IFY Mathematics Teaching Note

愛飛揚數學教學參考

當將你的事交托耶和華，并信靠他，他就必成全。
詩篇 37: 5
Commit your way to the Lord; trust in him and he will do this. Psalm 37:5

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Version: 1.0

目　　录

第 1 章 Numbers 数字

1.1 Basic concept 数字相关基础知识

1.1.1 Digits and numbers 数字与数

Digits are the building blocks of numbers. 数都是由数字所組成的。

1234567890

图 1.1: Digits 数字

The position of each digit in a number tells what the digit means. For example, 2143 can be expaned as $2000 + 100 + 40 + 3$.

数字在数的不同的位置代表了不同的意義。例如 2143 可以表示爲 $2000 + 100 + 40 + 3$.

billions period			millions period			thousands period			ones period		
hundred-billions	ten-billions	billions	hundred-millions	ten-millions	millions	hundred-thousands	ten-thousands	thousands	hundred	ten	ones
		7	5	5	0	2	6	2	1	0	1

表 1.1: World population (by July 1st, 2017) 世界人口（截止至 2017 年 7 月 1 日）7 billion, 755 million, 262 thousand, 101. 75 億 5026 萬 2101

Example 1.1: Tell what each number means. 解釋每個數

a. 4,823 b. 8,910 c. 219,181,193 d. 4,200,000,000

Example 1.2: Given the standard form 寫出每個表示代表的數

a. 4000+300+20+1

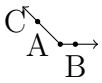
b. 8000+5

c. nine hundred six billion, three million

d. 27 million, 321 thousands, 456

1.1.2 Figure and Geometry 图形与几何

Digits are the building blocks of numbers. 数都是由数字所組成的。

	Point 點	Point A 點 A
	Line 直綫	Line AB 直綫 AB
	Segment 綫段	Segment AB 綫段 AB
	Ray 射綫	Ray AB 射綫 AB
	Intersecting lines 兩綫相交	Two lines meet in a point 兩綫有一個交點
	Parallel lines 平行綫	Line AB and CD never meet. 直綫 AB 和 CD 不相交
	Angle 角	Vertex: A, Sides: AB, AC 頂點：A，邊：AB，AC
	Right Angle 直角	Right angle looks like a square. 直角兩邊互相垂直
	Acute Angle 銳角	Acute angle is smaller than right angle. 銳角小于直角
	Obtuse Angle 鈍角	Obtuse angle is larger than right angle. 鈍角大于直角

1.1.3 Counting 数数

Counting is one of the basic skills when we learn numbers. And it is the basis of arithmetic. "Skip Counting" is counting by a number that is not 1. Learning to "Skip Count" helps you count many things quickly and learn your multiplication tables.

数数是我們在學習了數字以後所學習的，它是所有計算的基礎。除了連續的數數，我們也應該學會跳着數數。它可以幫助我們數的更快，也是學習乘法口訣的基礎。

For counting problems, you should be careful that miss anything and should not count something more than once.

對于數數的問題，最重要的就是不要漏掉，不應重復，如果漏掉了要加上，如果重復了要剪掉。

Example 1.3: How many line segments in the graph? 數一數下圖中有多少條綫段?

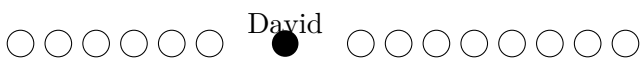
✎ **Solution:**

- Continuous points 連續兩點：AB, BC, CD
- Every other points 隔一個點：AC, BD
- Every other two points 隔兩點：AD

Therefore, there are 6 line segments. 因此，總共有 6 條綫段。 □

Example 1.4: Several people are in one row. There are 6 people in front of David and 8 people behind. How many people are there in this row? 一些人排成一列，在 David 的前面有 6 人，後面有 8 人，問這列隊共有多少人?

✎ **Solution:** As shown in the graph, 如圖所示：



There are 15 people in the line. 總人數是 15 人。 □

1.1.4 Comparing and ordering 比较与排序

There are three relationship between two numbers: equal, greater, or less. Sorting is ordering the numbers in a line by comparing the values, or arranging something with certain requirements.

兩個數之間有三種關系：相等，大于和小于。排序就是把互相不等的一些數通過比較按大小順序排列起來，或是按照一定的要求把一些東西排列起來。

Theorem 1.1 The Greater Than, Less Than, and Equal Symbols

The special symbols are used to show if a number is greater $>$, less $<$, or $=$.
$>$，$<$ 和 $=$ 用來表示大于、小于和等于關系。

$$\text{large number} > \text{small number} \qquad \text{大數} > \text{小數} \tag{1.1}$$
$$\text{small number} < \text{large number} \qquad \text{小數} < \text{大數} \tag{1.2}$$

Very often we need to compare numbers to see which is the greatest or the least. The basic rules for comparing the numbers are as follows

我們經常需要比較兩個數的大小。比較兩個數大小的方法如下

Algorithm 算法 1.1 Rules For Comparing Two Numbers 比较两个数大小的方法

1. The greater the number of digits, the greater is the number.
2. If two numbers have the same number of digits, the number with the bigger digit on the left hand side is greater.
3. If the leftmost digits are the same we compare the next digit to the right and keep doing this until the digits are different.

1. 兩個數位數多的數大
2. 如果兩個數的數字個數相同，最左邊的數字大的數大
3. 如果最左邊的數字相同，就比較它右邊的數字，如果還相同繼續比較右邊的數字，直到數字不同，數字大的數大。如果數字都相同則兩個數一樣大。

Example 1.5: Compare these numbers 比較下面的數字

$a)675,\quad 67 \quad b)2,148,\quad 3,147 \quad c)2,178,345,\quad 2,178,987 \quad d)492,789,100,\quad 492,789,100$

✎ **Solution:**

	Why? 原因
$675 > 67$	The first number is 3-digit number, while second is 2-digit. 第一個數有三位，第二個數有兩位
$2,148 < 3,147$	The first digit is smaller in the first number. 第一個數首位數小
$2,178,345 < 2,178,987$	The first digit is the same, but the fifth digit is smller in the first number. 首位相同，但第一個數從左起第五個位數字小
$492,789,100 = 492,789,100$	All digits are the same. 所有的位置數字相同

□

Example 1.6: List the numbers in order from least to greatest. 把下列數字從小到大排序。

1 5 35 27 14 36 63 69 78 87 99 100

✎ **Solution:** Based on the following rules: 根據以下的原則進行排序：

1. 1-digit numbers < 2-digit numbers < 3-digit numbers. 一位數小于兩位數小于三位數。
2. For 2-digit numbers, compare tens first. 兩位數比較，先比較十位；十位數字相同的比較個位。

The result is as follows 排列結果如下：

$$1 < 5 < 14 < 27 < 35 < 36 < 63 < 69 < 78 < 87 < 99 < 100.$$

□

Example 1.7: The following are some animals and their ages. List them from young to old. 下面是一些動物的年齡，請將他們按年齡從小到大排列。

Elephant: 80, 大象：80 歲； Giraffe: 25 長頸鹿：25 歲
Horse: 40, 馬：40 歲； Monkey: 30, 猴子：30 歲
Tiger: 20, 老虎：20 歲； Carp: 100, 鯉魚：100 歲
Turtle: 170, 烏龜：170 歲； Parrot: 104, 鸚鵡：104 歲

✎ **Solution:** List the numbers from least to greast: 把數字從小到大排序，我們得到：

$$20 < 25 < 30 < 40 < 80 < 100 < 104 < 170.$$

Therefore, the animals are listed as 所以動物按照以下順序排序

$$\text{tiger} < \text{Giraffe} < \text{Monkey} < \text{Horse} < \text{Elephant} < \text{Carp} < \text{Parrot} < \text{Turtle}.$$

$$\text{老虎} < \text{長頸鹿} < \text{猴子} < \text{馬} < \text{大象} < \text{鯉魚} < \text{鸚鵡} < \text{烏龜}.$$

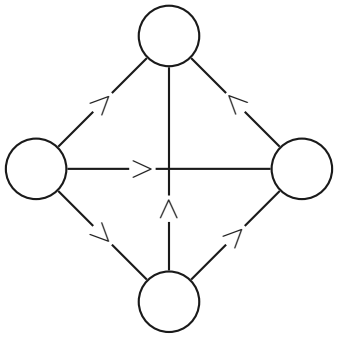
□

Note: *From here, it is not required in the normal class.* 以下爲補充內容，非課本要求。

Proposition 1.1 transitivity 传递性

There are three numbers. If the first number is less than the second and the second is less than the third, then the first number is less than the third number. Similarly, if the first number is greater than the second and the second is greater than the third, then the first number is greater than the third number.
如果有三個数。如果第一個数小于第二個，第二個数小于第三個数，則第一個数小于第三個数。同樣的如果第一個数大于第二個，第二個数大于第三個数，則第一個数大于第三個数。

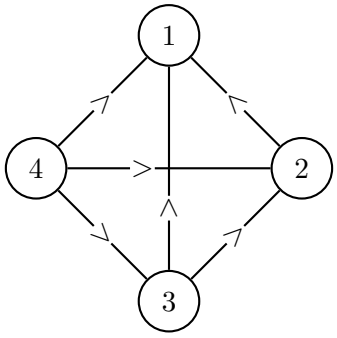
Example 1.8: Put $1, 2, 3, 4$ into the circles in the following graph to satisfy all of the relationships between each two circles. 把 $1, 2, 3, 4$ 填入下圖中的小圓圈裏是的圖中所示的不等關系成立。



✎ **Solution:**

- Since the number in the left circle is greater than the other three, it is 4.
- Since the number in the upper circle is less than the other three, it should be 1.
- The number in the lower circle is greater than the number in the right circle. Therefore, the lower one is 3 and the right one is 2.

- 最左端的小圓圈中應填的数都大于其他三個小圓圈中應填的数，所以應填最大的数 4；
- 最上面的小圓圈應填的数最小，所以應填 1；
- 最下面的小圓圈應填的数大于右邊的小圓圈應填的数，所以最下面圓圈應填 3，最右邊應填 2。



□

Example 1.9:

There are 6 friends working on one projects. Here are some information about their ages.

1. Peter and Andrew are the same age.
2. James is older than John, but younger than Peter.
3. Philip is younger than Peter and James, but older than John.
4. Andrew is younger than Thomas.

Please list them by age and find out who is the oldest and who is the youngest.

六個小伙伴一起在做一個任務。下面是關于他們年齡的一些綫索：

1. 彼得和安德烈同歲；
2. 雅各比約翰年齡大，但比彼得小；
3. 腓力比彼得和雅各小，但比約翰大；
4. 安德烈比多馬年紀小。

請按照年齡給他們排序。并回答誰年齡最大？誰年齡最小？

✎ **Solution:**

Based on the information, we have

1. Based on 1, Peter = Andrew.
2. Based on 2, Peter >James > John. Therefore, Peter = Andrew >James > John
3. Based on 3, Peter > James > Philip > John. Therefore, Peter = Andrew > James > Philip > John.
4. Based on 4, Thomas >Andrew.

We have listed them by ages as follows

根據他們年齡的綫索，我們可以判斷：

1. 由 1，彼得 = 安德烈；
2. 由 2，彼得 > 雅各 > 約翰。
   所以，彼得 = 安德烈 > 雅各 > 約翰；
3. 由 3，彼得 > 雅各 > 腓力 > 約翰。所以，彼得 = 安德烈 > 雅各 > 腓力 > 約翰；
4. 由 4，多馬 > 安德烈。

我們按照年齡給他們排序如下：

$$\text{Thomas} > \text{Peter} = \text{Andrew} > \text{James} > \text{Philip} > \text{John}$$

多馬 > 彼得 = 安德烈 > 雅各 > 腓力 > 約翰

Therefore, Thomas is the oldest and John is the youngest.

