Numerical Methods	**Problem Set 4**
Professor: Cezar Santos	**March 31, 2023**
TA: Rafael Vetromille	**Student:** Luan Borelli

Preliminary comments

Regarding the asset space. For all the exercises in this problem set, I consider a grid size of 500 for the asset space, linearly spaced in the interval $(-\phi, \phi]$, where $\phi = e^{z_1}/\bar{r}$ and $\bar{r} = 1/\beta - 1$. Of course, we could consider another asset space. However, any other choice would be *ad hoc*. The choice considered here, despite generating a very large asset space, is at least based on some theoretical *rationale*: the bounds on the asset space are derived from the loosest possible natural debt limit level — i.e., the natural debt limit associated with the risk-free interest rate level $1/\beta - 1$, which is the highest possible interest rate that could arise in equilibrium for a Huggett economy.

Regarding the stationary distribution. In order to obtain the stationary distribution I map the pair (a, z) of vectors into a single state vector x as follows: for $i = 1, \ldots, n$, $h = 1, \ldots, m$, let the j-th element of x be the pair (a_i, z_h), where $j = (i-1)m + h$. Denote

$$x' = [(\bar{a}_1, \bar{z}_1), (\bar{a}_1, \bar{z}_2), \ldots, (\bar{a}_1, \bar{z}_m), (\bar{a}_2, \bar{z}_1), \ldots, (\bar{a}_2, \bar{z}_m), \ldots, (\bar{a}_n, \bar{z}_1), \ldots, (\bar{a}_n, \bar{z}_m)].$$

The optimal policy function $a' = g(a, z)$ and the Markov chain Π on z induce a Markov chain for x via the formula

$$\begin{aligned}\mathbb{P}[(a_{t+1} = a', z_{t+1} = z')|(a_t = a, z_t = z)] &= \mathbb{P}[a_{t+1} = a'|a_t = a, z_t = z] \cdot \text{Prob}(z_{t+1} = z'|z_t = z)] \\ &= \mathcal{I}(a', a, z)\Pi(z, z'),\end{aligned}$$

where the indicator function $\mathcal{I}$ is defined as $\mathcal{I}(a', a, z) = 1$ if $a' = g(a, z)$, and 0 otherwise. This formula defines an $N \times N$ matrix Π_x, where $N = n \cdot m$. This is the Markov chain on the household's state vector x. The stationary distribution π_∞ is obtained by simply computing the eigenvector of Π_x associated with its unit eigenvalue, and then normalizing it to sum one. We can "unstack" the state vector x and use π_∞ to deduce the stationary probability measure $\lambda(\bar{a}_i, \bar{z}_h)$ over $(\bar{a}_i, \bar{z}_h)$ pairs, where

$$\lambda(\bar{a}_i, \bar{z}_h) = \mathbb{P}(a_t = \bar{a}_i, z_t = \bar{z}_h) = \pi_\infty(j),$$

and where $\pi_\infty(j)$ is the j-th component of the vector π_∞ , and $j = (i-1)m+h$. The marginal asset distribution can then be retrieved by computing $\sum_{h=1}^{m} \lambda(a, z_h)$ for all $a \in A_{\text{grid}}$, and the aggregate savings by computing $\mathbb{E}[a(r)] = \sum_{a,z} \lambda(a, z)g(a, z)$ for all $(a, z) \in A_{\text{grid}} \times Z_{\text{grid}}$.

Regarding the EEEs. The Euler equation errors are computed as in the previous problem sets, using the adapted formula for a Hugget economy derived from the Euler equation:

$$\text{EEE} = \log_{10} \left| 1 - \frac{[\beta(1+r)\mathbb{E}_{z'}[c^{-\gamma}(a', z')]]^{-1/\gamma}}{c(a, z)} \right|. \tag{1}$$

Regarding the code. The code used to solve this entire problem set basically revolves around one function:

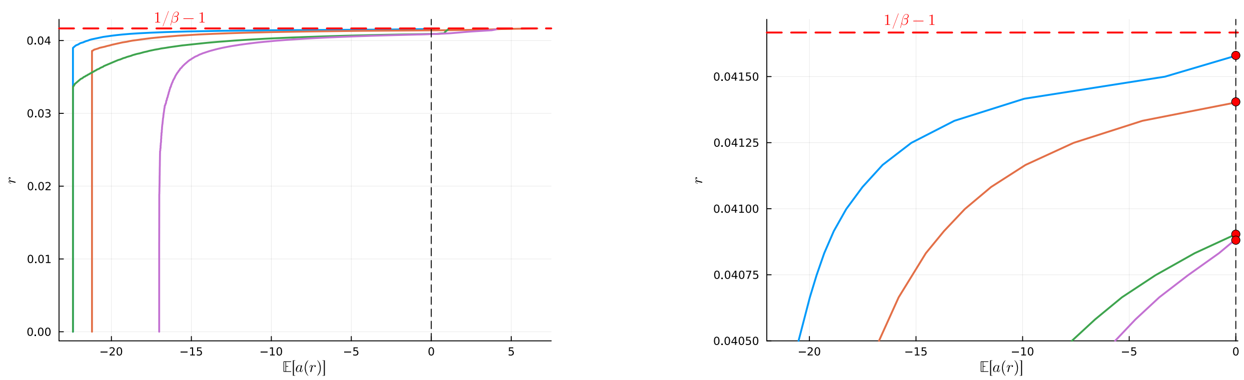
$$\texttt{solve\_individuals\_problem}(r).$$

Given an interest rate r, this function solves the individual's problem and returns the following objects: the aggregate asset demand of the economy, the value function, the asset policy function, the argmins of the asset policy function (necessary for later calculating the Euler equation errors), the stationary distribution and the (stationary) marginal asset distribution. To obtain the equilibrium interest rate for the economy, it suffices to apply some root-finding algorithm to this function — exactly what has been done throughout this problem set, using the `Roots.jl` Julia package with the `Bissection()` method.

Finally, it is worth mentioning my effort to optimize this function: for the value function iteration, acceleration is used and both the monotonicity and the concavity of the value function are exploited. In obtaining the stationary distribution, I use sparse matrices and Kronecker products in order to make matrix operations more efficient and faster, and I obtain the eigenvector associated with the unit eigenvalue directly, through the solution of a simple linear system.

Given this effort, the function ended up being quite quick to solve the individuals' problem. For a given interest rate r, the solution is found in just over one second, on average. To obtain the equilibrium interest rate, always considering an initial bracket from 0 to $1/\beta - 1$ for the bisection method, the code took on average about 60 seconds to converge.

Figure 1: Summary of results — $\mathbb{E}[a(r)]$ functions for the equilibrium interest rates.



(a) $\mathbb{E}[a(r)]$ curves from items (c), (d), (e) and (f) plotted together.

(b) Zoomed in $\mathbb{E}[a(r)]$ curves from items (c), (d), (e) and (f) plotted together. The red dots indicate the equilibrium interest rate levels.

