Iso $^{\smile}_{\mu}$ with casting

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1 Grammar

typ, A, B, C, Char, Bool, ?	::=		types
	|	X	type variable
	|	Int	int
	|	Top	top type
	|	$\mathsf{A} \to \mathsf{B}$	function type
	|	$\mu X.\mathsf{A}$	recursive type
exp, e, v, u, $\mathsf{v_f}$	::=		expressions
	|	x	variable
	|	$\top$	top
	|	i	lit
	|	$\lambda x : \mathsf{A}.\, \mathsf{e}$	abstraction with argument annotation
	|	$\mathsf{fix}\ x : \mathsf{A}.\, \mathsf{e}$	fixpoint
	|	$\mathsf{e}_1\ \mathsf{e}_2$	applications
	|	$\mathbf{cast}\ [\mathsf{c}]\mathsf{e}$	
mode, $\mathfrak{m}$	::=		modes
	|	$\oplus$	positive
	|	$\ominus$	negative
castop, c	::=		cast operators
	|	cx	cast variable
	|	id	id operator
	|	$\downarrow_{\mathsf{A}}$	unfold operator
	|	$\uparrow_{\mathsf{A}}$	castdn
	|	$\mathsf{c}_1 \to \mathsf{c}_2$	arrow operator
	|	$\mathsf{c}_1 \cdot \mathsf{c}_2$	composition of a sequence of ops
	|	$\mathsf{fix}\ cx.\, \mathsf{c}$	fixpoint
ctx, Γ	::=		term context
	|	.	
	|	$\Gamma, x : \mathsf{A}$	
tctx, Δ	::=		type context
	|	.	
	|	Δ, X	
cctx, $\mathbb{E}$	::=		cast context
	|	.	
	|	$\mathbb{E}, cx : \mathsf{A} \hookrightarrow \mathsf{B}$	

2 Subtyping

$\boxed{\vdash \Delta}$ *(Well Formed Type Environment)*

$$\textsc{WFTyE-empty}\ \frac{}{\vdash \cdot} \qquad \textsc{WFTyE-cons}\ \frac{\vdash \Delta \qquad X \notin \Delta}{\vdash \Delta, X}$$

3 Typing

$\boxed{\Delta \vdash \Gamma}$ *(Well Formed Term Environment)*

$$\textsc{WFTmE-empty}\ \frac{}{\Delta \vdash \cdot} \qquad \textsc{WFTmE-cons}\ \frac{\Delta \vdash \Gamma \qquad \Delta \vdash A \qquad x \notin \Gamma}{\Delta \vdash \Gamma, x : A}$$

$\boxed{\Delta \vdash A}$ *(Well Formed Type)*

$$\textsc{WFT-top}\ \frac{}{\Delta \vdash \mathsf{Top}} \qquad \textsc{WFT-int}\ \frac{}{\Delta \vdash \mathsf{Int}} \qquad \textsc{WFT-var}\ \frac{X \in \Delta}{\Delta \vdash X} \qquad \textsc{WFT-arrow}\ \frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \to B} \qquad \textsc{WFT-rec}\ \frac{\Delta, X \vdash A}{\Delta \vdash \mu X.A}$$

$\boxed{\Delta; \mathbb{E} \vdash A \hookrightarrow B : c}$ *(Typing Reduction Rules)*

$$\textsc{TCast-id}\ \frac{\Delta \vdash A \qquad \mathtt{uniq}\mathbb{E}}{\Delta; \mathbb{E} \vdash A \hookrightarrow A : \mathsf{id}} \qquad \textsc{TCast-arrow}\ \frac{\Delta; \mathbb{E} \vdash A_2 \hookrightarrow A_1 : c_1 \qquad \Delta; \mathbb{E} \vdash B_1 \hookrightarrow B_2 : c_2}{\Delta; \mathbb{E} \vdash A_1 \to B_1 \hookrightarrow A_2 \to B_2 : c_1 \to c_2}$$

$$\textsc{TCast-unfold}\ \frac{\Delta \vdash \mu X.A \qquad \mathtt{uniq}\mathbb{E}}{\Delta; \mathbb{E} \vdash \mu X.A \hookrightarrow A[X \mapsto \mu X.A] : \downarrow_{\mu X.A}} \qquad \textsc{TCast-fold}\ \frac{\Delta \vdash \mu X.A \qquad \mathtt{uniq}\mathbb{E}}{\Delta; \mathbb{E} \vdash A[X \mapsto \mu X.A] \hookrightarrow \mu X.A : \uparrow_{\mu X.A}}$$

$$\textsc{TCast-seq}\ \frac{\Delta; \mathbb{E} \vdash A \hookrightarrow B : c_1 \qquad \Delta; \mathbb{E} \vdash B \hookrightarrow C : c_2}{\Delta; \mathbb{E} \vdash A \hookrightarrow C : c_1 \cdot c_2} \qquad \textsc{TCast-var}\ \frac{\Delta \vdash A \qquad \Delta \vdash B \qquad \mathtt{uniq}\mathbb{E} \qquad A \hookrightarrow B : cx \in \mathbb{E}}{\Delta; \mathbb{E} \vdash A \hookrightarrow B : cx}$$

$$\textsc{TCast-fix}\ \frac{\Delta; \mathbb{E}, cx : A_1 \to B_1 \hookrightarrow A_2 \to B_2 \vdash A_2 \hookrightarrow A_1 : c_1 \qquad \Delta; \mathbb{E}, cx : A_1 \to B_1 \hookrightarrow A_2 \to B_2 \vdash B_1 \hookrightarrow B_2 : c_2}{\Delta; \mathbb{E} \vdash A_1 \to B_1 \hookrightarrow A_2 \to B_2 : \mathsf{fix}\ cx.\ (c_1 \to c_2)}$$

