Advanced Control Engineering

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

Stabilizability - I

Our familiar LTI state-space system

$$\underset{[N\times 1]}{\dot{\mathbf{x}}} = \underset{[N\times N]}{\mathbf{A}}\underset{[N\times 1]}{\mathbf{x}} + \underset{[N\times M]}{\mathbf{B}}\underset{[M\times 1]}{\mathbf{u}}$$

$$\underset{[P\times 1]}{\mathbf{y}} = \underset{[P\times N]}{\mathbf{C}}\underset{[N\times 1]}{\mathbf{x}} + \underset{[P\times M]}{\mathbf{D}}\underset{[M\times 1]}{\mathbf{u}}$$

is completely controllable iff the controllability matrix has rank N

$$\underset{[N\times NM]}{\mathbf{Q}} = \left[\underset{[N\times M]}{\mathbf{B}} \quad \underset{[N\times M]}{\mathbf{AB}} \quad \underset{[N\times M]}{\mathbf{A}^2\mathbf{B}} \quad \cdots \quad \underset{[N\times M]}{\mathbf{A}^{N-1}\mathbf{B}}\right]$$

Stabilizability - II

Remember that completely controllable means we can transfer the state of the LTI system from any initial state $\mathbf{x}(t_0)$ to any final state $\mathbf{x}(t_1)$ within a finite time.

What if the system is not completely controllable (i.e. $\operatorname{rank}(\mathbf{Q}) = m < N$)?

Is there anything we can do?

Stabilizability - III

There is something we can salvage if we first define the concept of a *controllable subspace*.

The <u>controllable subspace</u> of an LTI system is the linear subspace consisting of all of the states that can be reached from the zero state within a finite time.

Stabilizability - IV

Q. How do we find the controllable subspace of a given LTI system?

A. The columns of the controllability matrix span the controllable subspace!

If $\text{rank}(\mathbf{Q}) = m < N$, then $\mathbf{Q}$ has m linearly independent columns.

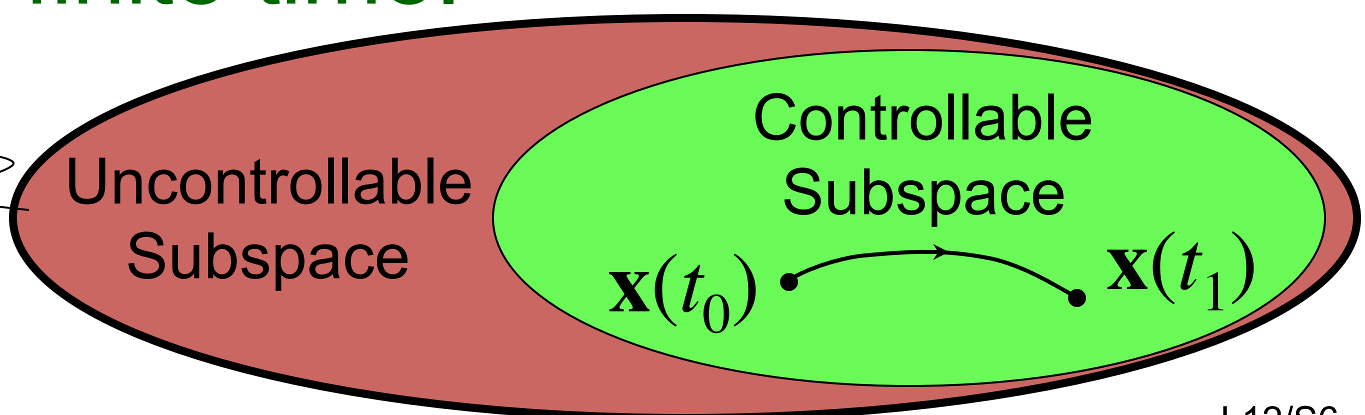
If $\mathbf{Q}$ is full rank, then it has N linearly independent columns which span the entire N-dimensional state space.

Stabilizability - V

Using the definition of controllable subspace:

We can always transfer the state of an LTI system from <u>any</u> initial state $\mathbf{x}(t_0)$ in the controllable subspace to <u>any</u> final state $\mathbf{x}(t_1)$ in the controllable subspace within a finite time.

