Best-of-Both-Worlds Fairness in Committee Voting

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Abstract

The best-of-both-worlds paradigm advocates an approach that achieves desirable properties both ex-ante and ex-post. We initiate a best-of-both-worlds fairness perspective for the important social choice setting of approval-based committee voting. To this end, we formalize a hierarchy of ex-ante properties including Individual Fair Share (IFS) and its strengthening Group Fair Share (GFS). We establish their relations with well-studied ex-post concepts such as extended justified representation (EJR) and proportional justified representation (PJR). Our central result is a polynomial-time algorithm that simultaneously satisfies ex-post EJR and ex-ante GFS. Our algorithm uses as a subroutine the first phase of the well-known Method of Equal Shares class of rules.

1 Introduction

Fairness is one of the central concerns when aggregating the preferences of multiple agents. Just as envy-freeness is viewed as a central fairness goal when allocating resources among agents [Foley, 1967; Moulin, 2019], proportional representation is the key fairness desideratum when making collective choice such as selecting a set of alternatives [Lackner and Skowron, 2023]. However, in both contexts, fairness is often too hard to achieve perfectly as an outcome satisfying their respective fairness notions may not exist.

Two successful approaches to counter the challenge of non-existence of fair outcomes are *relaxation* and *randomization*. The idea of relaxation is to weaken the ideal notion of fairness enough to get meaningful and guaranteed existence of fair outcomes. In the resource allocation context, a widely pursued relaxation of envy-freeness is envy-freeness up to one item (EF1) [Lipton et al., 2004; Budish, 2011; Caragiannis et al., 2019]. In the social choice context of approval-based committee voting, *core* is viewed as the strongest proportional representation concept. Since it is not known whether a core stable outcome is guaranteed to exist, researchers have focused on natural relaxations of the core that are based on the idea of *justified representation* [Aziz et al., 2018].

The second approach to achieve fairness is via randomization that specifies a probability distribution (or *lottery*) over ex-post integral outcomes. Randomization is one of the oldest tools to achieve fairness and has been applied to contexts of resource allocation [Bogomolnaia and Moulin, 2001] and collective choice [Bogomolnaia et al., 2005].

In most of the work on resource allocation and social choice, the two major approaches of relaxation and randomization are pursued separately. The concerted focus on pursuing both approaches simultaneously to achieve good fairness guarantees is a recent phenomenon [Aziz, 2019; Freeman et al., 2020]. The paradigm has been referred to as *best-of-both-worlds fairness*.

While the best-of-both-worlds fairness paradigm has been applied to resource allocation, it has not been explored as much in the contexts of social choice and public decision-making. Furthermore, this perspective has never been taken with respect to approval-based committee voting, which is the setting considered in this paper.

In approval-based committee voting, each voter approves of a subset of candidates. Based on these expressed approvals, a committee (i.e., a subset of candidates) of a target size is selected. Almost all of the papers on fairness in approval-based committee voting focus on ex-post fairness guarantees such as *justified representation (JR)*, *proportional justified representation (PJR)* [Sánchez-Fernández et al., 2017], and *extended justified representation (EJR)* [Aziz et al., 2018]. For instance, EJR says that for each positive integer ℓ, if a group of at least $\ell \cdot n/k$ voters approve at least ℓ common candidates, some voter in the group must have at least ℓ approved candidates in the committee. To the best of our knowledge, the only work that uses randomization to obtain ex-ante fairness in approval-based committee voting is that of Cheng et al. [2020]. However, despite studying outcomes which are lotteries over integral outcomes, this work ignores ex-post fairness guarantees.

We initiate a best-of-both-worlds fairness perspective in the context of social choice, particularly approval-based committee voting. We first motivate our approach by noting that without randomization, one cannot guarantee that each voter get strictly positive expected representation, and thus fails a property known as *positive share*.[1] In this paper, we will define a hierarchy of ex-ante fairness properties stronger

[1] This can be seen from a simple example with a target committee size of one and two voters with disjoint approvals.

than positive share and give a class of randomized algorithms which achieve these properties in addition to the existing ex-post fairness properties.

1.1 Our Contributions

Our first contribution is to broaden the best-of-both-worlds fairness paradigm, which has so far been limited to resource allocation, and explore it in the context of social choice problems, specifically committee voting. Whereas the relaxed notions of ex-post fairness have been examined at great length for committee voting, the literature on ex-ante fairness is less developed.

In Section 3, we formalize several natural axioms for ex-ante fairness that are careful extensions of similar concepts proposed in the restricted setting of single-winner voting. These include the following concepts in increasing order of strength: *positive share*, *individual fair share (IFS)*, *unanimous fair share (UFS)*, and *group fair share (GFS)*. For instance, positive share simply requires that each voter expects to have non-zero probability of having some approved candidate selected. At the other end of the spectrum, GFS gives a desirable level of ex-ante representation to *every* coalition of voters. In addition, we provide logical relations between fairness axioms for fractional and integral committees.