(a) *Let $\rho = 0.9$ and $\sigma = 0.01$. Use the Tauchen method to discretize the stochastic process in a Markov chain with 9 states. Use 3 standard deviations for each side.*

Solution. The labor income process is discretized using the Tauchen method exactly as in the previous problem sets. The resulting state space and transition matrix are reported below, in equations (2) and (3), respectively. Entries are rounded to three decimal places.

$$Z_{\text{grid}} \approx \begin{bmatrix} -0.069 \\ -0.052 \\ -0.034 \\ -0.017 \\ 0.0 \\ 0.017 \\ 0.034 \\ 0.052 \\ 0.069 \end{bmatrix} \tag{2}$$

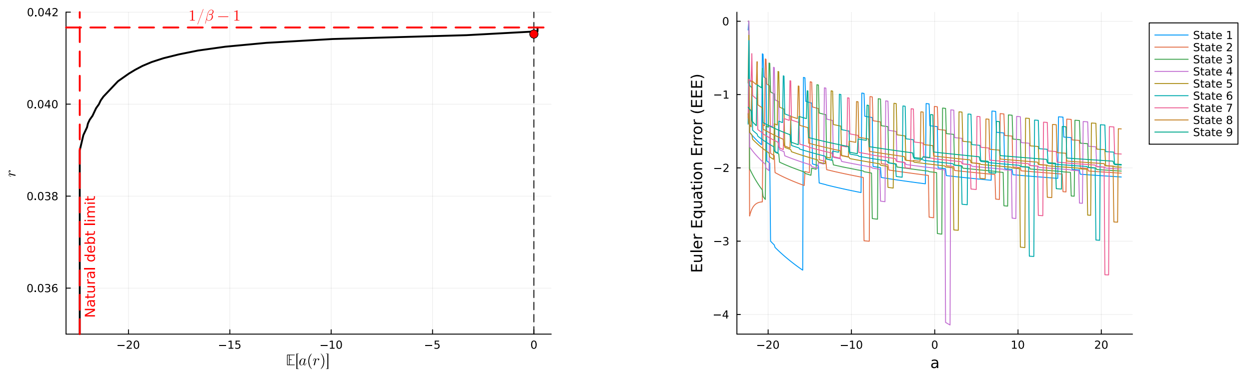
$$\Pi \approx \begin{bmatrix} 0.568 & 0.402 & 0.029 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.084 & 0.55 & 0.346 & 0.019 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.002 & 0.112 & 0.583 & 0.29 & 0.013 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.003 & 0.148 & 0.603 & 0.238 & 0.008 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.005 & 0.19 & 0.61 & 0.19 & 0.005 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.008 & 0.238 & 0.603 & 0.148 & 0.003 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.013 & 0.29 & 0.583 & 0.112 & 0.002 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.019 & 0.346 & 0.55 & 0.084 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.029 & 0.402 & 0.568 \end{bmatrix} \tag{3}$$

$\square$

(b) *Discretize the asset space using a grid and solve the individual's problem for each state variable.*

Solution. As the exercise did not inform which interest rate to use for this item, I decided to consider an *ad hoc* interest rate of $r = 0.03$. The code took 1.3 second to run. The resulting $\mathbb{E}[a(r)]$ function is reported in Figure 2a. The value function, the policy functions and the stationary distribution are reported in the panels of Figure 3. Finally, Euler equation errors are reported in Figure 2b, with some statistics on them reported in Table 1. Regarding these EEEs, focusing on the mean we can interpret that, on average, a \$1 mistake is made for every \$ $10^{1.87} \approx$ \$74 spent.

Figure 2: $\mathbb{E}[a(r)]$ function and Euler equation errors for an *ad hoc* interest rate level $r = 0.03$.



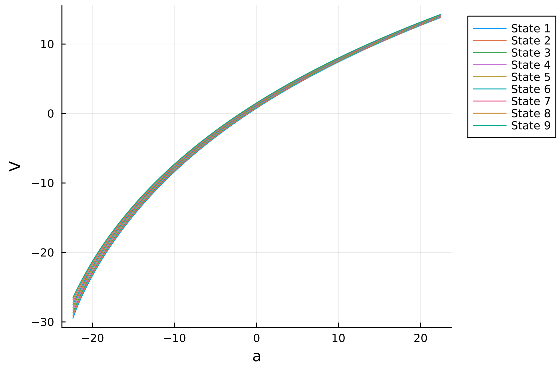
(a) $\mathbb{E}[a(r)]$ (b) Euler Equation Errors

Table 1: Euler Equation Errors (EEEs) statistics for an *ad hoc* interest rate level $r = 0.03$.

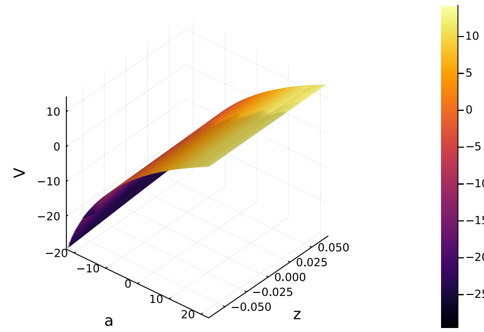
Euler Equation Errors' Statistics	
Mean	$\approx$ -1.87
Median	$\approx$ -1.92
Maximum	$\approx$ -3.40
Minimum	$\approx$ -4.14

□

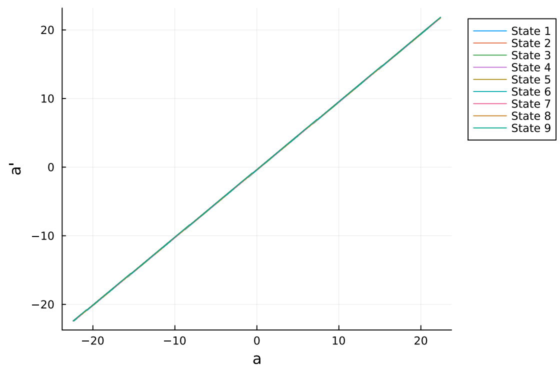
Figure 3: A Hugget economy — Value function, policy functions and stationary distribution for an *ad hoc* interest rate level $r = 0.03$.



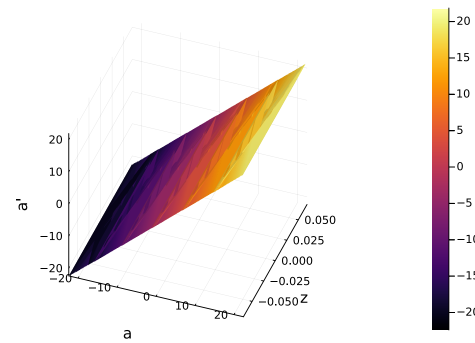
(a) Value function — 2D plot.



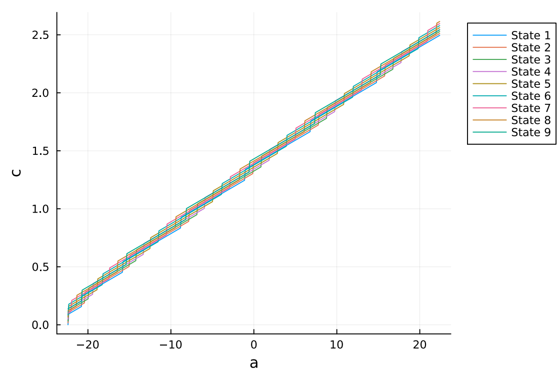
(b) Value function — 3D plot.