Complete State Space



Stabilizability - VI

We can always choose m basis vectors that span the controllable subspace:

$$\underset{[N\times m]}{\mathbf{T}_c} = \begin{bmatrix} \underset{[N\times 1]}{\mathbf{e}_1} & \underset{[N\times 1]}{\mathbf{e}_2} & \cdots & \underset{[N\times 1]}{\mathbf{e}_m} \end{bmatrix}$$

We can also choose $N-m$ linearly independent vectors to span the uncontrollable subspace:

$$\underset{[N\times(N-m)]}{\mathbf{T}_u} = \begin{bmatrix} \underset{[N\times 1]}{\mathbf{e}_{m+1}} & \underset{[N\times 1]}{\mathbf{e}_{m+2}} & \cdots & \underset{[N\times 1]}{\mathbf{e}_N} \end{bmatrix}$$

Stabilizability - VII

Next, we can construct a nonsingular transformation matrix:

$$\underset{[N\times N]}{\mathbf{T}} = \begin{bmatrix} \underset{[N\times m]}{\mathbf{T}_c} & \underset{[N\times(N-m)]}{\mathbf{T}_u} \end{bmatrix}$$

Q. Why is this matrix nonsingular?

The inverse of this transformation matrix can be partitioned as:

$$\underset{[N\times N]}{\mathbf{T}^{-1}} = \underset{[N\times N]}{\mathbf{R}} = \begin{bmatrix} \underset{[m\times N]}{\mathbf{R}_c} \\ \underset{[(N-m)\times N]}{\mathbf{R}_u} \end{bmatrix}$$

Stabilizability - VIII

From a basic definition of matrix inverse:

$$\mathbf{T}^{-1}\mathbf{T} = \begin{bmatrix} \mathbf{R}_c \\ \mathbf{R}_u \end{bmatrix} \begin{bmatrix} \mathbf{T}_c & \mathbf{T}_u \end{bmatrix} = \begin{bmatrix} \mathbf{R}_c\mathbf{T}_c & \mathbf{R}_c\mathbf{T}_u \\ \mathbf{R}_u\mathbf{T}_c & \mathbf{R}_u\mathbf{T}_u \end{bmatrix} = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{N-m} \end{bmatrix}$$

Therefore, we see that:

Since $\mathbf{T}_c$ is made of columns that span the entire controllable subspace, $\mathbf{R}_u$ maps any vector $\mathbf{x}_c$ in the controllable subspace to the zero vector.

Stabilizability - IX

Finally, we can transform our original LTI system using the transformation:

$$\mathbf{x} = \mathbf{T}\hat{\mathbf{x}} \quad \Rightarrow$$

Using the partitioned matrices we have:

$$\left[\mathbf{T}^{-1}\mathbf{A}\mathbf{T}\right] = \begin{bmatrix} \mathbf{R}_c \\ \mathbf{R}_u \end{bmatrix} \mathbf{A} \begin{bmatrix} \mathbf{T}_c & \mathbf{T}_u \end{bmatrix} = \begin{bmatrix} \mathbf{R}_c\mathbf{A}\mathbf{T}_c & \mathbf{R}_c\mathbf{A}\mathbf{T}_u \\ \mathbf{R}_u\mathbf{A}\mathbf{T}_c & \mathbf{R}_u\mathbf{A}\mathbf{T}_u \end{bmatrix}$$

$$\left[\mathbf{T}^{-1}\mathbf{B}\right] = \begin{bmatrix} \mathbf{R}_c\mathbf{B} \\ \mathbf{R}_u\mathbf{B} \end{bmatrix}$$

Stabilizability - X

Since $\mathbf{T}_c$ is in the controllable subspace, then $\mathbf{AT}_c$ is also in the controllable subspace (Why?), therefore:

The columns of $\mathbf{B}$ are in the controllable subspace because $\mathbf{B}$ is part of the controllability matrix, therefore:

Stabilizability - XI

We can now write the LTI system in <u>controllable canonical form</u> as:

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}_c \\ \dot{\hat{\mathbf{x}}}_u \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{cc} & \mathbf{A}_{cu} \\ \mathbf{0} & \mathbf{A}_{uu} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_c \\ \hat{\mathbf{x}}_u \end{bmatrix} + \begin{bmatrix} \mathbf{B}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