所以多馬年齡最大，約翰年齡最小。

□

Exercise 1.1:

1. 1
2. 2
3. 3
4. 4

1. 1
2. 2
3. 3
4. 4

1.1.5 Rounding 四舍五入

Rounding means making a number simpler but keeping its value close to what it was. The result is less accurate, but easier to use.

近似是尋找一個和原來的數接近的看起來比較“簡單”的數。它不精確，但是相對簡單。

There are several different methods for rounding. Here we look at the common method, the one used by most people.

有許多種不同的近似方法，通常我們用的方法是四捨五入。其方法如下：

Algorithm 算法 1.2 How to Round Numbers 四舍五入方法

1. Decide which is the last digit to keep
2. Leave it the same if the next digit is less than 5 (this is called rounding down)
3. But increase it by 1 if the next digit is 5 or more (this is called rounding up)

1. 决定要保持的位
2. 如果其下一位小于 5，保持這位的数字（向下近似）
3. 如果其下一位大于等于 5，則這位数字加 1（向上近似）

Example 1.10: Round 73 to the nearest 10 把 73 近似到十位

✎ **Solution:**

Since 7 is the tens position, we want to keep it. The next digit is 3, which is less than 5. Therefore, no change is needed. The answer is 70.

因爲十位上是 7，這是我們要保持的數。而其個位爲 3，小于 5。所以十位不用加一就是 7。所以，四捨五入到十位是 70。

□

Example 1.11: Round 191 to the nearest 100 把 191 近似到百位

✎ **Solution:**

Since 1 is the hundreds position, we want to keep it. The next digit is 9, which is greater than 5. Therefore, increase 1 to 2. The answer is 200.

因爲百位上是 1，這是我們要保持的數。而其十位爲 9，大于 5。所以百位加一就是 2。所以，四捨五入到百位是 200。

□

1.2 Addtion and subtraction 数的加减法

1.2.1 Basic facts of addition and subtraction 加减法基础

These basic strategies can help you remember addition facts for small numbers.

1. Count on from the greater number. For example, $6+3$. Count on from 6 3 times: $7, 8, 9$. So $6+3=9$.
2. Use doubles. For example, $8+7$. Since $7+7=14$, $8+7=14+1=15$.
3. Use 10 to add 8 or 9. For example, $9+5=10+4=14$.

對于小數的加法，有如下的技巧可以幫助我們：

1. 從大數開始數數、例如：$6+3$。從 6 開始數三次：$7, 8, 9$. 所以 $6+3=9$.
2. 翻倍。例如：$8+7$。因爲 $7+7=14$，$8+7=7+7+1=15$。
3. 利用 10 做 8 或 9 相關的加法。例如：$9+5=10+4=14$。

These basic strategies can help you remember subtraction facts for small numbers.

1. Count back from the greater number. For example, $6-2$. Count back from 6 twice: $5, 4$. So $6-2=4$.
2. Count up from the lesser number. For example, $7-5$. Count on from 5 to 7: $6, 7$. So $7-5=2$.
3. Use 10 to subtract 8 or 9. For example, $16-9=16-10+1=7$.

對于小數的減法，有如下的技巧可以幫助我們：

1. 從大數倒數。例如，$6-2$. 從 6 倒數兩個數: $5, 4$. 所以 $6-2=4$。
2. 從小數數。例如，$7-5$. 從 5 數到 7: $6, 7$. 所以 $7-5=2$。
3. 利用 10 計算減 8 或 9. 例如, $16-9=16-10+1=7$。

For the addition and subtraction of large numbers, the column method can be useful. The way to do the column method is as follows:

Step 1: Create columns for each place value. From right to left, we will create a ones column, tens column, hundreds column, and so on. And we align the numbers by the place value. For example, ones of the first

對于較大的數的加減法，我們會利用竪式來幫助我們計算。其方法如下：

第 1 步： 列竪式。數的每一位對其。例如第一個數的個位和第二個數的個位要對其。

第 2 步： 從個位起，諸位數字相加。

Step 3: 處理進位/借位。加法中如果某位数字加和大于 10，則要向下一位

number will align with ones of the second.

Step 2: Add/subtract the digits in each column starting from ones.

Step 3: Borrowing/ Carrying numbers. If the sum of the digits is 2-digit number, we have to carry a number to the next column. If the top digit in any of the columns is smaller than the bottom digit then we have to borrow from the next column.

进位。减法中如果某位数字被减数较小，则需要向上一位借位。

Example 1.12: 238+126

✎ **Solution:**

	hundreds 百位	*tens* 十位	*ones* 個位
	2	3	8
+	1	2	6

$\Rightarrow$

	hundreds 百位	*tens* 十位	*ones* 個位
	2	3	**8**
+	1	$2_{\mathbf{1}}$	**6**
			4

$\Rightarrow$

	hundreds 百位	*tens* 十位	*ones* 個位
	2	**3**	8
+	1	$\mathbf{2}_1$	6
		6	4

$\Rightarrow$

	hundreds 百位	*tens* 十位	*ones* 個位
	2	3	8
+	**1**	2_1	6
	3	6	4

$238 + 126 = 364$. □

Example 1.13: 126-109

✎ **Solution:**

	hundreds 百位	*tens* 十位	*ones* 個位
	1	2	6
−	1	0	9

$\Rightarrow$

	hundreds 百位	*tens* 十位	*ones* 個位
		1	16
	1	~~2~~	~~6~~
−	1	0	9
			7

		hundreds 百位	*tens* 十位	*ones* 個位
			1	16
⇒		1	~~2~~	~~6~~
	−	1	**0**	9
			1	7

⇒

		hundreds 百位	*tens* 十位	*ones* 個位
			1	16
		1	~~2~~	~~6~~
	−	**1**	0	9
		~~**0**~~	1	7

$126 - 109 = 17$. □

1.2.2 Three or more addends 多个数字加法

When there are thre or more addends, adding them one by one is the basic method. However there are also some tricks we can use to help solve the problem more quickly. All of the tricks are based on Theorem ??. We can add the easy addends first.

當計算三個或以上數的加法時，逐項相加是最基本的方法。但是對于一些有特殊特點的計算，有一些計算技巧可以來幫助我們。所有的技巧都是根據定理??。我們可以先計算簡單的加和。

Theorem 1.2 Commutative property of addtion 加法交换律

changing the order of the addends does not change the result.
改變加法各項的順序，計算結果不變。

Looking for a sum of 10 凑十法

We already know that the addition of the following 5 pairs is 10. We can use these results to do the computation quickly.

我們已經知道，下面的五組成對的數相加之和都等于 10。利用這些結果可以使計算又快又準。

$$1+9=9+1=10,\quad 2+8=8+2=10,\qquad 3+7=7+3=10$$
$$4+6=6+4=10,\quad 5+5=10.$$

Example 1.14: 1+2+3+4+5+6+7+8+9+10

✎ **Solution:**

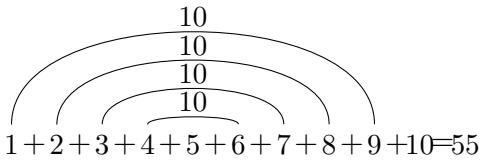
As the basic method, we can add them one by one.

對于這道題，基礎的方法是從左往右逐項相加。

$$1+2=3,\quad 3+3=6,\qquad 6+4=10,$$
$$10+5=15,\quad 15+6=21,\quad ,\quad 21+7=28,$$
$$28+8=36,\quad 36+9=45,\quad ,\quad 45+10=55.$$

The advantage of this method is that we can get all of the intermedia results. However it is complecated and easy to make mistakes. If one addition in one of the steps is wrong, it will be wrong after. And it is really difficult to find out where it goes wrong. The new method which looking for the sum of 10 could fix this problem.

這種逐項相加的方法，好處是可以得到每一步的結果，但缺點是麻煩，容易出錯；而且一步出錯，以後步步都錯。并且很難檢查錯誤。若是利用凑十法，就可以克服這種缺點。



□

Looking for a sum of a whole number 凑整法

Some number pairs could add up to tens or hundreds. For example

有些數相加之和是整十、整百的數，如：

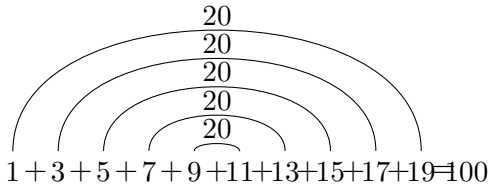
$$1+19=20, \qquad 11+19=30,$$
$$12+28=40, \qquad 12+38=50,$$
$$23+37=60, \qquad 33+37=70,$$
$$34+46=80, \qquad 44+46=90,$$
$$15+85=100, \qquad 45+55=100.$$

These results could help us in the computation.

巧用這些結果，可以使那些較大的數相加又快又準。

Example 1.15: $1+3+5+7+9+11+13+15+17+19$

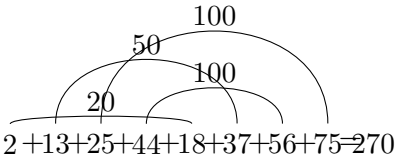
✎ **Solution:**



□

Example 1.16: $2+13+25+44+18+37+56+75$

✎ **Solution:**



□

1.2.3 Mixed calculation 加减混合运算

Changing the order of operations 改变运算顺序

Sometimes, it is convinient to change the order of operations in the mixed equation with only addition and subtraction.

在衹有加减運算的算式中，有時改變加、减的運算順序可使計算顯得十分巧妙!

Example 1.17: $10-9+8-7+6-5+4-3+2-1$

✎ **Solution:**

If this question is added or subtracted in order from left to right, it is possible to draw a correct result. However, it is more complicated and easier to make a mistake. If you change the order of the operations, doing the subtraction first, the computation will be easier. The formula in parentheses indicates calculating first.

這題如果從左到右按順序進行加减運算，是能够得出正確結果的。但因爲算式較長，多次加减又繁又慢且容易出錯。如果改變一下運算順序，先减後加，就使運算顯得非常“漂亮”。下式括號中的算式表示先算。

$$\begin{aligned}&10-9+8-7+6-5+4-3+2-1\\=&(10-9)+(8-7)+(6-5)+(4-3)+(2-1)\\=&1+1+1+1+1=5.\end{aligned}$$

□

Changing the order of terms with the sign 带着“+”、“-”号搬家

Example 1.18: $1-2+3-4+5-6+7-8+9-10+11$

✎ **Solution:**

How to compute $1-2$? To make it simple. We can move $+3$ to the front of -2. And similarly we can move $+5$ to the front of -4. We can simplify the computation with this technique. When you move the terms, just remember move with the sign.

這題衹有加减運算，而且 1-2 不够减。我們可以采用帶着加减號搬家的方法解决。要注意每個數自己的符號就是這個數前面的那個“+”號或“-”號，搬家時要帶着符號一起搬。，把“+3”搬到“-2”的前面，把“+5”搬到了“-4”的前面，……把“+11”搬到了“-10”的前面，這就叫帶着符號搬家。巧妙利用這種搬法，可以使計算簡便。

$$\begin{aligned}&1-2+3-4+5-6+7-8+9-10+11\\=&1+3-2+5-4+7-6+9-8+11-10\\=&1+(3-2)+(5-4)+(7-6)+(9-8)+(11-10)\\=&1+1+1+1+1+1=6\end{aligned}$$

□

1.3 Multiplication and division 数的乘除法

1.3.1 Basic facts of multiplication 乘法的基础

Multiplication could be thought as a repeated addition. The multiplication of two numbers is equivalent to adding as many copies of one of them, the multiplicand, as the value of the other one, the multiplier. Consider the following

乘法是可以用加法來表示的。兩個數相乘相當于把被乘數加乘數次。例如

$$8 \times 4 = 32 \Leftrightarrow 8 + 8 + 8 + 8 = 32.$$

Multiplicand and multiplier are called factors of the result.