In line with the goals of the best-of-both-worlds fairness paradigm, our central research question is to understand which combinations of ex-ante and ex-post fairness properties can be achieved simultaneously. In Section 4, we show that both EJR (one of the strongest known ex-post fairness notions) and GFS (one of the strongest ex-ante fairness notions) can be achieved simultaneously by devising a class of randomized algorithms. Our class of algorithms, which we call *Best-of-Both-Worlds MES* (*BW-MES*), uses as a subroutine the first phase of the well-known Method of Equal Shares (MES) class of rules [Peters and Skowron, 2020]. Lastly, we show that an outcome satisfying GFS and EJR simultaneously can be computed efficiently by giving an algorithm belonging to BW-MES which runs in polynomial time. More discussions and results can be found in the full version of our paper [Aziz et al., 2023c].

1.2 Related Work

In this paper, we examine approval-based committee voting, a generalization of the classical voting setting which has been studied at length, particularly from the 19th century to the present. One of the persistent questions within the committee voting setting is how to produce committees which proportionally represent groups of voters. Aziz et al. [2017] initiated an axiomatic study of approval-based committee voting based on the idea of "justified representation" for cohesive groups. The study has led to a hierarchy of axioms and a large body of work focusing on voting rules which produce committees satisfying these axioms and thus give some guarantee of fair representation [Aziz et al., 2018; Elkind et al., 2022; Brill et al., 2023]. For detailed survey of the recent work on approval-based committee voting, we refer the readers to the book of Lackner and Skowron [2023]. While this paper also targets committees satisfying these properties, we examine outcomes that are randomized committees which specify a probability distribution over integral committees.

In social choice theory, randomization is one of the oldest tools used to achieve stronger fairness properties and to bypass various impossibility results which apply to discrete outcomes [e.g., Gibbard, 1977]. For single-winner randomized voting (also known as *probabilistic voting*) with approval preferences, Bogomolnaia et al. [2005] defined ex-ante fairness notions, the individual fair share (IFS) and unanimous fair share (UFS), and provide rules satisfying them. They also proposed a group fairness property called group fair share (GFS) [Bogomolnaia et al., 2002], independently proposed by Duddy [2015], which is stronger than UFS and IFS but weaker than core fair share (CFS), a group fairness and stability property inspired by that of *core* from cooperative game theory [Scarf, 1967]. In a setting which generalizes probabilistic voting by allowing arbitrary endowments, Brandl et al. [2021] studied fair distribution rules and introduce GFS, which they show is equivalent to a notion called decomposability. None of their results imply those presented in this paper since their setting places no restriction on the distribution a single alternative can receive while our setting does.

Michorzewski et al. [2020] explored the trade-off between group fairness and utilitarian social welfare by measuring the "price of fairness" with respect to fairness axioms such as IFS and GFS. We formulate these ex-ante fairness properties for the more general committee voting setting for the first time. As mentioned, we search for outcomes which also give fair representation to groups ex-post, a desideratum which has no analogue in the classical voting setting.

Aziz [2019] proposed research directions regarding probabilistic decision making with desirable ex-ante and ex-post stability or fairness properties. Freeman et al. [2020] were the first to coin term of best-of-both-worlds fairness. They examined the compatibility of achieving ex-ante envy-freeness and ex-post near envy-freeness in the context of resource allocation. There have been several recent works on best-of-both-worlds fairness in resource allocation [Halpern et al., 2020; Babaioff et al., 2021, 2022; Aziz et al., 2023a,b; Hoefer et al., 2023; Feldman et al., 2023]. Other works that consider the problem of implementing a fractional allocation over deterministic allocations subject to constraints include [Budish et al., 2013; Akbarpour and Nikzad, 2020].

2 Preliminaries

For any positive integer $t \in \mathbb{N}$, let $[t] := \{1, 2, \ldots, t\}$. Let $C = [m]$ be the set of *candidates* (also called *alternatives*). Let $N = [n]$ be the set of voters. We assume that the voters have *approval* preferences (also known as *dichotomous* or *binary*), that is, each voter $i \in N$ approves a non-empty ballot $A_i \subseteq C$. We denote by N_c the set of voters who approve of candidate c, i.e., $N_c := \{i \in N \mid c \in A_i\}$. An *instance* I can be described by a set of candidates C, a list of ballots $\mathcal{A} = (A_1, A_2, \ldots, A_n)$, and a positive committee size $k \leq m$ which is an integer.

Integral and Fractional Committees As is standard in committee voting, an *(integral) winning committee* W is a subset of C having size k. A *fractional committee* is specified by an m-dimensional vector $\vec{p} = (p_c)_{c \in C}$ with $p_c \in [0, 1]$

for each $c \in C$, and $\sum_{c \in C} p_c = k$. Note an integral committee W can be alternatively represented by the vector $\vec{p}$ in which $p_c = 1$ for all $c \in W$ and $p_c = 0$ otherwise. For notational convenience, let $\vec{1}_W \in \{0,1\}^m$, whose j^{th} component is 1 if and only if $j \in W$, be the vector representation of an integral committee W. The utility of voter $i \in N$ for a (fractional or integral) committee $\vec{p}$ is given by $u_i(\vec{p}) \coloneqq \sum_{c \in A_i} p_c$.