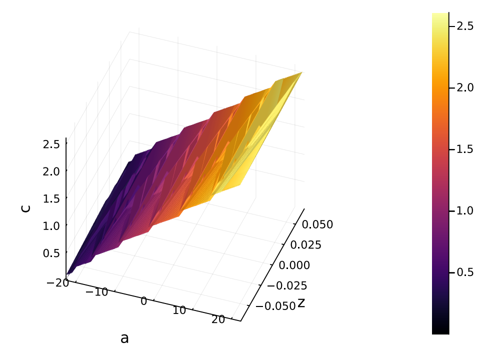
(c) Asset policy function — 2D plot.



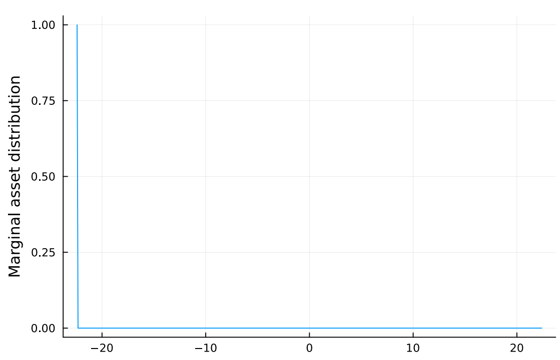
(d) Asset policy function — 3D plot.



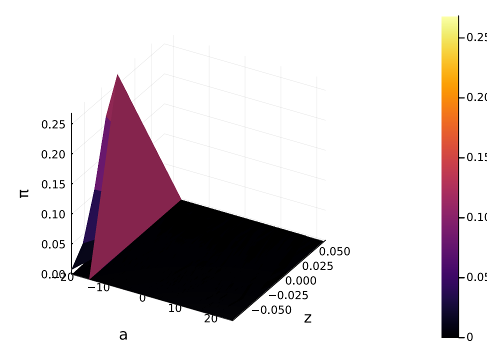
(e) Consumption policy function — 2D plot.



(f) Consumption policy function — 3D plot.



(g) Stationary marginal asset distribution.

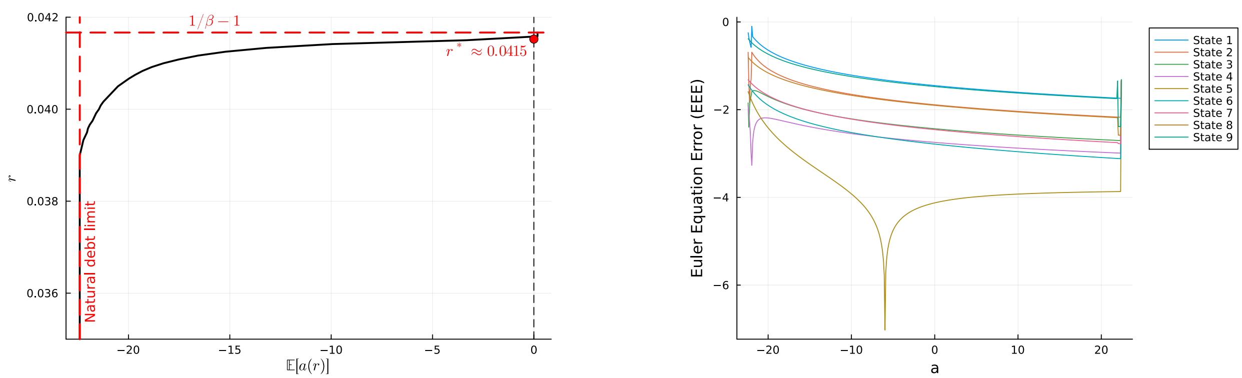


(h) Stationary asset distribution.

(c) *Find the stationary distribution $\pi(z, a)$ and use it to compute the aggregate savings in the economy. Find the equilibrium interest rate.*

Solution. The code took 55.13 seconds to run, using $[0, 1/\beta - 1]$ as the initial bracket for the bissection method. The equilibrium interest rate is $r \approx 0.04152388$. The resulting $\mathbb{E}[a(r)]$ function is reported in Figure 4a. The value function, the policy functions and the stationary distribution are reported in the panels of Figure 5. Finally, Euler equation errors are reported in Figure 4b, with some statistics on them reported in Table 2. Regarding these EEEs, focusing on the mean we can interpret that, on average, a \$1 mistake is made for every \$ $10^{2.24} \approx$ \$173.78 spent.

Figure 4: $\mathbb{E}[a(r)]$ function and Euler equation errors for the equilibrium interest rate level $r \approx 0.0415$.



(a) $\mathbb{E}[a(r)]$

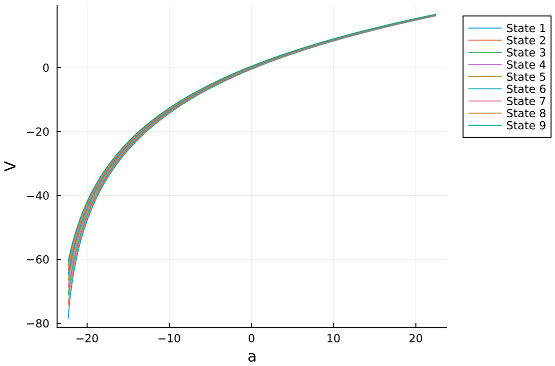
(b) Euler Equation Errors

Table 2: Euler Equation Errors (EEEs) statistics for the equilibrium interest rate level $r \approx 0.0415$.

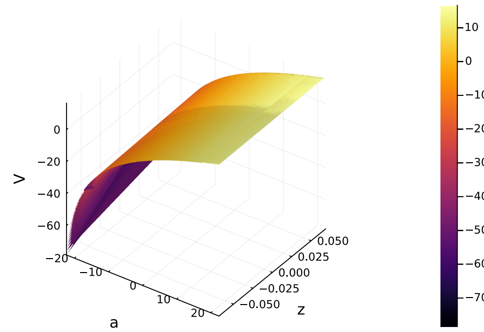
Euler Equation Errors' Statistics	
Mean	$\approx$ -2.24
Median	$\approx$ -2.15
Maximum	$\approx$ -0.10
Minimum	$\approx$ -7.02

□

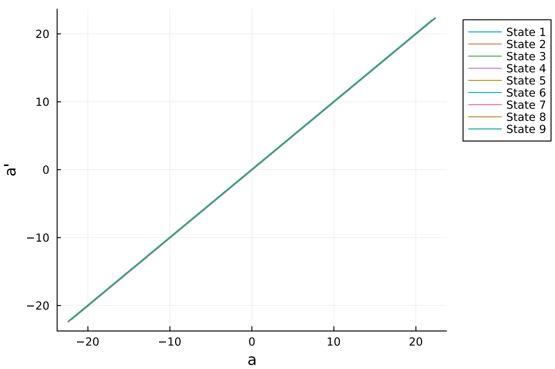
Figure 5: A Hugget economy — Value function, policy functions and stationary distribution for the equilibrium interest rate level $r \approx 0.0415$.



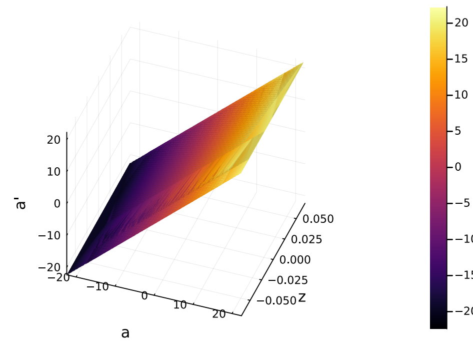
(a) Value function — 2D plot.



