frac{3}{2} \right) \Big|_{x=1} = -1 = f'_+(1), f'(1) = -1.$$

f 在 $x=1$ 可导.

2. 设函数 $f(x) = \begin{cases} 2x-2 & x < -1 \text{ 时} \\ Ax^3 + Bx^2 + Cx + D, & -1 \leq x \leq 1 \text{ 时} \\ 5x+7 & x > 1 \text{ 时} \end{cases}$

试确定常数 A, B, C, D 的值, 使 $f(x)$ 在 $(-\infty, +\infty)$ 可导.

解 $f(-1-0) = \lim_{x \rightarrow -1} (2x-2) = -4 = f(-1) = -A+B-C+D$.

$$f'_-(-1) = (2x-2)'|_{x=-1} = 2 = f'_+(-1) = (Ax^3 + Bx^2 + Cx + D)'|_{x=-1}$$

$$= (3Ax^2 + 2Bx + C)|_{x=-1} = 3A - 2B + C.$$

$$f(1-0) = A+B+C+D = f(1+0) = 12,$$

$$f'_-(1) = 3A+2B+C = f'_+(1) = 5.$$

$$\begin{cases} -A+B-C+D = -4 \\ 3A-2B+C = 2 \\ A+B+C+D = 12 \\ 3A+2B+C = 5. \end{cases}$$

$$\{A = -9/4, B = 3/4, C = 41/4, D = 13/4\}.$$

3. 设函数 $g(x) = (\sin 2x)f(x)$, 其中 $f(x)$ 在 $x=0$ 连续. 问 $g(x)$ 在 $x=0$ 是否可导, 若可导, 求出 $g'(0)$.

解 $\frac{g(\Delta x) - g(0)}{\Delta x} = 2 \frac{f(\Delta x) \sin 2\Delta x}{2\Delta x} \rightarrow 2f(0)(\Delta x \rightarrow 0), g'(0) = 2f(0).$

4. 问函数 $f(x) = \frac{x^2 + \sin^2 x}{1+x^2}$ 与 $g(x) = \frac{-\cos^2 x}{1+x^2}$ 为什么有相同得导数?

解 因为 $f(x) - g(x) = 1$.

5. 设函数 $f(x)$ 在 $[-1, 1]$ 上有定义, 且满足 $x \leq f(x) \leq x^2 + x, x \in [-1, 1]$. 证明存在且等于 1.

证 $0 \leq f(0) \leq 0, f(0) = 0. \Delta x > 0,$

$$\frac{f(\Delta x) - f(0)}{\Delta x} = \frac{f(\Delta x)}{\Delta x} \leq \frac{\Delta x^2 + \Delta x}{\Delta x} = \Delta x + 1 \rightarrow 1 (\Delta x \rightarrow 0+0), f'_+(0) = 1, \text{ 类似 } f'_-(0) = 1,$$

故 $f'(0) = 1$.

6. 设 $f(x) = |x^2 - 4|$, 求 $f'(x)$.

解 $|x| > 2$ 时, $f(x) = x^2 - 4$, $f'(x) = 2x$. $f'_+(2) = (x^2 - 4)'|_{x=2} = 4$,
 $f'_-(2) = (4 - x^2)'|_{x=2} = -4$, $f'(2)$ 不存在, 同理 $f'(-2)$ 不存在.

7. 设 $y = \frac{1+x}{1-x}$, 求 $\frac{d^2 y}{dx^2}$.

解 $y = -1 + \frac{2}{1-x}$, $\frac{dy}{dx} = \frac{2}{(1-x)^2}$, $\frac{d^2 y}{dx^2} = -\frac{4}{(1-x)^3}$.

8. 设函数 $f(x)$ 在 $(-\infty, +\infty)$ 上有定义, 且满足下列性质:

(1) $f(a+b) = f(a)f(b)$ (a, b 为任意实数); (2) $f(0) = 1$; (3) 在 $x=0$ 处可导. 证明: 对于任意 $x \in (-\infty, +\infty)$ 都有 $f'(x) = f'(0)gf(x)$.

证 $\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{f(x)f(\Delta x) - f(x)f(0)}{\Delta x}$
 $= f(x) \frac{f(\Delta x) - f(0)}{\Delta x} \rightarrow f'(0)gf(x) (\Delta x \rightarrow 0), f'(x) = f'(0)gf(x)$.

9. 设 $f(x) = \begin{cases} 1/2^{2n}, & x = 1/2^n, \\ 0, & x \neq 1/2^n \end{cases} (n=1, 2, \dots); g(x) = \begin{cases} 1/2^{n+1}, & x = 1/2^n, \\ 0, & x \neq 1/2^n \end{cases} (n=1, 2, \dots);$

问 $f(x)$ 在 $x=0$ 处是否可导? $g(x)$ 在 $x=0$ 处是否可导?

解 $\frac{f(1/2^n) - f(0)}{1/2^n} = \frac{1/2^{2n}}{1/2^n} = \frac{1}{2^n} \rightarrow 0 (n \rightarrow \infty),$

$\frac{f(x) - f(0)}{x} = 0 \rightarrow 0 (x \neq 1/2^n, x \rightarrow 0), \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = 0, f'(0) = 0.$

$\frac{g(1/2^n) - g(0)}{1/2^n} = \frac{1/2^{n+1}}{1/2^n} = \frac{1}{2} \rightarrow \frac{1}{2} (n \rightarrow \infty),$

$\frac{g(x) - g(0)}{x} = 0 \rightarrow 0 (x \neq 1/2^n, x \rightarrow 0), \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x} \text{ 不存在. } g'(0) \text{ 不存在.}$

10. 设 $y = f(x)$ 及 $y = g(x)$ 在 $[a, b]$ 上连续, 证明:

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

证 $\int_a^b [f(x) + tg(x)]^2 dx = \left(\int_a^b g^2(x)dx \right)t^2 + \left(2 \int_a^b f(x)g(x)dx \right)t + \int_a^b f^2(x)dx \geq 0 (*)$,

如果 $\int_a^b g^2(x)dx = 0$, 则由 g 的连续性 $g(x) = 0, x \in [a, b]$, 不等式两端都是 0.

如果 $\int_a^b g^2(x)dx > 0$, $(*)$ 左端的二次函数恒非负, 故其判别式非正,

$$\left(2 \int_a^b f(x)g(x)dx \right)^2 - 4 \left(\int_a^b g^2(x)dx \right) \int_a^b f^2(x)dx \leq 0,$$

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

11. 求出函数

$$f(x) = \frac{1}{2}x + \frac{1}{2^2}x^2 + L + \frac{1}{2^n}x^n$$

在点 $x=1$ 的导数, 再将函数 $f(x)$ 写成 $f(x) = \frac{x/2 - (x/2)^{n+1}}{1 - x/2}$ 的形式, 再求 $f'(1)$,

由此证明下列等式:

$$\frac{1}{2} + \frac{2}{2^2} + L + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$$

$$\text{证 } f'(x) = \frac{1}{2} + \frac{2}{2^2}x + L + \frac{n}{2^n}x^{n-1}, f'(1) = \frac{1}{2} + \frac{2}{2^2} + L + \frac{n}{2^n}.$$

$$f(x) = \frac{x/2 - (x/2)^{n+1}}{1 - x/2},$$

$$f'(x) = \frac{(1/2 - (n+1)(x/2)^n(1/2))(1 - x/2) + (1/2)(x/2 - (x/2)^{n+1})}{(1 - x/2)^2},$$

$$f'(1) = \frac{(1/2 - (n+1)(1/2^{n+1}))(1/2) + (1/2)(1/2 - 1/2^{n+1})}{1/2^2}$$

$$= (1 - (n+1)/2^n) + 1 - 1/2^n = 2 - \frac{n+2}{2^n}.$$

12. 由类似上题的办法证明 $1 + 2x + 3x^2 + L + nx^{n-1} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2} (x \neq 1)$.

$$\text{证 由等比级数求和公式 } x + x^2 + L + x^n = \frac{x - x^{n+1}}{1 - x},$$

两端求导得 $1 + 2x + 3x^2 + L + nx^{n-1}$

$$= \frac{(1 - (n+1)x^n)(1 - x) + (x - x^{n+1})}{(1 - x)^2} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1 - x)^2} (x \neq 1).$$

13. 设 $y = f(x)$ 在 $[0, 1]$ 连续且 $f(x) > 0$ 证明 $\int_0^1 \frac{1}{f(x)} dx \geq \frac{1}{\int_0^1 f(x) dx}$.

$$\text{证 } 1 = \int_0^1 1 dx = \int_0^1 \frac{\sqrt{f(x)}}{\sqrt{f(x)}} dx \leq \int_0^1 f(x) dx \int_0^1 \frac{1}{f(x)} dx.$$

$$14. \ln x = \int_1^x \frac{dt}{t}$$

$$(a) \frac{1}{n+1} < \ln \left(1 + \frac{1}{n} \right) < \frac{1}{n} (n > 0)$$

$$(b) \frac{1}{2} + \frac{1}{3} + L + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + L + \frac{1}{n-1}; (c) e^{\frac{1}{n+1}} < \left(1 + \frac{1}{n} \right)^n < e.$$

$$\text{证 } (1) \frac{1}{n+1} = \int_1^{1+1/n} \frac{dt}{1+1/n} \ln \left(1 + \frac{1}{n} \right) = \int_1^{1+1/n} \frac{dt}{t} < \int_1^{1+1/n} \frac{dt}{1} = \frac{1}{n}.$$

$$(2) \ln n = \ln \frac{2}{1} \frac{3}{2} \frac{4}{3} \cdots \frac{n}{n-1} = \ln \left(1 + \frac{1}{1} \right) + L + \left(1 + \frac{1}{n} \right) < 1 + \frac{1}{2} + L + \frac{1}{n},$$

$$\ln n = \ln\left(1 + \frac{1}{1}\right) + \ln\left(1 + \frac{1}{2}\right) + \ln\left(1 + \frac{1}{3}\right) + \dots + \ln\left(1 + \frac{1}{n}\right) > \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

$$(3) \left(1 + \frac{1}{n}\right)^n = e^{n \ln\left(1 + \frac{1}{n}\right)} > e^{n \cdot \frac{1}{n+1}} = e^{1 - \frac{1}{n+1}}.$$

第二章总练习题

1. 讨论函数 $f(x) = \begin{cases} |x-3| & x \geq 1 \text{ 时} \\ \frac{x^2}{4} - \frac{3}{2}x + \frac{13}{4}, & x < 1 \text{ 时} \end{cases}$ 的连续性和可导性.

解 $x \neq 1$ 时 $f(x)$ 可导. $f(1-0) = \lim_{x \rightarrow 1} \left(\frac{x^2}{4} - \frac{3}{2}x + \frac{13}{4} \right) = 2$;

$f(1+0) = \lim_{x \rightarrow 1} |x-3| = 2 = f(1-0) = f(1)$, f 在 $x=1$ 连续.

$$f'_+(1) = (3-x)'|_{x=1} = -1, f'_-(1) = \left(\frac{x^2}{4} - \frac{3}{2}x + \frac{13}{4} \right)' \Big|_{x=1} = \left(\frac{x}{2} - \frac{3}{2} \right)' \Big|_{x=1} = -1 = f'_+(1), f'(1) = -1.$$

f 在 $x=1$ 可导.

2. 设函数 $f(x) = \begin{cases} 2x-2 & x < -1 \text{ 时} \\ Ax^3 + Bx^2 + Cx + D, & -1 \leq x \leq 1 \text{ 时} \\ 5x+7 & x > 1 \text{ 时} \end{cases}$

试确定常数 A, B, C, D 的值, 使 $f(x)$ 在 $(-\infty, +\infty)$ 可导.

解 $f(-1-0) = \lim_{x \rightarrow -1} (2x-2) = -4 = f(-1) = -A+B-C+D$.

$$f'_-(-1) = (2x-2)'|_{x=-1} = 2 = f'_+(-1) = (Ax^3 + Bx^2 + Cx + D)'|_{x=-1}$$

$$= (3Ax^2 + 2Bx + C)|_{x=-1} = 3A - 2B + C.$$

$$f(1-0) = A+B+C+D = f(1+0) = 12,$$

$$f'_-(1) = 3A+2B+C = f'_+(1) = 5.$$

$$\begin{cases} -A+B-C+D = -4 \\ 3A-2B+C = 2 \\ A+B+C+D = 12 \\ 3A+2B+C = 5. \end{cases}$$

$$\{A = -9/4, B = 3/4, C = 41/4, D = 13/4\}.$$

3. 设函数 $g(x) = (\sin 2x)f(x)$, 其中 $f(x)$ 在 $x=0$ 连续. 问 $g(x)$ 在 $x=0$ 是否可导, 若可导, 求出 $g'(0)$.

解 $\frac{g(\Delta x) - g(0)}{\Delta x} = 2 \frac{f(\Delta x) \sin 2\Delta x}{2\Delta x} \rightarrow 2f(0)(\Delta x \rightarrow 0), g'(0) = 2f(0).$

4. 问函数 $f(x) = \frac{x^2 + \sin^2 x}{1+x^2}$ 与 $g(x) = \frac{-\cos^2 x}{1+x^2}$ 为什么有相同得导数?

解 因为 $f(x) - g(x) = 1$.

5. 设函数 $f(x)$ 在 $[-1, 1]$ 上有定义, 且满足 $x \leq f(x) \leq x^2 + x, x \in [-1, 1]$. 证明存在且等于 1.

证 $0 \leq f(0) \leq 0, f(0) = 0. \Delta x > 0,$

$$\frac{f(\Delta x) - f(0)}{\Delta x} = \frac{f(\Delta x)}{\Delta x} \leq \frac{\Delta x^2 + \Delta x}{\Delta x} = \Delta x + 1 \rightarrow 1 (\Delta x \rightarrow 0+0), f'_+(0) = 1, \text{ 类似 } f'_-(0) = 1,$$

故 $f'(0) = 1$.

6. 设 $f(x) = |x^2 - 4|$, 求 $f'(x)$.

解 $|x| > 2$ 时, $f(x) = x^2 - 4$, $f'(x) = 2x$. $f'_+(2) = (x^2 - 4)'|_{x=2} = 4$,
 $f'_-(2) = (4 - x^2)'|_{x=2} = -4$, $f'(2)$ 不存在, 同理 $f'(-2)$ 不存在.

7. 设 $y = \frac{1+x}{1-x}$, 求 $\frac{d^2 y}{dx^2}$.

解 $y = -1 + \frac{2}{1-x}$, $\frac{dy}{dx} = \frac{2}{(1-x)^2}$, $\frac{d^2 y}{dx^2} = -\frac{4}{(1-x)^3}$.

8. 设函数 $f(x)$ 在 $(-\infty, +\infty)$ 上有定义, 且满足下列性质:

(1) $f(a+b) = f(a)f(b)$ (a, b 为任意实数); (2) $f(0) = 1$; (3) 在 $x=0$ 处可导. 证明: 对于任意 $x \in (-\infty, +\infty)$ 都有 $f'(x) = f'(0)gf(x)$.

证 $\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{f(x)f(\Delta x) - f(x)f(0)}{\Delta x}$
 $= f(x) \frac{f(\Delta x) - f(0)}{\Delta x} \rightarrow f'(0)gf(x) (\Delta x \rightarrow 0), f'(x) = f'(0)gf(x)$.

9. 设 $f(x) = \begin{cases} 1/2^{2n}, & x = 1/2^n, \\ 0, & x \neq 1/2^n \end{cases} (n=1, 2, \dots); g(x) = \begin{cases} 1/2^{n+1}, & x = 1/2^n, \\ 0, & x \neq 1/2^n \end{cases} (n=1, 2, \dots);$

问 $f(x)$ 在 $x=0$ 处是否可导? $g(x)$ 在 $x=0$ 处是否可导?

解 $\frac{f(1/2^n) - f(0)}{1/2^n} = \frac{1/2^{2n}}{1/2^n} = \frac{1}{2^n} \rightarrow 0 (n \rightarrow \infty),$

$\frac{f(x) - f(0)}{x} = 0 \rightarrow 0 (x \neq 1/2^n, x \rightarrow 0), \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = 0, f'(0) = 0.$

$\frac{g(1/2^n) - g(0)}{1/2^n} = \frac{1/2^{n+1}}{1/2^n} = \frac{1}{2} \rightarrow \frac{1}{2} (n \rightarrow \infty),$

$\frac{g(x) - g(0)}{x} = 0 \rightarrow 0 (x \neq 1/2^n, x \rightarrow 0), \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x} \text{ 不存在. } g'(0) \text{ 不存在.}$

10. 设 $y = f(x)$ 及 $y = g(x)$ 在 $[a, b]$ 上连续, 证明:

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

证 $\int_a^b [f(x) + tg(x)]^2 dx = \left(\int_a^b g^2(x)dx \right)t^2 + \left(2 \int_a^b f(x)g(x)dx \right)t + \int_a^b f^2(x)dx \geq 0 (*)$,

如果 $\int_a^b g^2(x)dx = 0$, 则由 g 的连续性 $g(x) = 0, x \in [a, b]$, 不等式两端都是 0.

如果 $\int_a^b g^2(x)dx > 0$, $(*)$ 左端的二次函数恒非负, 故其判别式非正,

$$\left(2 \int_a^b f(x)g(x)dx \right)^2 - 4 \left(\int_a^b g^2(x)dx \right) \int_a^b f^2(x)dx \leq 0,$$

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

11. 求出函数

$$f(x) = \frac{1}{2}x + \frac{1}{2^2}x^2 + L + \frac{1}{2^n}x^n$$

在点 $x=1$ 的导数, 再将函数 $f(x)$ 写成 $f(x) = \frac{x/2 - (x/2)^{n+1}}{1 - x/2}$ 的形式, 再求 $f'(1)$,

由此证明下列等式:

$$\frac{1}{2} + \frac{2}{2^2} + L + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$$

$$\text{证 } f'(x) = \frac{1}{2} + \frac{2}{2^2}x + L + \frac{n}{2^n}x^{n-1}, f'(1) = \frac{1}{2} + \frac{2}{2^2} + L + \frac{n}{2^n}.$$

12. 由类似上题的办法证明

$$f(x) = \frac{x/2 - (x/2)^{n+1}}{1 - x/2},$$

$$f'(x) = \frac{(1/2 - (n+1)(x/2)^n(1/2))(1 - x/2) + (1/2)(x/2 - (x/2)^{n+1})}{(1 - x/2)^2},$$

$$f'(1) = \frac{(1/2 - (n+1)(1/2^{n+1}))(1/2) + (1/2)(1/2 - 1/2^{n+1})}{1/2^2}$$

$$= (1 - (n+1)/2^n) + 1 - 1/2^n = 2 - \frac{n+2}{2^n}.$$

$$\frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2} \quad (x \neq 1)$$

13. 设 $y = f(x)$ 在 $[0, 1]$ 连续且 $f(x) > 0$ 证明 $\int_0^1 \frac{1}{f(x)} dx \geq \frac{1}{\int_0^1 f(x) dx}.$

$$\text{证 } 1 = \int_0^1 1 dx = \int_0^1 \sqrt{f(x)} \frac{1}{\sqrt{f(x)}} dx \leq \int_0^1 f(x) dx \int_0^1 \frac{1}{f(x)} dx.$$

$$14. \ln x = \int_1^x \frac{dt}{t}$$

$$(a) \frac{1}{n+1} < \ln \left(1 + \frac{1}{n} \right) < \frac{1}{n} \quad (n > 0)$$

$$(b) \frac{1}{2} + \frac{1}{3} + L + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + L + \frac{1}{n-1};$$

$$(c) e^{\frac{1}{n+1}} < \left(1 + \frac{1}{n} \right)^n < e.$$

$$\text{证 } (1) \frac{1}{n+1} = \int_1^{1+1/n} \frac{dt}{t} \ln \left(1 + \frac{1}{n} \right) = \int_1^{1+1/n} \frac{dt}{t} < \int_1^{1+1/n} \frac{dt}{1} = \frac{1}{n}.$$

$$(2) \ln n = \ln \frac{2}{1} \frac{3}{2} \cdots \frac{n}{n-1} = \ln \left(1 + \frac{1}{1} \right) + L + \left(1 + \frac{1}{n} \right) < 1 + \frac{1}{2} + L + \frac{1}{n},$$

$$\ln n = \ln \left(1 + \frac{1}{1} \right) + L + \left(1 + \frac{1}{n} \right) > \frac{1}{2} + L + \frac{1}{n}.$$

$$(c) e^{\frac{1}{n+1}} < \left(1 + \frac{1}{n}\right)^n < e.$$

$$\text{证}(1) \frac{1}{n+1} = \int_1^{1+1/n} \frac{dt}{1+1/n} \ln\left(1 + \frac{1}{n}\right) = \int_1^{1+1/n} \frac{dt}{t} < \int_1^{1+1/n} \frac{dt}{1} = \frac{1}{n}.$$

$$(2) \ln n = \ln \frac{2}{1} \frac{3}{2} \cdots \frac{n}{n-1} = \ln\left(1 + \frac{1}{1}\right) + \ln\left(1 + \frac{1}{2}\right) + \cdots + \ln\left(1 + \frac{1}{n-1}\right) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1},$$

$$\ln n = \ln\left(1 + \frac{1}{1}\right) + \ln\left(1 + \frac{1}{2}\right) + \cdots + \ln\left(1 + \frac{1}{n-1}\right) > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}.$$

$$(3) \left(1 + \frac{1}{n}\right)^n = e^{n \ln\left(1 + \frac{1}{n}\right)} > e^{n \frac{1}{n+1}} = e^{\frac{1}{1 + \frac{1}{n}}}.$$

习题 3.1

求下列不定积分:

$$1. \int \sqrt{1+2x} dx = \frac{1}{2} \int \sqrt{1+2x} d(1+2x) = \frac{1}{3} (1+2x)^{3/2} + C.$$

$$2. \int \frac{3x}{(x^2+1)^2} dx = \frac{1}{2} \int \frac{3}{(x^2+1)^2} d(x^2+1) = -\frac{3}{2(x^2+1)} + C.$$

$$3. \int x\sqrt{2x^2+7} dx = \frac{1}{4} \int \sqrt{2x^2+7} d(2x^2+7) = \frac{1}{6} (2x^2+7)^{3/2} + C.$$

$$4. \int (2x^{3/2}+1)^{2/3} \sqrt{x} dx = \frac{2}{3} \int (2x^{3/2}+1)^{2/3} dx^{3/2} \\ = \frac{2}{3} \frac{1}{2} \int (2x^{3/2}+1)^{2/3} d(2x^{3/2}+1) = \frac{1}{5} (2x^{3/2}+1)^{5/3} + C.$$

$$5. \int \frac{e^{1/x}}{x^2} dx = -\int e^{1/x} d(1/x) = -e^{1/x} + C.$$

$$6. \int \frac{dx}{(2-x)^{100}} = -\int \frac{d(2-x)}{(2-x)^{100}} = \frac{1}{99(2-x)^{99}} + C.$$

$$7. \int \frac{dx}{3+5x^2} = \frac{1}{3} \int \frac{dx}{1+[(5/3)x]^2} = \frac{1}{3} \sqrt{\frac{3}{5}} \int \frac{d\sqrt{5/3}x}{1+[\sqrt{5/3}x]^2} = \frac{1}{\sqrt{15}} \arctan \sqrt{\frac{5}{3}}x + C.$$

$$8. \int \frac{dx}{\sqrt{7-3x^2}} = \int \frac{dx}{\sqrt{7}\sqrt{1-3/7x^2}} = \frac{1}{\sqrt{7}} \sqrt{\frac{7}{3}} \int \frac{d\sqrt{3/7}x}{\sqrt{7}\sqrt{1-\sqrt{3/7}x^2}} = \frac{1}{\sqrt{3}} \arcsin \sqrt{\frac{3}{7}}x + C.$$

$$9. \int \frac{dx}{\sqrt{x}(1+x)} = 2 \int \frac{d\sqrt{x}}{(1+x)} = 2 \arctan \sqrt{x} + C.$$

$$10. \int \frac{e^x}{2+e^{2x}} dx = \int \frac{1}{2+(e^x)^2} de^x = \frac{1}{\sqrt{2}} \arctan e^x + C.$$

$$11. \int \frac{dx}{\sqrt{e^{-2x}-1}} = \int \frac{de^x}{\sqrt{1-(e^x)^2}} = \arcsin e^x + C.$$

$$12. \int \frac{dx}{e^x - e^{-x}} = \int \frac{de^x}{e^{2x}-1} = \int \frac{du}{(u-1)(u+1)} = \frac{1}{2} \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du \\ = \frac{1}{2} \ln \frac{u-1}{u+1} + C = \frac{1}{2} \ln \frac{e^x-1}{e^x+1} + C.$$

$$13. \int \frac{\ln \ln x}{x \ln x} dx = \int \frac{\ln \ln x}{\ln x} d \ln x = \int \ln \ln x d \ln \ln x = \frac{1}{2} (\ln \ln x)^2 + C.$$

$$14. \int \frac{dx}{1+\cos x} = \int \frac{dx}{2 \sin^2 \frac{x}{2}} = \int \frac{d \frac{x}{2}}{\sin^2 \frac{x}{2}} = -\cot^2 \frac{x}{2} + C.$$

$$15. \int \frac{dx}{1 - \sin x} = \int \frac{d\left(x + \frac{\pi}{2}\right)}{1 + \cos\left(x + \frac{\pi}{2}\right)} = -\cot^2\left(\frac{x}{2} + \frac{\pi}{4}\right) + C.$$

$$\begin{aligned} 16. \int \frac{x^{14}}{(x^5 + 1)^4} dx &= \frac{1}{5} \int \frac{x^{10}}{(x^5 + 1)^4} dx^5 = \frac{1}{5} \int \frac{u^2}{(u + 1)^4} du (u = x^5) \\ &= \frac{1}{5} \int \frac{u^2 - 1 + 1}{(u + 1)^4} du = \frac{1}{5} \int \frac{(v - 1)^2}{v^4} dv (v = u + 1) \\ &= \frac{1}{5} \int \frac{v^2 - 2v + 1}{v^4} dv = \frac{1}{5} \int (v^{-2} - 2v^{-3} + v^{-4}) dv \\ &= \frac{1}{5} \left(-v^{-1} + v^{-2} - \frac{1}{3} v^{-3} \right) + C = \frac{1}{5} \left(-(x^5 + 1)^{-1} + (x^5 + 1)^{-2} - \frac{1}{3} (x^5 + 1)^{-3} \right) + C. \end{aligned}$$

$$\begin{aligned} 17. \int \frac{x^{2n-1}}{x^n - 1} dx &= \frac{1}{n} \int \frac{x^n}{x^n - 1} dx^n = \frac{1}{n} \int \frac{u}{u - 1} du (u = x^n) \\ &= \frac{1}{n} \int \left(1 + \frac{1}{u - 1} \right) du = \frac{1}{n} (u + \ln |u - 1|) + C = \frac{1}{n} (x^n + \ln |x^n - 1|) + C. \end{aligned}$$

$$\begin{aligned} 18. \int \frac{dx}{x(x^5 + 2)} &= \int \frac{x^4 dx}{x^5(x^5 + 2)} = \frac{1}{5} \int \frac{du}{u(u + 2)} (u = x^5) \\ &= \frac{1}{5} \cdot \frac{1}{2} \int \left(\frac{1}{u} - \frac{1}{u + 2} \right) du = \frac{1}{10} (\ln |u| - \ln |u + 2|) + C = \frac{1}{10} \ln \left| \frac{u}{u + 2} \right| + C. \end{aligned}$$

$$\begin{aligned} 19. \int \frac{\ln(x + 1) - \ln x}{x(x + 1)} dx &= \int (\ln(x + 1) - \ln x) \left(\frac{1}{x} - \frac{1}{x + 1} \right) dx \\ &= \int (\ln(x + 1) - \ln x) d(\ln x - \ln(x + 1)) = - \int (\ln(x + 1) - \ln x) d(\ln(x + 1) - \ln x) \\ &= -\frac{1}{2} \ln^2 \frac{x + 1}{x} + C. \end{aligned}$$

$$\begin{aligned} 20. \int \frac{e^{\arctan x} + x \ln(1 + x^2)}{1 + x^2} dx &= \int \frac{e^{\arctan x}}{1 + x^2} dx + \int \frac{x \ln(1 + x^2)}{1 + x^2} dx \\ &= \int e^{\arctan x} d \arctan x + \frac{1}{2} \int \ln(1 + x^2) d \ln(1 + x^2) \\ &= e^{\arctan x} + \frac{1}{4} \ln^2(1 + x^2) + C. \end{aligned}$$

$$21. \int \sin 2x \cos 2x dx = \frac{1}{2} \int \sin 2x d \sin 2x = \frac{1}{4} \sin^2 2x + C.$$

$$22. \int \sin^2 \frac{x}{2} \cos \frac{x}{2} dx = 2 \int \sin^2 \frac{x}{2} d \sin \frac{x}{2} = \frac{2}{3} \sin^3 \frac{x}{2} + C.$$

$$23. \int \sin 5x \sin 6x dx = \frac{1}{2} \int (\cos x - \cos 11x) dx = \frac{1}{2} \left(\sin x - \frac{1}{11} \sin 11x \right) + C.$$

$$24. \int \frac{2x-1}{\sqrt{1-x^2}} dx = \int \frac{2x}{\sqrt{1-x^2}} dx - \int \frac{1}{\sqrt{1-x^2}} dx \\ = - \int \frac{d(1-x^2)}{\sqrt{1-x^2}} - \arcsin x + C = -2\sqrt{1-x^2} - \arcsin x + C.$$

$$25. \int \frac{x^3+x}{\sqrt{1-x^2}} dx = \int \frac{x^3}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx \\ = \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx^2 - \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} \\ = \frac{1}{2} \int \sqrt{1-x^2} d(1-x^2) + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx^2 - \sqrt{1-x^2} \\ = \frac{1}{3} (1-x^2)^{3/2} - 2\sqrt{1-x^2} + C.$$

$$26. \int \frac{dx}{(a^2-x^2)^{3/2}} (a>0)$$

$$x = a \sin t, t \in (-\pi/2, \pi/2), dx = a \cos t dt,$$

$$(a^2-x^2)^{3/2} = a^3 \cos^3 t,$$

$$\int \frac{dx}{(a^2-x^2)^{3/2}} = \int \frac{dt}{a^2 \cos^2 t} dx = \frac{1}{a^2} \tan t + C$$

$$= \frac{1}{a^2} \frac{x/a}{\sqrt{1-(x/a)^2}} + C = \frac{x}{a^2 \sqrt{a^2-x^2}} + C.$$

$$x < 0 \text{ 时, 令 } x = -y, y > 0,$$

$$\int \frac{\sqrt{x^2-a^2}}{x} dx = \int \frac{\sqrt{y^2-a^2}}{y} dy = \sqrt{y^2-a^2} - a \arccos \frac{a}{y} + C \\ = \sqrt{x^2-a^2} - a \arccos \frac{a}{-x} + C = \sqrt{x^2-a^2} - \left(\pi - a \arccos \frac{a}{x} \right) + C \\ = \sqrt{x^2-a^2} + a \arccos \frac{a}{x} + C'.$$

$$27. \int \frac{\sqrt{x^2 - a^2}}{x} dx (a > 0). x > 0 \text{ 时, 令 } x = a \sec t, t \in (0, \pi/2).$$

$$dx = a \tan t \sec t dt, \sqrt{x^2 - a^2} = a \tan t,$$

$$\int \frac{\sqrt{x^2 - a^2}}{x} dx = a \int \tan^2 t dt = a \int (\sec^2 t - 1) dt = a(\tan t - t) + C$$

$$= a(\sqrt{\sec^2 t - 1} - \arccos \frac{a}{x}) + C = a(\sqrt{\left(\frac{x}{a}\right)^2 - 1} - \arccos \frac{a}{x}) + C$$

$$= \sqrt{x^2 - a^2} - a \arccos \frac{a}{x} + C.$$

$$28. \int \frac{x^2}{\sqrt{a^2 - x^2}} dx = - \int \sqrt{a^2 - x^2} dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= -\frac{a^2}{2} \arcsin \frac{x}{a} - \frac{1}{2} x \sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a} + C$$

$$= \frac{a^2}{2} \arcsin \frac{x}{a} - \frac{1}{2} x \sqrt{a^2 - x^2} + C.$$

$$29. \int \frac{dx}{\sqrt{1 + e^{3x}}} = \int \frac{e^{-3x/2} dx}{\sqrt{1 + e^{-3x}}} = -\frac{2}{3} \int \frac{de^{-3x/2}}{\sqrt{1 + e^{-3x}}} = -\frac{2}{3} \ln(e^{-3x/2} + \sqrt{1 + e^{-3x}}) + C$$

$$= -\frac{2}{3} \ln(1 + \sqrt{1 + e^{3x}}) + x + C = -\frac{2}{3} \ln \frac{(\sqrt{1 + e^{3x}} + 1)(\sqrt{1 + e^{3x}} - 1)}{\sqrt{1 + e^{3x}} - 1} + x + C$$

$$= \frac{2}{3} \ln(\sqrt{1 + e^{3x}} - 1) - x + C.$$

$$30. \int \frac{x^3}{\sqrt{1 + x^8}} dx = \frac{1}{4} \int \frac{dx^4}{\sqrt{1 + x^8}} = \frac{1}{4} \int \frac{du}{\sqrt{1 + u^2}} (u = x^4)$$

$$= \frac{1}{4} \ln(u + \sqrt{1 + u^2}) + C = \frac{1}{4} \ln(x^4 + \sqrt{1 + x^8}) + C.$$

$$\begin{aligned}
31. \int \frac{dx}{x^6 \sqrt{1+x^2}} &= \int \frac{dx}{x^7 \sqrt{1+x^{-2}}} = -\frac{1}{2} \int \frac{dx^{-2}}{x^4 \sqrt{1+x^{-2}}} = -\frac{1}{2} \int \frac{u^2 du}{\sqrt{1+u}} \left(u = \frac{1}{x^2}\right) \\
&= -\frac{1}{2} \int \frac{(v-1)^2}{v^{1/2}} dv = -\frac{1}{2} \int \frac{v^2 - 2v + 1}{v^{1/2}} dv \quad (v = 1+u) \\
&= -\frac{1}{2} \int (v^{3/2} - 2v^{1/2} + v^{-1/2}) dx \\
&= -\frac{1}{2} \left(\frac{2}{5} v^{5/2} - 2 \frac{2}{3} v^{3/2} + 2 v^{1/2} \right) \\
&= -\frac{1}{5} \left(1 + \frac{1}{x^2} \right)^{5/2} + \frac{2}{3} \left(1 + \frac{1}{x^2} \right)^{3/2} - \left(1 + \frac{1}{x^2} \right)^{1/2} + C \\
&= -\frac{\sqrt{1+x^2}^5}{5x^5} + \frac{\sqrt{1+x^2}^3}{3x^3} - \frac{\sqrt{1+x^2}}{x} + C.
\end{aligned}$$

$$\begin{aligned}
32. \int \frac{e^{2x}}{\sqrt[3]{1+e^x}} dx &= \int \frac{e^x}{\sqrt[3]{1+e^x}} de^x = \int \frac{u}{\sqrt[3]{1+u}} du \quad (u = e^x) \quad (\sqrt[3]{u+1} = v, u = v^3 - 1) \\
&= \int \frac{u}{\sqrt[3]{1+u}} du = \int \frac{v^3 - 1}{v} 3v^2 dv = 3 \int (v^4 - v) dv = 3 \left(\frac{v^5}{5} - \frac{v^2}{2} \right) + C \\
&= \frac{3}{5} (e^x + 1)^{5/3} - \frac{3}{2} (e^x + 1)^{2/3} + C.
\end{aligned}$$

$$\begin{aligned}
33. \int \frac{dx}{\sqrt{3+x-x^2}} &= \int \frac{dx}{\sqrt{3 - \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}}} = \int \frac{d\left(x - \frac{1}{2}\right)}{\sqrt{\frac{13}{4} - \left(x - \frac{1}{2}\right)^2}} \\
&= \arcsin \frac{x - \frac{1}{2}}{\frac{\sqrt{13}}{2}} + C = \arcsin \frac{2x-1}{\sqrt{13}} + C.
\end{aligned}$$

$$\begin{aligned}
34. \int \sqrt{7+x-x^2} dx &= \int \sqrt{7 - \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}} dx = \int \sqrt{\frac{29}{4} - \left(x - \frac{1}{2}\right)^2} d\left(x - \frac{1}{2}\right) \\
&= \frac{1}{2} \left(x - \frac{1}{2}\right) \sqrt{\frac{29}{4} - \left(x - \frac{1}{2}\right)^2} + \frac{29}{8} \arcsin \frac{x - \frac{1}{2}}{\frac{\sqrt{29}}{2}} + C \\
&= \frac{2x-1}{4} \sqrt{7+x-x^2} + \frac{29}{8} \arcsin \frac{2x-1}{\sqrt{29}} + C.
\end{aligned}$$

$$35. \int \frac{dx}{1+\sqrt{x-1}}, 1+\sqrt{x-1}=u, x=1+(u-1)^2, dx=2(u-1)du,$$

$$\int \frac{dx}{1+\sqrt{x-1}}, 1+\sqrt{x-1}=u, x=1+(u-1)^2, dx=2(u-1)du,$$

$$\begin{aligned} \int \frac{dx}{1+\sqrt{x-1}} &= \int \frac{2(u-1)du}{u} = 2(u - \ln u) + C = 2(1+\sqrt{x-1}) - \ln(1+\sqrt{x-1}) + C \\ &= 2\sqrt{x-1} - \ln(1+\sqrt{x-1}) + C'. \end{aligned}$$

习题 3.2

求下列不定积分：

$$\begin{aligned} 1. \int x \ln x dx &= \frac{1}{2} \int \ln x dx^2 = \frac{x^2}{2} \ln x - \frac{1}{2} \int x^2 d \ln x \\ &= \frac{x^2}{2} \ln x - \frac{1}{2} \int x^2 \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C. \end{aligned}$$

$$\begin{aligned} 2. \int x^2 e^{ax} dx &= \frac{1}{a} \int x^2 d e^{ax} = \frac{1}{a} x^2 e^{ax} - \frac{1}{a} \int e^{ax} dx^2 = \frac{1}{a} x^2 e^{ax} - \frac{2}{a} \int x e^{ax} dx \\ &= \frac{1}{a} x^2 e^{ax} - \frac{2}{a^2} \int x d e^{ax} = \frac{1}{a} x^2 e^{ax} - \frac{2x}{a^2} e^{ax} + \frac{2}{a^2} \int e^{ax} dx \\ &= \frac{1}{a} x^2 e^{ax} - \frac{2}{a^2} \int x d e^{ax} = \frac{1}{a} x^2 e^{ax} - \frac{2x}{a^2} e^{ax} + \frac{2}{a^3} e^{ax} + C \\ &= e^{ax} \left(\frac{1}{a} x^2 - \frac{2x}{a^2} + \frac{2}{a^3} \right) + C. \end{aligned}$$

$$\begin{aligned} 3. \int x \sin 2x dx &= -\frac{1}{2} \int x d \cos 2x = -\frac{1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x dx \\ &= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C. \end{aligned}$$

$$\begin{aligned} 4. \int \arcsin x dx &= x \arcsin x - \int x d \arcsin x = x \arcsin x - \int \frac{x dx}{\sqrt{1-x^2}} \\ &= x \arcsin x + \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + C. \end{aligned}$$

$$\begin{aligned} 5. \int \arctan x dx &= x \arctan x - \int x d \arctan x = x \arctan x - \int \frac{x dx}{1+x^2} \\ &= x \arctan x - \frac{1}{2} \int \frac{d(1+x^2)}{1+x^2} = x \arctan x - \frac{1}{2} \ln(1+x^2) + C. \end{aligned}$$

$$\begin{aligned} 6. I &= \int e^{2x} \cos 3x dx = \frac{1}{2} \int \cos 3x d e^{2x} = \frac{1}{2} e^{2x} \cos 3x - \frac{1}{2} \int e^{2x} d \cos 3x \\ &= \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} \int \sin 3x d e^{2x} \\ &= \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} \left(e^{2x} \sin 3x - 3 \int e^{2x} \cos 3x dx \right) \\ &= \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} I, \end{aligned}$$

$$I = \frac{4}{13} \left(\frac{1}{2} \cos 3x + \frac{3}{4} \sin 3x \right) e^{2x} + C = \frac{1}{13} (2 \cos 3x + 3 \sin 3x) e^{2x} + C.$$

$$\begin{aligned} 7. I &= \int \frac{\sin 3x}{e^x} dx = -\int \sin 3x d e^{-x} = -e^{-x} \sin 3x + 3 \int e^{-x} \cos 3x dx \\ &= -e^{-x} \sin 3x - 3 \int \cos 3x d e^{-x} = -e^{-x} \sin 3x - 3 \left(e^{-x} \cos 3x + 3 \int e^{-x} \sin 3x dx \right) \end{aligned}$$

$$= -e^{-x} \sin 3x - 3(e^{-x} \cos 3x + 3I),$$

$$I = \frac{1}{10} \left(-e^{-x} \sin 3x - 3e^{-x} \cos 3x \right) + C = -\frac{e^{-x}}{10} (\sin 3x + 3 \cos 3x) + C.$$

$$8. I = \int e^{ax} \sin bxdx = \frac{1}{a} \int \sin bxd e^{ax} = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int e^{ax} \cos bxdx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} \int \cos bxd e^{ax}$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} \left(e^{ax} \cos bx + b \int e^{ax} \sin bxdx \right)$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} \left(e^{ax} \cos bx + bI \right).$$

$$I = \frac{1}{1 + \frac{b^2}{a^2}} \left(\frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx \right),$$

$$I = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C.$$

$$9. I = \int \sqrt{1+9x^2} dx = x\sqrt{1+9x^2} - \int x d\sqrt{1+9x^2}$$

$$= x\sqrt{1+9x^2} - \int \frac{x \cdot 18x dx}{2\sqrt{1+9x^2}}$$

$$= x\sqrt{1+9x^2} - \left(\int \sqrt{1+9x^2} dx - \int \frac{dx}{\sqrt{1+9x^2}} \right)$$

$$= x\sqrt{1+9x^2} - \left(I - \int \frac{dx}{\sqrt{1+9x^2}} \right),$$

$$I = \frac{1}{2} x\sqrt{1+9x^2} + \frac{1}{2} \frac{1}{3} \ln(3x + \sqrt{1+9x^2}) + C$$

$$= \frac{1}{2} x\sqrt{1+9x^2} + \frac{1}{6} \ln(3x + \sqrt{1+9x^2}) + C.$$

$$10. \int x \cosh x dx = \int x d \sinh x = x \sinh x - \int \sinh x dx$$

$$= x \sinh x - \cosh x + C.$$

$$11. \int \ln(x + \sqrt{1+x^2}) dx = x \ln(x + \sqrt{1+x^2}) - \int x d \ln(x + \sqrt{1+x^2})$$

$$= x \ln(x + \sqrt{1+x^2}) - \int \frac{xdx}{\sqrt{1+x^2}} = x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

$$12. \int (\arccos x)^2 dx = x(\arccos x)^2 + 2 \int \frac{x \arccos x}{\sqrt{1-x^2}} dx$$

$$= x(\arccos x)^2 - 2 \int \arccos x d\sqrt{1-x^2}$$

$$= x(\arccos x)^2 - 2 \left(\sqrt{1-x^2} \arccos x + \int 1 dx \right)$$

$$= x(\arccos x)^2 - 2\sqrt{1-x^2} \arccos x - 2x + C.$$

$$\begin{aligned} 13. \int \frac{x \arccos x dx}{(1-x^2)^2} &= \frac{1}{2} \int \arccos x d \frac{1}{1-x^2} \\ &= \frac{\arccos x}{2(1-x^2)} + \frac{1}{2} \int \frac{dx}{(1-x^2)\sqrt{1-x^2}} \\ &= \frac{\arccos x}{2(1-x^2)} + \frac{1}{2} \frac{x}{\sqrt{1-x^2}} + C. \end{aligned}$$

$$\begin{aligned} 14. \int \arctan \sqrt{x} dx &= x \arctan \sqrt{x} - \int \frac{x dx}{2(1+x)\sqrt{x}} \\ &= x \arctan \sqrt{x} - \frac{1}{2} \int \frac{\sqrt{x} dx}{1+x} \cdot \sqrt{x} = u, x = u^2, dx = 2u du \\ \int \frac{\sqrt{x} dx}{1+x} &= \int \frac{u 2u du}{1+u^2} = 2(u - \arctan u) + C, \\ \int \arctan \sqrt{x} dx &= x \arctan \sqrt{x} - \frac{1}{2} 2(u - \arctan u) + C \\ &= x \arctan \sqrt{x} - (\sqrt{x} - \arctan \sqrt{x}) + C \\ &= (x+1) \arctan \sqrt{x} - \sqrt{x} + C. \end{aligned}$$

$$\begin{aligned} 15. \int \frac{\arcsin x}{x^2} dx &= - \int \arcsin x d \left(\frac{1}{x} \right) = - \frac{\arcsin x}{x} + \int \frac{dx}{x\sqrt{1-x^2}} \\ &= - \frac{\arcsin x}{x} + \int \frac{dx}{x^2 \sqrt{1/x^2 - 1}} (x > 0) \\ &= - \frac{\arcsin x}{x} - \int \frac{d(1/x)}{\sqrt{1/x^2 - 1}} = - \frac{\arcsin x}{x} - \ln |1/x + \sqrt{1/x^2 - 1}| + C \\ &= - \frac{\arcsin x}{x} + \ln(1 - \sqrt{1-x^2}) - \ln x + C \\ &= - \frac{\arcsin x}{x} + \ln(1 - \sqrt{1-x^2}) - \ln |x| + C (x \neq 0) \text{ (原函数为偶函数)}. \end{aligned}$$

$$\begin{aligned} 16. \int x^3 (\ln x)^2 dx &= \frac{1}{4} \int (\ln x)^2 dx^4 = \frac{x^4 (\ln x)^2}{4} - \frac{1}{4} \int \frac{x^4 \ln x dx}{x} \\ &= \frac{x^4 (\ln x)^2}{4} - \frac{1}{2} \int x^3 \ln x dx = \frac{x^4 (\ln x)^2}{4} - \frac{1}{8} \int \ln x dx^4 \\ &= \frac{x^4 (\ln x)^2}{4} - \frac{x^4}{8} \ln x + \frac{1}{2} \int x^3 dx = \frac{x^4 (\ln x)^2}{4} - \frac{x^4}{8} \ln x + \frac{1}{8} x^4 + C. \end{aligned}$$

$$17. \int \frac{x \arctan x dx}{(1+x^2)^{5/2}} = \frac{1}{2} \int \frac{\arctan x d(1+x^2)}{(1+x^2)^{5/2}} = \frac{1}{2} \left(-\frac{2}{3} \right) \int \arctan x d(1+x^2)^{-3/2}$$

$$= -\frac{\arctan x}{3(1+x^2)^{3/2}} + \frac{1}{3} \int \frac{dx}{(1+x^2)^{5/2}} \cdot x = \tan u, u \in (-\pi/2, \pi/2). dx = \sec^2 u du,$$

$$\int \frac{dx}{(1+x^2)^{5/2}} = \int \cos^3 u du = \int (1 - \sin^2 u) d \sin u =$$

$$= \sin u - \frac{1}{3} \sin^3 u + C = \frac{x}{\sqrt{1+x^2}} - \frac{1}{3} \left(\frac{x}{\sqrt{1+x^2}} \right)^3 + C,$$

$$\int \frac{x \arctan x dx}{(1+x^2)^{5/2}} = -\frac{\arctan x}{3(1+x^2)^{3/2}} + \frac{1}{3} \left(\frac{x}{\sqrt{1+x^2}} - \frac{1}{3} \left(\frac{x}{\sqrt{1+x^2}} \right)^3 \right) + C$$

$$= -\frac{\arctan x}{3(1+x^2)^{3/2}} + \frac{1}{3} \frac{x}{\sqrt{1+x^2}} - \frac{1}{9} \frac{x^3}{(1+x^2)^{3/2}} + C.$$

$$18. \int x \ln(x + \sqrt{1+x^2}) dx = \frac{1}{2} \int \ln(x + \sqrt{1+x^2}) dx^2$$

$$= \frac{1}{2} x^2 \ln(x + \sqrt{1+x^2}) - \frac{1}{2} \int \frac{x^2 dx}{\sqrt{1+x^2}}$$

$$= \frac{1}{2} x^2 \ln(x + \sqrt{1+x^2}) - \frac{1}{2} \int \frac{(x^2+1)-1 dx}{\sqrt{1+x^2}}$$

$$= \frac{1}{2} x^2 \ln(x + \sqrt{1+x^2}) - \frac{1}{2} \int \sqrt{1+x^2} dx + \frac{1}{2} \int \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{1}{2} x^2 \ln(x + \sqrt{1+x^2}) - \frac{1}{2} \left(\frac{x\sqrt{1+x^2}}{2} + \frac{\ln(x + \sqrt{1+x^2})}{2} \right) + \frac{1}{2} \ln(x + \sqrt{1+x^2}) + C$$

$$= \frac{1}{2} x^2 \ln(x + \sqrt{1+x^2}) - \frac{1}{4} x\sqrt{1+x^2} + \frac{1}{4} \ln(x + \sqrt{1+x^2}) + C.$$