乘數和被乘數都是乘積的因子。

For the mutiplication within 10, it is better to remember all of the results.

我們最好能都熟練的掌握 10 以内的數的乘法的結果。

MathATube.com

*,/	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

图 1.2: Multiplication chart

1.3.2 Long multiplication 乘法竖式

Long multiplication is a method used to solve the multiplication problems with large

乘法竖式是用來計算較大數的乘法。如果我們可以牢記上面的乘法表，可以幫助我

图 1.3: 九九乘法口訣表

numbers. One thing can really help you in long multiplication is if you know the multiplication chart by heart. This will speed up your work and make it more accurate.

們更快速準確的進行竪式運算。

Multi-digit number times 1-digit number 多位数乘一位数

The key of the multiplication is that multiplying each place value of the multi-digit number by the one-digit number and suming up the results.

如果是多位數乘一位數，就用多位數中的每一位分別乘那個一位數，然後把結果相加。

Step 1: write down the numbers on top of each other. Make sure align the numbers on the right.

第一步： 寫竪式，相同數位對齊；

Step 2: Multiply each place value of the multiplicand by the 1-digit multiplier.

第二步： 多位數每一位上的數分別與這個一位數相乘，從多位數個位起；

Step 3: Align the result with the corresponding place. For exampe, multiply ones by the 1-digit number, then align the result with ones. Add the carry on numbers from the previous.

第三步： 乘到哪一位就把結果寫在哪一位數相應的位置上。注意加上上次計算的進位。

Example 1.19: 284×3

✎ **Solution:**

$$\begin{array}{r} 2\ 8\ 4 \\ \times\quad\ 3 \\ \hline \end{array} \Rightarrow \begin{array}{r} 2\ 8\ \mathbf{4} \\ \times\quad {\scriptstyle 1}\ \mathbf{3} \\ \hline \mathbf{2} \end{array}$$

$$\Rightarrow \begin{array}{r} 2\ \mathbf{8}\ 4 \\ \times\ {\scriptstyle 2\ 1}\ \mathbf{3} \\ \hline \mathbf{5}\ 2 \end{array} \Rightarrow \begin{array}{r} \mathbf{2}\ 8\ 4 \\ \times\ 2\ 1\ \mathbf{3} \\ \hline \mathbf{8}\ 6\ 4 \end{array}$$

$284 \times 3 = 864$. □

Multi-digit number times multi-digit number 多位数乘多位数

The multiplication bewteen two multi-digit numbers is based on the multiplication between multi-digit number and 1-digit number. The rule is as follows.

多位數多位數乘法法則是基于多位數一位數的乘法。其法則如下：

Step 1: write down the numbers on top of each other. It is better to write down the number with more digits on the top.

第一步： 寫竪式，最好把數位較多的乘數寫在上面作爲被乘數，數位較少的寫在下面；

Step 2: Multiply the multiplicand by ones of the multiplier and align the ones of the results to the ones under the line. Then multiply the multiplicand by tens of the multiplier and align the ones of the results to the tens under the line. And so on.

第二步： 用被乘數與乘數的個位相乘，把相乘得到的積的末位寫在個位上；然後再與十位上的數相乘寫在十位上，... ...

Step 3: Sum all of the results up.

第三步： 把每次乘得的數相加得到最終結果。

Example 1.20: 125×124

✎ **Solution:**

$$\begin{array}{r} 1\ 2\ 5 \\ \times\ 1\ 2\ 4 \\ \hline \end{array} \Rightarrow \begin{array}{r} \mathbf{1}\ \mathbf{2}\ \mathbf{5} \\ \times\ 1\ 2\ \mathbf{4} \\ \hline \mathbf{5}\ \mathbf{0}\ \mathbf{0} \end{array}$$

$$\Rightarrow \begin{array}{r} \mathbf{1\ 2\ 5} \\ \times\quad 1\ \mathbf{2}\ 4 \\ \hline 5\ 0\ 0 \\ \mathbf{2\ 5\ 0}\ \ \\ \hline \end{array} \Rightarrow \begin{array}{r} \mathbf{1\ 2\ 5} \\ \times\quad \mathbf{1}\ 2\ 4 \\ \hline 5\ 0\ 0 \\ 2\ 5\ 0\ \ \\ \mathbf{1\ 2\ 5}\ \ \ \\ \hline \end{array} \Rightarrow \begin{array}{r} 1\ 2\ 5 \\ \times\quad 1\ 2\ 4 \\ \hline 5\ 0\ 0 \\ 2\ 5\ 0\ \ \\ 1\ 2\ 5\ \ \ \\ \hline \mathbf{1\ 5\ 5\ 0\ 0} \end{array}$$

$125 \times 124 = 15,500$. □

There are some tricks to help do the multiplication faster and easier.

1. If there are zeros at the end of the number, we can leave them and multiply the remaining numbers. Then add the zeros.

Example 1.21: 2800×130

✎ **Solution:**

$$\begin{array}{r} 2\ 8\ \vdots\ 0 \\ \times\quad 1\ 3\ \vdots\ 0 \\ \hline 8\ 4\ \vdots\ \ \\ 2\ 8\ \vdots\ \ \\ \hline 3\ 6\ 4\ \vdots\ 0\ 0 \end{array}$$

$2800 \times 130 = 36400$. □

2. If there are repeat numbers in the multiplier, you do not need to compute it again.

Example 1.22: 315×66

✎ **Solution:**

$$\begin{array}{r} 3\ 1\ 5 \\ \times\quad 6\ 6 \\ \hline 1\ 8\ 9\ 0 \\ 1\ 8\ 9\ 0\ \ \\ \hline 2\ 0\ 7\ 9\ 0 \end{array}$$

$315 \times 66 = 20790$. □

要提高計算速度有以下一些技巧。

1. 末尾有 0 的數的乘法。可以先把 0 前面的數相乘再添 0，得出結果。

2. 當乘數有重復的數字時，不必重復相乘，可以直接寫出後面的結果。衹是要注意對齊。

1.3.3 Basic facts of division 除法的基础

The division of two natural numbers is the process of calculating the number of times one number is contained within another one. Division is the inverse of multiplication. For example, since $4 \times 5 = 20$, $20 \div 5 = 4$. In division, the dividend is divided by the divisor to get a quotient. In the above example, 20 is the dividend, five is the divisor, and four is the quotient. In some cases, the divisor may not be contained fully by the dividend.

除法可以看成“重復的减法”，其的本質是計算一個數需要减另一個數多少次才能變成零。除法是减法的逆運算。例如：$4 \times 5 = 20$，所以 $20 \div 5 = 4$。除法中，是被除數除以除數，其結果爲商數。在上面的例子中 20 是被除數，5 是除數，4 是商數。

Sometimes this remainder is added to the quotient as a fractional part, but in the context of integer division, where numbers have no fractional part, the remainder is kept separately or discarded.

有時會出現被除數不能够被除數整除的情況，即有餘數。餘數可以繼續計算成爲商數的一部分，但現在（整數除法）我們會把它單獨寫出來或者忽略掉。

In order to do the division for large numbers, we have to know:

如果要學習較大數的除法，我們需要做到如下：

1. multiplication charts (at least fairly well)
2. basic division concept, based on multiplication tables (for example $28 \div 7$ or $56 \div 8$)
3. basic division with remainders (for example $54 \div 7$ or $23 \div 5$)

1. 熟練掌握九九乘法表
2. 利用乘法表計算除法 (例如 $28 \div 7$ or $56 \div 8$)
3. 知道如何計算基本的帶餘數的除法 (例如 $54 \div 7$ or $23 \div 5$)

1.3.4 Long division 除法竖式

Long division is an algorithm that repeats the basic steps of 1) Divide; 2) Multiply; 3) Subtract; 4) Drop down the next digit. Here are the basic procedure[1]:

除法竖式的算法是在重復 1) 除; 2) 乘; 3) 减; 4) 拉下一位這樣的過程。其基本算法如下 [1]:：

1. Divide

1. 除

(a) **Set up the equation.** Write the dividend on the right, under the division symbol, and the divisor to the left on the outside. The quotient will eventually go on top, right above the dividend. Leave yourself plenty of space below the equation to carry out multiple subtraction operations.

(a) **建立竖式。** 被除數寫在除號下面，除數寫在除號左邊。商數將寫在除號上邊。所以在建立竖式時在上方留些空間。

(b) **Divide the first digit.** Working from left to right, determine how many times the divisor can go into the first digit of the dividend without exceeding it.

(b) **除第一位。** 從被除數的最左一位開始。試驗出最大的數使得這個數乘以除數小于第一位。

(c) **Divide the first two digits.** If the divisor is a larger number than the first digit, determine how many times the divisor goes into the first two digits of the dividend without exceeding it. If your divisor has more than two digits, you'll have to expand out even further.

(c) **除前两位** 如果除數比第一位大, 就試驗出最大的數使得這個數乘以除數小于前兩位。注意，如果除數也是一個多位數，那麼我們就需要更多位使得比除數大。

(d) **Enter the first digit of the quotient.** Put the number of times the divisor goes into the first digit (or digits) of the dividend above the appropriate digit(s).

(d) **记下商数的第一位。** 把前面試驗得到的數字寫到商數，數字要與計算的被除數最後一位數字對齊。

2. Multiply

2. 乘

(a) **Multiply the divisor.** The divisor should be multiplied by the number you have just written above the dividend.

(a) **乘除数** 用除數乘以上步記下的數字。

(b) **Record the product.** Put the result of your multiplication in step 1 beneath the dividend.

(b) **记录结果** 把結果寫在被除數下面，結果要個上步使用的被除數數字對齊。

(c) **划线。** 畫一條綫在剛才計算的

(c) **Draw a line.** A line should be placed beneath the product of your multiplication

3. subtract

(a) **Subtract the product.** Subtract the number you just wrote below the dividend from the digits of the dividend directly above it. Write the result beneath the line you just drew.

(b) **Bring down the next digit.** Write the next digit of the dividend after result of your subtraction operation.

4. **Repeat the whole process.** Divide the new number by your divisor, and write the result above the dividend as the next digit of the quotient.

