The Boussinesq Equation

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1 Introduction

1.1 Goals

In this paper, I seek to describe the process by which the canonical Boussinesq Equation for long shallow-water waves is found and collect the core assumptions on which the derivations rests.

1.2 Context

Most of the process included here regarding the Boussinesq system of equations (nonlinear shallow water wave equations) and their combined Boussinesq Equation ("BE") came through two books by 20th-century British mathematicians, Gerald B. Whitham and D. Howell Peregrine. Gaining a complete picture of the PDE was difficult, partially due to naming conventions (Boussinesq has many other mathematical objects named after him) and multiple conflicting sources on the derivation.

Boussinesq first put forth his work in in shallow-water waves in 1871 and 1872 for one spatial dimension (as we will derive here). He also posited the soliton solution at the time. Since then, this methods have been refined, clarified, and built upon to account for more fluid effects in the equations.

2 Core Fluid Model

2.1 Navier-Stokes Equations

We will start with the Navier-Stokes equations for incompressible, inviscid Newtonian fluids. These are a set of canonical PDEs derived from physical modeling of fluids in three dimensions and accounting for conservation of mass and momentum. With $\mathbf{u}$ denoting fluid velocity, ρ fluid density, p pressure, and $\mathbf{F}$ external forces (in our case, only gravity), we have:

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p - \mathbf{F} \tag{2}$$

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For our purposes, we will focus on the "1D" model, where we have one horizontal dimension x, one vertical y, and time t. We also make two key assumptions about our fluid: that it can be described via a potential ϕ so that $\mathbf{u} = \nabla\phi$ and that the flow is irrotational, or $\nabla \times \mathbf{u} = 0$. With the velocity potential and Eqn. 1 we can easily find

$$\triangle\phi = 0 \tag{3}$$

or that our velocity is a harmonic function in x and y ($\mathbf{u}$ also has t dependence though).

2.2 Defining our Surfaces

We next define a surface for our waves, assuming water waves against open air with an interface defined as $y = \eta(x, t)$. We fix boundary conditions of fluid velocity normal to the surface defined as the velocity of the surface at that point. This is our kinematic boundary condition.

$$\frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + \phi_x\eta_x = \phi_y \tag{4}$$

Secondly, we can rewrite Eqn. 2 as

$$\frac{\partial\mathbf{u}}{\partial t} + \nabla(\frac{1}{2}\mathbf{u}^2) + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \mathbf{F}$$

Which, with our velocity potential and assumed constant pressure, we can integrate in each dimension into a scalar equation[4]. Inserting $y = \eta$ into the condition and using ϕ in place of integrated $\mathbf{u}$ terms, we find our dynamic boundary condition.

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0 \tag{5}$$

We next establish boundary conditions at our other surface—the basin. Here we make a fundamental assumption that we are modeling flow in a flat-bottomed, infinitely-long, shallow basin. At the bottom, we are not obeying the no-slip condition given our assumption of inviscid fluid, but we do obey a zero-flux condition. So when $y = -h_0$:

$$\phi_y = 0 \tag{6}$$

In-between these boundary conditions, our flow obeys $\triangle\phi = 0$.

3 Waves and Dispersion

3.1 Linearized Boundary Conditions and Plane Wave Assumption

Our next big assumption is that the water is initially at rest, and that η and ϕ will be small. With this, we can linearize our surface boundary conditions, starting with Eqn. 4 to

$$\eta_t = \phi_y$$

and Eqn. 5 to

$$\phi_t = -g\eta$$

Thus, at our surface $y = \eta$, we can eliminate η:

$$\phi_{tt} + g\phi_y = 0 \tag{7}$$

We are seeking to solve for ϕ to find η. To advance, let us start with a simple plane-wave ansatz:

$$\eta = A(x,t)\exp i(\kappa x - \omega t)$$

And assuming that ϕ does not exhibit wave behavior in the y direction,

$$\phi = Y(y)\exp i(\kappa x - \omega t)$$

This relatively arbitrary ansatz form for ϕ can be solved from our equations. From Eqn. 7 we find $Y(0) = \frac{Ag}{i\omega}$ and our bottom boundary condition yields $Y'(-h_0) = 0$. Now solving for ϕ through Laplace's Equation we find

$$Y(y) = \frac{Ag}{i\omega}\frac{\cosh(\kappa(y+h_0))}{\cosh(\kappa h_0)}$$

Now, our combined equation 7 (where $y = -h_0$) gives our dispersion relationship.

$$\omega^2 = g\kappa\tanh(\kappa h_0) \tag{8}$$

This relationship is immediately interesting as it has two modes

$$\omega(k) = \pm\sqrt{g\kappa\tanh\kappa h_0}$$

. This is unique to Boussinesq's model as opposed to similar models like Korteweg-de Vries. In addition, the dispersion relationship is satisfied with $(\kappa, \omega) = (0, 0)$ as a solution, so the governing equations must admit constant solutions.