习题 3.3

求下列不定积分:

$$1. \int \frac{x-1}{x^2+6x+8} dx = \int \frac{x-1}{(x+2)(x+4)} dx,$$

$$\frac{x-1}{(x+2)(x+4)} = \frac{A}{x+2} + \frac{B}{x+4},$$

$$A = \frac{-2-1}{-2+4} = -\frac{3}{2}, B = \frac{-4-1}{-4+2} = \frac{5}{2},$$

$$\frac{x-1}{(x+2)(x+4)} = \frac{-3/2}{x+2} + \frac{5/2}{x+4},$$

$$\int \frac{x-1}{x^2+6x+8} dx = -\frac{3}{2} \ln|x+2| + \frac{5}{2} \ln|x+4| + C.$$

$$2. I = \int \frac{3x^4+x^2+1}{x^2+x-6} dx.$$

$$\frac{3x^4+x^2+1}{x^2+x-6} = 3x^2-3x+22 + \frac{-40x+133}{x^2+x-6},$$

$$\frac{-40x+133}{x^2+x-6} = \frac{-40x+133}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2},$$

$$A = \frac{-40(-3)+133}{-3-2} = -\frac{253}{5}, B = \frac{-40(2)+133}{2+3} = \frac{53}{5}.$$

$$I = x^3 - \frac{3x^2}{2} + 22x - \frac{253}{5} \ln|x+3| + \frac{53}{5} \ln|x-2| + C.$$

$$3. I = \int \frac{2x^2-5}{x^4-5x^2+6} dx$$

$$\frac{2x^2-5}{x^4-5x^2+6} = \frac{2u-5}{u^2-5u+6} (u=x^2)$$

$$= \frac{2u-5}{(u-2)(u-3)} = \frac{A}{u-2} + \frac{B}{u-3},$$

$$A = \frac{2(2)-5}{2-3} = 1, B = \frac{2(3)-5}{3-2} = 1.$$

$$\frac{2x^2-5}{x^4-5x^2+6} = \frac{1}{x^2-\sqrt{2}^2} + \frac{1}{x^2-\sqrt{3}^2},$$

$$I = \frac{1}{2\sqrt{2}} \ln \frac{x-\sqrt{2}}{x+\sqrt{2}} + \frac{1}{2\sqrt{3}} \ln \frac{x-\sqrt{3}}{x+\sqrt{3}} + C.$$

$$4. I = \int \frac{dx}{(x-1)^2(x-2)}.$$

$$\frac{1}{(x-1)^2(x-2)} = \frac{1}{x-2} \left(\frac{1}{x-2} - \frac{1}{x-1} \right)$$

$$= \frac{1}{(x-2)^2} - \left(\frac{1}{x-2} - \frac{1}{x-1} \right),$$

$$I = -\frac{1}{x-2} + \ln \left| \frac{x-1}{x-2} \right| + C.$$

$$5. I = \int \frac{x^2}{1-x^4} dx.$$

$$\begin{aligned} \frac{x^2}{1-x^4} &= \frac{x^2}{(1-x^2)(1+x^2)} = \frac{1}{2} \frac{(1+x^2) - (1-x^2)}{(1-x^2)(1+x^2)} \\ &= \frac{1}{2} \left(\frac{1}{1-x^2} - \frac{1}{1+x^2} \right), \end{aligned}$$

$$I = \frac{1}{4} \ln \frac{1+x}{1-x} - \frac{1}{2} \arctan x + C.$$

$$6. I = \int \frac{dx}{x^3+1}.$$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1},$$

$$A = \frac{1}{1^2+1+1} = \frac{1}{3},$$

$$1 = \frac{x^2-x+1}{3} + (x+1)(Bx+C) = (B+\frac{1}{3})x^2 + (B+C-\frac{1}{3})x + C + \frac{1}{3},$$

$$C + \frac{1}{3} = 1, C = \frac{2}{3}, B + \frac{1}{3} = 0, B = -\frac{1}{3}.$$

$$\begin{aligned} \frac{1}{x^3+1} &= \frac{1}{3(x+1)} + \frac{-x+2}{3(x^2-x+1)} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)} \\ &= \frac{1}{3(x+1)} - \frac{2x-4}{6(x^2-x+1)} = \frac{1}{3(x+1)} - \frac{1}{6} \frac{(2x-1)-3}{(x^2-x+1)} \\ &= \frac{1}{3(x+1)} - \frac{1}{6} \frac{2x-1}{(x^2-x+1)} + \frac{1}{2} \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}, \end{aligned}$$

$$I = \frac{1}{3} \ln |x+1| - \frac{1}{6} \ln(x^2-x+1) + \frac{1}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} + C.$$

$$\begin{aligned}
7.I &= \int \frac{dx}{1+x^4} \cdot \frac{1}{1+x^4} = \frac{1}{(1+2x^2+x^4)-2x^2} = \frac{1}{(x^2+1)^2-2x^2} \\
&= \frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}, \\
1 &= (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1), \\
1 &= (A+C)x^3 + (B-\sqrt{2}A+D+\sqrt{2}C)x^2 + (A-\sqrt{2}B+C+\sqrt{2}D)x + B+D.
\end{aligned}$$

$$\begin{cases}
A+C=0 \\
B-\sqrt{2}A+D+\sqrt{2}C=0, \\
A-\sqrt{2}B+C+\sqrt{2}D=0, \\
B+D=1.
\end{cases}$$

$$A = \frac{1}{2\sqrt{2}}, B = \frac{1}{2}, C = -\frac{1}{2\sqrt{2}}, D = \frac{1}{2}.$$

$$\begin{aligned}
\frac{1}{1+x^4} &= \frac{\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2+\sqrt{2}x+1} + \frac{-\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2-\sqrt{2}x+1} \\
&= \frac{1}{2\sqrt{2}} \left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1} \right) \\
&= \frac{1}{4\sqrt{2}} \left(\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1} \right) \\
&= \frac{1}{4\sqrt{2}} \left(\frac{(2x+\sqrt{2})+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{(2x-\sqrt{2})-\sqrt{2}}{x^2-\sqrt{2}x+1} \right) \\
&= \frac{1}{4\sqrt{2}} \left(\frac{(2x+\sqrt{2})}{x^2+\sqrt{2}x+1} - \frac{(2x-\sqrt{2})}{x^2-\sqrt{2}x+1} \right) + \frac{1}{4} \frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\
&\quad + \frac{1}{4} \frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}.
\end{aligned}$$

$$I = \frac{1}{4\sqrt{2}} \ln \frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}}{4} \left(\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1) \right) + C.$$

$$8.I = \int \frac{x^3+x^2+2}{(x^2+2)^2} dx.$$

$$\begin{aligned}
\frac{x^3+x^2+2}{(x^2+2)^2} &= \frac{x(x^2+2)}{(x^2+2)^2} + \frac{x^2-2x+2}{(x^2+2)^2} \\
&= \frac{x}{(x^2+2)} + \frac{1}{(x^2+2)} - \frac{2x}{(x^2+2)^2}.
\end{aligned}$$

$$I = \frac{1}{2} \ln(x^2+2) + \frac{1}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + \frac{1}{x^2+2} + C.$$

$$\begin{aligned}
 9. \int \frac{e^x dx}{e^{2x} + 3e^x + 2} &= \int \frac{de^x}{e^{2x} + 3e^x + 2} = \int \frac{du}{u^2 + 3u + 2} = \\
 &= \int \frac{du}{(u+1)(u+2)} = \int \left(\frac{1}{u+1} - \frac{1}{u+2} \right) du = \ln \frac{u+1}{u+2} + C = \ln \frac{e^x + 1}{e^x + 2} + C.
 \end{aligned}$$

$$\begin{aligned}
 10. \int \frac{\cos x dx}{\sin^2 x + \sin x - 6} &= \int \frac{d \sin x}{\sin^2 x + \sin x - 6} = \int \frac{du}{u^2 + u - 6} \quad (u = \sin x) = \\
 &= \int \frac{du}{(u+3)(u-2)} = \frac{1}{5} \int \left(\frac{1}{u-2} - \frac{1}{u+3} \right) du = \ln \left| \frac{u-2}{u+3} \right| + C = \ln \left| \frac{\sin x - 2}{\sin x + 3} \right| + C.
 \end{aligned}$$

$$\begin{aligned}
 11. \int \frac{x^3 dx}{x^4 + x^2 + 2} &= \frac{1}{2} \int \frac{x^2 dx^2}{x^4 + x^2 + 2} = \frac{1}{2} \int \frac{udu}{u^2 + u + 2} \\
 &= \frac{1}{4} \int \frac{2udu}{u^2 + u + 2} = \frac{1}{4} \int \frac{(2u+1) - 1}{u^2 + u + 2} du = \\
 &= \frac{1}{4} \int \frac{d(u^2 + u + 2)}{u^2 + u + 2} du - \frac{1}{4} \int \frac{1}{\left(u + \frac{1}{2}\right)^2 + \frac{7}{4}} du \\
 &= \frac{1}{4} \ln(u^2 + u + 2) - \frac{1}{2\sqrt{7}} \arctan \frac{2u+1}{\sqrt{7}} + C \\
 &= \frac{1}{4} \ln(x^4 + x^2 + 2) - \frac{1}{2\sqrt{7}} \arctan \frac{2x^2+1}{\sqrt{7}} + C.
 \end{aligned}$$

$$\begin{aligned}
 12. I &= \int \frac{dx}{(x+2)(x^2 - 2x + 2)}. \\
 \frac{1}{(x+2)(x^2 - 2x + 2)} &= \frac{A}{x+2} + \frac{Bx+C}{x^2 - 2x + 2} \\
 A &= \frac{1}{(-2)^2 - 2(-2) + 2} = \frac{1}{10}. \\
 \frac{1}{(x+2)(x^2 - 2x + 2)} - \frac{1}{10(x+2)} &= \frac{Bx+C}{x^2 - 2x + 2} \\
 \frac{10 - (x^2 - 2x + 2)}{10(x+2)(x^2 - 2x + 2)} &= \frac{Bx+C}{x^2 - 2x + 2} \\
 \frac{-(x^2 - 2x - 8)}{10(x+2)(x^2 - 2x + 2)} &= \frac{Bx+C}{x^2 - 2x + 2} \\
 \frac{-(x+2)(x-4)}{10(x+2)(x^2 - 2x + 2)} &= \frac{Bx+C}{x^2 - 2x + 2} \\
 \frac{-(x-4)}{10(x^2 - 2x + 2)} &= \frac{Bx+C}{x^2 - 2x + 2}, B = -\frac{1}{10}, C = \frac{2}{5}. \\
 I &= \frac{1}{10} \ln |x+2| - \frac{1}{10} \int \frac{x-4}{x^2 - 2x + 2} dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{10} \ln |x+2| - \frac{1}{10} \int \frac{2x-8}{x^2-2x+2} dx \\
&= \frac{1}{10} \ln |x+2| - \frac{1}{20} \int \frac{(2x-2)-6}{x^2-2x+2} dx \\
&= \frac{1}{10} \ln |x+2| - \frac{1}{20} \ln(x^2-2x+2) + \frac{3}{10} \int \frac{dx}{(x-1)^2+1} \\
&= \frac{1}{10} \ln |x+2| - \frac{1}{20} \ln(x^2-2x+2) + \frac{3}{10} \arctan(x-1) + C
\end{aligned}$$

$$13. I = \int \frac{dx}{2+\sin x} \cdot \tan \frac{x}{2} = u, x = 2 \arctan u, dx = \frac{2du}{1+u^2}, \sin x = \frac{2 \tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{2u}{1+u^2}.$$

$$\begin{aligned}
I &= \int \frac{\frac{2du}{1+u^2}}{2+\frac{2u}{1+u^2}} = \int \frac{1}{u^2+u+1} du = \int \frac{1}{\left(u+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} du \\
&= \frac{2}{\sqrt{3}} \arctan \frac{2u+1}{\sqrt{3}} + C = \frac{2}{\sqrt{3}} \arctan \frac{2 \tan \frac{x}{2} + 1}{\sqrt{3}} + C.
\end{aligned}$$

$$14. I = \int \frac{dx}{1+\sin x + \cos x} \cdot \tan \frac{x}{2} = u, x = 2 \arctan u, dx = \frac{2du}{1+u^2},$$

$$\sin x = \frac{2u}{1+u^2}, \cos x = \frac{1-u^2}{1+u^2}.$$

$$\begin{aligned}
I &= \int \frac{\frac{2du}{1+u^2}}{1+\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} = 2 \int \frac{1}{1+u^2+2u+1-u^2} du = \int \frac{1}{u+1} du \\
&= \ln |u+1| + C = \ln \left| \tan \frac{x}{2} + 1 \right| + C.
\end{aligned}$$

$$\begin{aligned}
15. &\int \cot^4 x dx \\
&= \int \cot^2 x (\csc^2 x - 1) dx \\
&= \int \cot^2 x \csc^2 x dx - \int \cot^2 x dx \\
&= -\int \cot^2 x d \cot x - \int (\csc^2 x - 1) dx \\
&= -\frac{1}{3} \cot^3 x + \cot x + x + C.
\end{aligned}$$

$$16. \int \sec^4 x dx = \int (1 + \tan^2 x) d \tan x = \tan x + \frac{1}{3} \tan^3 x + C.$$

$$17. I = \int \frac{\cos x dx}{5-3\cos x} = -\frac{1}{3} \int \frac{-3\cos x dx}{5-3\cos x} = -\frac{1}{3} \int \frac{(-3\cos x + 5) - 5 dx}{5-3\cos x}$$

$$= -\frac{x}{3} + \frac{5}{3} \int \frac{dx}{5-3\cos x}.$$

$$\tan \frac{x}{2} = u, dx = \frac{2du}{1+u^2}, \cos x = \frac{1-u^2}{1+u^2},$$

$$I = -\frac{x}{3} + \frac{5}{3} \int \frac{\frac{2du}{1+u^2}}{5 - \frac{3(1-u^2)}{1+u^2}} = -\frac{x}{3} + \frac{5}{3} \int \frac{2du}{5(1+u^2) - 3(1-u^2)}$$

$$= -\frac{x}{3} + \frac{5}{3} \int \frac{2du}{8u^2 + 2} = -\frac{x}{3} + \frac{5}{3} \int \frac{du}{4u^2 + 1} = -\frac{x}{3} + \frac{5}{3} \frac{1}{2} \int \frac{d2u}{4u^2 + 1}$$

$$= -\frac{x}{3} + \frac{5}{6} \arctan 2u + C = -\frac{x}{3} + \frac{5}{6} \arctan \left(2 \tan \frac{x}{2} \right) + C.$$

$$18. I = \int \frac{\cos^3 x dx}{\sin x + \cos x} = \int \frac{\cos^2 x dx}{1 + \tan x} = \int \frac{dx}{(1 + \tan x)(1 + \tan^2 x)}.$$

$$\tan x = u, x = \arctan u, dx = \frac{du}{1+u^2},$$

$$I = \int \frac{\frac{du}{1+u^2}}{(1+u)(1+u^2)} = \int \frac{du}{(1+u)(1+u^2)^2},$$

$$\frac{1}{(1+u)(1+u^2)^2} = \frac{1}{2(1+u^2)} \left(\frac{1}{1+u} + \frac{1-u}{1+u^2} \right)$$

$$= \frac{1}{4} \left(\frac{1}{1+u} + \frac{1-u}{1+u^2} \right) + \frac{1-u}{2(1+u^2)^2},$$

$$I = \frac{1}{4} \ln |1 + \tan x| + \frac{1}{4} \arctan u - \frac{1}{8} \ln(1+u^2) + \frac{1}{4(1+u^2)} + \frac{1}{2} \left(\frac{1}{2} \arctan u + \frac{u}{2(1+u^2)} \right) + C$$

$$= \frac{1}{4} \ln |1 + \tan x| + \frac{x}{2} + \frac{1}{4} \ln |\cos u| + \frac{1}{4} \cos^2 x + \frac{1}{4} \tan x \cos^2 x + C.$$

$$19. \int \sin^5 x \cos^2 x dx = -\int \sin^4 x \cos^2 x d \cos x = -\int (1-u^2)^2 u^2 du$$

$$= -\int (u^2 - 2u^4 + u^6) dx = -\frac{1}{3} u^3 + \frac{2}{5} u^5 - \frac{1}{7} u^7 + C$$

$$= -\frac{1}{3} (\cos x)^3 + \frac{2}{5} (\cos x)^5 - \frac{1}{7} (\cos x)^7 + C.$$

$$20. \int \sin^6 x dx = \int \left(\frac{1 - \cos 2x}{2} \right)^3 dx$$

$$= \frac{1}{8} \int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x) dx$$

$$\begin{aligned}
&= \frac{x}{8} - \frac{3}{16} \sin 2x + \frac{3}{16} \int (1 + \cos 4x) dx - \frac{1}{16} \int (1 - \sin^2 2x) d \sin 2x \\
&= \frac{x}{8} - \frac{3}{16} \sin 2x + \frac{3}{16} \left(x + \frac{1}{4} \sin 4x \right) - \frac{1}{16} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right) + C \\
&= +C.
\end{aligned}$$

$$\begin{aligned}
21. \int \sin^2 x \cos^4 x dx &= \frac{1}{4} \int \sin^2 2x \cos^2 x dx = \frac{1}{4} \int \left(\frac{\sin 3x + \sin x}{2} \right)^2 dx \\
&= \frac{1}{16} \int (\sin^2 3x + \sin^2 x + 2 \sin 3x \sin x) dx \\
&= \frac{1}{16} \int \left(\frac{1 - \cos 6x}{2} + \frac{1 - \cos 2x}{2} + \cos 2x - \cos 4x \right) dx \\
&= \frac{1}{16} \left(x + \frac{1}{4} \sin 2x - \frac{1}{4} \sin 4x - \frac{1}{12} \sin 6x \right) + C.
\end{aligned}$$

$$\begin{aligned}
\text{另解: } \int \sin^2 x \cos^4 x dx &= \int \frac{1 - \cos 2x}{2} \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\
&= \frac{1}{8} \int (1 + \cos^2 2x + 2 \cos 2x)(1 - \cos 2x) dx \\
&= \frac{1}{8} \int (1 + \cos^2 2x + 2 \cos 2x - \cos 2x - \cos^3 2x - 2 \cos^2 2x) dx \\
&= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx \\
&= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x \right) - \frac{1}{16} \int (1 + \cos 4x) dx - \frac{1}{16} \int (1 - \sin^2 2x) d \sin 2x \\
&= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x \right) - \frac{1}{16} \left(x + \frac{1}{4} \sin 4x \right) - \frac{1}{16} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right) + C \\
&= \frac{1}{16} x - \frac{1}{64} \sin 4x + \frac{1}{48} \sin^3 2x + C.
\end{aligned}$$

$$22. I = \int \frac{dx}{\sin x + 2 \cos x} \cdot \tan \frac{x}{2} = u, x = 2 \arctan u, dx = \frac{2du}{1+u^2}.$$

$$I = \int \frac{\frac{2du}{1+u^2}}{\frac{2u}{1+u^2} + \frac{2(1-u^2)}{1+u^2}} = \int \frac{2du}{-2u^2 + 2u + 2} = -\int \frac{du}{u^2 - u - 1} = -\int \frac{du}{\left(u - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2} =$$

$$= \frac{1}{\sqrt{5}} \ln \left| \frac{u - \frac{1}{2} + \frac{\sqrt{5}}{2}}{u - \frac{1}{2} - \frac{\sqrt{5}}{2}} \right| + C = \ln \left| \frac{2u + \sqrt{5} - 1}{2u - \sqrt{5} - 1} \right| + C.$$

$$23. \int \frac{\sin x \cos x}{\sin^2 x + \cos^4 x} dx =$$

$$\begin{aligned}
&= \int \frac{\tan x}{\tan^2 x (1 + \tan^2 x) + 1} d \tan x = \int \frac{u}{u^2 (1 + u^2) + 1} du (u = \tan x) \\
&= \frac{1}{2} \int \frac{du^2}{u^2 (1 + u^2) + 1} = \frac{1}{2} \int \frac{dv}{v(1+v)+1} (v = u^2) \\
&= \frac{1}{2} \int \frac{dv}{\left(v + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \arctan \frac{2v+1}{\sqrt{3}} + C \\
&= \frac{1}{\sqrt{3}} \arctan \frac{2 \tan^2 x + 1}{\sqrt{3}} + C.
\end{aligned}$$

$$\begin{aligned}
\text{另解: } I &= \frac{1}{2} \int \frac{d \sin^2 x}{\sin^2 x + (1 - \sin^2 x)^2} = \frac{1}{2} \int \frac{dw}{w + (1-w)^2} (w = \sin^2 x) \\
&= \frac{1}{2} \int \frac{dw}{\left(w - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\sqrt{3}} \arctan \frac{2w-1}{\sqrt{3}} + C = \arctan \frac{2 \sin^2 x - 1}{\sqrt{3}} + C.
\end{aligned}$$

$$24. \int \frac{dx}{\sin^4 x} = -\int (1 + \cot^2 x) d \cot x = -\cot x - \frac{1}{3} \cot^3 x + C.$$

$$25. \int \sqrt{\frac{1-x}{1+x}} dx = \int \frac{1-x}{\sqrt{1-x^2}} dx = \arcsin x + \sqrt{1-x^2} + C.$$

$$26. I = \int \frac{1 - \sqrt{x-1}}{1 + \sqrt[3]{x-1}} dx. \sqrt[6]{x-1} = u, x = 1 + u^6, dx = 6u^5 du,$$

$$\begin{aligned}
I &= 6 \int \frac{(1-u^3)u^5 du}{1+u^2} = 6 \int \frac{u^5 - u^8}{1+u^2} du = -6 \int (u^6 - u^4 - u^3 + u^2 + u + 1 + \frac{-u+1}{1+u^2}) dx \\
&= -6 \left(\frac{1}{7} u^7 - \frac{1}{5} u^5 - \frac{1}{4} u^4 + \frac{1}{3} u^3 + \frac{1}{2} u^2 + u - \frac{1}{2} \ln(1+u^2) + \arctan u \right) + C.
\end{aligned}$$

$$\begin{aligned}
27. \int \frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} dx &= \int \frac{(\sqrt{x+1} + \sqrt{x-1})^2}{(\sqrt{x+1} - \sqrt{x-1})(\sqrt{x+1} + \sqrt{x-1})} dx \\
&= \int \frac{2x + 2\sqrt{x^2-1}}{2} dx = \frac{1}{2} x^2 + \frac{1}{2} x \sqrt{x^2-1} - \frac{1}{2} \ln(x + \sqrt{x^2-1}) + C.
\end{aligned}$$

$$28. I = \int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = \int \frac{dx}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}} \cdot \sqrt[3]{\frac{x-1}{x+1}} = u, \frac{x-1}{x+1} = u^3,$$

$$x-1 = (x+1)u^3, x = \frac{1+u^3}{1-u^3} = -1 + \frac{2}{1-u^3}, dx = \frac{6u^2 du}{(1-u^3)^2},$$

$$I = \int \frac{\frac{6u^2 du}{(1-u^3)^2}}{\left(\left(\frac{1+u^3}{1-u^3}\right)^2 - 1\right)u} = 6 \int \frac{u}{2(2u^3)} du = \frac{3}{2} \left(-\frac{1}{u}\right) + C = -\frac{3}{2} \sqrt[3]{\frac{x+1}{x-1}} + C.$$

$$\begin{aligned} 29. \int \frac{x dx}{\sqrt{x^2 - x + 3}} &= \frac{1}{2} \int \frac{2x dx}{\sqrt{x^2 - x + 3}} = \frac{1}{2} \int \frac{2x - 1 + 1 dx}{\sqrt{x^2 - x + 3}} = \\ &= \frac{1}{2} \int \frac{d(x^2 - x + 3)}{\sqrt{x^2 - x + 3}} + \frac{1}{2} \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2}} = \\ &= \sqrt{x^2 - x + 3} + \frac{1}{2} \ln \left(x - \frac{1}{2} + \sqrt{x^2 - x + 3}\right) + C. \end{aligned}$$

$$30. I = \int \frac{x}{(1+x^{1/3})^{1/2}} dx. (1+x^{1/3})^{1/2} = u, x = (u^2 - 1)^3, dx = 3(u^2 - 1)^2 (2u) du,$$

$$\begin{aligned} I &= 6 \int \frac{(u^2 - 1)^3 (u^2 - 1)^2 (u) du}{u} = 6 \int (u^6 - 3u^4 + 3u^2 - 1)(u^4 - 2u^2 + 1) du \\ &= 6 \int (u^{10} - 5u^8 + 10u^6 - 10u^4 + 5u^2 - 1) du \\ &= 6 \left(\frac{1}{11} u^{11} - \frac{5}{9} u^9 + \frac{10}{7} u^7 - 2u^5 + \frac{5}{3} u^3 - u \right) + C. \end{aligned}$$

$$31. I = \int \frac{\sqrt{x} dx}{\sqrt[4]{x^3 + 1}}. \sqrt[4]{x} = u, x = u^4, dx = 4u^3 du.$$

$$\begin{aligned} I &= \int \frac{u^2 4u^3 du}{u^3 + 1} = 4 \int \frac{u^5}{u^3 + 1} dx = 4 \int \frac{(u^5 + u^2) - u^2}{u^3 + 1} du \\ &= 4 \int \left(u^2 - \frac{u^2}{u^3 + 1} \right) du = \frac{4}{3} u^3 - \frac{4}{3} \ln(u^3 + 1) + C = \frac{4}{3} \sqrt[4]{x^3} - \frac{4}{3} \ln(\sqrt[4]{x^3} + 1) + C. \end{aligned}$$

$$\begin{aligned} 32. \int \frac{2x+3}{\sqrt{x^2+x}} dx &= \int \frac{(2x+1)+2}{\sqrt{x^2+x}} dx = \int \frac{1}{\sqrt{x^2+x}} d(x^2+x) + 2 \int \frac{1}{\sqrt{x^2+x}} dx \\ &= 2\sqrt{x^2+x} + 2 \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx \\ &= 2\sqrt{x^2+x} + 2 \ln \left| x + \frac{1}{2} + \sqrt{x^2+x} \right| + C. \end{aligned}$$

$$\begin{aligned} 33. \int \frac{2+x}{\sqrt{4x^2-4x+5}} dx &= \frac{1}{8} \int \frac{16+8x}{\sqrt{4x^2-4x+5}} dx \\ &= \frac{1}{8} \int \frac{8x-4+20}{\sqrt{4x^2-4x+5}} dx = \frac{1}{8} \int \frac{d(4x^2-4x+5)}{\sqrt{4x^2-4x+5}} dx + \frac{5}{2} \int \frac{dx}{\sqrt{4x^2-4x+5}} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\sqrt{4x^2-4x+5} + \frac{5}{4}\int \frac{dx}{\sqrt{x^2-x+5/4}} \\
&= \frac{1}{4}\sqrt{4x^2-4x+5} + \frac{5}{4}\int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2+1}} \\
&= \frac{1}{4}\sqrt{4x^2-4x+5} + \frac{5}{4}\ln\left(x-\frac{1}{2}+\sqrt{x^2-x+5/4}\right) + C \\
&= \frac{1}{4}\sqrt{4x^2-4x+5} + \frac{5}{4}\ln\left(2x-1+\sqrt{4x^2-4x+5}\right) + C'.
\end{aligned}$$

$$\begin{aligned}
34. \int \sqrt{5-2x+x^2} dx &= \int \sqrt{2^2+(x-1)^2} dx \\
&= \frac{(x-1)}{2}\sqrt{5-2x+x^2} + 2\ln(\sqrt{5-2x+x^2}) + C.
\end{aligned}$$

习题 3.4

求下列各定积分:

$$1. I = \int_{-1}^1 \frac{xdx}{\sqrt{5-4x}}. \sqrt{5-4x} = u, -1 \rightarrow 3, 1 \rightarrow 1.5 - 4x = u^2, x = \frac{1}{4}(5-u^2), dx = -\frac{1}{2}udu,$$

$$I = \int_3^{1.5} \frac{\frac{1}{4}(5-u^2)}{u} g\left(-\frac{1}{2}udu\right) = \frac{1}{8} \int_1^3 (5-u^2)dx = \frac{1}{8} \left(5u - \frac{1}{3}u^3\right) \Big|_1^3 = \frac{1}{6}.$$

$$2. \int_0^{\ln 2} xe^{-x} dx = -\int_0^{\ln 2} xde^{-x} = -xe^{-x} \Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-x} dx = -\frac{\ln 2}{2} - e^{-x} \Big|_0^{\ln 2} = \frac{1}{2}(1 - \ln 2).$$

$$3. \int_0^1 x^2 \sqrt{1-x^2} dx = \int_0^{\pi/2} \sin^2 t \cos^2 t dt (x = \sin t) \\ = \int_0^{\pi/2} \sin^2 t (1 - \sin^2 t) dt = I_2 - I_4 = \left(\frac{1}{2} - \frac{3g}{4g^2}\right) \frac{\pi}{2} = \frac{\pi}{16}.$$

$$4. \int_0^{\pi} x \sin x dx = -\int_0^{\pi} x d \cos x = -x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x dx = \pi + \sin x \Big|_0^{\pi} = \pi.$$

$$5. \int_0^4 \sqrt{x^2+9} dx = \left(\frac{x}{2} \sqrt{x^2+9} + \frac{9}{2} \ln(x + \sqrt{x^2+9})\right) \Big|_0^4 = 10 + \frac{9}{2} \ln 3.$$

$$6. \int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{6}} \sin^2 t dt = \frac{1}{2} \int_0^{\frac{\pi}{6}} (1 - \cos 2t) dt = \frac{1}{2} \left(t - \frac{1}{2} \sin 2t\right) \Big|_0^{\frac{\pi}{6}} = \frac{1}{2} \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right).$$

$$7. \int_0^1 \sqrt{4-x^2} dx = \left(\frac{x}{2} \sqrt{4-x^2} + 2 \arcsin \frac{x}{2}\right) \Big|_0^1 = \frac{\sqrt{3}}{2} + \frac{\pi}{3}.$$

$$8. \int_0^3 x \sqrt[3]{1-x^2} dx = \frac{1}{2} \int_0^3 \sqrt[3]{1-x^2} dx^2 = \frac{1}{2} \int_0^9 \sqrt[3]{1-u} du = -\frac{3}{8} (1-u)^{\frac{4}{3}} \Big|_0^9 = -\frac{45}{8}.$$

$$9. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x} dx = 2 \int_0^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x} dx = 2 \int_0^{\frac{\pi}{2}} \sqrt{\cos x} \sin x dx$$

$$= -2 \int_0^{\frac{\pi}{2}} \sqrt{\cos x} d \cos x = -\frac{4}{3} \cos^{\frac{3}{2}} x \Big|_0^{\frac{\pi}{2}} = \frac{4}{3}.$$

$$10. \int_0^{\frac{\pi}{2}} \cos^n 2x dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^n 2x d2x = \frac{1}{2} \int_0^{\pi} \cos^n u du = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^n \left(t + \frac{\pi}{2}\right) dt$$

$$= \frac{(-1)^n}{2} \int_{-\pi/2}^{\pi/2} \sin^n(t) dt = \begin{cases} 0, n=2k-1; \\ \int_0^{\pi/2} \sin^n(t) dt = \frac{(n-1)!!}{n!!} \frac{\pi}{2}. \end{cases}$$

$$11. \int_0^a (a^2 - x^2)^{\frac{n}{2}} dx (x = a \sin t) = \int_0^{\frac{\pi}{2}} \cos^{n+1} t dt = \begin{cases} \frac{n!!}{(n+1)!!} & n \text{ 是偶数}; \\ \frac{n!!}{(n+1)!!} \frac{\pi}{2} & n \text{ 是奇数}. \end{cases}$$

$$12. \int_0^{\pi/2} \sin^{11} x dx = \frac{10!!}{11!!} = \frac{156}{693}.$$

$$13. \int_0^{\pi} \sin^6 \frac{x}{2} dx = 2 \int_0^{\pi/2} \sin^6 u du = 2 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{\pi}{2} = \frac{5\pi}{16}.$$

$$\begin{aligned} 14. \int_0^{\pi} (x \sin x)^2 dx &= \frac{1}{2} \int_0^{\pi} x^2 (1 - \cos 2x) dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_0^{\pi} - \frac{1}{4} \int_0^{\pi} x^2 d \sin 2x \\ &= \frac{\pi^3}{6} - \frac{1}{4} x^2 \sin 2x \Big|_0^{\pi} + \frac{1}{2} \int_0^{\pi} x \sin 2x dx \\ &= \frac{\pi^3}{6} - \frac{1}{4} \int_0^{\pi} x d \cos 2x = \frac{\pi^3}{6} - \frac{1}{4} x \cos 2x \Big|_0^{\pi} + \frac{1}{4} \int_0^{\pi} \cos 2x dx \\ &= \frac{\pi^3}{6} - \frac{\pi}{4} + \frac{1}{8} \sin 2x \Big|_0^{\pi} = \frac{\pi^3}{6} - \frac{\pi}{4}. \end{aligned}$$

$$\begin{aligned} 15. \int_0^{\pi/4} \tan^4 x dx &= \int_0^{\pi/4} \tan^2 x (\sec^2 x - 1) dx \\ &= \int_0^{\pi/4} \tan^2 x \sec^2 x dx - \int_0^{\pi/4} \tan^2 x dx \\ &= \int_0^{\pi/4} \tan^2 x d \tan x - \int_0^{\pi/4} (\sec^2 x - 1) dx \\ &= \frac{1}{3} \tan^3 x \Big|_0^{\pi/4} - \tan x \Big|_0^{\pi/4} + \frac{\pi}{4} = \frac{1}{3} - 1 + \frac{\pi}{4} = -\frac{2}{3} + \frac{\pi}{4}. \end{aligned}$$

$$\begin{aligned} 16. \int_0^1 \arcsin x dx &= x \arcsin x \Big|_0^1 - \int_0^1 x d \arcsin x \\ &= \frac{\pi}{2} - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \frac{\pi}{2} + \sqrt{1-x^2} \Big|_0^1 = \frac{\pi}{2} - 1. \end{aligned}$$

$$\begin{aligned} 17. \int_0^{\pi} \ln(x + \sqrt{x^2 + a^2}) dx &= x \ln(x + \sqrt{x^2 + a^2}) \Big|_0^{\pi} - \int_0^{\pi} x d \ln(x + \sqrt{x^2 + a^2}) \\ &= \pi \ln(\pi + \sqrt{\pi^2 + a^2}) - \int_0^{\pi} \frac{x}{\sqrt{x^2 + a^2}} dx = \pi \ln(\pi + \sqrt{\pi^2 + a^2}) - \sqrt{x^2 + a^2} \Big|_0^{\pi} \\ &= \pi \ln(\pi + \sqrt{\pi^2 + a^2}) - \sqrt{\pi^2 + a^2} + |a|. \end{aligned}$$

$$18. \text{设 } f(x) \text{ 在 } [a, b] \text{ 连续, 证明 } \int_a^b f(x) dx = (b-a) \int_0^1 f(a + (b-a)x) dx.$$

证 令 $x = a + (b-a)t$, 则 $0 \rightarrow a, 1 \rightarrow b, dx = (b-a)dt$, 故

$$\int_a^b f(x) dx = (b-a) \int_0^1 f(a + (b-a)t) dt = (b-a) \int_0^1 f(a + (b-a)x) dx.$$

$$19. \text{证明 } \int_0^a x^3 f(x^2) dx = \frac{1}{2} \int_0^{a^2} x f(x) dx.$$

证 令 $x^2 = t$, 则 $x = 0$ 时, $t = 0$, $x = a$ 时, $t = a^2$ 故

$$\int_0^a x^3 f(x^2) dx = \frac{1}{2} \int_0^a x^2 f(x^2) dx^2 = \frac{1}{2} \int_0^{a^2} t f(t) dt = \frac{1}{2} \int_0^{a^2} x f(x) dx.$$

$$20. \text{证明 } \int_0^1 x^m (1-x)^n dx = \int_0^1 x^n (1-x)^m dx.$$

证 令 $x = 1-t$, 则 $x = 0$ 时, $x = 1$ 时, $t = 0, dx = -dt$, 故

$$\int_0^1 x^m (1-x)^n dx = -\int_1^0 (1-t)^m t^n dt = \int_0^1 (1-t)^m t^n dt = \int_0^1 x^n (1-x)^m dx.$$

21. 利用分部积分公式证明, 若 $f(x)$ 连续, 则

$$\int_0^x \int_0^t f(x) dx dt = \int_0^x f(t)(x-t) dx.$$

$$\begin{aligned} \text{证} \int_a^x \int_0^t f(x) dx dt &= t \int_0^t f(x) dx \Big|_0^x - \int_0^x t \left(\int_0^t f(x) dx \right)' dt \\ &= \int_0^x xf(x) dx - \int_0^x tf(t) dt = \int_0^x xf(t) dt - \int_0^x tf(t) dt \\ &= \int_0^x f(t)(x-t) dt. \end{aligned}$$

22. 利用换元积分法证明 $\int_0^\pi xf(\sin x) dx = \pi \int_0^{\pi/2} f(\sin x) dx$.

证 $x = \pi - t$, $x = 0$ 时, $t = \pi$, $x = \pi$ 时, $dx = -dt$, 故

$$\begin{aligned} \int_0^\pi xf(\sin x) dx &= -\int_\pi^0 (\pi - t) f(\sin(\pi - t)) dt \\ &= \int_0^\pi (\pi - t) f(\sin t) dt = \pi \int_0^\pi f(\sin t) dt - \int_0^\pi tf(\sin t) dt \\ &= \pi \int_0^\pi f(\sin t) dt - \int_0^\pi xf(\sin x) dx. \end{aligned}$$

$$2 \int_0^\pi xf(\sin x) dx = \pi \int_0^\pi f(\sin t) dt,$$

$$\int_0^\pi xf(\sin x) dx = \frac{1}{2} \pi \int_0^\pi f(\sin t) dt$$

$$= \frac{1}{2} \pi \int_0^{\pi/2} f(\sin t) dt + \frac{1}{2} \pi \int_{\pi/2}^\pi f(\sin t) dt$$

令 $u = \pi - t$, 则 $t = \pi/2$ 时, $u = \pi/2$, $t = \pi$ 时, $u = 0$, $du = -dt$,

$$\int_{\pi/2}^\pi f(\sin t) dt = -\int_{\pi/2}^0 f(\sin(\pi - u)) du = \int_0^{\pi/2} f(\sin u) du,$$

$$\int_0^\pi xf(\sin x) dx = \pi \int_0^{\pi/2} f(\sin x) dx.$$

23. 利用上题结果求 $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$.

$$\begin{aligned} \text{解} \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx &= \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx = - \int_0^{\pi/2} \frac{d \cos x}{1 + \cos^2 x} \\ &= -\arctan \cos x \Big|_0^{\pi/2} = \frac{\pi}{4}. \end{aligned}$$

24. 设函数 $f(x)$ 在 $(-\infty, +\infty)$ 上连续, 以 T 为周期, 证明:

(1) 函数 $F(x) = \frac{x}{T} \int_0^T f(x) dx - \int_0^x f(t) dt$ 也以 T 为周期;

$$(2) \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x f(t) dt = \frac{1}{T} \int_0^T f(x) dx.$$

$$\begin{aligned} \text{证} (1) F(x+T) &= \frac{x+T}{T} \int_0^T f(x) dx - \int_0^{x+T} f(t) dt \\ &= \frac{x}{T} \int_0^T f(x) dx + \int_0^T f(x) dx - \left(\int_0^x f(t) dt + \int_x^{x+T} f(t) dt \right) \\ &= \frac{x}{T} \int_0^T f(x) dx + \int_0^T f(x) dx - \left(\int_0^x f(t) dt + \int_0^T f(t) dt \right) \\ &= \frac{x}{T} \int_0^T f(x) dx - \int_0^x f(t) dt = F(x). \end{aligned}$$

$$\begin{aligned} (2) \frac{1}{x} \int_0^x f(t) dt - \frac{1}{T} \int_0^T f(x) dx \\ = -\frac{1}{x} \left(\frac{x}{T} \int_0^T f(x) dx - \int_0^x f(t) dt \right) = -\frac{F(x)}{x}. \end{aligned}$$

$F(x)$ 在 $(-\infty, +\infty)$ 上连续, 以 T 为周期, 故有界,

$$\lim_{x \rightarrow +\infty} \left(\frac{1}{x} \int_0^x f(t) dt - \frac{1}{T} \int_0^T f(x) dx \right) = \lim_{x \rightarrow +\infty} \frac{F(x)}{x} = 0.$$

$$\text{于是} \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x f(t) dt = \frac{1}{T} \int_0^T f(x) dx.$$

25. 设 $f(x)$ 是以 T 为周期的连续函数, $f(x_0) \neq 0$, 且 $\int_0^T f(x) dx = 0$, 证明:

$f(x)$ 在区间 $(x_0, x_0 + T)$ 内至少有两个根.

证为明确起见, 设 $f(x_0) > 0$. 如果 f 在 $(x_0, x_0 + T)$ 没有根, 则由连续函数的

中间值定理, f 在 $(x_0, x_0 + T)$ 恒正, 设其最小值为 m . 则 $m > 0$,

$$\int_{x_0}^{x_0+T} f(x) dx \geq \int_{x_0}^{x_0+T} m dx = mT > 0. \text{由周期性和假设} \int_{x_0}^{x_0+T} f(x) dx = \int_0^T f(x) dx = 0,$$

矛盾. 故 f 在 $(x_0, x_0 + T)$ 至少有一个根 x_1 . 若 f 在 $(x_0, x_0 + T)$ 再无其它根, 由于

$$f(x_0 + T) = f(x_0) > 0, f \text{ 在 } (x_0, x_1) \text{ 和 } (x_1, x_0 + T) \text{ 恒正,}$$

$$\int_{x_0}^{x_0+T} f(x) dx = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_0+T} f(x) dx > 0, \text{矛盾. 故 } f \text{ 在 } (x_0, x_1) \text{ 或 } (x_1, x_0 + T) \text{ 至少}$$

还有一个根, 即 $f(x)$ 在区间 $(x_0, x_0 + T)$ 内至少有两个根.

26. 求定积分

$$\int_0^{2m\pi} \frac{dx}{\sin^4 x + \cos^4 x}$$

其中 m 为正整数.

解 被积函数以 2π 为周期,故 $\int_0^{2m\pi} \frac{dx}{\sin^4 x + \cos^4 x} = m \int_0^{2\pi} \frac{dx}{\sin^4 x + \cos^4 x}$.

$$\int_0^{2\pi} \frac{dx}{\sin^4 x + \cos^4 x} = \int_0^{2\pi} \frac{dx}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x}$$

$$= \int_0^{2\pi} \frac{dx}{1 - \frac{1}{2}\sin^2 2x} = 4 \int_0^{\pi/2} \frac{dx}{1 - \frac{1}{2}\sin^2 2x} \quad (\sin^2 2x \text{ 周期为 } \frac{\pi}{2})$$

$$= 8 \int_0^{\pi/2} \frac{dx}{2 - \sin^2 2x} = -4 \int_0^{\pi/2} \frac{d \cot 2x}{2 \csc^2 2x - 1}$$

$$= -4 \int_0^{\pi/2} \frac{d \cot 2x}{2 \cot^2 2x + 1} = 4 \int_{-\infty}^{+\infty} \frac{du}{1 + 2u^2} = \frac{4}{\sqrt{2}} \arctan \sqrt{2}u \Big|_{-\infty}^{+\infty} = 2\sqrt{2}\pi.$$

$$\int_0^{2m\pi} \frac{dx}{\sin^4 x + \cos^4 x} = 2m\sqrt{2}\pi.$$

习题 3.

求下列曲线所围成的图形的面积：

1. $y = x^2$ 与 $x = y^2$.

解求交点： $\begin{cases} y = x^2 \\ x = y^2 \end{cases}, x = x^4,$

$x(1-x)(1+x+x^2) = 0, x = 0, x = 1.$

$S = \int_0^1 (\sqrt{x} - x^2) dx = \left(\frac{2}{3} x^{3/2} - \frac{1}{3} x^2 \right) \Big|_0^1 = \frac{1}{3}.$

2. $y = x, y = 1$ 与 $y = \frac{x^2}{4}.$

解 $S = \int_0^1 (y + 2\sqrt{y}) dy = \left(\frac{y^2}{2} + y^{3/2} \right) \Big|_0^1 = \frac{3}{2}.$

3. $y^2 = 2x + 1$ 与 $x - y = 1.$

解 $\begin{cases} y^2 = 2x + 1 \\ x - y = 1 \end{cases} \Rightarrow (x-1)^2 = 2x+1,$

$x^2 - 4x = 0, x = 0, y = -1; x = 4, y = 3.$

$S = \int_{-1}^3 \left(y + 1 - \frac{1}{2}(y^2 - 1) \right) dy$
 $= \left(\frac{3}{2}y + \frac{y^2}{2} - \frac{1}{6}y^3 \right) \Big|_{-1}^3 = \frac{16}{3}.$

4. $y = 0$ 与 $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} \quad 0 \leq t \leq 2\pi (a > 0)$

$S = \int_0^{2\pi} a(1 - \cos t) da(t - \sin t)$

$= a^2 \int_0^{2\pi} (1 - \cos t)^2 dt$

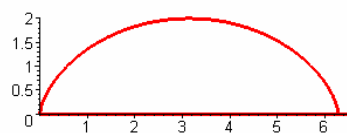
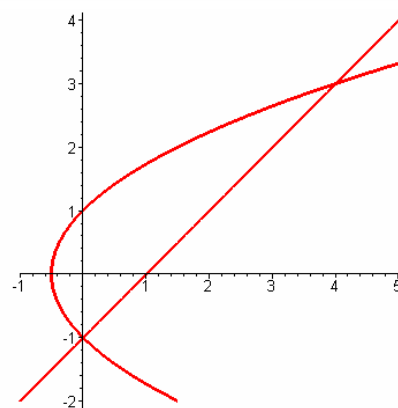
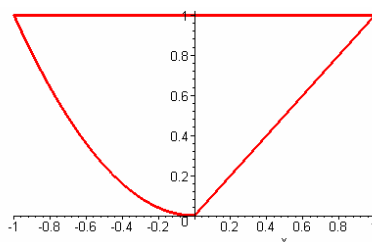
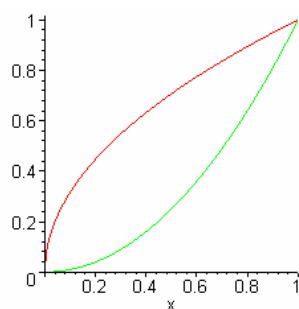
$= 4a^2 \int_0^{2\pi} \sin^4 \frac{t}{2} dt = 8a^2 \int_0^{\pi} \sin^4 u du$

$= 16a^2 \int_0^{\pi/2} \sin^4 u du = 16a^2 \frac{3}{4} \frac{\pi}{2}$

$= 3\pi a^2.$

5. $y = x^2 - 4$ 与 $y = -x^2 - 2x.$

解 $\begin{cases} y = x^2 - 4 \\ y = -x^2 - 2x \end{cases} \Rightarrow x^2 - 4 = -x^2 - 2x,$



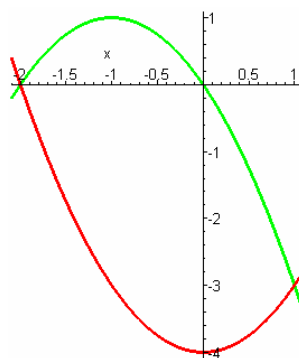
$$2x^2 + 2x - 4 = 0, (2x - 2)(x + 2) = 0,$$

$$x_1 = -2, x_2 = 1.$$

$$S = \int_{-2}^1 (-x^2 - 2x - x^2 + 4) dx$$

$$= \int_{-2}^1 (-2x^2 - 2x + 4) dx$$

$$= \left(-\frac{2}{3}x^3 - x^2 + 4x \right) \Big|_{-2}^1 = 9.$$



$$6. x^2 + y^2 = 8 \text{ 与 } y = \frac{1}{2}x^2 \text{ (分上下两部分).}$$

$$\text{解} \begin{cases} x^2 + y^2 = 8 \\ y = \frac{1}{2}x^2 \end{cases} \quad x^2 + \frac{1}{4}x^4 = 8$$

$$x^4 + 4x^2 - 32 = 0, x^2 = u$$

$$u^2 + 4u - 32 = 0, (u + 8)(u - 4) = 0$$

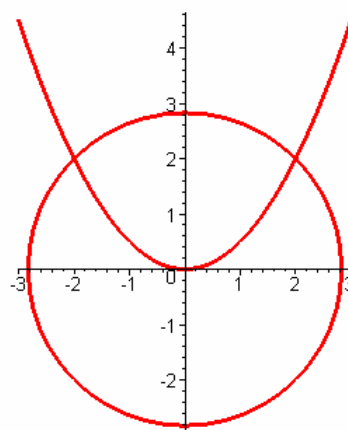
$$u_2 = -8 \text{ (舍)} u_2 = 4, x^2 = 4, x_1 = -2, x_2 = 2$$

$$S_1 = \int_{-2}^2 \left(\sqrt{8 - x^2} - \frac{1}{2}x^2 \right) dx$$

$$= 2 \int_0^2 \left(\sqrt{8 - x^2} - \frac{1}{2}x^2 \right) dx$$

$$= 2 \left(\frac{x}{2} \sqrt{8 - x^2} + 4 \arcsin \frac{x}{2\sqrt{2}} \right) \Big|_0^2 = 2\pi + \frac{4}{3}$$

$$S_2 = 8\pi - \left(2\pi + \frac{4}{3} \right) = 6\pi - \frac{4}{3}.$$



$$7. y = 4 - x^2 \text{ 与 } y = x + 2.$$

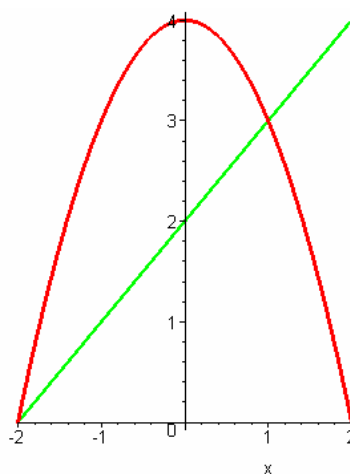
$$\text{解} \begin{cases} y = 4 - x^2 \\ y = x + 2 \end{cases} \quad 4 - x^2 = x + 2$$

$$x^2 + x - 2 = 0, (x + 2)(x - 1) = 0,$$

$$x_1 = -2, x_2 = 1.$$

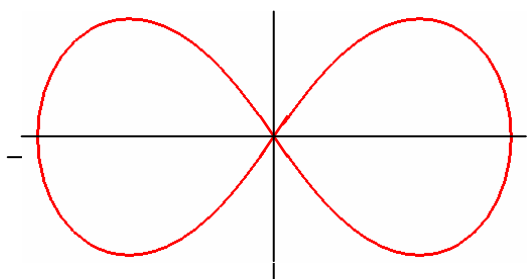
$$S = \int_{-2}^1 (4 - x^2 - x - 2) dx = 6 - \int_{-2}^1 (x^2 + x) dx$$

$$= 6 - \left(\frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_{-2}^1 = \frac{9}{2}.$$



8. 求双纽线 $r^2 = a^2 \cos 2\varphi (a > 0)$ 所围图形的面积.

$$\text{解 } S = 4 \int_0^{\pi/4} \frac{1}{2} a^2 \cos 2\varphi d\varphi = 2a^2 \left[\frac{1}{2} \sin 2\varphi \right]_0^{\pi/4} = a^2.$$



求下列曲线围成的平面图形绕轴旋转所成旋转体的体积:

9. $x^{2/3} + y^{2/3} = a^{2/3} (a > 0).$

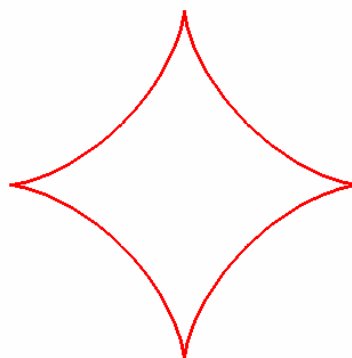
$$\text{解 } \begin{cases} x = a \cos^3 t \\ y = a \sin^3 t \end{cases}, 0 \leq t \leq 2\pi.$$

$$V = 2\pi \int_0^a y^2 dx = 2\pi \int_0^{\pi/2} a^2 \sin^6 t \cdot 3 \cos^2 t \sin t dt$$

$$= 6\pi a^3 \int_0^{\pi/2} \sin^7 t \cos^2 t dt$$

$$= 6\pi a^3 \int_0^{\pi/2} \sin^7 t (1 - \sin^2 t) dt$$

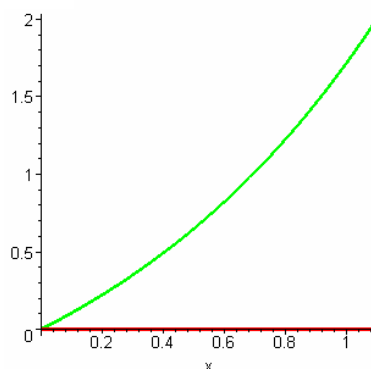
$$= 6\pi a^3 \left(\frac{64}{75} \left(1 - \frac{8}{9} \right) \right) = \frac{32}{105} \pi a^3.$$



10. $y = e^x - 1, x = \ln 3, y = e.$

$$V = \pi \int_0^{\ln 3} (e^x - 1)^2 dx = \pi \int_0^{\ln 3} (e^{2x} - 2e^x + 1) dx$$

$$= \pi \left(\frac{1}{2} e^{2x} - 2e^x + x \right) \Big|_0^{\ln 3} = \pi \ln 3.$$



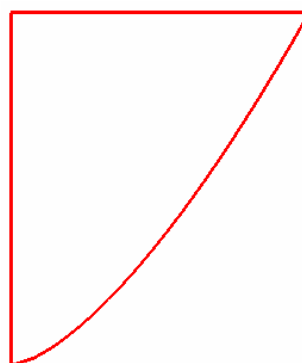
求下列平面曲线围成的平面图形绕轴旋转所成旋转体的体积:

11. $ay^2 = x^3, x = 0$ 及 $y = b (a > 0, b > 0).$

$$\text{解 } x = a^{1/3} y^{2/3},$$

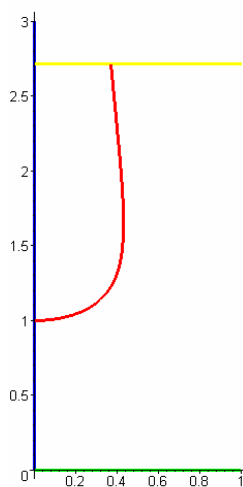
$$V = \pi \int_0^b (a^{1/3} y^{2/3})^2 dy$$

$$= \pi a^{2/3} \left[\frac{3}{7} y^{7/3} \right]_0^b = \frac{3}{7} \pi a^{2/3} b^{7/3}.$$



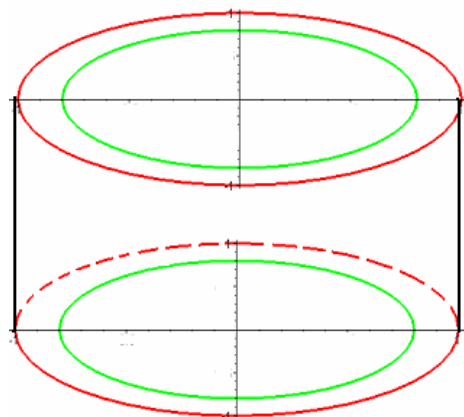
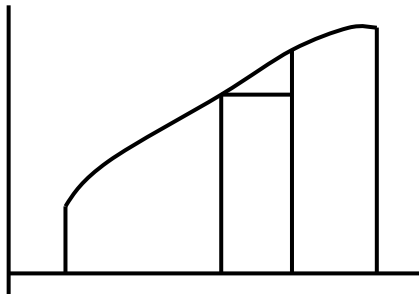
$$12. x = \frac{\sqrt{8 \ln y}}{y}, x=0, y=e.$$

$$\begin{aligned} \text{解 } V &= \pi \int_1^e \frac{8 \ln y}{y^2} dy \\ &= 8\pi \left[-\int_1^e \ln y dy^{-1} \right] \\ &= 8\pi \left[-y^{-1} \ln y \Big|_1^e + \int_1^e y^{-1} d \ln y \right] \\ &= 8\pi \left[-\frac{1}{e} + \int_1^e y^{-2} dy \right] \\ &= 8\pi \left[-\frac{1}{e} - y^{-1} \Big|_1^e \right] = 8\pi \left[1 - \frac{2}{e} \right]. \end{aligned}$$



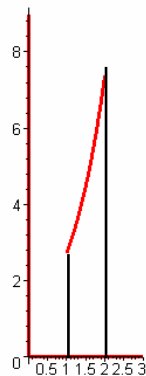
13. 设 $y = f(x)$ 在区间 $[a, b]$ ($a > 0$) 上连续且不取负值, 试用微元法推导: 由曲线 $y = f(x)$, 直线 $x = a$, $x = b$ 及轴围成的平面图形绕 y 轴旋转所成立体的体积为 $V = 2\pi \int_a^b xf(x)dx$.

解 厚度 dx 的圆筒的体积 $dV = 2\pi xf(x)dx$, $V = 2\pi \int_a^b xf(x)dx$.



14. 求曲线 $y = e^x$, $x = 1$, $x = 2$ 及 x 轴所围成的平面图形绕 y 轴旋转所成的立体的体积.

$$\begin{aligned} \text{解 } V &= 2\pi \int_1^2 xe^x dx = 2\pi \left[\int_1^2 xde^x \right] \\ &= 2\pi \left[xe^x \Big|_1^2 - \int_1^2 e^x dx \right] = 2\pi \left[2e^2 - e - e^x \Big|_1^2 \right] \\ &= 2\pi \left[2e^2 - e - (e^2 - e) \right] = 2\pi e^2. \end{aligned}$$

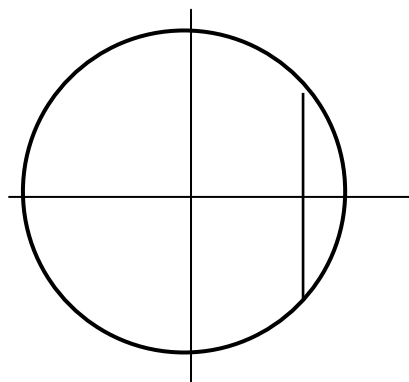


15. 证明: 半径为 a 高为 h 的球缺的体积为

$$V = \pi h^2 \left(a - \frac{h}{3} \right).$$

证 $y = f(x) = \sqrt{a^2 - x^2}, a - h \leq x \leq a$.