Randomized Committees A *randomized committee* $\mathbf{X}$ is a lottery over integral committees and specified by a set of $s \in \mathbb{N}$ tuples $\{(\lambda_j, W_j)\}_{j \in [s]}$ with $\sum_j \lambda_j = 1$, where for each $j \in [s]$, the integral committee $W_j \subseteq C$ is selected with probability $\lambda_j \in [0,1]$. The *support* of $\mathbf{X}$ is the set of integral committees $\{W_1, W_2, \ldots, W_s\}$. Unless specified otherwise, when we simply say "a committee", it will mean an integral committee.

A randomized committee $\{(\lambda_j, W_j)\}_{j \in [s]}$ is an *implementation* of (or "implements") a fractional committee $\vec{p}$ if $\vec{p} = \sum_{j \in [s]} \lambda_j \vec{1}_{W_j}$. Note that there may exist many implementations of any given fractional committee.

The fact that any fractional committee can be implemented by a probability distribution over integral committees of the same size is implied by various works on randomized rounding schemes in combinatorial optimization [Srinivasan, 2001; Grimmet, 2004; Gandhi et al., 2006; Aziz et al., 2019]. We explain this connection explicitly using the classical result of Gandhi et al. [2006]. Theorem 2.3 of Gandhi et al. [2006] states that there is a polynomial-time rounding scheme that satisfies three properties. Framed in our context, the first one ensures that the randomized committee is a valid implementation of the fractional committee. The second property ensures that each committee in the support of the implementation are of size k. We do not need the third property for our purposes.

2.1 Fairness for Integral Committees

Fairness properties for integral committees are well-studied in committee voting. A desideratum that has received significant attention is *justified representation (JR)* [Aziz et al., 2017]. In order to reason about JR and its strengthenings, an important concept is that of a cohesive group. For any positive integer ℓ, a set of voters $N^* \subseteq N$ is said to be *ℓ-cohesive* if $|N^*| \geq \ell \cdot n/k$ and $\left|\bigcap_{i \in N^*} A_i\right| \geq \ell$.

Definition 2.1 (JR)**.** A committee W is said to satisfy *justified representation (JR)* if for every 1-cohesive group of voters $N^* \subseteq N$, it holds that $A_j \cap W \neq \emptyset$ for some $j \in N^*$.

Two important strengthenings of JR have been proposed.

Definition 2.2 (PJR [Sánchez-Fernández et al., 2017])**.** A committee W is said to satisfy *proportional justified representation (PJR)* if for every positive integer ℓ and every ℓ-cohesive group of voters $N^* \subseteq N$, it holds that $\left|\left(\bigcup_{i \in N^*} A_i\right) \cap W\right| \geq \ell$.

Definition 2.3 (EJR [Aziz et al., 2017])**.** A committee W is said to satisfy *extended justified representation (EJR)* if for every positive integer ℓ and every ℓ-cohesive group of voters $N^* \subseteq N$, it holds that $|A_j \cap W| \geq \ell$ for some $j \in N^*$.

It follows from the definitions that EJR implies PJR, which in turn implies JR. A committee providing EJR (and therefore PJR / JR) always exists and can be computed in polynomial time [Aziz et al., 2017; Peters and Skowron, 2020].

3 Fairness for Fractional Committees

In this section, we first lay out fairness properties for fractional committees, followed by establishing the relations between the introduced fractional fairness notions and those integral fairness notions presented in Section 2.

3.1 Fairness Concepts

Whereas the literature on fairness concepts for integral committees is very well-developed, fairness properties for fractional committees are largely unexplored except for the special case of single-winner voting [Bogomolnaia et al., 2005; Duddy, 2015; Aziz et al., 2020]. We introduce a hierarchy of fairness notions for fractional committees in the committee voting setting by generalizing axioms from the single-winner context based around *fair share*. The weakest in the hierarchy of axioms is *individual fair share (IFS)*, the idea behind which is that "each one of the n voters 'owns' a $1/n$-th share of decision power, so she can ensure an outcome she likes with probability at least $1/n$", as Aziz et al. [2020, page 18:2] put it. This idea volunteers at least two distinct interpretations of the utility lower bound guaranteed by IFS:

(a) each voter is given $1/n$ probability to choose their favourite integral outcome, or

(b) each voter can select $1/n$ of the (fractional) outcome.

In probabilistic voting, both interpretations coincide. Critically, this is not the case in the committee voting setting. Instead, these interpretations diverge and lead to two alternative hierarchies of fair share axioms for committee voting, which we term *fair share* and *strong fair share*, respectively.

We begin by defining both generalizations of individual fair share (IFS). Both impose a natural lower bound on individual utilities stronger than that of *positive share*, which requires that $u_i(\vec{p}) > 0$. In the single-winner setting, IFS requires the probability that the (single) alternative selected is approved by any individual voter is no less than $1/n$. It is thus tempting to require $u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \frac{k}{n}$, which turns out to be too strong in our setting as a fractional committee satisfying it may not exist.[2] Intuitively speaking, this is because our only restriction on the voters' approval sets is that each voter approves of at least one candidate, just as is standard in the single-winner literature. However, whereas in the $k = 1$ special case this assumption is sufficient to ensure that a uniform cut-off utility lower bound for each voter is feasible, the same is not true for general k.