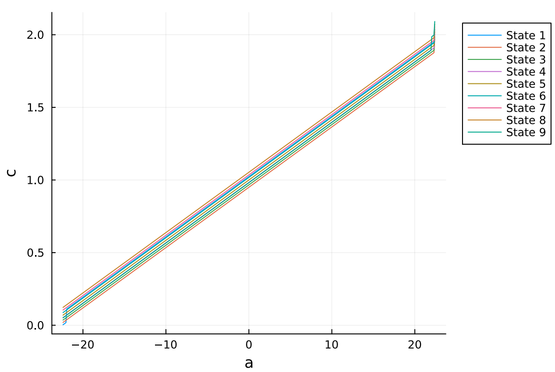
(b) Value function — 3D plot.



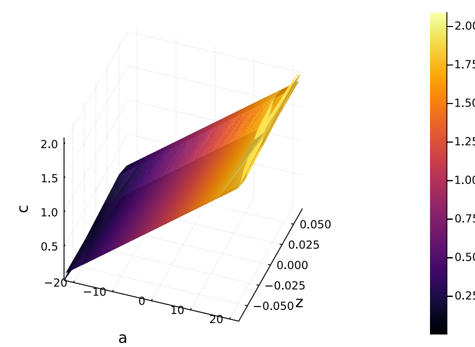
(c) Asset policy function — 2D plot.



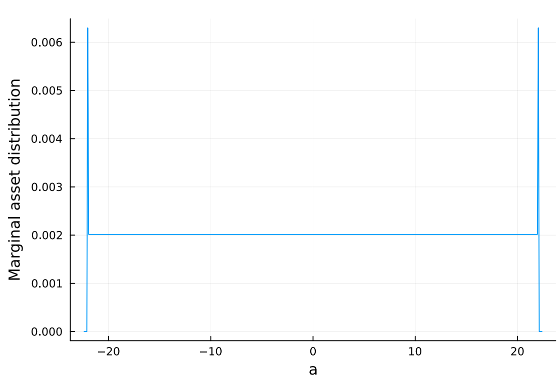
(d) Asset policy function — 3D plot.



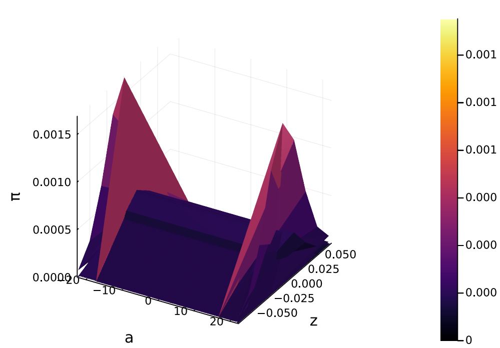
(e) Consumption policy function — 2D plot.



(f) Consumption policy function — 3D plot.



(g) Stationary marginal asset distribution.



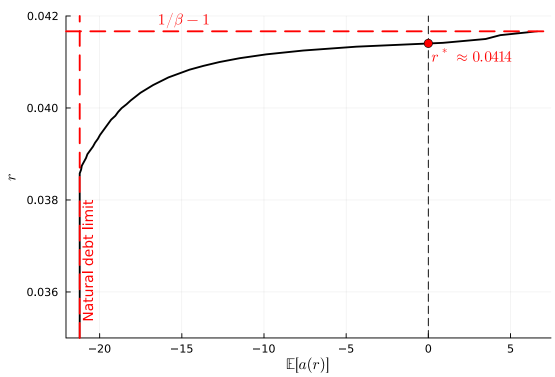
(h) Stationary asset distribution.

(d) *Suppose $\rho = .97$. Redo the analysis. How does the interest rate compare now? Explain.*

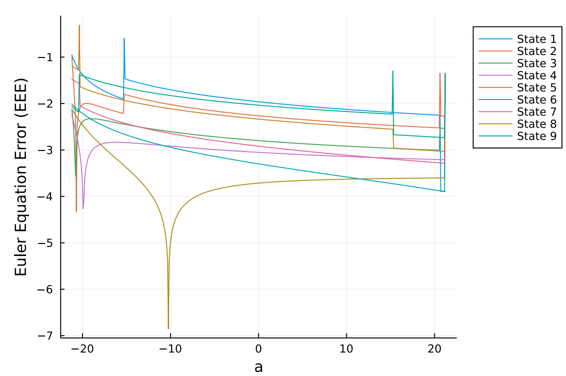
Solution. The code took 59.27 seconds to run, using $[0, 1/\beta - 1]$ as the initial bracket for the bissection method. The equilibrium interest rate is $r \approx 0.04140437$. The resulting $\mathbb{E}[a(r)]$ function is reported in Figure 6a. The value function, the policy functions and the stationary distribution are reported in the panels of Figure 7. Finally, Euler equation errors are reported in Figure 6b, with some statistics on them reported in Table 3. Regarding these EEEs, focusing on the mean we can interpret that, on average, a \$1 mistake is made for every \$ $10^{2.66} \approx$ \$457.08 spent.

In comparison to the result obtained with a persistence parameter of 0.9 in (c), the equilibrium interest rate is now slightly lower. When the persistence of the stochastic income process increases, households' income process become more volatile over time. Also, agents are aware that if a bad income realization occurs, they will remain in this situation for a longer period. As a result, risk-averse agents choose to save more to protect themselves against future income fluctuations, leading to a lower equilibrium interest rate which reflects the higher demand for saving and lower demand for borrowing.

Figure 6: $\mathbb{E}[a(r)]$ function and Euler equation errors, with $\rho = .97$, for the equilibrium interest rate level $r \approx 0.0414$.



(a) $\mathbb{E}[a(r)]$



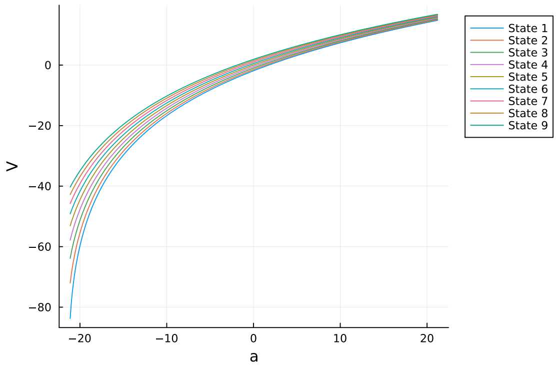
(b) Euler Equation Errors

Table 3: Euler Equation Errors (EEEs) statistics, with $\rho = .97$, for the equilibrium interest rate level $r \approx 0.0414$.