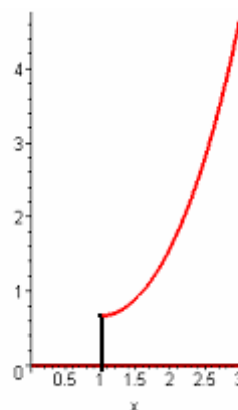
$$\begin{aligned} V &= \pi \int_{a-h}^a (a^2 - x^2) dx = \pi \left[a^2 h - \frac{1}{3} x^3 \Big|_{a-h}^a \right] \\ &= \pi \left[a^2 h - \frac{1}{3} (a^3 - (a-h)^3) \right] = \pi h^2 \left(a - \frac{h}{3} \right) \end{aligned}$$



16. 求曲线 $y = \frac{x^3}{6} + \frac{1}{2x}$ 在 $x=1$ 到 $x=3$ 之间的弧长.

$$\text{解 } y' = \frac{x^2}{2} - \frac{1}{2x^2}.$$

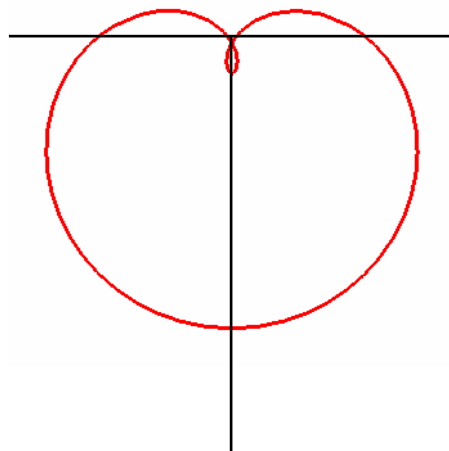
$$\begin{aligned} s &= \int_1^3 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2} \right)^2} dx \\ &= \int_1^3 \frac{x^4 + 1}{2x^2} dx = \left[\frac{x^3}{6} - \frac{1}{2x} \right] \Big|_1^3 = \frac{14}{3}. \end{aligned}$$



17. 求曲线 $r = a \sin^3 \frac{\theta}{3}$ 的全长.

$$\text{解 } r' = a \sin^2 \frac{\theta}{3} \cos \frac{\theta}{3},$$

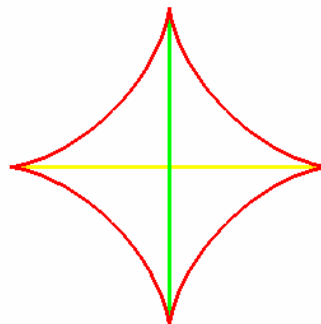
$$\begin{aligned} s &= a \int_0^{3\pi} \sqrt{\sin^6 \frac{\theta}{3} + \sin^4 \frac{\theta}{3} \cos^2 \frac{\theta}{3}} d\theta \\ &= a \int_0^{3\pi} \sin^2 \frac{\theta}{3} d\theta \\ &= 6a \int_0^{\pi/2} \sin^2 \theta d\theta = 6a \frac{1}{2} \frac{\pi}{2} = \frac{3}{2} \pi a. \end{aligned}$$



18. 求向星形线 $x = a \cos^3 t, y = a \sin^3 t$ 的弧长.

$$\text{解 } x' = 3a \cos^2 t (-\sin t), y' = 3a \sin^2 t \cos t$$

$$s = 4 \int_0^{\pi/2} \sqrt{x'^2 + y'^2} dt = 12a \int_0^{\pi/2} \sin^2 t \cos^2 t dt = 6a.$$

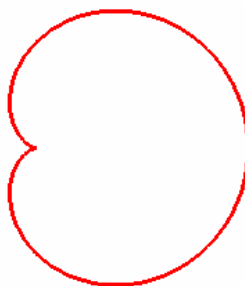


19.求心脏线 $r = a(1 + \cos \theta)$ 的全长.

解 $r' = a(-\sin \theta)$

$$s = 2a \int_0^\pi \sqrt{(1 + \cos \theta)^2 + \sin^2 \theta} d\theta$$

$$= 4a \int_0^\pi \cos \frac{\theta}{2} dx = 8a \sin \frac{\theta}{2} \Big|_0^\pi = 8a.$$



20.试证双纽线 $r^2 = 2a^2 \cos 2\theta (a > 0)$ 的全长 L 可表为 $L = 4\sqrt{2}a \int_0^1 \frac{dx}{\sqrt{1-x^4}}$. 20

证 $2rr' = -4a^2 \sin 2\theta, r' = -2a^2 \sin 2\theta / r,$

$$s = 4 \int_0^{\pi/4} \sqrt{2a^2 \cos 2\theta + \frac{4a^4 \sin^2 2\theta}{2a^2 \cos 2\theta}} d\theta$$

$$= 4\sqrt{2}a \int_0^{\pi/4} \frac{1}{\sqrt{\cos 2\theta}} d\theta$$

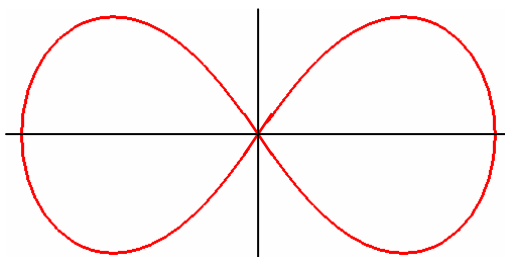
$$= 4\sqrt{2}a \int_0^{\pi/4} \frac{d\theta}{\sqrt{\cos^2 \theta - \sin^2 \theta}}$$

$$= 4\sqrt{2}a \int_0^{\pi/4} \frac{d\theta}{\sqrt{(\cos^2 \theta - \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta)}}$$

$$= 4\sqrt{2}a \int_0^{\pi/4} \frac{d\theta}{\sqrt{\cos^4 \theta - \sin^4 \theta}}$$

$$= 4\sqrt{2}a \int_0^{\pi/4} \frac{d \tan \theta}{\sqrt{1 - \tan^4 \theta}} (\tan \theta = x)$$

$$= 4\sqrt{2}a \int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$



21.求抛物线 $y = 1 + \frac{x^2}{4} (0 \leq x \leq 2)$ 绕 x 旋转所得的旋转体的侧面积.

解 $y' = \frac{x}{2}$.

$$S = 2\pi \int_0^2 \left(1 + \frac{x^2}{4}\right) \sqrt{1 + \left(\frac{x}{2}\right)^2} dx$$

$$= \frac{1}{4}\pi \int_0^2 \sqrt{4 + x^2}^3 dx = 4\pi \int_0^{\pi/4} \frac{dx}{\cos^5 x}$$

$$I_n = \int \sec^n x dx = \int \sec^{n-2} x d \tan x$$

$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan^2 x dx$$

$$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2},$$

$$I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}.$$

$$I_5 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} I_3 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \left(\frac{1}{2} \sec x \tan x + \frac{1}{2} I_1 \right) \\ = \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln(\tan x + \sec x) + C.$$

$$S = 4\pi \left(\frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln(\tan x + \sec x) \right) \Big|_0^{\pi/4} \\ = \frac{\pi}{2} [7\sqrt{2} + 3\ln(1+\sqrt{2})].$$

22. 求 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (0 < b \leq a)$ 分别绕长, 短轴旋转而成的椭球面的面积.

解 $\begin{cases} x = a \cos t \\ y = b \sin t \end{cases}, 0 \leq t \leq 2\pi, x' = -a \sin t, y' = b \cos t.$

$$S_a = 2\pi b \int_0^{\pi/2} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \sin t dt = \\ = -4\pi b \int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2) \cos^2 t} d \cos t = \\ = 4\pi b \int_0^1 \sqrt{a^2 - (a^2 - b^2) u^2} du \\ = 4\pi b \sqrt{a^2 - b^2} \int_0^1 \sqrt{\varepsilon^{-2} - u^2} du \\ = 4\pi ab \frac{\sqrt{a^2 - b^2}}{a} \left[\frac{u}{2} \sqrt{\varepsilon^{-2} - u^2} + \frac{\varepsilon^{-2}}{2} \arcsin \varepsilon u \right] \Big|_0^1 \\ = 2\pi ab \left(\sqrt{1 - \varepsilon^2} + \frac{\arcsin \varepsilon}{\varepsilon} \right). \\ S_b = 2\pi a \int_0^{\pi/2} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \cos t dt \\ = 4\pi a \int_0^{\pi/2} \sqrt{b^2 + (a^2 - b^2) \sin^2 t} d \sin t \\ = 4\pi a \int_0^1 \sqrt{b^2 + (a^2 - b^2) u^2} du \\ = 4\pi a \sqrt{a^2 - b^2} \int_0^1 \sqrt{\frac{b^2}{a^2 - b^2} + u^2} du \\ = 4\pi a \sqrt{a^2 - b^2} \left[\frac{u}{2} \sqrt{\frac{b^2}{a^2 - b^2} + u^2} + \frac{b^2}{2(a^2 - b^2)} \ln(u + \sqrt{\frac{b^2}{a^2 - b^2} + u^2}) \right] \Big|_0^1 \\ = 2\pi a^2 + \frac{2\pi b^2}{\varepsilon} \ln \left[\frac{a}{b} (1 + \varepsilon) \right].$$

23. 计算圆弧 $x^2 + y^2 = a^2 (a - h \leq y \leq a, 0 < h < a)$ 绕 y 轴旋转所得球冠的面积.

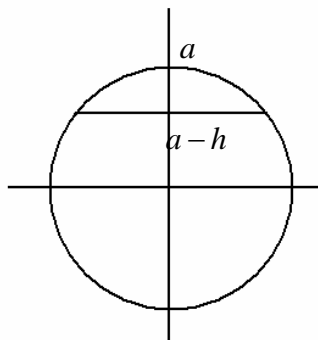
$$\text{解} \begin{cases} x = a \cos t \\ y = a \sin t \end{cases} \arcsin \frac{a-h}{a} \leq t \leq \frac{\pi}{2}.$$

$$S = 2\pi \int_{\arcsin \frac{a-h}{a}}^{\frac{\pi}{2}} x \sqrt{x'^2 + y'^2} dt$$

$$= 2\pi a^2 \int_{\arcsin \frac{a-h}{a}}^{\frac{\pi}{2}} \cos t dt$$

$$= \pi a^2 \left[\sin t \right]_{\arcsin \frac{a-h}{a}}^{\frac{\pi}{2}}$$

$$= 2\pi a^2 \left[1 - \frac{a-h}{a} \right] = 2\pi ah.$$



24. 求心脏线 $r = a(1 + \cos \theta)$ 绕极轴旋转所成的旋转体的侧面积.

$$\text{解} r' = -a \sin \theta.$$

$$S = 2\pi \int_0^\pi a(1 + \cos \theta) \sin \theta \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$$

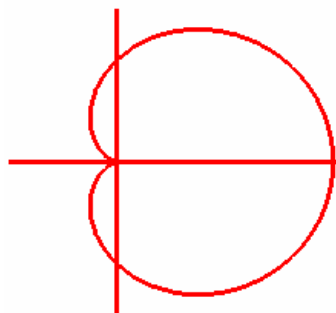
$$= 2\pi a^2 \sqrt{2} \int_0^\pi (1 + \cos \theta)^{3/2} \sin \theta d\theta$$

$$= -2\pi a^2 \sqrt{2} \int_0^\pi (1 + \cos \theta)^{3/2} d \cos \theta$$

$$= 2\pi a^2 \sqrt{2} \int_{-1}^1 (1+x)^{3/2} dx$$

$$= 2\pi a^2 \sqrt{2} \frac{2}{5} (1+x)^{5/2} \Big|_{-1}^1$$

$$= \frac{32}{5} \pi a^2.$$



25. 有一细棒长10m 已知距左端点x处的线密度是 $\rho(x) = (7 + 0.2x)$ kg/m 求这细棒的质量.

$$\text{解} m = \int_0^{10} (7 + 0.2x) dx = \left[7x + 0.1x^2 \right]_0^{10} = 80(\text{kg}).$$

26. 求半径为a的均匀半圆周的重心坐标.

$$\text{解} \text{由对称性, } x_0 = 0. \begin{cases} x = a \cos t \\ y = a \sin t \end{cases}, 0 \leq t \leq \pi$$

$$y_0 = \frac{\int_0^\pi a \sin t dt}{\pi a} = \frac{a}{\pi} [-\cos t]_0^\pi = \frac{2a}{\pi}.$$

$$\text{重心坐标} (0, \frac{2a}{\pi}).$$

27. 有一均匀细杆, 长为l. 质量为M. 计算细杆绕距离一端l/5处的转动惯量.

$$\text{解} \rho = M/l. J = \int_0^{l/5} \frac{M}{l} x^2 dx + \int_0^{4l/5} \frac{M}{l} x^2 dx$$

$$= \frac{M}{l} \frac{x^3}{3} \Big|_0^{l/5} + \frac{M}{l} \frac{x^3}{3} \Big|_0^{4l/5} = \frac{13}{75} Ml^2.$$

28. 设有一均匀圆盘, 半径为 a , 质量为 M , 求它对于通过其圆心且与盘垂直的轴之转动惯量.

$$\text{解 } \rho = \frac{M}{\pi a^2}, dm = \frac{M}{\pi a^2} 2\pi x dx = \frac{2Mx dx}{a^2}.$$

$$J = \int_0^a x^2 \frac{2Mx dx}{a^2} = \frac{2M}{a^2} \frac{x^4}{4} \Big|_0^a = \frac{1}{2} Ma^2.$$

29. 有一均匀的圆锥形陀螺, 质量为 M , 底半径为 a , 高为 h , 试求此陀螺关于其对称轴的转动惯量.

$$\text{解 } y = \frac{a}{h} x, \rho = \frac{M}{\frac{1}{3}\pi a^2 h} = \frac{3M}{\pi a^2 h}, dm = \rho \pi \left(\frac{a}{h} x\right)^2 dx = \frac{3M}{h^3} x^2 dx$$

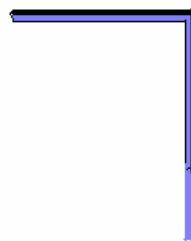
$$dJ = \frac{1}{2} dm \left(\frac{a}{h} x\right)^2 = \frac{1}{2} \frac{3a^2 M}{h^5} x^4 dx$$

$$J = \int_0^h \frac{1}{2} \frac{3a^2 M}{h^5} x^4 dx = \frac{1}{2} \frac{3a^2 M}{h^5} \frac{x^5}{5} \Big|_0^h = \frac{3}{10} Ma^2.$$

30. 楼顶有一绳索沿墙壁下垂, 该绳索的密度为 2kg/m . 若绳索下垂部分长为 5m , 求将下垂部分全部拉到楼顶所需做的功.

$$\text{解 } dW = 2g \cdot 8x dx.$$

$$W = \int_0^5 2g \cdot 8x dx = 9.8x^2 \Big|_0^5 = 25g \cdot 8(J).$$



31. 设 $y = f(x)$ 在 $[a, b]$ 上连续, 非负, 将由 $y = f(x)$, $x = a$, $x = b$ 及 x 轴围成的曲边梯形垂直放置于水中, 使 y 轴与水平面相齐, 求水对此曲边梯形的压力.

$$\text{解 } dS = f(x) dx, dF = \rho p dS = g \rho x f(x) dx,$$

$$F = g \rho \int_a^b x f(x) dx.$$

32. 一水闸门的边界线为一抛物线, 沿水平面的宽度为 48m , 最低处在水面下 64m , 求水对闸门的压力.

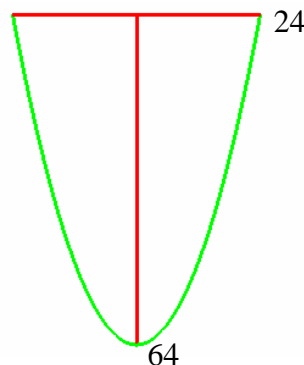
$$\text{解 } y = 64 - ax^2, 0 = 64 - ag24^2, a = \frac{1}{9}, x = \pm 3\sqrt{(64 - y)}.$$

$$F = 6g\rho \int_0^{64} y \sqrt{64 - y} dy. \sqrt{64 - y} = u, y = 64 - u^2,$$

$$y = 0 \text{ 时 } u = 8, y = 64 \text{ 时 } u = 0.$$

$$F = 6g\rho \int_0^8 (64 - u^2) u (2u) dy$$

$$= 12g\rho \left[64g \frac{u^3}{3} - \frac{u^5}{5} \right] \Big|_0^8 = 52428.8g\rho.$$



习题 3.6

1. 利用定积分近似计算 π 的值:

(1) 证明公式 $\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$;

(2) 令 $f(x) = \frac{1}{1+x^2}$, 给出 $|f'(x)|$ 在 $(0,1)$ 的上界;

(3) 在使用矩形法近似计算上述积分时, 欲使公式误差小于 5×10^{-5} , 应取矩形法中分点个数 $n > ?$

(4) 用电脑计算 π 到小数点后4位.

解 (1) $\int_0^1 \frac{dx}{1+x^2} = \arctan x \Big|_0^1 = \frac{\pi}{4}$.

(2) $f'(x) = -\frac{2x}{(1+x^2)^2}$, $|f'(x)| \leq 2$.

(3) $|R_n| \leq \frac{1}{2n} \cdot 2 = \frac{1}{n} < 5 \times 10^{-5}$, $n > \frac{1}{5 \times 10^{-5}} = 2 \times 10^4$.

(4) $n:=2.0*10^4$; assume(m, integer): J:=4*sum(1/(1.0+(m/n)^2), m=1..n)/n;
J := 3.141542654 + 0. I

2. 自河的一岸开始沿河的横截面方向, 每隔5m测量一次水深, 一直测到河对岸, 依次得到如下21个数据(单位:m)

0, 0.9, 1.2, 3.5, 2.8, 4.6, 8.8, 7.5, 9.6, 12.1, 13.8,
20.1, 18.2, 15.6, 11.9, 9.2, 7.6, 5.3, 4.5, 2.7, 0.

假定河宽为100m. 试用simpson法计算河床的横截面面积.

解 $n:=10$; $d:=[0,0.9,1.2,3.5,2.8,4.6,8.8,7.5,9.6,12.1,13.8,20.1,18.2,15.6,11.9,9.2,7.6,5.3,4.5,2.7,0]$;

$S:=((100)/(6*n))*(d[1]+d[21]+2*sum(d[2*i+1], i=1..n-1)+4*sum(d[2*i], i=1..n))$;
(Maple程序)

$S := 804.6666667$

第三章总练习题

1. 为什么用Newton-Leibniz公式于下列积分会得到不正确结果?

$$(1) \int_{-1}^1 \frac{d}{dx} \left(e^{\frac{1}{x}} \right) dx = \frac{d}{dx} \left(e^{\frac{1}{x}} \right) = - \left(e^{\frac{1}{x}} \right) \frac{1}{x^2} [-1, 1] \text{ 无界, 从而不可积.}$$

$$(2) \int_0^{2\pi} \frac{d \tan x}{2 + \tan^2 x} dx, u = \tan x \text{ 在 } (0, 2\pi) \text{ 的一些点不可导.}$$

2. 证明奇连续函数的原函数为偶函数, 而偶连续函数的原函数之一为奇函数.

证设奇连续函数 f 的原函数为 F , 现在证明 F 是偶函数.

$$F'(x) = f(x), (F(-x) - F(x))' = -F'(-x) - F'(x) = -f(-x) - f(x) = 0,$$

$$F(-x) - F(x) = C, C = F(-0) - F(0) = 0, F(-x) - F(x) = 0.$$

设偶连续函数 f 的原函数为 F , 现在证明 F 是奇函数.

$$F'(x) = f(x), (F(-x) + F(x))' = -F'(-x) + F'(x) = -f(-x) + f(x) = 0,$$

$$F(-x) + F(x) = C. \text{ 设 } F(0) = 0, \text{ 则 } C = F(-0) + F(0) = 0, F(-x) + F(x) = 0.$$

$$3. f(x)f(x) = \begin{cases} \sin x, & x \geq 0, \\ x^3, & x < 0, \end{cases} \text{ 求定积分 } \int_a^b f(x) dx = ? \text{ 其中 } a < 0, b > 0.$$

$$\text{解 } \int_a^b f(x) dx = \int_a^0 f(x) dx + \int_0^b f(x) dx = \int_a^0 x^3 dx + \int_0^b \sin x dx$$

$$= \frac{x^4}{4} \Big|_0^a - \cos x \Big|_0^b = 1 + \frac{a^4}{4} - \cos b.$$

$$4. \text{求微商 } \frac{d}{dx} \int_0^1 \sin(x+t) dx.$$

$$\text{解 } \frac{d}{dx} \int_0^1 \sin(x+t) dx = \frac{d}{dx} \int_x^{x+1} \sin(u) du = \sin(x+1) - \sin(x).$$

5. 试证明 $\lim_{h \rightarrow 0} \int_0^1 f(x+ht) dx = f(x)$, 其中 $f(x)$ 是实轴上的连续函数.

$$\text{证 } \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(x+ht) du = \left(\int_0^u f(t) dt \right)' \Big|_{u=x} = f(x).$$

$$6. \text{求极限 } \lim_{n \rightarrow \infty} \int_0^1 (1-x^2)^n dx.$$

$$\text{解 } \int_0^1 (1-x^2)^n dx = \int_0^{\pi/2} \cos^{2n+1} t dt = I_{2n+1} = \frac{(2n)!!}{(2n+1)!!}.$$

$$(I_{2n+1})^2 < \frac{2(2n)!!}{(2n+1)!!} \frac{(2n+1)!!}{(2n+2)!!} = \frac{1}{n+1},$$

$$0 < I_{2n+1} < \frac{1}{\sqrt{n+1}} \rightarrow 0 (n \rightarrow \infty), \lim_{n \rightarrow \infty} \int_0^1 (1-x^2)^n dx = 0.$$

$$7. \int \frac{\sin x + \cos x}{2 \sin x - 3 \cos x} dx.$$

$$\text{解令 } \sin x + \cos x = A(2 \sin x - 3 \cos x) + B(2 \sin x - 3 \cos x)'$$

$$= A(2 \sin x - 3 \cos x) + B(2 \cos x + 3 \sin x) = (2A + 3B) \sin x + (-3A + 2B) \cos x,$$

$$\begin{cases} 2A+3B=1 \\ -3A+2B=1 \end{cases}, A=-\frac{1}{13}, B=\frac{5}{13}.$$

$$\begin{aligned} & \int \frac{\sin x + \cos x}{2 \sin x - 3 \cos x} dx = \\ & = \int \frac{A(2 \sin x - 3 \cos x) + B(2 \sin x - 3 \cos x)'}{2 \sin x - 3 \cos x} dx \\ & = Ax + B \ln |2 \sin x - 3 \cos x| + C \\ & = -\frac{1}{13}x + \frac{5}{13} \ln |2 \sin x - 3 \cos x| + C. \end{aligned}$$

8.通过适当的有理化或变量替换求下列积分:

$$(1) \int \sqrt{e^x - 2} dx. \sqrt{e^x - 2} = u, x = \ln(2 + u^2), dx = \frac{2u du}{2 + u^2}.$$

$$\begin{aligned} \int \sqrt{e^x - 2} dx &= 2 \int \frac{u^2 du}{2 + u^2} = 2 \left(u - 2 \int \frac{du}{2 + u^2} \right) \\ &= 2 \left(u - \sqrt{2} \arctan \frac{u}{\sqrt{2}} \right) + C = 2 \left(\sqrt{e^x - 2} - \sqrt{2} \arctan \frac{\sqrt{e^x - 2}}{\sqrt{2}} \right) + C. \end{aligned}$$

$$\begin{aligned} (2) \int \frac{x e^x}{\sqrt{e^x - 2}} dx &= \int \frac{x}{\sqrt{e^x - 2}} d(e^x - 2) = 2 \int x d\sqrt{e^x - 2} \\ &= 2x\sqrt{e^x - 2} - 2 \int \sqrt{e^x - 2} dx \\ &= 2x\sqrt{e^x - 2} - 4 \left(\sqrt{e^x - 2} - \sqrt{2} \arctan \frac{\sqrt{e^x - 2}}{\sqrt{2}} \right) + C. \\ &= 2\sqrt{e^x - 2}(x - 2) + 4\sqrt{2} \arctan \frac{\sqrt{e^x - 2}}{\sqrt{2}} + C. \end{aligned}$$

$$\begin{aligned} (3) \int \sqrt{\frac{x}{1 - x\sqrt{x}}} dx &= \frac{2}{3} \int \frac{dx\sqrt{x}}{\sqrt{1 - x\sqrt{x}}} = -\frac{2}{3} \times 2\sqrt{1 - x\sqrt{x}} + C \\ &= -\frac{4}{3}\sqrt{1 - x\sqrt{x}} + C. \end{aligned}$$

$$\begin{aligned} (4) \int \frac{dx}{1 + \sqrt{x} + \sqrt{1 + x}} &= \int \frac{(1 + \sqrt{x} - \sqrt{1 + x})dx}{(1 + \sqrt{x} + \sqrt{1 + x})(1 + \sqrt{x} - \sqrt{1 + x})} \\ &= \int \frac{(1 + \sqrt{x} - \sqrt{1 + x})dx}{2\sqrt{x}} = \frac{1}{2} \left(2\sqrt{x} + x - \sqrt{x(1 + x)} + \ln(\sqrt{x} + \sqrt{1 + x}) \right) + C. \end{aligned}$$

$$\begin{aligned}
9. \int \frac{dx}{\sin^4 x + \cos^4 x} &= \int \frac{\sec^2 x d \tan x}{1 + \tan^4 x} = \int \frac{(1+u^2)du}{1+u^4}. \\
\frac{1+u^2}{1+u^4} &= \frac{1+u^2}{(1+u^2+\sqrt{2}u)(1+u^2-\sqrt{2}u)} = \frac{1}{2} \left(\frac{1}{1+u^2+\sqrt{2}u} + \frac{1}{1+u^2-\sqrt{2}u} \right) \\
&= \frac{1}{2} \left(\frac{1}{\left(u+\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} + \frac{1}{\left(u-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \right). \\
\int \frac{dx}{\sin^4 x + \cos^4 x} &= \frac{1}{\sqrt{2}} \left(\arctan(\sqrt{2}u+1) + \arctan(\sqrt{2}u-1) \right) + C.
\end{aligned}$$

10. 设函数 $f(x)$ 在 $(-\infty, +\infty)$ 上连续, 以 T 为周期, 令 $g(x) = f(x) - \frac{1}{T} \int_0^T f(x) dx$, 证明:

函数 $h(x) = \int_0^x g(t) dt$ 也以 T 为周期.

证(此即习题3.4第24题)

11. 设函数 $f(x)$ 在区间 $[a, b]$ 上连续, 且 $\int_a^b f(x) dx = 0$. 证明: 在 (a, b) 内至少存在一点 c ,

使 $f(c) = 0$.

证 若不然, $f(x)$ 在 (a, b) 没有零点, 由 f 的连续性和连续函数的中间值定理,

f 在 (a, b) 不变号. 不妨设 $f(x) > 0, x \in (a, b)$. 取 c, d 满足, $a < c < d < b$, 则 f 在 $[c, d]$ 取最小值 $m > 0$. 于是

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^d f(x) dx + \int_d^b f(x) dx \geq m(d-c) > 0.$$

矛盾.

12. 设函数 f 在区间 $[a, b]$ 上连续, 且 $\int_a^b f^2(x) dx = 0$, 证明: $f(x) \equiv 0, x \in [a, b]$.

证 若不然, 存在 $c \in [a, b], f(c) \neq 0$. 由 f 在 c 的连续性, 存在区间 $[d, e] \subseteq [a, b]$,

$$|f(x)|^2 > \frac{|f(c)|^2}{2}, x \in [d, e].$$

$$\int_a^b f^2(x) dx \geq \int_d^e f^2(x) dx > \frac{|f(c)|^2}{2} (d-e) > 0.$$

矛盾.

13. 设 $f(x)$ 在 $(-\infty, +\infty)$ 上可积, 证明

(1) 对于任意实数 a , 有 $\int_0^a f(x) dx = \int_0^a f(a-x) dx$;

$$(2) \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4};$$

$$(3) \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx = \frac{1}{\sqrt{2}} \ln(1 + \sqrt{2}).$$

证 (1) $\int_0^a f(x) dx (x = a-t) = -\int_a^0 f(a-t) dt = \int_0^a f(a-t) dt = \int_0^a f(a-x) dx$.

$$(2) I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = - \int_0^{\pi} \frac{(x - \pi) \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx - I,$$

$$I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x} = - \frac{\pi}{2} \int_0^{\pi} \frac{d \cos x}{1 + \cos^2 x} = \int_0^1 \frac{\pi du}{1 + u^2} = \pi \arctan u \Big|_0^1 = \frac{\pi^2}{4}.$$

$$(3) I = \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x)}{\cos(\pi/2 - x) + \sin(\pi/2 - x)} dx \\ = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx, 2I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx + \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx \\ = \int_0^{\pi/2} \frac{dx}{\cos x + \sin x} = \int_0^{\pi/2} \frac{dx}{\sqrt{2} \sin(x + \pi/4)} = \frac{1}{\sqrt{2}} \ln |\csc(x + \pi/4) - \cot(x + \pi/4)| \Big|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} \left[\ln \left(\frac{1}{\cos \frac{\pi}{4}} + 1 \right) \right] - \ln \left(\frac{1}{\sin \frac{\pi}{4}} - 1 \right) = \sqrt{2} \ln(\sqrt{2} + 1),$$

$$I = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1).$$

14. 一质点作直线运动, 其加速度 $a(t) = (2t - 3)\text{m/s}^2$. 若 $t = 0$ 时 $x = 0$ 且 $v = -4\text{m/s}$, 求

(1) 质点改变运动方向的时刻;

(2) 头5秒钟内质点所走的总路程.

$$\text{解}(1) x''(t) = 2t - 3, x' = t^2 - 3t + C_1, -4 = C_1, x' = t^2 - 3t - 4, x = \frac{t^3}{3} - \frac{3}{2}t^2 - 4t + C_2,$$

$$0 = C_2, x(t) = \frac{t^3}{3} - \frac{3}{2}t^2 - 4t, x' = t^2 - 3t - 4 = (t - 4)(t + 1) = 0, t_0 = 4.$$

$$s = x(5) - x(4) + |x(4)| = \left(\frac{t^3}{3} - \frac{3}{2}t^2 - 4t \right) \Big|_{t=5} - 2 \left(\frac{t^3}{3} - \frac{3}{2}t^2 - 4t \right) \Big|_{t=4} = \frac{43}{2} \text{m}.$$

15. 一运动员跑完100m, 共用了10.2s, 在跑头25m时以等加速度进行, 然后保持等速运动跑完了剩余路程. 求跑头25m时的加速度.

$$\text{解} v(t) = \begin{cases} at, & 0 \leq t \leq t_0; \\ at_0, & t_0 \leq t \leq 10.2. \end{cases}$$

$$s(t) = \begin{cases} \frac{at^2}{2}, & 0 \leq t \leq t_0; \\ at_0 t + C, & t_0 \leq t \leq 10.2. \end{cases}$$

$$\begin{cases} at_0^2/2 = at_0^2 + C \\ at_0^2/2 = 25 & a \approx 3\text{m/s}^2. \\ 100 = 10.2at_0 + C_2 \end{cases}$$

16.(1) 利用积分的几何意义证明:

$$\frac{1}{n+1} < \ln \frac{n+1}{n} < \frac{1}{n}, n=1, 2, \dots$$

$$(2) \text{ 令 } x_n = 1 + \frac{1}{2} + \frac{1}{n-1} - \ln n,$$

$$y_n = 1 + \frac{1}{2} + \frac{1}{n-1} + \frac{1}{n} - \ln n,$$

证明序列 x_n 单调上升, 而序列 y_n 单调下降.

(3) 证明极限 $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{n-1} + \frac{1}{n} - \ln n \right)$ 存在 (此极限称为 Euler 常数).

$$\text{证 (1)} \quad \frac{1}{n+1} = \int_n^{n+1} \frac{dx}{n+1} < \int_n^{n+1} \frac{dx}{x} = \ln x \Big|_n^{n+1}$$

$$= \ln(n+1) - \ln n = \ln \frac{n+1}{n} < \int_n^{n+1} \frac{dx}{n} = \frac{1}{n}.$$

$$(2) x_{n+1} - x_n = \left(1 + \frac{1}{2} + \frac{1}{n} - \ln(n+1) \right) - \left(1 + \frac{1}{2} + \frac{1}{n-1} - \ln n \right)$$

$$= \frac{1}{n} - \ln \left(1 + \frac{1}{n} \right) > 0 \text{ (由(1))}.$$

$$y_{n+1} - y_n = \left(1 + \frac{1}{2} + \frac{1}{n} + \frac{1}{n+1} - \ln(n+1) \right) - \left(1 + \frac{1}{2} + \frac{1}{n} - \ln n \right)$$

$$= \frac{1}{n+1} - \ln \left(1 + \frac{1}{n} \right) < 0 \text{ (由(1))}.$$

(3) $y_n > x_n > x_2 = 1 - \ln 2 > 0 (n > 2)$, y_n 单调下降有下界, 故有极限 $\lim_{n \rightarrow \infty} y_n$.

17. 证明: 当 $x > 0$ 时,

$$\int_x^1 \frac{1}{1+t^2} dt = \int_1^{1/x} \frac{1}{1+t^2} dt.$$

$$\text{证} \quad \int_x^1 \frac{1}{1+t^2} dt (x = 1/u) = \int_1^{1/x} \frac{1}{1+1/u^2} \times \frac{1}{u^2} dx = \int_1^{1/x} \frac{1}{1+t^2} dt.$$

18. 设 $f(x)$ 在 $(-\infty, +\infty)$ 上连续 (书上为可积, 欠妥), 且对一切实数 x , 均有

$$f(2-x) = -f(x). \text{ 求实数 } a \neq 2, \text{ 使 } \int_a^2 f(x) dx = 0.$$

解 (条件 $f(2-x) = -f(x)$ 相当 f 关于 $x=1$ 为奇函数 $f(1+1-x) = -f(1+x-1)$)

$$\int_0^2 f(x) dx = \int_0^2 f(2-u) du = -\int_0^2 f(u) du, \int_0^2 f(x) dx = 0. \text{ 取 } a=0 \text{ 即可}.$$

19. 利用定积分的性质, 证明不等式 $\ln(1+x) \leq \arctan x, 0 \leq x \leq 1$.

$$\text{证} \quad \frac{1}{1+t} \leq \frac{1}{1+t^2}, t \in [0, 1], \text{ 在 } [0, x] \text{ 上积分得 } \int_0^x \frac{dt}{1+t} \leq \int_0^x \frac{dt}{1+t^2},$$

$$\ln(1+x) \leq \arctan x, 0 \leq x \leq 1.$$

$$20. (1) \text{ 设 } f(x) \text{ 在 } [0, a] \text{ 上可积, 证明 } \int_0^a \frac{f(x) dx}{f(x) + f(a-x)} dx = \frac{a}{2};$$

(2) 利用 (1) 中的公式求下列积分的值:

$$\int_0^2 \frac{x^2}{x^2 - 2x + 2} dx; \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

$$\text{证(1)} I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx = \int_0^a \frac{f(a-u)}{f(u) + f(a-u)} du$$

$$\begin{aligned} 2I &= \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx + \int_0^a \frac{f(a-u)}{f(u) + f(a-u)} du \\ &= \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx + \int_0^a \frac{f(a-x)}{f(x) + f(a-x)} dx = \int_0^a 1 dx = a, I = \frac{a}{2}. \end{aligned}$$

$$\text{解(2)} \int_0^2 \frac{x^2}{x^2 - 2x + 2} dx = 2 \int_0^2 \frac{x^2}{x^2 + (2-x)^2} dx = 2 \times \frac{2}{2} = 2.$$

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \sin(\pi/2 - x)} dx = \frac{\pi/2}{2} = \frac{\pi}{4}.$$

$$21. \text{设 } f(x) = \int_{\sin x}^{\tan x} (1 + xt^2) dt \text{ 求 } \frac{df(x)}{dx}.$$

$$\text{解 } f(x) = \int_{\sin x}^{\tan x} (1 + xt^2) dt = \tan x - \sin x + x \int_{\sin x}^{\tan x} t^2 dt,$$

$$\frac{df(x)}{dx} = \sec^2 x - \cos x + x \tan^2 x \sec^2 x - x \sin^2 x \cos x + \int_{\sin x}^{\tan x} t^2 dt$$

$$= \sec^2 x - \cos x + x \tan^2 x \sec^2 x - x \sin^2 x \cos x + \frac{t^3}{3} \Big|_{\sin x}^{\tan x}$$

$$= \sec^2 x - \cos x + x \tan^2 x \sec^2 x - x \sin^2 x \cos x + \frac{1}{3} (\tan^3 x - \sin^3 x)$$

$$= \sec^2 x (1 + x \tan^2 x) - \cos x (1 + x \sin^2 x) + \frac{1}{3} (\tan^3 x - \sin^3 x).$$

$$22. \text{求定积分 } I = \int_0^{\pi/2} \cos^2 3\theta d\theta \text{ 的值.}$$

$$\text{解 } I = \int_0^{\pi/2} \cos^2 3\theta d\theta = \frac{1}{2} \int_0^{\pi/2} (1 + \cos 6\theta) d\theta = \frac{\pi}{4} + \frac{1}{12} \sin 6\theta \Big|_0^{\pi/2} = \frac{\pi}{4}.$$

$$23. \text{求定积分 } I = \int_0^{2\pi} |\sin x - \cos x| dx \text{ 的值.}$$

$$\text{解 } I = 2 \int_0^{\pi} |\sin x - \cos x| dx$$

$$= 2 \left(\int_0^{\pi/2} |\sin x - \cos x| dx + \int_{\pi/2}^{\pi} |\sin x - \cos x| dx \right)$$

$$= 2 \left(\int_0^{\pi/2} |\sin x - \cos x| dx + \int_0^{\pi/2} \left| \sin\left(\frac{\pi}{2} + t\right) - \cos\left(\frac{\pi}{2} + t\right) \right| dx \right)$$

$$= 2 \left(\int_0^{\pi/2} |\sin x - \cos x| dx + \int_0^{\pi/2} |\cos t + \sin t| dx \right)$$

$$= 2 \left(\int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx + \int_0^{\pi/2} (\cos t + \sin t) dx \right)$$

$$= 2 \left((\sin x + \cos x) \Big|_0^{\pi/4} + (-\cos x - \sin x) \Big|_{\pi/4}^{\pi/2} + (\sin x - \cos x) \Big|_0^{\pi/2} \right) = 4\sqrt{2}.$$

24. 设 $0 < x_0 < x_1$, 求定积分 $I = \int_{x_0}^{x_1} \sqrt{(x-x_0)(x_1-x)} dx$ 的值.

$$\begin{aligned}
 \text{解 } I &= \int_{x_0}^{x_1} \sqrt{(x-x_0)(x_1-x)} dx \\
 &= \int_{x_0}^{x_1} \sqrt{-x^2 + (x_1+x_0)x - x_0x_1} dx \\
 &= \int_{x_0}^{x_1} \sqrt{-\left(x - \frac{x_1+x_0}{2}\right)^2 + \frac{(x_1+x_0)^2}{4} - x_0x_1} dx \\
 &= \int_{x_0}^{x_1} \sqrt{-\left(x - \frac{x_1+x_0}{2}\right)^2 + \frac{(x_1-x_0)^2}{4}} dx \left(u = x - \frac{x_1+x_0}{2} \right) \\
 &= \int_{-(x_1-x_0)/2}^{(x_1-x_0)/2} \sqrt{-(u)^2 + \frac{(x_1-x_0)^2}{4}} du \\
 &= 2 \int_0^a \sqrt{a^2 - u^2} dx \left(a = \frac{x_1-x_0}{2} \right) \\
 &= \left[u\sqrt{a^2 - u^2} + a^2 \arcsin \frac{u}{a} \right]_0^a \\
 &= \frac{\pi a^2}{2} = \frac{\pi(x_1-x_0)^2}{8}.
 \end{aligned}$$

25. 求下列曲线所围图形的面积:

(1) $y = x^2 - 6x + 8$ 与 $y = 2x - 7$.

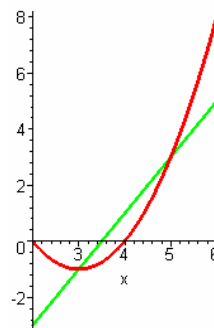
$$\text{解 } \begin{cases} y = x^2 - 6x + 8 \\ y = 2x - 7 \end{cases} \quad 2x - 7 = x^2 - 6x + 8,$$

$$x^2 - 8x + 15 = 0, (x-3)(x-5) = 0.$$

$$x_1 = 3, x_2 = 5.$$

$$S = \int_3^5 (2x - 7 - (x^2 - 6x + 8)) dx = \int_3^5 (-x^2 + 8x - 15) dx$$

$$= \left(-\frac{x^3}{3} + 4x^2 - 15x \right) \Big|_3^5 = \frac{4}{3}.$$



(2) $y = x^4 + x^3 + 16x - 4$ 与 $y = x^4 + 6x^2 + 8x - 4$.

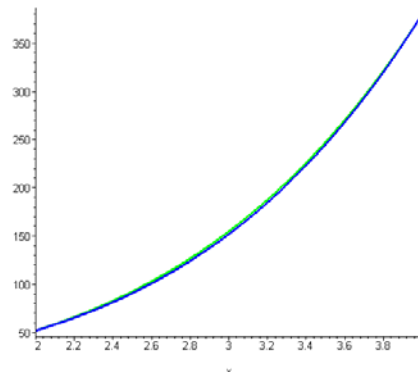
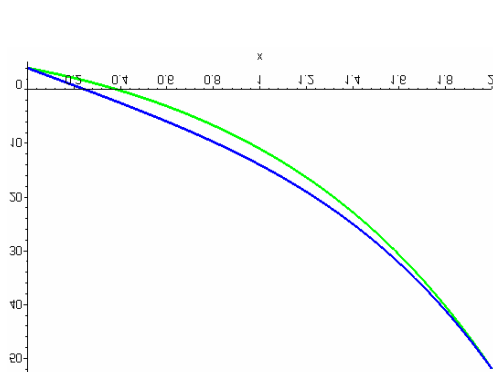
$$\text{解 } \begin{cases} y = x^4 + x^3 + 16x - 4 \\ y = x^4 + 6x^2 + 8x - 4 \end{cases} \quad x^3 + 16x - 4 = 6x^2 + 8x - 4, x^3 - 6x^2 + 8x = 0,$$

$$x = 0, x^2 - 6x + 8 = 0, (x-2)(x-4) = 0, x = 2, 4.$$

$$S = \int_0^2 [(x^4 + x^3 + 16x - 4) - (x^4 + 6x^2 + 8x - 4)] dx$$

$$+ \int_2^4 [(x^4 + 6x^2 + 8x - 4) - (x^4 + x^3 + 16x - 4)] dx$$

$$\begin{aligned}
 &= \int_0^2 (x^3 - 6x^2 + 8x) dx + \int_2^4 (-x^3 + 6x^2 - 8x) dx \\
 &= \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 + \left[-\frac{x^4}{4} + 2x^3 - 4x^2 \right]_2^4 = 8.
 \end{aligned}$$



(3) $y^2 = x - 1$ 与 $y = x - 3$.

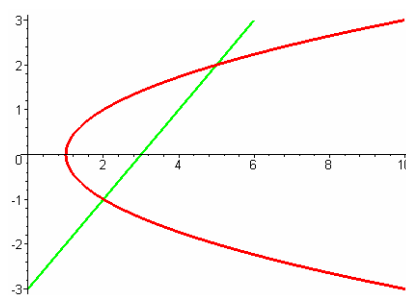
解 $(x - 3)^2 = x - 1, x^2 - 7x + 10 = 0,$

$(x - 2)(x - 5) = 0,$

$x = 2, 5, y = -1, 2.$

$$S = \int_{-1}^2 [(y + 3) - (1 + y^2)] dy$$

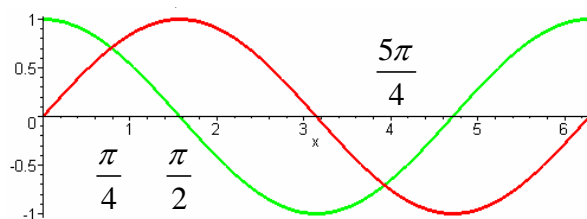
$$= \left(-\frac{y^3}{3} + \frac{y^2}{2} + 2y \right) \Big|_{-1}^2 = \frac{9}{2}.$$



(4) $y = \sin x, y = \cos x$ 与 $x = \pi/2$.

解 $S = \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx = (-\cos x - \sin x) \Big|_{\pi/4}^{\pi/2} = \sqrt{2} - 1;$

$S = \int_{\pi/2}^{5\pi/4} (\sin x - \cos x) dx = (-\cos x - \sin x) \Big|_{\pi/2}^{5\pi/4} = \sqrt{2} + 1.$



26. 设区域 σ 由曲线 $y = \cos x, y = 1$ 及 $x = \pi/2$ 所围成, 将 σ 绕 x 轴旋转一周, 得一旋转体 V . 试用两种不同的积分表示体积 V , 并且求 V 的值.

解 $V = \pi \int_0^{\pi/2} (1 - \cos^2 x) dx = 2\pi \int_0^1 y \left(\frac{\pi}{2} - \arccos y \right) dy = 2\pi \int_0^1 y \arcsin y dy =$

$$V = \frac{\pi}{2} \int_0^{\pi/2} (1 - \cos 2x) dx = \frac{\pi^2}{4}.$$

$$V = 2\pi \int_0^1 y \arcsin y dy = \pi \int_0^1 \arcsin y dy^2$$

$$= \pi \arcsin y (y^2) \Big|_0^1 - \pi \int_0^1 y^2 \times \frac{1}{\sqrt{1-y^2}} dx$$

$$= \frac{\pi^2}{2} - \pi \left[\frac{y}{2} \sqrt{1-y^2} + \frac{1}{2} \arcsin y \right] \Big|_0^1 = \frac{\pi^2}{2} - \frac{\pi^2}{4} = \frac{\pi^2}{4}.$$

27.求下列定积分的值:

$$(1) \int_{\sqrt{2}}^2 \frac{du}{u\sqrt{u^2-1}} = \int_{\sqrt{2}}^2 \frac{du}{u^2\sqrt{1-1/u^2}} = \int_{1/2}^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{1/2}^{1/\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

$$(2) \int_{-200}^{200} (91x^{21} - 80x^{33} + 5580x^{97} + 1) dx = 400.$$

28.设 $f(x)$ 在 $[0, 7]$ 上可积,且一直已知 $\int_0^2 f(x)dx = 5, \int_2^5 f(x)dx = 6, \int_0^7 f(x)dx = 3$.

(1)求 $\int_0^5 f(x)dx$ 的值;

(2)求 $\int_5^7 f(x)dx$ 的值.

(3)证明:在 $(5, 7)$ 内至少存在一点,使 $f(x) < 0$.

$$\text{解}(1) \int_0^5 f(x)dx = \int_0^2 f(x)dx + \int_2^5 f(x)dx = 5 + 6 = 11.$$

$$(2) \int_5^7 f(x)dx = \int_0^7 f(x)dx - \int_0^5 f(x)dx = 3 - 11 = -8.$$

证(3)若不然, $f(x) \geq 0, x \in (5, 7)$,

$$\int_5^7 f(x)dx \geq 0, \text{但是} \int_5^7 f(x)dx = -8 < 0, \text{矛盾}.$$

29.设 $f(x) = \sin x, h(x) = \frac{1}{x^2}, g(x) = \begin{cases} 1, & -\pi \leq x \leq 2, \\ 2, & 2 < x \leq \pi. \end{cases}$ 试求下列定积分的值或表达式:

$$(1) \int_{-\pi/2}^{\pi/2} f(x)g(x)dx; (2) \int_1^3 g(x)h(x)dx; (3) \int_{\pi/2}^x f(t)g(t)dx.$$

$$\text{解}(1) \int_{-\pi/2}^{\pi/2} f(x)g(x)dx = \int_{-\pi/2}^{\pi/2} \sin x dx = 0.$$

$$(2) \int_1^3 g(x)h(x)dx = \int_1^2 g(x)h(x)dx + \int_2^3 g(x)h(x)dx$$

$$= \int_1^2 \frac{1}{x^2} dx + \int_2^3 \frac{2}{x^2} dx = -\frac{1}{x} \Big|_1^2 - \frac{2}{x} \Big|_2^3 = \frac{5}{6}.$$

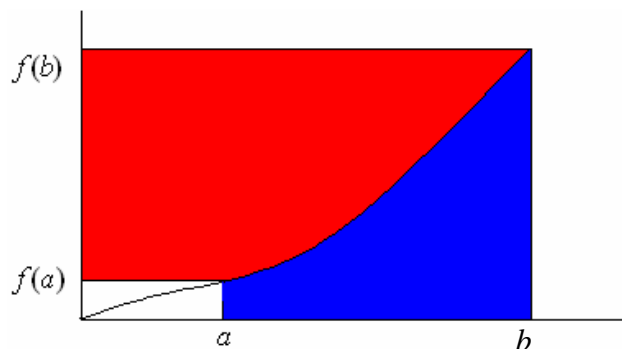
$$(3) \int_{\pi/2}^x f(t)g(t)dx = \begin{cases} \int_{\pi/2}^x \sin t dt = -\cos x, & t - \pi \leq x \leq 2 \\ \int_{\pi/2}^2 \sin t dt + \int_2^x 2 \sin t dt = \cos 2 - 2 \cos x, & 2 < x \leq \pi. \end{cases}$$

30.设函数 $f(x)$ 在区间 $[a, b]$ 上连续,严格单调递增($a > 0$), $g(y)$ 是 $f(x)$ 的反函数,利用定积分的几何意义证明下列公式

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy.$$

并作图解释这一公式.

解



31.(1) 设函数 $\varphi(x)$ 在 $[0, +\infty)$ 上连续且严格单调递增, 又 $\lim_{x \rightarrow +\infty} \varphi(x) = +\infty$ 且 $\varphi(0)=0$. 证明: 对于任意实数 $a \geq 0, B \geq 0$, 下列不等式成立:

$$aB \leq \int_0^a \varphi(x) dx + \int_0^B \varphi^{-1}(x) dx$$

其中 $\varphi^{-1}(x)$ 是 $\int_0^a \varphi(x)$ 的反函数.

证由30题, $\int_0^a \varphi(x) dx + \int_0^{\varphi(a)} \varphi^{-1}(x) dx = a\varphi(a)$ (*).

$B=0$ 时不等式显然成立. 设 $B > 0 = \varphi(0)$, 由于 $x \rightarrow +\infty$ 时 $\varphi(x) \rightarrow +\infty$, 存在 $a' > 0$, $\varphi(a') > B$, φ 在 $[0, a']$ 连续, 根据连续函数的中间值定理, 存在 $a_1 > 0, \varphi(a_1) = B$.

若 $a_1 = a$, 则由 (*) 得 $aB = \int_0^a \varphi(x) dx + \int_0^B \varphi^{-1}(x) dx$.

$$\begin{aligned} \text{若 } a_1 > a, \text{ 则 } & \int_0^a \varphi(x) dx + \int_0^B \varphi^{-1}(x) dx \\ &= \int_0^a \varphi(x) dx + \int_0^{\varphi(a)} \varphi^{-1}(x) dx + \int_{\varphi(a)}^B \varphi^{-1}(x) dx \\ &= a\varphi(a) + \int_{\varphi(a)}^B \varphi^{-1}(x) dx \end{aligned}$$

$$\geq a\varphi(a) + \varphi^{-1}(\varphi(a))(B - \varphi(a)) = aB.$$

$$\begin{aligned} \text{若 } a_1 < a, \text{ 则 } & \int_0^a \varphi(x) dx + \int_0^B \varphi^{-1}(x) dx \\ &= \int_0^a \varphi(x) dx + \int_0^{\varphi(a)} \varphi^{-1}(x) dx - \int_B^{\varphi(a)} \varphi^{-1}(x) dx \\ &= a\varphi(a) - \int_B^{\varphi(a)} \varphi^{-1}(x) dx \end{aligned}$$

$$\geq a\varphi(a) - \varphi^{-1}(\varphi(a))(\varphi(a) - B) = aB.$$

(2) 利用(1)中的不等式, 对于任意实数 $a, b \geq 0, p, q \geq 1, \frac{1}{p} + \frac{1}{q} = 1$, 证明下列Minkowski

$$\text{不等式 } ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

证不妨设 $p > 1$. 在(1)中取 $\varphi(x) = x^{p-1}$, 则 $\varphi^{-1}(x) = x^{1/(p-1)}$.

$$ab \leq \int_0^a x^p dx + \int_0^b x^{1/p} dx = \frac{a^p}{p} + \frac{b^{1/(p-1)+1}}{1/(p-1)+1} = \frac{a^p}{p} + \frac{b^{p/(p-1)}}{p/(p-1)} = \frac{a^p}{p} + \frac{b^q}{q}.$$

32. 设 $a > 0$, 求 a 的值, 使由曲线 $y = 1 + \sqrt{x}e^{x^2}$, $y = 1$ 及 $x = a$ 所围成的区域绕直线 $y = 1$ 旋转所得之旋转体的体积等于 2π .

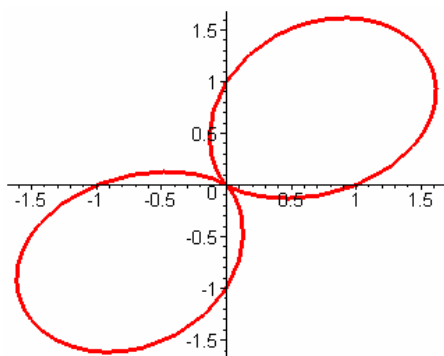


解 $\pi \int_0^a (y-1)^2 dx = 2\pi \cdot \int_0^a (\sqrt{x}e^{x^2})^2 dx = 2,$

$\int_0^a xe^{2x^2} dx = 2, \frac{1}{4} \int_0^a e^{2x^2} d2x^2 = 2, \frac{1}{4} \int_0^{2a^2} e^u du = 2, e^{2a^2} - 1 = 8, 2a^2 = \ln 9, a = \sqrt{\ln 3}.$

33. 作由极坐标方程 $r = 1 + \sin 2\theta$ 所确定的函数的图形, 并求它所围区域的面积.

解 $S = \int_0^\pi (1 + \sin 2\theta)^2 d\theta = \int_0^\pi (1 + 2\sin 2\theta + \frac{1 - \cos 4\theta}{2}) d\theta = \frac{3\pi}{2}.$



习题 4.1

1. 验证函数 $f(x) = x^3 - 3x^2 + 2x$ 在区间 $[0, 1]$ 及 $[1, 2]$ 上满足 Rolle 定理的条件并分别求出导数为 0 的点.

解 f 处处可导, $f(0) = f(1) = f(2) = 0$, 故 $f(x)$ 在区间 $[0, 1]$ 及 $[1, 2]$ 上满足 Rolle 定理的条件.

$$f'(x) = 3x^2 - 6x + 2 = 0, x = \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{3 \pm \sqrt{3}}{3}, x$$

$$x_1 = \frac{3 - \sqrt{3}}{3} \in (0, 1), x_2 = \frac{3 + \sqrt{3}}{3} \in (1, 2), f'(x_1) = f'(x_2) = 0.$$

2. 讨论下列函数 $f(x)$ 在区间 $[-1, 1]$ 上是否满足 Rolle 定理的条件, 若满足, 求 $c \in (-1, 1)$, 使 $f'(c) = 0$.

(1) $f(x) = (1+x)^m(1-x)^n$, m, n 为正整数;

(2) $f(x) = 1 - \sqrt[3]{x^2}$.

解 (1) $f'(x) = m(1+x)^{m-1}(1-x)^n - n(1+x)^m(1-x)^{n-1}$

$$= (1+x)^{m-1}(1-x)^{n-1}(m-mx-n-nx) = 0, c = \frac{m-n}{m+1} \in (-1, 1), f'(c) = 0.$$

(2) $f'(x) = -\frac{2}{3}x^{-1/3}$, $f'(0)$ 不存在.

3. 写出函数 $f(x) = \ln x$ 在区间 $[1, e]$ 上的微分中值公式, 并求出其中的 $c = ?$

解 $f'(x) = \frac{1}{x}$, $f(e) - f(1) = \ln e - \ln 1 = 1 = \frac{1}{c}(e-1)$, $c = e-1$.