結果下面。

3. 減

(a) **減。** 用被除數對應的位置減去剛才計算的結果。把結果寫到綫的下方。

(b) **写下被除数后面的数字** 把被除數後面的數字寫到剛才的計算結果後面。

4. **重复前面的过程** 重新用除數除新上面寫下的數字。把新試驗出的數字寫到商數位置的上一個數字右面。

Dividing a number by a 1-digit number 一位数除法

Example 1.23: $148 \div 21$

✎ **Solution:**

$$3\overline{)\,2\ \ 8\ \ 5} \Rightarrow 3\overline{)\,2\ \ 8\ \ 5} \Rightarrow 3\overline{)\,2\ \ 8\ \ 5}^{\,9}$$

$$\Rightarrow 3\overline{)\,2\ \ 8\ \ 5}^{\,9} \Rightarrow 3\overline{)\,2\ \ 8\ \ 5}^{\,9} \Rightarrow 3\overline{)\,2\ \ 8\ \ 5}^{\,9}$$
$$\underline{2\ \ 7} \qquad \underline{2\ \ 7} \qquad \underline{2\ \ 7\ \downarrow}$$
$$1 \qquad\qquad 1\ \ 5$$

$$\begin{array}{r} \ 9\ 5 \\ 3\overline{)\ 2\ 8\ 5} \\ \Rightarrow\ \underline{\ 2\ 7\ \downarrow} \\ \ 1\ 5 \end{array} \qquad \begin{array}{r} \ 9\ 5 \\ 3\overline{)\ 2\ 8\ 5} \\ \Rightarrow\ \underline{\ 2\ 7\ \downarrow} \\ \ 1\ 5 \\ \underline{\ 1\ 5} \\ \ 0 \end{array}$$

$285 \div 3 = 95$. □

Dividing a number by a multi-digit number 多位数除法

Example 1.24: $148 \div 21$

✎ **Solution:**

$$\begin{array}{r} \ 7 \\ 21\overline{)\ 148} \\ \underline{\ 147} \\ \ 1 \end{array}$$

$148 \div 21 = 7\ R\ 1$. □

1.3.5 Exponentiation 指数运算

Exponentiation is a special operation for writing a product in which all of the numbers being multiplied are the same. For example, $2 \times 2 \times 2 \times 2$ can be written as 2^4. We call the entire expression 2^4 a **power**, specifically the power of 2. The number on the bottom is the base. The number on top is the exponent.

指數運算是多個相同數字的乘積的另一種表達方式。例如：$2 \times 2 \times 2 \times 2$ 可以寫成 2^4。2^4 被稱爲指數運算，或 2 的冪。指數運算中，在下面的數字被稱爲底數，在上面的數字被稱爲指數。

When the exponent is 2, we call it a square which means the product of a number ans itself. For example, 4^2 is called "four squared". When the exponent is 3, we call it a cube. So 6^3 is called "six cubed".

當指數爲 2 時，我們稱爲平方。一個數的平方是此數與它的本身相乘所得的乘積。例如：4^2 稱爲 4 的平方。當指數爲 3 時，我們稱爲立方。如 6^3 稱爲 6 的立方。

Example 1.25: a. 10^2 b. 3^5 c. 5^6 d. 10^7

✎ **Solution:**

a. $10^2 = 10 \times 10 = 100.$

b. $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243.$

c. $5^6 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 125 \times 125 = 15625.$

d. $10^7 = 10,000,000$

□

Proposition 1.2 Product of powers 冪相乘

Same exponent: The product of two powers with the same exponent is the product of the two base with the same exponent. For example,
同指數冪相乘，指數不變，底數相乘。例如：

$$3^3 \times 4^3 = (3 \times 4)^3$$

Same base: The exponent of the product of two powers with the same base is the sum of the two powers. The base is the same.
同底數冪相乘，底數不變，指數相加

$$4^3 \times 4^5 = 4^{3+5}$$

Example 1.26: Explain why

$$3^3 \times 4^3 = (3 \times 4)^3, \quad 4^3 \times 4^5 = 4^{3+5}$$

✎ **Solution:**

$$\begin{aligned} & 3^3 \times 4^3 \\ =&(3 \times 3 \times 3) \times (4 \times 4 \times 4) \\ =&(3 \times 4) \times (3 \times 4) \times (3 \times 4) \\ =&(3 \times 4)^3 \end{aligned}$$

$$\begin{aligned} & 4^3 \times 4^5 \\ =&(4 \times 4 \times 4) \times (4 \times 4 \times 4 \times 4 \times 4) \\ =&4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \\ =&4^8 \\ =&4^{3+5} \end{aligned}$$

□

Proposition 1.3 Quotient of powers 冪相除

Same exponent: The quotient of two powers with the same exponent is the quotient of the two base with the same exponent. For example,
同指數冪相除，指數不變，底數相除。例如：

$$6^3 \div 2^3 = (6 \div 2)^3$$

Same base: The exponent of the quotient of two powers with the same base is the subtraction of the two powers. The base is the same.
同底數冪相除，底數不變，指數相減。例如：

$$4^5 \div 4^3 = 4^{5-3}$$

Example 1.27: Explain why

$$6^3 \div 2^3 = (6 \div 2)^3, \quad 4^5 \div 4^3 = 4^{5-3}$$

✎ **Solution:**

$$\begin{aligned}&6^3 \div 2^3\\ =&(6\times 6\times 6)\div(2\times 2\times 2)\\ =&6\times 6\times 6\div 2\div 2\div 2\\ =&(6\div 2)\times(6\div 2)\times(6\div 2)\\ =&(6\div 2)^3\end{aligned}$$

$$\begin{aligned}&4^5 \div 4^3\\ =&(4\times 4\times 4\times 4\times 4)\div(4\times 4\times 4)\\ =&4\times 4\times 4\times 4\times 4\div 4\div 4\div 4\\ =&4^2\\ =&4^{5-3}\end{aligned}$$

□

Proposition 1.4 Power of powers 冪的冪

When raising a power to a power in an exponential expression, you find the new power by multiplying the two powers together. For example,
一個数的冪的冪等于這兩個冪相乘。例如：

$$(6^3)^3 = 6^{3\times 3}$$

Example 1.28: Simplify. Rewrite the expression in the form 5^*. 化簡。把下面表達式寫成 5^* 的形式。

$$(5^3)^2$$

✎ **Solution:**

$$(5^3)^2 = 5^{3\times 2} = 5^6.$$

□

Proposition 1.5 Zero-Exponent Rule 零次冪

Any number raised to the zero power is 1. For example,
任何数的零次冪等于 1。例如：

$$181^0 = 1.$$

1.3.6 factoring 因数分解

Factors are the numbers you multiply together to get another number. There can be many factors of a number.

兩個正整數相乘，那麼這兩個數都叫做積的因数，或稱爲約数。一個数可以有許多的因数。

Example 1.29: Find all factors of 12 找到 12 的所有因数

✎ **Solution:**

Since　　因爲

$$1 \times 12 = 12, \quad 2 \times 6 = 12, \quad 3 \times 4 = 12$$

1, 2, 3, 4, 6 and 12 are factors of 12.　　1, 2, 3, 4, 6, 12 是 12 的因数。

□

Definition 1.1 Prime and composite number 质数（素数）和合数

A **Prime number** is a whole number greater than 1 that cannot be made by multiplying two smaller whole numbers. A whole number greater than 1 that is not prime is called a **composite number.**
質数（素数）爲在大于 1 的自然数中，除了 1 和此整数自身外兩個因数，無法被其他自然数整除的数。合数是除了 1 和它本身還有其它正因数。

Note: 1 *is neither prime number nor composite number.* 1 既不是質數也不是合數。

Here is some examples of prime numbers.

下面是一些小的質数：

$$2, \quad 3 \quad 5, \quad 7, \quad 9, \quad 11, \quad 13, \quad 17, \quad 19, \quad 23, \quad 29, \quad 31, \quad 37, \quad 41, \cdots$$

Integer factorization is the decomposition of a composite number into a product of smaller integers. If these integers are further restricted to prime numbers, the process is called prime factorization.

整数分解是將一個正整数寫成幾個因数的乘積。如果所有的因数是質数則稱爲質因数分解。

Example 1.30: What are the prime factors of 30? 把 30 分解質因数。

✎ **Solution:**

It is best to start working from the smallest prime number, which is 2, so

要找到所有的的質因数，最好從最小的質数開始試。

$$30 \div 2 = 15.$$

15 is a composite number, but it cannot be divided by 2. Therefore, let us try 3.

15 是合數，但它不能被 2 整除，所以我們試着除以 3。

$$15 \div 3 = 5.$$

So we have the answer:

所以題目的答案是：

$$30 = 2 \times 3 \times 5.$$

□

Example 1.31: The sum of two prime numbers is 40. In all of the possible choices of these two numbers, what is the maximum product? 兩個質數的和是 40，求這兩個質數的乘積的最大值是多少?

✎ **Solution:**

All possible prime number pairs are as follows

把 40 表示爲兩個質數的和，共有三種形式

$$40 = 17 + 23 = 11 + 29 = 3 + 37.$$

Since

因爲

$$17 \times 23 = 391 > 11 \times 29 = 319 > 3 \times 37 = 111.$$

The maximum product is 391.

這兩個質數的最大乘積是 391。

□

Greatest common divisor and least common multiple 最大公约数和最小公倍数

Definition 1.2 Greatest common divisor 最大公约数

The greatest common divisor (gcd) of two or more integers, which are not all zero, is the largest positive integer that divides each of the integers.
幾個數公有的約數，叫做這幾個數的公約數；其中最大的一個，叫做這幾個數的最大公約數。

♡

There are several ways to find the gcd.

計算兩個數的最大公因數有許多不同的方法：

- Method of exhaustion
- 窮舉法
- Prime factorization
- 因數分解

Definition 1.3 Least common multiple 最小公倍数

The least common multiple (lcm) of two or more integers is the smallest positive integer that is divisible by each of the integers.
幾個數公有的倍數，叫做這幾個數的公倍數；其中最小的一個，叫做這幾個數的最小公倍数。

Definition 1.4 Coprimes 互质数

Coprimes is a pair of numbers not having any common factors other than 1.
如果兩個數的最大公約數是 1，那麼這兩個數叫做互質數。

- Euclid's algorithm
- 輾轉相除法

Method of exhaustion: List all of the factors for each of the integers and choose the largest one as gcd. In practice, this method is only feasible for small numbers.

穷举法：分別列出兩整數的所有約數，并找出最大的公約數。這種方法衹針對較小的數。

Example 1.32: 用一個數去除 60、75，都能整除，這個數最大是多少?

✎ **Solution:**[Method of exhaustion 窮舉法]

The factors of these three numbers are

這兩個數的因子分別爲：

$$60 : 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$$
$$75 : 1, 3, 5, 15, 25, 75$$

Therefore, the largest number is 15.

這個數最大是 15。

□

Prime factorization: Compute the prime factorization of each of the integers and find the "overlap" of the expressions. gcd is the product of the common factors. In practice, this method is only feasible for small numbers; computing prime factorizations in general takes far too long.

因数分解：分別列出兩數的素因數分解式，并計算共同項的乘積。這種方法衹針對較小的數，對于大數的因式分解通常比較困難。

✎ **Solution:**[Prime factorization 因數分解]

The prime factorizationof these three numbers are

這三個數的因式分解分別爲：

$$60 = 2 \times 2 \times 3 \times 5$$
$$75 := 3 \times 5 \times 5$$

The common factors are 3 and 5. Therefore, the largest number is $3 \times 5 = 15$.

共同的因子是：3，5。這個數最大是 $3 \times 5 = 15$。

□

Euclid's algorithm: Divide the large number by the small number to get the remainder. And divide the divisor by the remainder to get the new remainder. Repeat this prodecure until the remainder is zero. Then gcd is the last divisor. Euclid's algorithm is very efficient for large numbers. The total number of steps needed is less or equal than the number of digits of the smaller number.

辗转相除法：兩數相除，取餘數重復進行相除，直到餘數爲 0 時，前一個除數即爲最大公約數。輾轉相除法處理大數時非常高效，它需要的步驟不會超過較小數的位數（十進制下）的五倍。

Algorithm 算法 1.3 Euclid's algorithm 辗转相除法

1. Divide the large number by the small number to get the remainder;
2. Divide the divisor by the remainder to get the new remainder;
3. Repeat step 2, until the remainder is 0;
4. The gcd is the divisor of the last step.