$\boxed{\Delta; \mathbb{E}; \Gamma \vdash e : A}$ *(Typing rules)*

$$\textsc{Typing-int}\ \frac{\vdash \Delta \qquad \Delta \vdash \Gamma}{\Delta; \mathbb{E}; \Gamma \vdash i : \mathsf{Int}} \qquad \textsc{Typing-var}\ \frac{\vdash \Delta \qquad \Delta \vdash \Gamma \qquad x : A \in \Gamma}{\Delta; \mathbb{E}; \Gamma \vdash x : A} \qquad \textsc{Typing-abs}\ \frac{\Delta; \mathbb{E}; \Gamma, x : A_1 \vdash e : A_2}{\Delta; \mathbb{E}; \Gamma \vdash \lambda x : A_1.\, e : A_1 \to A_2}$$

$$\textsc{Typing-app}\ \frac{\Delta; \mathbb{E}; \Gamma \vdash e_1 : A_1 \to A_2 \qquad \Delta; \mathbb{E}; \Gamma \vdash e_2 : A_1}{\Delta; \mathbb{E}; \Gamma \vdash e_1\ e_2 : A_2} \qquad \textsc{Typing-fix}\ \frac{\Delta; \mathbb{E}; \Gamma, x : A \vdash e : A}{\Delta; \mathbb{E}; \Gamma \vdash \mathsf{fix}\ x : A.\, e : A} \qquad \textsc{Typing-cast}\ \frac{\Delta; \mathbb{E}; \Gamma \vdash e : A \qquad \Delta; \mathbb{E} \vdash A \hookrightarrow B : c}{\Delta; \mathbb{E}; \Gamma \vdash \mathbf{cast}\,[c] e : B}$$

4 Semantics

$\boxed{\mathsf{value}\ e}$ *(Values)*

$$\textsc{V-lit}\ \frac{}{\mathsf{value}\ i} \qquad \textsc{V-abs}\ \frac{}{\mathsf{value}\ (\lambda x : \mathsf{A}.\, e)} \qquad \textsc{V-fold}\ \frac{\mathsf{value}\ e}{\mathsf{value}\ (\mathbf{cast}\ [\uparrow_{\mathsf{A}}]e)} \qquad \textsc{V-arrow}\ \frac{\mathsf{value}\ e}{\mathsf{value}\ (\mathbf{cast}\ [\mathsf{c}_1 \to \mathsf{c}_2]e)}$$

$\boxed{e \hookrightarrow e'}$ *(Reduction rules)*

$$\textsc{Red-beta}\ \frac{}{(\lambda x : \mathsf{A}.\, e)\, e' \hookrightarrow e[x \mapsto e']} \qquad \textsc{Red-appl}\ \frac{e_1 \hookrightarrow e_1'}{e_1\, e_2 \hookrightarrow e_1'\, e_2} \qquad \textsc{Red-appr}\ \frac{\mathsf{value}\ v_1 \qquad e_2 \hookrightarrow e_2'}{v_1\, e_2 \hookrightarrow v_1\, e_2'}$$

$$\textsc{Red-fix}\ \frac{}{\mathsf{fix}\ x : \mathsf{A}.\, e \hookrightarrow e[x \mapsto \mathsf{fix}\ x : \mathsf{A}.\, e]} \qquad \textsc{Red-cast-arr}\ \frac{}{(\mathbf{cast}\ [\mathsf{c}_1 \to \mathsf{c}_2]e_1)\, e_2 \hookrightarrow \mathbf{cast}\ [\mathsf{c}_2](e_1\, (\mathbf{cast}\ [\mathsf{c}_1]e_2))}$$

$$\textsc{Red-cast-seq}\ \frac{}{\mathbf{cast}\ [\mathsf{c}_1 \cdot \mathsf{c}_2]e \hookrightarrow \mathbf{cast}\ [\mathsf{c}_2](\mathbf{cast}\ [\mathsf{c}_1]e)} \qquad \textsc{Red-cast}\ \frac{e \hookrightarrow e'}{\mathbf{cast}\ [\mathsf{c}]e \hookrightarrow \mathbf{cast}\ [\mathsf{c}]e'} \qquad \textsc{Red-castelim}\ \frac{\mathsf{value}\ v}{\mathbf{cast}\ [\downarrow_{\mathsf{A}}](\mathbf{cast}\ [\uparrow_{\mathsf{A}}]v) \hookrightarrow v}$$

$$\textsc{Red-castid}\ \frac{\mathsf{value}\ v}{\mathbf{cast}\ [\mathsf{id}]v \hookrightarrow v} \qquad \textsc{Red-castfix}\ \frac{}{\mathbf{cast}\ [\mathsf{fix}\ cx.\, \mathsf{c}]\mathsf{e} \hookrightarrow \mathbf{cast}\ [\mathsf{c}[cx \mapsto \mathsf{fix}\ cx.\, \mathsf{c}]]\mathsf{e}}$$