where $\underset{[m\times 1]}{\hat{\mathbf{x}}_c}$ are controllable states

$\underset{[(N-m)\times 1]}{\hat{\mathbf{x}}_u}$ are uncontrollable states

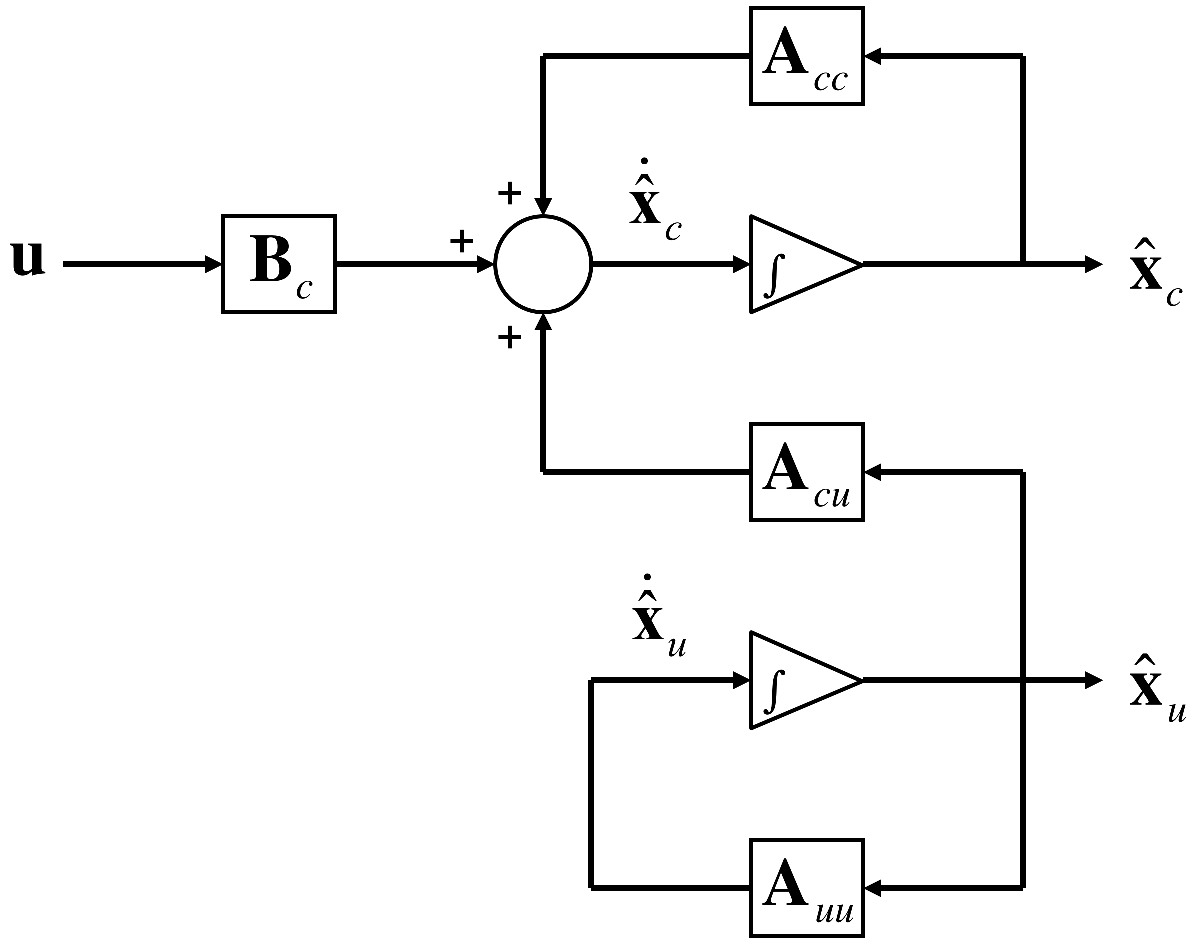
and

$$\underset{[m\times m]}{\mathbf{A}_{cc}} = \mathbf{R}_c \mathbf{A} \mathbf{T}_c \qquad \underset{[m\times(N-m)]}{\mathbf{A}_{cu}} = \mathbf{R}_c \mathbf{A} \mathbf{T}_u$$

$$\underset{[m\times M]}{\mathbf{B}_c} = \mathbf{R}_c \mathbf{B} \qquad \underset{[(N-m)\times(N-m)]}{\mathbf{A}_{uu}} = \mathbf{R}_u \mathbf{A} \mathbf{T}_u$$

Stabilizability - XII

In block diagram form:



Stabilizability - XIII

We now have the following facts:

- The pair $(\mathbf{A}_{cc}, \mathbf{B}_c)$ is completely controllable
- The uncontrollable states behave independently
- The controllable states are dependent on the input <u>and</u> on the uncontrollable states
- The controllability canonical form is NOT unique (Why?)
- A natural choice for the transformation matrix is the set of LTI eigenvectors

Stabilizability - XIV

Just like we partitioned the state space into controllable and uncontrollable subspaces, we can also partition it into stable and unstable subspaces.