1. 大數除以小數取餘數；
2. 上一步的除數除以上一步的餘數取餘數；
3. 重復上一步知道餘數爲零；
4. 前一個除數即爲最大公約數。

✎ **Solution:**[Euclid's algorithm 輾轉相除法]

$$75 \div 60 = 1 \ldots 15$$
$$60 \div 15 = 4$$

Therefore, the largest number is 15.
這個數最大是 15。

□

1.3.7 Even and odd numbers 奇数和偶数

Even numbers can be divided evenly into groups of two. Therefore, Any whole number that can be divided exactly by 2 is an even number. Odd numbers can NOT be divided evenly into groups of two. Any whole number that cannot be divided exactly by 2 is an odd number.

整數可以分成奇數和偶數兩大類. 能被 2 整除的數叫做偶數，不能被 2 整除的數叫做奇數。

Note: *Zero is even numbers becasue* $0 \div 2 = 0$. 因爲 *0* 能被 *2* 整除，所以 *0* 是偶數。

The arithmetic of even and odd numbers are as following

奇數與偶數的運算性質如下：

Even+Even=Even	偶數 + 偶數 = 偶數	Even−Even=Even	偶數 − 偶數 = 偶數
odd+ odd=Even	奇數 + 奇數 = 偶數	odd− odd=Even	奇數 − 奇數 = 偶數
Even+ odd= odd	偶數 + 奇數 = 奇數	Even− odd= odd	偶數 − 奇數 = 奇數
odd+Even= odd	奇數 + 偶數 = 奇數	odd−Even= odd	奇數 − 偶數 = 奇數
Even×Even=Even	偶數 × 偶數 = 偶數	odd× odd= odd	奇數 × 奇數 = 奇數
odd×Even=Even	奇數 × 偶數 = 偶數	Even× odd=Even	偶數 × 奇數 = 偶數
The sum of even numbers of odd is even		偶數個奇數相加得偶數	
The sum of odd numbers of odd is even		奇數個奇數相加得奇數	

Example 1.33: Is $1 + 2 + \cdots + 2018$ even or odd?

$1 + 2 + \cdots + 2018$ 是奇數還是偶數?

Solution:

Since $2018 \div 2 = 1009$. From 1 to 2018, there are 1009 even number and 1009 odd number. The sum of 1009 even number is even and the sum of 1009 odd number is odd. Therefore, $1 + 2 + \cdots + 2018$ is odd.

因爲 $2018 \div 2 = 1009$. 從 1 到 2018, 有 1009 個偶數，1009 個奇數。1009 個偶數的和是偶數，1009 個奇數的和是奇數。因爲奇數加偶數是奇數，所以原式的和一定是奇數。

□

1.4 Arithmetic of integers 四则混合运算

1.4.1 Order of operations 运算顺序

It is important to know the order of operations when you have different arithmetic in one equation.

對于四則混合運算，掌握運算順序是十分重要的。

P	E	M	D	A	S
Parenthesis	Exponents	Multiplication	Division	Addition	Subtraction
括號	指數	乘法	除法	加法	減法
From left to right →	*From left to right* →	*From left to right* →		*From left to right* →	
自左至右計算	自左至右計算	自左至右計算		自左至右計算	
(), [], {}	^{2}, $\sqrt{}$	×, *, ·	÷, /	+	−

Example 1.34: $150-50\times 2+18$

✎ **Solution:**

$$\begin{aligned} & 150-50\times 2+18 && \text{Compute multiplication first 先計算乘法}\\ =&150-100+18 && \text{Compute addition and subtraction}\\ =&50+18 && \text{from left to right 從左至右計算加減}\\ =&68 \end{aligned}$$

□

Example 1.35: $800-600\div(25\times 4)$

✎ **Solution:**

$$\begin{aligned} & 800-600\div(25\times 4) && \text{Parenthesis 括號}\\ =&800-600\div 100 && \text{Multiplication\& Division 乘法 除法}\\ =&800-6 && \text{Addition \& Subtraction 加法減法}\\ =&796 \end{aligned}$$

□

1.4.2 Properties 四则运算的运算律

The summary of the properties of the arithmetics are as follows.

四則運算的規律如下：

Proposition 1.6 Commutative property of addition 加法交换律

If the order of the addends changes, the sum will stay the same.
加法各項交换順序，其和不變。

$$8+7=15, \quad 7+8=15.$$

Example 1.36: Find the missing number 空格中填入缺少的数。

$$5+6=\_\_+5 \qquad 50+4=\_\_+50$$
$$125+70=\_\_+125 \qquad 17655+6=\_\_+17655$$

✎ **Solution:**

$$5+6=\underline{6}+5 \qquad 50+4=\underline{4}+50$$
$$125+70=\underline{70}+125 \qquad 17655+6=\underline{6}+17655$$

□

Proposition 1.7 Associative property of addition 加法结合律

If the grouping of the addends changes, the sum will stay the same.
三個数相加，先把前兩個数相加，或者先把後兩個数相加，和不變。

$$(7+8)+9=24, \quad 7+(8+9)=24.$$

Example 1.37: Find the missing number 空格中填入缺少的数。

$$(2+3)+4=\_\_+(3+4) \qquad (50+4)+7=50+(\_\_+7)$$
$$12+(2+5)=(12+\_\_)+5 \qquad 1+(6+2)=(1+6)+\_\_$$

✎ **Solution:**

$$(2+3)+4=\underline{2}+(3+4) \qquad (50+4)+7=50+(\underline{4}+7)$$
$$12+(2+5)=(12+\underline{2})+5 \qquad 1+(6+2)=(1+6)+\underline{2}$$

□

Proposition 1.8　Commutative property of multiplication 乘法交換律

If the order of the factors changes, the product will stay the same.
交換乘數和被乘數的位置，乘積不變。

$$3 \times 5 = 15, \quad 5 \times 3 = 15.$$

Example 1.38: Find the missing number 空格中填入缺少的數。

$$5 \times 6 = \_\_ \times 5 \qquad 50 \times \_\_ = 7 \times \_\_$$
$$125 \times 70 = 70 \times \_\_ \qquad \_\_ \times 6 = \_\_ \times 17$$

✎ **Solution:**

$$5 \times 6 = \underline{6} \times 5 \qquad 50 \times \underline{7} = 7 \times \underline{50}$$
$$125 \times 70 = 70 \times \underline{125} \qquad \underline{17} \times 6 = \underline{6} \times 17$$

□

Proposition 1.9　Associative property of multiplication 乘法结合律

If the grouping of the factors changes, the sum will stay the same.
三個數相乘，先把前兩個數相乘，或者先把後兩個數相乘，積不變。

$$(3 \times 4) \times 5 = 60, \quad 3 \times (4 \times 5) = 60.$$

Example 1.39: Find the missing number 空格中填入缺少的數。

$$(2 \times 3) \times 4 = \_\_ \times (3 \times 4) \qquad (50 \times 4) \times \_\_ = 50 \times (\_\_ \times 7)$$
$$12 \times (7 \times 5) = (\_\_ \times \_\_) \times 5 \qquad 10 \times (6 \times 2) = (\_\_ \times 6) \times \_\_$$

✎ **Solution:**

$$(2 \times 3) \times 4 = \underline{2} \times (3 \times 4) \qquad (50 \times 4) \times \underline{7} = 50 \times (\underline{4} \times 7)$$
$$12 \times (7 \times 5) = (\underline{12} \times \underline{7}) \times 5 \qquad 10 \times (6 \times 2) = (\underline{10} \times 6) \times \underline{2}$$

□

Proposition 1.10　Distributive law 乘法分配律

Multiplying a number by a group of numbers added together is the same as doing each multiplication separately.
兩個數的和與一個數相乘，可以先把它們分別與這個數相乘，再將積相加。

$$3 \times (4 + 5) = 3 \times 4 + 3 \times 5 = 27.$$

Example 1.40: $8 \times (10 + 2)$

✎ **Solution:**

$$\begin{aligned} & 8 \times (10 + 2) \\ =& 8 \times 10 + 8 \times 2 \\ =& 80 + 16 \\ =& 96 \end{aligned}$$

□

Proposition 1.11 Identity property 恒等性质

When we add 0 to any number, the number does not change.
任何一個數加零，等于這個數本身。

$$5 + 0 = 5.$$

When we multiply any number by 1, the number does not change.
任何數乘以零，等于這個數本身。

$$5 \times 1 = 5.$$

We are going to simplify the calculation by using these properties. But sometimes, it might be confusing. We need to do more practice to fully understand the concept.

靈活運用各種運算律是爲了幫助我們更快的計算。然而，這也很容易産生一些混淆和疑惑。建議理解和熟練要交互運用，通過練習來掌握這些計算技巧。

It is important to notice that there is no commutative and associative properties for subtraction and division. When applying the associative property, we need to transfer subtraction and division to addition and multiplication.

另外，需要注意减法和除法没有交换律和結合律。如果我們需要使用結合律，我們要記住變號（减號變加號，除號變乘號）

Example 1.41: $1600 \div 10 \div 4 \div 2$

✎ **Solution:** It is ok to solve it step by step. 我們可以從左到右依次計算。

$$\begin{aligned} & 1600 \div 10 \div 4 \div 2 \\ =& 160 \div 4 \div 2 \\ =& 40 \div 2 \\ =& 20 \end{aligned}$$

We can also combine the divisor. 我們也可以把除數一起整合後一起除。

$$\begin{aligned} & 1600 \div 10 \div 4 \div 2 \\ =& 1600 \div (10 \times 4 \times 2) \\ =& 1600 \div 80 \\ =& 20 \end{aligned}$$

□

1.5 Number patterns 数字的规律

1.5.1 Number patterns 数字的规律

Patterns are all around us! Finding and understanding patterns gives us great power. With patterns we can learn to predict the future, discover new things and better understand the world around us. Number pattern is a list of numbers that follow a certain sequence or pattern.

生活中處處有規律存在。能够找到并理解其中的規律能够幫助我們處理很多事情。利用這些規律可以幫助我們進行預測分析，也能够幫助我們更好的理解這個世界。数字的規律就是給出一系列有一定的規律的数字。

Example 1.42: Look at the following number sequence and fill in the missing gaps 觀察下面的数列，找出規律并填下缺少的数字。

1) $1, 2, 3, 4, 5, \_\_, \_\_$

2) $2, 4, 6, \_\_, 10, \_\_, \_\_$

3) $2, 4, 8, \_\_, 32, \_\_, 128$

4) $1, 2, \_\_, 7, 11, \_\_, 22$

5) $1600, \_\_, 400, 200, \_\_, 50, \_\_$

6) $0, 15, \_\_, 0, 15, 30$

✎ **Solution:**

1) Pattern: Addition of one. 規律：下一個是前一個数字加一。

$$1, 2, 3, 4, 5, \underline{\ 6\ }, \underline{\ 7\ }$$

2) Pattern: Addition of two. 規律：下一個是前一個数字加二。

$$2, 4, 6, \underline{\ 8\ }, 10, \underline{\ 12\ }, \underline{\ 14\ }$$

3) Pattern: Doubling each time. 規律：下一個是前一個数字的兩倍。

$$2, 4, 8, \underline{\ 16\ }, 32, \underline{\ 64\ }, 128$$

4) Pattern: Adding one more each time. (Fibonacci Numbers) 規律：相鄰兩個数字的差逐次加一。（斐波那切数列）

$$1, 2, \underline{\ 4\ }, 7, 11, \underline{\ 16\ }, 22$$

5) Pattern: Halving each time. 規律：下一個是前一個数字的一半。

$$1600, \underline{\ 800\ }, 400, 200, \underline{\ 100\ }, 50, \underline{\ 25\ }$$

6) Pattern: Repeated pattern.　　規律：三個數一組重復出現。

$$0, 15, \underline{30}, 0, 15, 30$$

□

1.5.2 Common number sequences 常见的一些数列

An **Arithmetic Sequence** is a sequence that the difference between one term and the next is a constant. The difference between the terms is called common difference.