Definition 3.1 (IFS)**.** A fractional committee $\vec{p}$ satisfies *IFS* if for each $i \in N$,

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \frac{1}{n} \cdot \min\{k, |A_i|\}.$$

[2]For instance, let $k > n$ and consider the case where voter i only approves a single candidate. Then, the above inequality cannot hold for i as the left-hand side is upper bounded by $|A_i| = 1$ while the right-hand side is greater than one and can be arbitrarily large.

While IFS captures interpretation (a) of fair share, Strong IFS reflects interpretation (b) which says that each voter should control $1/n$ of the fractional outcome.

Definition 3.2 (Strong IFS)**.** A fractional committee $\vec{p}$ satisfies *Strong IFS* if for each $i \in N$,

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \min\left\{\frac{k}{n}, |A_i|\right\}.$$

Next, we strengthen IFS (resp., Strong IFS) to *unanimous fair share (UFS)* (resp., *Strong UFS*), which guarantees any group of like-minded voters an influence proportional to its size.

Definition 3.3 (UFS)**.** A fractional committee $\vec{p}$ is said to provide *UFS* if for any $S \subseteq N$ where $A_i = A_j$ for any $i, j \in S$, then the following holds for each $i \in S$:

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \frac{|S|}{n} \cdot \min\{k, |A_i|\}.$$

Definition 3.4 (Strong UFS)**.** A fractional committee $\vec{p}$ is said to provide *Strong UFS* if for any $S \subseteq N$ where $A_i = A_j$ for any $i, j \in S$, then the following holds for each $i \in S$:

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \min\left\{|S| \cdot \frac{k}{n}, |A_i|\right\}.$$

Our focus in this paper is a stronger notion—*group fair share (GFS)*—which gives a non-trivial ex-ante representation guarantee to *every* coalition of voters.

Definition 3.5 (GFS)**.** A fractional committee $\vec{p}$ is said to provide *GFS* if the following holds for every $S \subseteq N$:

$$\sum_{c \in \bigcup_{i \in S} A_i} p_c \geq \frac{1}{n} \cdot \sum_{i \in S} \min\{k, |A_i|\}.$$

We note that a GFS fractional committee always exists and can be computed by a very natural algorithm called *Random Dictator*, which selects each voter's favourite integral committee (breaking ties arbitrarily) with probability $1/n$.

Proposition 3.6. *Random Dictator computes a randomized committee that is ex-ante GFS in polynomial time.*

Proof. First, it is clear that Random Dictator runs in polynomial time. Let $\{(\frac{1}{n}, W_i)\}_{i \in N}$ be the randomized committee returned by Random Dictator for an instance of our problem. Let $\vec{p}$ be the fractional committee it implements. Note that $p_c = \sum_{i \in N} \frac{1}{n} \cdot \mathbb{1}_{\{c \in W_i\}}$ for all $c \in C$, where $\mathbb{1}_{\{\cdot\}}$ is an indicator function.

Fix any $S \subseteq N$. Substituting to the LHS of the GFS guarantee, we get

$$\begin{aligned}\sum_{c \in \bigcup_{i \in S} A_i} p_c &= \sum_{c \in \bigcup_{i \in S} A_i} \Big(\sum_{j \in N} \frac{1}{n} \cdot \mathbb{1}_{\{c \in W_j\}}\Big) \\ &\geq \frac{1}{n} \cdot \sum_{j \in S} \sum_{c \in \bigcup_{i \in S} A_i} \mathbb{1}_{\{c \in W_j\}} \\ &= \frac{1}{n} \cdot \sum_{j \in S} \Big|W_j \cap \bigcup_{i \in S} A_i\Big| \geq \frac{1}{n} \cdot \sum_{j \in S} \min\{k, |A_i|\},\end{aligned}$$

where the last transition holds as W_j is one of j's most preferred committees by the definition of Random Dictator. $\square$

However, Random Dictator does not satisfy Strong IFS.[3] Indeed, this is the principal reason we chose to name the respective axiom hierarchies as we did. There is significant precedent to treat Random Dictator as the utility lower bound for fair share axioms, including by the authors [Bogomolnaia et al., 2005, page 167] who introduced fair share:

> *Fair welfare share uses the random dictator mechanism as the disagreement option that each participant is entitled to enforce.*

Furthermore, the natural extensions of Strong UFS to Strong GFS are not guaranteed to exist. For instance, following [Bogomolnaia et al., 2005; Brandl et al., 2021] and our own Definition 3.5, we may be tempted to formulate the RHS of Strong GFS as the sum of the Strong IFS guarantees, i.e., $\sum_{i \in S} \min\left\{\frac{k}{n}, |A_i|\right\}$. However, as Example 3.7 will show, a fractional committee satisfying this notion may not always exist.[4] Another natural generalization would be:

$$\sum_{c \in \bigcup_{i \in S} A_i} p_c \geq \min\left\{\frac{|S|k}{n}, \left|\bigcup_{i \in S} A_i\right|\right\} \qquad (1)$$

Equation (1) captures the spirit of strong fair share well by affording each coalition of voters control over the outcome proportional to their size, upper bounded by the number of candidates they collectively approve. Example 3.7 shows the formulation of Strong GFS given by Equation (1) is also impossible to satisfy.

