The generation procedure of particle images (figure 3) is similar to what is recommended in the literature [1]. The generation process of images is here described in detail for the case of second order velocity fields. The cases of Rankine vortex and Rankine vortex with shear are similar.

1. Second order velocity fields

The mathematical structure of these fields is inspired by Rabault et al. [2]: second order polynomials are created for the u and v components of the velocity. The resulting velocity fields represent 2-dimensional and stationary motion:

$$\begin{cases} u = u_0 + \overline{J_u} dr + dr^T \overline{\overline{H_u}} dr \\ v = v_0 + \overline{J_v} dr + dr^T \overline{\overline{H_v}} dr \end{cases}$$

where:

- $\overline{J_u}$ and $\overline{J_v}$ represent the gradients of the horizontal and vertical velocity components respectively,
- $\overline{\overline{H_u}}$ and $\overline{\overline{H_v}}$ represent the Hessian matrices related to the horizontal and vertical velocity components respectively.

Since:

$$\overline{J_\iota} = \begin{bmatrix} J_{i_{11}} & J_{i_{12}} \end{bmatrix} = \begin{bmatrix} \frac{\partial i}{\partial x} & \frac{\partial i}{\partial y} \end{bmatrix},$$

$$\overline{\overline{H_\iota}} = \begin{bmatrix} H_{i_{11}} & H_{i_{21}} \\ H_{i_{21}} & H_{i_{22}} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 i}{\partial x^2} & \frac{\partial^2 i}{\partial x\, \partial y} \\ \frac{\partial^2 i}{\partial y\, \partial x} & \frac{\partial^2 i}{\partial y^2} \end{bmatrix},$$

where $i = \{u, v\}$, then the velocity field can be rewritten as follow:

$$\begin{cases} u = u_0 + \begin{bmatrix} J_{u_{11}} & J_{u_{12}} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} + \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} H_{u_{11}} & H_{u_{21}} \\ H_{u_{21}} & H_{u_{22}} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \\ v = v_0 + \begin{bmatrix} J_{v_{11}} & J_{v_{12}} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} + \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} H_{v_{11}} & H_{v_{21}} \\ H_{v_{21}} & H_{v_{22}} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} u = u_0 + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} + \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x\, \partial y} \\ \frac{\partial^2 u}{\partial y\, \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \\ v = v_0 + \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} + \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial x\, \partial y} \\ \frac{\partial^2 v}{\partial y\, \partial x} & \frac{\partial^2 v}{\partial y^2} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \end{cases}$$

The origin of the reference system is located at the center of the particle image and the axes are parallel to the image edges, so that u_0 and v_0 correspond to the velocity field at the center of the image (figure 1).

Each time a new image pair is generated, new values are randomly assigned to the coefficients of the second order polynomial so that each image pair is generated independently from the others.

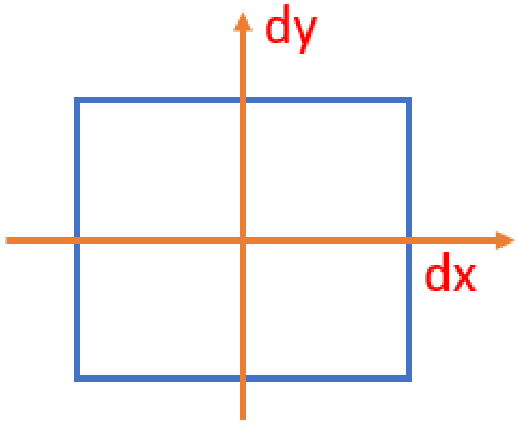

Figure 1: reference system for particle images

Once the random velocity field is generated, a random uniform distribution of material points representing the particle centers is placed in a matrix of size $[xmax, ymax]$ that is slightly larger than the resolution of the images $[resx, resy]$. This distribution is integrated half a time-step backward and forward in time to determine the position of the particle centers in both the frames of the image pair. Integration corresponds to the multiplication for the time step, since the velocity fields are constant in time between the first and the second frame.

Sufficiently small particles recorded by a camera form circular patterns known as Airy disks. The main lobe of diffraction is normally well approximated by a Gaussian bell curve. Gaussian distributions are therefore used to generate the synthetic particle images [1]:

$$I = I_0 \exp\left(\frac{-(x-x_0)^2 - (y-y_0)^2}{\left(\frac{1/8}{d_\tau^2}\right)}\right),$$

where:

- I_0 is the particle image luminosity,
- (x_0, y_0) indicates the position of each particle center,
- d_τ is the effective diameter of the particles in pixels.

