Isotropic and Gaussian random vectors

1 Isotropic random vectors

1. Let $p_1, \ldots, p_k$ be strictly positive numbers summing to 1 and let $X = (X_1, \ldots, X_k)^\top$ be a random vector in $\mathbb{R}^k$ taking the value e_j with probability p_j, $j = 1, \ldots, k$ where $e_1, \ldots, e_k$ is the canonical basis of $\mathbb{R}^k$. Define the random vector $Y = (Y_1, \ldots, Y_k)^\top$ by
$$Y_j = \frac{X_j - p_j}{\sqrt{p_j}}, \quad j = 1, \ldots, k.$$
Show that Y has expectation 0 and variance operator $Q_r = P_{r^\perp}$ where r is the vector in $\mathbb{R}^k$ with j^{th} coordinate $\sqrt{p_j}$, $j = 1, \ldots, k$.

2. Let $V, \langle \cdot, \cdot \rangle$ be an euclidean space. Show that Y is isotropic in V iif, for some $v \in V$, the random vectors $Y - v$ and $\mathcal{O}(Y - v)$ have the same mean and variance for all orthogonal linear transformation $\mathcal{O} : V \to V$.

2 Gaussian vectors I

1. Let X, Y be independent gaussian random variables in $\mathbb{R}$. Give a necessary and sufficient condition for $X + Y$ and $X - Y$ to be independent.

2. Let X be a standard Gaussian in $\mathbb{R}$ and ϵ be a real random variable independent from X such that $\mathbb{P}(\epsilon = 1) = \mathbb{P}(\epsilon = -1) = 1/2$.
   (a) What is the distribution of $Y = \epsilon X$? (Hint: use characteristic function).
   (b) Give the covariance matrix of X, Y.
   (c) Compare $\mathbb{E}(X^2Y^2)$ to $\mathbb{E}(X^2)\mathbb{E}(Y^2)$. Are X and Y independent ?
   (d) What can you deduce on the pair (X, Y) ?

3. Let $(X, Y) \in \mathbb{R}^2$ be a gaussian vector with covariance matrix
$$\Gamma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$
where $\rho \in [-1, 1]$. Compute the covariance of X and $Z = Y - \rho X$. Is (X, Z) a gaussian vector ?

4. Consider the Gaussian vector
$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \quad \text{with mean} \quad \mathbb{E}(X) = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

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and covariance matrix

$$\Gamma = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & 12 \end{pmatrix}.$$

(a) Compute the marginal distribution of X_1 and $Z_2 = (X_2, X_3)^\top$

(b) Compute the conditional distribution of $Z_2 | X_1 = x_1$.

(c) Define

$$Y = \begin{pmatrix} X_1 - X_2 + X_3 \\ 2X_2 - X_3 \\ X_1 + X_2 - 2X_3 \end{pmatrix}.$$

Determine the distribution of Y.

5. Let $(X, Y) \in \mathbb{R}^2$ admits the density $f(x, y) = \frac{1}{4\pi} \exp\left(-\frac{1}{2} T(x, y)\right)$ with

$$T(x, y) = \frac{1}{2}x^2 + y^2 - xy + 4x - 7y + \frac{25}{2}.$$

(a) What is the distribution of $Z = 2Y - X$?

(b) Are the random variables Z and X independent ?

6. Let $X_1, \ldots, X_n$ be random variables satisfying

$$\begin{cases} X_1 &= b_1 + U_1 \\ X_j &= b_j + \theta U_{j-1} + U_j, \quad \forall 2 \leq j \leq n, \end{cases}$$

where $U_1, \ldots, U_n$ are i.i.d. $N(0, \sigma^2)$ and $\sigma^2 > 0$, $b_1, \ldots, b_n, \theta \geq 0$.

(a) Show that $X = (X_1, \ldots, X_n)^\top$ is a Gaussian vector.

(b) Determine the distribution of X.

