PA 5.3 Quantification d'incertitudes en optimisation stochastique

October 16, 2020

1 Portfolio optimization

1.1 Mean-variance portfolio

Mean Variance Portfolio: build the allocation that provides the highest returns for a limited risk, measured via the volatility.

Mean and variance: estimated with historical samples and are subject to uncertainties. They also have to be updated sequentially. This is why UQSA framework looks adapted to this problem.

Formalization:

Consider d assets with return $R = (R_1, \dots, R_n)'$ with $\mu := \mathbb{E}[R]$, $S := \mathbb{E}[(R-\mu)(R-\mu)']$. Suppose the proportion of asset i in the portfolio is w_i.

We consider the following mean-variance portfolio optimization formulation:

$$\min_w \frac{1}{2} w'Sw - t\mu'w \tag{1.1}$$
$$\text{u.c.} \begin{cases} \sum_{i=1}^n w_i = 1, & \text{(budget constraint)} \\ (w_i \geq 0, i = 1, \dots n) & \text{(positivity constraint).} \end{cases}$$

where t is the "risk tolerance".

It has a unique solution (strictly convex problem on a convex set) that is not explicit.

1.2 Minimum Variance portfolio

$$\min_w \frac{1}{2} w'Sw \tag{1.2}$$
$$\text{u.c.} \begin{cases} \sum_{i=1}^n w_i = 1, & \text{(budget constraint)} \\ (w_i \geq 0, i = 1, \dots n) & \text{(positivity constraint).} \end{cases}$$

First order condition under budget constraint only (by deriving the Lagrangian):

$$Sw + \lambda \mathbf{1}_d = \mathbf{0}_d.$$

2 Objectives

The objective is to solve (1.1) or , e.g. get the optimal weights $w(\mu, S)$.

- If μ and S are known, we can hope to solve (1.1) with standard optimization algorithms.
- But in practice, μ and S are estimated with uncertainty. In this project, we assume that the uncertainty is held by S: we suppose that S depends on an uncertain parameter θ of known distribution π and we want to get the distribution of $w(\mu, S^\theta) = w(\theta)$.

Two big questions (to use UQSA algo):

1. **Transformation of (P_f) in a (P_H) type of problem:**

$$(P_f) := \min_{w \in S_{n-1}} f(w),$$
$$(P_H) := \{z \in \mathbb{R}^p,\ \mathbb{E}\left[H(z, V)\right] = 0\}.$$

   where $f(w) = \frac{1}{2}w'Sw - t\mu'w$ the objective function to minimize; $S_{n-1} = \{w \in \mathbb{R}^n :\ w_i \geq 0, i = 1, \ldots n,\ \sum_{i=1}^n w_i = 1\}$ the simplex.

2. **Uncertainty modelization**

   Focus on S^θ: large dimension ($\frac{d(d+1)}{2}$) but not so many data (n).

   How to model the uncertainty source $\theta \in \mathbb{R}^p$, $p < \frac{d(d+1)}{2}$?

   - simple Gaussian model,
   - GARCH model.

3 First toy example

3.1 Simple Robbins Monro

Objective: write minimum variance problem (2) under the form '(P_H)'. Verify Robbins Monro assumptions and implement the algorithm.

Assume that $R \sim \mathcal{N}(\mu, S)$.

Begin with the following parameters:

- $n = 3$,
- $\mu = [5\%, 7\%, 6\%]$,
- $S = D\Gamma D$, where $D = \text{Diag}(\sigma_1, \sigma_2, \sigma_3)$, $\Gamma = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}$.

  Take $D = \text{Diag}[10\%, 14\%, 02\%]$, $\rho = 50\%$.

Compare your result with the true solution $w = \frac{S^{-1}\mathbf{1}_d}{\mathbf{1}_d' S^{-1}\mathbf{1}_d}$. (Remark: you can chose the Lagrangian multiple arbitrarily then renormalize w.)

3.2 UQSA

Implement UQSA algorithm: projection of $w(\rho)$ on multidimensional Legendre basis, assuming the following uncertainty on ρ: $\rho \sim U[\rho_{\min}, \rho_{\max}]$.

- Test first the case $\rho_{\min} = 0.6, \rho_{\max} = 0.8$.
- Then the case when $\rho_{\min}$ and $\rho_{\max}$ have different values, for example ± 0.5.

4 Impact of backtest size and estimates update

Investor seeks for an allocation w which minimizes a certain function f parameterized by $\boldsymbol{\mu}$ and S

$$w^*(\boldsymbol{\mu}, S) := \arg\min_{w \in W} f(w; \boldsymbol{m}u, S)$$

In practice, there is a model on $\boldsymbol{r}_t$ which might be **uncertain**: example: GARCH models.

GARCH models are very popular to model financial returns (mainly because they reflect the fat tails and the dependence in time of the volatility of the returns). One of simplest is the GARCH-CCC model, under which the (Conditionnal) Correlation matrix is assumed fixed:

This model assumes the following structure for the centered returns $\boldsymbol{r}_t$:

$$\begin{aligned}\boldsymbol{r}_t &= D_t \Gamma^{1/2} \boldsymbol{\eta}_t = D_t \tilde{\boldsymbol{\eta}}_t,\\ V_t &= D_t \Gamma D_t\end{aligned}$$

- $D_t = \mathrm{Diag}\left(\sigma_{1,t}, \ldots, \sigma_{n,t}\right)$,
- $\sigma_{i,t}^2$ unidimensional GARCH volatilities
- $\Gamma = \{\rho_{ij}\}_{1 \le i,j \le n}$ positive definite matrix (the Constant Conditional Correlation matrix)
- $\boldsymbol{\eta}_t$ n-dimensional independent process, $\mathbb{E}\left[\eta_{i,t}\eta_{j,t}\right] = 0\ \forall i \neq j$, $\mathbb{E}\left[\eta_{i,t}^2\right] = 1$, and $\boldsymbol{\eta}_t$ independent from D_t.

The original GARCH-CCC assumes a simple GARCH volatility recursion for the $\sigma_{i,t}$:

$$\sigma_{i,t}^2 = w_i + \alpha_i r_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2, \quad i = 1, \ldots, d,$$

$w_i, \alpha_i, \beta_i \in \mathbb{R}$.

In practice, it is estimated via Maximum Likelihood, assuming the $\boldsymbol{\eta}_t$ gaussian. The estimated parameters $\theta_i = \begin{pmatrix} w_i \\ \alpha_i \\ \beta_i \end{pmatrix}$ satisfy a Central Limit theorem:

$$\sqrt{T}\left(\hat{\theta}_i - \theta_i\right) \to \mathcal{N}(0, (\kappa_\eta - 1) J_i^{-1})$$

where $J_i = \mathbb{E}_{\theta_0}\left[\frac{\partial^2 \ell_t}{\partial\theta\theta'}\right] = \mathbb{E}_{\theta_0}\left[\left(\frac{1}{\sigma_t^2}\frac{\partial\sigma_{i,t}^2}{\partial\theta}\right)\left(\frac{1}{\sigma_{i,t}^2}\frac{\partial\sigma_{i,t}^2}{\partial\theta'}\right)\right]$, $\kappa_\eta = 3$.

- GARCH model gives the law of the future return $\boldsymbol{r}_t \sim \mathcal{N}(0, V_t)$ ...
- ... with uncertain parameters $V_t \leftarrow \hat{V}_t(\hat{\theta}, T)$

$\rightarrow$ interesting to see the distribution of the **future allocation** while taking into account the uncertainty on the parameters

	std w	std α	std β
regression method	0.2849043	0.58938122	0.49413832