Perhaps the most characterizing assumption in the entire derivation of Boussinesq's PDE is that $\kappa h_0 \ll 1$. That is, for wavelength $\lambda = \frac{1}{\kappa}$, we know $h_0 \ll \lambda$. This is where the denominations "long wave" and "shallow water" come from. In addition, we assume $A \ll \lambda$, or that our wave amplitudes are small in comparison to their length. This necessitates that wave slope η_x must be small. For a more in-depth discussion on what these conditions allow, see Peregrine[3]. I will refer to these two statements as our "long-wave assumptions." With this, we can asymptotically expand our dispersion relationship:

$$\omega^2 \sim {c_0}^2\kappa^2 - \frac{1}{3}{c_0}^2{h_0}^2\kappa^4 + \ldots \tag{9}$$

where $c_0 = \sqrt{gh_0}$. Note that if we look only at the first term, the phase speed will be independent of κ and we lose dispersive effects. Boussinesq's insight was that we can find an equation that satisfies this relationship (to the first two asymptotic approximation terms) by going beyond our simple linearization to arrive at Eqn. 7.

3.2 Nonlinear Shallow Wave Equations

Let $h = h_0 + \eta$ be our total depth, and we can rewrite our initial descriptive equations 1 and 2 as:

$$h_t + (uh)_x = 0 \tag{10}$$

$$u_t + uu_x + gh_x = 0 \tag{11}$$

Where the first equation comes from integrating the continuity equation from $-h_0$ to η in y and applying our boundary conditions. Whitham[4] details this transition on page 455. These are our shallow water equations. Now, through careful nondimensionalization and an expansion of ϕ, we will find our "nonlinear" or "coupled" shallow wave equations. Our asymptotic expansion for ϕ, where $Y = h_0 + \eta$ or total depth (still assumed to be a small parameter), looks like this:

$$\phi = \sum_{n=0}^{\infty} Y^n f_n(x,t)$$

Let us follow the following nondimensionalization and parameter definitions. Note that Y will represent a nondimensional total depth (rather than depth from $y = 0$. "Old" scaled variables are primed.

$$\begin{aligned} \alpha &= \frac{A}{h_0} \ll 1 \text{ and } \beta = \frac{{h_0}^2}{\lambda^2} \ll 1 \\ x &= \frac{x'}{\lambda} \text{ and } Y = \frac{h}{h_0} \\ t &= \frac{c_0 t'}{\lambda} \\ \phi &= \frac{c_0 \phi'}{g\lambda A} \text{ and } \eta = \frac{\eta'}{A} \end{aligned} \tag{12}$$

Now, by applying Laplace's equation and our bottom boundary condition (at $Y = 0$), our expansion takes the form[4]:

$$\phi \sim f - \frac{Y^2}{2!}\beta f_{xx} + \frac{Y^4}{4!}\beta^2 f_{xxxx} + O(\beta^3) \tag{13}$$

We can now also find our governing equations in rescaled coordinates. Eqn. 3 becomes

$$\beta\phi_{xx} + \phi_{YY} = 0 \text{ for } 0 < Y < 1 + \alpha\eta$$

Eqn. 6 becomes

$$\phi_Y = 0 \text{ for } Y = 0$$

And our kinematic and dynamic boundary conditions, Eqns. 4, 5, become

$$\left.\begin{aligned}\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_Y = 0,\\ \eta + \phi_t + \frac{1}{2}\alpha{\phi_x}^2 + \frac{\alpha}{2\beta}{\phi_Y}^2 = 0\end{aligned}\right\} \quad \text{for } Y = 1 + \alpha\eta \tag{14}$$

Substituting our expansion for ϕ into Eqns. 14, recognizing $\frac{\partial f}{\partial Y} = 0$, removing terms of $O(\beta^2)$ and $O(\alpha\beta)$, and evaluating at $Y \sim 1$ yields

$$\eta_t + \alpha\eta_x f_x + f_{xx} - \frac{\beta}{6}f_{xxxx} = \eta_t + [(1+\alpha\eta)f_x]_x - \frac{\beta}{6}f_{xxxx} = 0 \tag{15}$$

and

$$\eta + f_t - \frac{\beta}{2}f_{xxt} + \frac{\alpha}{2}{f_x}^2 = 0$$

which we can differentiate with respect to x once to find

$$\eta_x + f_{xt} - \frac{\beta}{2}f_{xxxt} + \frac{\alpha}{2}({f_x}^2)_x = 0 \tag{16}$$

up to $O(\beta^2, \alpha\beta, \alpha\eta)$. Equations 15 and 16 are our Boussinesq system for coupled shallow water waves.