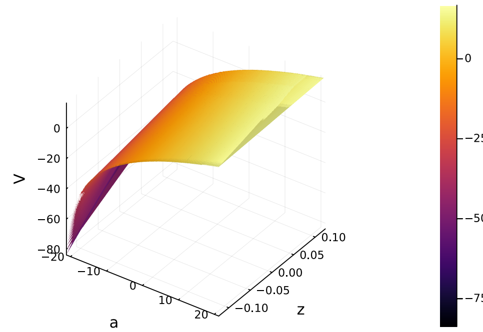
Euler Equation Errors' Statistics	
Mean	≈ -2.66
Median	≈ -2.65
Maximum	≈ -0.31
Minimum	≈ -6.86

□

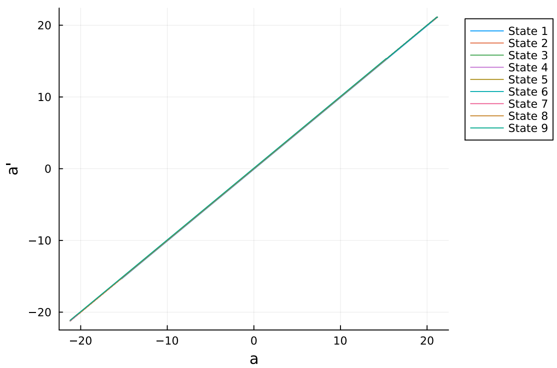
Figure 7: A Hugget economy — Value function, policy functions and stationary distribution, with $\rho = .97$, for the equilibrium interest rate level $r \approx 0.0414$.



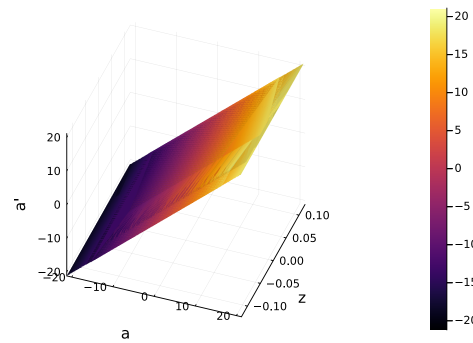
(a) Value function — 2D plot.



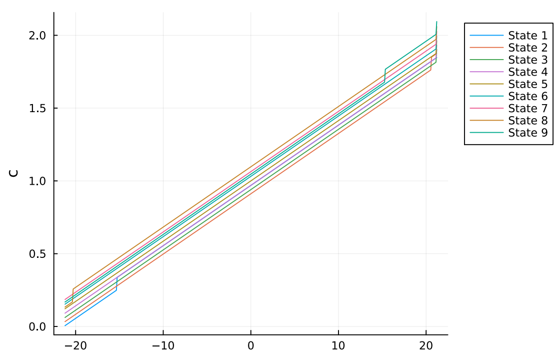
(b) Value function — 3D plot.



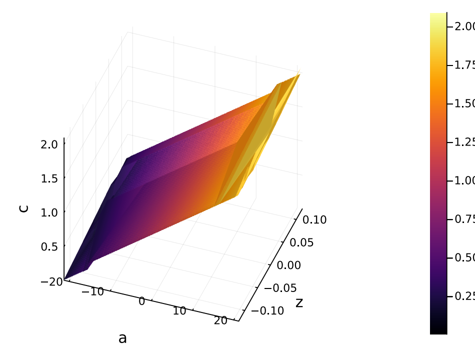
(c) Asset policy function — 2D plot.



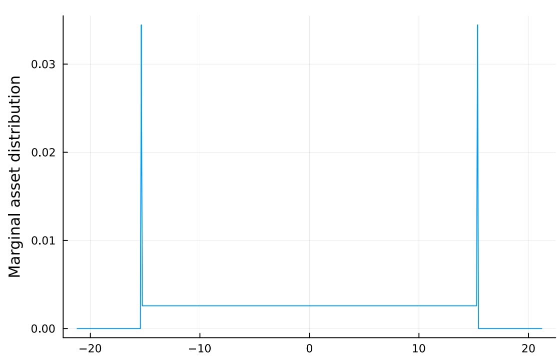
(d) Asset policy function — 3D plot.



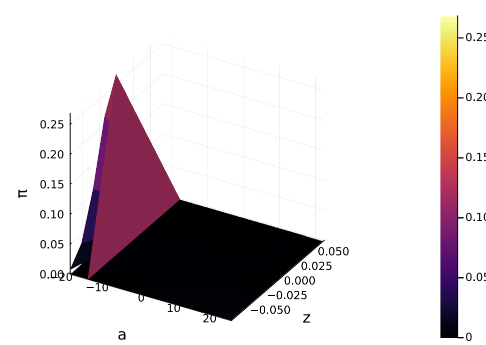
(e) Consumption policy function — 2D plot.



(f) Consumption policy function — 3D plot.



(g) Stationary marginal asset distribution.



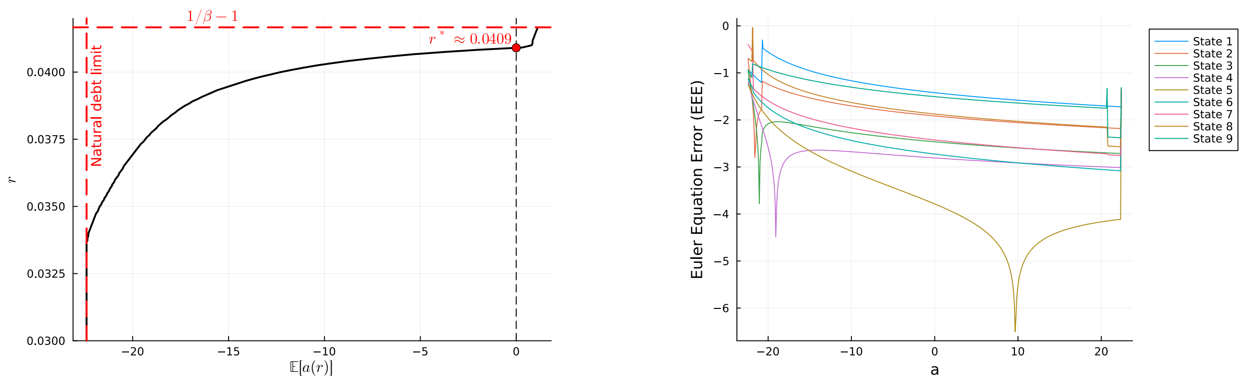
(h) Stationary asset distribution.

(e) *Suppose $\gamma = 5$. Redo the analysis. How does the interest rate compare now? Explain.*

Solution. The code took 79.96 seconds to run, using $[0, 1/\beta - 1]$ as the initial bracket for the bissection method. The equilibrium interest rate is $r \approx 0.04090359$. The resulting $\mathbb{E}[a(r)]$ function is reported in Figure 8a. The value function, the policy functions and the stationary distribution are reported in the panels of Figure 9. Finally, Euler equation errors are reported in Figure 8b, with some statistics on them reported in Table 4. Regarding these EEEs, focusing on the mean we can interpret that, on average, a \$1 mistake is made for every \$ $10^{2.24} \approx$ \$173.78 spent.

In comparison to the result obtained with $\gamma = 1$ in (c), the equilibrium interest rate is now significantly lower. Since γ is the risk-aversion parameter of the CRRA utility function, when γ increases, it means that agents become more risk-averse. This higher risk aversion of agents lead them to save more, reducing the demand for loans and increasing the supply of loanable funds in the economy. As a result, the interest rate would decrease until the market for loanable funds clears. Therefore, an increase in γ ultimately results in a decrease in the equilibrium interest rate.