等差數列：如果一個數列從第二項起，每一項與它的前一項的差等于同一個常數，這個數列就叫做等差數列，而這個常數叫做等差數列的公差。

Proposition 1.12 通项公式

We can write an Arithmetic Sequence as a rule:

$$\text{k-th term} = \text{first term} + \text{Common difference} \times (k-1)$$

$$\text{第幾項} = \text{首項} + \text{公差} \times (\text{項數} - 1)$$

Proposition 1.13 项数公式

The number of terms of the Arithmetic Sequence as can be computed by

$$\text{Number of terms} = (\text{last term} - \text{first term}) \div \text{Common difference} + 1$$

$$\text{項數} = (\text{末項} - \text{首項}) \div \text{公差} + 1$$

Proposition 1.14 Summing an Arithmetic Series 求和公式

To sum up the terms of this arithmetic sequence, use this formula

$$\text{sum} = (\text{fisrt term} + \text{last term}) \times \text{Number of terms} \div 2$$

$$\text{總和} = (\text{首項} + \text{末項}) \times \text{項數} \div 2$$

An **Geometric Sequence** each term is found by multiplying the previous term

等比數列：從第二項起，每一項與前一項的比都是一個常數，這個數列就叫做

by a constant. The difference between the terms is called common ratio.

等比數列，而這個常數叫做等差數列的公比。

The **Fibonacci Sequence** is found by adding the two numbers before it together.

斐波那契數列：數列從第三項開始，每一項都等于前兩項之和。

第 2 章 Decimals and fractions 小数和分数

2.1 Decimals and fractions 小数与分数

2.1.1 Decimals 小数

Decimals is a representation of a non-negative real numbers. The dots in the number is called decimal separator. A decimal separator is a symbol used to separate the integer part from the fractional part of a number written in decimal form.

小數，是實數的一種特殊的表現形式。小數中的圓點叫做小數點，它是一個小數的整數部分和小數部分的分界號。其中整數部分是零的小數叫做純小數，整數部分不是零的小數叫做帶小數。

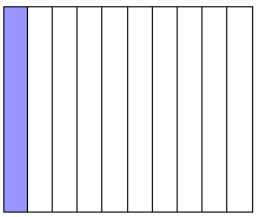
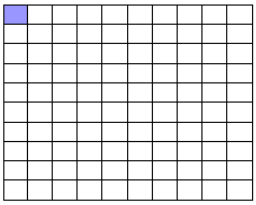
Visual Representation	Decimal Representation
	one tenth 0.1
	one hundredth 0.01

图 2.1: Visual representation of decimals and fractions 小数和分数的圖像表示

Finite decimals: A regular number, also called a finite decimal, is a positive number that has a finite decimal expansion,like 3.1465, 0.364, 8.3218798456,

有限小数： 小数部分後有有限個数位的小数。如 3.1465, 0.364, 8.3218798456 等。

thousands	hundred	ten	ones	tenths	hundredths	thousandths
7	5	5	0	2	6	2

表 2.1: 7550.262: Seven thousand five hundred fifty and two hundred sixty-two thousandths
七千五百五十五又千分之二百六十二

and so on.

Repeating decimals: A repeating decimal,
also called a recurring decimal, is a number whose decimal representation eventually becomes periodic (i.e., the same sequence of digits repeats indefinitely), like $0.\overline{142857} = 0.142857142857\ldots$ and $1.8\overline{3} = 1.833\ldots$. The minimum number of digits that repeats in such a number is known as the decimal period.

循环小数： 從小數部分的某一位起，一個數字或幾個數字，依次不斷地重復出現的小數叫做循環小數。如 $0.\overline{142857} = 0.142857142857\ldots, 1.8\overline{3} = 1.833\ldots$ 等。這一節數字稱爲循環節。

Nonperiodic and nonrepeating decimals:

Nonperiodic and nonrepeating decimal is an irrational number. An irrational number is a number that cannot be expressed as a fraction. For example, the ratio of a circle's circumference to its diameter is $\pi = 3.141592653589793\ldots$. Euler's number is $e = 2.71828\ldots$.

无限不循环小数： 小數部分有無限多個數字，且没有依次不斷地重復出現的一個數字或幾個數字的小數叫做無限不循環小數，如圓周率 $\pi = 3.14159265358979323\ldots$，自然對數的底數 $e = 2.71828182845904\ldots$。

2.1.2 Fractions 分数

A fraction represents a part of a whole or, more generally, any number of equal parts. A fraction (examples: $\frac{1}{2}$ and 17/3) consists

分數可視爲將某件事物平均分成的幾份中占據其中的幾份。分數可以用分式來表示，例如 $\frac{1}{2}$ 和 17/3。其中，中間的綫稱爲分

of an numerator displayed above a line (or before a slash), and a non-zero denominator. The numerator represents a number of equal parts, and the denominator indicates how many of those parts make up a unit or a whole.

數綫。分數綫上的稱爲分子，分數綫下的數稱爲分母。

Irreducible fraction: An irreducible fraction is a fraction in which the numerator and denominator are integers that have no other common divisors than 1.

最简分数（既约分数）： 分子是整數，分母是正整數，且分子和分母互素的分數。

Proper Fraction: The fraction is called proper if the numerator is less than the denominator. In general, a common fraction is said to be a proper fraction if the absolute value of the fraction is strictly less than one.

真分数： 除商小于 1、大于 0 的分數，即分子小于分母的分數。

Top-heavy/Improper Fraction: It is said to be an improper fraction, or sometimes top-heavy fraction,[12] if the absolute value of the fraction is greater than or equal to 1.

假分数： 假分數是指除商不小于 1 的分數，即分子等于或大于分母的分數。

mixed numeral: A mixed numeral is a traditional denotation of the sum of a non-zero integer and a proper fraction. Any such mixed fractions can be converted to an improper fraction. For example, $1\frac{1}{2} = \frac{1\times2+1}{2} = \frac{3}{2}$.

带分数： 一個整數加一個真分數，帶分數可以寫成假分數。例如 $1\frac{1}{2}$，就是一又二分之一，與 $\frac{1\times2+1}{2} = \frac{3}{2}$ 等價。

unit fraction: A unit fraction is a number written as a fraction where the numerator is one and the denominator is a positive integer.

单位分数： 分子爲 1，分母是整數的分數。

Equivalent fractions 约分、扩分及通分

Multiplying the numerator and denominator of a fraction by the same (non-zero) number results in a fraction that is equivalent to the original fraction. This is true because multiplying the numerator and denominator of a fraction by the same number is equivalent to multiplying by one, and any number multiplied by one has the same value as the original number. Dividing the numerator and denominator of a fraction by the same non-zero number will also yield an equivalent fraction. This is called reducing or simplifying the fraction. A common fraction can be reduced to lowest terms by dividing both the numerator and denominator by their greatest common divisor.

一個分數約分後或擴分後，其分數與原來之分數的值相等，稱爲等值分數。“擴分”是將一個分數的分子和分母同乘以比 1 大的數。擴分後的分數和原來分數的值相等。“約分”是將一個分數的分子和分母同除以一個比 1 大的整數（它們的公約數）。約分後的分數和原來分數的值相等。“通分”是利用約分或擴分，將兩個分母不同的分數，分別化爲同分母的分數。

2.1.3 Converting between decimals and fractions 小数与分数的转化

To change a common fraction to a decimal, do a long division of the decimal representations of the numerator by the denominator (this is idiomatically also phrased as ”divide the denominator into the numerator”), and round the answer to the desired accuracy. To change a decimal to a fraction, we need to consider by the different types of decimals.

把分數化爲小數衹需要做一個除法。用分子除以分母，然後取到小數點後相應的位數。根據小數類型的不同，有不同的方法把小數化爲分數：

Convert finite decimals: To change a decimal to a fraction, write in the denominator a 1 followed by as many zeroes as there are digits to the right of the decimal point, and write in the numerator all the digits of the orig-

有限小数化分数： 把有限小數去掉小數點後作爲分子；分母是以“1”開頭，有限小數的小數點後有幾位，就在 1 後跟幾個零。例如：$12.3456 = \frac{123456}{10000}$。

inal decimal, just omitting the decimal point. For example, $12.3456 = \frac{123456}{10000}$.

Converting repeating decimals to fractions

For repeating patterns where the repeating pattern begins immediately after the decimal point, a simple division of the pattern by the same number of nines as numbers it has will suffice. For example, $0.\overline{9} = \frac{9}{9} = 1$, $0.\overline{25} = \frac{25}{99}$, $0.\overline{3} = \frac{3}{9} = \frac{1}{3}$. In case leading zeros precede the pattern, the nines are suffixed by the same number of trailing zeros. In case a non-repeating set of decimals precede the pattern, we can write it as the sum of the non-repeating and repeating parts, respectively: $0.1\overline{3} = 0.1 + 0.0\overline{3} = \frac{3}{30} + \frac{1}{10}\frac{1}{3} = \frac{4}{30} = \frac{2}{15}$.

循环小数化分数： 如果是純循環小數則把循環節作爲分子，循環節如果有一位，分母爲 9；循環節有兩位，分母爲 99；循環節有三位，分母爲 999，依次類推。如 $0.\overline{9} = \frac{9}{9} = 1$，$0.\overline{25} = \frac{25}{99}$，$0.\overline{3} = \frac{3}{9} = \frac{1}{3}$。混循環小數化分數則需要化爲有限小數和純循環小數之和後化簡，如 $0.1\overline{3} = 0.1 + 0.0\overline{3} = \frac{3}{30} + \frac{1}{10}\frac{1}{3} = \frac{4}{30} = \frac{2}{15}$。

Nonperiodic and nonrepeating decimals

Nonperiodic and nonrepeating decimal is an irrational number. Therefore, it cannot be converted to a fraction.

无限不循环小数 無限不循環小數爲無理數，不可以化爲分數。

2.1.4 Comparing and ordering 小数和分数的比较

The method to compare decimals is as follows.

比較兩個小數的方法如下：

Algorithm 算法 2.1 Comparison of decimals 小数的比较方法

1. **Compare the whole numbers to the left of the decimal point.** If they are not the same, the smaller decimal number is the one with the smaller whole number.
2. **Compare the whole number to the right of the decimal point.** Starting from the digit on the right of the decimal point, compare the digits. The smaller digit number is the one with the smaller whole number.

1. 比较整数部分。如果整数部分不同，则整数部分小的数小。
2. 比较小数部分。如果整数部分相同，再比较小数部分。小数部分第一位大的那个数就大；如果第一位上的数相同，就比较第二位上的数……依次比下去。

Comparing fractions with the same denominator yields the same result as comparing the numerators. One way to compare fractions with different numerators and denominators is to find a common denominator. If two positive fractions have the same numerator, then the fraction with the smaller denominator is the larger number.