Example 3.7. *Consider an instance with $n = 4$, $k = 4$, and the following approval sets:*

$$A_1 = A_2 = \{c_1\} \quad A_3 = \{c_1, c_2, c_3\} \quad A_4 = \{c_1, c_4, c_5\}.$$

For the group $T = \{1, 2, 3\}$, Equation (1) requires that

$$\sum_{c \in \bigcup_{i \in T} A_i} p_c \geq \min\left\{\frac{|T|k}{n}, \left|\bigcup_{i \in T} A_i\right|\right\} = 3.$$

This means that each candidate in A_3 must receive probability one. By symmetry, the same for the group $\{1, 2, 4\}$ and thus A_4. However, since $|A_3 \cup A_4| = 5$ and $k = 4$, this is an impossibility.

It follows directly from the definitions that GFS implies UFS, which in turn implies IFS, and that each of our generalizations of IFS, UFS, and GFS correspond to their definitions in the single-winner voting scenarios. The relations between these axioms are pictured in Figure 1. We would also like to point out that Strong IFS (or Strong UFS) and GFS do not imply each other. First, recall that Random Dictator satisfies GFS but not Strong IFS, let alone Strong UFS. The following example shows that Strong UFS (and thus Strong IFS) does

[3] To see this, consider an instance with $k = 2$, three candidates $\{c_1, c_2, c_3\}$, and two voters with $A_1 = \{c_1\}$ and $A_2 = \{c_2, c_3\}$. Since each voter must select an integral committee, voter 1 allocates some of her probability to a candidate she does not approve, and thus $\sum_{c \in A_1} p_c = p_{c_1} = 1/2 < \min\left\{\frac{k}{n}, |A_1|\right\} = 1$.

[4] This demonstrates that the setting of Brandl et al. [2021] deviates from our setting as Algorithm 1 outputs a decomposable outcome satisfying Strong IFS, but Strong GFS is not satisfied.

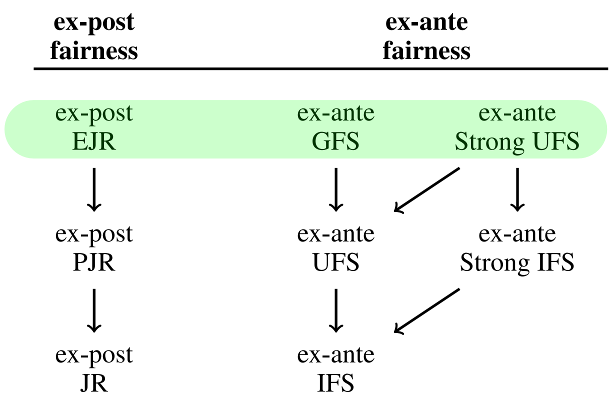

Figure 1: Visualization of ex-ante and ex-post fairness hierarchies studied in this paper. An arrow from (A) to (B) denotes that (A) implies (B). The properties in green are simultaneously satisfied by our algorithm.

not imply GFS: Consider an instance with $n = 3$, $k = 2$, and approval preferences $A_1 = \{c_1, c_2\}$, $A_2 = \{c_1, c_3\}$, and $A_3 = \{c_4, c_5\}$. Observe that the fractional committee $\vec{p} = \left(\frac{2}{3}, 0, 0, 1, \frac{1}{3}\right)$ satisfies Strong UFS. However, $\vec{p}$ does not satisfy GFS with respect to the group $S = \{1, 2\}$ since $\frac{2}{3} = \sum_{c \in \bigcup_{i \in S} A_i} p_c < \sum_{i \in S} \frac{1}{n} \cdot \min\{k, |A_i|\} = \frac{4}{3}$.

3.2 Relations between Fractional and Integral Fairness Concepts

Before describing and proving our approach to best-of-both-worlds fairness in this setting, we first investigate the logical relations between our ex-ante and ex-post properties for integral committees. In doing so, we begin to build skepticism toward some naive approaches to our problem of interest and illustrate the usefulness of our ex-ante properties. We begin by remarking that, as mentioned in Section 1, there may not exist an integral committee satisfying any of our ex-ante fairness properties.

Remark 3.8. An integral committee satisfying positive share may not exist.

As mentioned, this fact is the principal motivation for studying randomized committees. However, we would also like to understand what our fairness concepts for fractional committees can tell us about the space of integral committees satisfying our ex-post fairness properties. The following example and summarizing remark show that our fractional fairness concepts can help in reasoning about which outcomes satisfying our ex-post properties are more desirable.