Any initial state can be factored into components from each subspace:

In order to control the system properly, we must be able to completely control the unstable part!

Stabilizability - XV

An LTI system is stabilizable if its unstable subspace is contained in its controllable subspace.

In other words, all state vectors $\mathbf{x}_{unstable}$ in the unstable subspace must be in the controllable subspace

This is equivalent to saying that an LTI system is stabilizable iff $\mathbf{A}_{uu}$ is asymptotically stable.

Stabilizability Example - I

Start with the following LTI system:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -2 & -2 & 1 \\ 0 & -3 & 0 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

The open-loop poles are at $\{-1 \pm j, -3\}$

and the controllability matrix is:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

Stabilizability Example - II

The controllability matrix only has two linearly independent columns (the third is a linear combination of the first two).

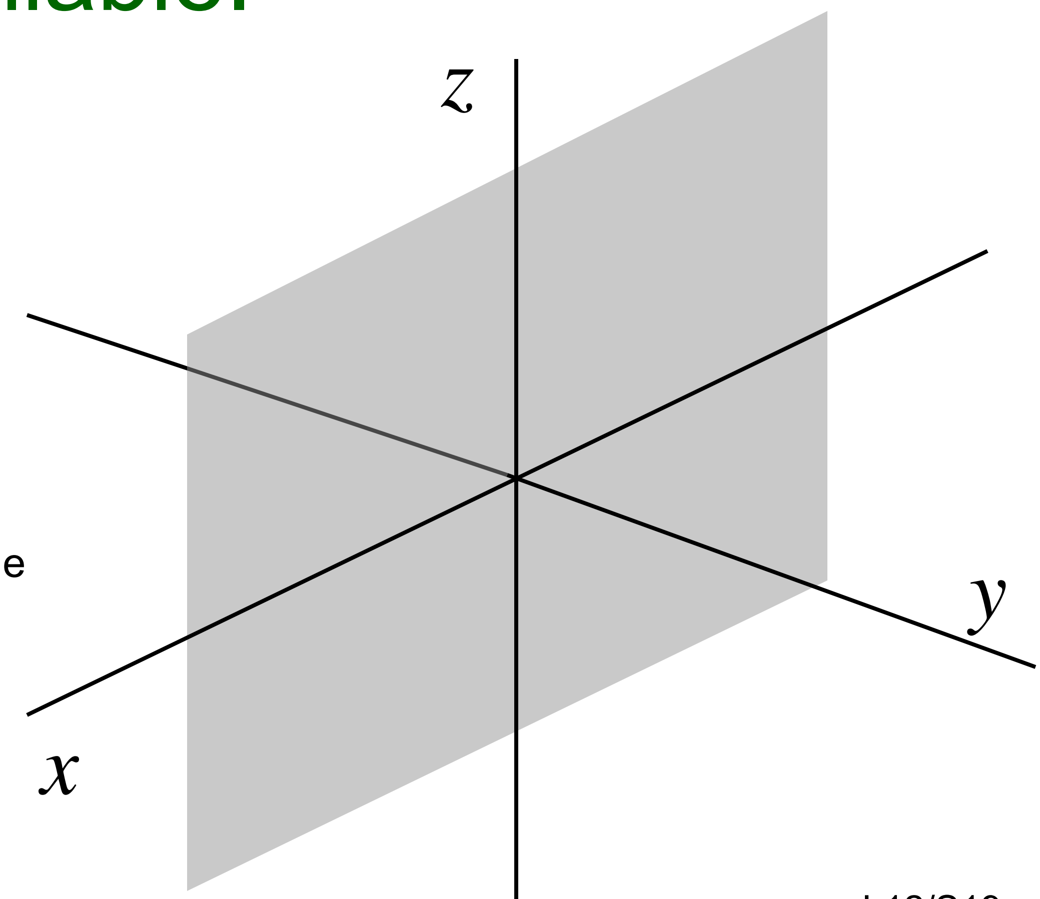
We can arbitrarily choose the first two columns of $\mathbf{Q}$ to span the controllable subspace (Why?).

$$\mathbf{T}_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{T}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Stabilizability Example - III

Geometrically, we see that the x-z plane is the controllable subspace and the y-axis is uncontrollable.