如果分数的分子相同，则祇需要比較兩個分數的分子，分子大的分數比較大。所以分數通常的比較方法是先通分，使得兩個分數的分母相同，然後比較分子。另外，如果分數的分子相同，則分母大的分數比較小。

Example 2.1: Compare these numbers 比較下面的數字

$$a)2.9,\quad 3.11 \quad b)6.51,\quad 6.38 \quad c)2.7 \quad 2.70 \quad d)\frac{6}{11},\quad \frac{8}{11} \quad e)\frac{6}{5},\quad \frac{8}{7}$$

✎ **Solution:**

	Why? 原因
$2.9 < 3.11$	The numbers to the left of the decimal point of the first number is smaller. 第一個數的整數部分小
$6.51 > 6.38$	The tenth of the first number is greater. 第一個數十分位大
$2.7 = 2.70$	All place value are the same. 兩個數各位都相同。
$\frac{6}{11} < \frac{8}{11}$	The denominators are the same and the numerator of the first number is smaller. 分母相同，第一個數的分子較小
$\frac{6}{5} > \frac{8}{7}$	$\frac{6}{5} = \frac{42}{35} > \frac{8}{7} = \frac{40}{35}$

□

Exercise 2.1:

1. 1

2. 2

3. 3

4. 4

1. 1

2. 2

3. 3

4. 4

2.2 Arithmetic with decimals and fractions 小数和分数的四则运算

2.2.1 Arithmetic with decimals 小数的四则运算

Addition 加法

We assume that you are familiar with the decimal number system and arithmetical operations on integers (whole numbers). Two decimal numbers may be added as follows:

1. The numbers are written one below the other with the decimal points vertically aligned
2. The numbers are added as if they are whole numbers
3. A decimal point is placed in the sum such that it is directly below the decimal points in the numbers being added

小數的加法和整數的加法類似，小數加法的竪式有以下的要點：

1. 列竪式時小數點要對齊；
2. 和整數的加法一樣，從最低位加起，滿十向前一位進一；
3. 添加和的小數點，和的小數點和加項對齊。

Example 2.2: 123.45+37.50

✎ **Solution:**

$$\begin{array}{rcccccc} & & 1 & & & & \\ & 1 & 2 & 3 & . & 4 & 5 \\ + & & 3 & 7 & . & 5 & \\ \hline & 1 & 6 & 0 & . & 9 & 5 \end{array}$$

$123.45 + 37.50 = 160.95.$ □

Subtraction 减法

Subtracting one decimal from the other is similar to addition. Two positive decimal numbers may be subtracted as follows:

1. The numbers are written one below

小數的減法和整數的減法類似，小數減法的竪式有以下的要點：

1. 列竪式時小數點要對齊；

the other with the decimal points vertically aligned

2. Mentally place a zero in any places where only one of the numbers has a digit.
3. The numbers are subtracted as if they are whole numbers
4. A decimal point is placed in the sum such that it is directly below the decimal points in the numbers being subtracted

2. 如果竪式中衹有一個數假設另一個數位零；
3. 和整數的減法一樣，從最低位減起，不够減借一還十；
4. 添加結果的小數點，小數點和減項對齊。

Example 2.3: 123.4-37.45

✎ **Solution:**

$$\begin{array}{rccccc} & 0 & 11 & 12 & . & 13 & \\ & 0 & \not{1} & \not{2} & . & \not{3} & 10 \\ & \not{1} & \not{2} & \not{3} & . & \not{4} & \not{0} \\ - & & 3 & 7 & . & 4 & 5 \\ \hline & & 8 & 5 & . & 9 & 5 \end{array}$$

$123.4 - 37.45 = 85.95.$ □

Multiplication 乘法

Two decimal numbers may be multiplied as follows:

1. The numbers are are written one below the other with the rightmost digit aligned.
2. Starting on the right, multiply each digit in the top number by each digit in the bottom number, just as with whole numbers (ignore the decimal points).
3. Add the products.
4. Place the decimal point in the answer

小數乘法的竪式有以下的要點：

1. 列竪式時數字靠右對齊；
2. 忽略小數點，從乘數的最低位起，乘以被乘數；
3. 把所有乘得的結果相加；
4. 看因數中一共有幾位小數，就從積的右邊數幾位點上小數點。

by starting at the right and moving a number of places equal to the sum of the decimal places in both numbers multiplied.

Example 2.4: 23.45×2.3

✎ **Solution:**

$$\begin{array}{cccccc} & & 2 & 3. & 4 & 5 \\ \times & & & & 2. & 3 \\ \hline & & 7 & 0 & 3 & 5 \\ 4 & 6 & 9 & 0 & & \\ \hline 5 & 3. & 9 & 3 & 5 & \end{array}$$

$23.45 \times 2.3 = 53.935$. □

Division 除法

The way to do the division by a decimal number is as follows:

1. Convert the divisor to a whole number first if it is not by multiplying both numbers by 10, 100, etc. .
2. Ignore the decimal point of the dividend; use Long Division as with whole numbers. If there is remainder, add 0 and continue to do the division until certain accurarcy.
3. Put the decimal point upon the answer above in the same spot as the dividend (number being divided).

小數除法的竪式的方法如下：

1. 列竪式時小數點要對齊；
2. 忽略被除數的小數點，按照整數的除法計算。如果除到被除數的末尾有餘數，就在餘數後面添 0 再繼續除；
3. 商的小數點和被除數的小數點對齊。

Example 2.5: Perform the following division without using a calculator, rounding to 2 decimal places:

$$1.27 \div 0.7$$

✎ **Solution:**

$$1.27 \div 0.7 = 12.7 \div 7$$

$$
\begin{array}{r}
 & & 1. & 8 & 1 & 4 \\
\cline{3-6}
7\big) & 1 & 2. & 7 & & \\
 & & 7 & \downarrow & & \\
\cline{3-6}
 & & 5 & 7 & & \\
 & & 5 & 6 & & \\
\cline{3-6}
 & & & 1 & 0 & \\
 & & & & 7 & \\
\cline{3-6}
 & & & & 3 & 0 \\
 & & & & 2 & 8 \\
\cline{3-6}
 & & & & & 2
\end{array}
$$

$12.7 \div 0.7 \approx 1.81$. □

.

2.2.2 Arithmetic with fractions 分数的四则运算

Addition and subtraction 加減法

Two fractions may be added or subtracted as follows:

分數的加減法如下：

1. Write each fraction using a common denominator
2. Add or subtract the numerators and keep the same denominator
3. Simplify the fractions if needed

1. 通分，將兩個數寫成分母相同的形式；
2. 分子相加或者相減；
3. 如果需要約分的就約分。

Example 2.6: $\frac{1}{4}+\frac{5}{6}$

✎ **Solution:** Method 1: Use the product of the two denominators as a common denominator. 方法一：利用兩個分母的乘積作爲共同的分母。

$$\frac{1}{4}+\frac{5}{6}=\frac{1\times 6}{4\times 6}+\frac{5\times 4}{6\times 4}=\frac{6}{24}+\frac{20}{24}=\frac{26}{24}=\frac{13}{12}.$$

Method2: Use the LCM of the denominators as a common denominator. This method is preferred to Method 1 as, in general, the final solution is further reduced than the solution using Method 1. 方法二：用兩個分母的最小公倍數作爲分母。相比于前一種方法，這樣可以減少一些約分簡化的過程。

$$\frac{1}{4}+\frac{5}{6}=\frac{1\times 3}{4\times 3}+\frac{5\times 2}{6\times 2}=\frac{3}{12}+\frac{10}{12}=\frac{13}{12}.$$

□

Example 2.7: $\frac{5}{6}-\frac{3}{4}$

✎ **Solution:**

$$\frac{5}{6}-\frac{3}{4}=\frac{10}{12}-\frac{9}{12}=\frac{1}{12}.$$

□

Multiplication 乘法

To multiply two fractions we simply multiply their numerators and multiply their denominators, reducing the answer if pos-

分數乘分數, 用分子相乘的積做分子，分母相乘的積做分母, 能約分的要約成最簡分數。分數乘整數時，用分數的分子和

sible. To multiply an integer by a fraction, first write the integer as a fraction and then multiply as usual. Or simply multiply the integer with the numerator.

整數相乘的積做分子，分母不變。（要約成最簡分數）

Example 2.8: $\frac{2}{3} \times \frac{3}{4}$

✎ **Solution:**

$$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12} = \frac{1}{2}.$$

Effectively the answer is reduced to lowest terms during multiplication. 我們也可以在乘的時候約分。

$$\frac{2}{3} \times \frac{3}{4} = \frac{\cancel{2}^{1}}{\cancel{3}^{1}} \times \frac{\cancel{3}^{1}}{\cancel{4}^{2}} = \frac{1 \times 1}{1 \times 2} = \frac{1}{2}.$$

□

Division 除法

To divide a fraction by a whole number, you may either divide the numerator by the number, if it goes evenly into the numerator, or multiply the denominator by the number. To divide a number by a fraction, multiply that number by the reciprocal of that fraction.

如果是分數除以一個整數，我們可以用分子除以這個數作爲分子，也可以用分母乘以這個數做分母。如果是分數除以一個分數等于被除數乘以除數的的倒數。

Example 2.9: $\frac{10}{3} \div 5$

✎ **Solution:**

$$\frac{10}{3} \div 5 = \frac{10 \div 5}{3} = \frac{2}{3}.$$

Or 或者

$$\frac{10}{3} \div 5 = \frac{10}{3 \times 5} = \frac{10}{15} = \frac{2}{3}.$$

□

Example 2.10: $\frac{1}{2} \div \frac{3}{4}$

✎ **Solution:**

$$\frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \times \frac{4}{3} = \frac{1 \times 4}{2 \times 3} = \frac{2}{3}.$$

□

2.2.3 Arithmetic 四则混合运算

Like whole numbers, decimals and fractions obey the commutative, associative, and distributive laws, and the rule against division by zero.

像整數一樣，小數和分數的四則運算滿足交換律結合律和交換律，以及不能除以零。

In the calculation, if the fractions can transfer to finite decimals, do the calculation after transferring all fractions to decimals. Otherwise, do the calculation after transferring all decimals to fractions.

在小數和分數的四則混合運算中，如果分數能化成有限小數，通常把分數化成有限小數後再計算，如果分數不能化成有限小數，就把小數化成分數後再計算。

If there is division by fractions, transfer the division to multiplication first, then do the calculation.

分數乘除混合運算時，一般先把除法變成乘法，然後進行計算。

Example 2.11: $4\frac{3}{4}-0.74+2\frac{3}{5}$

✎ **Solution:**

$$\begin{aligned}&4\frac{3}{4}-0.74+2\frac{3}{5}\\=&4.75-0.74+2.6\\=&4.01+2.6\\=&6.61\end{aligned}$$

□

Example 2.12: $3\frac{2}{3}+4.2-2\frac{1}{7}$

✎ **Solution:**

$$\begin{aligned}&3\frac{2}{3}+4.2-2\frac{1}{7}\\=&3\frac{2}{3}+4\frac{1}{5}-2\frac{1}{7}\\=&7\frac{13}{15}-2\frac{1}{7}\\=&5\frac{76}{105}\end{aligned}$$

□

Example 2.13: $\frac{6}{5}\div\frac{6}{25}\times\frac{1}{4}$

✎ **Solution:**

$$\begin{aligned}&\frac{6}{5}\div\frac{6}{25}\times\frac{1}{4}\\=&\frac{6}{5}\times\frac{25}{6}\times\frac{1}{4}\\=&\frac{6\times25\times1}{5\times6\times4}\\=&\frac{5}{4}\end{aligned}$$

□

第 3 章 Application 应用

3.0.1 Measurement 测度

Time 时间

A clock is a device or instrument for measuring or displaying the current time.