Figure 8: $\mathbb{E}[a(r)]$ function and Euler equation errors, with $\gamma = 5$, for the equilibrium interest rate level $r \approx 0.0409$.



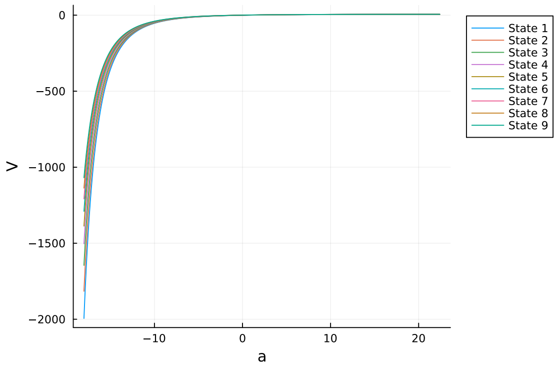
(a) $\mathbb{E}[a(r)]$ (b) Euler Equation Errors

Table 4: Euler Equation Errors (EEEs) statistics, with $\gamma = 5$, for the equilibrium interest rate level $r \approx 0.0409$.

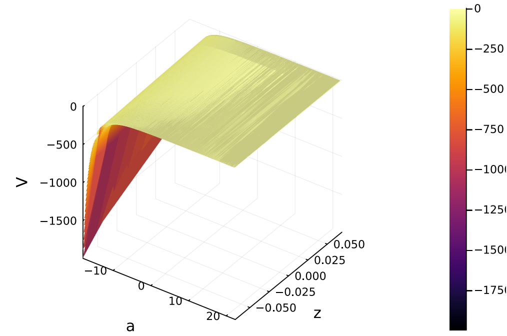
Euler Equation Errors' Statistics	
Mean	≈ -2.24
Median	≈ -2.15
Maximum	≈ -0.03
Minimum	≈ -6.50

□

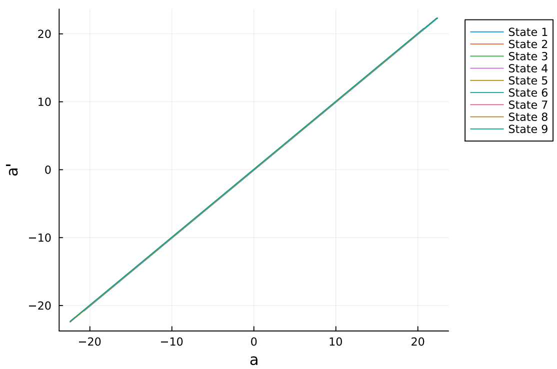
Figure 9: A Hugget economy — Value function, policy functions and stationary distribution, with $\gamma = 5$, for the equilibrium interest rate level $r \approx 0.0409$.



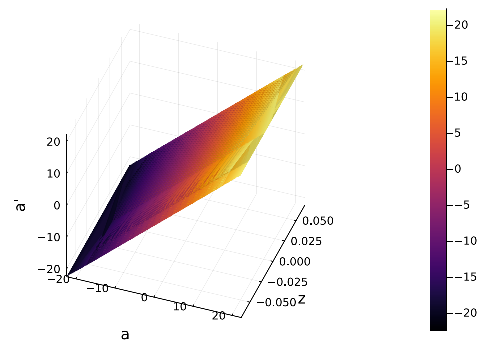
(a) Value function — 2D plot.



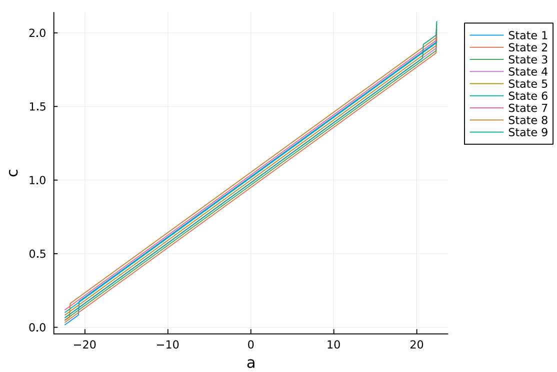
(b) Value function — 3D plot.



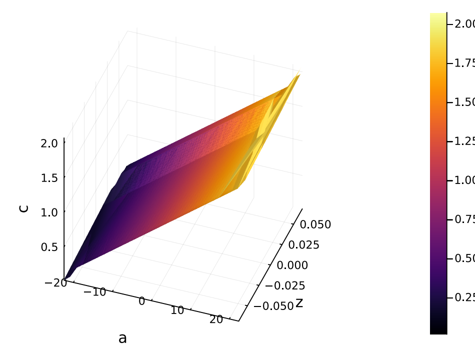
(c) Asset policy function — 2D plot.



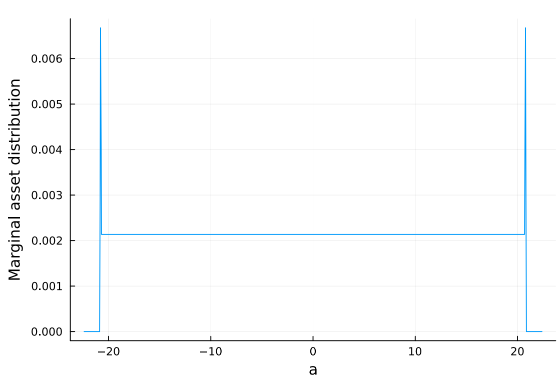
(d) Asset policy function — 3D plot.



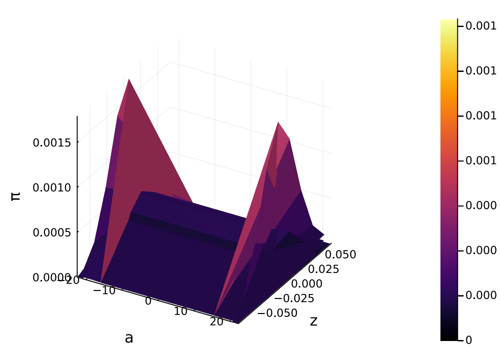
(e) Consumption policy function — 2D plot.



(f) Consumption policy function — 3D plot.



(g) Stationary marginal asset distribution.



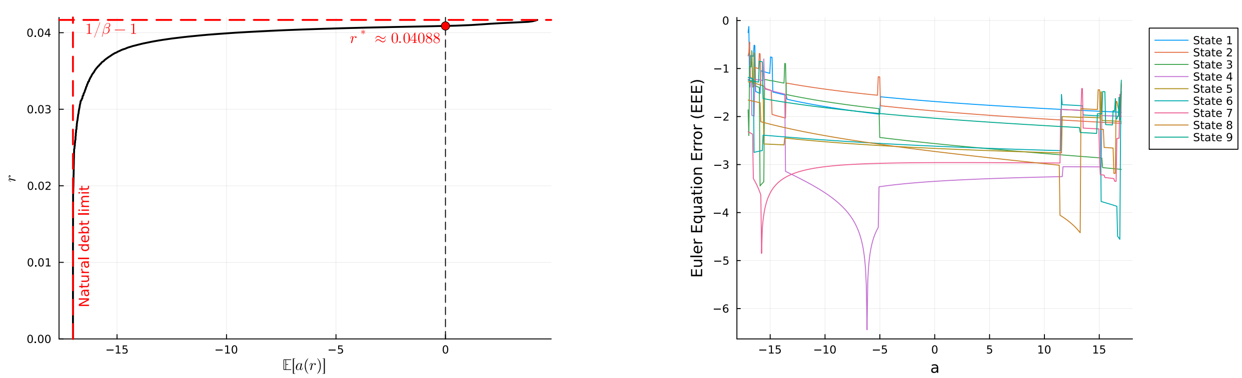
(h) Stationary asset distribution.