$$\mathbf{x} = \underbrace{\alpha_z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{Controllable Subspace}} + \underbrace{\alpha_y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{Uncontrollable Subspace}}$$



Stabilizability Example - IV

Substituting our basis vectors into the transformation matrix, we have

$$\mathbf{T} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} \quad \Rightarrow \quad \mathbf{T}^{-1} = \mathbf{R} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{bmatrix}$$

and the transformed LTI system in controllable canonical form is

$$\begin{bmatrix} \begin{bmatrix} \dot{\hat{x}}_{c1} \\ \dot{\hat{x}}_{c2} \end{bmatrix} \\ \begin{bmatrix} \dot{\hat{x}}_{u} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix} & \begin{bmatrix} -2 \\ -2 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} -3 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \hat{x}_{c1} \\ \hat{x}_{c2} \end{bmatrix} \\ \begin{bmatrix} \hat{x}_{u} \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \end{bmatrix} \end{bmatrix} u$$

Stabilizability Example - V

Expanding the controllable canonical form, we see that

$$\begin{bmatrix} \dot{\hat{x}}_{c1} \\ \dot{\hat{x}}_{c2} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_{c1} \\ \hat{x}_{c2} \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \hat{x}_u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\dot{\hat{x}}_u = -3\hat{x}_u$$

It is easy to check that $(\mathbf{A}_{cc}, \mathbf{B}_c)$ is completely controllable (How?).

The uncontrollable state is stable (Why?) so this system is stabilizable!

Stabilizability Example - VI

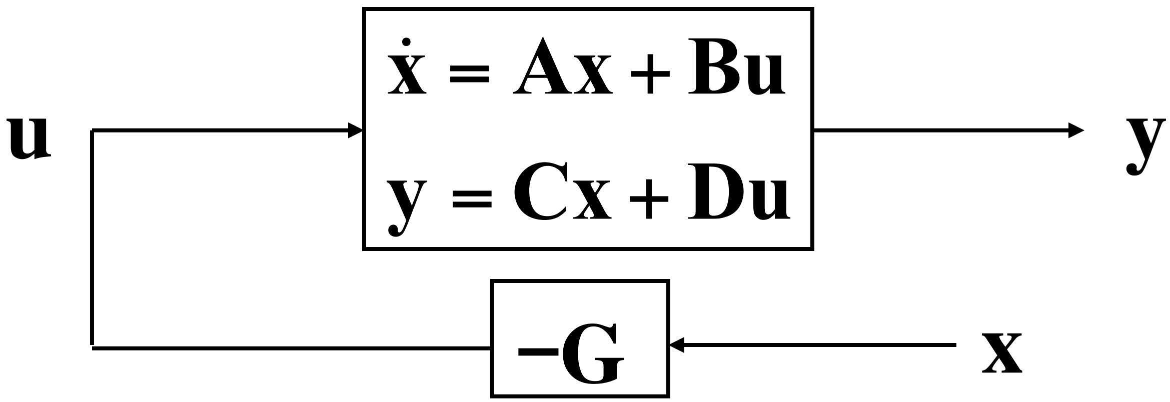
Because of the simple structure of the transformation matrix chosen in this example

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{c1} \\ \hat{x}_{c2} \\ \hat{x}_{u} \end{bmatrix} =$$

we see that y is the uncontrollable state. (This is not a typical result.)

State Estimation - I

We have only considered the case where all states are available for state feedback control. The only inputs are $\mathbf{u}$ and the only outputs are $\mathbf{y}$



Full state feedback using direct states is often <u>impractical</u> or <u>impossible</u>.

State Estimation - II

How do we design the feedback compensator if:

- We only have outputs available for feedback
- There are fewer outputs than states

Both of these conditions represent the "normal" or "typical" situation.

Instead of <u>full state feedback</u> control, we really need <u>output feedback</u> control.

