Course Outline - 1st Half

- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response, Matrix Exponential
- Transfer Functions & Eigenvalues
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

Design Methodology

A general design methodology for pole placement is:

- Model the system in state-space
- Transform the design specifications into pole locations of the s-plane that will yield *acceptable* performance
- Compute full-state feedback gains
- Simulate the response
- Iterate on the pole locations if necessary

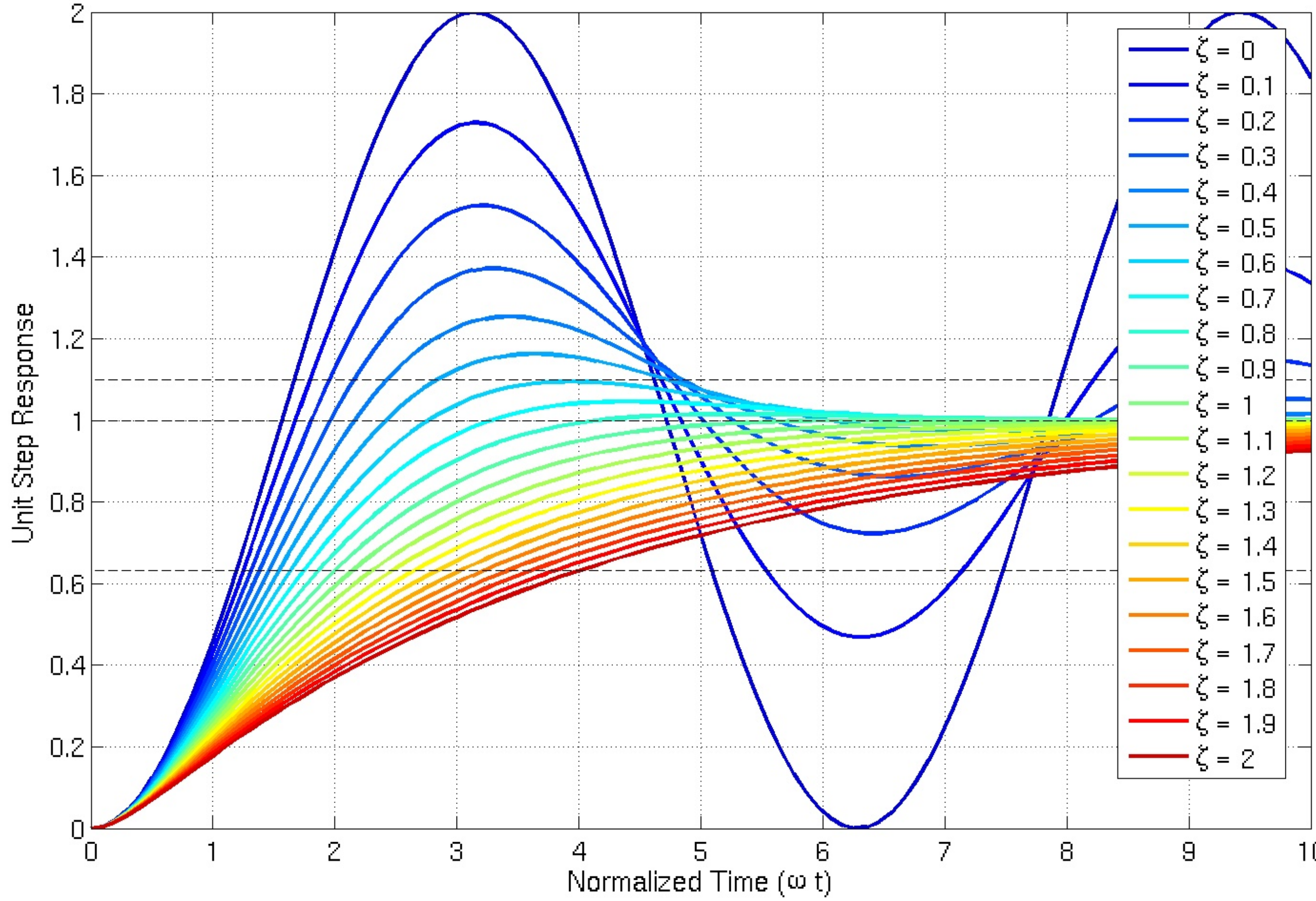
In-Class Assignment

Determine the region of the s-plane to place the poles of a <u>second-order</u> system to have a rise time of < 300 ms. and an overshoot of < 10%.

