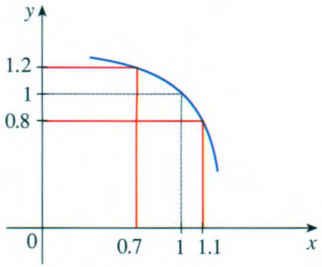
1.7 Exercises

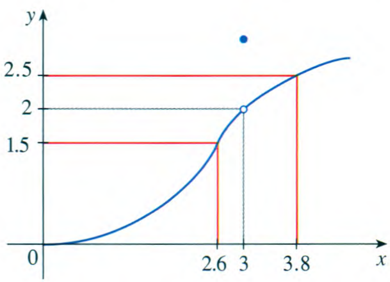
1. Use the given graph of f to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |f(x) - 1| < 0.2$$



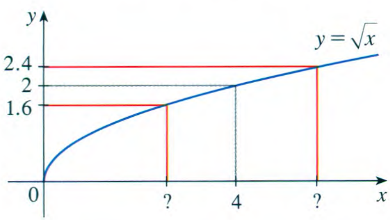
2. Use the given graph of f to find a number δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |f(x) - 2| < 0.5$$



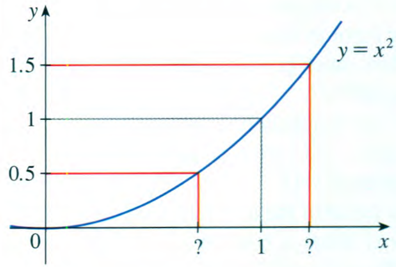
3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

$$\text{if} \quad |x - 4| < \delta \quad \text{then} \quad |\sqrt{x} - 2| < 0.4$$



4. Use the given graph of $f(x) = x^2$ to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |x^2 - 1| < \tfrac{1}{2}$$



5. Use a graph to find a number δ such that

$$\text{if} \quad |x - 2| < \delta \quad \text{then} \quad \left|\sqrt{x^2 + 5} - 3\right| < 0.3$$

6. Use a graph to find a number δ such that

$$\text{if} \quad \left|x - \frac{\pi}{6}\right| < \delta \quad \text{then} \quad \left|\cos^2 x - \frac{3}{4}\right| < 0.1$$

7. For the limit

$$\lim_{x \to 2} (x^3 - 3x + 4) = 6$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.2$ and $\varepsilon = 0.1$.

8. For the limit

$$\lim_{x \to 2} \frac{4x + 1}{3x - 4} = 4.5$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

9. (a) Use a graph to find a number δ such that

$$\text{if} \quad 4 < x < 4 + \delta \quad \text{then} \quad \frac{x^2 + 4}{\sqrt{x - 4}} > 100$$

(b) What limit does part (a) suggest is true?

10. Given that $\lim_{x\to\pi} \csc^2 x = \infty$, illustrate Definition 6 by finding values of δ that correspond to (a) $M = 500$ and (b) $M = 1000$.

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.
(a) What radius produces such a disk?
(b) If the machinist is allowed an error tolerance of $\pm 5 \text{ cm}^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
(c) In terms of the ε, δ definition of $\lim_{x\to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

12. Crystal growth furnaces are used in research to determine how best to manufacture crystals used in electronic components. For proper growth of a crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.
(a) How much power is needed to maintain the temperature at 200°C?
(b) If the temperature is allowed to vary from 200°C by up to ±1°C, what range of wattage is allowed for the input power?
(c) In terms of the ε, δ definition of $\lim_{x\to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

13. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.
(b) Repeat part (a) with $\varepsilon = 0.01$.

14. Given that $\lim_{x\to 2}(5x - 7) = 3$, illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.1$, $\varepsilon = 0.05$, and $\varepsilon = 0.01$.

15–18 Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 9.

15. $\lim_{x\to 4} \left(\frac{1}{2}x - 1\right) = 1$

16. $\lim_{x\to 2} (2 - 3x) = -4$

17. $\lim_{x\to -2} (-2x + 1) = 5$

18. $\lim_{x\to 1} (2x - 5) = -3$

19–32 Prove the statement using the ε, δ definition of a limit.

19. $\lim_{x\to 9} \left(1 - \frac{1}{3}x\right) = -2$

20. $\lim_{x\to 5} \left(\frac{3}{2}x - \frac{1}{2}\right) = 7$

21. $\lim_{x\to 4} \dfrac{x^2 - 2x - 8}{x - 4} = 6$

22. $\lim_{x\to -1.5} \dfrac{9 - 4x^2}{3 + 2x} = 6$

23. $\lim_{x\to a} x = a$

24. $\lim_{x\to a} c = c$

25. $\lim_{x\to 0} x^2 = 0$

26. $\lim_{x\to 0} x^3 = 0$

27. $\lim_{x\to 0} |x| = 0$

28. $\lim_{x\to -6^+} \sqrt[8]{6 + x} = 0$

29. $\lim_{x\to 2} (x^2 - 4x + 5) = 1$

30. $\lim_{x\to 2} (x^2 + 2x - 7) = 1$

31. $\lim_{x\to -2} (x^2 - 1) = 3$

32. $\lim_{x\to 2} x^3 = 8$

33. Verify that another possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ in Example 3 is $\delta = \min\{2, \varepsilon/8\}$.

34. Verify, by a geometric argument, that the largest possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ is $\delta = \sqrt{9 + \varepsilon} - 3$.

35. (a) For the limit $\lim_{x\to 1} (x^3 + x + 1) = 3$, use a graph to find a value of δ that corresponds to $\varepsilon = 0.4$.
(b) By solving the cubic equation $x^3 + x + 1 = 3 + \varepsilon$, find the largest possible value of δ that works for any given $\varepsilon > 0$.
(c) Put $\varepsilon = 0.4$ in your answer to part (b) and compare with your answer to part (a).

36. Prove that $\lim_{x\to 2} \dfrac{1}{x} = \dfrac{1}{2}$.

37. Prove that $\lim_{x\to a} \sqrt{x} = \sqrt{a}$ if $a > 0$.

$$\left[\textit{Hint}\text{: Use } \left|\sqrt{x} - \sqrt{a}\right| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}}. \right]$$

38. If H is the Heaviside function defined in Section 1.5, prove, using Definition 2, that $\lim_{t\to 0} H(t)$ does not exist. [*Hint:* Use an indirect proof as follows. Suppose that the limit is L. Take $\varepsilon = \frac{1}{2}$ in the definition of a limit and try to arrive at a contradiction.]

39. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x)$ does not exist.

40. By comparing Definitions 2, 3, and 4, prove Theorem 1.6.1:

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

41. How close to -3 do we have to take x so that

$$\frac{1}{(x + 3)^4} > 10{,}000$$

42. Prove, using Definition 6, that $\lim_{x\to -3} \dfrac{1}{(x + 3)^4} = \infty$.

43. Prove that $\lim_{x\to -1^-} \dfrac{5}{(x + 1)^3} = -\infty$.

44. Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$, where c is a real number. Prove each statement.
(a) $\lim_{x\to a} [f(x) + g(x)] = \infty$
(b) $\lim_{x\to a} [f(x)g(x)] = \infty \quad \text{if } c > 0$
(c) $\lim_{x\to a} [f(x)g(x)] = -\infty \quad \text{if } c < 0$