(f) *Suppose $\sigma = .05$. Redo the analysis. How does the interest rate compare now? Explain.*

Solution. The code took 57.35 seconds to run, using $[0, 1/\beta - 1]$ as the initial bracket for the bissection method. The equilibrium interest rate is $r \approx 0.04088093$. The resulting $\mathbb{E}[a(r)]$ function is reported in Figure 10a. The value function, the policy functions and the stationary distribution are reported in the panels of Figure 11. Finally, Euler equation errors are reported in Figure 10b, with some statistics on them reported in Table 5. Regarding these EEEs, focusing on the mean we can interpret that, on average, a \$1 mistake is made for every \$ $10^{2.40} \approx$ \$251.19 spent.

Once again, in comparison to the result obtained with $\sigma = .01$ in (c), the equilibrium interest rate becomes significantly lower. The channel is similar to that explained in (d): as the standard deviation of the income process increases, the riskiness of future income increases, and the precautionary saving motive strengthens. In other words, individuals will tend to save more to protect themselves against the risk of future income fluctuations. Therefore, an increase in the uncertainty of the stochastic income process will ultimately lead to a decrease in the equilibrium interest rate. It is worth noting how with a higher standard deviation the stationary distribution looks more interesting than in the previous exercises.

Figure 10: $\mathbb{E}[a(r)]$ function and Euler equation errors, with $\sigma = .05$, for the equilibrium interest rate level $r \approx 0.04088$.



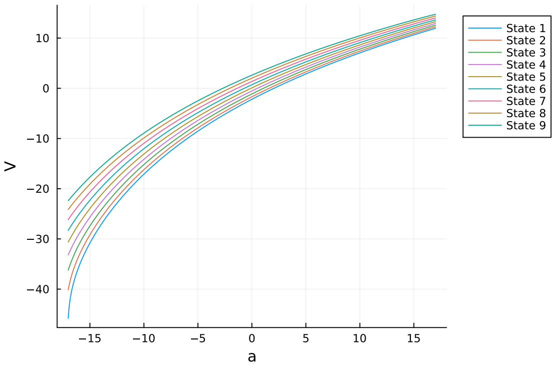
(a) $\mathbb{E}[a(r)]$ (b) Euler Equation Errors

Table 5: Euler Equation Errors (EEEs) statistics, with $\sigma = .05$, for the equilibrium interest rate level $r \approx 0.04088$.

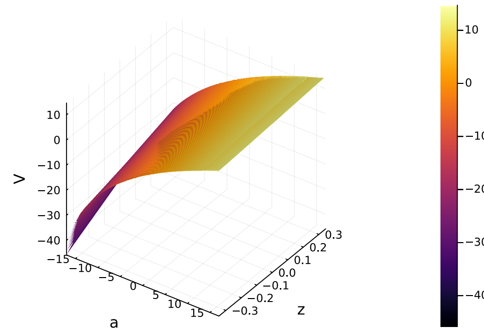
Euler Equation Errors' Statistics	
Mean	≈ -2.40
Median	≈ -2.47
Maximum	≈ -0.12
Minimum	≈ -6.44

□

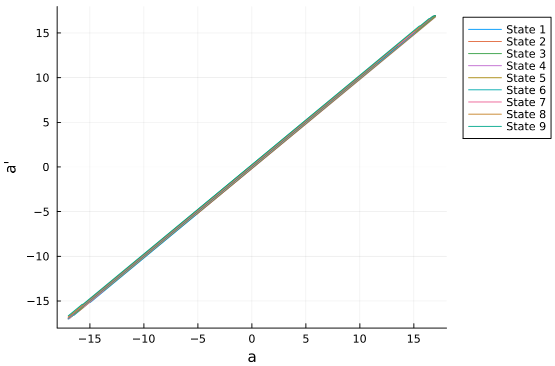
Figure 11: A Hugget economy — Value function, policy functions and stationary distribution, with $\sigma = .05$, for the equilibrium interest rate level $r \approx 0.04088$.



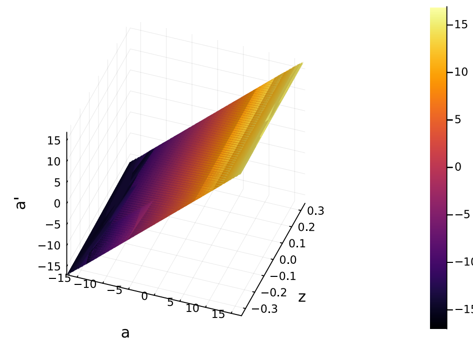
(a) Value function — 2D plot.



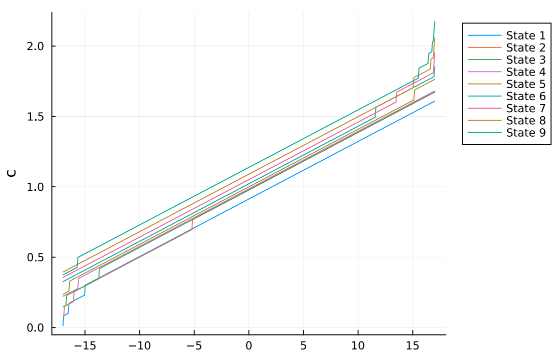
(b) Value function — 3D plot.



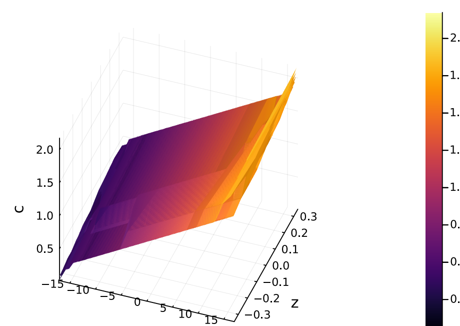
(c) Asset policy function — 2D plot.



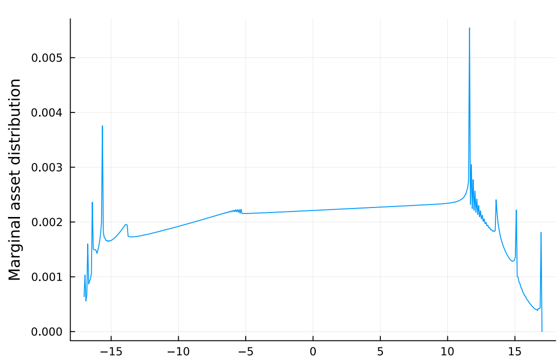
(d) Asset policy function — 3D plot.



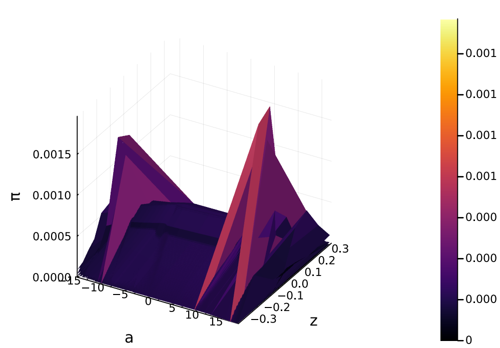
(e) Consumption policy function — 2D plot.