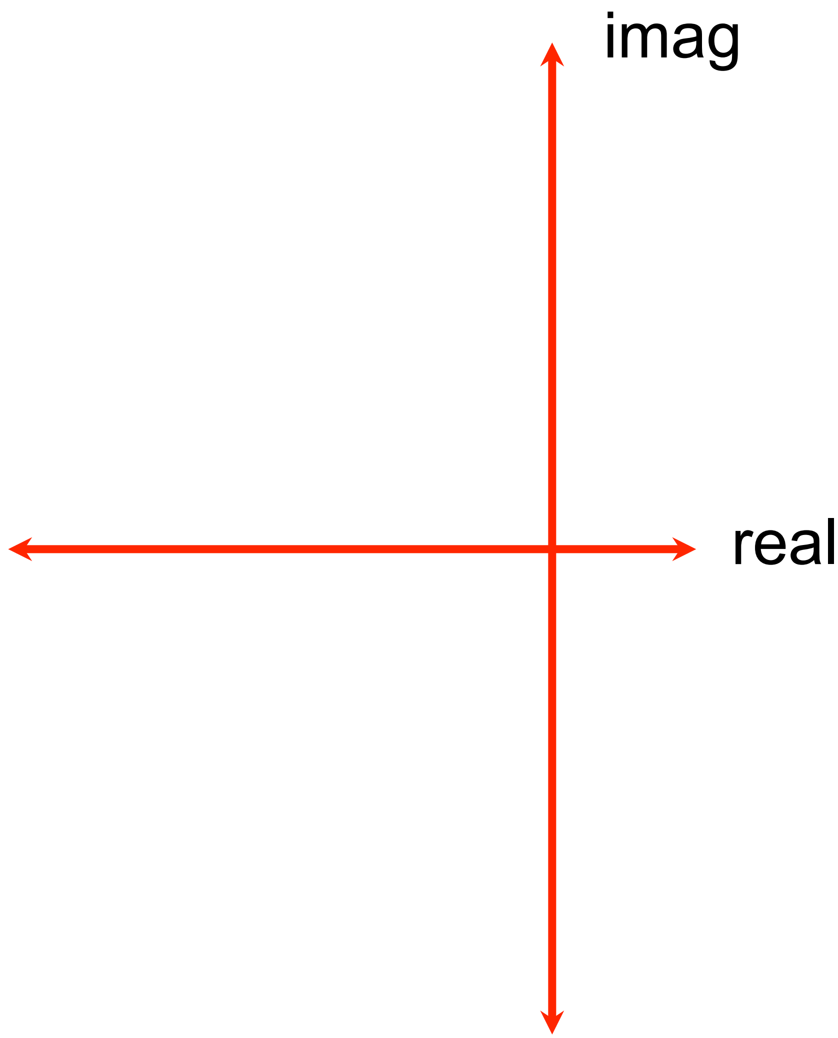
- <u>Rise Time</u> = Time for the output to reach a specified fraction (usually 1 time constant; $(1\text{-}e^{-1})=0.632$ or 63.2%) of the distance between the initial and steady-state values
- <u>Overshoot</u> = Difference between peak value and steady-state value

In-Class Assignment

Use a published normalized step response plot, or generate your own



In-Class Assignment



Time Domain Response - I

Many closed-loop design requirements are related to or are directly specified in terms of the system time response, for example, the settling time or convergence rate.

Pole placement is equivalent to "shaping" the time-domain response.

Selecting desired pole locations (design) requires a good understanding of the effect of poles on the time response.

Time Domain Response - II

For a homogeneous (unforced) LTI dynamic system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

we learned (L5) that the time response evolves from the initial condition by the following matrix exponential solution:

Time Domain Response - III

We also learned from the “method of diagonalization” that the matrix exponential can be expanded as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{V}e^{\Lambda t}\mathbf{V}^{-1}\mathbf{x}(0)$$

where the eigenvectors & eigenvalues are

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_N \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{bmatrix} \qquad \mathbf{A}\mathbf{v}_n = \mathbf{v}_n\lambda_n$$

Time Domain Response - IV

Now define a vector of constants:

$$\underset{[N\times 1]}{\mathbf{c}_0} = \underset{[N\times N]}{\mathbf{V}^{-1}}\,\underset{[N\times 1]}{\mathbf{x}(0)} = \begin{bmatrix} c_1 & c_2 & \cdots & c_N \end{bmatrix}^T$$

For convenience, let's assume all the eigenvalues are real and stable (all poles on the negative real axis). The total time response is:

Time Domain Response - V

The modal expansion theorem tells us that any LTI system response to initial conditions can be represented as a summation of contributions from each individual mode.

$$\mathbf{x}(t) = \sum_{n=1}^{N} c_n \mathbf{v}_n e^{\lambda_n t} = \underbrace{c_1 \mathbf{v}_1 e^{\lambda_1 t}}_{\text{Mode } 1} + \cdots + \underbrace{c_N \mathbf{v}_N e^{\lambda_N t}}_{\text{Mode } N}$$

Depending on the initial condition, some modes may or may not be present in the time response (How?)