4. 应用 Lagrange 中值定理, 证明下列不等式:

(1) $|\sin y - \sin x| \leq |x - y|$;

(2) $|\tan x - \tan y| \geq |y - x|$, $x, y \in (-\pi/2, \pi/2)$;

(3) $\frac{b-a}{b} < \ln \frac{b}{a} < \frac{b-a}{a}$ ($0 < a < b$).

证 (1) $|\sin x - \sin y| = |(\sin x)'|_{x=c} (x-y)| = |\cos c| |x-y| \leq |x-y|$.

(2) $|\tan y - \tan x| = |(\tan x)'|_{x=c} (y-x)| = \sec^2 c |y-x| \geq |y-x|$.

(3) $\frac{b-a}{a} < \ln \frac{b}{a} = \ln b - \ln a = (\ln x)'|_{x=c} (b-a) = \frac{b-a}{c}$ ($c \in (a, b)$) $< \frac{b-a}{a}$.

5. 证明多项式 $P(x) = (x^2 - 1)(x^2 - 4)$ 的导函数的三个根都是实根, 并指出它们的范围.

证 $P(x)$ 有四个实根 $\pm 1, \pm 2$, 根据 Rolle 定理, 它的导函数有三个实根, 又作为四次多项式的导函数, 是三次多项式, 最多三个实根, 故 $P(x)$ 的导函数的三个根都是实根, 分别在区间 $(-2, -1), (-1, 1), (1, 2)$.

6. 设 c_1, c_2, \dots, c_n 为任意实数, 证明: 函数 $f(x) = c_1 \cos x + c_2 \cos 2x + \dots + c_n \cos nx$ 在 $(0, \pi)$ 内必有根.

证 $g(x) = c_1 \sin x + \frac{1}{2} c_2 \sin 2x + \dots + \frac{1}{n} c_n \sin nx$ 在 $[0, \pi]$ 满足定理的条件

($g(0) = g(\pi) = 0$), 故其导函数 $f(x)$ 在 $(0, \pi)$ 内必有根.

7. 设函数 $f(x)$ 与 $g(x)$ 在 (a, b) 内可微, $g(x) \neq 0$, 且 $\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = 0, x \in (a, b)$.

证明: 存在常数 k , 使 $f(x) = kg(x), x \in (a, b)$.

$$\text{证} \left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} = \frac{\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}}{g^2(x)} = 0,$$

根据公式的一个推论, 存在常数 k , 使 $\frac{f(x)}{g(x)} = k$, 即 $f(x) = kg(x), x \in (a, b)$.

8. 设 $f(x)$ 在 $(-\infty, +\infty)$ 上可微且 $f'(x) = k, -\infty < x < +\infty$. 证明: $f(x) = kx + b, -\infty < x < +\infty$, 其中 k, b 为常数.

证 $(f(x) - kx)' = f'(x) - k = k - k = 0, -\infty < x < +\infty, f(x) - kx = b, -\infty < x < +\infty$.

9. 证明下列等式:

$$(1) \arcsin x + \arccos x = \pi/2, -1 \leq x \leq 1;$$

$$(2) \arctan x = \arcsin \frac{x}{\sqrt{1+x^2}}, -\infty < x < +\infty.$$

证 (1) $(\arcsin x + \arccos x)' = (\arcsin x)' + (\arccos x)'$

$$= \frac{1}{\sqrt{1-x^2}} + \left(-\frac{1}{\sqrt{1-x^2}} \right) = 0, x \in (-1, 1), \arcsin x + \arccos x \text{ 在 } [-1, 1] \text{ 连续, 故}$$

$$\arcsin x + \arccos x = C, C = \arcsin 0 + \arccos 0 = \frac{\pi}{2}, \arcsin x + \arccos x = \frac{\pi}{2}.$$

$$(2) \left(\arctan x - \arcsin \frac{x}{\sqrt{1+x^2}} \right)'$$

$$= \frac{1}{1+x^2} - \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} \cdot \frac{\sqrt{1+x^2} - x \times \frac{x}{\sqrt{1+x^2}}}{1+x^2}$$

$$= \frac{1}{1+x^2} - \frac{\sqrt{1+x^2} \left(\sqrt{1+x^2} - x \times \frac{x}{\sqrt{1+x^2}} \right)}{1+x^2} = 0,$$

$$\arctan x - \arcsin \frac{x}{\sqrt{1+x^2}} = C, \text{ 以 } x=0 \text{ 代入得 } C=0, \text{ 故 } \arctan x - \arcsin \frac{x}{\sqrt{1+x^2}} = 0,$$

$$x \in (-\infty, +\infty).$$

10. 证明不等式: $\frac{2}{\pi}x < \sin x < x, 0 < x < \pi/2$.

证 $f(x) = \frac{\sin x}{x} (0 < x \leq \pi/2), f(0) = 1, f$ 在 $[0, \pi/2]$ 连续,

f 在 $(0, \pi/2)$ 可导, $f'(x) = \frac{x \cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2} < 0$.

f 在 $[0, \pi/2]$ 严格单调递减, $= \frac{2}{\pi} f(\frac{\pi}{2}) < f(x) < f(0) = 1, 0 < x < \pi/2$.

11. 设函数 $f(x)$ 在 (a, b) 内可微, 对于任意一点 $x_0 \in (a, b)$, 若 $\lim_{x \rightarrow x_0} f'(x)$ 存在, 则

$$\lim_{x \rightarrow x_0} f'(x) = f'(x_0).$$

$$\text{证 } f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f'(x_0 + \theta \Delta x) \Delta x}{\Delta x} \quad (0 < \theta < 1)$$

$$= \lim_{\Delta x \rightarrow 0} f'(x_0 + \theta \Delta x) = \lim_{x \rightarrow x_0} f'(x).$$

12. (Darboux 中值定理) 设 $y = f(x)$ 在 (A, B) 区间中可导, 又设 $[a, b] \subset (A, B)$, 且 $f'(a) < f'(b)$. 证明: 对于任意给定的 $\eta: f'(a) < \eta < f'(b)$, 都存在 $c \in (a, b)$ 使得 $f'(c) = \eta$.

证 先设 $f'(a) < 0 < f'(b)$. $f'(a) = \lim_{\Delta x \rightarrow 0+} \frac{f(a + \Delta x) - f(a)}{\Delta x} < 0$, 存在 $(b-a)/2 > \delta_1 > 0$,

使得 $0 < \Delta x \leq \delta_1$ 时 $\frac{f(a + \Delta x) - f(a)}{\Delta x} < 0$, 即 $f(a + \Delta x) - f(a) < 0$. 特别 $f(a + \delta_1) < f(a)$.

类似存在 $\delta_2: 0 < \delta_2 < (b-a)/2, f(b - \delta_2) < f(b)$. $f[a, b]$ 某点 c 取最小值 $f(c)$,

$f(c) \leq f(a + \delta_1) < f(a), c \neq a$, 同理, $c \neq b$. $c \in (a, b)$, c 是极小值点, 由 Fermat 引理,

$f'(c) = 0$. 再设 $\eta: f'(a) < \eta < f'(b)$. 考虑 $g(x) = f(x) - \eta x, g'(x) = f'(x) - \eta$,

$g'(a) = f'(a) - \eta < 0, g'(b) = f'(b) - \eta > 0$, 由前面的结果, 存在 $c \in (a, b)$ 使得

$g'(c) = f'(c) - \eta = 0$, 即 $f'(c) = \eta$.

习题 4.2

用L'Hospital法则求下列极限:

$$1. \lim_{x \rightarrow 0} \frac{2^x - 1}{3^x - 1} = \lim_{x \rightarrow 0} \frac{2^x \ln 2}{3^x \ln 3} = \frac{\ln 2}{\ln 3}.$$

$$2. \lim_{x \rightarrow 0} \frac{\cos x - 1}{x - \ln(1+x)} = \lim_{x \rightarrow 0} \frac{-\sin x}{1 - 1/(1+x)} = -\lim_{x \rightarrow 0} \frac{\sin x}{x} = -1.$$

$$\begin{aligned} 3. & \lim_{x \rightarrow 0} \left(\frac{1}{\ln(x + \sqrt{1+x^2})} - \frac{1}{\ln(1+x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\ln(1+x) - \ln(x + \sqrt{1+x^2})}{\ln(x + \sqrt{1+x^2}) \ln(1+x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1/(1+x) - 1/\sqrt{1+x^2}}{1/\sqrt{1+x^2} \times \ln(1+x) + \ln(x + \sqrt{1+x^2})1/(1+x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sqrt{1+x^2} - 1 - x}{(1+x)\ln(1+x) + \sqrt{1+x^2}\ln(x + \sqrt{1+x^2})} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{x/\sqrt{1+x^2} - 1}{\ln(1+x) + 1 + (x/\sqrt{1+x^2})\ln(x + \sqrt{1+x^2}) + 1} \right) = -\frac{1}{2}. \end{aligned}$$

$$4. \lim_{x \rightarrow \pi/2} \frac{\tan 3x}{\tan x} = \lim_{x \rightarrow \pi/2} \frac{3\sec^2 3x}{\sec^2 x} = 3.$$

$$5. \lim_{x \rightarrow 0} \frac{\ln(\cos ax)}{\ln(\cos bx)} = \lim_{x \rightarrow 0} \frac{(1/(\cos ax))(-\sin ax)a}{(1/(\cos bx))(-\sin bx)b} = \frac{a^2}{b^2}.$$

$$6. \lim_{x \rightarrow 0+0} x^\alpha \ln x (\alpha > 0) = \lim_{x \rightarrow 0+0} \frac{\ln x}{x^{-\alpha}} = \lim_{x \rightarrow 0+0} \frac{1/x}{(-\alpha)x^{-\alpha-1}} = -\frac{1}{\alpha} \lim_{x \rightarrow 0+0} x^\alpha = 0.$$

$$7. \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^{100}} = \lim_{y \rightarrow +\infty} \frac{y^{50}}{e^y} = \lim_{y \rightarrow +\infty} \left(\frac{y}{e^{y/50}} \right)^{50} = \left(\lim_{y \rightarrow +\infty} \frac{y}{e^{y/50}} \right)^{50} = \left(\lim_{y \rightarrow +\infty} \frac{50}{e^{y/50}} \right)^{50} = 0.$$

$$\begin{aligned} 8. & \lim_{x \rightarrow \frac{\pi}{2}-0} (\tan x)^{2x-\pi} \cdot y = (\tan x)^{2x-\pi}, \lim_{x \rightarrow \frac{\pi}{2}-0} \ln y = \lim_{x \rightarrow \frac{\pi}{2}-0} (2x-\pi) \ln \tan x \\ &= \lim_{x \rightarrow \frac{\pi}{2}-0} \frac{\ln \tan x}{\frac{1}{2x-\pi}} = \lim_{x \rightarrow \frac{\pi}{2}-0} \frac{\sec^2 x / \tan x}{-\frac{2}{(2x-\pi)^2}} = -2 \lim_{z \rightarrow 0-0} \frac{z^2 \tan z}{\sin^2 z} = 0, \lim_{x \rightarrow \frac{\pi}{2}-0} y = \lim_{x \rightarrow \frac{\pi}{2}-0} e^{\ln y} \\ &= e^{\lim_{x \rightarrow \frac{\pi}{2}-0} \ln y} = e^0 = 1. \end{aligned}$$

$$9. \lim_{x \rightarrow \infty} (a^{1/x} - 1)x (a > 0) = \lim_{y \rightarrow 0} \frac{a^y - 1}{y} = \lim_{y \rightarrow 0} \frac{a^y \ln a}{1} = \ln a.$$

$$10. \lim_{y \rightarrow 0} \frac{y - \arcsin y}{\sin^3 y} = \lim_{y \rightarrow 0} \frac{y - \arcsin y}{y^3} = \lim_{y \rightarrow 0} \frac{1 - \frac{1}{\sqrt{1-y^2}}}{3y^2}$$

$$= \frac{1}{3} \lim_{y \rightarrow 0} \frac{\sqrt{1-y^2}-1}{y^2} = -\frac{1}{3} \lim_{y \rightarrow 0} \frac{\frac{y}{\sqrt{1-y^2}}}{2y} = -\frac{1}{6}.$$

$$\begin{aligned} 11. \lim_{y \rightarrow 1} \left(\frac{y}{y-1} - \frac{1}{\ln y} \right) &= \lim_{y \rightarrow 1} \left(\frac{y \ln y - y + 1}{(y-1) \ln y} \right) \\ &= \lim_{y \rightarrow 1} \left(\frac{\ln y + 1 - 1}{\ln y + (y-1)/y} \right) = \lim_{y \rightarrow 1} \left(\frac{\ln y}{y \ln y + (y-1)} \right) \\ &= \lim_{y \rightarrow 1} \left(\frac{1/y}{\ln y + 2} \right) = \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} 12. \lim_{x \rightarrow 0} \frac{1-x^2-e^{-x^2}}{x \sin^3 x} &= \lim_{x \rightarrow 0} \frac{1-x^2-e^{-x^2}}{x^4} = \lim_{y \rightarrow 0} \frac{1-y-e^{-y}}{y^2} \\ &= \lim_{y \rightarrow 0} \frac{-1+e^{-y}}{2y} = \lim_{y \rightarrow 0} \frac{-e^{-y}}{2} = -\frac{1}{2}. \end{aligned}$$

$$13. \lim_{x \rightarrow 0} \left(\frac{\arctan x}{x} \right)^{1/x^2}, y = \left(\frac{\arctan x}{x} \right)^{1/x^2},$$

$$\begin{aligned} \lim_{x \rightarrow 0} \ln y &= \lim_{x \rightarrow 0} \frac{\ln \frac{\arctan x}{x}}{\frac{x}{x^2}} = \lim_{x \rightarrow 0} \frac{(x/\arctan x) \times \frac{\frac{x}{1+x^2} - \arctan x}{x^2}}{2x} \\ &= \lim_{x \rightarrow 0} \frac{x - (1+x^2) \arctan x}{2x^3} = \lim_{x \rightarrow 0} \frac{1-1-2x \arctan x}{6x^2} = -\frac{1}{3} \lim_{x \rightarrow 0} \frac{\arctan x}{x} = -\frac{1}{3}, \end{aligned}$$

$$\lim_{x \rightarrow 0} \left(\frac{\arctan x}{x} \right)^{1/x^2} = e^{-1/3}.$$

$$14. \lim_{x \rightarrow +\infty} \left(\frac{\pi}{2} - \arctan x \right)^{\frac{1}{\ln x}}. y = \left(\frac{\pi}{2} - \arctan x \right)^{\frac{1}{\ln x}}.$$

$$\lim_{x \rightarrow +\infty} \ln y = \lim_{x \rightarrow +\infty} \frac{\ln \left(\frac{\pi}{2} - \arctan x \right)}{\ln x} = -\lim_{x \rightarrow +\infty} \frac{x}{\left(\frac{\pi}{2} - \arctan x \right) (1+x^2)}$$

$$= -\lim_{x \rightarrow +\infty} \frac{x}{\left(\arctan \frac{1}{x} \right) (1+x^2)} = -\lim_{x \rightarrow +\infty} \frac{x}{\left(\frac{1}{x} \right) (1+x^2)} = -1, \lim_{x \rightarrow +\infty} \left(\frac{\pi}{2} - \arctan x \right)^{\frac{1}{\ln x}} = e^{-1}.$$

$$15. \lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{\tan^2 x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{x^2}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{2x}{\sin x} = 2.$$

$$16. \lim_{x \rightarrow 0} \frac{\cosh x - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{\sinh x + \sin x}{2x} = \lim_{x \rightarrow 0} \frac{\cosh x + \cos x}{2} = 1.$$

$$17. \lim_{x \rightarrow 1} \frac{x^x - x}{\ln x - x + 1} = \lim_{x \rightarrow 1} \frac{x^x (\ln x + 1) - 1}{1/x - 1} = \lim_{x \rightarrow 1} \frac{x^x (\ln x + 1) - 1}{1 - x}$$

$$= \lim_{x \rightarrow 1} \frac{x^x (\ln x + 1)^2 + x^{x-1}}{-1} = -2.$$

$$18. \lim_{x \rightarrow +\infty} \left(\frac{2}{\pi} \arctan x \right)^x \cdot y = \left(\frac{2}{\pi} \arctan x \right)^x.$$

$$\lim_{x \rightarrow +\infty} \ln y = \lim_{x \rightarrow +\infty} \frac{\ln\left(\frac{2}{\pi} \arctan x\right)}{\frac{1}{x}} = \lim_{x \rightarrow +\infty} \frac{(1/\arctan x) \times \frac{1}{1+x^2}}{-\frac{1}{x^2}} = -\frac{2}{\pi},$$

$$\lim_{x \rightarrow +\infty} \left(\frac{2}{\pi} \arctan x \right)^x = e^{-2/\pi}.$$

习题 4.3

1. 求下列函数在 $x=0$ 点的局部 Taylor 公式:

$$(1) \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{2} \left(\left(1 + x + \frac{x^2}{2!} + L + \frac{x^{2n+1}}{(2n+1)!} \right) - \left(1 - x + \frac{x^2}{2!} + L - \frac{x^{2n+1}}{(2n+1)!} \right) \right) + o(x^{2n+2})$$

$$= x + \frac{x^3}{3!} + L + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}).$$

$$(2) \frac{1}{2} \ln \frac{1-x}{1+x} = \frac{1}{2} \left(\left(-x - \frac{x^2}{2} + L - \frac{x^{2n}}{2n} - \frac{x^{2n-1}}{2n-1} \right) - \left(x - \frac{x^2}{2} + L - \frac{x^{2n}}{2n} + \frac{x^{2n-1}}{2n-1} \right) \right) + o(x^{2n})$$

$$= - \left(x + \frac{x^3}{3} + L + \frac{x^{2n-1}}{2n-1} \right) + o(x^{2n}).$$

$$(3) \sin^2 x = \frac{1}{2} (1 - \cos 2x) = \frac{1}{2} \left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + L + (-1)^{n-1} \frac{(2x)^{2n}}{(2n)!} \right) + o(x^{2n+1}).$$

$$(4) \frac{x^2 + 2x - 1}{x - 1} = -(x^2 + 2x - 1)(1 + x + L + x^n + o(x^n))$$

$$= -(x^2 + x^3 + L + x^{n+2} + o(x^{n+2})) - 2(x + x^2 + L + x^{n+1} + o(x^{n+1})) + (1 + x + L + x^n + o(x^n))$$

$$= 1 - x - 2x^2 - 2x^3 - L - 2x^n + o(x^n).$$

$$(5) \cos x^3 = 1 - \frac{x^6}{2!} + L + (-1)^n \frac{x^{6n}}{(2n)!} + o(x^{6n+3}).$$

2. 求下列函数在 $x=0$ 点的局部 Taylor 公式至所指定的阶数:

$$(1) e^x \sin x (x^4)$$

$$\text{解 } e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) \right) \left(x - \frac{x^3}{6} + o(x^4) \right) = x + x^2 + \frac{x^3}{3} + o(x^4).$$

$$(2) \sqrt{1+x} \cos x (x^4)$$

$$\text{解 } \sqrt{1+x} \cos x = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + o(x^4) \right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \right)$$

$$= 1 + \frac{1}{2}x - \frac{5}{8}x^2 - \frac{3}{16}x^3 + \frac{25}{384}x^4 + o(x^4).$$

$$(3) \sqrt{1-2x+x^3} - \sqrt{1-3x+x^2} (x^3)$$

$$\text{解 } \sqrt{1-2x+x^3} - \sqrt{1-3x+x^2}$$

$$= \left(1 + \frac{1}{2}(-2x+x^3) - \frac{1}{8}(-2x+x^3)^2 + \frac{1}{16}(-2x+x^3)^3 \right)$$

$$- \left(1 + \frac{1}{2}(-3x+x^2) - \frac{1}{8}(-3x+x^2)^2 + \frac{1}{16}(-3x+x^2)^3 \right)$$

$$\begin{aligned}
&= \left(1 + \frac{1}{2}(-2x + x^3) - \frac{1}{8}(4x^2) + \frac{1}{16}(-8x^3) \right) \\
&\quad - \left(1 + \frac{1}{2}(-3x + x^2) - \frac{1}{8}(9x^2 - 6x^3) + \frac{1}{16}(-27x^3) \right) + o(x^3) \\
&= \frac{1}{2}x + \frac{1}{8}x^2 + \frac{15}{16}x^3 + o(x^3).
\end{aligned}$$

3.求下列函数在点 $x=0$ 的局部Taylor公式:

(1) $\arctan x$.

$$\text{解 } \frac{1}{1+x^2} = 1 - x^2 + L + (-1)^n x^{2n} + o(x^{2n})$$

$$\begin{aligned}
(2) \arcsin x &= \int_0^x \frac{1}{1+t^2} dt = \sum_{k=0}^n \frac{(-1)^k}{2k+1} x^{2k+1} + o(x^{2n+1}) \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} + L + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+1}).
\end{aligned}$$

$$\text{解 } \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum_{k=0}^n \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)L\left(-\frac{1}{2}-k+1\right)}{k!} x^k + o(x^n)$$

$$= \sum_{k=0}^n (-1)^k \frac{(2k-1)!!}{(2k)!!} x^k + o(x^n)$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^n \frac{(2k-1)!!}{(2k)!!} x^{2k} + o(x^{2n}),$$

$$\arcsin x = \sum_{k=0}^n \frac{(2k-1)!!}{(2k)!!} \int_0^x t^{2k} dt + \int_0^x o(t^{2n}) dt$$

$$= \sum_{k=0}^n \frac{(2k-1)!!}{(2k)!!(2k+1)} x^{2k+1} + o(x^{2n+1}).$$

4.利用Taylor公式求下列极限:

$$(1) \lim_{x \rightarrow 0} \frac{1-x^2-e^{-x^2}}{x \sin^3 2x} = \lim_{x \rightarrow 0} \frac{1-x^2-\left(1-x^2+\frac{x^4}{2}+o(x^4)\right)}{8x^4} = -\frac{1}{16}.$$

$$(2) \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + o(x^2)}{x(x + o(x))} = \frac{1}{2}.$$

$$\begin{aligned}
(3) \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{\cos x}{\sin x} \right) \frac{1}{\sin x} &= \lim_{x \rightarrow 0} \left(\frac{\sin x - x \cos x}{x \sin x} \right) \frac{1}{\sin x} \\
&= \lim_{x \rightarrow 0} \frac{\sin x - x \cos x}{x^3} = \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{6}\right) - x \left(1 - \frac{x^2}{2}\right) + o(x^3)}{x^3} = \frac{1}{3}.
\end{aligned}$$

5. 当 x 较小时, 可用 $\sin a + x \cos a$ 近似代替 $\sin(a+x)$, 其中 a 为常数, 试证其误差不超过 $|x|^2/2$.

$$\text{证 } f(x) = \sin(a+x) - (\sin a + x \cos a)$$

$$f(0) = 0, f'(x) = \cos(a+x) - \cos a,$$

$$f''(x) = -\sin(a+x).$$

$$f(x) = f(0) + f'(0)x + \frac{f''(c)}{2}x^2 = \frac{-\sin(a+c)}{2}x^2,$$

$$|f(x)| = |\sin(a+x) - (\sin a + x \cos a)| \leq \frac{x^2}{2}.$$

6. 设 $0 < x \leq 1/3$, 按公式 $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ 计算 e^x 的近似值, 试证公式误差不超过 8×10^{-4} .

$$\text{证 } e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{e^{\theta x}}{24}x^4, \left| e^x - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \right) \right| = \frac{e^{\theta x}}{24}x^4 \leq \frac{e^{1/3}}{24} \times \left(\frac{1}{3} \right)^4$$

$$= .000717L < 8 \times 10^{-4}.$$

习题 4.4

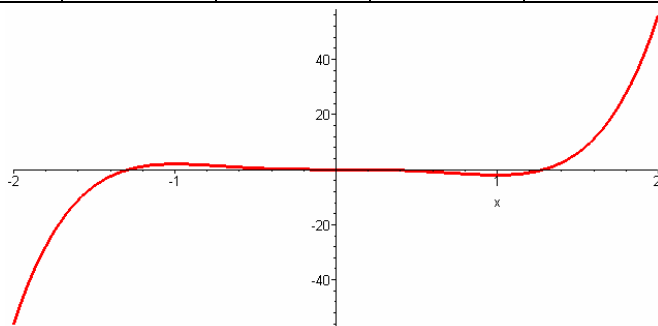
1. 求下列函数的单调性区间与极值点：

(1) $y = 3x^5 - 5x^3$.

解 $y' = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$,

$y' = 15x^2(x^2 - 1) = 15x^2(x - 1)(x + 1) = 0, x_1 = -1, x_2 = 0, x_3 = 1$.

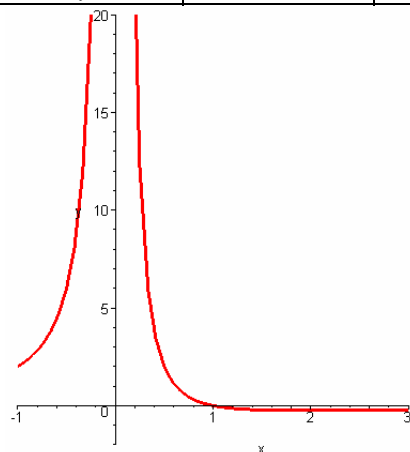
x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
y'	$+$	0	$-$	0	$-$	0	$+$
y	\nearrow	极大值	\searrow	无极值	\searrow	极小值	\nearrow



(2) $y = \frac{1}{x^2} - \frac{1}{x}, x \neq 0$.

$y' = -\frac{2}{x^3} + \frac{1}{x^2} = \frac{x-2}{x^3} = 0, x_1 = 2$.

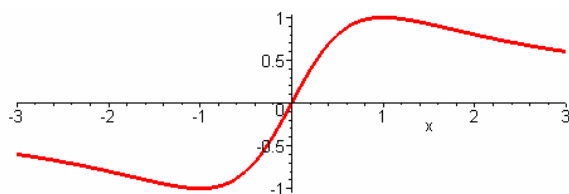
x	$(-\infty, 0)$	$(0, 2)$	2	$(2, +\infty)$
y'	$+$	$-$	0	$+$
y	\nearrow	\searrow	极小值	\nearrow



$$(3) y = \frac{2x}{1+x^2}, x \in (-\infty, +\infty).$$

$$y' = 2 \times \frac{1+x^2-2x^2}{(1+x^2)^2} = 2 \times \frac{1-x^2}{(1+x^2)^2} = 0, x = \pm 1.$$

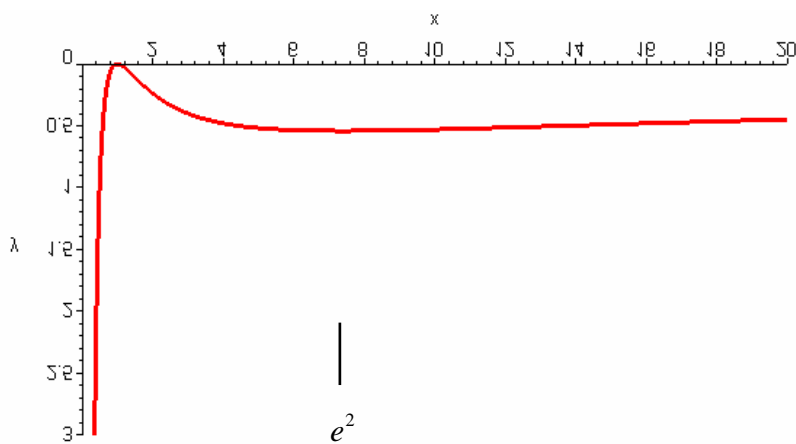
x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, +\infty)$
y'	$-$	0	$+$	0	$-$
y	\searrow	极小值-1	\nearrow	极大值 1	\searrow



$$(4) y = \frac{1}{x} \ln^2 x, x > 0.$$

$$y' = \frac{2(\ln x)(1/x)x - \ln^2 x}{x^2} = \frac{2(\ln x) - \ln^2 x}{x^2} = \frac{\ln x[2 - \ln x]}{x^2} = 0, x = 1, x = e^2.$$

x	$(0, 1)$	1	$(1, e^2)$	e^2	$(e^2, +\infty)$
y'	$-$	0	$+$	0	$-$
y	\searrow	极小值	\nearrow	极大值	\searrow

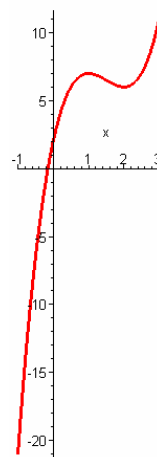


2. 求函数 $f(x) = 2x^3 - 9x^2 + 12x + 2$ 在区间 $[-1, 3]$ 上的最大值与最小值, 并指明最大值点与最小值点.

$$\text{解 } f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) \\ = 6(x-1)(x-2) = 0, x = 1, 2.$$

$$f(-1) = -21, f(1) = 7, f(2) = 6, f(3) = 11.$$

$f(-1) = -21$ 是最小值, $f(3) = 11$ 是最大值.



3.将周长为 $2p$ 的等腰三角形绕其底边旋转一周,求使所得旋转体体积最大的等腰三角形的底边长度.

解设腰长为 x , 则

$$V = \frac{2\pi}{3} (x^2 - (p-x)^2)(p-x) = \frac{2\pi}{3} (2px - p^2)(p-x), p/2 \leq x \leq p.$$

$$V' = \frac{2\pi}{3} (2p(p-x) - (2px - p^2)) = \frac{2\pi}{3} (-4px + 3p^2) = 0,$$

$$x_0 = \frac{3}{4}p, V(p/2) = V(p) = 0, V(\frac{3}{4}p) \text{ 是最大值.}$$

$$\text{等腰三角形的底边长度} = 2p - \frac{3}{2}p = \frac{1}{2}p.$$

4.求出常数 l 与 k 的值,使函数 $f(x) = x^3 + lx^2 + kx$ 在

$x = -1$ 处有极值2,并求出在这样

的 l 与 k 之下 $f(x)$ 的所有极值点,以及在 $[0, 3]$ 上的

最小值和最大值.

$$\text{解 } f'(x) = 3x^2 + 2lx + k, 3 - 2l + k = 0, -1 + l - k = 2.$$

$$k = -3, l = 0.$$

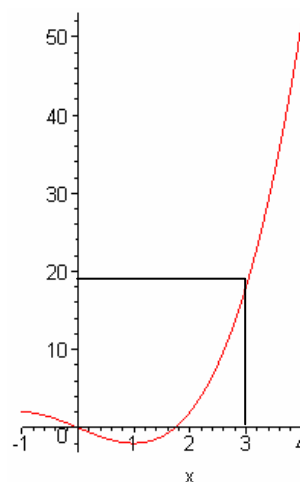
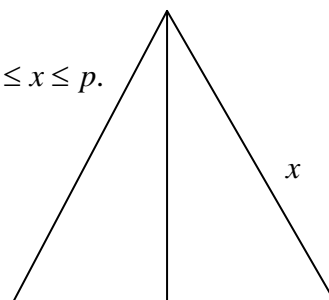
$$f(x) = x^3 - 3x, f'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0,$$

$$x = \pm 1, f''(x) = 6x, f''(\pm 1) = \pm 6,$$

$f(1)$ 是极小值, $f(-1)$ 是极大值.

$$f(0) = 0, f(1) = -2, f(3) = 18. f(1) = -2 \text{ 是最小值,}$$

$$f(3) = 18 \text{ 是最大值.}$$



5.设一电灯可以沿垂直线 OB 移动, OA 是

一条水平线,长度为 a .问灯距离 O 点多

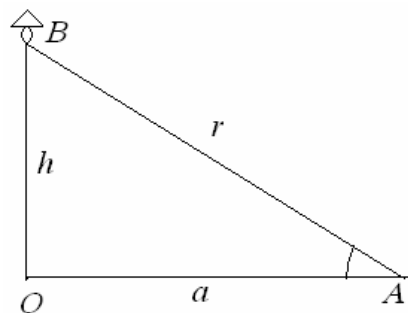
高时, A 点有最大的照度.

$$\text{解 } J = K \frac{\sin \varphi}{a^2 + a^2 \tan^2 \varphi} = \frac{K}{a^2} \sin \varphi \cos^2 \varphi, 0 \leq \varphi \leq \frac{\pi}{2}.$$

$$J' = \frac{K}{a^2} (\cos^3 \varphi - 2 \sin^2 \varphi \cos \varphi) = 0, \tan \varphi_0 = \frac{1}{\sqrt{2}}.$$

$$J(0) = J(\pi/2) = 0, J(\varphi_0) \text{ 是最大值, 这时灯的}$$

$$\text{高度 } h = a \tan \varphi_0 = \frac{a}{\sqrt{2}}.$$



6.若两条宽分别为 a 及 b 的河垂直相交,若一船从一河

转入另一河,问其最大的长度是多少?

解设船与一岸夹角为 θ ,则船长为

$$l = a \csc \theta + b \sec \theta, 0 < \theta < \frac{\pi}{2}.$$

$$l' = -a \csc \theta \cot \theta + b \sec \theta \tan \theta = 0, \frac{\sec \theta \tan \theta}{\csc \theta \cot \theta} = \frac{a}{b},$$

$$\tan^3 \theta = \frac{a}{b}, \tan \theta = \sqrt[3]{\frac{a}{b}}, \theta_0 = \arctan \sqrt[3]{\frac{a}{b}}.$$

$\lim_{\theta \rightarrow 0} l(\theta) = +\infty, \lim_{\theta \rightarrow \pi/2} l(\theta) = +\infty, l$ 在 $\left(0, \frac{\pi}{2}\right)$ 有最小值, θ_0 是最小值点.

$$\text{此时船长 } l = a \sqrt{1 + \left(\sqrt[3]{\frac{b}{a}}\right)^2} + b \sqrt{1 + \left(\sqrt[3]{\frac{a}{b}}\right)^2}$$

$$= a^{2/3} \sqrt{a^{2/3} + b^{2/3}} + a^{2/3} \sqrt{a^{2/3} + b^{2/3}} = \sqrt{a^{2/3} + b^{2/3}}^3.$$

7.在半径为 a 的球内作一内接圆锥体, 要使锥体体积最大, 问其高及底半径应是多少?

解设球心到内接圆锥体底的距离为 x , 则锥体体积

$$V = \frac{\pi}{3} (a^2 - x^2)(a + x), 0 \leq x \leq a.$$

$$V' = \frac{\pi}{3} (-2x(a + x) + a^2 - x^2) = \frac{\pi}{3} (-3x^2 - 2ax + a^2)$$

$$= -\frac{\pi}{3} (3x^2 + 2ax - a^2) = -\frac{\pi}{3} (3x - a)(x + a) = 0, x_0 = \frac{a}{3}.$$

$$V(0) = \frac{\pi}{3} a^3, V(a) = 0, V\left(\frac{a}{3}\right) = \frac{\pi}{3} a^3 \times \frac{32}{27}. V\left(\frac{a}{3}\right) \text{为最大值}.$$

$$\text{底半径} = \sqrt{a^2 - x_0^2} = \sqrt{a^2 - \left(\frac{a}{3}\right)^2} = \frac{2\sqrt{2}}{3} a, \text{高 } h = a + x_0 = a + \frac{a}{3} = \frac{4a}{3}.$$

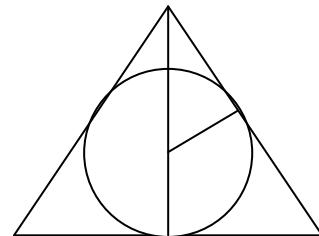
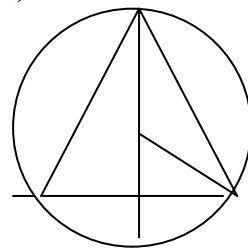
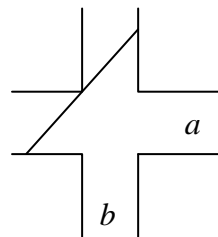
8.在半径为 a 的球外作一外切圆锥体, 要问其高及底半径取多少才能使锥体体积最小?

$$\text{解设锥的高为 } h, \frac{r}{h} = \frac{a}{\sqrt{(h-a)^2 - a^2}}, r = \frac{ah}{\sqrt{(h-a)^2 - a^2}}.$$

$$V = V(h) = \frac{\pi}{3} \frac{a^2 h^3}{(h-a)^2 - a^2}.$$

$$V' = \frac{\pi a^2}{3} \frac{3h^2((h-a)^2 - a^2) - 2h^3(h-a)}{((h-a)^2 - a^2)^2} = \frac{\pi a^2}{3} \frac{h^2[3((h-a)^2 - a^2) - 2h(h-a)]}{((h-a)^2 - a^2)^2}$$

$$= \frac{\pi a^2}{3} \frac{h^2[h^2 - 4ah]}{((h-a)^2 - a^2)^2} = \frac{\pi a^2}{3} \frac{h^3[h - 4a]}{((h-a)^2 - a^2)^2} = 0, h_0 = 4a.$$



当 $h < 4a$ 时, $V' < 0$, 当 $h > 4a$ 时, $V' > 0$, $V(4a)$ 为最小值, 此时 $r_0 = \frac{4a^2}{\sqrt{8a^2}} = \sqrt{2}a$.

9. 在曲线 $y^2 = 4x$ 上求出到点 $(18, 0)$ 的距离最短的点.

解 $d^2 = f(y) = \left(\frac{y^2}{4} - 18\right)^2 + y^2 = \left(\frac{z}{4} - 18\right)^2 + z = g(z), 0 \leq z < +\infty (z = y^2)$.

$\lim_{z \rightarrow +\infty} g(z) = +\infty$, $g(z)$ 在 $[0, +\infty)$ 有最小值.

$$g'(z) = 2\left(\frac{z}{4} - 18\right) \frac{1}{4} + 1 = \frac{z}{8} - 8 = 0, z = 64, g(0) = 324, g(64) = 68 < g(0),$$

$g(64)$ 为最小值. $y = \sqrt{z} = \pm 8, x = \frac{y^2}{4} = 16$.

曲线 $y^2 = 4x$ 上到点 $(18, 0)$ 的距离最短的点 $(16, 8), (16, -8)$.

10. 试求内接于已知圆锥且有最大体积的正圆柱的高度.

解 设已知圆锥的高度为 H , 底半径为 R . 设内接正圆柱的底半径为 x , 则其体积为

$$V = \pi x^2 (R - x) \frac{H}{R}, 0 \leq x \leq R.$$

$$V' = \pi (2x(R - x) - x^2) = \pi (2Rx - 3x^2) = \pi x (2R - 3x) = 0, x = 0, \frac{2}{3}R.$$

$$V(0) = V(R) = 0, V\left(\frac{2}{3}R\right) \text{为最大值. 此时内接正圆柱的高度 } h = \left(R - \frac{2}{3}R\right) \frac{H}{R} = \frac{H}{3}.$$

11. 试求内接于椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 且其底平行于 x 轴的最大等腰三角形的面积.

$$\text{解 } \begin{cases} x = a \cos t \\ y = b \sin t \end{cases}, 0 \leq t \leq 2\pi.$$

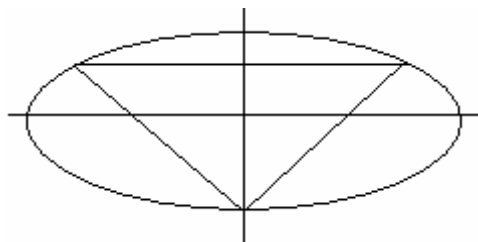
设内接等腰三角形的顶点在 $(-b, 0)$, 而底边上的一个顶点在第一象限.

$$\text{内接三角形面积 } S = ab \cos t (1 + \sin t), 0 \leq t \leq \frac{\pi}{2}.$$

$$S' = ab[-\sin t (1 + \sin t) + \cos^2 t] = ab[1 - \sin t - 2\sin^2 t] (\sin t = z)$$

$$= -ab(2z^2 + z - 1) = -ab(2z - 1)(z + 1) = 0, z = \sin t_0 = \frac{1}{2}.$$

$$S(0) = ab, S\left(\frac{\pi}{2}\right) = 0, S(t_0) = ab\sqrt{1 - \frac{1}{4}}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}ab \text{为最大值.}$$



12. 设动点 A 自平面坐标的原点 O 开始以速度 8m/min 沿 y 轴正向前进, 而点 B 在 x 轴的正向距离原点 50m 处, 同时沿 x 轴向原点作匀速运动, 速度为 6m/min . 问何时 A 与 B 距离最近? 最近的距离是多少?

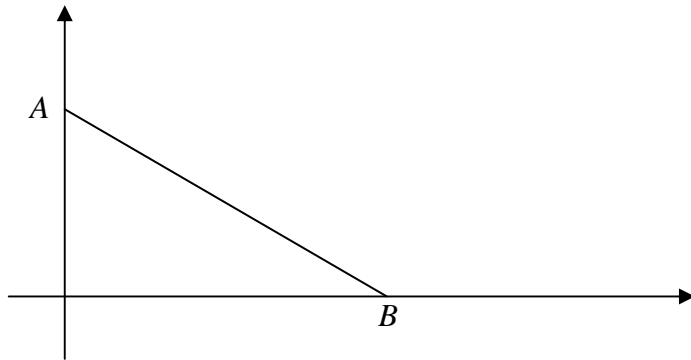
解 $s^2 = f(t) = (8t)^2 + (50 - 6t)^2, t \geq 0$.

$\lim_{t \rightarrow +\infty} f(t) = +\infty$, $f(t)$ 在 $t \geq 0$ 取最小值.

$$f'(t) = 128t - 12(50 - 6t) = 200t - 600 = 0, t_0 = 3.$$

$$f(0) = 50, f(3) = 24^2 + 32^2 = 1600 = d^2, d = 40.$$

开始后3分钟达到最近距离40m.

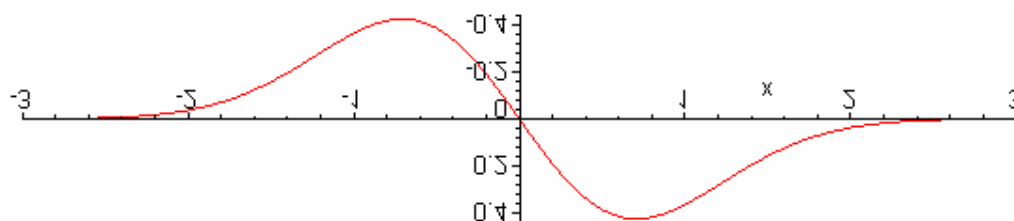


习题 4.5

1. 求函数 $f(x) = xe^{-x^2}$ 的凸凹性区间及拐点.

解 $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2)$, $f''(x) = e^{-x^2}(1 - 2x^2)(-2x) - 4xe^{-x^2}$
 $= e^{-x^2}(-6x + 4x^3) = 2xe^{-x^2}(-3 + 2x^2) = 0, x = 0, \pm\sqrt{\frac{3}{2}}.$

x	$(-\infty, -\sqrt{\frac{3}{2}})$	$-\sqrt{\frac{3}{2}}$	$(-\sqrt{\frac{3}{2}}, 0)$	0	$(0, \sqrt{\frac{3}{2}})$	$\sqrt{\frac{3}{2}}$	$(\sqrt{\frac{3}{2}}, +\infty)$
f''	-	0	+	0	-	0	+
f	\cap	拐点	\cup	拐点	\cap	拐点	\cup

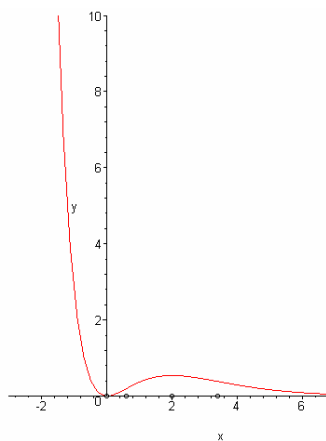


作下列函数的图形:

2. $y = x^2 - \frac{1}{3}x^3, x \in (-\infty, \infty).$

$y' = 2x - x^2 = x(2 - x) = 0, x = 0, 2.$

$y'' = 2 - 2x = 0, x = 1.$



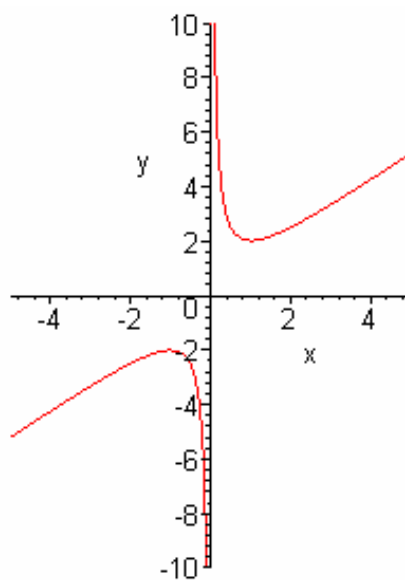
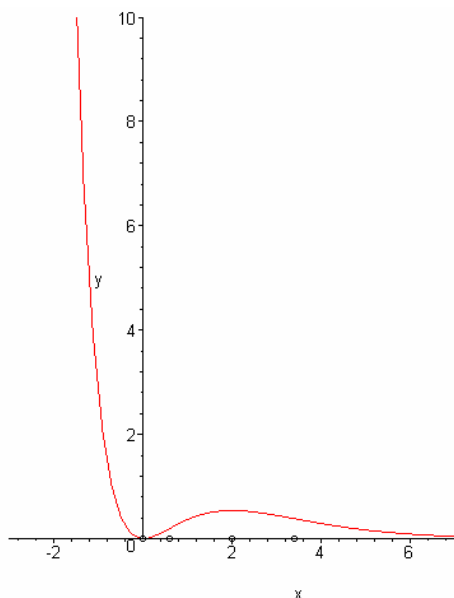
x	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, 2)$	2	$(2, +\infty)$
y'	-	0	+		+	0	-
y''	+		+		-		-
y	$\searrow \cup$	极小值	$\cup \nearrow$	拐点	$\nearrow \cap$	极大值	$\searrow \cap$

$$3. y = x^2 e^{-x}, x \in (-\infty, +\infty). y' = 2xe^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2) = e^{-x}x(2 - x) = 0, x = 0, 2;$$

$$y'' = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) = e^{-x}(x^2 - 4x + 2) = 0,$$

$$x = 2 \pm \sqrt{2}.$$

x	$(-\infty, 0)$	0	$(0, 2 - \sqrt{2})$	$2 - \sqrt{2}$	$(2 - \sqrt{2}, 2)$	2	$(2, 2 + \sqrt{2})$	$2 + \sqrt{2}$	$(2 + \sqrt{2}, +\infty)$
y'	-	0	+		+	0	-		-
y''	+		+	0	-		-	0	+
y	$\downarrow \cup$	极小值	$\downarrow \cup$	拐点	$\downarrow \cap$	极大值	$\downarrow \cap$	拐点	$\downarrow \cup$



$$4. y = x + \frac{1}{x}, x \neq 0.$$

$$y' = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} = 0,$$

$$x = \pm 1; y'' = \frac{2}{x^3}.$$

x	$(-\infty, -1)$	-1	$(-1, 0)$	$(0, 1)$	1	$(1, +\infty)$
y'	+	0	-	-	0	+
y''	-		-	+		+
y	$\nearrow \cap$	极大值	$\searrow \cap$	$\searrow \cup$	极小值	$\nearrow \cup$

$$5. y = \frac{(x+1)^3}{(x-1)^2}, x \neq 1.$$

$$y' = \frac{3(x+1)^2(x-1)^2 - 2(x+1)^3(x-1)}{(x-1)^4}$$

$$= \frac{(x+1)^2(x-1)(3x-3-2x-2)}{(x-1)^4} = \frac{(x+1)^2(x-1)(x-5)}{(x-1)^4} = \frac{(x+1)^2(x-5)}{(x-1)^3},$$

$$y' = 0, x = -1, 5.$$

$$y'' = \frac{[2(x+1)(x-5) + (x+1)^2](x-1)^3 - 3(x+1)^2(x-5)(x-1)^2}{(x-1)^6}$$

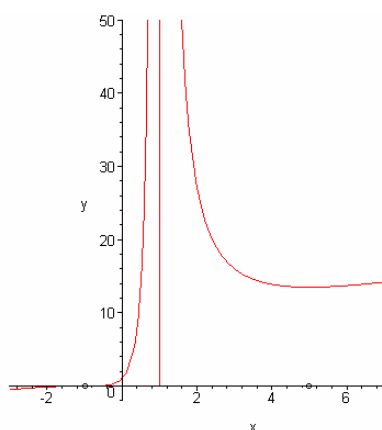
$$= \frac{[2(x+1)(x-5) + (x+1)^2](x-1) - 3(x+1)^2(x-5)}{(x-1)^4}$$

$$= \frac{(x+1)\{[2(x-5) + (x+1)](x-1) - 3(x+1)(x-5)\}}{(x-1)^4}$$

$$= \frac{(x+1)\{(3x-9)(x-1) - 3(x^2-4x-5)\}}{(x-1)^4}$$

$$= \frac{(x+1)\{(3x^2-12x+9) - 3(x^2-4x-5)\}}{(x-1)^4} = \frac{24(x+1)}{(x-1)^4} = 0, \quad x = -1.$$

x	$(-\infty, -1)$	-1	$(-1, 1)$	$(1, 5)$	5	$(5, +\infty)$
y'	+	0	+	-	0	+
y''	-	0	+	+		+
y	\searrow	拐点	\searrow	\searrow	极小值	\searrow



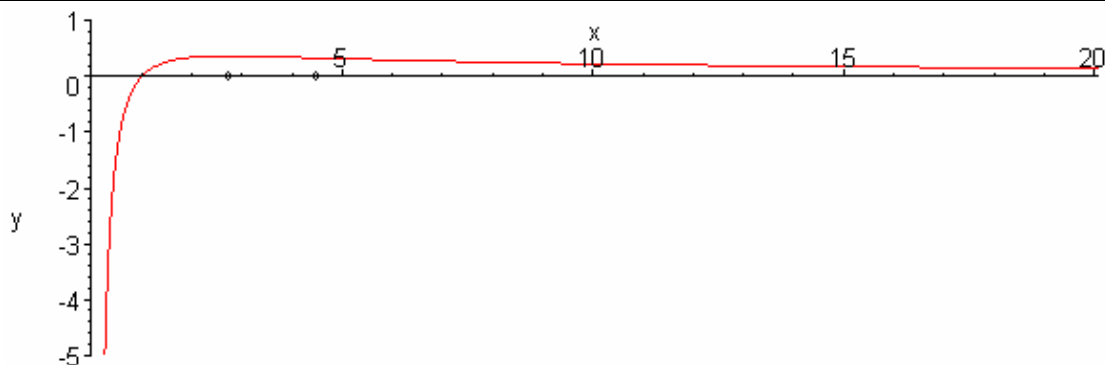
$$6. y = \frac{\ln x}{x}, x > 0.$$

$$y' = \frac{1 - \ln x}{x^2} = 0, x = e.$$

$$y'' = \frac{-\frac{1}{x} \times x^2 - 2x(1 - \ln x)}{x^4} = -\frac{1 + 2(1 - \ln x)}{x^3} = -\frac{3 - 2\ln x}{x^3},$$

$$y'' = 0, x = e^{3/2}.$$

x	$(-\infty, e)$	e	$(e, e^{3/2})$	$e^{3/2}$	$(e^{3/2}, +\infty)$
y'	-	0	+		+
y''	+		+	0	-
y	\cap	极小值	\cup	拐点	\cap



7. 设函数 $y = f(x)$ 在 (a, b) 内有二阶导数 $f''(x)$ 且在 (a, b) 内向上凸. 证明 $f''(x) \leq 0, x \in (a, b)$.

证 $y = f(x)$ 在 (a, b) 内向上凸, 故对于任意 $x_1, x_2 \in (a, b), x_1 < x_2$,

$$f(x_1) \leq f(x_2) + f'(x_2)(x_1 - x_2), f(x_2) \leq f(x_1) + f'(x_1)(x_2 - x_1).$$

两式相加得

$$0 \leq (f'(x_1) - f'(x_2))(x_2 - x_1),$$

消去 $x_2 - x_1 > 0$ 得 $0 \leq f'(x_1) - f'(x_2)$, 即 $f'(x_2) \leq f'(x_1)$, $f'(x)$ 是单调递减函数, 故 $f''(x) \leq 0, x \in (a, b)$.

习题 4.6

1. 求下列曲线在指定点的曲率:

(1) $y = 3x^3 - x + 1$ 在 $\left(-\frac{1}{3}, \frac{11}{9}\right)$ 处;

(2) $y = \frac{x^2}{x-1}$ 在 $\left(3, \frac{9}{2}\right)$ 处;

(3) $x(t) = a(t - \sin t)$, $y(t) = a(1 - \cos t)$, 其中 a 为常数, 在 $t = \pi/2$ 处.

解 (1) $y' = 9x^2 - 1$, $y'' = 18x$, $K = \frac{|y''|}{(1 + y'^2)^{3/2}} = \frac{|-6|}{(1 + 0^2)^{3/2}} = 6$.

(2) $y = x + 1 + \frac{1}{x-1}$, $y' = 1 - \frac{1}{(x-1)^2}$, $y'' = \frac{2}{(x-1)^3}$, $K = \frac{\frac{1}{4}}{(1 + \frac{9}{16})^{3/2}} = \frac{16}{125}$.

(3) $x' = a(1 - \cos t)$, $x'' = a \sin t$, $y' = a \sin t$, $y'' = a \cos t$, $K = \frac{a^2}{(a^2 + a^2)^{3/2}} = \frac{1}{2\sqrt{2}a}$.

2. 求曲线 $y = 2x^2 + 1$ 在点 $(0, 1)$ 处的曲率圆方程.

解 $y' = 4x$, $y'' = 4$. $\alpha = x_0 - \frac{y'(1 + y'^2)}{y''} = 0$, $\beta = y_0 + \frac{(1 + y'^2)}{y''} = 1 + \frac{1}{4} = \frac{5}{4}$,

$K = \frac{|y''|}{(1 + y'^2)^{3/2}} = 4$, $R = \frac{1}{4}$, 曲率圆方程: $x^2 + (y - \frac{5}{4})^2 = \left(\frac{1}{4}\right)^2$.

3. 问曲线 $y = 2x^2 - 4x + 3$ 上哪一点处曲率最大? 并对其作几何解释.

解 $y' = 4x - 4$, $y'' = 4$. $K = \frac{|y''|}{(1 + y'^2)^{3/2}} = \frac{4}{(1 + (4x - 4)^2)^{3/2}}$ 当 $x = 1$ 时最大, 对应点 $(1, 1)$

恰是抛物线的顶点.

18. 设函数 $f(x)$ 在 $(-\infty, +\infty)$ 内可导, 且 a, b 是方程 $f(x) = 0$ 的两个实根. 证明方程 $f(x) + f'(x) = 0$ 在 (a, b) 内至少有一个实根.

证 设 $g(x) = e^x f(x)$, $g(a) = g(b) = 0$, g 在 $[a, b]$ 连续, 在 (a, b) 可导, .