A threshold of 255 is imposed on image luminosity, so eventual higher values originating by the overlap of different particles are set to 255. Then, image luminosity is normalized in [0-1] and Gaussian white noise of variance 1% of the maximum image intensity is added to produce non-perfect PIV images.

Finally, images are cropped from $[xmax, ymax]$ to $[resx, resy]$ to reproduce situations in which particles leave and enter the image domain between the first and the second frame. An example of synthetic particle images is shown in figure 2.

The number of decimals is set to 4 for both the velocity field and the particle images.

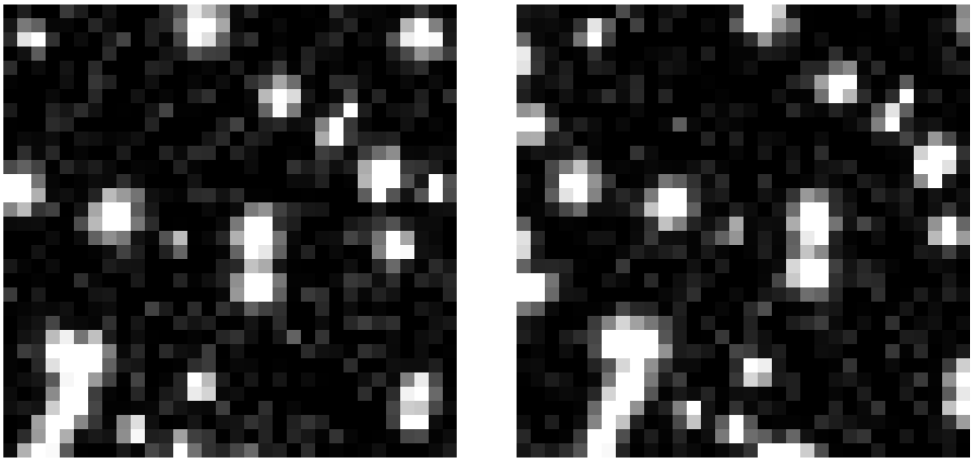

Figure 2: example of particle images, size [32x32]: first exposition (sx) and second exposition (dx)