3 Gaussian vectors II

1. Suppose that S is a random variable distributed according to the $\chi^2_{\nu,\theta}$ distribution. Show that

$$\mathbb{E}(S) = \nu + \theta^2, \quad \mathrm{Var}(S) = 2\nu + 4\theta^2.$$

2. Show that: (1) if $X_1, \ldots, X_\nu$ are independent normal random variables with unit variances, then $\sum_{i=1}^{\nu} X_i^2 \sim \chi^2_{\nu,\theta}$ with $\theta = \sqrt{\sum_i (\mathbb{E}X_i)^2}$; (2) if S_1, S_2 are independent random variables with $S_i \sim \chi^2_{\nu_i,\theta_i}$ for $i = 1, 2$, then $S_1 + S_2 \sim \chi^2_{\nu_1+\nu_2,\sqrt{\theta_1^2+\theta_2^2}}$.

3. Let $X^{(1)}, \ldots, X^{(n)}$ be independent random vectors in $\mathbb{R}^k$, each having the distribution of the random vector X in Exercise 1. Define

$$N = \sum_{m=1}^{n} X^{(m)}.$$

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The distribution of N is the multinomial distribution for n trials and k cells having occupancy probabilities $p_1, \dots, p_k$. Show that the limiting distribution, taken as $n \to \infty$, of Pearson's goodness of fit statistic

$$G_n = \sum_{j=1}^{k} \frac{(N_j - np_j)^2}{np_j}$$

is χ^2_{k-1}.

4. Let $X_1, \dots, X_n$ be i.i.d. random variables with finite second moment such that the empirical mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and empirical variance $S_n^2 = \frac{1}{n-1}\sum_{j=1}^n (X_j - \bar{X}_n)^2$ are independent.

   (a) Check that $S_n^2 = \frac{1}{n}\sum_{j=1}^n X_j^2 - \frac{1}{n(n-1)}\sum_{j,k=1,k\neq j}^n X_j X_k$ and deduce $\mathbb{E}(S_n^2)$.

   (b) Exploiting the independence assumption, compute $\mathbb{E}(S_n^2 e^{iu\bar{X}_n})$ in function of $\mathrm{var}(X_1)$ and the common characteristic function of the X_js denoted as Φ_X.

   (c) Show that

   $$\mathbb{E}(S_n^2 e^{iu\bar{X}_n}) = \mathbb{E}(X_1^2 e^{iuX_1})(\Phi_X(u))^{n-1} - (\mathbb{E}(X_1 e^{iuX_1}))^2(\Phi_X(u))^{n-2}.$$

   (d) Explicit the relation between $\mathbb{E}(X_1 e^{iuX_1})$ and $\mathbb{E}(X_1^2 e^{iuX_1})$ and the derivatives of Φ_X. Deduce a differential equation satisfied by Φ_X.

   (e) Set $f(u) = \Phi_X'(u)/\Phi_X(u)$ and compute $f'(u)$. Deduce Φ_X and the common distribution of the X_j.

5. For any $\nu \geq 1$ and $\theta \geq 0$, show that

$$\chi^2_{\nu,\theta} = \sum_{k\geq 0} e^{-\lambda}\frac{\lambda^k}{k!}\chi^2_{\nu+2k},$$

where $\lambda = \theta^2/2$.

6. Suppose that $S \sim \chi^2_{\nu,\theta}$. Show that

$$\mathbb{E}\left(\frac{1}{S}\right) = \begin{cases} \sum_{k\geq 0} e^{-\lambda}\frac{\lambda^k}{k!}\frac{1}{\nu-2+2k}, & \text{if } \nu \geq 3, \\ \infty, & \text{if } \nu \leq 2. \end{cases}$$

Hint: Recall that the χ^2_ν distribution has density $x^{\nu/2-1}e^{-x/2}/(2^{\nu/2}\Gamma(\nu/2))$ for $x > 0$.