根据 Rolle 定理, 存在 $c \in (a, b)$, 使得 $g'(c) = e^c (f(c) + f'(c)) = 0$, 即 $f(c) + f'(c) = 0$.

19. 决定常数 A 的范围, 使方程 $3x^4 - 8x^3 - 6x^2 + 24x + A$ 有四个不相等的实根.

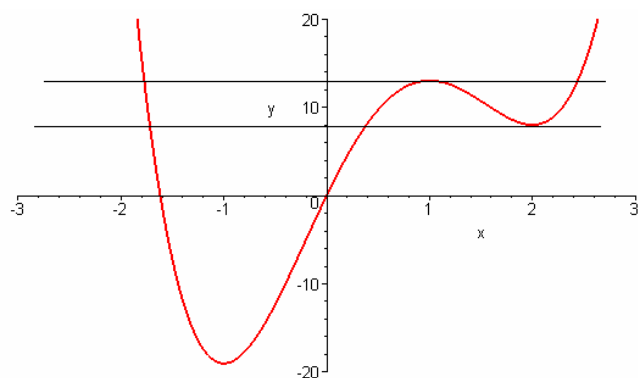
解 $P(x) = 3x^4 - 8x^3 - 6x^2 + 24x$, $P'(x) = 12x^3 - 24x^2 - 12x + 24$

$$= 12(x^3 - 2x^2 - x + 2) = 12[x^2(x-2) - (x-2)] = 12(x-2)(x^2-1) = 12(x-2)(x-1)(x+1) = 0,$$

$$x_1 = -1, x_2 = 1, x_3 = 2. P(x_1) = -19, P(1) = 13, P(2) = 8.$$

根据这些数据画图, 由图易知当在区间 $(-P(1), -P(2)) = (-13, -8)$ 时

$3x^4 - 8x^3 - 6x^2 + 24x + A$ 有四个不相等的实根.



20. 设 $f(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + L + (-1)^n \frac{x^n}{n}$. 证明: 方程 $f(x) = 0$ 当 n 为奇数时有一个实根, 当 n 为偶数时无实根.

证 当 $x \leq 0$ 时 $f(x) > 0$, 故 f 只有正根, 当 $n = 2k - 1$ 为奇数时, $\lim_{x \rightarrow -\infty} f(x) = +\infty$,

$\lim_{x \rightarrow +\infty} f(x) = -\infty$, 存在 $a, b, a < b, f(a) > 0, f(b) < 0$.

根据连续函数的中间值定理, 存在 $x_0 \in (a, b)$, 使得 $f(x_0) = 0$.

$$f'(x) = -1 + x - x^2 + L - x^{2k-2} = \frac{x^{2k-1} + 1}{-x-1} < 0 (x > 0), \text{ 当 } x > 0 \text{ 时, } f \text{ 严格单调递减, 故}$$

实根唯一.

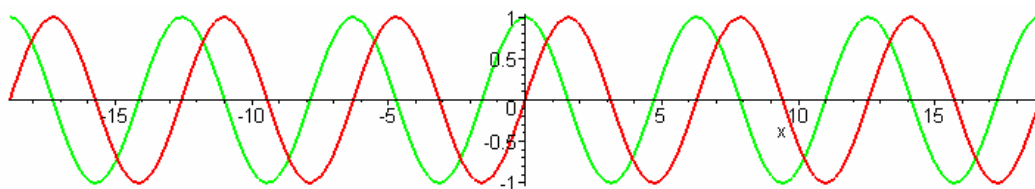
$$\text{当 } n = 2k \text{ 为偶数时, } f'(x) = -1 + x - x^2 + L + x^{2k-1} = \frac{-x^{2k} + 1}{-x-1} = 0, x = 1.$$

$0 < x < 1, f'(x) < 0, x > 1, f'(x) > 0$, $f(1)$ 是 $x > 0$ 时的最小值, $f(1) > 0$, 故当 n 为偶数时 $f(x)$ 无实根.

21. 设函数 $u(x)$ 与 $v(x)$ 以及它们的导函数 $u'(x)$ 与 $v'(x)$ 在区间 $[a, b]$ 上都连续, 且 $uv' - u'v$ 在 $[a, b]$ 上恒不等于零. 证明 $u(x)$ 在 $v(x)$ 的相邻根之间必有一根, 反之也对. 即有 $u(x)$ 与 $v(x)$ 的根互相交错地出现. 试句举处满足上述条件的 $u(x)$ 与 $v(x)$.

证设 x_1, x_2 是 $u(x)$ 的在 $[a, b]$ 的两个根, $x_1 < x_2$. 由于 $u'v - uv' \neq 0, v(x_1) \neq 0, v(x_2) \neq 0$. 如果 $v(x)$ 在 $[x_1, x_2]$ 上没有根, 则 $w = \frac{u}{v}$ 在 $[a, b]$ 连续, $w(x_1) = w(x_2) = 0$, 由Rolle定理, 存在 $c \in [x_1, x_2]$, 使得 $w'(c) = \frac{u'v - uv'}{v^2}(c) = 0$, 即 $(u'v - uv')(c) = 0$, 此与 $u'v - uv'$ 恒不等于零的假设矛盾. 故 $v(x)$ 在 $[x_1, x_2]$ 上有根.

例如 $u = \cos(x), v = \sin x, u'v - uv' = -1 \neq 0, \sin x \cos x$ 的根交错出现.



22. 证明: 当 $x > 0$ 时函数 $f(x) = \frac{\arctan x}{\tanh x}$ 单调'递增, 且 $\arctan x < \frac{\pi}{2} (\tanh x)$.

$$\text{证 } f'(x) = \left(\frac{\arctan x}{\tanh x} \right)' = \frac{\frac{\tanh x}{1+x^2} - \frac{\arctan x}{\cosh^2 x}}{\tanh^2 x} = \frac{\sinh x \cosh x - (1+x^2) \arctan x}{(1+x^2) \tanh^2 x \cosh^2 x}$$

$$= \frac{\frac{1}{2} \sinh 2x - (1+x^2) \arctan x}{(1+x^2) \tanh^2 x \cosh^2 x} = \frac{g(x)}{(1+x^2) \tanh^2 x \cosh^2 x}.$$

$$g(0) = 0.$$

$$g'(x) = \cosh 2x - 1 - 2x \arctan x, g'(0) = 0,$$

$$g''(x) = 2 \sinh 2x - 2 \arctan x - \frac{2x}{1+x^2}, g''(0) = 0,$$

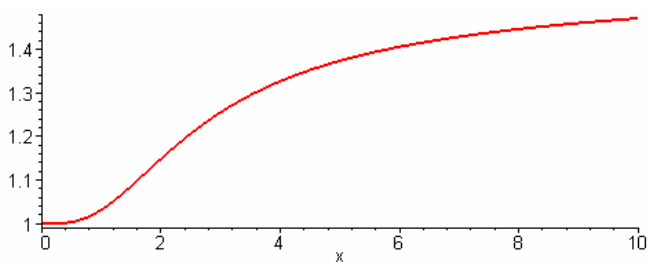
$$g'''(x) = 4 \cosh 2x - \frac{2}{1+x^2} - 2 \times \frac{(1+x^2) - 2x^2}{(1+x^2)^2} = 4 \cosh 2x - \frac{2}{1+x^2} - \frac{2(1-x^2)}{1+x^2}$$

$$= 4 \cosh 2x - \frac{4}{1+x^2} + \frac{4x^2}{1+x^2} > 0 (\text{当 } x > 0 \text{ 时 } \cosh x > 1),$$

由Taylor公式, 对于 $x > 0$ 有

$$g(x) = \frac{g'''(\theta x)}{3!} x^3 > 0, f'(x) > 0, f \text{ 严格单调递增.}$$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{\arctan x}{\tanh x} = \frac{\pi}{2}, \text{ 故对于 } x > 0 \text{ 有 } \frac{\arctan x}{\tanh x} < \frac{\pi}{2}.$$



23. 证明: 当 $0 < x < \frac{\pi}{2}$ 时有 $\frac{x}{\sin x} < \frac{\tan x}{x}$.

证 $f(x) = \sin x \tan x - x^2$,

$$f'(x) = \cos x \tan x + \sin x \sec^2 x - 2x = \sin x + \sin x \sec^2 x - 2x,$$

$$f''(x) = \cos x + \sec x + 2 \sin x \sec^2 x \tan x - 2 = (\cos x + \sec x - 2) + 2 \sin^2 x \sec x - 2 > 0$$

$$(\cos x + \sec x = \cos x + \frac{1}{\cos x} \geq 2, x \in (0, \pi/2)).$$

$f(0) = f'(0) = 0$, 根据Taylor公式,

$$f(x) = \frac{f''(\theta x)}{2} x^2 > 0, \sin x \tan x - x^2 > 0, \frac{x}{\sin x} < \frac{\tan x}{x} (x \in (0, \pi/2)).$$

24. 证明下列不等式:

$$(1) e^x > 1 + x, x \neq 0.$$

$$(2) x - \frac{x^2}{2} < \ln(1+x), x > 0.$$

$$(3) x - \frac{x^3}{6} < \sin x < x, x > 0.$$

$$\text{证 } (1) e^x = 1 + x + \frac{e^{\theta x}}{2} x^2 > 1 + x, x \neq 0.$$

$$(2) \ln(1+x) = x - \frac{1}{(1+\theta x)^2} x^2 < x, x > 0.$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{1}{3(1+\theta x)^3} x^3 > x - \frac{x^2}{2}, x > 0.$$

(3) $f(x) = x - \sin x, f(0) = 0, f'(x) = 1 - \cos x \geq 0$, 仅当 $x = 2n\pi$ 时 $f'(x) = 0$, 故当 $x > 0$ 时 f 严格单调递增, $f(x) > f(0) = 0, x > 0$.

$$g(x) = \sin x - \left(x - \frac{x^3}{6} \right),$$

$$g'(x) = \cos x - \left(1 - \frac{x^2}{2} \right), g''(x) = -\sin x + x > 0, x > 0. g \text{ 当 } x > 0 \text{ 时}$$

严格单调递增, $g(x) > g(0) = 0, x > 0$.

25. 设 $x_n = (1+q)(1+q^2)\cdots(1+q^n)$, 其中常数 $q \in [0, 1)$. 证明序列 x_n 有极限.

$$\text{证 } x_n \text{ 单调递增. } \ln x_n = \sum_{i=1}^n \ln(1+q^i) < \sum_{i=1}^n q^i = \frac{q-q^{n+1}}{1-q} < \frac{q}{1-q},$$

$$x_n = e^{\ln x_n} < e^{\frac{q}{1-q}}. x_n \text{ 有上界. 故 } x_n \text{ 有极限.}$$

26. 求函数 $f(x) = \tan x$ 在 $x = \pi/4$ 处的三阶Taylor多项式, 并由此估计 $\tan(50^\circ)$ 的值.

$$\text{解 } f'(x) = \sec^2 x, f''(x) = 2 \sec^2 x \tan x, f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x.$$

$$f\left(\frac{\pi}{4}\right) = 1, f'\left(\frac{\pi}{4}\right) = 2, f''\left(\frac{\pi}{4}\right) = 4, f'''\left(\frac{\pi}{4}\right) = 16.$$

$$f(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + o\left(\left(x - \frac{\pi}{4}\right)^3\right).$$

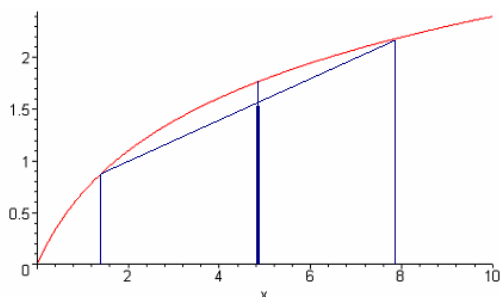
$$\tan(50^\circ) = \tan\left(\frac{\pi}{4} + \frac{\pi}{36}\right) \approx 1 + 2 \times \frac{\pi}{36} + 2\left(\frac{\pi}{36}\right)^2 + \frac{8}{3}\left(\frac{\pi}{36}\right)^3 \approx 1.191536480.$$

27. 设 $0 < a < b$, 证明 $(1+a)\ln(1+a) + (1+b)\ln(1+b) < (1+a+b)\ln(1+a+b)$.

$$\text{证 } f(x) = \ln(1+x), f'(x) = \frac{1}{1+x}, f''(x) = -\frac{1}{(1+x)^2} < 0,$$

f 在 $x > 0$ 上凸,

$$\begin{aligned} & \frac{(1+a)}{(1+a+b)}\ln(1+a) + \frac{(1+b)}{(1+a+b)}\ln(1+b) \\ & < \ln\left(1 + \frac{(1+a)a}{(1+a+b)} + \frac{(1+b)b}{(1+a+b)}\right) \\ & < \ln\left(1 + \frac{(1+a+b)a}{(1+a+b)} + \frac{(1+a+b)b}{(1+a+b)}\right) = \ln(1+a+b). \end{aligned}$$



28. 设有三个常数 a, b, c , 满足

$$a < b < c, a+b+c=2, ab+bc+ca=1. \text{ 证明: } 0 < a < \frac{1}{3}, \frac{1}{3} < b < 1, 1 < c < \frac{4}{3}.$$

$$\text{证 考虑多项式 } f(x) = (x-a)(x-b)(x-c) = x^3 - 2x^2 + x - abc.$$

$$f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1) = 0, x_1 = \frac{1}{3}, x_2 = 1.$$

当 $x < \frac{1}{3}$ 或 $x > 1$ 时 $f'(x) > 0$, f 严格单调递增, 当 $\frac{1}{3} < x < 1$ 时 $f'(x) < 0$, f 严格单调递减.

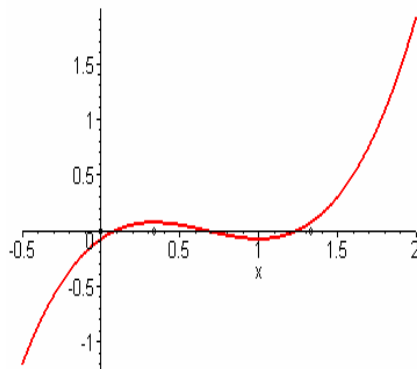
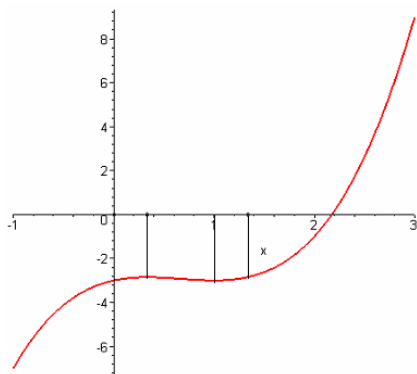
如果 $f(0) = f(1) = -abc \geq 0$, f 将至多有两个

实根. 如果 $f(\frac{1}{3}) = f(\frac{4}{3}) = \frac{4}{27} - abc \leq 0$, f 也将至多有两个

根 (见附图). 而 f 实际有根 a, b, c . 故 $f(0) = f(1) = -abc < 0$, 并且 $f(\frac{1}{3}) = f(\frac{4}{3}) = \frac{4}{27} - abc > 0$.

考虑到严格单调性, 于是 f

在 $(0, \frac{1}{3}), (\frac{1}{3}, 1), (1, \frac{4}{3})$ 各有一实根, 正是 a, b, c , 故结论成立.



29. 设函数 $f(x)$ 的二阶导数 $f''(x)$ 在 $[a, b]$ 上连续, 且对于每一点 $x \in [a, b]$, $f''(x)$ 与 $f(x)$ 同号. 证明: 若有两点 $c, d \in [a, b]$, 使 $f(c) = f(d) = 0$, 则 $f(x) \equiv 0, x \in [c, d]$.

证 由于 $f''(x)$ 与 $f(x)$ 同号, $(f(x)f'(x))' = f'^2(x) + f(x)f''(x) \geq 0$, $g(x) = f(x)f'(x)$ 单调, $g(c) = g(d) = 0$, 故 $f(x)f'(x) \equiv 0, x \in [c, d]$. $(f^2(x))' = 2f(x)f'(x) \equiv 0, x \in [c, d]$.

$f^2(x) \equiv C, x \in [c, d]$. $f^2(c) = 0$, 故 $f^2(x) \equiv 0, x \in [c, d]$, 即 $f(x) \equiv 0, x \in [c, d]$.

30. 求多项式 $P_3(x) = 2x^3 - 7x^2 + 13x - 9$ 在 $x=1$ 处的 Taylor 公式.

解 $P_3'(x) = 6x^2 - 14x + 13, P_3''(x) = 12x - 14, P_3'''(x) = 12$.

$P_3(1) = -1, P_3'(1) = 5, P_3''(1) = -2, P_3'''(1) = 12$.

$P_3(x) = -1 + 5(x-1) - (x-1)^2 + 2(x-1)^3$.

31. 设 $P_n(x)$ 是一个 n 次多项式.

(1) 证明: $P_n(x)$ 在任一点 x_0 处的 Taylor 公式为

$$P_n(x) = P_n(x_0) + P_n'(x_0)(x-x_0) + \frac{1}{2!}P_n''(x_0)(x-x_0)^2 + \cdots + \frac{1}{n!}P_n^{(n)}(x_0)(x-x_0)^n.$$

(2) 若存在一个数 a , 使 $P_n(a) > 0, P_n^{(k)}(a) \geq 0 (k=1, 2, \dots, n)$. 证明 $P_n(x)$ 的所有实根都不超过 a .

证 (1) $P_n(x)$ 是一个 n 次多项式.

(1) 证明: 因为 $P_n(x)$ 是一个 n 次多项式, $P_n^{(n+1)}(x) \equiv 0, x \in (-\infty, +\infty)$. 故在任一点 x_0 处, 根据带 Lagrange 余项的 Taylor 公式

$$\begin{aligned} P_n(x) &= P_n(x_0) + P_n'(x_0)(x-x_0) + \frac{1}{2!}P_n''(x_0)(x-x_0)^2 + \cdots + \frac{1}{(n+1)!}P_n^{(n+1)}(\xi)(x-x_0)^{n+1} \\ &= P_n(x_0) + P_n'(x_0)(x-x_0) + \frac{1}{2!}P_n''(x_0)(x-x_0)^2 + \cdots + \frac{1}{n!}P_n^{(n)}(x_0)(x-x_0)^n. \end{aligned}$$

$$(2) P_n(x) = P_n(a) + P_n'(a)(x-a) + \frac{1}{2!}P_n''(a)(x-a)^2 + \cdots + \frac{1}{n!}P_n^{(n)}(a)(x-a)^n \geq P_n(a) > 0 (x \geq a),$$

故 $P_n(x)$ 的所有实根都小于 a .

32. 设函数 $f(x)$ 在 $(0, +\infty)$ 上有二阶导数, 又知对于一切 $x > 0$, 有

$|f(x)| \leq A, |f''(x)| \leq B$ 其中 A, B 为常数. 证明: $|f'(x)| \leq 2\sqrt{AB}, x \in (0, +\infty)$.

证 任意取 $x \in (0, +\infty), h > 0$. $f(x+h) = f(x) + f'(x)h + \frac{f''(c)}{2}h^2$,

$$f'(x) = \frac{1}{h}(f(x+h) - f(x)) - \frac{f''(c)}{2}h.$$

$$|f'(x)| \leq \frac{2A}{h} + \frac{B}{2}h (*).$$

当 $\frac{2A}{h} = \frac{B}{2}h$ 时 (*) 右端取最小值. 在 (*) 中取 $h = 2\sqrt{\frac{A}{B}}$, 即得 $|f'(x)| \leq 2\sqrt{AB}$.

第四章总练习题

1. 设 $y=f(x)$ 在 $[x_0-h, x_0+h]$ ($h>0$) 内可导. 证明存在 $\theta, 0<\theta<1$ 使得

$$f(x_0+h)-f(x_0-h)=[f'(x_0+\theta h)+f'(x_0-\theta h)]h.$$

证 令 $g(x)=f(x_0+x)-f(x_0-x), x \in [0, h]$. $g(x)$ 在 $[0, h]$ 内可导,

$$g'(x)=f'(x_0+x)+f'(x_0-x), g(0)=0.$$

根据 Lagrange 公式, 存在 $\theta \in (0, 1)$ 使得

$$g(h)-g(0)=g'(\theta h)h, \text{ 即 } f(x_0+h)-f(x_0-h)=[f'(x_0+\theta h)+f'(x_0-\theta h)]h.$$

2. 证明: 当 $x \geq 0$ 时, 等式 $\sqrt{x+1}-\sqrt{x}=\frac{1}{2\sqrt{x+\theta(x)}}$

中的 $\theta(x)$ 满足 $1/4 \leq \theta(x) \leq 1/2$ 且 $\lim_{x \rightarrow 0} \theta(x) = 1/4, \lim_{x \rightarrow +\infty} \theta(x) = 1/2$.

$$\text{证 } \sqrt{x+1}-\sqrt{x}=\frac{1}{2\sqrt{x+\theta(x)}}, 2\sqrt{x+\theta(x)}=\frac{1}{\sqrt{x+1}-\sqrt{x}}=\sqrt{x+1}+\sqrt{x},$$

$$4(x+\theta(x))=2x+1+2\sqrt{x(x+1)},$$

$$\theta(x)=\frac{1}{4}(1+2\sqrt{x(x+1)}-2x).$$

$$\theta(x) \geq \frac{1}{4}(1+2\sqrt{x(x)}-2x)=\frac{1}{4},$$

由算术-几何平均不等式得

$$\theta(x)=\frac{1}{4}(1+2\sqrt{x(x+1)}-2x) \leq \frac{1}{4}(1+(x+x+1)-2x)=\frac{1}{2}.$$

$$\lim_{x \rightarrow 0} \theta(x) = \lim_{x \rightarrow 0} \frac{1}{4}(1+2\sqrt{x(x+1)}-2x) = \frac{1}{4}.$$

$$\lim_{x \rightarrow +\infty} \theta(x) = \lim_{x \rightarrow +\infty} \frac{1}{4}(1+2\sqrt{x(x+1)}-2x)$$

$$= \frac{1}{4} \lim_{x \rightarrow +\infty} \frac{(1+2\sqrt{x(x+1)}-2x)(1+2\sqrt{x(x+1)}+2x)}{(1+2\sqrt{x(x+1)}+2x)}$$

$$= \frac{1}{4} \lim_{x \rightarrow +\infty} \frac{1+4x+4\sqrt{x(x+1)}}{(1+2\sqrt{x(x+1)}+2x)} = \frac{1}{4} \lim_{x \rightarrow +\infty} \frac{1/x+4+4\sqrt{1+1/x}}{(1/x+2\sqrt{(1/x+1)}+2)} = \frac{1}{2}.$$

3. 设 $f(x) = \begin{cases} \frac{3-x^2}{2}, & 0 \leq x \leq 1 \\ \frac{1}{x}, & 1 < x < +\infty \end{cases}$ 求 $f(x)$ 在闭区间 $[0, 2]$ 上的微分中值定理的中间值.

解 $f'(x) = \begin{cases} -x, & 0 \leq x \leq 1 \\ -\frac{1}{x^2}, & 1 < x < +\infty \end{cases} \cdot \frac{f(2)-f(0)}{2-0} = \frac{1/2-3/2}{2} = -\frac{1}{2}.$

$-x = -\frac{1}{2}, x = \frac{1}{2}; -\frac{1}{x^2} = -\frac{1}{2}, x = \sqrt{2}. f(x)$ 在闭区间 $[0, 2]$ 上的微分中值定理的中间值为 $\frac{1}{2}$ 或 $\sqrt{2}$.

4.在闭区间 $[-1,1]$ 上Cauchy中值定理对于函数 $f(x)=x^2$ 与 $g(x)=x^3$ 是否成立?并说明理由.

解由于 $g'(x)=3x^2$ 有零点 $0 \in (-1,1)$,Cauchy中值定理的条件不满足. 其实其结论也不成立.

因为若 $\frac{f(1)-f(-1)}{g(1)-g(-1)}=0=\frac{f'(c)}{g'(c)}$, $f'(c)=2c=0$, $c=0$,但 $g'(0)=0$, $\frac{f'(c)}{g'(c)}$ 无意义.

5.设 $f(x)$ 在 $[a,b]$ 上连续,在 (a,b) 上有二阶导数,且 $f''(x) \neq 0$, $x \in (a,b)$ 又 $f(a)=f(b)=0$,证明当 $x \in (a,b)$ 时 $f(x) \neq 0$.

证一若存在 $c \in (a,b)$, $f(c)=0$,则由Rolle定理,存在 $c_1 \in (a,c)$, $c_2 \in (c,b)$ 使得 $f'(c_1)=f'(c_2)=0$.

对于 $f'(x)$ 在 $[c_1,c_2]$ 应用定理, 存在 $\xi \in (c_1,c_2)$,使得 $f''(\xi)=0$,此与条件 $f''(x) \neq 0$, $x \in (a,b)$ 矛盾.

证二由假设, $f''(x) \neq 0$, $x \in (a,b)$,根据Darboux定理, $f''(x)$ 恒正或恒负.不妨设 $f''(x)$ 恒正,于是 f 下凸, 曲线严格在连结 $(a, f(a))=(a,0)$ $(b, f(b))=(b,0)$ 的弦下方,故 $f(x) < 0$, $x \in (a,b)$.

6.设 $f(x)$ 在 $[a,b]$ 上有二阶导数,且 $f(a)=f(b)=0$,又存在 $c \in (a,b)$ 使 $f(c) > 0$.证明:在 (a,b) 内至少存在一点 x_0 使 $f''(x_0) < 0$.

证一由公式,存在 $c_1 \in (a,c)$,满足 $f'(c_1)=\frac{f(c)-f(a)}{c-a}=\frac{f(c)}{c-a} > 0$,

存在 $c_2 \in (c,b)$,满足 $f'(c_2)=\frac{f(b)-f(c)}{b-c}=\frac{-f(c)}{c-a} < 0$.

对于 $f'(x)$ 在 $[c_1,c_2]$ 应用Lagrange公式, 存在 $x_0 \in (c_1,c_2)$,使得

$$f''(x_0)=\frac{f'(c_2)-f'(c_1)}{c_2-c_1} < 0.$$

证二若不然, $f''(x) \geq 0$, $x \in (a,b)$, f 在 $[a,b]$ 下凸,曲线在连结 $(a, f(a))=(a,0)$ $(b, f(b))=(b,0)$ 的弦下方,故 $f(x) \leq 0$, $x \in (a,b)$.

7.证明方程 $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1}$

在0与1之间有一个根.

证考虑函数

$$f(x) = \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \frac{a_2x^{n-1}}{n-1} + \dots + \frac{a_nx}{1} - \left(\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1} \right) x,$$

$$f'(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n - \left(\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1} \right)$$

$f(0)=f(1)=0$.由Rolle定理,存在 $c \in (0,1)$, $f'(c)=0$,即 c 是

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1}$$

在0与1之间的一个根.

8. 设函数 $f(x)$ 在有限区间 (a, b) 内可导, 但无界, 证明 $f'(x)$ 在 (a, b) 内也无界.

逆命题是否成立? 试举例说明.

证 若不然, 设 $f'(x)$ 在 (a, b) 内有界 M , 取定 $x_0 \in (a, b)$, 则对于任意 $x \in (a, b)$,

根据 Lagrange 公式, $f(x) - f(x_0) = f'(c)(x - x_0)$,

$$|f(x)| = |f(x_0) + f'(c)(x - x_0)| \leq |f(x_0)| + |f'(c)| |x - x_0| \leq |f(x_0)| + M |b - a|.$$

逆命题不成立. 例如 \sqrt{x} 在 $(0, 1)$ 内有界, $0 < \sqrt{x} < 1$, 但是 $\sqrt{x}' = \frac{1}{2\sqrt{x}}$ 在 $(0, 1)$ 内无界.

9. 若函数 $f(x)$ 在区间 $[a, b]$ 上有 n 个根 (一个 k 重根算作 k 个根), 且存在 $f^{(n-1)}(x)$,

证明 $f^{(n-1)}(x)$ 在 $[a, b]$ 至少有一个根. (注意: 若 $f(x)$ 可以表示成 $f(x) = (x - x_0)^k g(x)$ 且 $g(x_0) \neq 0$, 则称 x_0 为 $f(x)$ 的 k 重根).

证 我们对于 n 作归纳法证明. 函数 $f(x)$ 在区间 $[a, b]$ 上有 2 个根. 如果 x_0 是 2 重根, 则

$f(x) = (x - x_0)^2 g(x)$ 且 $g(x_0) \neq 0$, 则 $f'(x) = 2(x - x_0)g(x) + (x - x_0)^2 g'(x)$, $f'(x)$ 有根 x_0 .

如果 $f(x)$ 在区间 $[a, b]$ 上有 2 个不同的根 $x_1, x_2, x_1 < x_2$, 在 $[x_1, x_2]$ 应用 Rolle 定理, 存在 $x_0 \in (x_1, x_2)$, 使得 $f'(x_0) = 0$. 设结论对于 n 个根的情况成立. 现在假定 $f(x)$ 在区间 $[a, b]$ 上有 $n+1$ 个根. 如果 f 有 $n+1$ 重根 x_0 , 则

$f(x) = (x - x_0)^{n+1} g(x)$ 且 $g(x_0) \neq 0$, 则

$$f'(x) = (n+1)(x - x_0)^n g(x) + (x - x_0)^{n+1} g'(x) = (x - x_0)^n ((n+1)g(x) + (x - x_0)g'(x)),$$

$(n+1)g(x) + (x - x_0)g'(x) = g_1(x)$, $g_1(x_0) = (n+1)g(x_0) \neq 0$, $f'(x)$ 有 n 重根 x_0 .

如果 f 有 $n+1$ 个单重根 x_1, \dots, x_{n+1} , 在区间 $[x_1, x_2], \dots, [x_n, x_{n+1}]$ 上应用 Rolle 定理, 存在 $c_1 \in (x_1, x_2), \dots, c_n \in (x_n, x_{n+1})$ 使得 $f'(c_1) = \dots = f'(c_n) = 0$, $f'(x)$ 至少有 n 个根.

如果 f 有不同的根 x_1, \dots, x_k , 重数分别为 $n_1, \dots, n_k, n+1 > k > 1, \sum_{i=1}^k n_i = n+1$. 在 $[x_1, x_2],$

$\dots, [x_{k-1}, x_k]$ 上应用 Rolle 定理, 存在 $c_1 \in (x_1, x_2), \dots, c_{k-1} \in (x_{k-1}, x_k)$ 使得

$f'(c_1) = \dots = f'(c_{k-1}) = 0$. $f'(x)$ 至少有根 $k-1 + \sum_{i=1}^k (n_i - 1) = n$ 个. 对 $f'(x)$ 用归纳假设,

$(f'(x))^{(n)} = f^{(n+1)}(x)$ 至少有一个根.

10. 证明: Legendre 多项式 $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$ 在 $(-1, 1)$ 内有 n 个根.

证 $f(x) = \frac{1}{2^n n!} (x^2 - 1)^n, f(1) = f(-1) = 0$, 对于 f 在 $[-1, 1]$ 应用 Rolle 定理, 存在

$c_1^1 \in (-1, 1)$, 使得 $f'(c_1^1) = 0$. $f'(-1) = f'(1) = 0$ (当 $n > 1$ 时), 对于 f' 在 $(-1, c_1^1)$

$(c_1^1, 1)$ 应用 Rolle 定理, 存在

$c_1^2 \in (-1, c_1^1), c_2^2 \in (c_1^1, 1)$ 使得 $f'(c_1^2) = f'(c_2^2) = 0$. 如此下去, $f^{(n-1)}(x)$ 在

$(-1, 1)$ 有零点 $c_1^{n-1}, \dots, c_{n-1}^{n-1}, f^{(n-1)}(-1) = f^{(n-1)}(1) = 0$, 在 $(-1, c_1^{n-1}), (c_1^{n-1}, c_2^{n-1}),$

$\dots, (c_{n-1}^{n-1}, 1)$ 应用 Rolle 定理, 得到 $x_1, x_2, \dots, x_n \in (-1, 1)$ 使得 $f^{(n)}(x) = P_n(x) = 0$.

$P_n(x)$ 是 n 次多项式, 至多有 n 个零点, 故 $P_n(x)$ 恰有 n 个零点.

11. 设函数 f 在 $(-\infty, +\infty)$ 内可导, 且 $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x)$. 证明: 必存在一点 $c \in (-\infty, +\infty)$, 使得 $f'(c) = 0$.

证若 $f(x) \equiv \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x) = A$. $x \in (-\infty, +\infty)$, 取任意一点 $c \in (-\infty, +\infty)$, 都有 $f'(c) = 0$.

设存在 $f(x_0) \neq A$, 不妨设 $f(x_0) > A$. 根据极限不等式, 存在 a, b , 满足: $a < b$, $x_0 \in (a, b)$, $f(a) < f(x_0)$, $f(b) < f(x_0)$. f 在 $[a, b]$ 连续, 必在一点 $c \in [a, b]$ 取最大值. $f(c) \geq f(x_0) > f(a)$, $f(c) \geq f(x_0) > f(b)$, 故 $x_0 \in (a, b)$, x_0 为极大值点, 根据 Fermat 引理, $f'(c) = 0$.

12. 设函数 $f(x)$ 在无穷区间 $(x_0, +\infty)$ 可导, 且 $\lim_{x \rightarrow +\infty} f'(x) = 0$, 证明 $\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = 0$.

证由于 $\lim_{x \rightarrow +\infty} f'(x) = 0$, 根据极限定义, 存在正数 $x_1 > x_0$, 使得 $x > x_1$ 时 $|f'(x)| < \varepsilon$.

$$\left| \frac{f(x)}{x} \right| = \left| \frac{f(x) - f(x_1) + f(x_1)}{x} \right| = \left| \frac{f'(c)(x - x_1) + f(x_1)}{x} \right| \leq \frac{\varepsilon(x - x_1) + |f(x_1)|}{x}$$

$$< \varepsilon + \frac{|f(x_1)|}{x}. \text{ 为使 } \frac{|f(x_1)|}{x} < \varepsilon, \text{ 只需 } x > \frac{|f(x_1)|}{\varepsilon}. \text{ 令 } X = \max\{x_1, \frac{|f(x_1)|}{\varepsilon}\},$$

当 $x > X$ 时, 必有 $\left| \frac{f(x)}{x} \right| < 2\varepsilon$, 故 $\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = 0$.

13. 设函数 $f(x)$ 在无穷区间 $[a, +\infty)$ 内连续, 且当 $x > a$ 时 $f'(x) > l > 0$,

其中 l 为常数. 证明: 若 $f(a) < 0$, 则在区间 $\left(a, a - \frac{f(a)}{l}\right)$ 内方程

$f(x) = 0$ 有唯一实根.

证 $f(a) < 0$,

$$\left(a - \frac{f(a)}{l}\right) = f(a) + f'(c) \left(-\frac{f(a)}{l}\right) > f(a) + l \left(-\frac{f(a)}{l}\right) = 0,$$

f 在 $\left[a, a - \frac{f(a)}{l}\right]$ 连续, 由连续函数中间值定理,

在区间 $\left(a, a - \frac{f(a)}{l}\right)$ 内方程 $f(x) = 0$ 至少有一实根. 若有两个实根, 根据

Rolle 定理, $f'(x)$ 将在 $\left(a, a - \frac{f(a)}{l}\right)$ 有一零点, 这与条件 $f'(x) > l > 0$ 矛盾.

14. 设函数 $f(x)$ 在 $(-\infty, +\infty)$ 上可导, 且 $\lim_{x \rightarrow \infty} f'(x) = 0$. 现令 $g(x) = f(x+1) - f(x)$, 证明

$\lim_{x \rightarrow \infty} g(x) = 0$.

证 $\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} (f(x+1) - f(x)) = \lim_{x \rightarrow \infty} f'(x+\theta)(0 < \theta < 1) = 0$.

15. 称函数 $f(x)$ 在 $[a, b]$ 满足 Lipschitz 条件, 若存在常数 $L > 0$, 使对于任意 $x_1, x_2 \in [a, b]$, 都有 $|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$.

(1) 若 $f'(x)$ 在 $[a, b]$ 连续, 则 $f(x)$ 在 $[a, b]$ 满足 Lipschitz 条件

(2) (1) 中所述事实的逆命题是否成立?

(3) 举一个在 $[a, b]$ 上连续但不满足 Lipschitz 条件的函数.

解 (1) $f'(x)$ 在 $[a, b]$ 连续, 存在常数 $L > 0$, 使得 $|f'(x)| \leq L, x \in [a, b]$.

根据中值公式, 对于任意 $x_1, x_2 \in [a, b], x_1 < x_2$, 存在 $c \in [x_1, x_2]$, 使得

$$|f(x_1) - f(x_2)| = |f'(c)(x_1 - x_2)| = |f'(c)|(x_2 - x_1) \leq L(x_2 - x_1).$$

(2) 否. $f(x)$ 在 $[a, b]$ 满足 Lipschitz 条件, 未必处处可导, 更谈不到 $f'(x)$ 在 $[a, b]$ 连续. 例如, $f(x) = |x|$ 在 $[-1, 1]$ 满足 Lipschitz 条件, 但在 0 不可导.

(3) $f(x) = \sqrt{x}$ 在 $[0, 1]$ 连续, 但不满足 Lipschitz 条件, 因其导函数

$$f'(x) = \frac{1}{2\sqrt{x}} \text{ 在 } (0, 1] \text{ 无界.}$$

16. 设 $F(x)$ 在 $[a, b]$ 可导, 且其导函数 $F'(x) = f(x)$ 在 $[a, b]$ 上可积, 证明

$$\int_a^b f(x) dx = F(b) - F(a).$$

$$\text{证 } F(b) - F(a) = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = \sum_{i=1}^n F'(\xi_i)(x_i - x_{i-1})$$

$$\sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \rightarrow \int_a^b f(x) dx (\lambda(\Delta) \rightarrow 0).$$

$\{x_i\}$ 为 $[a, b]$ 的分割.

17. 设多项式 $P(x) - a$ 与 $P(x) - b$ 的全部根都是单实根, 证明对于任意实数 $c \in (a, b)$, 多项式 $P(x) - c$ 的根也全都是单实根.

证不妨设 $a=0, b>0, c \in (0, b), P(x)$ 是 n 次多项式, 且首项系数为正.

$P(x)$ 有单实根 $x_1 < x_2 < \dots < x_n$, 则这些根把实轴分为 $n+1$ 个区间, 每个区间保持固定正负号, 且正负相间. 否则某个根将为极值点, 导数为零, 此与单实根矛盾. 在两个无穷区间保持正号, 且严格单调递增或递减, 在每个有穷区间有一个最值点, 且在其两侧分别递增和递减, 设 $n = 2k$ 为偶数,

则 $\lim_{x \rightarrow \infty} P(x) = +\infty$. 设 $b > 0$ 且 $P(x) = b$ 有 n 个单实根 $x'_1 < x'_2 < \dots < x'_n$. 必有

$$x'_1 < x_1, x'_2, x'_3 \in (x_2, x_3), x'_4 < x_4, x'_5 \in (x_6, x_7), x'_8 \in (x_{2k}, +\infty), P(x'_i) = b.$$

根据连续函数的中间值定理, 对于 $c \in (0, b)$, 存在 $c_1 \in (-\infty, x'_1), c_2 \in (x'_2, x'_3),$

$$c_3 \in (x'_3, x_3), c_{2k-2} \in (x_{2k-2}, x'_{2k-2}), c_{2k-1} \in (x'_{2k-1}, x_{2k-1} + \infty), c_{2k} \in (x_{2k}, +\infty),$$

使得 $P(c_i) = c$. P 为 n 次多项式, c_i 是 $P(x) = c$ 的所有单实根.

第四章总练习题

1. 设 $y=f(x)$ 在 $[x_0-h, x_0+h]$ ($h>0$) 内可导. 证明存在 $\theta, 0<\theta<1$ 使得

$$f(x_0+h)-f(x_0-h)=[f'(x_0+\theta h)+f'(x_0-\theta h)]h.$$

证 令 $g(x)=f(x_0+x)-f(x_0-x), x \in [0, h]$. $g(x)$ 在 $[0, h]$ 内可导,

$$g'(x)=f'(x_0+x)+f'(x_0-x), g(0)=0.$$

根据Lagrange公式, 存在 $\theta \in (0, 1)$ 使得

$$g(h)-g(0)=g'(\theta h)h, \text{ 即 } f(x_0+h)-f(x_0-h)=[f'(x_0+\theta h)+f'(x_0-\theta h)]h.$$

2. 证明: 当 $x \geq 0$ 时, 等式 $\sqrt{x+1}-\sqrt{x}=\frac{1}{2\sqrt{x+\theta(x)}}$

中的 $\theta(x)$ 满足 $1/4 \leq \theta(x) \leq 1/2$ 且 $\lim_{x \rightarrow 0} \theta(x) = 1/4, \lim_{x \rightarrow +\infty} \theta(x) = 1/2$.

$$\text{证 } \sqrt{x+1}-\sqrt{x}=\frac{1}{2\sqrt{x+\theta(x)}}, 2\sqrt{x+\theta(x)}=\frac{1}{\sqrt{x+1}-\sqrt{x}}=\sqrt{x+1}+\sqrt{x},$$

$$4(x+\theta(x))=2x+1+2\sqrt{x(x+1)},$$

$$\theta(x)=\frac{1}{4}(1+2\sqrt{x(x+1)}-2x).$$

$$\theta(x) \geq \frac{1}{4}(1+2\sqrt{x(x)}-2x)=\frac{1}{4},$$

由算术-几何平均不等式得

$$\theta(x)=\frac{1}{4}(1+2\sqrt{x(x+1)}-2x) \leq \frac{1}{4}(1+(x+x+1)-2x)=\frac{1}{2}.$$

$$\lim_{x \rightarrow 0} \theta(x) = \lim_{x \rightarrow 0} \frac{1}{4}(1+2\sqrt{x(x+1)}-2x) = \frac{1}{4}.$$

$$\lim_{x \rightarrow +\infty} \theta(x) = \lim_{x \rightarrow +\infty} \frac{1}{4}(1+2\sqrt{x(x+1)}-2x)$$

$$= \frac{1}{4} \lim_{x \rightarrow +\infty} \frac{(1+2\sqrt{x(x+1)}-2x)(1+2\sqrt{x(x+1)}+2x)}{(1+2\sqrt{x(x+1)}+2x)}$$

$$= \frac{1}{4} \lim_{x \rightarrow +\infty} \frac{1+4x+4\sqrt{x(x+1)}}{(1+2\sqrt{x(x+1)}+2x)} = \frac{1}{4} \lim_{x \rightarrow +\infty} \frac{1/x+4+4\sqrt{1+1/x}}{(1/x+2\sqrt{(1/x+1)}+2)} = \frac{1}{2}.$$

3. 设 $f(x) = \begin{cases} \frac{3-x^2}{2}, & 0 \leq x \leq 1 \\ \frac{1}{x}, & 1 < x < +\infty \end{cases}$ 求 $f(x)$ 在闭区间 $[0, 2]$ 上的微分中值定理的中间值.

解 $f'(x) = \begin{cases} -x, & 0 \leq x \leq 1 \\ -\frac{1}{x^2}, & 1 < x < +\infty \end{cases} \cdot \frac{f(2)-f(0)}{2-0} = \frac{1/2-3/2}{2} = -\frac{1}{2}.$

$-x = -\frac{1}{2}, x = \frac{1}{2}; -\frac{1}{x^2} = -\frac{1}{2}, x = \sqrt{2}. f(x)$ 在闭区间 $[0, 2]$ 上的微分中值定理的中间值为 $\frac{1}{2}$ 或 $\sqrt{2}$.

4.在闭区间 $[-1,1]$ 上Cauchy中值定理对于函数 $f(x)=x^2$ 与 $g(x)=x^3$ 是否成立?并说明理由.

解由于 $g'(x)=3x^2$ 有零点 $0 \in (-1,1)$,Cauchy中值定理的条件不满足. 其实其结论也不成立.

因为若 $\frac{f(1)-f(-1)}{g(1)-g(-1)}=0=\frac{f'(c)}{g'(c)}$, $f'(c)=2c=0$, $c=0$,但 $g'(0)=0$, $\frac{f'(c)}{g'(c)}$ 无意义.

5.设 $f(x)$ 在 $[a,b]$ 上连续,在 (a,b) 上有二阶导数,且 $f''(x) \neq 0$, $x \in (a,b)$ 又 $f(a)=f(b)=0$,证明当 $x \in (a,b)$ 时 $f(x) \neq 0$.

证一若存在 $c \in (a,b)$, $f(c)=0$,则由Rolle定理,存在 $c_1 \in (a,c)$, $c_2 \in (c,b)$ 使得 $f'(c_1)=f'(c_2)=0$.

对于 $f'(x)$ 在 $[c_1,c_2]$ 应用定理, 存在 $\xi \in (c_1,c_2)$,使得 $f''(\xi)=0$,此与条件 $f''(x) \neq 0$, $x \in (a,b)$ 矛盾.

证二由假设, $f''(x) \neq 0$, $x \in (a,b)$,根据Darboux定理, $f''(x)$ 恒正或恒负.不妨设 $f''(x)$ 恒正,于是 f 下凸, 曲线严格在连结 $(a, f(a))=(a,0)$ $(b, f(b))=(b,0)$ 的弦下方,故 $f(x) < 0$, $x \in (a,b)$.

6.设 $f(x)$ 在 $[a,b]$ 上有二阶导数,且 $f(a)=f(b)=0$,又存在 $c \in (a,b)$ 使 $f(c) > 0$.证明:在 (a,b) 内至少存在一点 x_0 使 $f''(x_0) < 0$.

证一由公式,存在 $c_1 \in (a,c)$,满足 $f'(c_1)=\frac{f(c)-f(a)}{c-a}=\frac{f(c)}{c-a} > 0$,

存在 $c_2 \in (c,b)$,满足 $f'(c_2)=\frac{f(b)-f(c)}{b-c}=\frac{-f(c)}{c-a} < 0$.

对于 $f'(x)$ 在 $[c_1,c_2]$ 应用Lagrange公式, 存在 $x_0 \in (c_1,c_2)$,使得

$$f''(x_0)=\frac{f'(c_2)-f'(c_1)}{c_2-c_1} < 0.$$

证二若不然, $f''(x) \geq 0$, $x \in (a,b)$, f 在 $[a,b]$ 下凸,曲线在连结 $(a, f(a))=(a,0)$ $(b, f(b))=(b,0)$ 的弦下方,故 $f(x) \leq 0$, $x \in (a,b)$.

7.证明方程 $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1}$

在0与1之间有一个根.

证考虑函数

$$f(x) = \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \frac{a_2x^{n-1}}{n-1} + \dots + \frac{a_nx}{1} - \left(\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1} \right) x,$$

$$f'(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n - \left(\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1} \right)$$

$f(0)=f(1)=0$.由Rolle定理,存在 $c \in (0,1)$, $f'(c)=0$,即 c 是

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1}$$

在0与1之间的一个根.

8. 设函数 $f(x)$ 在有限区间 (a, b) 内可导, 但无界, 证明 $f'(x)$ 在 (a, b) 内也无界.

逆命题是否成立? 试举例说明.

证 若不然, 设 $f'(x)$ 在 (a, b) 内有界 M , 取定 $x_0 \in (a, b)$, 则对于任意 $x \in (a, b)$,

根据 Lagrange 公式, $f(x) - f(x_0) = f'(c)(x - x_0)$,

$$|f(x)| = |f(x_0) + f'(c)(x - x_0)| \leq |f(x_0)| + |f'(c)| |x - x_0| \leq |f(x_0)| + M |b - a|.$$

逆命题不成立. 例如 \sqrt{x} 在 $(0, 1)$ 内有界, $0 < \sqrt{x} < 1$, 但是 $\sqrt{x}' = \frac{1}{2\sqrt{x}}$ 在 $(0, 1)$ 内无界.

9. 若函数 $f(x)$ 在区间 $[a, b]$ 上有 n 个根 (一个 k 重根算作 k 个根), 且存在 $f^{(n-1)}(x)$,

证明 $f^{(n-1)}(x)$ 在 $[a, b]$ 至少有一个根. (注意: 若 $f(x)$ 可以表示成 $f(x) = (x - x_0)^k g(x)$ 且 $g(x_0) \neq 0$, 则称 x_0 为 $f(x)$ 的 k 重根).

证 我们对于 n 作归纳法证明. 函数 $f(x)$ 在区间 $[a, b]$ 上有 2 个根. 如果 x_0 是 2 重根, 则

$f(x) = (x - x_0)^2 g(x)$ 且 $g(x_0) \neq 0$, 则 $f'(x) = 2(x - x_0)g(x) + (x - x_0)^2 g'(x)$, $f'(x)$ 有根 x_0 .

如果 $f(x)$ 在区间 $[a, b]$ 上有 2 个不同的根 $x_1, x_2, x_1 < x_2$, 在 $[x_1, x_2]$ 应用 Rolle 定理, 存在 $x_0 \in (x_1, x_2)$, 使得 $f'(x_0) = 0$. 设结论对于 n 个根的情况成立. 现在假定 $f(x)$ 在区间 $[a, b]$ 上有 $n+1$ 个根. 如果 f 有 $n+1$ 重根 x_0 , 则

$f(x) = (x - x_0)^{n+1} g(x)$ 且 $g(x_0) \neq 0$, 则

$$f'(x) = (n+1)(x - x_0)^n g(x) + (x - x_0)^{n+1} g'(x) = (x - x_0)^n ((n+1)g(x) + (x - x_0)g'(x)),$$

$(n+1)g(x) + (x - x_0)g'(x) = g_1(x)$, $g_1(x_0) = (n+1)g(x_0) \neq 0$, $f'(x)$ 有 n 重根 x_0 .

如果 f 有 $n+1$ 个单重根 x_1, \dots, x_{n+1} , 在区间 $[x_1, x_2], \dots, [x_n, x_{n+1}]$ 上应用 Rolle 定理, 存在 $c_1 \in (x_1, x_2), \dots, c_n \in (x_n, x_{n+1})$ 使得 $f'(c_1) = \dots = f'(c_n) = 0$, $f'(x)$ 至少有 n 个根.

如果 f 有不同的根 x_1, \dots, x_k , 重数分别为 $n_1, \dots, n_k, n+1 > k > 1, \sum_{i=1}^k n_i = n+1$. 在 $[x_1, x_2],$

$\dots, [x_{k-1}, x_k]$ 上应用 Rolle 定理, 存在 $c_1 \in (x_1, x_2), \dots, c_{k-1} \in (x_{k-1}, x_k)$ 使得

$f'(c_1) = \dots = f'(c_{k-1}) = 0$. $f'(x)$ 至少有根 $k-1 + \sum_{i=1}^k (n_i - 1) = n$ 个. 对 $f'(x)$ 用归纳假设,

$(f'(x))^{(n)} = f^{(n+1)}(x)$ 至少有一个根.

10. 证明: Legendre 多项式 $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$ 在 $(-1, 1)$ 内有 n 个根.

证 $f(x) = \frac{1}{2^n n!} (x^2 - 1)^n, f(1) = f(-1) = 0$, 对于 f 在 $[-1, 1]$ 应用 Rolle 定理, 存在

$c_1^1 \in (-1, 1)$, 使得 $f'(c_1^1) = 0$. $f'(-1) = f'(1) = 0$ (当 $n > 1$ 时), 对于 f' 在 $(-1, c_1^1)$

$(c_1^1, 1)$ 应用 Rolle 定理, 存在

$c_1^2 \in (-1, c_1^1), c_2^2 \in (c_1^1, 1)$ 使得 $f'(c_1^2) = f'(c_2^2) = 0$. 如此下去, $f^{(n-1)}(x)$ 在

$(-1, 1)$ 有零点 $c_1^{n-1}, \dots, c_{n-1}^{n-1}, f^{(n-1)}(-1) = f^{(n-1)}(1) = 0$, 在 $(-1, c_1^{n-1}), (c_1^{n-1}, c_2^{n-1}),$

$\dots, (c_{n-1}^{n-1}, 1)$ 应用 Rolle 定理, 得到 $x_1, x_2, \dots, x_n \in (-1, 1)$ 使得 $f^{(n)}(x) = P_n(x) = 0$.

$P_n(x)$ 是 n 次多项式, 至多有 n 个零点, 故 $P_n(x)$ 恰有 n 个零点.

11. 设函数 f 在 $(-\infty, +\infty)$ 内可导, 且 $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x)$. 证明: 必存在一点 $c \in (-\infty, +\infty)$, 使得 $f'(c) = 0$.

证若 $f(x) \equiv \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x) = A$. $x \in (-\infty, +\infty)$, 取任意一点 $c \in (-\infty, +\infty)$, 都有 $f'(c) = 0$.

设存在 $f(x_0) \neq A$, 不妨设 $f(x_0) > A$. 根据极限不等式, 存在 a, b , 满足: $a < b$, $x_0 \in (a, b)$, $f(a) < f(x_0)$, $f(b) < f(x_0)$. f 在 $[a, b]$ 连续, 必在一点 $c \in [a, b]$ 取最大值. $f(c) \geq f(x_0) > f(a)$, $f(c) \geq f(x_0) > f(b)$, 故 $x_0 \in (a, b)$, x_0 为极大值点, 根据 Fermat 引理, $f'(c) = 0$.

12. 设函数 $f(x)$ 在无穷区间 $(x_0, +\infty)$ 可导, 且 $\lim_{x \rightarrow +\infty} f'(x) = 0$, 证明 $\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = 0$.

证由于 $\lim_{x \rightarrow +\infty} f'(x) = 0$, 根据极限定义, 存在正数 $x_1 > x_0$, 使得 $x > x_1$ 时 $|f'(x)| < \varepsilon$.

$$\left| \frac{f(x)}{x} \right| = \left| \frac{f(x) - f(x_1) + f(x_1)}{x} \right| = \left| \frac{f'(c)(x - x_1) + f(x_1)}{x} \right| \leq \frac{\varepsilon(x - x_1) + |f(x_1)|}{x}$$

$$< \varepsilon + \frac{|f(x_1)|}{x}. \text{ 为使 } \frac{|f(x_1)|}{x} < \varepsilon, \text{ 只需 } x > \frac{|f(x_1)|}{\varepsilon}. \text{ 令 } X = \max\{x_1, \frac{|f(x_1)|}{\varepsilon}\},$$

当 $x > X$ 时, 必有 $\left| \frac{f(x)}{x} \right| < 2\varepsilon$, 故 $\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = 0$.

13. 设函数 $f(x)$ 在无穷区间 $[a, +\infty)$ 内连续, 且当 $x > a$ 时 $f'(x) > l > 0$,

其中 l 为常数. 证明: 若 $f(a) < 0$, 则在区间 $\left(a, a - \frac{f(a)}{l}\right)$ 内方程

$f(x) = 0$ 有唯一实根.

