single-sided power spectral densities (PSD) of this signal computed by the Fourier transformation and by the method of maximum entropy. In the ranges of a corresponding to chaotic dynamics the spectrum has no prominent peaks and is practically continuous, which is one of the landmarks of chaos.

For individual values of a, compute the auto-correlations

$$R_{xx}(\tau) = \langle x(t)x(\tau + t)\rangle_t - \left(\langle x(t)\rangle_t\right)^2$$

of the sequences $\{x(t)\}_{i=0}^{N-1}$ and determine an approximation of the correlation length ξ as a function of a. Choose N large enough that $R_{xx}(\tau)/R_{xx}(0)$ will be precise to three digits. Assume that the auto-correlation has the form

$$R_{xx}(\tau) = A\mathrm{e}^{-|\tau|/\xi}.$$

It can be shown that

$$\xi = \log\left(\frac{|S_1^2 - S_2|}{S_1^2 + S_2}\right)^{-1}, \quad S_n = \sum_{t=0}^{\infty} R_{xx}(t)^n.$$

Systems in which all correlations between quantities converge to zero at long times, are said to *mix the space*, which is one of the key properties permitting a statistical analysis of dynamics.

6.9.2 Diffusion and Chaos in the Standard Map

In some Hamiltonian systems, continuous dynamics can be translated to discrete dynamics. This applies in particular to systems on which we act with a periodic external force in the form of short pulses. The point (p, q) of the trajectory of the system (position and momentum) at time $t + 1$ is related to the point at time t when the system receives the pulse, by the equations

$$\begin{aligned} p(t+1) &= p(t) + F\big(q(t), p(t)\big), \\ q(t+1) &= q(t) + G\big(p(t+1)\big), \end{aligned}$$

where F is the force pulse and G represents the dynamics between the pulses. In such systems a very clear connection between diffusion and auto-correlation exists. The diffusion coefficient is defined as

$$D = \lim_{t\to\infty} \frac{1}{2t}\left\langle\big[q(t) - q(0)\big]^2\right\rangle,$$

where $\langle\cdot\rangle$ denotes the phase average (average over initial points $(q(0), p(0))$ distributed uniformly in the region of phase space which is invariant with respect to the dynamics). The time auto-correlation function G is defined as

$$R_{GG}(\tau) = \big\langle G\big(p(\tau)\big)G\big(p(0)\big)\big\rangle - \big\langle G(p(0)\big\rangle^2.$$

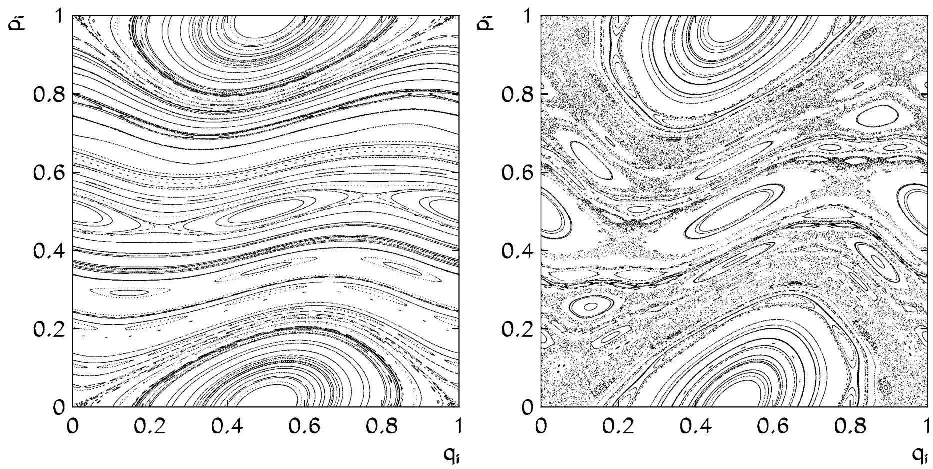

Fig. 6.19 Phase portraits of the standard map for different values of the parameter K. [*Left*] $K = 0.5$. [*Right*] $K = 1.0$