State Estimation - III

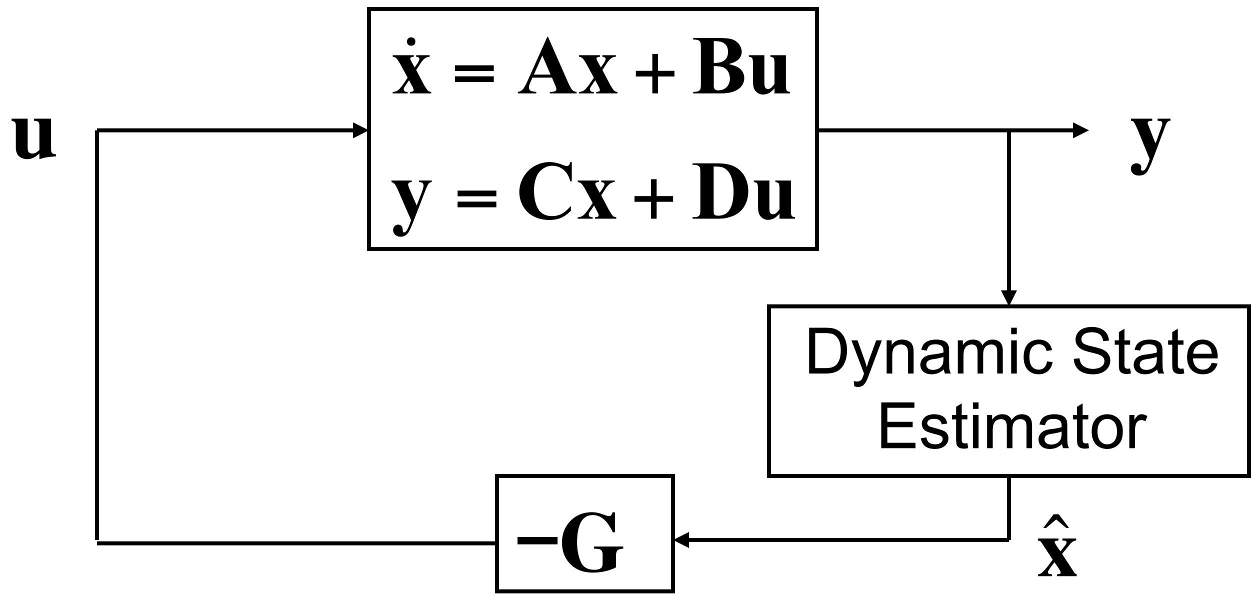
We also want to take advantage of everything we learned about full state feedback.

Here is the approach we will take:

- Design a <u>state estimator</u> or <u>dynamic observer</u> to estimate the full state vector using the actual outputs $\mathbf{y}$
- Use the estimated state vector in a full state feedback control law

State Estimation - IV

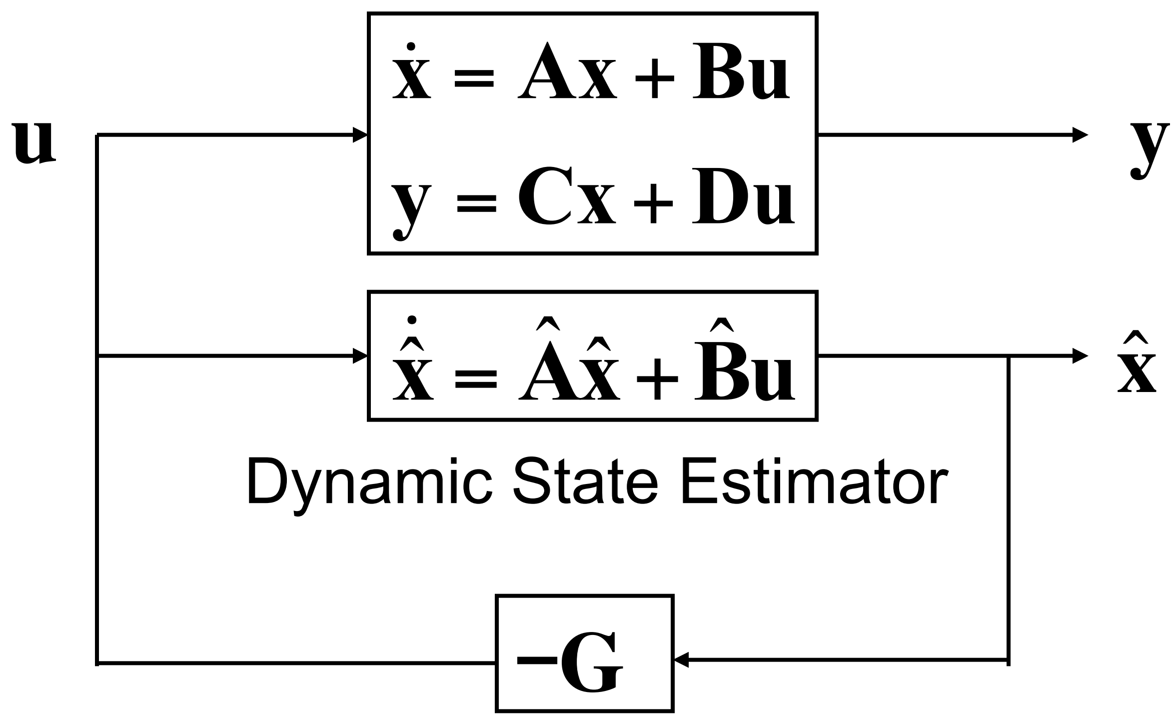
In block diagram form:



This only works if we can guarantee that the estimated states converge to the actual states!