证 $f(a) < 0$,

$$\left(a - \frac{f(a)}{l}\right) = f(a) + f'(c) \left(-\frac{f(a)}{l}\right) > f(a) + l \left(-\frac{f(a)}{l}\right) = 0,$$

f 在 $\left[a, a - \frac{f(a)}{l}\right]$ 连续, 由连续函数中间值定理,

在区间 $\left(a, a - \frac{f(a)}{l}\right)$ 内方程 $f(x) = 0$ 至少有一实根. 若有两个实根, 根据

Rolle 定理, $f'(x)$ 将在 $\left(a, a - \frac{f(a)}{l}\right)$ 有一零点, 这与条件 $f'(x) > l > 0$ 矛盾.

14. 设函数 $f(x)$ 在 $(-\infty, +\infty)$ 上可导, 且 $\lim_{x \rightarrow \infty} f'(x) = 0$. 现令 $g(x) = f(x+1) - f(x)$, 证明

$\lim_{x \rightarrow \infty} g(x) = 0$.

证 $\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} (f(x+1) - f(x)) = \lim_{x \rightarrow \infty} f'(x+\theta)(0 < \theta < 1) = 0$.

15. 称函数 $f(x)$ 在 $[a, b]$ 满足 Lipschitz 条件, 若存在常数 $L > 0$, 使对于任意 $x_1, x_2 \in [a, b]$, 都有 $|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$.

(1) 若 $f'(x)$ 在 $[a, b]$ 连续, 则 $f(x)$ 在 $[a, b]$ 满足 Lipschitz 条件

(2) (1) 中所述事实的逆命题是否成立?

(3) 举一个在 $[a, b]$ 上连续但不满足 Lipschitz 条件的函数.

解 (1) $f'(x)$ 在 $[a, b]$ 连续, 存在常数 $L > 0$, 使得 $|f'(x)| \leq L, x \in [a, b]$.

根据中值公式, 对于任意 $x_1, x_2 \in [a, b], x_1 < x_2$, 存在 $c \in [x_1, x_2]$, 使得

$$|f(x_1) - f(x_2)| = |f'(c)(x_1 - x_2)| = |f'(c)|(x_2 - x_1) \leq L(x_2 - x_1).$$

(2) 否. $f(x)$ 在 $[a, b]$ 满足 Lipschitz 条件, 未必处处可导, 更谈不到 $f'(x)$ 在 $[a, b]$ 连续. 例如, $f(x) = |x|$ 在 $[-1, 1]$ 满足 Lipschitz 条件, 但在 0 不可导.

(3) $f(x) = \sqrt{x}$ 在 $[0, 1]$ 连续, 但不满足 Lipschitz 条件, 因其导函数

$$f'(x) = \frac{1}{2\sqrt{x}} \text{ 在 } (0, 1] \text{ 无界.}$$

16. 设 $F(x)$ 在 $[a, b]$ 可导, 且其导函数 $F'(x) = f(x)$ 在 $[a, b]$ 上可积, 证明

$$\int_a^b f(x) dx = F(b) - F(a).$$

$$\text{证 } F(b) - F(a) = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = \sum_{i=1}^n F'(\xi_i)(x_i - x_{i-1})$$

$$\sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \rightarrow \int_a^b f(x) dx (\lambda(\Delta) \rightarrow 0).$$

$\{x_i\}$ 为 $[a, b]$ 的分割.

17. 设多项式 $P(x) - a$ 与 $P(x) - b$ 的全部根都是单实根, 证明对于任意实数 $c \in (a, b)$, 多项式 $P(x) - c$ 的根也全都是单实根.

证不妨设 $a=0, b>0, c \in (0, b), P(x)$ 是 n 次多项式, 且首项系数为正.

$P(x)$ 有单实根 $x_1 < x_2 < \dots < x_n$, 则这些根把实轴分为 $n+1$ 个区间, 每个区间保持固定正负号, 且正负相间. 否则某个根将为极值点, 导数为零, 此与单实根矛盾. 在两个无穷区间保持正号, 且严格单调递增或递减, 在每个有穷区间有一个最值点, 且在其两侧分别递增和递减, 设 $n = 2k$ 为偶数,

则 $\lim_{x \rightarrow \infty} P(x) = +\infty$. 设 $b > 0$ 且 $P(x) = b$ 有 n 个单实根 $x'_1 < x'_2 < \dots < x'_n$. 必有

$$x'_1 < x_1, x'_2, x'_3 \in (x_2, x_3), x'_4 < x_4, x'_5 \in (x_6, x_7), x'_8 \in (x_{2k}, +\infty), P(x'_i) = b.$$

根据连续函数的中间值定理, 对于 $c \in (0, b)$, 存在 $c_1 \in (-\infty, x'_1), c_2 \in (x'_2, x'_3),$

$$c_3 \in (x'_3, x_3), c_{2k-2} \in (x_{2k-2}, x'_{2k-2}), c_{2k-1} \in (x'_{2k-1}, x_{2k-1} + \infty), c_{2k} \in (x_{2k}, +\infty),$$

使得 $P(c_i) = c$. P 为 n 次多项式, c_i 是 $P(x) = c$ 的所有单实根.

习题 5.1

1. 设 $ABCD$ 为一平行四边形, $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AD} = \mathbf{b}$. 试用 \mathbf{a}, \mathbf{b} 表示 $\overrightarrow{AC}, \overrightarrow{DB}, \overrightarrow{MA}$ (M 为平行四边形对角线的交点).

解 $\overrightarrow{AC} = \mathbf{a} + \mathbf{b}$, $\overrightarrow{DB} = \mathbf{a} - \mathbf{b}$, $\overrightarrow{MA} = -\overrightarrow{AM} = -\frac{1}{2}\overrightarrow{AC} = -\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

2. 设 M 为线段 AB 的中点, O 为空间中的任意一点, 证明

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}).$$

$$\begin{aligned} \text{证 } \overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA}) \\ &= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}). \end{aligned}$$

3. 设 M 为三角形 ABC 的重心, O 为空间中任意一点,

$$\text{证明 } \overrightarrow{OM} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}).$$

$$\begin{aligned} \text{证 } \overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AD} = \overrightarrow{OA} + \frac{2}{3} \times \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) \\ &= \overrightarrow{OA} + \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC}), \end{aligned}$$

$$\overrightarrow{OM} = \overrightarrow{OB} + \frac{1}{3}(\overrightarrow{BA} + \overrightarrow{BC}), \overrightarrow{OM} = \overrightarrow{OC} + \frac{1}{3}(\overrightarrow{CA} + \overrightarrow{CB}).$$

$$3\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}, \overrightarrow{OM} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}).$$

4. 设平行四边形 $ABCD$ 的对角线交点为 M , O 为空间中的

$$\text{任意一点, 证明 } \overrightarrow{OM} = \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}).$$

$$\text{证 } \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AD}),$$

$$\overrightarrow{OM} = \overrightarrow{OB} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AD}), \overrightarrow{OM} = \overrightarrow{OC} + \frac{1}{2}(\overrightarrow{CA} + \overrightarrow{DA}),$$

$$\overrightarrow{OM} = \overrightarrow{OD} + \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{DA}).$$

$$4\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}, \overrightarrow{OM} = \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}).$$

5. 对于任意三个向量 a, b 与 c , 判断下列各式是否成立?

$$(1) (a \cdot b)c = (b \cdot a)c;$$

$$(2) (a \cdot b)^2 = a^2 b^2;$$

$$(3) a \cdot (b \times c) = (c \times a) \cdot b.$$

解(1)不成立. 例如: $a = b = i, c = j, (a \cdot b)c = j, (b \cdot a)c = 0$.

(2)不成立. 例如: $a = i, b = j, (a \cdot b)^2 = 0, a^2 b^2 = 1$.

(3)成立, 都是 a, b 与 c 组成的平行六面体的有向体积.

6. 利用向量证明三角形两边中点的连线平行于第三边, 并且等于第三边长度之半.

$$\begin{aligned} \text{证 } DE &= DA + AE = \frac{1}{2}BA + \frac{1}{2}AC \\ &= \frac{1}{2}(BA + AC) = \frac{1}{2}BC. \end{aligned}$$

7. 利用向量证明:

(1)菱形的对角线互相垂直, 且平分顶角; (2)勾股弦定理.

$$\begin{aligned} \text{证(1)} \quad AC \cdot BD &= (AB + BC) \cdot (BC + CD) \\ &= (AB + BC) \cdot (BC - CD) = |BC|^2 - |CD|^2 = 0. \end{aligned}$$

$$\cos \alpha = \frac{AB \cdot AC}{|AB| |AC|} = \frac{AB \cdot (AB + AD)}{|AB| |AC|} = \frac{AB \cdot AB + AB \cdot AD}{|AB| |AC|} = \frac{a^2 + AB \cdot AD}{|a| |AC|},$$

$$\cos \beta = \frac{AD \cdot (AB + AD)}{|AB| |AC|} = \frac{AD \cdot AB + AD \cdot AD}{|AB| |AC|} = \frac{a^2 + AB \cdot AD}{|a| |AC|} = \cos \alpha.$$

α 与 β 都是锐角, 故 $\alpha = \beta$.

$$\begin{aligned} (2) \quad |AC|^2 &= AC \cdot AC = (AB + BC) \cdot (AB + BC) \\ &= |AB|^2 + |BC|^2 + 2AB \cdot BC = |AB|^2 + |BC|^2. \end{aligned}$$

8. 证明恒等式 $(a \times b)^2 + (a \cdot b)^2 = |a|^2 |b|^2$.

$$\begin{aligned} \text{证} \quad (a \times b)^2 + (a \cdot b)^2 &= |a|^2 |b|^2 \cos^2 \alpha + |a|^2 |b|^2 \sin^2 \alpha \\ &= |a|^2 |b|^2 (\cos^2 \alpha + \sin^2 \alpha) = |a|^2 |b|^2. \end{aligned}$$

9. 试用向量 AB 与 AC 表示三角形 ABC 的面积.

$$\text{解} \quad \triangle ABC \text{ 的面积} = \frac{1}{2} |AB \times AC|.$$

10. 给定向量 a , 记 $a \cdot a$ 为 a^2 , 即 $a^2 = a \cdot a$. 现设 a, b 为任意向量, 证明:

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2).$$

$$\begin{aligned} \text{证} \quad (a+b)^2 + (a-b)^2 &= (a+b) \cdot (a+b) + (a-b) \cdot (a-b) \\ &= a \cdot a + b \cdot b + 2a \cdot b + a \cdot a + b \cdot b - 2a \cdot b = 2(a^2 + b^2). \end{aligned}$$

11. 对于任意向量 \mathbf{a}, \mathbf{b} , 证明: $(\mathbf{a} \times \mathbf{b})^2 \leq a^2 b^2$. 问: 等号成立的充分必要条件是什么?

证 $(\mathbf{a} \times \mathbf{b})^2 = |\mathbf{a} \times \mathbf{b}|^2 = (|\mathbf{a}| |\mathbf{b}| \sin \alpha)^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 \alpha \leq |\mathbf{a}|^2 |\mathbf{b}|^2 = a^2 b^2$.

等号成立的充分必要条件是 \mathbf{a}, \mathbf{b} 正交.

习题 5.2

1. 写出点 (x, y, z) 分别到 x 轴, y 轴, z 轴, Oxy 平面, Oyz 平面以及原点的距离.

解 $d_x = \sqrt{y^2 + z^2}, d_y = \sqrt{x^2 + z^2}, d_z = \sqrt{x^2 + y^2}, d_{xy} = |z|, d_{yz} = |x|, d_o = \sqrt{x^2 + y^2 + z^2}.$

2. 已知三点 $A = (-1, 2, 1), B = (3, 0, 1), C = (2, 1, 2)$, 求 $\overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AC}, \overrightarrow{BC}$ 的坐标与模.

解 $\overrightarrow{AB} = (3, 0, 1) - (-1, 2, 1) = (4, -2, 0), |\overrightarrow{AB}| = \sqrt{20} = 2\sqrt{5},$

$|\overrightarrow{BA}| = -\overrightarrow{AB} = -(4, -2, 0) = (-4, 2, 0) = -2\sqrt{5},$

$\overrightarrow{AC} = (2, 1, 2) - (-1, 2, 1) = (3, -1, 1), |\overrightarrow{AC}| = \sqrt{11},$

$\overrightarrow{BC} = (2, 1, 2) - (3, 0, 1) = (-1, 1, 1), |\overrightarrow{BC}| = \sqrt{3}.$

3. $\mathbf{a} = (3, -2, 2), \mathbf{b} = (1, 3, 2), \mathbf{c} = (8, 6, -2),$

$3\mathbf{a} - 2\mathbf{b} + \frac{1}{2}\mathbf{c} = (9, -6, 6) + (-2, -6, -4) + (4, 3, -1) = (11, -9, 1).$

4. 设 $\mathbf{a} = (2, 5, 1), \mathbf{b} = (1, -2, 7)$, 分别求出沿 \mathbf{a} 和 \mathbf{b} 方向的单位向量, 并求常数 k , 使 $k\mathbf{a} + \mathbf{b}$ 与 xy 平面平行.

解 $\mathbf{a}^\circ = \frac{1}{\sqrt{30}}(2, 5, 1), \mathbf{b}^\circ = \frac{1}{3\sqrt{6}}(1, -2, 7).$

$k\mathbf{a} + \mathbf{b} = (2k, 5k, k) + (1, -2, 7) = (2k+1, 5k, -2, k+7), k+7=0, k=-7.$

5. 设 A, B 两点的坐标分别为 (x_1, y_1, z_1) 和 (x_2, y_2, z_2) , 求 A, B 连线中点 C 的坐标.

解 $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}((x_1, y_1, z_1) + (x_2, y_2, z_2)) = \frac{1}{2}(x_1 + x_2, y_1 + y_2, z_1 + z_2).$

6. 设 $\mathbf{a} = (1, -2, 3), \mathbf{b} = (5, 2, -1)$, 求

(1) $2\mathbf{a} \cdot \mathbf{b}$ (2) $\mathbf{a} \cdot \mathbf{j}$ (3) $\cos \langle \mathbf{a}, \mathbf{b} \rangle.$

解 (1) $2\mathbf{a} \cdot \mathbf{b} = 6\mathbf{a} \cdot \mathbf{b} = 6 \times (-2) = -12.$

(2) $\mathbf{a} \cdot \mathbf{j} = 1.$

(3) $\cos \langle \mathbf{a}, \mathbf{b} \rangle = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{-2}{\sqrt{14}\sqrt{30}} = -\frac{1}{\sqrt{105}},$

7. 设 $|\mathbf{a}| = 1, |\mathbf{b}| = 3, |\mathbf{c}| = 2, |\mathbf{a} + \mathbf{b} + \mathbf{c}| = \sqrt{17 + 6\sqrt{3}}$ 且 $\mathbf{a} \perp \mathbf{c} \langle \mathbf{a}, \mathbf{b} \rangle = \pi/3$, 求 $\langle \mathbf{b}, \mathbf{c} \rangle = ?$

解 $17 + 6\sqrt{3} = |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$

$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c}) =$

$= 1 + 9 + 4 + 2(3 \times \frac{1}{2} + \mathbf{b} \cdot \mathbf{c}),$

$\mathbf{b} \cdot \mathbf{c} = 3\sqrt{3}, \cos \langle \mathbf{b}, \mathbf{c} \rangle = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}| |\mathbf{c}|} = \frac{3\sqrt{3}}{3 \times 2} = \frac{\sqrt{3}}{2}. \langle \mathbf{b}, \mathbf{c} \rangle = \frac{\pi}{6}.$

8. 设 $|a| = 2, |b| = 6$, 试求常数 k , 使 $a + kb \perp a - kb$.

解 $(a + kb) \cdot (a - kb) = |a|^2 - k^2 |b|^2 = 4 - 36k^2 = 0, k = \pm 1/3$.

9. $a = (1, -2, 1), b = (1, -1, 3), c = (2, 5, -3)$

(1) $a \times b$ (2) $c \times j$ (3) $(a \times b) \cdot c$ (4) $(a \times b) \times c$ (5) $a \times (b \times c)$.

解 (1) $a \times b = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 1 & -1 & 3 \end{vmatrix} = (-5, -2, 1),$

(2) $c \times j = \begin{vmatrix} i & j & k \\ 2 & 5 & -3 \\ 0 & 1 & 0 \end{vmatrix} = (3, 0, 2).$

(3) $(a \times b) \cdot c = \begin{vmatrix} 1 & -2 & 1 \\ 1 & -1 & 3 \\ 2 & 5 & -3 \end{vmatrix} = -23. (4) (a \times b) \times c = \begin{vmatrix} i & j & k \\ -5 & -2 & 1 \\ 2 & 5 & -3 \end{vmatrix} = (1, -13, -21).$

(5) $b \times c = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ 2 & 5 & -3 \end{vmatrix} = (-12, 9, 7), a \times (b \times c) = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ -12 & 9 & 7 \end{vmatrix} = (-23, -19, -15).$

10. 在平行四边形 $ABCD$ 中, $\vec{AB} = (2, 1, 0), \vec{AD} = (0, -1, 2)$, 求两对角线的夹角 $\angle AC, BD$.

解 $\vec{AC} = \vec{AB} + \vec{AD} = (2, 1, 0) + (0, -1, 2) = (2, 0, 2),$

$\vec{BD} = \vec{AD} - \vec{AB} = (0, -1, 2) - (2, 1, 0) = (-2, -2, 2).$

$\cos \angle AC, BD = \frac{\vec{AC} \cdot \vec{BD}}{|\vec{AC}| |\vec{BD}|} = \frac{0}{\sqrt{8} \sqrt{12}} = 0, \angle AC, BD = \frac{\pi}{2}.$

解二 $|\vec{AB}| = |\vec{AD}| = \sqrt{5}$, 平行四边形 $ABCD$ 为菱形, 故两对角线的夹角 $\angle AC, BD = \frac{\pi}{2}.$

11. 已知三点 $A(3, 4, 1), B(2, 3, 0), C(3, 5, 1)$, 求三角形 ABC 的面积.

解 $\vec{AB} = (-1, -1, -1) = -(1, 1, 1), \vec{AC} = (0, 1, 0), \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1, 0, 1),$

三角形 ABC 的面积 $= \frac{1}{2} \times \sqrt{2}.$

12.证明向量 $\mathbf{a} = (3, 4, 5)$, $\mathbf{b} = (1, 2, 2)$ 和 $\mathbf{c} = (9, 14, 16)$ 是共面的.

证因为 $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 2 \\ 9 & 14 & 16 \end{vmatrix} = 0$, 故 \mathbf{a}, \mathbf{b} 和 \mathbf{c} 是共面的.

13.已知 $|\mathbf{a}| = 1, |\mathbf{b}| = 5, \mathbf{a} \cdot \mathbf{b} = -3$, 求 $|\mathbf{a} \times \mathbf{b}|$.

解 $\cos \langle \mathbf{a}, \mathbf{b} \rangle = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{-3}{5}, \sin \langle \mathbf{a}, \mathbf{b} \rangle = \frac{4}{5}, |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \langle \mathbf{a}, \mathbf{b} \rangle = 1 \times 5 \times \frac{4}{5} = 4.$

14. 设向量 \mathbf{a} 的方向余弦 $\cos \alpha, \cos \beta, \cos \gamma$, 在下列各情况下, 指出 \mathbf{a} 的方向特征.

(1) $\cos \alpha = 0, \cos \beta \neq 0, \cos \gamma \neq 0$;

(2) $\cos \alpha = \cos \beta = 0, \cos \gamma \neq 0$;

(3) $\cos \alpha = \cos \beta = \cos \gamma$.

解(1) \mathbf{a} 与 x 轴垂直.

(2) \mathbf{a} 是沿 z 轴的向量.

(3) \mathbf{a} 与三个轴的夹角相等, 都是 $\arccos \frac{1}{\sqrt{3}}$ 或 $\pi - \arccos \frac{1}{\sqrt{3}}$.

15. 设 $|\mathbf{a}| = \sqrt{2}$, \mathbf{a} 的三个方向角满足 $\alpha = \beta = \frac{1}{2}\gamma$, 求 \mathbf{a} 的坐标.

解 $2\cos^2 \alpha + \cos^2 2\alpha = 1, 2\cos^2 \alpha + (2\cos^2 \alpha - 1)^2 = 1$.

$\cos^2 \alpha = x, 2x + (2x - 1)^2 = 1, 4x^2 - 2x + 1 = 1, 2x(2x - 1) = 0, x = 0, x = \frac{1}{2}$.

$\cos^2 \alpha = 0, \alpha = \frac{\pi}{2}, \mathbf{a} = (0, 0, -\sqrt{2})$.

$\cos^2 \alpha = \frac{1}{2}, \cos \alpha = \pm \frac{1}{\sqrt{2}}, \alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \mathbf{a} = (1, 1, 0)$.

16. 设 \mathbf{a}, \mathbf{b} 为两非零向量, 且 $(7\mathbf{a} - 5\mathbf{b}) \perp (\mathbf{a} + 3\mathbf{b}), (\mathbf{a} - 4\mathbf{b}) \perp (7\mathbf{a} - 2\mathbf{b})$,

求 $\cos \langle \mathbf{a}, \mathbf{b} \rangle$.

解 $(7\mathbf{a} - 5\mathbf{b}) \cdot (\mathbf{a} + 3\mathbf{b}) = 0, 7|\mathbf{a}|^2 - 15|\mathbf{b}|^2 + 16|\mathbf{a}||\mathbf{b}|\cos \langle \mathbf{a}, \mathbf{b} \rangle = 0$,

$(\mathbf{a} - 4\mathbf{b}) \cdot (7\mathbf{a} - 2\mathbf{b}) = 0, 7|\mathbf{a}|^2 + 8|\mathbf{b}|^2 - 30|\mathbf{a}||\mathbf{b}|\cos \langle \mathbf{a}, \mathbf{b} \rangle = 0$.

$$\begin{cases} -15\frac{|\mathbf{b}|^2}{|\mathbf{a}|^2} + 16\frac{|\mathbf{b}|}{|\mathbf{a}|}\cos \langle \mathbf{a}, \mathbf{b} \rangle = -7, \\ 8\frac{|\mathbf{b}|^2}{|\mathbf{a}|^2} - 30\frac{|\mathbf{b}|}{|\mathbf{a}|}\cos \langle \mathbf{a}, \mathbf{b} \rangle = -7. \end{cases}$$

$$\frac{|\mathbf{b}|^2}{|\mathbf{a}|^2} = \frac{\begin{vmatrix} -7 & 16 \\ -7 & -30 \end{vmatrix}}{\begin{vmatrix} -15 & 16 \\ 8 & -30 \end{vmatrix}} = 1, \frac{|\mathbf{b}|}{|\mathbf{a}|} = 1$$

$$\cos \langle \mathbf{a}, \mathbf{b} \rangle = \frac{\begin{vmatrix} -15 & -7 \\ 8 & -7 \end{vmatrix}}{\begin{vmatrix} -15 & 16 \\ 8 & -30 \end{vmatrix}} = \frac{1}{2}.$$

习题 5.3

1. 指出下列平面位置的特点:

(1) $5x - 3z + 1 = 0$ (2) $x + 2y - 7z = 0$ (3) $y + 5 = 0$ (4) $2y - 9z = 0$ (5) $x - y - 5 = 0$ (6) $x = 0$.

解 (1) 平行于 y 轴. (2) 过原点. (3) 平行于 Oxz 平面.

(4) 过 x 轴. (5) 平行于 z 轴. (6) Oyz 平面.

2. 求下列各平面的方程:

(1) 平行于 y 轴且通过点 $(1, -5, 1)$ 和 $(3, 2, -2)$;

(2) 平行于 Oxz 平面且通过点 $(5, 2, -8)$;

(3) 垂直于平面 $x - 4y + 5z = 1$ 且通过点 $(-2, 7, 3)$ 及 $(0, 0, 0)$;

(4) 垂直于 Oyz 平面且通过点 $(5, -4, 3)$ 及 $(-2, 1, 8)$.

解 (1) $\mathbf{a} = (0, 1, 0), \mathbf{b} = (2, 7, -3), \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 2 & 7 & -3 \end{vmatrix} = (-3, 0, -2).$

$$-3(x-1) - 2(z-1) = 0, 3x + 2z - 5 = 0.$$

(2) $y = 2$.

(3) $\mathbf{a} = (1, -4, 5), \mathbf{b} = (-2, 7, 3), \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 5 \\ -2 & 7 & 3 \end{vmatrix} = (-47, -13, -1).$

$$47x + 13y + 1 = 0.$$

(4) $\mathbf{a} = (1, 0, 0), \mathbf{b} = (-7, 5, 5), \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ -7 & 5 & 5 \end{vmatrix} = (0, -5, 5) = 5(0, -1, 1).$

$$-(y+4) + (z-3) = 0, y - z + 7 = 0.$$

3. 求通过点 $A(2, 4, 8), B(-3, 1, 5)$ 及 $C(6, -2, 7)$ 的平面方程.

解 $\mathbf{a} = (-5, -3, -3), \mathbf{b} = (4, -6, -1).$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -3 & -3 \\ 4 & -6 & -1 \end{vmatrix} = (-15, -17, 42),$$

$$-15(x-2) - 17(y-4) + 42(z-8) = 0, 15x + 17y - 42z + 238 = 0.$$

4. 设一平面在各坐标轴上的截距都不等于零并相等, 且过点 $(5, -7, 4)$, 求此平面的方程.

解 $\frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1, \frac{5}{a} + \frac{-7}{a} + \frac{4}{a} = 1, a = 2, x + y + z - 2 = 0.$

5. 已知两点 $A(2, -1, -2)$ 及 $B(8, 7, 5)$, 求过 B 且与线段 AB 垂直的平面.

解 $\mathbf{n} = (6, 8, 7). 6(x-8) + 8(y-7) + 7(z-5) = 0, 6x + 8y + 7z - 139 = 0.$

6.求过点(2,0,-3)且与 $2x-2y+4z+7=0, 3x+y-2z+5=0$ 垂直的平面方程.

$$\text{解 } \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ 3 & 1 & -2 \end{vmatrix} = (0, 16, 8) = 8(0, 2, 1). 2y + (z+3) = 0, y + z + 3 = 0.$$

7.求通过 x 轴且与平面 $9x-4y-2z+3=0$ 垂直的平面方程.

$$\text{解 } By + Cz = 0, -4B - 2C = 0, \text{取 } B = 1, C = -2, y - 2z = 0.$$

8.求通过直线 $l_1: \begin{cases} x+2z-4=0 \\ 3y-z+8=0 \end{cases}$ 且与直线 $l_2: \begin{cases} x-y-4=0 \\ y-z-6=0 \end{cases}$ 平行的平面方程.

$$\text{解 } \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 3 & -1 \end{vmatrix} = (-6, 1, 3), \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1),$$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = (-2, 9, -7). \text{用 } z_0 = 0 \text{ 代入 } l_1 \text{ 的方程, 得 } x_0 = 4, y_0 = -8/3.$$

$$-2(x-4) + 9(y+8/3) - 7(z) = 0, -2x + 9y - 7z + 32 = 0.$$

9.求直线 $l_1: \frac{x+3}{3} = \frac{y+1}{2} = \frac{z-2}{4}$ 与直线 $l_2: \begin{cases} x=3t+8 \\ y=t+1 \\ z=2t+6 \end{cases}$ 的交点坐标,

并求通过此两直线的平面方程.

解求两条直线交点坐标:

$$\frac{3t+8+3}{3} = \frac{t+1+1}{2} = \frac{2t+6-2}{4}, t + \frac{11}{3} = \frac{t}{2} + 1 = \frac{t}{2} + 1, t = -\frac{16}{3},$$

$$x_0 = -8, y_0 = -\frac{13}{3}, z_0 = -\frac{14}{3}, \text{交点}(-8, -\frac{13}{3}, -\frac{14}{3}).$$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 3 & 1 & 2 \end{vmatrix} = (0, 6, -3) = 3(0, 2, -1). 2(y+1) - (z-2) = 0, 2y - z + 4 = 0.$$

10.求通过两直线 $l_1: \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z+1}{1}$ 和 $l_2: \frac{x+2}{-4} = \frac{y-2}{2} = \frac{z}{-2}$ 的平面方程.

$$\text{解 两直线平行. 平面过点}(1, -1, -1)\text{和}(-2, 2, 0). \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -3 & 3 & 1 \end{vmatrix} = (-4, -5, 3).$$

$$-4(x-1) - 5(y+1) + 3(z+1) = 0, -4x - 5y + 3z + 2 = 0.$$

11. 证明两直线 $l_1: \frac{x-1}{-1} = \frac{y}{2} = \frac{z+1}{1}$ 和 $l_2: \frac{x+2}{0} = \frac{y-1}{1} = \frac{z-2}{-2}$ 是异面直线.

证首先, 两直线的方向向量 $(-1, 2, 1)$ 和 $(0, 1, -2)$ 不平行.

$$l_2 \begin{cases} x = -2 \\ y = 1+t \\ z = 2-2t \end{cases} \frac{-2-1}{-1} = \frac{1+t}{2} = \frac{-2t+3}{1}, t=5, t=0, \text{矛盾. 故两直线无公共点.}$$

两直线不平行, 又无交点, 故是异面直线.

12. 将下列直线方程化为标准方程及参数方程:

$$(1) \begin{cases} 2x + y - z + 1 = 0 \\ 3x - y + 2z - 8 = 0 \end{cases}; (2) \begin{cases} x - 3z + 5 = 0 \\ y - 2z + 8 = 0 \end{cases}.$$

$$\text{解(1)} \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix} = (1, -7, -5).$$

$$(1) \text{中令 } x_0 = 0, \begin{cases} y - z + 1 = 0 \\ -y + 2z - 8 = 0 \end{cases} \text{解之得 } y_0 = 6, z_0 = 7.$$

$$\text{标准方程 } \frac{x}{1} = \frac{y-6}{-7} = \frac{z-7}{-5}.$$

$$\text{参数方程: } \begin{cases} x = t \\ y = 6 - 7t, -\infty < t < +\infty. \\ z = 7 - 5t \end{cases}$$

$$(2)(1) \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = (3, 2, 1).$$

$$(2) \text{中令 } z_0 = 0, \text{直接得 } x_0 = -5, y_0 = -8.$$

$$\text{标准方程 } \frac{x+5}{3} = \frac{y+8}{2} = \frac{z}{1}.$$

$$\text{参数方程: } \begin{cases} x = -5 + 3t \\ y = -8 + 2t, -\infty < t < +\infty. \\ z = t \end{cases}$$

13. 求通过点(3, 2, -5)及x轴的平面与平面 $3x - y - 7z + 9 = 0$ 的交线方程.

解地第一个平面的法向量 $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 3 & 2 & -5 \end{vmatrix} = (0, 5, 2),$

平面方程 $5y + 2z = 0.$

直线方程 $\begin{cases} 5y + 2z = 0 \\ 3x - y - 7z + 9 = 0. \end{cases}$

直线的方向向量 $\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 2 \\ 3 & -1 & -7 \end{vmatrix} = (-33, 6, -15) = 3(-11, 2, -5).$

$z_0 = 0, \begin{cases} 5y = 0 \\ 3x - y + 9 = 0. \end{cases} y_0 = 0, x_0 = -3.$

直线方程: $\frac{x+3}{-11} = \frac{y}{2} = \frac{z}{-5}.$

14. 当 D 为何值时, 直线 $\begin{cases} 3x - y + 2z - 6 = 0 \\ x + 4y - z + D = 0 \end{cases}$ 与 Oz 轴相交?

解直线 $\begin{cases} 3x - y + 2z - 6 = 0 \\ x + 4y - z + D = 0 \end{cases}$ 与 Oz 轴相交 \Leftrightarrow 存在 $(0, 0, z_0)$ 在此直线上,

$\Leftrightarrow \begin{cases} 2z_0 - 6 = 0 \\ -z_0 + D = 0 \end{cases} \Leftrightarrow D = z_0 = 3.$

15. 试求通过直线 $l_1: \begin{cases} x - 2z - 4 = 0 \\ 3y - z + 8 = 0 \end{cases}$ 并与直线 $l_2: \begin{cases} x - y - 4 = 0 \\ z - y + 6 = 0 \end{cases}$ 平行的平面方程.

解 l_1 的方向向量 $\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 3 & -1 \end{vmatrix} = (6, 1, 3).$

l_2 的方向向量 $\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = (-1, -1, -1) = -(1, 1, 1).$

平面的法向量 $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = (-2, -3, 5).$

在的方程中令 $z_0 = 0$ 得 $x_0 = 4, y_0 = -\frac{8}{3}.$

所求平面方程: $-2(x - 4) - 3(y + \frac{8}{3}) + 5z = 0$, 即 $2x + 3y - 5z = 0.$

16.求点(1,2,3)到直线 $\frac{x}{1} = \frac{y-4}{-3} = \frac{z-3}{-2}$ 的距离.

解过点(1,2,3)垂直于直线的平面:

$$(x-1)-3(y-2)-2(z-3)=0.$$

$$\text{直线参数方程: } \begin{cases} x=t \\ y=4-3t \\ z=3-2t \end{cases}$$

代入平面方程得对应交点的参数:

$$(t-1)-3(4-3t-2)-2(3-2t-3)=0, t_0=\frac{1}{2},$$

直线与平面交点为 $(\frac{1}{2}, \frac{5}{2}, 2)$.

$$\text{所求距离 } d = \sqrt{(1-\frac{1}{2})^2 + (2-\frac{5}{2})^2 + (3-2)^2} = \frac{\sqrt{6}}{2}.$$

17.求点(2,1,3)到平面 $2x-2y+z-3=0$ 的距离与投影.

解过点(2,1,3)垂直于平面 $2x-2y+z-3=0$ 的直线方程的参数方程:

$$\begin{cases} x=2+2t \\ y=1-2t, -\infty < t < +\infty. \text{代入平面方程} \\ z=3+t \end{cases}$$

$$2(2+2t)-2(1-2t)+(3+t)-3=0, t_0=-\frac{2}{9}.$$

$$x_0=\frac{14}{9}, y_0=\frac{13}{9}, z_0=\frac{25}{9}.$$

点(2,1,3)在平面 $2x-2y+z-3=0$ 上的投影为 $(\frac{14}{9}, \frac{13}{9}, \frac{25}{9})$.

点(2,1,3)在平面 $2x-2y+z-3=0$ 的距离为

$$\sqrt{(2-\frac{14}{9})^2 + (1-\frac{13}{9})^2 + (3-\frac{25}{9})^2} = \frac{2}{3}.$$

18. 求两平行直线 $\frac{x-1}{1} = \frac{y+1}{-2} = \frac{z}{3}$ 与 $\frac{x}{1} = \frac{y+1}{-2} = \frac{z-1}{3}$ 的距离.

解所求的就是点 $(1, -1, 0)$ 到直线 $\frac{x}{1} = \frac{y+1}{-2} = \frac{z-1}{3}$ 的距离.

作法与16题雷同. 过点 $(1, -1, 0)$ 垂直于直线 $\frac{x}{1} = \frac{y+1}{-2} = \frac{z-1}{3}$ 的平面:

$$(x-1) - 2(y+1) + 3z = 0.$$

直线的参数方程 $\begin{cases} x = t \\ y = -1 - 2t \\ z = 1 + 3t \end{cases}$, 代入平面方程

$$(t-1) - 2(-1-2t) + 3(1+3t) = 0, t_0 = -\frac{1}{7}.$$

直线与平面交点 $(-\frac{1}{7}, -\frac{5}{7}, \frac{4}{7})$.

$$\text{所求距离 } d = \sqrt{(1+\frac{1}{7})^2 + (-1+\frac{5}{7})^2 + (0-\frac{4}{7})^2} = 2\sqrt{\frac{3}{7}}.$$

19. 求过点 $A(2, 1, 3)$ 并与直线 $l_1: \frac{x+1}{3} = \frac{y-1}{2} = \frac{z}{-1}$ 垂直且相交的直线方程.

解过点 A 垂直于直线 l_1 的平面方程 $3(x-2) + 2(y-1) - (z-3) = 0$.

直线 l_1 的参数方程 $\begin{cases} x = -1 + 3t \\ y = 1 + 2t \\ z = -t \end{cases}$

代入平面方程求交点对应的参数他 t :

$$3(-3+3t) + 2(2t) - (-t-3) = 0, t_0 = \frac{3}{7}.$$

交点 $B(\frac{2}{7}, \frac{13}{7}, -\frac{3}{7})$.

连结点 A, B 的直线的方向向量

$$\vec{AB} = (\frac{2}{7} - 2, \frac{13}{7} - 1, -\frac{3}{7} - 3) = (-\frac{12}{7}, \frac{6}{7}, -\frac{24}{7}) = -\frac{6}{7}(2, -1, 4).$$

所求直线方程: $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-3}{4}$.

20.求两平行平面 $3x+6y-2z-7=0$ 与 $3x+6y-2z+14=0$ 之间的距离.

解点 $A(0,0,-\frac{7}{2})$ 在第一张平面上.

过 A 垂直于第二张平面的直线的参数方程:
$$\begin{cases} x=3t \\ y=6t \\ z=-7/2-2t \end{cases}$$

求直线与第二张平面的交点: $3(3t)+6(6t)-2(-7/2-2t)+14=0,$

$$t_0=-\frac{3}{7},(-\frac{9}{7},-\frac{18}{7},-\frac{37}{14}).$$

$$\text{所求距离}=\sqrt{(\frac{9}{7})^2+(\frac{18}{7})^2+(\frac{6}{7})^2}=3.$$

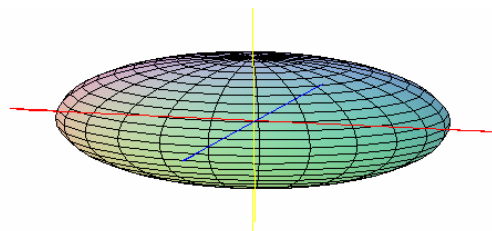
习题 5.4

1. 求椭球面 $2x^2 + 3y^2 + 4z^2 - 4x - 6y + 16z + 16 = 0$ 的中心的坐标及三个半轴之长度.

$$\begin{aligned} \text{解 } & 2x^2 + 3y^2 + 4z^2 - 4x - 6y + 16z + 16 = 0, \\ & 2x^2 + 3y^2 + 4z^2 - 4x - 6y + 16z + 17 \\ & = 2(x-1)^2 - 2 + 3(y-1)^2 - 3 + 4(z+2)^2 - 16 + 16 \\ & = 2(x-1)^2 + 3(y-1)^2 + 4(z+2)^2 - 5 = 0. \end{aligned}$$

$$\frac{(x-1)^2}{\sqrt{\frac{5}{2}}^2} + \frac{(y-1)^2}{\sqrt{\frac{5}{3}}^2} + \frac{(z+2)^2}{\left(\frac{\sqrt{5}}{2}\right)^2} = 1,$$

中心坐标: $(1, 1, -2)$, 半轴: $\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{3}}, \frac{\sqrt{5}}{2}$.



2. 说出下列曲面的名称, 并画出略图:

(1) $8x^2 + 11y^2 + 24z^2 = 1$; 椭球面.

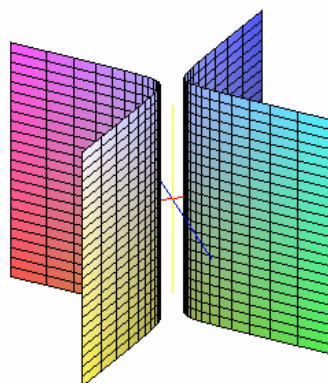
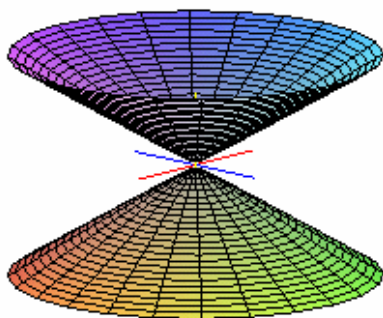
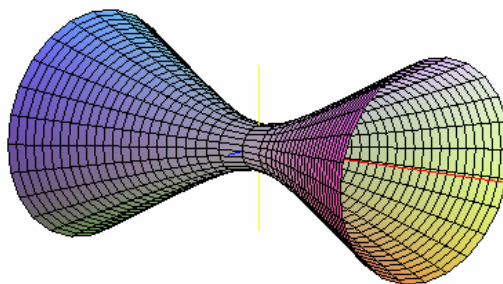
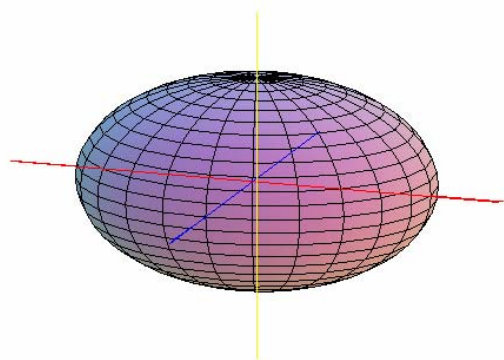
(2) $4x^2 - 9y^2 - 14z^2 = -25$; 单叶双曲面.

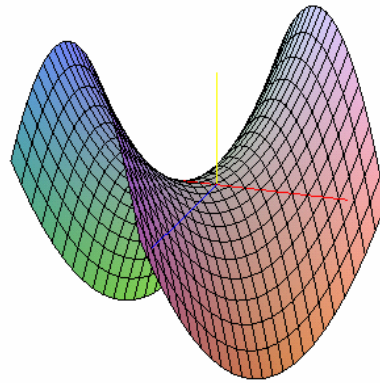
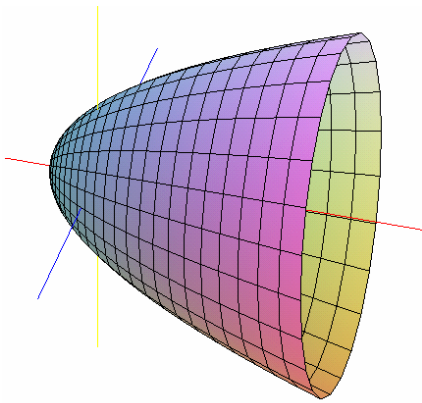
(3) $2x^2 + 9y^2 - 16z^2 = -9$; 双叶双曲面.

(4) $x^2 - y^2 = 2x$; 双曲柱面.

(5) $2y^2 + z^2 = x$; 椭圆抛物面.

(6) $z = xy$. 双曲抛物面.





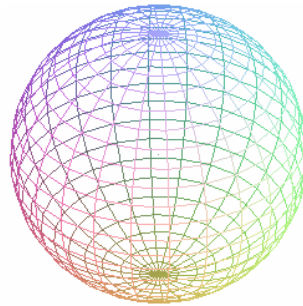
3.求下列曲面的参数方程:

$$(1)(x-1)^2 + (y+1)^2 + (z-3)^2 = R^2;$$

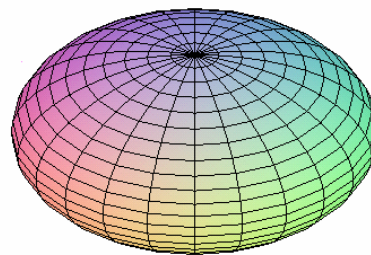
$$(2)x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1; (3)\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1;$$

$$(4)z = \frac{x^2}{a^2} - \frac{y^2}{b^2}; (5)z = \frac{z^2}{a^2} + \frac{y^2}{b^2}.$$

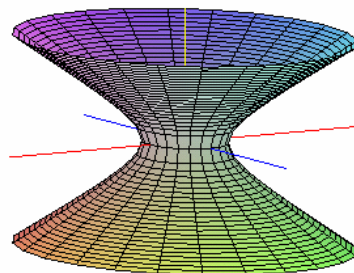
$$\text{解}(1) \begin{cases} x = 1 + R \sin \varphi \cos \theta \\ y = -1 + R \sin \varphi \sin \theta \\ z = 3 + R \cos \varphi \end{cases} 0 \leq \varphi \leq \pi, 0 \leq \theta < 2\pi;$$



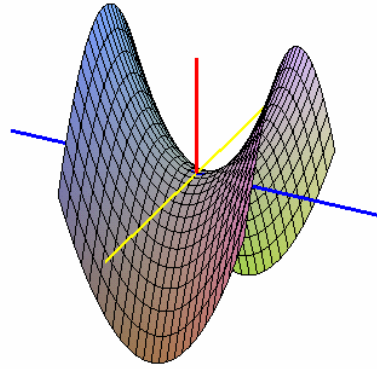
$$(2) \begin{cases} x = \sin \varphi \cos \theta \\ y = 3 \sin \varphi \sin \theta \\ z = 2 \cos \varphi \end{cases} 0 \leq \varphi \leq \pi, 0 \leq \theta < 2\pi;$$



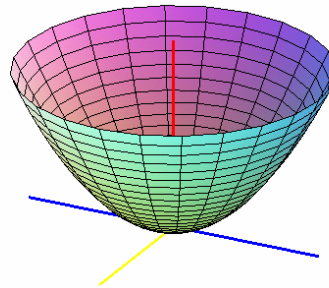
$$3) \begin{cases} x = 2 \cosh \varphi \cos \theta \\ y = 3 \cosh \varphi \sin \theta \\ z = 4 \sinh \varphi \end{cases} -\infty < \varphi < +\infty, 0 \leq \theta < 2\pi;$$



$$(4) \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = r^2 \cos \theta \end{cases} \quad 0 \leq r < +\infty, 0 \leq \theta \leq 2\pi$$



$$(5) \begin{cases} x = ar \cos \theta \\ y = br \sin \theta \\ z = r^2 \end{cases} \quad 0 \leq r < +\infty, 0 \leq \theta \leq 2\pi$$



习题 5.5

1. 求下列曲线在指定点 P_0 的切线方程和法平面方程:

(1) $x = t, y = t^2, z = t^3, P_0 = (1, 1, 1)$;

(2) 曲面 $z = x^2$ 与 $y = x$ 的交线, $P_0 = (2, 2, 4)$;

(3) 柱面 $x^2 + y^2 = R^2 (R > 0)$ 与平面 $z = x + y$ 的交线 $P_0 = (R, 0, R)$.

解 (1) $x' = 1, y' = 2t, z' = 3t^2, t = (1, 2, 3)$, 切线方程: $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$,

法平面方程: $(x-1) + 2(y-1) + 3(z-1) = 0, x + 2y + 3z - 6 = 0$.

(2) $x = x, y = x, z = x^2, x' = 1, y' = 1, z' = 2x, t = (1, 1, 4)$. 切线方程: $\frac{x-2}{1} = \frac{y-2}{1} = \frac{z-4}{4}$,

法平面方程: $(x-2) + (y-2) + 4(z-4) = 0, x + y + 4z - 20 = 0$.

(3) $\mathbf{n}_1 = (2x, 2y, 0) = (2R, 0, 0), \mathbf{n}_1 = (1, 1, -1), \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2R & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (0, 2R, 2R) = 2R(0, 1, 1)$,

切线方程: $\frac{x-R}{0} = \frac{y}{1} = \frac{z-R}{1}$, 法平面方程: $y + z - R = 0$.

2. 求出螺旋线 $\begin{cases} x = R \cos t \\ y = R \sin t \quad (R > 0, b > 0, 0 \leq t \leq 2\pi) \\ z = bt \end{cases}$ 在任意一

点处的切线的

方向余弦, 并证明切线与 z 轴之夹角为常数.

解 $(x', y', z') = (-R \sin t, R \cos t, b)$,

$\mathbf{t} = (\cos \alpha, \cos \beta, \cos \gamma) = \frac{1}{\sqrt{R^2 + b^2}} (-R \sin t, R \cos t, b)$,

$\cos \langle \mathbf{t}, \mathbf{k} \rangle = \frac{b}{\sqrt{R^2 + b^2}} = \text{常数}. 0 < \langle \mathbf{t}, \mathbf{k} \rangle < \pi, \langle \mathbf{t}, \mathbf{k} \rangle = \text{常数}.$

3. 设 $\mathbf{a} = \mathbf{a}(t)$ 与 $\mathbf{b} = \mathbf{b}(t)$ 是两个可导的向量函数, $\alpha < t < \beta$. 证明

$$\frac{d}{dt} \mathbf{a}(t) \cdot \mathbf{b}(t) = \mathbf{a}'(t) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot \mathbf{b}'(t).$$

证 设 $\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t))$, $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))$,

$$\mathbf{a}(t) \cdot \mathbf{b}(t) = a_1(t)b_1(t) + a_2(t)b_2(t) + a_3(t)b_3(t),$$

$$\frac{d}{dt} \mathbf{a}(t) \cdot \mathbf{b}(t) = \frac{d}{dt} [a_1(t)b_1(t) + a_2(t)b_2(t) + a_3(t)b_3(t)]$$

$$= a_1'(t)b_1(t) + a_1(t)b_1'(t) + a_2'(t)b_2(t) + a_2(t)b_2'(t) + a_3'(t)b_3(t) + a_3(t)b_3'(t)$$

$$= [a_1'(t)b_1(t) + a_2'(t)b_2(t) + a_3'(t)b_3(t)] + [a_1(t)b_1'(t) + a_2(t)b_2'(t) + a_3(t)b_3'(t)]$$

$$= \mathbf{a}'(t) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot \mathbf{b}'(t).$$

4. 设 $\mathbf{r} = \mathbf{r}(t)$ ($\alpha < t < \beta$) 是一条光滑曲线, 切 $|\mathbf{r}(t)| = C$ (常数). 证明 $\mathbf{r}(t)$ 与切线垂直, 即 $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

$$\text{证 } \mathbf{r}(t) \cdot \mathbf{r}(t) = C^2, \frac{d}{dt} \mathbf{r}(t) \cdot \mathbf{r}(t) = \frac{d}{dt} C^2, \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0, 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0,$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0.$$

第五章总练习题

1. 设 a, b 为两个非零向量, 指出下列等式成立的充分必要条件:

(1) $|a+b|=|a-b|$; (2) $|a+b|=|a|-|b|$; (3) $a+b$ 与 $a-b$ 共线.

解(1) $|a+b|=|a-b| \Leftrightarrow |a+b|^2=|a-b|^2 \Leftrightarrow |a|^2+|b|^2+2a \cdot b=|a|^2+|b|^2-2a \cdot b$
 $\Leftrightarrow a \cdot b=0 \Leftrightarrow a, b$ 正交.

(2) $|a+b|=|a|-|b| \Leftrightarrow |a+b|^2=(|a|-|b|)^2 \Leftrightarrow |a|^2+|b|^2+2a \cdot b=|a|^2+|b|^2-2|a||b|$
 $|a|^2+|b|^2-2|a||b| \Leftrightarrow a \cdot b=|a||b| \Leftrightarrow |a||b|\cos\langle a, b \rangle = -|a||b| \Leftrightarrow \cos\langle a, b \rangle = -1 \Leftrightarrow a, b$ 共线且方向相反.

(3) $a+b$ 与 $a-b$ 共线 $\Leftrightarrow (a+b) \times (a-b) = 0 \Leftrightarrow b \times a - a \times b = 0 \Leftrightarrow a \times b = 0$
 $\Leftrightarrow a, b$ 共线.

2. 设 a, b, c 为非零向量, 判断下列等式是否成立:

(1) $(a \cdot b)c = a(b \cdot c)$; (2) $(a \cdot b)^2 = a^2 b^2$; (3) $a \cdot (b \times c) = (a \times b) \cdot c$.

解(1) 不成立. 例如: $(i \cdot j)j = j \neq i(i \cdot j) = 0$.

(2) 不成立. 例如: $(i \cdot j)^2 = 0 \neq i^2 j^2 = 1$.

(3) 成立. $a \cdot (b \times c)$ 和 $(a \times b) \cdot c$ 都是 a, b, c 的有向体积, 且定向相同.

3. 设 a, b 为非零向量, 且 $7a-5b$ 与 $a+3b$ 正交, 与 $a-4b$ 与 $7a-2b$ 正交, 求 a^2-b^2 .

解 $(7a-5b) \cdot (a+3b) = 0, (a-4b) \cdot (7a-2b) = 0$.

$$\begin{cases} 7a^2 - 15b^2 + 16a \cdot b = 0 & (1) \\ 7a^2 + 8b^2 - 30a \cdot b = 0 & (2) \end{cases}$$

$$(1) \times 15 + (2) \times 8$$

$$161(a^2 - b^2) = 0, a^2 - b^2 = 0.$$

4. 利用向量运算, 证明下列几何命题: 射影定理. 考虑直角三角形 ABC , 其中 $\angle A$ 为

直角, AD 是斜边上的高, 则 $\overline{AD}^2 = \overline{BD} \cdot \overline{CD}, \overline{AB}^2 = \overline{BD} \cdot \overline{BC}, \overline{AC}^2 = \overline{CD} \cdot \overline{CB}$.

证 $\overline{AB} = \overline{AD} + \overline{DB}, \overline{AC} = \overline{AD} + \overline{DC}$,

$$\begin{aligned} 0 &= \overline{AB} \cdot \overline{AC} = (\overline{AD} + \overline{DB}) \cdot (\overline{AD} + \overline{DC}) = \overline{AD}^2 + \overline{AD} \cdot \overline{DC} + \overline{DB} \cdot \overline{AD} + \overline{DB} \cdot \overline{DC} \\ &= \overline{AD}^2 + \overline{DB} \cdot \overline{DC}, \overline{AD}^2 = -\overline{DB} \cdot \overline{DC} = \overline{BD} \cdot \overline{DC} = \overline{BD} \times \overline{DC} \quad (\overline{BD}, \overline{DC} \text{ 同向}). \end{aligned}$$

$$\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 = \overline{BD} \cdot \overline{DC} + \overline{BD}^2 = \overline{BD}(\overline{CD} + \overline{BD}) = \overline{BD} \cdot \overline{BC}.$$

$$\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 = \overline{BD} \cdot \overline{DC} + \overline{CD}^2 = \overline{CD}(\overline{BD} + \overline{CD}) = \overline{CD} \cdot \overline{BC}.$$

5. 已知三点 A, B, C 的坐标分别为 $(1, 0, 0), (1, 1, 0), (1, 1, 1)$. 若 $ACDBD$ 是一平行四边形, 求点 D 的坐标.

解 $A = (1, 0, 0), B = (1, 1, 0), C = (1, 1, 1). \overline{AC} = (0, 1, 1), \overline{AB} = (0, 1, 0), \overline{AD} = \overline{AB} + \overline{AC} = (0, 2, 1),$
 $\overline{OD} = \overline{OA} + \overline{AD} = (1, 0, 0) + (0, 2, 1) = (1, 2, 1).$ 点 D 的坐标 $(1, 2, 1)$.