By assuming that this system sufficiently quickly mixes the phase space, a relation between the diffusion coefficient and auto-correlation can be derived:

$$D = R_{GG}(0) + 2\sum_{k=1}^{\infty} R_{GG}(k). \tag{6.39}$$

⊙ Analyze the dynamical system of the standard (Chirik's) map

$$p(t+1) = p(t) + \frac{K}{2\pi}\sin\bigl(2\pi q(t)\bigr) \quad \mathrm{mod}\ 1,$$
$$q(t+1) = q(t) + p(t+1) \quad \mathrm{mod}\ 1,$$

which is defined on the torus $(q(t), p(t)) \in [0,1)^2$. This map played a major role in the development of the theory of classical and quantum chaos [63]. Examples of phase portraits obtained by propagating a number of uniformly distributed points in the case of $K = 0.5$ and $K = 1.0$ are shown in Fig. 6.19. By increasing K above ≈ 0.97 the chaotic region of the phase space, where the points are distributed uniformly, becomes ever larger, until diffusion occurs throughout.

Compute the time auto-correlation $R_{pp}(\tau) = \langle p(\tau + t)p(t)\rangle_t - \langle p(t)\rangle_t^2$ of the momentum samples $\{p(t)\}_{t=0}^{N-1}$ for the values $K = 1$, 2, 5, and 10, with the initial point $(q(0), p(0))$ in the chaotic region of the phase space. The expected absolute error of $R_{pp}(\tau)$ should be less than 10^{-3}. If the auto-correlation decays rapidly, show it in the rescaled form $R_{pp}(\tau)/R_{pp}(0)$ in logarithmic scale. If you observe an exponential fall-off of the auto-correlation,

$$\bigl|R_{pp}(\tau)\bigr| \sim \mathrm{e}^{-\tau/\xi},$$

estimate the correlation length ξ which represents the time after which the trajectories become statistically independent. To compute the auto-correlation use the FFT algorithm (Sect. 4.2.5).

⊕ Observe the dependence of the diffusion coefficient D on the parameter K from the interval $[2, 20]$. Compute D by summing the auto-correlations of momentum (i.e. by using (6.39) in which R_{GG} is replaced by R_{pp}). The initial points of the map should be located in the chaotic region of the phase space. An analytic approximation for the diffusion coefficient can be found in [63].

6.9.3 Phase Transitions in the Two-Dimensional Ising Model

Some properties of systems consisting of magnetic dipoles with local interactions can be nicely explained by the *Ising model*. This model assumes that the dipoles are arranged in the nodes of a two-dimensional mesh and that the spins s_i are either "up" ($s_i = 1$) or "down" ($s_i = -1$). The Hamiltonian (the energy) of such a system in the presence of an external magnetic field H can be written in the form

$$E = -J \sum_{\langle i,j \rangle} s_i s_j - H \sum_i s_i , \tag{6.40}$$

where the indices i and j represent the coordinates of the spins on the mesh (in two dimensions $i \in \mathbb{N}_0^2$). The first sum in (6.40) runs over neighboring points only. The parameter J determines how the spins interact. For $J > 0$ the adjacent dipoles tend to point in the same direction, while for $J < 0$ they tend in the opposite directions. In the following we set $J > 0$. The total magnetization of the system is

$$M = \sum_i s_i .$$

At different temperatures the system may reside in ferromagnetic or paramagnetic phase. The temperature T_c of the phase transition between these phases in the absence of external field ($H = 0$) is given by the equation $\sinh(2J/(k_\mathrm{B} T_\mathrm{c})) = 1$, which has the solution $T_\mathrm{c} \approx 2.269185\ J/k_\mathrm{B}$.

⊙ At the book's website you can find the data for the thermodynamic equilibrium state of spins in the two-dimensional Ising model at various temperatures. A few examples are shown in Fig. 6.20. Compute the auto-correlation function of the orientation of spins on a $N \times N$ mesh,

$$R_{ss}(i, j) = \frac{1}{N^2} \sum_{k,l} s_{k+i,l+j} s_{k,l} ,$$

where we assume periodic boundary conditions. Average the auto-correlation over the neighborhood of the points with approximately the same radius r. Assume that the auto-correlation has the form

$$\left| R_{ss}(i, j) \right| \sim C \mathrm{e}^{-r/L}, \quad r = \sqrt{i^2 + j^2},$$

and estimate the correlation length L. Show L as a function of temperature: you should observe a strong increase in L near the critical temperature T_c.