State Estimation - V

What if we try to use an open-loop model of our plant as a dynamic observer?



Will this work?

State Estimation - VI

The state estimation error dynamics can tell us:

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} \quad \Rightarrow \quad \dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}$$

$$\dot{\mathbf{e}} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) - (\hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u})$$

$$\dot{\mathbf{e}} = (\mathbf{A}\mathbf{x} - \hat{\mathbf{A}}\hat{\mathbf{x}}) + (\mathbf{B} - \hat{\mathbf{B}})\mathbf{u}$$

Assuming we have a perfect state model

$$\left.\begin{aligned} \hat{\mathbf{A}} &= \mathbf{A} \\ \hat{\mathbf{B}} &= \mathbf{B} \end{aligned}\right\} \quad \Rightarrow$$

State Estimation - VII

This shows that the state error WILL converge to zero for this method, but only IF the open-loop state matrix $\mathbf{A}$ is asymptotically stable.

Unfortunately, the convergence rate of the error is defined by the open-loop dynamics (poles) of the plant.

We really need to be able to control the state estimation convergence rate, but this method does not allow it!

In Class Assignment

Estimate the convergence rate of the state estimation error for the following LTI system:

$$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\mathbf{u}$$

State Estimation - VIII

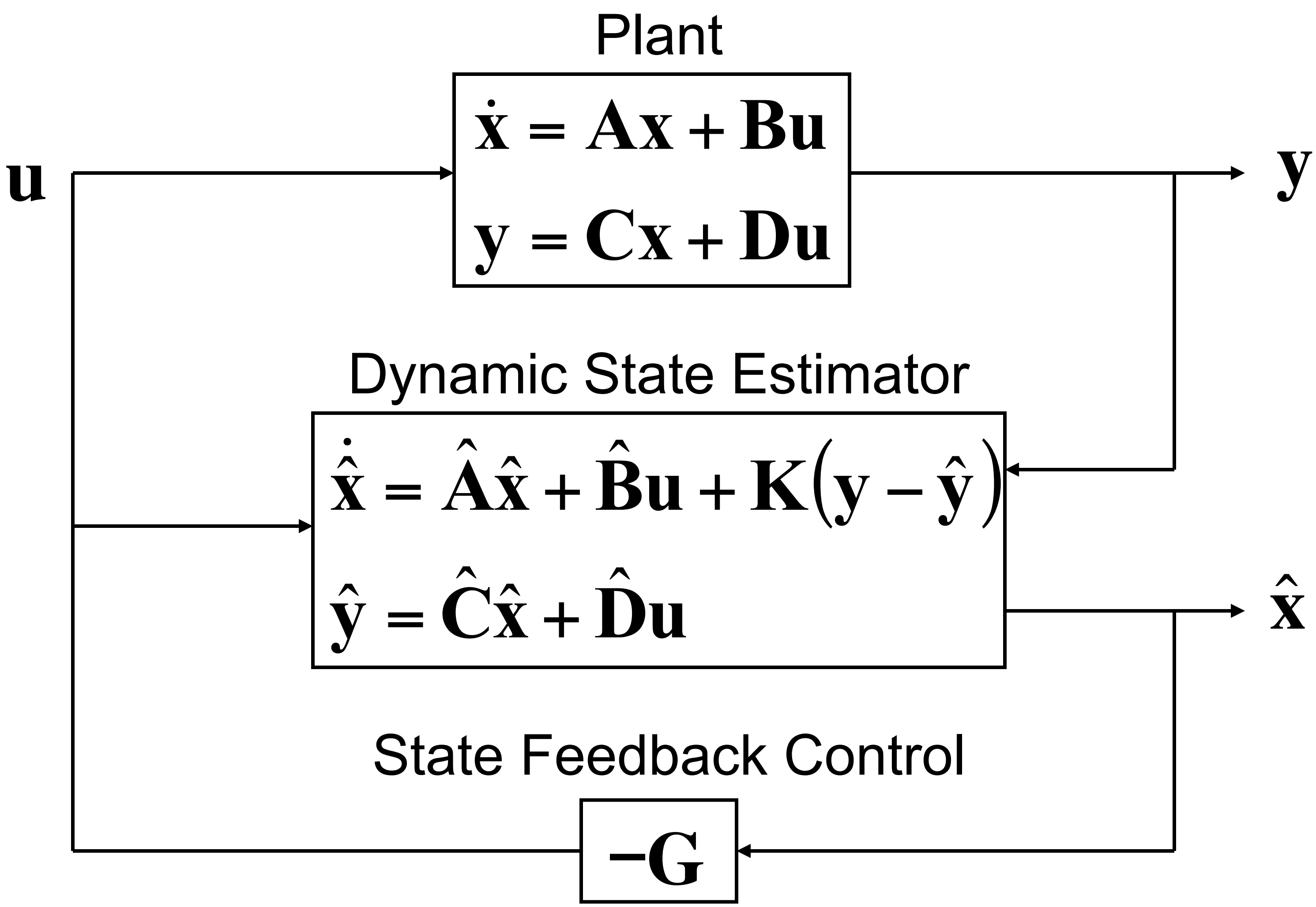
We can modify this observer to ensure that the state error converges to zero in a finite and desired amount of time.