6. 设 a, b 为非零向量, 证明 $(a \times b)^2 = a^2 b^2 - (a \cdot b)^2$.

$$\text{证 } (a \times b)^2 = |a|^2 |b|^2 \sin^2 \langle a, b \rangle = |a|^2 |b|^2 (1 - \cos^2 \langle a, b \rangle)$$

$$= |a|^2 |b|^2 - |a|^2 |b|^2 \cos^2 \langle a, b \rangle = a^2 b^2 - (a \cdot b)^2.$$

7. 设有两直线 $L_1: \frac{x-1}{-1} = \frac{y}{2} = \frac{z+1}{1}$, $L_2: \frac{x+2}{0} = \frac{y-1}{1} = \frac{z-2}{-2}$, 求平行于 L_1, L_2 且与它们等距的平面方程.

$$\text{解 } n = \begin{vmatrix} i & j & k \\ -1 & 2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = (-5, -2, -1), \text{ 所求平面过点 } A = (-1/2, 1/2, 1/2),$$

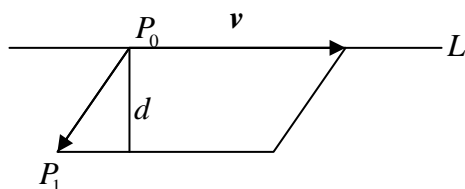
$$\text{所求平面: } -5(x+1/2) - 2(y-1/2) - (z-1/2) = 0, 5x + 2y + z + 1 = 0.$$

8. 设直线 L 通过点 P_0 且其方向向量为 v , 证明 L 外

$$\text{一点 } P_1 \text{ 到 } L \text{ 的距离 } d \text{ 可表为 } d = \frac{|\overrightarrow{P_0 P_1} \times v|}{|v|}.$$

证 平行四边形 $P_0 P_1 A B$ 的面积

$$= d \times |v| = |\overrightarrow{P_0 P_1} \times v|.$$



9. 设两直线 L_1, L_2 分别通过点 P_1, P_2 , 且它们的方向向量为 v_1, v_2 . 证明 L_1 与 L_2 共面的充分必要条件为 $\overrightarrow{P_1 P_2} \cdot (v_1 \times v_2) = 0$.

$$\text{证 } L_1 \text{ 与 } L_2 \text{ 共面} \Leftrightarrow \overrightarrow{P_1 P_2}, v_1, v_2 \text{ 共面} \Leftrightarrow \overrightarrow{P_1 P_2} \cdot (v_1 \times v_2) = 0.$$

10. 设两直线 L_1, L_2 分别通过点 P_1, P_2 , 且它们的方向向量为 v_1, v_2 . L_1 与 L_2 之间的距离定

$$\text{义为 } d = \min_{\substack{Q_1 \in L_1 \\ Q_2 \in L_2}} |\overrightarrow{Q_1 Q_2}| \text{ 证明: (1) 当 } L_1 \text{ 与 } L_2 \text{ 平行时, 它们之间的距离可表示为 } d = \frac{|\overrightarrow{P_1 P_2} \times v_1|}{|v_1|}$$

$$(2) \text{ 当 } L_1 \text{ 与 } L_2 \text{ 为异面直线时, 它们之间的距离可表示为 } d = \frac{|\overrightarrow{P_1 P_2} \cdot (v_1 \times v_2)|}{|v_1 \times v_2|}.$$

证 (1) 当 L_1 与 L_2 平行时, 它们之间的距离为 L_1 上任意一点到 L_2 的距离, 由第8题,

$$d = \frac{|\overrightarrow{P_1 P_2} \times v_1|}{|v_1|}.$$

$$(2) \frac{|\overrightarrow{P_1 P_2} \cdot (v_1 \times v_2)|}{|v_1 \times v_2|} = \overrightarrow{P_1 P_2} \cdot (v_1 \times v_2)^\circ \text{ 是 } \overrightarrow{P_1 P_2} \text{ 在 } L_1 \text{ 与 } L_2 \text{ 的公垂线方向的单位向量上的投影,}$$

$$\text{故其长度 } |\overrightarrow{P_1 P_2} \cdot (v_1 \times v_2)^\circ| = \frac{|\overrightarrow{P_1 P_2} \cdot (v_1 \times v_2)|}{|v_1 \times v_2|} \text{ 是异面直线 } L_1 \text{ 与 } L_2 \text{ 之间的距离.}$$

11. 设直线 L 的方程为 $L: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$

证明: (1) 对于任意两个不全为零的常数 λ_1, λ_2 , 方程

$$\lambda_1(A_1x + B_1y + C_1z + D_1) + \lambda_2(A_2x + B_2y + C_2z + D_2) = 0$$

表示一个通过直线 L 的平面;

(2) 任意给定一个通过直线 L 的平面 π , 必存在两个不全为零的实数 λ_1, λ_2 , 使平面 π 的方程为 $\lambda_1(A_1x + B_1y + C_1z + D_1) + \lambda_2(A_2x + B_2y + C_2z + D_2) = 0$.

证 (1) 向量 (A_1, B_1, C_1) 与 (A_2, B_2, C_2) 不共线, 故对于两个不全为零的常数 λ_1, λ_2 ,

$$\lambda_1(A_1x + B_1y + C_1z + D_1) + \lambda_2(A_2x + B_2y + C_2z + D_2) = 0 \text{ 的主系数}$$

$\lambda_1(A_1, B_1, C_1) + \lambda_2(A_2, B_2, C_2) \neq (0, 0, 0)$, 是一个平面的方程, 并且 L 上点的坐标

$$(x, y, z) \text{ 满足 } \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}, \text{ 故满足}$$

$$\lambda_1(A_1x + B_1y + C_1z + D_1) + \lambda_2(A_2x + B_2y + C_2z + D_2) = 0.$$

(2) 设平面 π 通过直线 L , 其方程为

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = Ax + By + Cz + D = 0.$$

(x_0, y_0, z_0) 在 L 上. 三个向量 (A, B, C) (A_1, B_1, C_1) 与 (A_2, B_2, C_2) 均垂直于 L 的方向向量, 故共面, 又 (A_1, B_1, C_1) 与 (A_2, B_2, C_2) 都是非零向量, 故存在两个不全为零的常数 λ_1, λ_2 , 使得

$$(A, B, C) = \lambda_1(A_1, B_1, C_1) + \lambda_2(A_2, B_2, C_2).$$

$$D = -Ax_0 - By_0 - Cz_0 = -(\lambda_1A_1 + \lambda_2A_2)x_0 - (\lambda_1B_1 + \lambda_2B_2)y_0 - (\lambda_1C_1 + \lambda_2C_2)z_0$$

$$= -\lambda_1(A_1x_0 + B_1y_0 + C_1z_0) - \lambda_2(A_2x_0 + B_2y_0 + C_2z_0) = \lambda_1D_1 + \lambda_2D_2.$$

故 π 表示为 $\lambda_1(A_1x + B_1y + C_1z + D_1) + \lambda_2(A_2x + B_2y + C_2z + D_2) = 0$.

12. 试求通过直线 $L_1: \begin{cases} x - 2z - 4 = 0 \\ 3y - z + 8 = 0 \end{cases}$ 且与直线 $L_2: x - 1 = y + 1 = z - 3$ 平行的平面方程.

解 根据11题的结论, 所求平面方程有形式

$$\lambda_1(x - 2z - 4) + \lambda_2(3y - z + 8) = 0, \lambda_1x + 3\lambda_2y + (-2\lambda_1 - \lambda_2)z - 4\lambda_1 + 8\lambda_2 = 0.$$

由于平面与 L_2 平行, $(\lambda_1, 3\lambda_2, -2\lambda_1 - \lambda_2) \cdot \mathbf{g}(1, 1, 1) = 0, \lambda_1 + 3\lambda_2 - 2\lambda_1 - \lambda_2 = 0, -\lambda_1 + 2\lambda_2 = 0$.

令 $\lambda_1 = 2, \lambda_2 = 1$, 得所求平面方程 $2x + 3y - 5z = 0$.

13. 已知曲面 S 的方程为 $S: x^2 + \frac{y^2}{4} + \frac{z^2}{2} = 1$, 平面 π 的方程为 $\pi: 2x + y + 2z + 6 = 0$.

(1) 求曲面 S 的平行于 π 的切平面方程;

(2) 在曲面 S 上求到平面 π 距离为最短及最长的点, 并求最短及最长的距离.

解 (1) S 的法向量 $(2x, \frac{y}{2}, z)$. $4x + \frac{y}{2} + 2z = 0$.

$$2x(X - x) + \frac{y}{2}(Y - y) + z(Z - z) = 0$$

13. 已知曲面 S 的方程为 $S: x^2 + \frac{y^2}{4} + \frac{z^2}{2} = 1$, 平面 π 的方程为 $\pi: 2x + y + 2z + 6 = 0$.

(1) 求曲面 S 的平行于 π 的切平面方程;

(2) 在曲面 S 上求到平面 π 距离为最短及最长的点, 并求最短及最长的距离.

解 (1) S 上的点记为 (x, y, z) . S 的法向量 $(2x, \frac{y}{2}, z)$.

切平面与 π 平行, 则法向量对应坐标成比例: $\frac{2x}{2} = \frac{y/2}{1} = \frac{z}{2}$. $z = 2x, y = z$.

与曲面方程联立: $x^2 + x^2 + 2x^2 = 1, x = \pm \frac{1}{2}, y = \pm 1, z = \pm 1$.

切平面方程: $2x(X - x) + \frac{y}{2}(Y - y) + z(Z - z) = 0$,

利用曲面方程得 $2xX + \frac{y}{2}Y + zZ = 2, \pm X \pm \frac{1}{2}Y \pm Z = 2$.

平面 π 过点 $A = (-3, 0, 0)$. $\vec{P_1A} =$

点 $P_1 = (\frac{1}{2}, 1, 1), \vec{P_1A} = (-\frac{7}{2}, -1, -1)$,

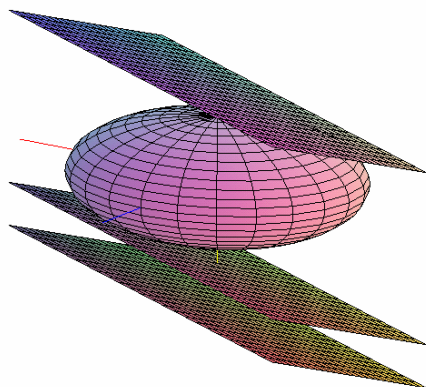
P_1 到平面 π 的距离 $d_1 = \frac{|\vec{P_1A} \cdot \vec{n}|}{|\vec{n}|} = \frac{|(-\frac{7}{2}, -1, -1) \cdot (2, 1, 2)|}{3} = \frac{10}{3}$.

点 $P_2 = (-\frac{1}{2}, -1, -1), \vec{P_2A} = (-\frac{5}{2}, 1, 1)$,

P_2 到平面 π 的距离 $d_2 = \frac{|\vec{P_2A} \cdot \vec{n}|}{|\vec{n}|} = \frac{|(-\frac{5}{2}, 1, 1) \cdot (2, 1, 2)|}{3} = \frac{2}{3}$.

在曲面 S 上到平面 π 距离为最短及最长的点分别是 $(-\frac{1}{2}, -1, -1)$ 和 $(\frac{1}{2}, 1, 1)$,

并求最短及最长的距离分别是 $\frac{2}{3}$ 和 $\frac{10}{3}$.

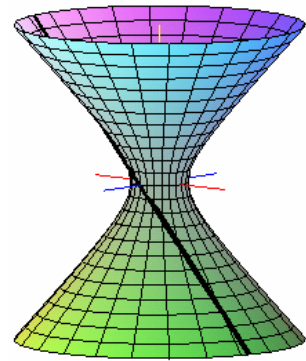


14. 直线 $\frac{x}{1} = \frac{y-1}{0} = \frac{z}{1}$ 绕 z 轴旋转一周, 求所得旋转曲面的方程.

解 直线参数方程
$$\begin{cases} x = z \\ y = 1 - \infty < z < +\infty. \\ z = z \end{cases}$$

直线 $\frac{x}{1} = \frac{y-1}{0} = \frac{z}{1}$ 绕 z 轴旋转, 对于固定的 z , 旋转曲面上的点组成一个圆, 其半径为 $\sqrt{1+z^2}$,

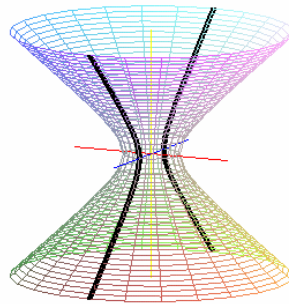
故旋转曲面的方程
$$\begin{cases} x = \sqrt{1+z^2} \cos \theta \\ y = \sqrt{1+z^2} \sin \theta - \infty < z < +\infty, 0 \leq \theta \leq 2\pi. \\ z = z \end{cases}$$



15. 求双曲线 $\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 (b, c > 0), \\ x = 0 \end{cases}$ 绕 z 轴

旋转一周所得曲面的方程.

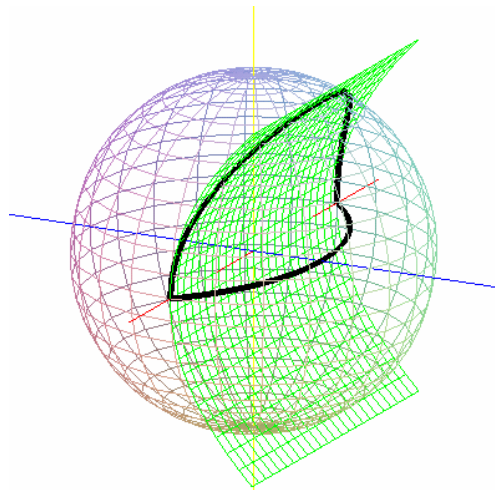
解 $\frac{x^2 + y^2}{b^2} - \frac{z^2}{c^2} = 1.$



16. 求曲线 $\begin{cases} x^2 + y^2 + z^2 = 1 \\ z^2 = 2y \end{cases}$ 在 Oxy

平面上的投影曲线的方程.

解 $x^2 + y^2 + 2y = 1, x^2 + (y+1)^2 = 2.$



习题 6.1

1. 确定下列函数的定义域并且画出定义域的图形:

(1) $z = (x^2 + y^2 - 2x)^{1/2} + \ln(4 - x^2 - y^2); x^2 + y^2 - 2x \geq 0, x^2 - y^2 < 4.$

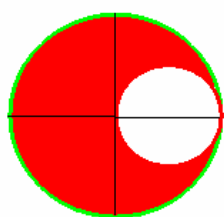
(2) $z = (x^2 - y^2)^{-1}; x^2 \neq y^2.$

(3) $z = \ln(y - x^2) + \ln(1 - y); y - x^2 > 0, y < 1.$

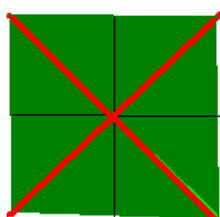
(4) $z = \arcsin \frac{x}{a} + \arccos \frac{y}{b} (a > 0, b > 0); |x| \leq a, |y| \leq b.$

(5) $z = \sqrt{1 - x^2 - y^2} + \ln(x + y); x^2 + y^2 \leq 1, x + y > 0.$

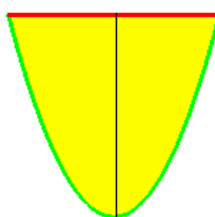
(6) $z = \arcsin(x^2 + y^2) + \sqrt{xy}; x^2 + y^2 \leq 1, xy \geq 0.$



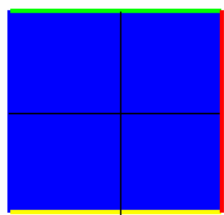
1(1)



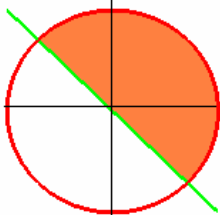
1(2)



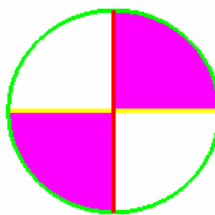
1(3)



1(4)



1(5)



1(6)

2. 指出下列集合中哪些集合在中是开集, 哪些是区域? 哪些是有界区域? 哪些是有界闭区域?

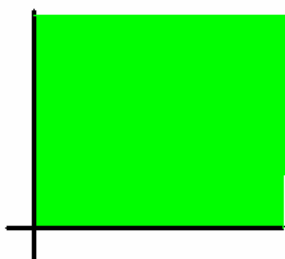
(1) $E_1 = \{(x, y) | x > 0, y > 0\}$; 开集, 区域.

(2) $E_2 = \{(x, y) | |x| < 1, |y - 1| < 2\}$; 开集, 区域, 有界区域.

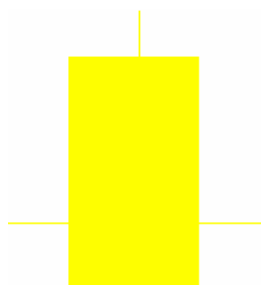
(3) $E_3 = \{(x, y) | y \geq x^2, x \geq y^2\}$; 有界闭区域.

(4) $E_4 = \{(x, y) | y \neq \sin \frac{1}{x} \text{ 且 } x \neq 0\}$. 区域, 边界点集合

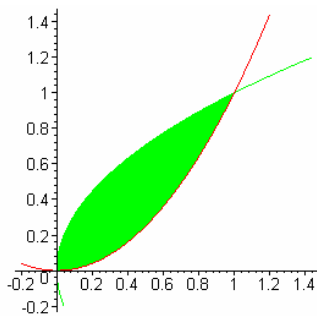
$\partial E_4 = \{(x, \sin \frac{1}{x}) | x \neq 0\} \cup \{(0, y) | -1 \leq y \leq 1\}.$



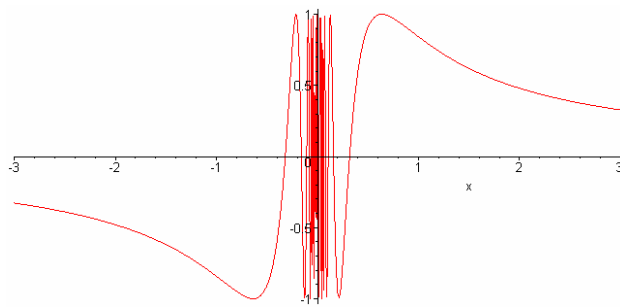
2(1)



2(2)



2(3)



2(4)

3. 设 $E \subset \mathbf{R}^n$, ∂E 为 E 的边界点集合. 试证明 $\bar{E} = E \cup \partial E$ 是一个闭集.

证 设 $P_0 \notin \bar{E}$, 则 $P_0 \notin E$ 且 $P_0 \notin \partial E$. 于是存在 $r > 0$, 使得 $U_r(P_0)$ 不含 E 的点, 从而不含 ∂E 的点. 否则, 存在 $Q \in U_r(P_0) \cap \partial E$, Q 作为 E 的边界点, 存在 $U_\rho(Q) \subseteq U_r(P_0)$, $U_\rho(Q)$ 含 E 的点, 于是 $U_r(P_0)$ 含 E 的点, 矛盾. 因此, $U_r(P_0)$ 不含 $E \cup \partial E = \bar{E}$ 的点, P_0 不是 \bar{E} 的边界点. 这表明 \bar{E} 的边界点全属于 \bar{E} . 故 \bar{E} 是闭集合.

4. 像在 \mathbf{R}^2 中一样, 我们把 \mathbf{R}^n 中的点 (x_1, \mathbf{L}, x_n) 同时也视作一个向量, 并定义两个向量 $\alpha = (x_1, \mathbf{L}, x_n)$ 及 $\beta = (y_1, \mathbf{L}, y_n)$ 的加法运算

$$\alpha + \beta = (x_1 + y_1, \mathbf{L}, x_n + y_n)$$

及数乘运算

$$\lambda \alpha = (\lambda x_1, \mathbf{L}, \lambda x_n), \forall \lambda \in \mathbf{R}.$$

此外, 我们也可以定义两个向量之内积

$$\alpha \mathbf{g} \beta = x_1 y_1 + \mathbf{L} + x_n y_n, \text{ 并规定}$$

$\sqrt{\alpha \mathbf{g} \alpha} = |\alpha|$ 作为向量的模. 试证明

$$(1) |\alpha \mathbf{g} \beta| \leq |\alpha| |\beta|, \forall \alpha, \beta \in \mathbf{R}^n;$$

$$(2) |\alpha - \beta| \leq |\alpha - \gamma| + |\gamma - \beta|, \forall \alpha, \beta, \gamma \in \mathbf{R}^n;$$

(3) 将点 $P(x_1, \mathbf{L}, x_n)$ 及 $Q(y_1, \mathbf{L}, y_n)$ 分别看成向量 α 及 β , 则有 P 到 Q 的距离

$d(P, Q) = |\alpha - \beta|$. 由此, 可由(2)中之不等式导出三角不等式.

证 (1) $\beta = \mathbf{0}$ 时结论显然成立. 设 $\beta \neq \mathbf{0}$. 考虑二次函数

$$|\alpha + \lambda \beta|^2 = |\beta|^2 \lambda^2 + 2\alpha \mathbf{g} \beta \lambda + |\alpha|^2 \geq 0, \forall \lambda \in \mathbf{R}.$$

其判别式 $|\alpha \mathbf{g} \beta|^2 - |\alpha|^2 |\beta|^2 \leq 0, |\alpha \mathbf{g} \beta| \leq |\alpha| |\beta|$.

$$(2) |\alpha + \beta|^2 = |\alpha|^2 + |\beta|^2 + 2\alpha \mathbf{g} \beta \leq |\alpha|^2 + |\beta|^2 + 2|\alpha| |\beta| = (|\alpha| + |\beta|)^2,$$

$$|\alpha + \beta| \leq |\alpha| + |\beta|.$$

$$|\alpha - \beta| = |(\alpha - \gamma) - (\beta - \gamma)| \leq |\alpha - \gamma| + |\beta - \gamma| = |\alpha - \gamma| + |\gamma - \beta|.$$

$$(3) P = \alpha, Q = \beta, R = \gamma, d(P, R) = |\alpha - \gamma| \leq |\alpha - \beta| + |\beta - \gamma| = d(P, Q) + d(Q, R).$$

习题 6.2

1. 求下列极限:

$$(1) \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{5}{2}.$$

$$(2) \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1.$$

$$(3) \lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin \frac{1}{x^2 + y^2} = 0 \text{ (有界变量乘无穷小量得无穷小量)}.$$

$$\begin{aligned} (4) \lim_{(x,y) \rightarrow (0,1)} \frac{x^3 + (y-1)^3}{x^2 + (y-1)^2} &= \lim_{(x,y) \rightarrow (0,1)} \frac{x^3}{x^2 + (y-1)^2} + \lim_{(x,y) \rightarrow (0,1)} \frac{(y-1)^3}{x^2 + (y-1)^2} \\ &= \lim_{(x,y) \rightarrow (0,1)} x \frac{x^2}{x^2 + (y-1)^2} + \lim_{(x,y) \rightarrow (0,1)} (y-1) \frac{(y-1)^2}{x^2 + (y-1)^2} = 0 + 0 = 0 \\ &\left(0 \leq \frac{x^2}{x^2 + (y-1)^2} \leq 1, 0 \leq \frac{(y-1)^2}{x^2 + (y-1)^2} \leq 1 \right). \end{aligned}$$

$$(5) \lim_{(x,y) \rightarrow (1,1)} \frac{xy - y - 2x + 2}{x - 1} = \lim_{(x,y) \rightarrow (1,1)} \frac{y(x-1) - 2(x-1)}{x-1} = \lim_{(x,y) \rightarrow (1,1)} (y-2) = -1.$$

$$(6) \lim_{(x,y,z) \rightarrow (1-2,0)} \ln \sqrt{x^2 + y^2 + z^2} = \ln \sqrt{5}.$$

2. 证明: 当 $(x, y) \rightarrow (0, 0)$ 时下列函数无极限:

$$(1) f(x, y) = \frac{x^4 - y^2}{x^4 + y^2} \text{ 由于}$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x^2}} \frac{x^4 - y^2}{x^4 + y^2} = 0, \quad \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x}} \frac{x^4 - y^2}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{x^4 - x^2}{x^4 + x^2} = \lim_{x \rightarrow 0} \frac{x^2 - 1}{x^2 + 1} = -1 \neq 0,$$

故当 $(x, y) \rightarrow (0, 0)$ 时上述函数无极限.

$$(2) f(x, y) = \begin{cases} \frac{x+y}{x-y}, & y \neq x, \\ 0, & y = x. \end{cases}$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x}} f(x, y) = 0, \quad \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=2x}} f(x, y) = \lim_{x \rightarrow 0} \frac{3x}{-x} = -3 \neq 0.$$

3. 讨论当 $(x, y) \rightarrow (0, 0)$ 时下列函数是否有极限, 若有极限, 求出其值:

$$(1) f(x, y) = (x + 2y) \ln(x^2 + y^2) = x \ln(x^2 + y^2) + 2y \ln(x^2 + y^2),$$

$$|x \ln(x^2 + y^2)| \leq 2\sqrt{|x|^2 + |y|^2} |\ln \sqrt{|x|^2 + |y|^2}| \rightarrow 0 ((x, y) \rightarrow (0, 0)),$$

$$\lim_{(x,y) \rightarrow (0,0)} x \ln(x^2 + y^2) = 0. \text{ 类似有 } \lim_{(x,y) \rightarrow (0,0)} 2y \ln(x^2 + y^2) = 0.$$

$$\text{故 } \lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0.$$

(2)

$$(2) f(x, y) = \frac{1 - \cos(x^2 + y^2)}{(x^2 + y^2)x^2 y^2} \cdot 1 - \cos(x^2 + y^2) \sim \frac{1}{2}(x^2 + y^2)^2.$$

只需讨论 $\frac{(x^2 + y^2)^2}{(x^2 + y^2)x^2 y^2} = \frac{x^2 + y^2}{x^2 y^2} = \frac{1}{y^2} + \frac{1}{x^2}$ 极限存在与否.

$$\lim_{(x,y) \rightarrow (0,0)} \left(\frac{1}{y^2} + \frac{1}{x^2} \right) = +\infty, \quad \lim_{(x,y) \rightarrow (0,0)} f(x, y) = +\infty.$$

不存在有限极限.

$$(3) f(x, y) = (x^2 + y^2)^{x^2 y^2} = e^{x^2 y^2 \ln(x^2 + y^2)},$$

$$|x^2 y^2 \ln(x^2 + y^2)| = x^2 y^2 |\ln(x^2 + y^2)| \leq \frac{1}{2}(x^2 + y^2) |\ln(x^2 + y^2)| \rightarrow 0$$

$$((x, y) \rightarrow (0, 0)). \quad \lim_{(x,y) \rightarrow (0,0)} x^2 y^2 \ln(x^2 + y^2) = 0,$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} e^{x^2 y^2 \ln(x^2 + y^2)} = e^{\lim_{(x,y) \rightarrow (0,0)} x^2 y^2 \ln(x^2 + y^2)} = e^0 = 1.$$

$$(4) f(x, y) = \frac{P_n(x, y)}{\rho^{n-1}}, n \geq 1, \text{ 其中 } \rho = \sqrt{x^2 + y^2}, P_n(x, y) \text{ 为 } n \text{ 次齐次多项式.}$$

$$0 \leq \alpha \leq n, \quad \left| \frac{x^\alpha y^{n-\alpha}}{\rho^{n-1}} \right| = \frac{|x|^\alpha |y|^{n-\alpha}}{\rho^{n-1}} \leq \frac{\rho^\alpha \rho^{n-\alpha}}{\rho^{n-1}} = \rho \rightarrow 0 (\rho \rightarrow 0).$$

故极限存在, 并且等于零.

4. 求下列函数的累次极限 $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y)$ 及 $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y)$:

$$(1) f(x, y) = \frac{|x| - |y|}{|x| + |y|}.$$

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{|x| - |y|}{|x| + |y|} = \lim_{x \rightarrow 0} \frac{|x|}{|x|} = 1, \quad \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{|x| - |y|}{|x| + |y|} = \lim_{y \rightarrow 0} \frac{-|y|}{|y|} = -1.$$

$$(2) f(x, y) = \frac{y^3 + \sin x^2}{x^2 + y^2}.$$

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{y^3 + \sin x^2}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = 1,$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{y^3 + \sin x^2}{x^2 + y^2} = \lim_{y \rightarrow 0} \frac{y^3}{y^2} = 0.$$

$$(3) f(x, y) = (1 + x)^{\frac{y}{x}} (x \neq 0), f(0, y) = 1.$$

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} (1 + x)^{\frac{y}{x}} = \lim_{x \rightarrow 0} 1 = 1,$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} (1 + x)^{\frac{y}{x}} = \lim_{y \rightarrow 0} e^y = 1.$$

习题 6.3

1. 下列函数在哪些点连续?

$$(1) z = \frac{1}{x^2 + y^2}, (x, y) \neq (0, 0).$$

$$(2) z = \frac{1}{\sin x} + \frac{1}{\cos y}, x \neq m\pi, y \neq \frac{\pi}{2}(2n+1), m, n \in \mathbf{Z}.$$

$$(3) z = \frac{y^2 + x}{y^2 - 2x}, y^2 - 2x \neq 0.$$

2. 设 \bar{D} 是平面 Oxy 上的有界闭区域, $P_0(x_0, y_0)$ 是 D 的外点. 证明: 在 \bar{D} 内一定存在与 P_0 距离最长的点, 也存在与 P_0 距离最近的点.

证 考虑函数 $f(x, y) = d((x, y), (x_0, y_0)) = \sqrt{(x - x_0)^2 + (y - y_0)^2}, (x, y) \in \bar{D}$.

f 在 \bar{D} 上连续, \bar{D} 是平面 Oxy 上的有界闭区域. 根据有界闭区域上连续函数的最值定理, 存在 $P_1, P_2 \in \bar{D}$, 使得 $f(P_2) \leq f(P) \leq f(P_1)$, P_1, P_2 分别是 \bar{D} 内与 P_0 距离最长和最近的点.

3. 设函数 $f(x, y)$ 在区域 D 内连续, 又点 $(x_i, y_i) \in D (i = 1, 2, \dots, n)$. 证明: 在 D 内存在点 (ξ, η) , 使

$$f(\xi, \eta) = \frac{1}{n} [f(x_1, y_1) + \dots + f(x_n, y_n)].$$

证 写出连结点 $(x_i, y_i) \in D (i = 1, 2, \dots, n)$ 的折线方程:

$$L: \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \quad \alpha \leq t \leq \beta, L \subset D.$$

设 $(\varphi(t_i), \psi(t_i)) = (x_i, y_i), (i = 1, 2, \dots, n)$.

考虑一元函数 $g(t) = f(\varphi(t), \psi(t)) \in C([\alpha, \beta])$.

根据一元函数的介值定理, 存在 $\tau \in [\alpha, \beta]$, 使得

$$g(\tau) = \frac{1}{n} [g(t_1) + \dots + g(t_n)],$$

令 $(\varphi(\tau), \psi(\tau)) = (\xi, \eta)$, 则有

$$f(\xi, \eta) = \frac{1}{n} [f(x_1, y_1) + \dots + f(x_n, y_n)].$$

4. 已知二元函数 $f(x, y)$ 在 (x_0, y_0) 处连续, 证明函数 $u = f(x, y)$ 在 x_0 处连续.

证 $\forall \varepsilon > 0, \exists \delta > 0$, 使得当 $|x - x_0| < \delta, |y - y_0| < \delta$ 时, $|f(x, y) - f(x_0, y_0)| < \varepsilon$.

特别有, 当 $|x - x_0| < \delta$ 时, $|f(x, y_0) - f(x_0, y_0)| < \varepsilon$, 即 $u = f(x, y_0)$ 在 x_0 处连续.

5. 将区间套原理推广到 \mathbf{R}^2 中, 也即证明下列命题:

设 $R_n = \{(x, y) | a_n \leq x \leq b_n, c_n \leq y \leq d_n\}$, 其中 $0 < b_n - a_n \rightarrow 0$ 且 $0 < d_n - c_n \rightarrow 0$,

$R_{n+1} \subseteq R_n, \forall n = 1, 2, \dots$. 则存在唯一的一个点 $(\xi, \eta) \in R_n, \forall n = 1, 2, \dots$.

证

6. 举出一个例子说明一个二元函数 $u = f(x, y)$ 在 $D = \{(x, y) \mid x^2 + y^2 < 1\}$ 连续, 但它在 D 中是无界的.

解 $f(x, y) = \frac{1}{1 - (x^2 + y^2)}, (x, y) \in D$ 在 D 连续, 但它在 D 中是无界的.

7. 设 $z = f(P)$ 在区域 D 中连续, 且 D 内有两点 P_1 与 P_2 . 证明: 对于任意 η , $f(P_1) \leq \eta \leq f(P_2)$, 在 D 内存在一点 P_0 , 使得 $f(P_0) = \eta$.

证由于 D 是连通的, 存在折线

$$L: \begin{cases} x = x(t) \\ y = y(t) \end{cases}, \alpha \leq t \leq \beta,$$

$$L \subset D, (x(\alpha), y(\alpha)) = P_1, (x(\beta), y(\beta)) = P_2.$$

$\varphi(t) = f(x(t), y(t))$ 在 $[\alpha, \beta]$ 连续, 并且 $\varphi(\alpha) \leq \eta \leq \varphi(\beta)$.

根据连续函数的介值定理, 存在 $t_0 \in [\alpha, \beta]$, 使得 $\varphi(t_0) = \eta$.

记 $(x(t_0), y(t_0)) = P_0$, 则有 $f(P_0) = \eta$.

习题 6.4

1. 求下列函数的一阶偏导数:

$$(1) z = \ln(x + \sqrt{x^2 + y^2}).$$

$$\frac{\partial z}{\partial x} = \frac{1 + \frac{x}{\sqrt{x^2 + y^2}}}{x + \sqrt{x^2 + y^2}} = \frac{1}{\sqrt{x^2 + y^2}},$$

$$\frac{\partial z}{\partial y} = \frac{\frac{y}{\sqrt{x^2 + y^2}}}{x + \sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}(x + \sqrt{x^2 + y^2})}.$$

$$(2) z = \frac{x}{\sqrt{x^2 + y^2}}.$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}} - \frac{x^2}{\sqrt{x^2 + y^2}^3} = \frac{y^2}{\sqrt{x^2 + y^2}^3},$$

$$\frac{\partial z}{\partial y} = -\frac{xy}{\sqrt{x^2 + y^2}^3}.$$

$$(3) z = x^{x^y}, \ln z = x^y \ln x.$$

$$\frac{1}{z} = yx^{y-1} \ln x + x^{y-1}, \frac{\partial z}{\partial x} = z(yx^{y-1} \ln x + x^{y-1}),$$

$$\frac{1}{z} \frac{\partial z}{\partial y} = x^y \ln x \ln x, \frac{\partial z}{\partial y} = z(x^y \ln^2 x).$$

$$(4) z = \frac{xy}{x-y}.$$

$$\frac{\partial z}{\partial x} = y \left(\frac{x-y-x}{(x-y)^2} \right) = \frac{-y^2}{(x-y)^2},$$

$$\frac{\partial z}{\partial y} = x \left(\frac{x-y+y}{(x-y)^2} \right) = \frac{x^2}{(x-y)^2}.$$

$$(5) z = \arcsin(x\sqrt{y}).$$

$$\frac{\partial z}{\partial x} = \frac{\sqrt{y}}{\sqrt{1-x^2y}}, \frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}\sqrt{1-x^2y}}.$$

$$(6) z = xe^{-xy}.$$

$$\frac{\partial z}{\partial x} = e^{-xy} + xe^{-xy}(-y) = e^{-xy}(1-xy), \frac{\partial z}{\partial y} = -x^2e^{-xy}.$$

$$(7) u = \frac{y}{x} + \frac{z}{y} - \frac{x}{z}.$$

$$\frac{\partial u}{\partial x} = -\frac{y}{x^2} - \frac{1}{z}, \frac{\partial u}{\partial y} = \frac{1}{x} - \frac{z}{y^2}, \frac{\partial u}{\partial z} = \frac{1}{y} + \frac{x}{z^2}.$$

$$(8) u = (xy)^z.$$

$$\frac{\partial u}{\partial x} = yz(xy)^{z-1}, \frac{\partial u}{\partial y} = xz(xy)^{z-1}, \frac{\partial u}{\partial z} = (xy)^z \cdot \ln(xy)$$

2求下列函数在指定点的偏导数:

$$(1) z = \frac{x \arccos(y-1) - (y-1) \cos x}{1 + \sin x + \sin(y-1)}, \text{求} \left. \frac{\partial z}{\partial x} \right|_{(0,1)} \text{及} \left. \frac{\partial z}{\partial y} \right|_{(0,1)}.$$

$$\left. \frac{\partial z}{\partial x} \right|_{(0,1)} = \left. \frac{d}{dx} \frac{x}{1 + \sin x} \right|_{x=0} = \left. \frac{d}{dx} \frac{1 + \sin x - x \cos x}{(1 + \sin x)^2} \right|_{x=0} = 1,$$

$$\left. \frac{\partial z}{\partial y} \right|_{(0,1)} = \left. \frac{d}{dy} \frac{-(y-1)}{1 + \sin(y-1)} \right|_{y=1} = \left. \frac{d}{dy} \frac{-(1 + \sin(y-1)) + (y-1) \cos(y-1)}{(1 + \sin(y-1))^2} \right|_{y=1} = -1.$$

$$(2) z = \frac{2y}{y + \cos x}, \text{求} \left. \frac{\partial z}{\partial x} \right|_{(\frac{\pi}{2}, 1)} \text{及} \left. \frac{\partial z}{\partial y} \right|_{(\frac{\pi}{2}, 1)}.$$

$$\frac{\partial z}{\partial x} = \frac{2y \sin x}{(y + \cos x)^2}, \left. \frac{\partial z}{\partial y} \right| = 2 \times \frac{y + \cos x - y}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}$$

$$\left. \frac{\partial z}{\partial x} \right|_{(\frac{\pi}{2}, 1)} = 2, \left. \frac{\partial z}{\partial y} \right|_{(\frac{\pi}{2}, 1)} = 0.$$

$$(3) f(x, y, z) = \ln(xy + z), \text{求} f_x(2, 1, 0), f_y(2, 1, 0), f_z(2, 1, 0).$$

$$f_x(x, y, z) = \frac{y}{xy + z}, f_y(x, y, z) = \frac{x}{xy + z}, f_z(x, y, z) = \frac{1}{xy + z}.$$

$$f_x(2, 1, 0) = \frac{1}{2}, f_y(2, 1, 0) = 1, f_z(2, 1, 0) = \frac{1}{2}.$$

$$3. \text{证明函数} f(x, y) = \begin{cases} \frac{x^2 + y^2}{|x| + |y|}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0) \end{cases}$$

在(0,0)连续,但是 $f_x(0,0)$ 不存在.

$$\text{证} \left| f(x, y) \right| = \frac{x^2 + y^2}{|x| + |y|} \leq |x| + |y| \rightarrow 0 ((x, y) \rightarrow (0, 0)),$$

$$f(x, y) \rightarrow f(0, 0) = 0 ((x, y) \rightarrow (0, 0)),$$

$f(x, y)$ 在(0,0)连续.

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x^2}{\Delta x}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} \text{不存在.}$$

4. 设 $z = \sqrt{x} \sin \frac{y}{x}$, 证明 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{z}{2}$.

证为齐1/2次函数, 根据关于齐次函数微分的一个定理, 立得结论.
直接计算如下.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{1}{2\sqrt{x}} \sin \frac{y}{x} + \sqrt{x} \cos \frac{y}{x} \left(-\frac{y}{x^2} \right), \quad \frac{\partial z}{\partial y} = \sqrt{x} \cos \frac{y}{x} \left(\frac{1}{x} \right), \\ x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= \frac{\sqrt{x}}{2} \sin \frac{y}{x} - \frac{y}{\sqrt{x}} \cos \frac{y}{x} + \frac{y}{\sqrt{x}} \cos \frac{y}{x} = \frac{\sqrt{x}}{2} \sin \frac{y}{x} = \frac{z}{2}.\end{aligned}$$

5.求下列函数的二阶混合偏导数 f_{xy} :

(1) $f(x, y) = \ln(2x + 3y)$.

$$f_x = \frac{2}{2x+3y}, f_{xy} = \frac{-6}{(2x+3y)^2}.$$

(2) $f(x, y) = y \sin x + e^x$.

$$f_x = y \cos x + e^x, f_{xy} = \cos x.$$

(3) $f(x, y) = x + xy^2 + 4x^3 - \ln(x^2 + 1)$.

$$f_x = 1 + y^2 + 12x^2 - \frac{2x}{x^2+1}, f_{xy} = 2y.$$

(4) $f(x, y) = x \ln(xy) = x \ln x + x \ln y$.

$$f_x = \ln y + \ln x + 1, f_{xy} = \frac{1}{y}.$$

6. 设 $u = e^{-3y} \cos 3x$, 证明 u 满足平面 Laplace 方程 $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

证 Q $\frac{\partial u}{\partial x} = -3e^{-3y} \sin 3x, \frac{\partial^2 u}{\partial x^2} = -9e^{-3y} \cos 3x,$

$$\frac{\partial u}{\partial y} = -3e^{-3y} \cos 3x, \frac{\partial^2 u}{\partial y^2} = 9e^{-3y} \cos 3x,$$

$$\therefore \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

7. 证明函数 $u(x, t) = e^{x+ct} + 4 \cos(3x + 3ct)$ 满足波动方程 $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.

证 $\frac{\partial u}{\partial t} = ce^{x+ct} - 12c \sin(3x + 3ct), \frac{\partial^2 u}{\partial t^2} = c^2 e^{x+ct} - 36c^2 \cos(3x + 3ct),$

$$\frac{\partial u}{\partial x} = e^{x+ct} - 12 \sin(3x + 3ct), \frac{\partial^2 u}{\partial x^2} = e^{x+ct} - 36 \cos(3x + 3ct),$$

故 $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$

8. 设 $u = u(x, y)$ 及 $v = v(x, y)$ 在 D 内又连续的二阶偏导数, 且满足方程组

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \text{ 证明 } u \text{ 及 } v \text{ 在 } D \text{ 内满足平面 Laplace 方程 } \Delta u = \Delta v = 0,$$

其中 $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$

证 $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y} \frac{\partial v}{\partial x} = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x \partial y} (\frac{\partial^2 v}{\partial y \partial x} \text{ 和 } \frac{\partial^2 v}{\partial x \partial y} \text{ 连续}),$

故 $\Delta u = 0$. 类似证 $\Delta v = 0$.

9. 已知函数 $z(x, y)$ 满足 $\frac{\partial z}{\partial x} = -\sin y + \frac{1}{1-xy}$ 以及 $z(0, y) = 2 \sin y + y^2$. 试求 z 的表达式.

$$\text{解 } z = \int \left(-\sin y + \frac{1}{1-xy} \right) dx = -x \sin y - \frac{1}{y} \ln(1-xy) + C,$$

$$z(0, y) = C = 2 \sin y + y^2, \quad z(x, y) = -x \sin y - \frac{1}{y} \ln |1-xy| + 2 \sin y + y^2$$

$$= (2-x) \sin y + y^2 - \frac{1}{y} \ln |1-xy|.$$

10. 求下列函数的全微分:

$$(1) z = e^{y/x}.$$

$$dz = e^{y/x} d \frac{y}{x} = e^{y/x} \frac{xdy - ydx}{x^2}.$$

$$(2) z = \frac{x+y}{x-y}, dz = \frac{(dx+dy)(x-y) - (x+y)(dx-dy)}{(x-y)^2} = \frac{(-2y)dx + (2x)dy}{(x-y)^2}.$$

$$(3) z = \arctan \frac{y}{x} + \arctan \frac{x}{y} = \arctan \frac{y}{x} + \operatorname{arccot} \frac{y}{x} = \frac{\pi}{2}, dz = 0.$$

$$(4) u = \sqrt{x^2 + y^2 + z^2}, du = \frac{d(x^2 + y^2 + z^2)}{\sqrt{x^2 + y^2 + z^2}} = \frac{2(xdx + ydy + zdz)}{\sqrt{x^2 + y^2 + z^2}}.$$

11. 已知函数 $z(x, y)$ 的全微分

$dz = (4x^3 + 10xy^3 - 3y^4)dx + (15x^2y^2 - 12xy^3 + 5y^4)dy$, 求 $f(x, y)$ 的表达式.

$$\text{解 } \frac{\partial z}{\partial x} = 4x^3 + 10xy^3 - 3y^4, \frac{\partial z}{\partial y} = 15x^2y^2 - 12xy^3 + 5y^4.$$

$$z = \int (4x^3 + 10xy^3 - 3y^4)dx = x^4 + 5x^2y^3 - 3xy^4 + C(y),$$

$$\frac{\partial z}{\partial y} = 15x^2y^2 - 12xy^3 + C'(y) = 15x^2y^2 - 12xy^3 + 5y^4,$$

$$C'(y) = 5y^4, C(y) = y^5 + C. f(x, y) = x^4 + 5x^2y^3 - 3xy^4 + y^5 + C.$$

12. 已知函数 $z = f(x, y)$ 的全微分

$$dz = \left(x - \frac{y}{x^2 + y^2} \right) dx + \left(y + \frac{x}{x^2 + y^2} \right) dy, \text{ 求 } z(x, y) \text{ 的表达式.}$$

$$\text{解 } dz = \left(x - \frac{y}{x^2 + y^2} \right) dx + \left(y + \frac{x}{x^2 + y^2} \right) dy$$

$$= xdx + ydy + \frac{xdy - ydx}{x^2 + y^2}$$

$$= \frac{1}{2} d(x^2 + y^2) + \frac{\frac{xdy - ydx}{x^2 + y^2}}{1 + \left| \frac{y}{x} \right|^2} = \frac{1}{2} d(x^2 + y^2) + \frac{d \frac{y}{x}}{1 + \left| \frac{y}{x} \right|^2} = \frac{1}{2} d(x^2 + y^2) + d \arctan \frac{y}{x}$$

$$= d \left(\frac{1}{2} (x^2 + y^2) + \arctan \frac{y}{x} \right).$$

$$z = \frac{1}{2} (x^2 + y^2) + \arctan \frac{y}{x} + C.$$

13. $z = f(x, y)$ $D: \{(x - x_0)^2 + (y - y_0)^2 < R^2\}$ $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$. 证明: $f(x, y)$

在区域上恒等于常数.

证 $\forall (x, y) \in D$,

$$f(x, y) - f(x_0, y_0) = [f(x, y) - f(x_0, y)] + [f(x_0, y) - f(x_0, y_0)]$$

$$= f_x(\xi, y)(x - x_0) + f_y(x_0, \eta)(y - y_0) = 0. f(x, y) = f(x_0, y_0), (x, y) \in D.$$

14. 证明: 函数 $f(x, y) = \sqrt{|xy|}$ 在点 $(0, 0)$ 处连续, $f_x(0, 0), f_y(0, 0)$ 存在, 但 $f(x, y)$ 在 $(0, 0)$ 处不可微.

证 $|f(x, y)| = \sqrt{|xy|} \rightarrow 0 = f(0, 0) ((x, y) \rightarrow (0, 0))$, $f(x, y) = \sqrt{|xy|}$ 在点 $(0, 0)$ 处连续.

$f_x(0, 0) = 0, f_y(0, 0) = 0$. 若 $f(x, y)$ 在 $(0, 0)$ 处可微, 将有

$$f(x, y) = o(\sqrt{x^2 + y^2}) (\sqrt{x^2 + y^2} \rightarrow 0), \text{ 特别应有}$$

$$f(x, x) = |x| = o(\sqrt{2}|x|) (x \rightarrow 0),$$

但此式显然不成立.

15. 设 $P(x, y)dx + Q(x, y)dy$ 在区域 D 中是某个函数 $u(x, y)$ 之全微分, 且 $P, Q \in C^1(D)$.

证明 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

证 由假设 $du = P(x, y)dx + Q(x, y)dy$. $\frac{\partial u}{\partial x} = P, \frac{\partial u}{\partial y} = Q$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x \partial y},$$

由 $P, Q \in C^1(D)$ 得 $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x} \in C(D)$, 故 $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$, 即 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$$16. \text{ 设函数 } f(x, y) = \begin{cases} \frac{(x^2 - y^2)xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

(1) 计算 $f_x(0, y) (y \neq 0)$;

(2) 根据偏导数定义证明 $f_x(0, 0) = 0$;

(3) 在上述结果的基础上证明 $f_{xy}(0, 0) = -1$;

(4) 重复上述步骤于 $f_y(x, 0)$, 并证明 $f_{yx}(0, 0) = 1$.

证 (1) 设 $y \neq 0$, 则 $f_x(x, y) = \frac{[2x^2y + (x^2 - y^2)y](x^2 + y^2) - 2x(x^2 - y^2)xy}{(x^2 + y^2)^2}$,

$$f_x(0, y) = \frac{-y^5}{y^4} = -y.$$

(2) $f(x, 0) = 0, f_x(0, 0) = 0$.

(3) $f_{xy}(0, 0) = (-y)'|_{y=0} = -1$.

(4) 设 $x \neq 0$, 则 $f_y(x, y) = \frac{[-2xy^2 + (x^2 - y^2)x](x^2 + y^2) - 2y(x^2 - y^2)xy}{(x^2 + y^2)^2}$,

$$f_y(x, 0) = x, f(0, y) = 0, f_y(0, 0) = 0, f_{yx}(0, 0) = x'|_{x=0} = 1.$$

17. 设 $z = x \ln(xy)$, 求 $\frac{\partial^3 z}{\partial x^3} \frac{\partial^3 z}{\partial x \partial y^2}$.

解 $\frac{\partial z}{\partial x} = \ln(xy) + x \frac{y}{xy} = \ln(xy) + 1, \frac{\partial^2 z}{\partial x^2} = \frac{1}{x}, \frac{\partial^3 z}{\partial x^3} = -\frac{1}{x^2},$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{y}, \frac{\partial^3 z}{\partial x \partial y^2} = -\frac{1}{y^2}.$$

习题 6.5

在下面的习题中,出现的函数 $f(u, v)$ 或 $F(u)$ 一律假定有连续的一阶偏导数或导数.

1.求下列复合函数的偏导数或导数:

(1) $z = \sqrt{u^2 + v^2}$, 其中 $u = xy, v = y^2$.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{uy}{\sqrt{u^2 + v^2}} = \frac{xy^2}{\sqrt{x^2 y^2 + y^4}} = \frac{x|y|}{\sqrt{x^2 + y^2}} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{ux}{\sqrt{u^2 + v^2}} + \frac{2vy}{\sqrt{u^2 + v^2}} \\ &= \frac{x^2 y}{\sqrt{x^2 y^2 + y^4}} + \frac{2y^3}{\sqrt{x^2 y^2 + y^4}} = \frac{x^2 \operatorname{sgn} y}{\sqrt{x^2 + y^2}} + \frac{2y^2 \operatorname{sgn} y}{\sqrt{x^2 + y^2}} = \frac{(x^2 + 2y^2) \operatorname{sgn} y}{\sqrt{x^2 + y^2}}.\end{aligned}$$

(2) $z = \frac{u^2}{v}$, 其中 $u = ye^x, v = x \ln y$.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{2u}{v} \times ye^x - \frac{u^2}{v^2} \ln y = \frac{2ye^x}{x \ln y} \times ye^x - \frac{(ye^x)^2}{(x \ln y)^2} \ln y = \frac{2y^2 e^{2x}}{x \ln y} - \frac{(ye^x)^2}{x^2 \ln y} = \frac{(2x-1)y^2 e^{2x}}{x^2 \ln y} \\ \frac{\partial z}{\partial y} &= \frac{2ue^x}{v} - \frac{u^2 x}{v^2 y} = \frac{2ye^{2x}}{x \ln y} - \frac{y^2 e^{2x} x}{x^2 (\ln^2 y)y} = \frac{ye^{2x}(2 \ln y - 1)}{x(\ln^2 y)}.\end{aligned}$$

(3) $z = f(u, v)$, 其中 $u = \sqrt{xy}, v = x + y$.

$$\frac{\partial z}{\partial x} = f_u(u, v) \frac{y}{2\sqrt{xy}} + f_v(u, v), \quad \frac{\partial z}{\partial y} = f_u(u, v) \frac{x}{2\sqrt{xy}} + f_v(u, v).$$

$$(4) z = f\left(xy, \frac{x}{y}\right), \quad \frac{\partial z}{\partial x} = f'_1 g y + f'_2 g \frac{1}{y}, \quad \frac{\partial z}{\partial y} = f'_1 g x - f'_2 g \frac{x}{y^2}.$$

$$(5) z = f(x^2 - y^2, e^{xy}), \quad \frac{\partial z}{\partial x} = f'_1 g 2x + f'_2 g e^{xy} y, \quad \frac{\partial z}{\partial y} = f'_1 g (-2y) + f'_2 g e^{xy} x.$$

2. 设 $u = f(x + y + z, x^2 + y^2 + z^2)$, 求 $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

$$\frac{\partial u}{\partial x} = f'_1 + 2xf'_2,$$

$$\frac{\partial^2 u}{\partial x^2} = f''_{11} + 2xf''_{12} + 2f'_2 + 2x(f''_{21} + 2xf''_{22}) = f''_{11} + 4xf''_{12} + 4x^2 f''_{22} + 2f'_2,$$

$$\frac{\partial^2 u}{\partial y^2} = f''_{11} + 4yf''_{12} + 4y^2 f''_{22} + 2f'_2, \quad \frac{\partial^2 u}{\partial z^2} = f''_{11} + 4zf''_{12} + 4z^2 f''_{22} + 2f'_2,$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3f''_{11} + 4(x + y + z)f''_{12} + 4(x^2 + y^2 + z^2)f''_{22} + 6f'_2.$$

4. 设 $z = x^n f\left(\frac{y}{x^2}\right)$ 其中函数 f 可微. 证明 z 满足下列方程

$$x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y} = nz.$$

$$\text{证 } \frac{\partial z}{\partial x} = nx^{n-1}f\left(\frac{y}{x^2}\right) + x^n f'\left(\frac{y}{x^2}\right)\left(-\frac{2y}{x^3}\right), \frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x^2}\right)\left(\frac{1}{x^2}\right).$$

$$\begin{aligned} x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y} &= x \left(nx^{n-1}f\left(\frac{y}{x^2}\right) + x^n f'\left(\frac{y}{x^2}\right)\left(-\frac{2y}{x^3}\right) \right) + 2yx^n f'\left(\frac{y}{x^2}\right)\left(\frac{1}{x^2}\right) \\ &= nx^n f\left(\frac{y}{x^2}\right) = nz. \end{aligned}$$

$$5. \text{ 设 } z = \frac{y}{F(x^2 - y^2)}, \text{ 试证明 } \frac{1}{x} \frac{\partial z}{\partial x} + \frac{1}{y} \frac{\partial z}{\partial y} = \frac{z}{y^2}.$$

$$\text{证 } \frac{\partial z}{\partial x} = -\frac{2xyF'(x^2 - y^2)}{(F(x^2 - y^2))^2}, \frac{\partial z}{\partial y} = \frac{F(x^2 - y^2) + 2y^2F'(x^2 - y^2)}{(F(x^2 - y^2))^2}.$$

$$\begin{aligned} \frac{1}{x} \frac{\partial z}{\partial x} + \frac{1}{y} \frac{\partial z}{\partial y} &= -\frac{2yF'(x^2 - y^2)}{(F(x^2 - y^2))^2} + \frac{F(x^2 - y^2) + 2y^2F'(x^2 - y^2)}{y(F(x^2 - y^2))^2} \\ &= \frac{1}{yF(x^2 - y^2)} = \frac{1}{y^2} \frac{y}{F(x^2 - y^2)} = \frac{z}{y^2}. \end{aligned}$$

$$6. \text{ 设函数 } u(x, y) \text{ 有二阶连续偏导数且满足 Laplace 方程 } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

证明, 作变量替换 $x = e^s \cos t, y = e^s \sin t$ 后, u 依然满足关于 s, t 的 Laplace 方程

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = 0.$$

$$\text{证 } \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t,$$

$$\begin{aligned} \frac{\partial^2 u}{\partial s^2} &= e^s \cos t \left(\frac{\partial^2 u}{\partial x^2} e^s \cos t + \frac{\partial^2 u}{\partial x \partial y} e^s \sin t \right) + \frac{\partial u}{\partial x} e^s \cos t + e^s \sin t \left(\frac{\partial^2 u}{\partial x \partial y} e^s \cos t + \frac{\partial^2 u}{\partial y^2} e^s \sin t \right) \\ &\quad + e^s \sin t \frac{\partial u}{\partial y}, \end{aligned}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} e^s \sin t + \frac{\partial u}{\partial y} e^s \cos t,$$

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= -e^s \sin t \left(-\frac{\partial^2 u}{\partial x^2} e^s \sin t + \frac{\partial^2 u}{\partial x \partial y} e^s \cos t \right) - \frac{\partial u}{\partial x} e^s \cos t + e^s \cos t \left(-\frac{\partial^2 u}{\partial x \partial y} e^s \sin t + \frac{\partial^2 u}{\partial y^2} e^s \cos t \right) \\ &\quad - e^s \sin t \frac{\partial u}{\partial y}. \end{aligned}$$

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = e^s \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

7. 验证下列各式:

$$(1) u = F(x^2 + y^2) \text{ 则, } y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0;$$

$$(2) u = F(x - ct), c \text{ 为常数, 则 } \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0.$$

7.验证下列各式:

$$(1) u = F(x^2 + y^2) \text{ 则, } y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0;$$

$$(2) u = F(x - ct), c \text{ 为常数, 则 } \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0.$$

$$\text{证}(1) \frac{\partial u}{\partial x} = F'(x^2 + y^2)2x, \frac{\partial u}{\partial y} = F'(x^2 + y^2)2y,$$

$$y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = yF'(x^2 + y^2)2x - xF'(x^2 + y^2)2y = 0.$$

$$(2) \frac{\partial u}{\partial t} = F'(x - ct)(-c), \frac{\partial u}{\partial x} = F'(x - ct),$$

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = F'(x - ct)(-c) + cF'(x - ct) = 0.$$

8.若 $f(x, y, z)$ 满足关系式 $f(tx, ty, tz) = t^n f(x, y, z)$, 其中 t 为任意实数, 则称 f 为 n 次齐次函数. 证明, 任意一个可微的 n 次齐次函数满足下列方程

$$xf_x + yf_y + zf_z = nz.$$

证 $f(tx, ty, tz) = t^n f(x, y, z)$, 对 t 求导,

$$f_1'(tx, ty, tz)x + f_2'(tx, ty, tz)y + f_3'(tx, ty, tz)z = nt^{n-1}f(x, y, z),$$

$$\text{令 } t=1 \text{ 得 } xf_x + yf_y + zf_z = nz.$$

9.设 $z = f(x, y)$ 在一个平面区域 D 中有定义. 假定 D 有这样的性质, 对于其中任意一点 (x_0, y_0) , 区域 D 与直线 $y = y_0$ 之交是一个区间. 又设 $z = f(x, y)$ 在区域 D 内有连续

的一阶偏导数, 若 $f(x, y)$ 对 x 的偏导数恒为零, 也即 $\frac{\partial f(x, y)}{\partial x} = 0, \forall (x, y) \in D$.