Example 3.9. *Consider an instance with ten voters and a desired committee size of four. Suppose eight of the voters approve of candidates $\{a, b, c, d\}$ and the remaining two voters approve of candidates $\{e, f, g, h\}$.*

Note that the committee $W = \{a, b, c, d\}$ satisfies EJR. However, the two voters approving $\{e, f, g, h\}$ do not approve of any candidate in W, violating positive share. The alternative committee of $\{a, b, c, e\}$ also satisfies EJR, and additionally satisfies IFS.

Remark 3.10. As shown by Example 3.9, even when an integral committee satisfying IFS exists, some EJR outcomes may not satisfy positive share.

From this, we conclude that a successful algorithm must select carefully from the space of outcomes satisfying our ex-post properties. We next explore to what extent our ex-ante properties imply our ex-post properties in the integral case.

Proposition 3.11. *If an integral committee satisfies IFS, then it satisfies JR.*

Proof. Let W be an integral committee which satisfies IFS and let $\vec{p} = \vec{1}_W$. Then, for all $i \in N$, we have

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c = |A_i \cap W| \geq \frac{\min(|A_i|, k)}{n} > 0.$$

Thus, since $|A_i \cap W|$ is an integer, $|A_i \cap W| \geq 1$ for all $i \in N$ and it follows that W is JR. □

While Proposition 3.11 hints at a synergy between our ex-ante and ex-post properties, Proposition 3.12 below shows that even the strongest ex-ante property in our hierarchy does not imply the next strongest ex-post property.

Proposition 3.12. *If an integral committee satisfies GFS, it does not necessarily satisfy PJR.*

Proof. Consider an instance with $k = 4$ and $n = 2$ and the following approval profile:

$$A_1 = \{a, b\} \qquad A_2 = \{b, c, d, e\}.$$

The committee $W = \{b, c, d, e\}$ satisfies GFS as $|W \cap A_1| = 1 = \frac{1}{n} \cdot \min\{k, |A_1|\}$ (and voter 2 receives their most preferred committee). Now see that $\{1\}$ is 2-cohesive; however, $|A_1 \cap W| < 2$, meaning that W does not satisfy PJR. □

Despite this negative finding, in the following section, we will present a class of algorithms for randomized committees which simultaneously satisfy ex-ante GFS and ex-post EJR.

4 Best of Both Worlds: GFS + Strong UFS + EJR

In this section, we present a family of rules called *Best-of-Both-Worlds MES* (or *BW-MES* for short), which obtains best-of-both-worlds fairness. Our main result is:

Theorem 4.1. *BW-MES (Algorithm 1) outputs a randomized committee that is ex-ante GFS, ex-ante Strong UFS, and ex-post EJR. Furthermore, the algorithm can be implemented in polynomial time.*

4.1 A Family of Rules: BW-MES

We start by providing an intuition behind our family of rules BW-MES, whose pseudocode can be found in Algorithm 1. At a high level, BW-MES follows in spirit the idea of the Method of Equal Shares (MES) of Peters and Skowron [2020]. To be more precise, we follow the MES algorithm description of Lackner and Skowron [2023, Rule 11], and make use of its first phase. For ease of exposition, we simply refer

Algorithm 1: BW-MES

Input: Voters $N = [n]$, candidates $C = [m]$, approval ballots $(A_i)_{i \in N}$, and committee size k.
Output: A GFS (fractional) committee $\vec{p} = (p_c)_{c \in C}$ and its implementation as a lottery over EJR (integral) committees.

```
1  Initialize b_i ← k/n for each i ∈ N, i.e., the budget voter i can spend on buying candidates.
2  Initialize y_ij ← 0 for each i ∈ N and j ∈ C, i.e., the amount voter i spends on candidate j.
   // Obtain an integral EJR committee via the Method of Equal of Shares (MES).
3  Let W_MES be an integral EJR committee returned by the first phase of MES [see, e.g., Lackner and Skowron, 2023,
     Rule 11] with initial budget (b_i)_{i∈N} = (k/n, ..., k/n).          // Each candidate costs 1.
4  Update (b_i)_{i∈N} to be the remaining budgets of the voters after executing MES.
5  Update y_ij for each i ∈ N and j ∈ W_MES to be the amount each voter i spent on candidate j during MES.
6  p⃗ = (p_1, p_2, ..., p_m) ← 1⃗_{W_MES}                                  // Initialize a fractional committee.
   // Extend the integral EJR committee to a fractional GFS committee.
7  N' ← {i ∈ N | A_i \ W_MES ≠ ∅}
8  foreach i ∈ N' do  Voter i spends an arbitrary amount of y_ic on each c ∈ A_i \ W_MES such that Σ_{c∈A_i\W_MES} y_ic = b_i.
9  foreach i ∈ N \ N' do  Voter i spends b_i in any fashion provided p_c ≤ 1 for any c ∈ C; update y_ic accordingly.
   // Implementation.
10 Apply a randomized rounding scheme [e.g., Gandhi et al., 2006] to p⃗, which outputs a lottery over integral committees
     of size k; let {(λ_j, W_j)}_{j∈[s]} denote the randomized committee.
11 return p⃗ and its implementation {(λ_j, W_j)}_{j∈[s]}
```

to this first phase as "MES". Specifically, each voter is initially given a budget of k/n, which can be spent on buying candidates—each candidate costs 1. In each round of MES, a candidate that incurs the smallest cost per utility for voters who approve it is chosen, and these voters pay as equally as possible. MES stops once no more candidate is affordable and returns an *integral* EJR committee W_{MES} in line 3. Denote by $(b_i)_{i \in N}$ the *remaining* budget of the voters after executing MES (line 4).