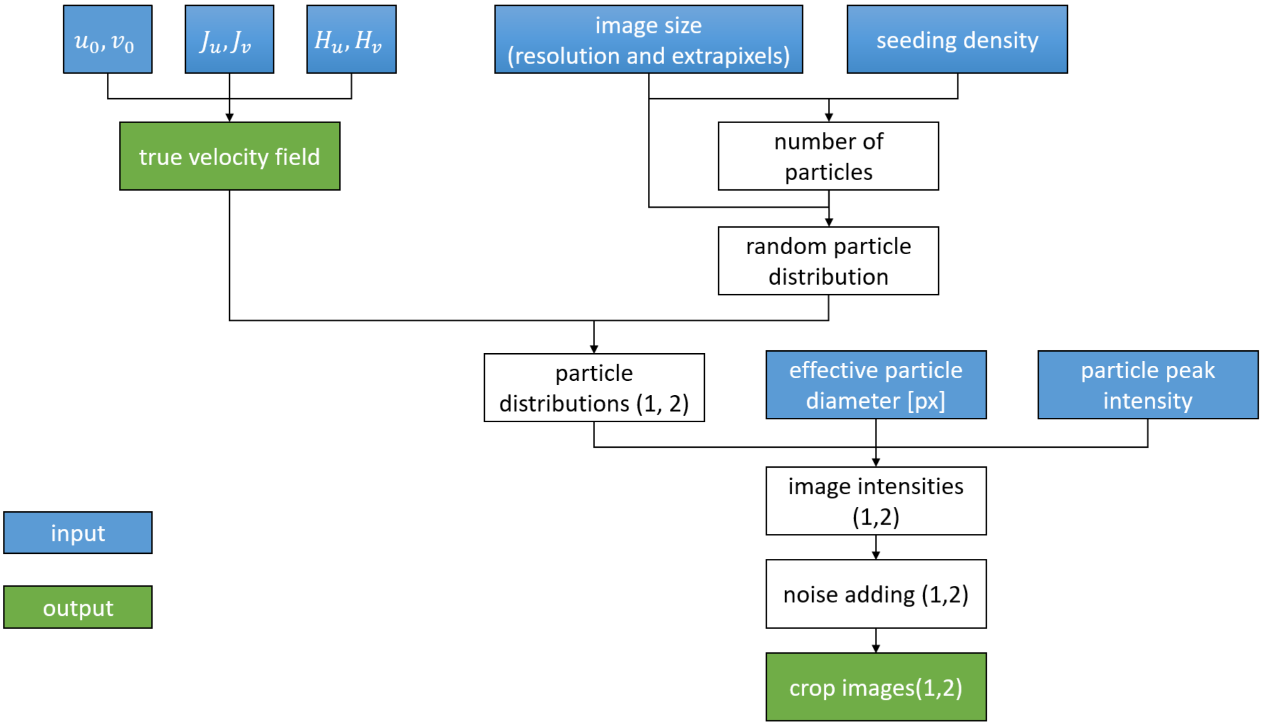

Figure 3: generation process of particle images

2. Rankine vortex and Rankine vortex with shear

The generation process is similar to that used for second order fields (described before).

The main difference is that the same velocity field is used to generate all the particle images (in section 1, the same mathematical model is used, but random values are selected for each image pair). You can easily modify the code to use different values for each image pair.

The Rankine vortex is modelled as follows:

$$V = \begin{cases} v_0 \cdot r/r_0, & r < r_0 \\ v_0 \cdot \dfrac{r_0}{r}, & r > r_0 \end{cases},$$

where:

- v_0 is the maximum velocity,
- r is the distance from the vortex center,
- r_0 is a parameter called core radius (figure 2).

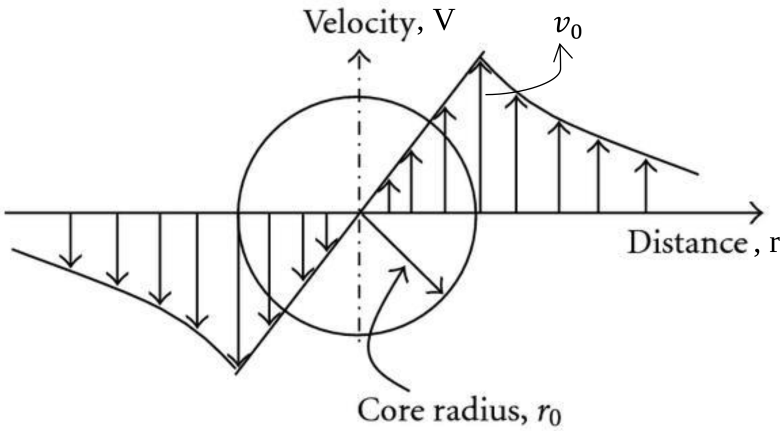

Figure 4: Rankine vortex

The Rankine vortex with shear is computed by adding the following shear effect (figure 5) to the Rankine vortex:

$$V = (u, v) = \left(\frac{\partial u}{\partial y} y, 0\right)$$

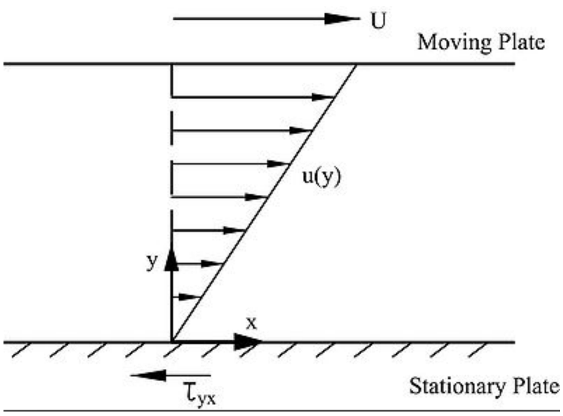

Figure 5: shear flow

3. References

[1] Raffel M., Willert C., Wereley S., Kompenhas J., Particle image velocimetry: a practical guide, Springer, 2007

[2] Rabault J., Kolaas J., ensen A., Performing particle image velocimetry using artificial neural networks: a proof-of-concept, Meas Sci Technol 28: 125301, 2017