4 The Boussinesq Long-Wave Equation

4.1 Derivation from Coupled Shallow Water Waves

Following a carefully constructed change in variables, we can capture the behavior of both Eqns. 15 and 16 in a partial differential equation we will call the Boussinesq Equation. To arrive here, we will be removing higher-order terms at multiple stages. To quote Dutykh and Dias[2], we find "one can transform higher-order terms by invoking lower-order asymptotic relations." The following derivations comes closely from a document published at the Universiti Teknologi Malaysia by Mukheta, Siam, Isa, and Mohamed[6].

We define new perturbed variables:

$$U = \eta - \alpha\eta^2 + O(\alpha^2)$$
$$X = x + \alpha\int_{-\infty}^{x}\eta dx$$
$$T = t$$

Accounting for the change in variables, we find Eqn. 15 becomes

$$\eta_T + f_{XX} + 2\alpha\eta f_{XX} - \frac{\beta}{6}f_{XXXX} = 0 \tag{17}$$

Which, following differentiation in T, becomes

$$\eta_{TT} + f_{XXT} + 2\alpha(\eta f_{XX})_T - \frac{\beta}{6}f_{XXXXT} = 0 \tag{18}$$

only removing terms beyond $O(1)$, $O(\alpha)$, and $O(\beta)$.

Similarly, Eqn. 16 becomes

$$f_{XT} + \eta_X + \frac{\alpha}{2}(\eta^2)_X - \frac{\beta}{2} f_{XXXT} = 0 \tag{19}$$

The third term in Eqn. 19 comes in the form of a perfect derivative.

Now we invoke the Dutykh and Dias quote in full, recursively substituting f_{XX} into Eqn. 17.

$$f_{XX} = -\eta_T - 2\alpha\eta(\eta_T - 2\alpha\eta f_{XX} + \frac{\beta}{6} f_{XXXX}) + \frac{\beta}{6}(-\eta_T - 2\alpha\eta f_{XX} + \frac{\beta}{6} f_{XXXX})_{XX}$$

Which, removing negligible terms, gives

$$f_{XX} = \eta_T + 2\alpha\eta\eta_T - \frac{\beta}{6}\eta_{XXT}$$

Similarly, Eqn. 19 allows us

$$f_{XT} = \eta_X - \frac{\alpha}{2}(\eta^2)_X + \frac{\beta}{2}(f_{XT})_{XX}$$

Which becomes

$$f_{XT} = \eta_X - \frac{\alpha}{2}(\eta^2)_X - \frac{\beta}{2}\eta_{XXXX} \tag{20}$$

Differentiation with X thrice, at $O(1)$ gives us

$$f_{XXXXT} = -\eta_{XXXX}$$

Substituting Eqn. 20 into 18 after X differentiation begets

$$\eta_{TT} - \eta_{XX} - \frac{\alpha}{2}(\eta^2)_{XX} - \frac{\beta}{2}\eta_{XXXX} + 2\alpha(-\eta_T{}^2 - \eta\eta_{TT}) + \frac{\beta}{6}\eta_{XXXX} + O(\alpha\beta, \alpha^2, \beta^2) = 0 \tag{21}$$

Finally, using our variable definitions, $\eta = U + \alpha\eta^2 + \ldots = U + \alpha U^2 + O(\alpha^2)$, Eqn. 21 becomes

$$U_{TT} - U_{XX} - \frac{3\alpha}{2}(U^2)_{XX} - \frac{\beta}{3}U_{XXXX} = 0 \tag{22}$$

Which is our Boussinesq PDE for long waves in shallow water. It is relatively intuitive in form: the first two terms establish it as a wave equation, the third term is a nonlinearity caused by our physical constraints and expansion, and the fourth is a dispersive term. Thus we find ourselves with a weakly nonlinear wave. It is sometimes seen with the $(U^2)_{XX}$ term expanded into $2(U_X{}^2 + UU_{XX})$ but the first term of this expansion is assumed negligibly small from our long wave assumptions.

As Ragna Eide explained in her presentation for this class, there are two forms of the equation. Eqn. 22 was Boussinesq's original, now known as the "bad" Boussinesq due to difficulties in finding descriptive solutions[1]. The "good" (or "modified") Boussinesq equation simply has a positive U_{XXXX} term instead of negative. The two can be transferred between by a nondimensional change in temporal variable with complex coefficients[6].