Our next step is to extend W_{MES} to a *fractional* GFS committee of size k using voters' remaining budget. We first initialize a fractional committee $\vec{p}$ using W_{MES} in line 6. It is worth noting that for any $c \in C \setminus W_{\text{MES}}$, $\sum_{i \in N_c} b_i < 1$; otherwise candidate c would have been included in W_{MES} in line 3. The key idea behind our completing method for the fractional committee in this family rule is the following:

- We first let each $i \in N$ such that $A_i \setminus W_{\text{MES}} \neq \emptyset$ spend her remaining budget b_i on candidates $A_i \setminus W_{\text{MES}}$, in an arbitrary way.
- Next, for any other voter, her remaining budget can be spent on any candidate $c \in C$ provided $p_c \leq 1$.

Finally, for the implementation step, we can use any rounding method that implements the fractional committee $\vec{p}$ by randomizing over integral committees of the same size; see, e.g., the ALLOCATIONFROMSHARES method of Aziz et al. [2019], the stochastic method of Grimmet [2004], or the dependent randomized rounding scheme of Gandhi et al. [2006].

In the following, we use an illustrative example to demonstrate our algorithm.

Example 4.2. *The following committee voting instance is used in Example 2.12 of Lackner and Skowron [2023] to illustrate MES. Let $k = 3$. Consider the following approval preferences which involve four candidates:*

$$A_1 = A_2 = A_3 = \{c, d\} \qquad A_4 = A_5 = \{a, b\}$$
$$A_6 = A_7 = \{a, c\} \qquad A_8 = \{b, d\}.$$

The voters start with a budget of $3/8$. Line 3 of Algorithm 1 returns candidates $\{a, c\}$ (alternatively, $\vec{p} = (1, 0, 1, 0)$); see Lackner and Skowron [2023, Example 2.12] for more details of this step. The starting budget of the voters for completing the fractional committee $\vec{p}$ is as follows:

$$b_1 = b_2 = b_3 = 7/40 \qquad b_4 = b_5 = 1/20$$
$$b_6 = b_7 = 0 \qquad b_8 = 3/8.$$

Then, in line 8 of Algorithm 1, each voter $i \in [8]$ spends b_i on candidates $\{b, d\} \cap A_i$ lexicographically. We therefore obtain the fractional committee $\vec{p} = (1, 19/40, 1, 21/40)$, which can be implemented by the randomized committee $\left\{\left(\frac{19}{40}, \{a, b, c\}\right), \left(\frac{21}{40}, \{a, c, d\}\right)\right\}$.

4.2 Analysis of BW-MES

Before proving Theorem 4.1, we first show a lower bound on voters' utilities provided by BW-MES.

Claim 4.3. *In Algorithm 1, for each $i \in N$, it holds that*

$$\sum_{j \in A_i} y_{ij} \geq \frac{1}{n} \cdot \min\{k, |A_i|\}.$$

Proof. Recall that each $i \in N$ is given an initial budget of k/n. Fix any $i \in N$ such that $A_i \setminus W_{\text{MES}} \neq \emptyset$. By the construction of Algorithm 1, voter i spends her budget k/n on candidates that she approves, in line 3 when executing MES or in line 8 when completing the fractional committee $\vec{p}$. It thus follows that $\sum_{j \in A_i} y_{ij} = \frac{k}{n} \geq \frac{1}{n} \cdot \min\{k, |A_i|\}$.

Now, fix any $i \in N$ such that $A_i \setminus W_{\text{MES}} = \emptyset$ (alternatively, $A_i \subseteq W_{\text{MES}}$). In other words, all candidates approved by voter i are already *fully* included in the fractional committee $\vec{p}$. Clearly, $|A_i| \leq k$. Fix any $c \in A_i$. Recall that N_c consists of voters who approve candidate c. If voter i spends the remainder of their budget on candidate c for any $c \in A_i$, then the claim holds trivially. Otherwise, by the construction of MES, voter i pays an amount of at least $1/|N_c|$ for candidate c, meaning that $y_{ic} \geq 1/|N_c| \geq 1/n$. We thus have $\sum_{j \in A_i} y_{ij} \geq \frac{1}{n} \cdot |A_i| = \frac{1}{n} \cdot \min\{k, |A_i|\}$, as desired. □

We are now ready to establish our main result.

Proof of Theorem 4.1. We break the proof into the following five parts.

Feasibility For each $c \in C$, note that whenever a (positive) amount p_c of the candidate is added to the fractional committee $\vec{p}$, the voters together pay a total of p_c. Since the voters have a total starting budget of k and each spends their entire budget, Algorithm 1 returns a fractional committee of size k. Next, due to the randomized rounding scheme we use in line 10, each integral committee in the returned randomized committee is of size k. In short, the fractional committee $\vec{p}$ and each integral committee in the randomized committee returned by Algorithm 1 respect the size constraint.