鐘表是測量時間的基本工具。

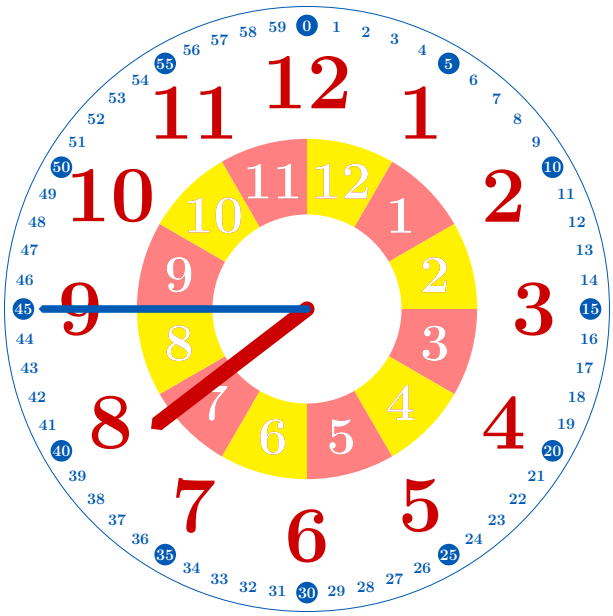


图 3.1: 7:45A.M.

There are 12 numbers on the clock. It seperates to 5 parts between two numbers. The short hand is the hour hand. It moves from one number to the next in 60 minutes. The long hand is the minute hand. It mores from one number to the next in 5 minutes. The minute hand moves from one small mark to the next in 1 minute. Sometimes, there are also second hand on the clock.

鐘表的圓周被 12 個數碼分成 12 個相等的大格。每個大格又分成 5 個相等的小格。時針走 1 個大格是 1 小時；分針走 1 個大格是 5 分鐘，1 個小格是 1 分鐘。有時鐘表上還會有秒針。

The relationship between different time units is as follows.

不同的時間單位換算如下：

60 seconds	= 1 minute,	60 秒	= 1 分
60 minutes	= 1 hour,	60 分	= 1 小時
24 hours	= 1 day,	24 小時	= 1 天
7 days	= 1 week,	7 天	= 1 周 (1 星期)
12 months	= 1 year,	12 個月	= 1 年
365 days	= 1 year,	365 天	= 1 年 (平年)
366 days	= 1 leap year,	365 天	= 1 潤年
10 years	= 1 decade,	10 年	= 1 十年
100 years	= 1 century,	100 年	= 1 世紀

Length 长度

The metric units of length are as follows.

不同的長度單位換算如下：

10 mm (millimeter)	= 1 cm (centimeter),	10 毫米	= 1 厘米
10 cm (centimeter)	= 1 dm (decimeter),	10 厘米	= 1 分米
10 dm(decimeter)	= 1 m (meter),	10 分米	= 1 米
1000 m (meter)	= 1 km (kilometer),	1000 米	= 1 千米 (公裏)
1 ft. (Foot)	= 12 in. (inch),	1 英尺	= 12 英寸
1 in. (inch)	= 2.54 cm (centimeter),	1 英寸	= 2.54 厘米
1 ft. (Foot)	= 0.3048 m (meter),	1 英尺	= 0.3048 米
3 ft. (Foot)	= 1 yd (yard),	3 英尺	= 1 碼
1 mi (Mile)	= 1760 yd (yard),	1 英裏	= 1760 碼

Weight 重量

The metric units of weight are as follows.

不同的重量單位換算如下：

1000 g (gram)	= 1 kg (kilogram),	1000 克	= 1 千克 (公斤)
1000 kg (kilogram)	= 1 T(tonne),	1000 千克	= 1 噸
28.3495 g (gram)	= 1 oz (ounce),	28.3495 克	= 1 盎司 (安士)
16 oz (ounce)	= 1 lb (pound),	16 盎司	= 1 磅
2000 lb (pound)	= 1 ton,	2000 磅	= 1 英噸

Volumn 体积

The metric units of volumn are as follows.

不同的體積單位換算如下：

1000 mL (milliliter)	= 1 L (liter),	1000 毫升	= 1 升
237 mL (milliliter)	= 1 c. (cup),	237 毫升	= 1 杯
2 c. (cup)	= 1 pt. (pint),	2 杯	= 1 品脱
2 pt. (pint)	= 1 qt. (quart),	2 品脱	=1 誇脱
4 qt. (quart)	= 1 gal. (gallon),	4 誇脱	= 1 加侖

Temperature 温度

Temperature is a physical quantity expressing hot and cold. Temperature is measured with a thermometer, historically calibrated in various temperature scales and units of measurement. The most commonly used scales are the Celsius scale, denoted in °C(informally, degrees centigrade), the Fahrenheit scale (°F), and the Kelvin scale.

温度是表示物體冷熱程度的物理量，而用來量度物體温度數值的標尺叫温度計。目前國際上用得較多的温標有攝氏温標（°C( )、華氏温標（°F）和熱力學温標（K）。

$$[°\mathrm{F}] = [°\mathrm{C}] \times \frac{9}{5} + 32, \quad [°\mathrm{C}] = ([°\mathrm{F}] - 32) \times \frac{5}{9}.$$

$$[K] = [°\mathrm{C}] + 273.15, \quad [°\mathrm{C}] = [K] - 273.15.$$

3.0.2 Basic concept of statistics 统计基础知识

Statistics is a branch of mathematics dealing with the collection, analysis, interpretation, presentation, and organization of data. ”Data” usually refer to a specific group of numbers. Sometimes, data can also refer to information as a general concept rather than to specific collections of numbers. One way to provide information about a list of numbers is to use a single number to describe some feature of the numbers in the list. Here are some basic ways to choose that representative value.

統計學是一種科學的数學分支，是關于收集、分析、解釋、陳述數據的科學。数據通常指的是某一組数字，有時我們也用数據這個詞指一個整體的概念。一種體現一組数據的特點的方式是找到一個数字來表達。下面就是幾個特殊的表達：

- **Average:** The average of a group of numbers is the sum of the numbers divided by the number of numbers in the group. The average is also called the mean or the arithmetic mean.
- **平均数：** 平均数是一組数的和除以這組数的数量。平均数又可成爲算術平均数或均值。
- **Median:** The median is the value separating the higher half of a group of numers from the lower half. The median of a finite list of numbers can be found by arranging all the numbers from smallest to greatest. If there is an odd number of numbers, the middle one is picked. If there is an even number of observations, the median is then usually defined to be the mean of the two middle values.
- **中位数：** 中位数是可將一組数劃分爲相等的上下兩部分。對于有限的数集，可以通過把所有觀察值高低排序後找出正中間的一個作爲中位数。如果觀察值有偶数個，通常取最中間的兩個数值的平均数作爲中位数。
- **Mode:** The mode of a set of data values is the value that appears most often.
- **众数：** 衆数指一組数中出現次数最多的数值。

Example 3.1: Suppose the following are your grades on tests: 下面是你多次考試得

到的成績：

$$70, 84, 91, 100, 96, 94, 60, 80, 96, 97$$

1. What is the average of your test scores? 平均成績是多少分?
2. What is the median of your test scores? 成績的中位數是多少?
3. What is the mode of your test scores? 成績的衆數是多少?

✎ **Solution:** 1. To find the average, we divide the sum of the scores y the number of scores: 計算平均值，用所有分數的和除以成績的個數。

$$\frac{70+84+91+100+96+94+60+80+96+97}{10} = \frac{868}{10} = 86.8$$

2. We list the grades in order: 首先我們把所有分數排序：

$$60, 70, 80, 84, 91, 94, 96, 96, 97, 100.$$

The median is 所以中位數爲：

$$\frac{91+94}{2} = 92.5.$$

3. The grade 96 occurs twice, and other grade only occurs once. Therefore, 96 is the mode. 分數 96 出現了兩次，而其他的分數衹出現了一次，所以衆數爲 96。 □

3.1 Arithmetic in geometry 几何中的计算

3.1.1 Primeter 周长

The **perimeter** of a closed figure is the total length of its boundary. The general way to find the perimeter of any shape is to add up the length of all its sides. For certain shapes, such as rectangles and circles, there are specific formulas you can use to simplify the process.

周長指一塊封閉的圖形邊緣長度的總和。通常計算周長的方法是把圖形每條邊長加在一起求和，即爲周長。對于一些特殊的形狀，例如長方形，正方形，圓形，根據其特點可以有計算周長的公式。我們也可以利用這些公式來幫助我們計算一些圖形的周長。

Proposition 3.1 The perimeter of rectangles 长方形的周长

The perimeter of rectangles can be calculated by

$$\textbf{Perimeter} = 2 \times (\textbf{width} + \textbf{height})$$

長方形周長可以用以下公式計算：

$$\text{周长} = 2 \times (\text{宽} + \text{高})$$

Proposition 3.2 The perimeter of squares 正方形的周长

The perimeter of squares can be calculated by

$$\textbf{Perimeter} = 4 \times \textbf{side}$$

正方形周長可以用以下公式計算：

$$\text{周长} = 4 \times \text{边长}$$

Proposition 3.3 The perimeter of circle 圆形的周长

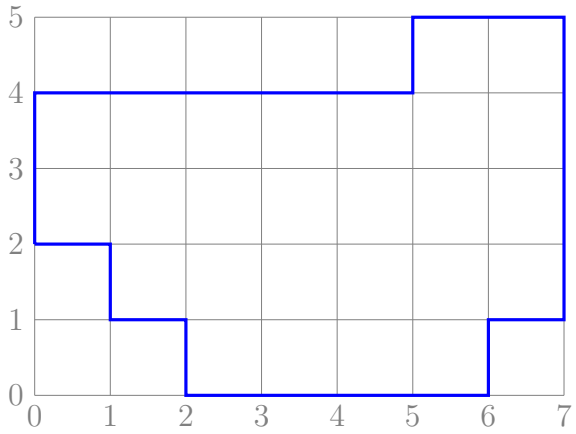
The perimeter of circle can be calculated by

$$\textbf{Perimeter} = 2 \times \pi \times \textbf{radius}$$

圓形周長可以用以下公式計算：

$$\text{周长} = 2 \times \pi \times \text{半径}$$

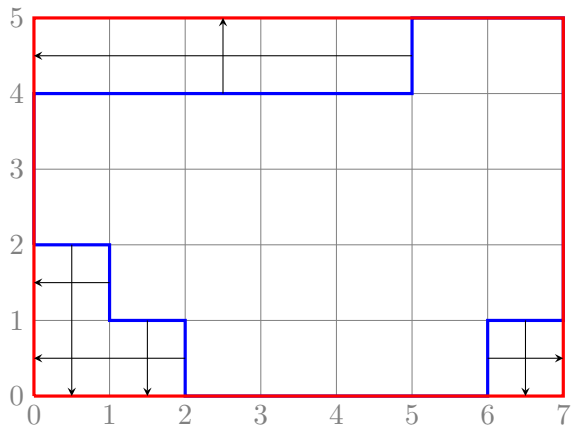
Example 3.2: Farmer Fred wants to fence the region shown in bold in the graph. If each of the squares shown has sides taht are 10 feet long, and the fence costs $ 7 per foot, then how much will Fred's fence cost?



✎ **Solution:**

We could figure out the perimeter by adding up all of the edges. But there is a more clever solution.

我們可以把多邊形每段邊加起來。但我們還有一種比較簡便的方式來求周長：把求這個多邊形的周長就轉化爲求一個正方形的周長，這個多邊形的周長就可以巧妙地求出來了。



$$2 \times (5 + 7) = 24.$$

Since each of the 24 side lengths is 10 feet. The permeter is $24 \times 10 = 240(feet)$. The fence cost $ 7, so Fred's fence costs $240 \times 7 = \$1680$.

因爲每段的長度是 10 英尺，所以周長是 $24 \times 10 = 240$(英尺). 每英尺圍欄價格爲 $ 7, 所以 Fred 建造圍欄的總花費爲：$240 \times 7 = \$1680$。

□

3.1.2 Area 面积

第 4 章 Geometry 几何

参考文献

[1] “How to do long division,” Feb 2018.