4.2 Miller's Argument

Peter Miller, in his book Applied Asymptotic Analysis, gives a physically intuitive derivation for why the equation must have its given form. He writes that physical wave systems for an amplitude variable u are of the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x} F[u] = 0$$

where $F[u]$ is an expression in u and $\frac{\partial^n u}{\partial x^n}$. Assuming symmetry with respect to the origin, the equation should be invariant if x is flipped sign, and likewise for u. Under Galilean invariance, the equation should not change with substitution $u \to u + b$ for a constant b. Assuming weak nonlinearity for multiple scales and rescaling our variables to $U = \epsilon u$, $X = \epsilon x$, and $T = \epsilon t$ we find $F[u]$ must take the form

$$F[u] = a\frac{\partial u}{\partial x} + b\epsilon^2 \frac{\partial^3 u}{\partial x^3} + c\epsilon^2 (\frac{\partial u}{\partial x})^2 + O(\epsilon^3)$$

Given proper choices on constants a, b, and c, with rescaling, this also yields Eqn. 22.

5 Solutions

My focus was on the construction of the Boussinesq equation and the process of finding solutions is intensive and a field of its own, but I will make note of it here. Miller details the process by which two new temporal and one new spatial scale are introduced, and we anticipate a PDE as an amplitude equation or solvability condition. Defining $X_0 = X$, $X_1 = \epsilon x$, $T_1 = \epsilon T$, and $T_2 = \epsilon^2 T$, and expanding $U \sim U_0 + \epsilon U_1 + \epsilon^2 U_2 + \ldots$, solving at $O(1)$ permits a solution in the form

$$U_0 = A \exp i\theta + \overline{A} \exp -i\theta$$

where $\theta = \kappa X_0 - \omega T_0$ and κ and ω satisfying our dispersion relationship Eqn. 9, where $A = A(X_1, T_1, T_2)$. If we proceed with this expansion, $O(\epsilon^2)$ finds us with an irreconcilable "mean" term with $\theta = 0$, which we should expect from $(\kappa, \omega) = (0, 0)$ solving our dispersion relationship. Thus, instead of our recent ansatz for U_0, we take

$$U_0 = A \exp i\theta + \overline{A} \exp -i\theta + M(X_1, T_1, T_2)$$

Through removal of resonant terms and a change of variables with $\xi = X_1 - \omega'(\kappa) T_1$, where $\omega(\kappa)$ is the group velocity, our solvability condition becomes

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(\kappa)}{2}\frac{\partial^2 A}{\partial \xi^2} + \beta |A|^2 A = 0 \quad (23)$$

which is the cubic nonlinear Schrödinger equation (NLSE). Due to the appearance of the NLSE, the Boussinesq equation 22 is often referred to as a *Universal Differential Equation.* Other PDEs, such as Korteweg-de Vries (derivable from Boussinesq's equations 15 and 16 and other methods, with some conditions), Rubel's Equation, the sine-Gordon equation, and more[5]. Solutions to NLSE (and thus BE) can take many forms depending on the approach for solving. Li[9] details various solution forms based on different parameter values. Methods include the tanh method, Bäcklund transformation, Painlevé, and inverse scattering transform[1].

6 Numerical Integration and Applications

Numerically solving the BE is an active research area with most applications falling into the category of modeling atmospheric and coastal tidal flows. Both of these examples exhibit long horizontal scales and waves with otherwise shallow vertical depths. A.G. Bratsos has done much research in the field, starting with finite difference methods to represent derivative terms, such that our differential equations become nonlinear algebraic equations, which can be solved iteratively through Newton's method or higher-order schemes like Runge-Kutta[8]. Numerican schemes are of interest for generating CFD models (such as FLUENT) for use in, as aforementioned, atmospheric and geophysical fluid approximations, as well as in computer graphics, where they can be more efficient to solve under our long-wave assumptions than Navier-Stokes[7].

7 Summary

The Boussinesq Equation captures all behavior of interface waves as described by the Navier-Stokes equations for incomopressible, irrotational, inviscid, potential flows, under our long-wave (shallow-water) assumptions. The equation can be derived through manipulation of continuity and momentum equations, analysis of the position of the surface η and the applicable boundary conditions, and asymptotic expansion of our dispersion relationship, surface height, and velocity potential. These descriptive PDEs are useful when seeking numerical integration schemes for use in research and industry. When seeking solutions, the BE yields the Cubic Nonlinear Schrödinger Equation as a solvability condition.

References

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