Solution:

Feed back an ESTIMATE of the output and compare it to the actual output of the system.

State Estimation - IX

Block diagram of a Luenberger Observer



State Estimation - X

This dynamic observer…

- has two vector inputs, $\mathbf{y}$ and $\mathbf{u}$
- has one vector output, the state estimate
- is a dynamic system, i.e. it has internal states which evolve in time
- has a fixed gain (static) feedback matrix $\mathbf{K}$ which must be designed
- requires a state-space plant model
- was first published in IEEE T-AC 16(6) 1971, by David Luenberger

State Estimation - XI

Let' s look at the state estimation error dynamics for the Luenberger observer.

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} \quad \Rightarrow \quad \dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}$$

$$\dot{\mathbf{e}} = (\mathbf{Ax} + \mathbf{Bu}) - \left(\hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} + \mathbf{K}(\mathbf{y} - \hat{\mathbf{y}})\right)$$

$$\dot{\mathbf{e}} = \left(\mathbf{Ax} - \hat{\mathbf{A}}\hat{\mathbf{x}}\right) + \left(\mathbf{B} - \hat{\mathbf{B}}\right)\mathbf{u} - \mathbf{K}\left((\mathbf{Cx} + \mathbf{Du}) - \left(\hat{\mathbf{C}}\hat{\mathbf{x}} + \hat{\mathbf{D}}\mathbf{u}\right)\right)$$

$$\dot{\mathbf{e}} = \left(\mathbf{Ax} - \hat{\mathbf{A}}\hat{\mathbf{x}}\right) - \mathbf{K}\left(\mathbf{Cx} - \hat{\mathbf{C}}\hat{\mathbf{x}}\right) + \left(\mathbf{B} - \hat{\mathbf{B}}\right)\mathbf{u} - \mathbf{K}\left(\mathbf{D} - \hat{\mathbf{D}}\right)\mathbf{u}$$

If we now assume a perfect plant model:

$$\left.\begin{array}{ll} \hat{\mathbf{A}} = \mathbf{A} & \hat{\mathbf{C}} = \mathbf{C} \\ \hat{\mathbf{B}} = \mathbf{B} & \hat{\mathbf{D}} = \mathbf{D} \end{array}\right\} \quad \Rightarrow$$

State Estimation - XII

The state estimation error will converge to zero if the matrix $[\mathbf{A}-\mathbf{KC}]$ is asymptotically stable.

The poles of $[\mathbf{A}-\mathbf{KC}]$ will govern the convergence rate of the estimation error.

We can choose the gain matrix $\mathbf{K}$ to place the poles of $[\mathbf{A}-\mathbf{KC}]$ wherever we want them.

Or can we? Does this problem look familiar?

Summary

We learned that even if an LTI system is not controllable, we can still stabilize it if it is stabilizable and we can still control states in the controllable subspace.

We learned that a Luenberger observer can be used to estimate the states of an LTI system using output feedback, and that this leads to a pole placement problem very similar to the control problem.