Time Domain Response - VI

For stable complex conjugate poles,

$$\lambda_n = -\sigma_n + j\omega_n \qquad \mathbf{v}_n = \alpha_n + j\beta_n$$

$$\lambda_{n+1} = -\sigma_n - j\omega_n \qquad \mathbf{v}_{n+1} = \alpha_n - j\beta_n$$

the modal expansion theorem still holds

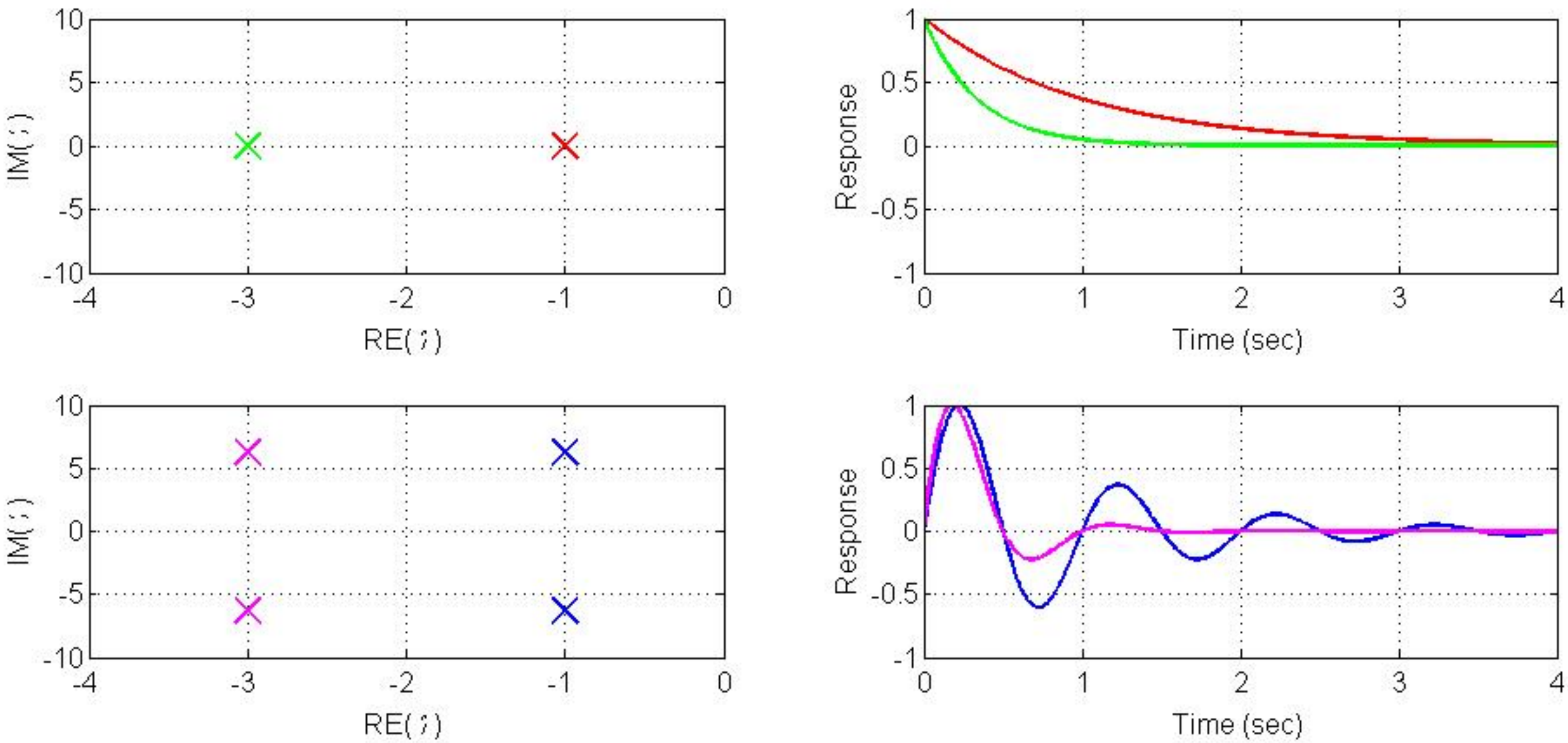
$$\mathbf{x}(t) = \cdots + c_n \mathbf{v}_n e^{\lambda_n t} + c_{n+1} \mathbf{v}_{n+1} e^{\lambda_{n+1} t} + \cdots$$

$$= \cdots + e^{-\sigma_n t}\left[\mathbf{w}_n \cos(\omega_n t) + \mathbf{w}_{n+1} \sin(\omega_n t)\right] + \cdots$$

and gives a real response characterized by sinusoidal oscillation with an exponentially decaying magnitude.