Ex-ante GFS Recall that y_{ij} denotes the amount each voter $i \in N$ spent on each candidate $j \in C$ in Algorithm 1. The fractional committee $\vec{p} = (p_1, p_2, \ldots, p_m)$ can thus be alternatively expressed by $p_j = \sum_{i \in N} y_{ij}$ for each $j \in C$.

Given this, for any $S \subseteq N$, we now have

$$\begin{aligned}\sum_{j \in \bigcup_{v \in S} A_v} p_j &= \sum_{j \in \bigcup_{v \in S} A_v} \sum_{i \in N} y_{ij} \\ &\geq \sum_{i \in S} \sum_{j \in \bigcup_{v \in S} A_v} y_{ij} \\ &\geq \sum_{i \in S} \sum_{j \in A_i} y_{ij} \geq \sum_{i \in S} \frac{1}{n} \cdot \min\{k, |A_i|\},\end{aligned}$$

where the last transition is due to Claim 4.3.

Ex-ante Strong UFS Consider any $S \subseteq N$ such that $A_i = A$ for all $i \in S$, where $A \subseteq C$. First, if $A \subseteq W_{\text{MES}}$, Strong UFS follows immediately. In the following, we consider the case where $A \setminus W_{\text{MES}} \neq \emptyset$. In this case, each voter in S spends the entirety of their budget in line 8. Thus, since voters only pay for candidates they approve, for each $i \in S$, we have $u_i(\vec{p}) \geq \sum_{j \in A} \sum_{i \in S} y_{ij} = |S| \cdot \frac{k}{n}$.

Ex-post EJR According to the first property of Gandhi et al. [2006, Theorem 2.3], W_{MES} is included in every realization. Since MES satisfies EJR [Peters and Skowron, 2020], we have that the lottery outputted is ex-post EJR.

Polynomial-time Computation To begin, note that both MES of Peters and Skowron [2020] and the randomized rounding scheme of Gandhi et al. [2006] used in lines 3 and 10 run in polynomial time. Thus, the computational complexity of any rule in the BW-MES family is dominated by how it completes the fractional committee in lines 8 to 9, which can be done in polynomial time as follows. Iterate through candidates $C \setminus W_{\text{MES}}$ in an arbitrary order, allocating to each candidate the remaining budgets of those agents who approve the candidate, i.e., for $c \in C \setminus W_{\text{MES}}$, $p_c \leftarrow \sum_{i \in N_c} b_i$, and zero the agents' budgets accordingly. □

4.3 Completing MES

Because MES may return an EJR committee W_{MES} of size *less than* k, several ways of extending W_{MES} to an *integral* committee of size exactly k have been discussed [see, e.g., Peters and Skowron, 2020; Lackner and Skowron, 2023]. As we have seen previously (Remark 3.8), an EJR committee may not provide positive share, let alone GFS. We provide a novel perspective on the completion of MES. Specifically, we define a family of rules which extends an integral committee returned by MES to a *randomized* committee providing GFS.

Due to the flexibility in the definition of the BW-MES family, the number of BW-MES rules obtaining distinct outcomes can be quite large. For example, one BW-MES rule that seems quite natural is that which continues in the spirit of MES: for the candidate whose supporters have the most collective budget leftover, this budget is spent on the candidate, and we continue in this fashion sequentially. It is an interesting future direction to further identify specific algorithms in the BW-MES family which provide additional desiderata such as high social welfare or additional ex-ante properties.

5 Discussion

In this work, we have initiated the best-of-both-worlds paradigm in the context of committee voting, which allows us to achieve both ex-ante and ex-post fairness.

We first generalized *fair share* axioms from the single-winner randomized voting literature. Future work can continue this effort by generalizing core fair share (CFS), which is stronger than GFS [Aziz et al., 2020], to the committee voting setting. A natural research question would then be whether CFS can be achieved by a lottery over committees satisfying some ex-post fairness guarantee. On the ex-post side, *fully justified representation (FJR)* is an axiom intermediate between EJR and core [Peters et al., 2021]. In the full version of our paper [Aziz et al., 2023c], we investigate the strongest ex-ante fairness notion we can satisfy if we require that our integral committees satisfy FJR.

Furthermore, the best-of-both-worlds perspective can be used to search for randomized committees which obtain other desirable ex-ante and ex-post properties. Fain et al. [2016] and Cheng et al. [2020] defined distinct ex-ante notions of core and proved existence of fractional and randomized committees (respectively) satisfying them. Can their existence results be extended to lotteries over JR committees? What if we additionally require ex-ante or ex-post Pareto efficiency?

Approval-based committee voting is but one of many social choice settings of interest. Others include multiple referenda [Brams et al., 1997], public decision making [Conitzer et al., 2017], and participatory budgeting [Aziz and Shah, 2020; Rey and Maly, 2023]. What new challenges does implementation present in more complex settings such as participatory budgeting, which involves candidate costs and budget constraints? We hope that our work serves as an invitation for further research applying the best-of-both-worlds perspective to social choice problems.

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