# CS258 Information Theory Homework 9

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**Exercise 1** (Channel with two independent looks at $Y$) Let $Y_1$ and $Y_2$ be conditionally independent and conditionally identically distributed given $X$

1. Show that $I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2)$
2. Conclude that the capacity of the channel $X \mapsto Y_1, Y_2$ is less than twice the capacity of the channel $X \mapsto Y_1$.

*Proof.*

1.
$$\begin{aligned} I(X; Y_1, Y_2) &= H(Y_1, Y_2) - H(Y_1, Y_2|X) \\ &= H(Y_1, Y_2) - H(Y_1|X) - H(Y_2|X) \\ &= H(Y_1) + H(Y_2) - I(Y_1; Y_2) - H(Y_1|X) - H(Y_2|X) \\ &= I(X; Y_1) + I(X; Y_2) - I(Y_1, Y_2) \\ &= 2I(X; Y_1) - I(Y_1, Y_2) \end{aligned} \tag{1}$$

2.
$$\begin{aligned} C_1 &= \max_{p(x)} I(X; Y_1, Y_2) \\ &= \max_{p(x)} (2I(X; Y_1) - I(Y_1, Y_2)) \\ &\leq \max_{p(x)} 2I(X; Y_1) \\ &= 2C_2 \end{aligned} \tag{2}$$

□

**Exercise 2** (Two-look Gaussian channel) Given $X \mapsto Y_1, Y_2$. Consider the ordinary Gaussian channel with two correlated looks at $X$, that is, $Y = (Y_1, Y_2)$, where

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2 \end{aligned}$$

with a power constraint $P$ on $X$, and $(Z_1, Z_2) \sim \mathcal{N}_2(0, K)$, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}$$

Find the capacity $C$ for

1. $\rho = 1$
2. $\rho = 0$
3. $\rho = -1$

*Solution.* From Theorem 8.6.5 [Cover] we know that the Gaussian distribution maximizes the entropy over all distributions with the same variance. Hence it is clear that normally distributed $X \sim \mathcal{N}(0, P)$ will maximize the mutual information. In this case $(Y_1, Y_2) \sim \left(0, \begin{bmatrix} P+N & P+\rho N \\ P+\rho N & P+N \end{bmatrix}\right)$

$$
\begin{aligned}
\max I(X; Y_1, Y_2) &= h(Y_1, Y_2) - h(Y_1, Y_2 | X) \\
&= h(Y_1, Y_2) - h(Z_1, Z_2) \\
&= \frac{1}{2} \log (2\pi e)^2 \begin{vmatrix} P+N & P+\rho N \\ P+\rho N & P+N \end{vmatrix} - \frac{1}{2} \log (2\pi e)^2 \begin{vmatrix} N & N\rho \\ N\rho & N \end{vmatrix} \\
&= \frac{1}{2} \log \left(1 + \frac{2P}{(1+\rho)N}\right)
\end{aligned}
\tag{3}
$$

1. $\rho = 1$, $C = \frac{1}{2} \log \left(1 + \frac{P}{N}\right)$
2. $\rho = 0$, $C = \frac{1}{2} \log \left(1 + \frac{2P}{N}\right)$
3. $\rho = -1$, $C = +\infty$.

□

**Exercise 3** (Output power constraint) Consider an additive white Gaussian noise channel with an expected output power constraint $P$. Thus, $Y = X + Z, Z \sim N\left(0, \sigma^2\right), Z$ is independent of $X$, and $EY^2 \leq P$ Find the channel capacity.

*Solution.*

$$
\begin{aligned}
I(X; Y) &= h(Y) - h(Y|X) \\
&= h(Y) - h(Z) \\
&= h(Y) - \frac{1}{2} \log \left(2\pi e \sigma^2\right) \\
&\leq \frac{1}{2} \log (2\pi e P) - \frac{1}{2} \log \left(2\pi e \sigma^2\right) \\
&= \frac{1}{2} \log \frac{P}{\sigma^2}
\end{aligned}
\tag{4}
$$

The equailty holds when $Y$ is normally distributed. In this case $X \sim \mathcal{N}(0, P - \sigma^2)$. □

**Exercise 4** (Exponential noise channels) $Y_i = X_i + Z_i$, where $Z_i$ is i.i.d. exponentially distributed noise with mean $\mu$. Assume that we have a mean constraint on the signal (i.e., $EX_i \leq \lambda$ ). Show that the capacity of such a channel is $C = \log \left(1 + \frac{\lambda}{\mu}\right)$

*Proof.*

$$
\begin{aligned}
I(X; Y) &= h(Y) - h(Y|X) \\
&= h(Y) - h(Z) \\
&= h(Y) - \sum_i h(Z_i) \\
&\leq \sum_i \left(h(Y_i) - h(Z_i)\right)
\end{aligned}
\tag{5}
$$

The equailty holds when $Y_i$s are independent, which can be obtained if $X_i$s are independent. Hence we can only consider the channel for the input and output to be single-valued. Still, $I(X; Y) = h(Y) - h(Z)$ holds.

Note for exponentially distributed $Z$,

$$\begin{aligned}
h(Z) &= -\int_0^{+\infty} g(z)\ln\frac{1}{\mu}e^{-\frac{z}{\mu}}dz \\
&= -\int_0^{+\infty} g(z)\ln\frac{1}{\mu}dz - \int_0^{+\infty} g(z)\frac{z}{\mu}dz \\
&= 1+\ln\mu
\end{aligned} \tag{6}$$

Note that $EY = EX + EZ \leq \lambda + \mu$. For mean-value bounded $Y$, by Theorem 12.1.1 and Example 12.2.5 [Cover], the maximizing differential entropy is $h^*(Y) = 1 + \ln(\lambda+\mu)$, with distribution $p^*(y) = \frac{1}{\lambda+\mu}e^{-\frac{y}{\lambda+\mu}}$. Therefore

$$I(X;Y) \leq \sum_{i=1}^{n} \left((1+\ln(\lambda+\mu)) - (1+\ln(\mu))\right) = n\ln\frac{\lambda+\mu}{\mu} \tag{7}$$

The equailty holds when $X_i$ are independent with mean value $\lambda$ and $Y_i \sim \exp\left(\frac{1}{\lambda+\mu}\right)$. We need to find such distribution for $X_i$. Since $X_i$ and $Z_i$ are independent and $Y_i = X_i + Z_i$, it follows that the characteristc functions hold the following relation.

$$\phi_Y(t) = \phi_X(t)\cdot\phi_Z(t) \tag{8}$$

Therefore

$$\begin{aligned}
\phi_X(t) &= \frac{\phi_Y(t)}{\phi_Z(t)} \\
&= \frac{(1-i(\lambda+\mu)t)^{-1}}{(1-i\mu t)^{-1}} \\
&= \frac{1}{\lambda+\mu}\frac{[\mu - i\mu(\lambda+\mu)t]+\lambda}{1-i(\lambda+\mu)t} \\
&= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}(1-i(\lambda+\mu)t)^{-1}
\end{aligned} \tag{9}$$

The characteristc function above is a linear combination of two kinds of distribution. We can set every $X_i$ to be 0 with the probability of $\frac{\mu}{\lambda+\mu}$, and to be exponentially distributed with mean value $\lambda+\mu$ by the probability of $\frac{\lambda}{\mu+\lambda}$. Then the channel capacity $n\ln\frac{\lambda+\mu}{\mu}$ can be obtained.

$\square$