(f) Consumption policy function — 3D plot.



(g) Stationary marginal asset distribution.



(h) Stationary asset distribution.

(g) *Relate your results with Table 2 in Aiyagari (1994).*

A. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.2$)

$\rho\backslash\mu$	1	3	5
0	4.1666/23.67	4.1456/23.71	4.0858/23.83
0.3	4.1365/23.73	4.0432/23.91	3.9054/24.19
0.6	4.0912/23.82	3.8767/24.25	3.5857/24.86
0.9	3.9305/24.14	3.2903/25.51	2.5260/27.36

B. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.4$)

$\rho\backslash\mu$	1	3	5
0	4.0649/23.87	3.7816/24.44	3.4177/25.22
0.3	3.9554/24.09	3.4188/25.22	2.8032/26.66
0.6	3.7567/24.50	2.7835/26.71	1.8070/29.37
0.9	3.3054/25.47	1.2894/31.00	−0.3456/37.63

Figure 12: Table 2 in Aiyagari (1994).

Solution. In summary, the preceding exercises reveal that, all else being equal, an increase in both the persistence parameter ρ, the CRRA risk-aversion parameter γ, and the standard deviation of the stochastic income process σ lead to a decrease in the economy's equilibrium interest rate. This outcome aligns with the findings reported in Table 2 of Aiyagari (1994). Indeed, as depicted in parts A and B of the table, upsurges in both ρ and μ (CRRA parameter in Aiyagari, 1994) correspond to a decrease in the net return to capital — i.e., the equilibrium interest rate. Additionally, a comparison between parts A and B of Table 2 also indicates that an increase in σ results in a reduction of the equilibrium interest rate. Therefore, results from Table 2 of Aiyagari (1994) qualitatively mirror the outcomes obtained in this problem set.

Also related to this table, the main point Aiyagari (1994) emphasized is that the differences between the saving rates with and without full insurance are quite small for moderate and empirically plausible values of σ, ρ and μ. However, for high values of σ, ρ, and μ, the presence of idiosyncratic risk can raise the saving rate quite significantly. These findings also align with the results presented in this problem set. Indeed, the small change in ρ from item (c) to (d) did not significantly impact the equilibrium interest rate relative to the full insurance interest rate $1/\beta - 1 \approx 0.04167$. Conversely, when γ and σ underwent more pronounced changes in the subsequent items, we noted more significant shifts in the interest rate.

□

A Plotting all the $\mathbb{E}[a(r)]$ curves together

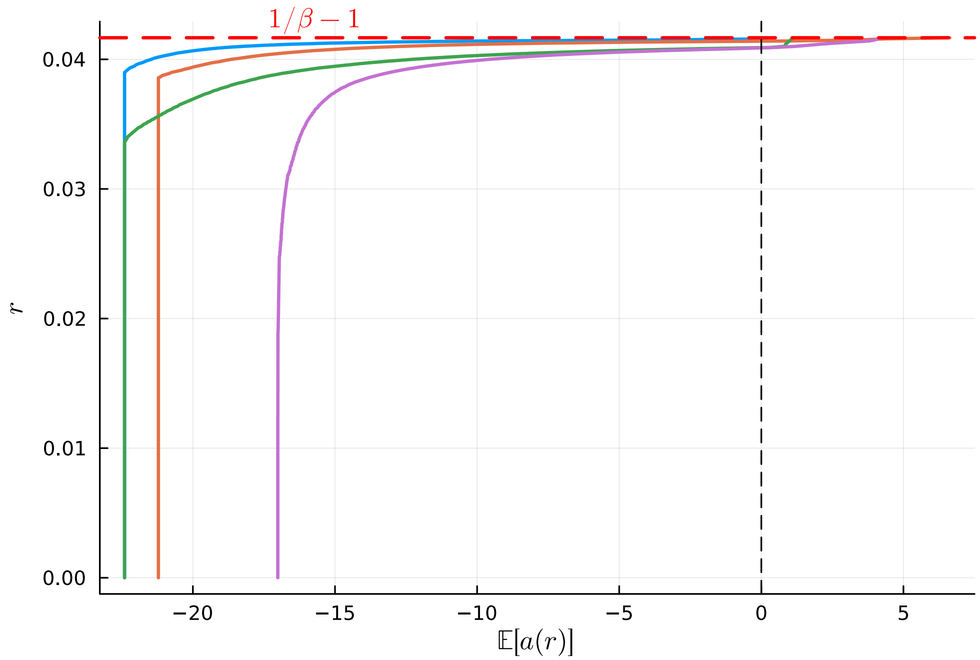


Figure 13: $\mathbb{E}[a(r)]$ curves from items (c), (d), (e) and (f) plotted together.

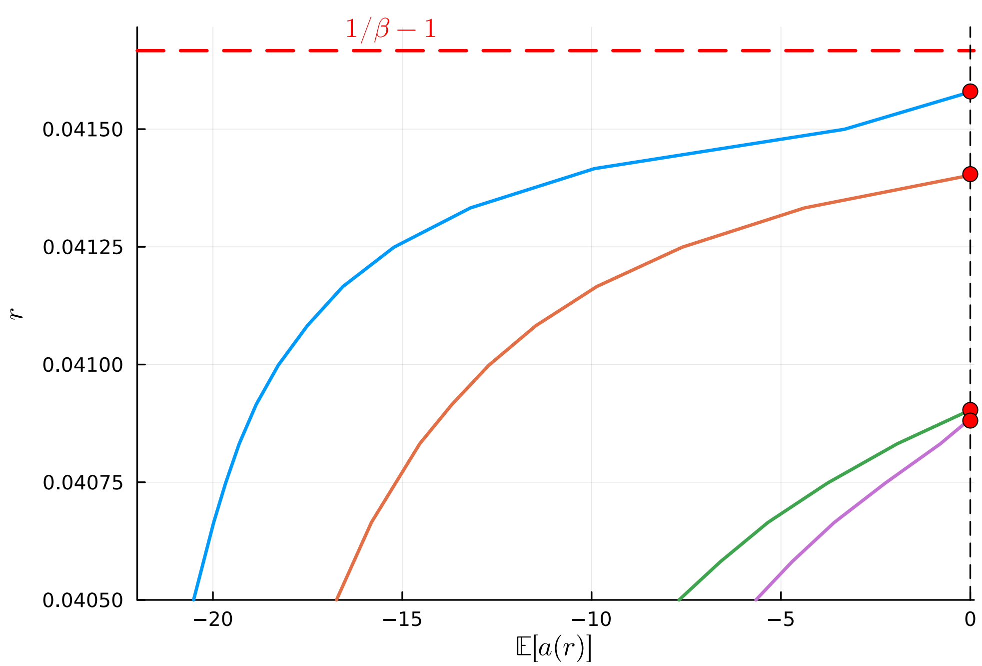


Figure 14: Zoomed in $\mathbb{E}[a(r)]$ curves from items (c), (d), (e) and (f) plotted together. The red dots indicate the equilibrium interest rate levels.