证明: $f(x, y)$ 可以表示成 y 的函数, 也即存在一个函数 $F(y)$, 使得

$$f(x, y) = F(y), \forall (x, y) \in D.$$

证设 $(x, y) \in D, (x_0, y) \in D, x_0 < x$. 由Lagrange中值公式,

$$f(x, y) - f(x_0, y) = \frac{\partial f(\xi, y)}{\partial x}(x - x_0) = 0.$$

即 $f(x, y)$ 的值不依赖 x , 只依赖 y , 其值记为 $F(y)$, 则有 $f(x, y) = F(y), \forall (x, y) \in D$.

10.设 $z = f(x, y)$ 在全平面上有定义, 且有连续的一阶偏导数, 满足方程

$$xf_x(x, y) + yf_y(x, y) = 0. \text{ 证明: 存在一个函数 } F(\theta),$$

使得 $f(r \cos \theta, r \sin \theta) = F(\theta)$.

$$\text{证 } x = r \cos \theta, y = r \sin \theta. \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}(r \cos \theta) + \frac{\partial z}{\partial y}(r \sin \theta)$$

$$= \frac{\partial z}{\partial x}(x) + \frac{\partial z}{\partial y}(y) = 0.$$

由上题, 存在一个函数 $G(r)$, 使得 $f(r \cos \theta, r \sin \theta) = F(\theta)$.

11. 设 $z = f(x, y)$ 在全平面上有定义, 且有连续的一阶偏导数, 满足方程 $y f_x(x, y) - x f_y(x, y) = 0$. 证明: 存在一个函数 $G(r)$,

使得 $f(r \cos \theta, r \sin \theta) = G(r)$.

$$\begin{aligned} \text{证 } x = r \cos \theta, y = r \sin \theta. \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}(-r \sin \theta) + \frac{\partial z}{\partial y}(r \cos \theta) \\ &= \frac{\partial z}{\partial x}(-y) + \frac{\partial z}{\partial y}(x) = 0. \end{aligned}$$

由9题, 存在一个函数 $G(r)$, 使得 $f(r \cos \theta, r \sin \theta) = G(r)$.

习题 6.6

1. 求函数 $f(x, y) = x^2 - xy + y^2$ 在点 $P_0(2 + \sqrt{3}, 1 + 2\sqrt{3})$ 处沿极角为 θ 的方向 l 的方向导数. 并问 θ 取何值时, 对应的方向导数 (1) 达到最大值; (2) 达到最小值; (3) 等于 0.

解 (1) $\nabla f(x, y) = (2x - y, -x + 2y)$, $\nabla f(2 + \sqrt{3}, 1 + 2\sqrt{3}) = (3, 3\sqrt{3}) = 3(1, \sqrt{3})$.

$$\frac{\partial f}{\partial l} = 6\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right) = 6\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right) = 6\cos\left(\frac{\pi}{3} - \theta\right).$$

$$\theta = \frac{\pi}{3}. (2) \theta = \frac{4\pi}{3}. (3) \theta = \frac{5\pi}{6}, \frac{11\pi}{6}.$$

2. 求函数 $f(x, y) = x^3 - 3x^2y + 3xy^2 + 2$ 在点 $P_0(3, 1)$ 处沿从 P_0 到 $P(6, 5)$ 方向的方向导数.

解 $\nabla f(x, y) = (3x^2 - 6xy + 3y^2, -3x^2 + 6xy)$,

$$\nabla f(3, 1) = (12, -9) = 3(4, -3) = 3\sqrt{5}\left(\frac{4}{\sqrt{5}}, -\frac{3}{\sqrt{5}}\right).$$

$$l = (6, 5) - (3, 1) = (3, 4) = 5\left(\frac{3}{5}, \frac{4}{5}\right), \frac{\partial f}{\partial l}(3, 1) = 3\sqrt{5}\left(\frac{4}{\sqrt{5}}, -\frac{3}{\sqrt{5}}\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = 0.$$

3. 求函数 $f(x, y) = \ln(x + y)$ 在点 $(1, 2)$ 沿抛物线 $y = 2x^2$ 在该点的切线方向的方向导数.

解 $y' = 4x$, 切线斜率 $k = 4$, 方向 $l = \pm\left(\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$. $\nabla f(x, y) = \left(\frac{1}{x+y}, \frac{1}{x+y}\right)$,

$$\nabla f(1, 2) = \left(\frac{1}{3}, \frac{1}{3}\right). \frac{\partial f}{\partial l} = \pm\left(\frac{1}{3\sqrt{17}} + \frac{4}{3\sqrt{17}}\right) = \pm\frac{5}{3\sqrt{17}}.$$

4. 求函数 $u(x, y, z) = xy + yz + zx$ 在点 $P_0(2, 1, 3)$ 沿着与各坐标轴构成等角的方向的方向导数.

解 设方向 l 与各坐标轴构成等角 α , $3\cos^2\alpha = 1$, $\cos\alpha = \pm\frac{1}{\sqrt{3}}$.

$$\nabla u(x, y, z) = (y + z, x + z, x + y), \nabla u(2, 1, 3) = (4, 5, 3).$$

$$\frac{\partial u}{\partial l} = \pm\frac{12}{\sqrt{3}} = \pm 4\sqrt{3}.$$

5. 求 $z = f(x, y) = x^2 + 2xy + y^2$ 在点 $(1, 2)$ 处的梯度.

解 $\nabla f(x, y) = (2x + 2y, 2x + 2y) = 2(x + y, x + y)$,

$$\nabla f(1, 2) = 2(3, 3) = 6(1, 1).$$

6. 求 $z = f(x, y) = \arctan \frac{y}{x}$ 在点 (x_0, y_0) 的梯度, 并求沿向量 (x_0, y_0) 的方向导数.

$$\text{解 } \nabla f(x, y) = \left(\frac{-\frac{y}{x^2}}{1 + \left(\frac{y}{x}\right)^2}, \frac{\frac{1}{x^2}}{1 + \left(\frac{y}{x}\right)^2} \right) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right),$$

$$\nabla f(x_0, y_0) = \left(-\frac{y_0}{x_0^2 + y_0^2}, \frac{x_0}{x_0^2 + y_0^2} \right),$$

$$\frac{\partial f}{\partial l}(x_0, y_0) = -\frac{y_0}{x_0^2 + y_0^2} \frac{x_0}{\sqrt{x_0^2 + y_0^2}} + \frac{x_0}{x_0^2 + y_0^2} \frac{y_0}{\sqrt{x_0^2 + y_0^2}} = 0.$$

7. 求函数 $z = f(x, y) = \ln \frac{y}{x}$ 分别在点 $A\left(\frac{1}{3}, \frac{1}{10}\right)$ 及点 $B\left(1, \frac{1}{6}\right)$ 处的两个梯度之间的夹角余弦.

解 $z = \ln |y| - \ln |x|$. $\nabla f(x, y) = \left(-\frac{1}{x}, \frac{1}{y} \right)$, $\nabla f\left(\frac{1}{3}, \frac{1}{10}\right) = (-3, 10)$,

$$\nabla f\left(1, \frac{1}{6}\right) = (-1, 6).$$

$$\langle \nabla f\left(\frac{1}{3}, \frac{1}{10}\right), \nabla f\left(1, \frac{1}{6}\right) \rangle = \frac{(-3, 10) \cdot (-1, 6)}{\sqrt{109}\sqrt{37}} = \frac{63}{\sqrt{109}\sqrt{37}}.$$

8. 求函数 $f(x, y) = x(x - 2y) + x^2 y^2$ 在点 $(1, 1)$ 处沿方向 $(\cos \alpha, \cos \beta)$ 的方向导数, 并求出最大的与最小的方向导数, 它们各沿什么方向?

解 $\nabla f(x, y) = (2x - 2y + 2xy^2, -2x + 2x^2 y)$, $\nabla f(1, 1) = (2, 0)$.

$$\frac{\partial f}{\partial l}(1, 1) = 2 \cos \alpha.$$

最大的与方向导数: 2, 最小的方向导数: -2, 分别沿方向 x 轴方向和负 x 轴方向.

9. 证明函数 $f(x, y) = \frac{y}{x^2}$ 在椭圆周 $x^2 + 2y^2 = 1$ 上任一点处沿椭圆周法方向的方向导数等于 0.

证 $\nabla f(x, y) = \left(-\frac{2y}{x^3}, \frac{1}{x^2} \right)$. 椭圆周法方向 $n(x, y) = (2x, 4y)$.

$$\text{方向导数} = \frac{1}{|n|} \left(-\frac{2y}{x^3} \times 2x + \frac{1}{x^2} \times 4y \right) = \frac{2(-2xy + 2xy)}{|n|x^3} = 0.$$

习题 6.7

1. 求函数 $f(x, y) = xy - y$ 在点 $(1, 1)$ 的二阶 Taylor 多项式.

$$\begin{aligned}\text{解 } f(x, y) &= xy - y = (x-1+1)(y-1+1) - (y-1) - 1 \\ &= (x-1) + (x-1)(y-1).\end{aligned}$$

2. 在点 $(0, 0)$ 的邻域内, 将下列函数按带 Peano 型余项展开成 Taylor 公式 (到二阶):

$$\begin{aligned}(1) f(x, y) &= \frac{\cos x}{\cos y} = \frac{1 - \frac{x^2}{2} + o(x^2)}{1 - \frac{y^2}{2} + o(y^2)} = \left(1 - \frac{x^2}{2} + o(x^2)\right) \left(1 + \frac{y^2}{2} + o(y^2)\right) \\ &= 1 - \frac{x^2}{2} + \frac{y^2}{2} + o(x^2 + y^2) (\sqrt{x^2 + y^2} \rightarrow 0).\end{aligned}$$

$$\begin{aligned}(2) f(x, y) &= \ln(1+x+y) = x+y - \frac{1}{2}(x+y)^2 + o(x^2 + y^2) \\ &= x+y - \frac{1}{2}(x^2 + 2xy + y^2) + o(x^2 + y^2) (\sqrt{x^2 + y^2} \rightarrow 0).\end{aligned}$$

$$(3) f(x, y) = \sqrt{1-x^2-y^2} = 1 - \frac{1}{2}(x^2 + y^2) + o(x^2 + y^2) (\sqrt{x^2 + y^2} \rightarrow 0).$$

$$(4) f(x, y) = \sin(x^2 + y^2) = x^2 + y^2 + o(x^2 + y^2) (\sqrt{x^2 + y^2} \rightarrow 0).$$

3. 在点 $(0, 0)$ 的邻域内, 将函数 $f(x, y) = \ln(1+x+y)$ 按 Lagrange 余项展开成 Taylor 公式 (到一阶).

$$\text{解 } \ln(1+x) = x - \frac{1}{2(1+\theta x)^2} x^2.$$

$$\ln(1+x+y) = x+y - \frac{1}{2(1+\theta x+\theta y)^2} (x+y)^2.$$

4. 利用 Taylor 公式证明: 当 $|x|, |y|, |z|$ 充分小时, 有近似公式

$$\cos(x+y+z) - \cos x \cos y \cos z \approx -(xy + yx + zx).$$

证 由于 $\cos(x+y+z) - \cos x \cos y \cos z$

$$\begin{aligned}&= 1 - \frac{1}{2}(x+y+z)^2 + o(\rho^2) - \left(1 - \frac{x^2}{2} + o(\rho^2)\right) \left(1 - \frac{y^2}{2} + o(\rho^2)\right) \left(1 - \frac{z^2}{2} + o(\rho^2)\right) \\ &= -(xy + yx + zx) + o(\rho^2) (\rho \rightarrow 0).\end{aligned}$$

故当 $|x|, |y|, |z|$ 充分小时, 有近似公式

$$\cos(x+y+z) - \cos x \cos y \cos z \approx -(xy + yx + zx).$$

5. 设 D 是单位圆, 即 $D = \{(x, y) \mid x^2 + y^2 < 1\}$, 又设函数 $f(x, y)$ 在 D 内有连续的偏导数且满足 $xf_x(x, y) + yf_y(x, y) = 0, (x, y) \in D$. 证明: $f(x, y)$ 在 D 内是一常数.

$$\text{证 } f(x, y) - f(0, 0) = f_x(\theta x, \theta y)x + f_y(\theta x, \theta y)y$$

$$= \frac{1}{\theta} [f_x(\theta x, \theta y)\theta x + f_y(\theta x, \theta y)\theta y] = 0.$$

$$f(x, y) = f(0, 0), (x, y) \in D.$$

习题 6.8

在本节习题中所涉及的函数 f 或 F 都是有连续一阶偏导数的函数.

1. 求由下列方程确定的隐函数 $z = z(x, y)$ 的所有一阶偏导数:

(1) $x^3 z + z^3 x - 2yz = 0$.

$$3x^2 z + x^3 \frac{\partial z}{\partial x} + 3z^2 \frac{\partial z}{\partial x} x + z^3 - 2y \frac{\partial z}{\partial x} = 0,$$

$$x^3 \frac{\partial z}{\partial y} + 3z^2 \frac{\partial z}{\partial y} x - 2z - 2y \frac{\partial z}{\partial y} = 0.$$

$$= -\frac{3x^2 z + z^3}{x^3 + 3xz^2 - 2y}, \frac{\partial z}{\partial x} = \frac{2z}{x^3 + 3xz^2 - 2y}.$$

(2) $yz - \ln z = x + y$.

$$y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1, z + y \frac{\partial z}{\partial y} - \frac{1}{z} \frac{\partial z}{\partial y} = 1.$$

$$\frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}} = \frac{z}{yz - 1}, \frac{\partial z}{\partial y} = \frac{1 - z}{y - \frac{1}{z}} = \frac{z - z^2}{yz - 1}.$$

(3) $x + z - \varepsilon \sin z = y (0 < \varepsilon < 1)$.

$$1 + (1 - \varepsilon \cos z) \frac{\partial z}{\partial x} = 0, (1 - \varepsilon \cos z) \frac{\partial z}{\partial y} = 1.$$

$$\frac{\partial z}{\partial x} = -\frac{1}{1 - \varepsilon \cos z}, \frac{\partial z}{\partial y} = \frac{1}{1 - \varepsilon \cos z}.$$

(4) $z^x = y^z$.

$$z^x \ln z + xz^{x-1} \frac{\partial z}{\partial x} = y^z \ln y \frac{\partial z}{\partial x}, xz^{x-1} \frac{\partial z}{\partial y} = zy^{z-1} + y^z \ln y \frac{\partial z}{\partial y}.$$

$$\frac{\partial z}{\partial x} = -\frac{z^x \ln z}{xz^{x-1} - y^z \ln y} = -\frac{z^x \ln z}{xz^{x-1} - z^x \ln y} = -\frac{z \ln z}{x - z \ln y},$$

$$\frac{\partial z}{\partial y} = \frac{zy^{z-1}}{xz^{x-1} - y^z \ln y} = \frac{zy^z}{xyz^{x-1} - y^z y \ln y} = \frac{zz^x}{xyz^{x-1} - z^x y \ln y} = \frac{z^2}{xy - zy \ln y}.$$

(5) $x \cos y + y \cos z + z \cos x = 1$.

$$\cos y - z \sin x + (-y \sin z + \cos x) \frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial x} = \frac{\cos y + z \sin x}{y \sin x - \cos x}.$$

$$-x \sin y + \cos z + (-y \sin z + \cos x) \frac{\partial z}{\partial y} = 0, \frac{\partial z}{\partial y} = \frac{x \sin y - \cos z}{\cos x - y \sin z}.$$

2. 设由方程 $f(xy^2, x+y) = 0$ 确定隐函数为 $y = y(x)$, 求 $\frac{dy}{dx}$.

解 $f_1'(xy^2, x+y)(y^2 + 2xyy') + f_2'(xy^2, x+y)(1+y') = 0,$

$$\frac{dy}{dx} = -\frac{y^2 f_1'(xy^2, x+y) + f_2'(xy^2, x+y)}{2xy f_1'(xy^2, x+y) + f_2'(xy^2, x+y)}.$$

3. 设 $z + \cos xy = e^z$, 求 $\frac{\partial z}{\partial x}$ 及 $\frac{\partial^2 z}{\partial x^2}$.

解 $z + \cos xy - e^z = 0$. $-y \sin xy + (1 - e^z) \frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial x} = \frac{y \sin xy}{1 - e^z}$.

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{y^2(1 - e^z) \cos xy - y \sin xy(-e^z \frac{\partial z}{\partial x})}{(1 - e^z)^2} \\ &= \frac{y^2(1 - e^z) \cos xy - y \sin xy(-e^z \frac{y \sin xy}{1 - e^z})}{(1 - e^z)^2} \\ &= y^2 \frac{(1 - e^z)^2 \cos xy + e^z \sin^2 xy}{(1 - e^z)^3}. \end{aligned}$$

4. 设 $F(x, x + y, x + y + z) = 0$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解 $F'_1 + F'_2 + F'_3(1 + \frac{\partial z}{\partial x}) = 0$, $\frac{\partial z}{\partial x} = -\frac{F'_1 + F'_2 + F'_3}{F'_3}$.

$$F'_2 + F'_3(1 + \frac{\partial z}{\partial y}) = 0, \frac{\partial z}{\partial y} = -\frac{F'_2 + F'_3}{F'_3}.$$

另解 $dF(x, x + y, x + y + z) = 0$,

$$F'_1 dx + F'_2(dx + dy) + F'_3(dx + dy + dz) = 0,$$

$$dz = -\frac{F'_1 + F'_2 + F'_3}{F'_3} dx - \frac{F'_2 + F'_3}{F'_3} dy, \frac{\partial z}{\partial x} = -\frac{F'_1 + F'_2 + F'_3}{F'_3}, \frac{\partial z}{\partial y} = -\frac{F'_2 + F'_3}{F'_3}.$$

5. 设 $z = z(x, y)$ 是方程 $F(x, y, z) = 0$ 确定的隐函数, 利用一阶微分形式的不变型,

证明 $dz = -\frac{F_x}{F_z} dx - \frac{F_y}{F_z} dy$ ($F_z \neq 0$), 并求

$F(x^2 + y^2 + z^2, xy - z^2) = 0$ 确定的隐函数 $z = z(x, y)$ 的8微分 dz .

证 $dF(x, y, z) = F_x dx + F_y dy + F_z dz = 0$, $dz = -\frac{F_x}{F_z} dx - \frac{F_y}{F_z} dy$ ($F_z \neq 0$).

记 $dF(x^2 + y^2 + z^2, xy - z^2) = 0$.

$$F'_1(2x dx + 2y dy + 2z dz) + F'_2(y dx + x dy - 2z dz) = 0,$$

$$(2xF'_1 + yF'_2)dx + (2yF'_1 + xF'_2)dy + (2zF'_1 - 2zF'_2)dz = 0,$$

$$dz = \frac{(2xF'_1 + yF'_2)dx + (2yF'_1 + xF'_2)dy}{2z(F'_1 - F'_2)}.$$

6. 证明球坐标变换的Jacobi行列式 $J = r^2 \sin \varphi$.

$$\text{证} \begin{cases} x = r \sin \varphi \cos \theta \\ y = r \sin \varphi \sin \theta \\ z = r \cos \varphi \end{cases}$$

$$\text{证 } dx = \sin \varphi \cos \theta dr + r \cos \varphi \cos \theta d\varphi - r \sin \varphi \sin \theta d\theta,$$

$$dy = \sin \varphi \sin \theta dr + r \cos \varphi \sin \theta d\varphi + r \sin \varphi \cos \theta d\theta,$$

$$dz = \cos \varphi dr - r \sin \varphi d\varphi.$$

$$J = \begin{vmatrix} \sin \varphi \cos \theta & r \cos \varphi \cos \theta & -r \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & r \cos \varphi \sin \theta & r \sin \varphi \cos \theta \\ \cos \varphi & -r \sin \varphi & 0 \end{vmatrix}$$

$$= r^2 \cos^2 \varphi \sin \varphi + r^2 \sin^3 \varphi = r^2 \sin \varphi.$$

7. 设由 $x = u + v$, $y = u^2 + v^2$, $z = u^3 + v^3$ 确定函数 $z = z(x, y)$, 求当

$$x = 0, y = u = \frac{1}{2}, v = -\frac{1}{2} \text{ 时, } \frac{\partial z}{\partial x} \text{ 与 } \frac{\partial z}{\partial y} \text{ 的值.}$$

$$\text{解 } dz = 3u^2 du + 3v^2 dv.$$

$$\begin{cases} du + dv = dx \\ 2udu + 2vdv = dy \end{cases} \quad du = \frac{2vdx - dy}{2v - 2u}, dv = \frac{dy - 2udx}{2v - 2u}.$$

$$x = 0, y = u = \frac{1}{2}, v = -\frac{1}{2}$$

$$du = \frac{-dx - dy}{-2}, dv = \frac{dy - dx}{-2},$$

$$dz = \frac{3}{4} \times \frac{1}{2} (dx + dy) + \frac{3}{4} \times \frac{1}{2} (dx - dy) = \frac{3}{4} dx,$$

$$\frac{\partial z}{\partial x} = \frac{3}{4}, \frac{\partial z}{\partial y} = 0.$$

$$8. \text{ 设 } \begin{cases} xu + yv = 0, \\ uv - xy = 5. \end{cases}$$

求当 $x=1, y=-1, u=v=2$ 时 $\frac{\partial^2 u}{\partial x^2}$ 与 $\frac{\partial^2 u}{\partial x \partial y}$ 的值.

$$\text{解 } \begin{cases} udx + xdu + vdy + ydv = 0, \\ vdu + udv - ydx - xdy = 0. \end{cases}$$

$$\begin{cases} xdu + ydv = -udx - vdy, \\ vdu + udv = ydx + xdy. \end{cases}$$

$$du = \frac{u(-udx - vdy) - y(ydx + xdy)}{xu - yv} = \frac{(-u^2 - y^2)dx + (-uv - xy)dy}{xu - yv}$$

$$\frac{\partial u}{\partial x} = -\frac{u^2 + y^2}{xu - yv}, \quad \frac{\partial u}{\partial y} = -\frac{uv + xy}{xu - yv}.$$

$$dv = \frac{x(ydx + xdy) + v(udx + vdy)}{xu - yv} = \frac{(xy + uv)dx + (x^2 + v^2)dy}{xu - yv},$$

$$\frac{\partial v}{\partial x} = \frac{(xy + uv)}{xu - yv}.$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial}{\partial x} \frac{u^2 + y^2}{xu - yv} = -\frac{\left(2u \frac{\partial u}{\partial x}\right)(xu - yv) - (u^2 + y^2)\left(u + x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x}\right)}{(xu - yv)^2}.$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x} \frac{\partial u}{\partial y} = -\frac{\partial}{\partial x} \frac{uv + xy}{xu - yv} \\ &= -\frac{\left(\frac{\partial u}{\partial x}v + \frac{\partial v}{\partial x}u + y\right)(xu - yv) - (uv + xy)\left(u + x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x}\right)}{(xu - yv)^2}. \end{aligned}$$

当 $x=1, y=-1, u=v=2$ 时,

$$\frac{\partial u}{\partial x} = -\frac{u^2 + y^2}{xu - yv} = -\frac{5}{4}, \quad \frac{\partial v}{\partial x} = \frac{(xy + uv)}{xu - yv} = \frac{3}{4}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\left(2u \frac{\partial u}{\partial x}\right)(xu - yv) - (u^2 + y^2)\left(u + x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x}\right)}{(xu - yv)^2}$$

$$= -\frac{4 \times (-5) - 5 \times \left(-\frac{5}{4} + \frac{3}{4}\right)}{16} = \frac{55}{32}.$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\left(\frac{\partial u}{\partial x}v + \frac{\partial v}{\partial x}u + y\right)(xu - yv) - (uv + xy)\left(u + x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x}\right)}{(xu - yv)^2} = \frac{25}{32}.$$

9. 设 $x^2 + y^2 = \frac{1}{2}z^2$, $x + y + z = 2$, 求当 $x = 1, y = -1, z = 2$ 时 $\frac{dx}{dz}$ 与 $\frac{dy}{dz}$ 的值.

$$\text{解} \begin{cases} 2xdx + 2ydy = zdz \\ dx + dy + dz = 0 \end{cases}$$

$$\begin{cases} 2xdx + 2ydy = zdz \\ dx + dy = -dz \end{cases}$$

$$dx = \frac{z+2y}{2-2y}dz, dy = \frac{-2x-z}{2-2y}dz,$$

$$\frac{dx}{dz} = \frac{z+2y}{2-2y}, \frac{dy}{dz} = \frac{-2x-z}{2-2y}.$$

当 $x = 1, y = -1, z = 2$ 时

$$\frac{dx}{dz} = \frac{z+2y}{2-2y} = \frac{0}{4} = 0, \frac{dy}{dz} = \frac{-2x-z}{2-2y} = \frac{-4}{4} = -1.$$

10. 设 $x = \cos \varphi \cos \theta, y = \cos \varphi \sin \theta, z = \sin \varphi$, 求 $\frac{\partial z}{\partial x}$.

$$\text{解} \begin{cases} -\sin \varphi \cos \theta d\varphi - \cos \varphi \sin \theta d\theta = dx \\ -\sin \varphi \sin \theta d\varphi + \cos \varphi \cos \theta d\theta = dy \\ \cos \varphi d\varphi = dz \end{cases}$$

由前两个方程解出

$$d\varphi = \frac{\begin{vmatrix} dx & -\cos \varphi \sin \theta \\ dy & \cos \varphi \cos \theta \end{vmatrix}}{\begin{vmatrix} -\sin \varphi \cos \theta & -\cos \varphi \sin \theta \\ -\sin \varphi \sin \theta & \cos \varphi \cos \theta \end{vmatrix}} = -\frac{\cos \varphi \cos \theta dx + \cos \varphi \sin \theta dy}{\sin \varphi \cos \varphi}$$

$$= -\frac{\cos \theta}{\sin \varphi} dx - \frac{\sin \theta}{\sin \varphi} dy,$$

$$dz = \cos \varphi d\varphi = -\frac{\cos \varphi \cos \theta}{\sin \varphi} dx - \frac{\cos \varphi \sin \theta}{\sin \varphi} dy$$

$$\frac{\partial z}{\partial x} = -\frac{\cos \varphi \cos \theta}{\sin \varphi} = -\frac{x}{z}.$$

另解 $x^2 + y^2 + z^2 = 1, 2xdx + 2zdz = 0,$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}.$$

再解 $z = \pm \sqrt{1 - x^2 - y^2},$

$$\frac{\partial z}{\partial x} = \frac{-x}{\pm \sqrt{1 - x^2 - y^2}} = -\frac{x}{z}.$$

11. 设 $u = u(x, y)$ 及 $v = v(x, y)$ 有连续一阶偏导数, 又设 $x = x(\xi, \eta)$ 及 $y = y(\xi, \eta)$ 也有连续一阶偏导数, 且使复合函数 $u = u(x(\xi, \eta), y(\xi, \eta))$ 及 $v = v(x(\xi, \eta), y(\xi, \eta))$ 有定义. 证明

$$\frac{D(u, v)}{D(\xi, \eta)} = \frac{D(u, v)}{D(x, y)} \frac{D(x, y)}{D(\xi, \eta)}.$$

$$\text{证 } \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi}, \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta},$$

$$\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \xi}, \frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \eta},$$

$$\frac{D(u, v)}{D(\xi, \eta)} = \frac{\begin{vmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi} & \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta} \\ \frac{\partial v}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \xi} & \frac{\partial v}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}}$$

$$\frac{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}} = \frac{D(u, v)}{D(x, y)} \frac{D(x, y)}{D(\xi, \eta)}.$$

习题 6.9

1. 求下列函数的极值:

$$(1) z = x^2(x-1)^2 + y^2.$$

$$\frac{\partial z}{\partial x} = 2x(x-1)^2 + 2(x-1)x^2$$

$$= x(x-1)(2x-2+2x) = x(x-1)(4x-2) = 0,$$

$$x = 0, \frac{1}{2}, 1.$$

$$\frac{\partial z}{\partial y} = 2y = 0, y = 0.$$

三个稳定点 $(0, 0), (\frac{1}{2}, 0), (1, 0)$. $2x(x-1)^2 + 2(x-1)x^2$

$$A = \frac{\partial^2 z}{\partial x^2} = 2(x-1)^2 + 4x(x-1) + 2x^2 + 4x(x-1) = 2(x-1)^2 + 8x(x-1) + 2x^2,$$

$$C = \frac{\partial^2 z}{\partial y^2} = 2, B = \frac{\partial^2 z}{\partial x \partial y} = 0.$$

$(0, 0), A = 2 > 0, B = 0, C = 2, AC - B^2 = 4$, 极小值点, 极小值 $z(0, 0) = 1$.

$(\frac{1}{2}, 0), A = -1, B = 0, C = 2, AC - B^2 = -2$, 非极小值点.

$(1, 0), A = 2, B = 0, C = 2, AC - B^2 = 4 > 0$, 极小值点. 极小值 $z(1, 0) = 0$.

$$(2) z = 2xy - 5x^2 - 2y^2 + 4x + 4y - 1.$$

$$\frac{\partial z}{\partial x} = 2y - 10x + 4 = 2(y - 5x + 2),$$

$$\frac{\partial z}{\partial y} = 2x - 4y + 4 = 2(x - 2y + 2).$$

$$\begin{cases} -5x + y = -2 \\ x - 2y = -2 \end{cases} \quad x = \frac{2}{3}, y = \frac{4}{3}.$$

稳定点 $(\frac{2}{3}, \frac{4}{3})$.

$$A = \frac{\partial^2 z}{\partial x^2} = -10 < 0, C = \frac{\partial^2 z}{\partial y^2} = -4, B = \frac{\partial^2 z}{\partial x \partial y} = 2.$$

$AC - B^2 = 36 > 0, (\frac{2}{3}, \frac{4}{3})$ 极大值点.

极大值 $= z(\frac{2}{3}, \frac{4}{3}) = 3$.

$$(3) z = 6x^2 - 2x^3 + 3y^2 + 6xy + 1.$$

$$\frac{\partial z}{\partial x} = 12x - 6x^2 + 6y = 6(2x - x^2 + y)$$

$$\frac{\partial z}{\partial y} = 6y + 6x = 6(x + y)$$

$$\begin{cases} 2x - x^2 + y = 0 \\ x + y = 0 \end{cases} \quad x = 0, 1, \text{相应地 } y = 0, -1.$$

稳定点(0,0), (1,-1).

$$\text{在点}(0,0), A = \frac{\partial^2 z}{\partial x^2} = 12 - 12x = 12 > 0, C = \frac{\partial^2 z}{\partial y^2} = 6, B = \frac{\partial^2 z}{\partial x \partial y} = 6.$$

$$AC - B^2 = 66 > 0, (0,0) \text{极小值点, 极小值 } z(0,0) = 1.$$

$$\text{在点}(1,-1), A = \frac{\partial^2 z}{\partial x^2} = 12 - 12x = 0, C = \frac{\partial^2 z}{\partial y^2} = 6, B = \frac{\partial^2 z}{\partial x \partial y} = 6.$$

$$AC - B^2 = -36 < 0, z \text{不取极值.}$$

$$(4) z = 4xy - x^4 - y^4 + 5.$$

$$\frac{\partial z}{\partial x} = 4y - 4x^3 = 4(y - x^3),$$

$$\frac{\partial z}{\partial y} = 4x - 4y^3 = 4(x - y^3).$$

$$\begin{cases} y - x^3 = 0 \\ x - y^3 = 0 \end{cases} \quad x = 0, \pm 1, \text{相应地 } y = 0, \pm 1. \text{稳定点}(0,0), (1,1), (-1,-1).$$

$$\text{在点}(0,0), A = \frac{\partial^2 z}{\partial x^2} = -12x^2 = 0, C = \frac{\partial^2 z}{\partial y^2} = -12y^2 = 0, B = \frac{\partial^2 z}{\partial x \partial y} = 4.$$

$$AC - B^2 = -16 < 0, (0,0) \text{不是极值点.}$$

$$\text{在点}(1,1), A = \frac{\partial^2 z}{\partial x^2} = -12 < 0, C = \frac{\partial^2 z}{\partial y^2} = -12, B = \frac{\partial^2 z}{\partial x \partial y} = 4.$$

$$AC - B^2 = 128 > 0, z \text{取极大值 } z(1,1) = 7.$$

$$\text{在点}(-1,-1), A = \frac{\partial^2 z}{\partial x^2} = -12 < 0, C = \frac{\partial^2 z}{\partial y^2} = -12, B = \frac{\partial^2 z}{\partial x \partial y} = 4.$$

$$AC - B^2 = 128 > 0, z \text{取极大值 } z(-1,-1) = 7.$$

$$(5) z = x^3 y^2 (6 - x - y) (x > 0, y > 0).$$

$$\frac{\partial z}{\partial x} = 3x^2 y^2 (6 - x - y) - x^3 y^2 = x^2 y^2 (18 - 3x - 3y - x) = x^2 y^2 (18 - 4x - 3y)$$

$$\frac{\partial z}{\partial y} = 2x^3 y (6 - x - y) - x^3 y^2 = x^3 y (12 - 2x - 2y - y) = x^3 y (12 - 2x - 3y).$$

$$\begin{cases} 4x + 3y = 18 \\ 2x + 3y = 12 \end{cases} \text{ 在 } \{(x, y) | x > 0, y > 0\} \text{ 的稳定点 } (x, y) = (3, 2).$$

$$\text{在稳定点 } (3, 2), A = \frac{\partial^2 z}{\partial x^2} = 2xy^2 (18 - 4x - 3y) - 4x^2 y^2 = -144,$$

$$C = \frac{\partial^2 z}{\partial y^2} = x^3 (12 - 2x - 3y) - 3x^3 y = -162,$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = 2x^2 y (18 - 4x - 3y) - 3x^2 y^2 = -108.$$

$$AC - B^2 = 144 \times 162 - 108^2 = 11664 > 0, (3, 2) \text{ 极大值点, 极大值 } z(3, 2) = 108.$$

2. 确定下列函数在所给条件下的最大值及最小值:

(1) $z = x^2 + y^2$, 当 $\frac{x}{2} + \frac{y}{3} = 1$ 时.

解 由于 $\sqrt{x^2 + y^2} \rightarrow +\infty$ 时, $z \rightarrow +\infty$, 又 $z = x^2 + y^2$ 是连续函数, 故在平面 $\frac{x}{2} + \frac{y}{3} = 1$

上取极小值. $z =$ 代入法. $z = x^2 + (3(1 - \frac{x}{2}))^2 = x^2 + \frac{9}{4}(2 - x)^2 = \frac{13}{4}x^2 - 9x + 9$

$= \frac{1}{4}(13x^2 - 36x + 36) = f(x), x \in (-\infty, +\infty).$

$f'(x) = \frac{1}{4}(26x - 36) = 0, x_0 = \frac{18}{13}, y_0 = 3(1 - \frac{9}{13}) = \frac{12}{13}.$

$f''(x) = \frac{13}{2} > 0. \frac{18}{13}$ 是唯一极值点, 且是极小值点, 故是最小值点.

最小值 $f(\frac{18}{13}) = \frac{36}{13}.$

对二次函数 f 用配方法当然得到同一结果.

再解 Lagrange 乘子法. 考虑 Lagrange 函数

$F(x, y, \lambda) = x^2 + y^2 + \lambda \left(\frac{x}{2} + \frac{y}{3} - 1 \right).$

$$\begin{cases} 2x + \frac{\lambda}{2} = 0, \\ 2y + \frac{\lambda}{3} = 0, \\ \frac{x}{2} + \frac{y}{3} - 1 = 0. \end{cases} \quad x = -\frac{\lambda}{4}, y = -\frac{\lambda}{6}, -\frac{\lambda}{8} - \frac{\lambda}{18} - 1 = 0, \lambda_0 = -\frac{72}{13}.$$

得到满足条件的唯一点 $x_0 = \frac{18}{13}, y_0 = \frac{12}{13}. z(x_0, y_0)$ 是最小值.

3. 在某一行星表面要安装一个无线电望远镜, 为了减少干扰, 要将望远镜装在磁场最弱的位置. 设该行星为一球体, 半径为 6 个单位. 若以球心为坐标原点建立坐标系 $Oxyz$, 则行星表面上点 (x, y, z) 处的磁场强度为 $H(x, y, z) = 6x - y^2 + xz + 60$. 问, 应将望远镜安装在何处?

解 球面方程: $x^2 + y^2 + z^2 = 36. F(x, y, z, \lambda) = H(x, y, z) + \lambda(x^2 + y^2 + z^2 - 36).$

$$\begin{cases} \frac{\partial H}{\partial x} = 6 + z + 2\lambda x = 0 \end{cases} \quad (1)$$

$$\begin{cases} \frac{\partial H}{\partial y} = -2y + 2\lambda y = 2y(\lambda - 1) = 0 \end{cases} \quad (2)$$

$$\begin{cases} \frac{\partial H}{\partial z} = x + 2\lambda z = 0 \end{cases} \quad (3)$$

$$\begin{cases} x^2 + y^2 + z^2 = 36 \end{cases} \quad (4)$$

由(2), $y = 0$ 或 $\lambda = 1$.

$$\text{设 } y = 0, \text{ 则有 } \begin{cases} 6 + z + 2\lambda x = 0 \\ x + 2\lambda z = 0 \\ x^2 + y^2 + z^2 = 36 \end{cases}$$

解之得 $(\pm 5, 0, 3), (0, 0, -6)$, 相应 H 值为 105, 15 和 60.

设 $\lambda = 1$, 则

$$\begin{cases} 6 + z + 2x = 0 \\ x + 2z = 0 \\ x^2 + y^2 + z^2 = 36 \end{cases}$$

解之得 $(-4, \pm 4, 2)$, 相应 H 值为 12. 各条件极值比较得 $(x, y, z) = (-4, \pm 4, 2)$ 时 H 取最小值 12.

4. 已知三角形的周长为 $2p$, 问怎样的三角形绕自己的一边旋转所得的体积最大?

解 设三角形底边上的高为 x , 垂足分底边的长度为 y, z . 设三角形绕底边旋转, 旋转体体积

$$V = \frac{\pi}{3} x^2 (y + z), \quad y + z + \sqrt{x^2 + y^2} + \sqrt{x^2 + z^2} = 2p, \quad x \geq 0, y \geq 0, z \geq 0.$$

V 在有界闭集上取最大值.

$$L(x, y, z, \lambda) = x^2 (y + z) + \lambda (y + z + \sqrt{x^2 + y^2} + \sqrt{x^2 + z^2} - 2p),$$

$$\begin{cases} 2x(y + z) + \lambda \left(\frac{x}{\sqrt{x^2 + y^2}} + \frac{x}{\sqrt{x^2 + z^2}} \right) = 0, & (1) \end{cases}$$

$$\begin{cases} x^2 + \lambda \left(1 + \frac{y}{\sqrt{x^2 + y^2}} \right) = 0, & (2) \end{cases}$$

$$\begin{cases} x^2 + \lambda \left(1 + \frac{z}{\sqrt{x^2 + z^2}} \right) = 0. & (3) \end{cases}$$

$$\begin{cases} y + z + \sqrt{x^2 + y^2} + \sqrt{x^2 + z^2} - 2p = 0. & (4) \end{cases}$$

$$(2) - (3) \Rightarrow \lambda \left(\frac{y}{\sqrt{x^2 + y^2}} - \frac{z}{\sqrt{x^2 + z^2}} \right) = 0.$$

$$\text{若 } \lambda = 0, \text{ 将有 } x = 0, \text{ 不可能. 故 } \frac{y}{\sqrt{x^2 + y^2}} - \frac{z}{\sqrt{x^2 + z^2}} = 0.$$

由于 $y > 0, z > 0$, 易得 $y = z$.

$$\begin{cases} 2xy + \frac{\lambda x}{\sqrt{x^2 + y^2}} = 0, \\ x^2 + \lambda \left(1 + \frac{y}{\sqrt{x^2 + y^2}} \right) = 0, \\ y + \sqrt{x^2 + y^2} = p. \end{cases} \quad \begin{cases} 2y + \frac{\lambda}{\sqrt{x^2 + y^2}} = 0, \\ x^2 + \lambda \left(1 + \frac{y}{\sqrt{x^2 + y^2}} \right) = 0, \\ y + \sqrt{x^2 + y^2} = p. \end{cases}$$

解之得 $y = z = \frac{p}{4}$, 底边长 $= \frac{p}{2}$, 两腰长 $= \frac{1}{2}(2p - \frac{p}{2}) = \frac{3p}{4}$.

5. 在两平面有 $y + 2z = 12$ 及 $x + y = 6$ 的交线上求到原点距离最近的点.

解 $u = x^2 + y^2 + z^2$,

$$z = 6 - \frac{y}{2}, x = 6 - y, u = (6 - y)^2 + y^2 + \left(6 - \frac{y}{2}\right)^2 = \frac{9}{4}y^2 - 18y + 72.$$

$z' = \frac{9}{2}y - 18 = 0, z'' = \frac{9}{2} \cdot y_0 = 4$ 是唯一极值点, 且是极小值点, 故是最小值点.

$x_0 = 2, z_0 = 4$. 所求的点为 $(2, 4, 4)$.

6. 求椭球面 $x^2 + y^2 + \frac{z^2}{4} = 1$ 与平面 $x + y + z = 0$ 的交线上到坐标原点的最大距离与最小距离.

解 $L(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(x^2 + y^2 + \frac{z^2}{4} - 1) + \mu(x + y + z)$.

$$(*) \begin{cases} L_x = 2x + 2\lambda x + \mu = 0, \\ L_y = 2y + 2\lambda y + \mu = 0, \\ L_z = 2z + \frac{1}{2}\lambda z + \mu = 0, \\ x^2 + y^2 + \frac{z^2}{4} = 1, \\ x + y + z = 0. \end{cases}$$

由前三个方程得

$$(**) \begin{cases} 2x(1 + \lambda) = 2z + \frac{1}{2}\lambda z, \\ 2y(1 + \lambda) = 2z + \frac{1}{2}\lambda z. \end{cases}$$

下面分两种情况求解.

(1) $\lambda = -1$. 由方程组(**)得 $z = 0$, 再由(*)的后两个方程得 $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$,

$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. 这两点与原点距离为 1.

(2) $\lambda \neq -1$. 由方程组(**)得 $x = y$, 再由(*)的后两个方程得 $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$,

$(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}})$. 这两点与原点距离为 2.

在 $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$ 和 $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ 有最小距离1, 在 $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$ 和 $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ 有最大距离2.

7. 在已知圆锥体内做一内接长方体, 长方体的底面在圆锥体的底面上, 求使体积最大的那个长方体的边长.

解 设圆锥体高为 H , 底半径为 R . 取其底面为 xy 平面, 底面中心为坐标原点. 设内接长方体底面边长为 $2x, 2y$, 高为 z , 则长方体体积

$$V = 4xyz, x \geq 0, y \geq 0, z \geq 0 \text{ 满足圆锥面方程 } (H - z)^2 = \frac{H^2}{R^2}(x^2 + y^2).$$

$$L(x, y, z, \lambda) = xyz + \lambda \left((H - z)^2 - \frac{H^2}{R^2}(x^2 + y^2) \right).$$

$$(*) \begin{cases} L_x = yz - 2\lambda \frac{H^2}{R^2} x = 0, \\ L_y = xz - 2\lambda \frac{H^2}{R^2} y = 0, \\ L_z = xy - 2\lambda(H - z) = 0, \\ (H - z)^2 = \frac{H^2}{R^2}(x^2 + y^2). \end{cases}$$

由(*)的前两个方程易得 $x = y$. 由(*)的前三个方程易得 $x^2 = y^2 = \frac{R^2}{H^2} z(H - z)$.

再与第四个方程联立得 $(H - z)^2 = 2z(H - z)$, $z = \frac{H}{3}$, $x = y = \frac{\sqrt{2}}{3} R$.

8. 当 n 个正数 x_1, L, x_n 的和等于常数 l 时, 求它们的乘积的最大值. 并证明: n 个正数

a_1, L, a_n 的几何平均值不超过算术平均值, 即 $\sqrt[n]{a_1 L a_n} \leq \frac{a_1 + L + a_n}{n}$.

解 $f(x_1, L, x_n) = x_1 L x_n, x_1 + L + x_n = l$.

$F(x_1, L, x_n, \lambda) = x_1 L x_n + \lambda(x_1 + L + x_n - l)$,

$$\begin{cases} F_{x_1} = x_2 x_3 L x_n + \lambda = 0 \\ F_{x_2} = x_1 x_3 L x_n + \lambda = 0 \\ L L L L L L L L \\ F_{x_n} = x_1 x_2 L x_{n-1} + \lambda = 0. \end{cases}$$

$$\begin{cases} x_1 x_2 x_3 L x_n + \lambda x_1 = 0 \\ x_1 x_2 x_3 L x_n + \lambda x_2 = 0 \\ L L L L L L L L \\ x_1 x_2 L x_{n-1} x_n + \lambda x_n = 0. \end{cases} \quad \lambda x_1 = \lambda x_2 L = \lambda x_n.$$

若 $\lambda = 0$, 将有 $x_1 x_2 L x_{n-1} x_n = 0$, 不会是最大值. 若 $\lambda \neq 0$, 则有 $x_1 = x_2 L = x_n = \frac{l}{n}$.

$$x_1 x_2 L x_{n-1} x_n = \left(\frac{l}{n}\right)^n, \sqrt[n]{x_1 x_2 L x_{n-1} x_n} = \left(\frac{l}{n}\right) = \frac{x_1 + L + x_n}{n}.$$

10. 求函数 $f(x, y) = \frac{1}{2}(x^n + y^n)$ ($n > 1$ 是常数, $x \geq 0, y \geq 0$.) 在条件 $x + y = A$ ($A > 0$) 下的最小值, 并由此证明

$$\frac{1}{2}(x^n + y^n) \geq \left(\frac{x + y}{2}\right)^n \quad (x > 0, y > 0).$$

9. 在椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 上哪些点处, 其切线与坐标轴构成的三角形面积最大?

解 $\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$, 切线斜率: $y' = -\frac{b^2x}{a^2y}$, 切线的点 (X, Y) 满足方程:

$$Y - y = -\frac{b^2x}{a^2y}(X - x). Y_0 = 0, X_0 = x + \frac{a^2y^2}{b^2x}. X_1 = 0, Y_1 = y + \frac{b^2x^2}{a^2y}.$$

三角形面积 $f(x, y) = \left(x + \frac{a^2y^2}{b^2x}\right)\left(y + \frac{b^2x^2}{a^2y}\right) = \frac{a^2b^2}{xy}$, (x, y) 满足

$$x > 0, y > 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

由于 $x \rightarrow 0, y \rightarrow 0$ 时 $f(x, y) \rightarrow +\infty$, 故 f 在所述条件下取极小值.

$$\text{令 } L(x, y, \lambda) = \frac{a^2b^2}{xy} + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right),$$

$$\begin{cases} L_x = -\frac{a^2b^2}{x^2y} + 2\frac{\lambda x}{a^2} = 0, \\ L_y = -\frac{a^2b^2}{xy^2} + 2\frac{\lambda y}{b^2} = 0, \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \end{cases} \begin{cases} -\frac{a^2b^2}{x^2y^2} + 2\frac{\lambda x}{a^2y} = 0 \\ \frac{\lambda x}{a^2y} = \frac{\lambda y}{b^2x} \\ -\frac{a^2b^2}{x^2y^2} + 2\frac{\lambda y}{b^2x} = 0 \end{cases}$$

易见 $\lambda \neq 0$, 故 $\frac{x}{a^2y} = \frac{y}{b^2x}$, $\frac{y}{x} = \frac{b}{a}$, $y = \frac{b}{a}x$, 代入椭圆方程得

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} = 1, x = \frac{a}{\sqrt{2}}, y = \frac{b}{\sqrt{2}}.$$

在第一象限, $(x, y) = (\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$ 时, 该点切线与坐标轴构成的三角形面积

最小. 由对称性, $(-\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}), (\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}), (-\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$ 也满足要求.

习题 6.10

在指定的各点求曲面的切平面:

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 (a > 0, b > 0, c > 0)$, 在 $\left(0, \frac{b}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$ 点.

$$\mathbf{n} = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right) = \left(0, \frac{\sqrt{2}}{b}, \frac{\sqrt{2}}{c} \right),$$

$$\frac{\sqrt{2}}{b} \left(x - \frac{b}{\sqrt{2}} \right) + \frac{\sqrt{2}}{c} \left(x - \frac{c}{\sqrt{2}} \right) = 0,$$

$$\frac{\sqrt{2}}{b} x + \frac{\sqrt{2}}{c} y - 2 = 0.$$

(2) $z = x^2 - y^2, (2, 1, 3). x^2 - y^2 - z = 0$

$$\mathbf{n} = (2x, -2y, -1) = (4, -2, -1),$$

$$4(x-2) - 2(y-1) - (z-3) = 0, 4x - 2y - z = 0.$$

(3) $x = \cosh \rho \cos \theta, y = \cosh \rho \sin \theta, z = \rho (\rho > 0, 0 \leq \theta \leq 2\pi), \rho = 1, \theta = \frac{\pi}{2}.$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh \rho \cos \theta & \sinh \rho \sin \theta & 1 \\ -\cosh \rho \sin \theta & \cosh \rho \cos \theta & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \sinh 1 & 1 \\ -\cosh 1 & 0 & 0 \end{vmatrix} = (0, -\cosh 1, \sinh 1 \cosh 1),$$

$$(0, \cosh 1, 1),$$

$$-\cosh 1(y - \cosh 1) + \sinh 1 \cosh 1(z - 1) = 0.$$

(4) $e^z - 2z + xy = 3, (2, 1, 0)$

$$\mathbf{n} = (y, x, e^z - 2) = (1, 2, -1),$$

$$(x-2) + 2(y-1) - z = 0, x + 2y - z - 4 = 0.$$

2. 试证明曲面 $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a} (a > 0)$ 上任一点的切平面在各坐标轴上截距之和等于 a .

证 $\mathbf{n} = \left(\frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}}, \frac{1}{2\sqrt{z}} \right),$

$$\frac{1}{\sqrt{x}}(X-x) + \frac{1}{\sqrt{y}}(Y-y) + \frac{1}{\sqrt{z}}(Z-z) = 0,$$

$$\frac{1}{\sqrt{x}}X + \frac{1}{\sqrt{y}}Y + \frac{1}{\sqrt{z}}Z - \sqrt{a} = 0,$$

$$x \text{ 轴上截距 } X_0 = \sqrt{x}\sqrt{a}, Y_0 = \sqrt{y}\sqrt{a}, Z_0 = \sqrt{z}\sqrt{a},$$

$$X_0 + Y_0 + Z_0 = \sqrt{x}\sqrt{a} + \sqrt{y}\sqrt{a} + \sqrt{z}\sqrt{a} = (\sqrt{x} + \sqrt{y} + \sqrt{z})\sqrt{a} = \sqrt{a}\sqrt{a} = a.$$