Last Time

- Definition of stochastic process
- Sample paths
- Finite-dimensional distributions
- Versions and modifications

Today's lecture: Section 3.2

Linear Algebra Notation

- The dot product of two column vectors $x = (x_1, \ldots, x_n)'$ and $y = (y_1, \ldots, y_n)'$ is

$$(x, y) = x_1 y_1 + \cdots + x_n y_n$$

- A symmetric $n \times n$ matrix A is nonnegative definite if

$$(x, Ax) = x' Ax = \sum_{j=1}^{n} \sum_{k=1}^{n} x_j A_{jk} x_k \geq 0 \text{ for all vectors } x \in I\!R^n$$

- A symmetric $n \times n$ matrix A is positive definite if

$$(x, Ax) > 0 \text{ for all non-zero column vectors } x \in I\!R^n$$

Random Vectors

- Given a measurable space $(\Omega, \mathcal{F})$ an $\mathcal{F}$-measurable map $X : \Omega \mapsto \mathbb{R}^n$ is a **random vector**
- $X = (X_1, \ldots, X_n)$ is a random vector if and only if X_i is a random variable for all $i = 1, \ldots, n$
- Any random vector has a distribution/law on $(\mathbb{R}^n, \mathcal{B}^n)$
- A random vector X has a probability density function f_X if

$$\mathbb{P}(a_i \leq X_i \leq b_i) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f_X(x_1, \ldots, x_n) dx_n \cdots dx_1,$$

for all $a_i < b_i \in \mathbb{R}, i = 1, \ldots, n$

Characteristic Function and its Properties

- The **characteristic function** of a random vector $X = (X_1, \ldots, X_n)$ is a function $\Phi_X : I\!R^n \to \mathbb{C}$ defined by

$$\begin{aligned} \Phi_X(\theta) &= I\!E(e^{i(\theta,X)}) \\ &= I\!E(\cos((\theta, X))) + iI\!E(\sin((\theta, X))), \end{aligned} \tag{1}$$

  where $i = \sqrt{-1}$ and $\theta = (\theta_1, \ldots, \theta_n) \in I\!R^n$
- $|\Phi_X(\theta)| \leq 1$ for all $\theta \in I\!R^n$
- $\Phi_X(\theta)$ is continuous as a function of θ

Characteristic Function and its Properties (cont.)

- The RV's $X_1, \ldots, X_n$ are independent if and only if

$$\Phi_X(\theta) = \prod_{k=1}^{n} \Phi_{X_k}(\theta_k),$$

  for all $\theta = (\theta_1, \ldots, \theta_n)$
- Two random vectors have the same law if and only if they have the same characteristic function
- X_n converges to X in distribution as $n \to \infty$ if and only if $\Phi_{X_n}(\theta) \to \Phi_X(\theta)$ for all $\theta \in I\!R^n$.

Gaussian Random Vector

- A random vector X has a **Gaussian (or Multivariate Normal) Distribution** if its characteristic function has the form

$$\Phi_X(\theta) = e^{i(\theta,\mu)-\frac{1}{2}(\theta,\Sigma\theta)}, \quad \theta \in I\!R^n,$$

  for some nonnegative definite $n \times n$ matrix Σ and some vector $\mu \in I\!R^n$

- A RV X is Gaussian (or Normal) if its characteristic function is

$$\Phi_X(\theta) = e^{i\theta\mu-\frac{1}{2}\theta^2\sigma^2}, \quad \theta \in I\!R,$$

  for some $\mu \in I\!R$ and $\sigma^2 \geq 0$

Mean and Covariance of Gaussian Random Vector

- If X has a Gaussian Distribution with characteristic function

$$\Phi_X(\theta) = e^{i(\theta,\mu)-\frac{1}{2}(\theta,\Sigma\theta)}, \quad \theta \in I\!R^n,$$

- Then μ is the mean vector of X, i.e.

$$I\!E(X_k) = \mu_k, \quad k = 1, \ldots, n$$

- And Σ is the covariance matrix of X, i.e.

$$\text{Cov}(X_j, X_k) \doteq I\!E[(X_j - \mu_j)(X_k - \mu_k)] = \Sigma_{jk}, \quad j, k = 1, \ldots, n$$

- Thus, a Gaussian random vector is completely characterized by its mean vector and covariance matrix

Non-degenerate Gaussian Random Vector

- A random vector X has a **non-degenerate Gaussian Distribution** if Σ is positive definite
- A random vector X with non-degenerate Gaussian distribution has density

$$f_X(x) = \frac{1}{(2\pi)^{n/2}(\mathrm{det}\Sigma)^{1/2}} e^{-\frac{1}{2}(x-\mu,\Sigma^{-1}(x-\mu))}, \quad x \in \mathbb{R}^n$$

- A Gaussian RV with $\sigma > 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

Convergence of Gaussian Random Vectors

- Let $X^{(m)}, m = 1, 2, \ldots$ be a sequence of Gaussian random vectors with mean $\mu^{(m)}$ and covariance matrix $\Sigma^{(m)}$. Let X be another random vector.
- If $X^{(m)} \to X$ in L^2 as $m \to \infty$
- Then X is a Gaussian random vector with mean μ and covariance matrix Σ given by

$$\begin{aligned} \mu &= \lim_{m\to\infty} \mu^{(m)} \\ \Sigma &= \lim_{m\to\infty} \Sigma^{(m)} \end{aligned}$$

Alternate Definitions of Gaussian Random Vector

- A random vector $X = (X_1, \ldots, X_n)$ is Gaussian if and only if for all real numbers $a_1, \ldots, a_n$, the random variable $a_1 X_1 + \cdots + a_n X_n$ has a Gaussian distribution
- A random vector $X = (X_1, \ldots, X_n)$ is Gaussian if and only if for all real numbers $b_{11}, b_{12}, \ldots, b_{mn}$, the m-dimensional random vector

$$(b_{11}X_1 + \cdots + b_{1n}X_n \ , \ \ldots \ , \ b_{m1}X_1 + \cdots + b_{mn}X_n)$$

  is Gaussian
- Example: if (X, Y) is a Gaussian random vector then
  - X is a Gaussian random variable
  - $X + Y$ is a Gaussian random variable
  - $(X + Y, X - Y)$ is a Gaussian random vector

Gaussian Random Vectors and Independence

- If $X = (X_1, \ldots, X_n)$ is a Gaussian random vector with uncorrelated coordinates (that is, $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j]$ for all $i \neq j$)
- Then X has independent coordinates, that is, $X_1, \ldots, X_n$ are independent random variables
- Essential that X is a Gaussian random vector for above relationship to hold
- In particular, if X and Y are uncorrelated Gaussian random variables, then X and Y need not be independent (see Exercise 3.2.12)
- If X and Y are Gaussian random variables then (X, Y) is not necessarily a Gaussian random vector

Gaussian Stochastic Process

- A stochastic process $\{X_t : t \in \mathcal{I}\}$ is a **Gaussian SP** if its FDD's are Gaussian
- That is, for all integers $n < \infty$ and times $t_1, \ldots, t_n \in \mathcal{I}$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ has a Gaussian distribution
- All distributional properties of a Gaussian process are determined by its mean and autocovariance functions:

$$\begin{aligned} \mu(t) &= \mathbb{E}(X_t) \\ \rho(t, s) &= \mathbb{E}[(X_t - \mu(t))(X_s - \mu(s))] \end{aligned}$$