Time Domain Response - VII

ALL stable responses are represented by a summation of 1st order (real) and/or 2nd order (complex) poles



Time Domain Response - VIII

If the pole is stable ($RE(\lambda_n) < 0$), then the time constant is directly related to the real part of the eigenvalue.

$$e^{RE\left(\lambda_n\right)t} = e^{-\left(t/\tau_n\right)} \qquad \Rightarrow$$

This result is valid for real poles and/or complex conjugate poles.

Time Domain Response - IX

Some of the poles might be "fast"

$|\tau| << 1$ or $RE(\lambda_n) << 0$

Some might be "slow"

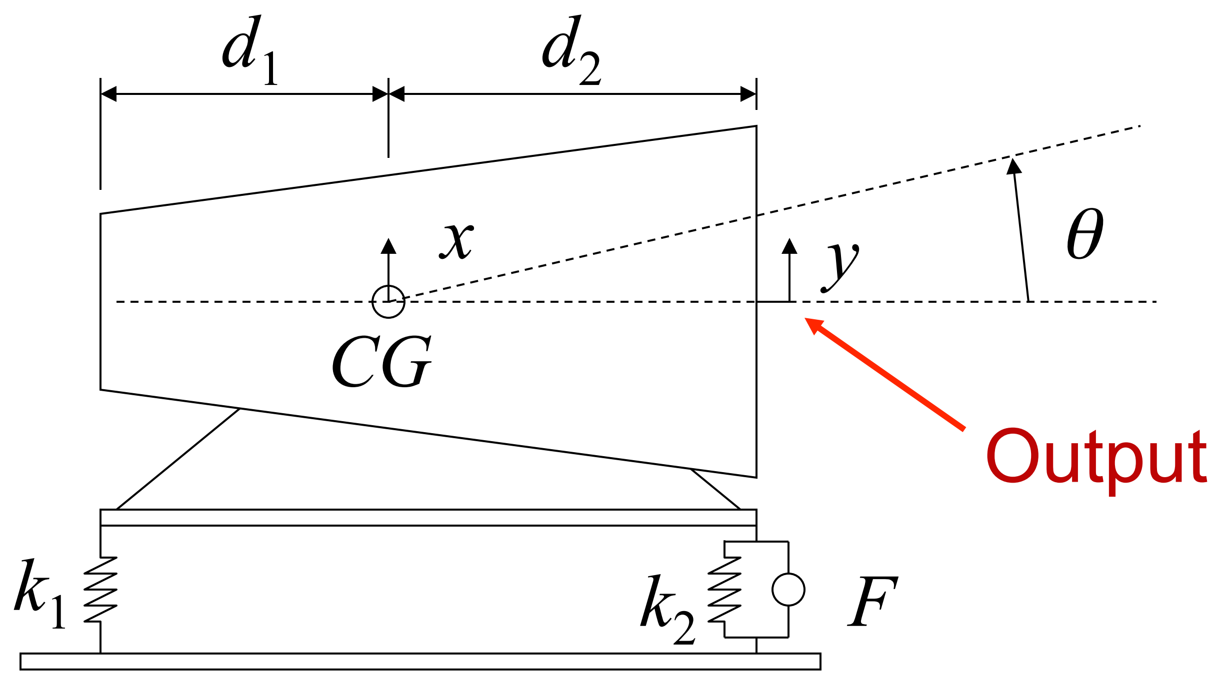
$|\tau| >> 1$ or $-1 << RE(\lambda_n) < 0$

In the modal expansion, the "fast" modes will decay quickly and the "slow" modes will dominate the response.

This is why poles close to the imaginary axis are called <u>dominant</u> poles.

FSF Design Example - I

Control the initial condition response of a flexible instrument platform using full-state feedback.



FSF Design Example - II

The design requirements are:

- Minimize the settling time of the output (5% of initial value) to an initial condition of $\theta_0 = 100$ micro-radians
- Peak output no greater than 100 microns (1 micron = 1e-6 meters)
- Peak control force no greater than 5 N

FSF Design Example - III

The second-order equations of motion are:

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}(t) \end{bmatrix}+\begin{bmatrix} k_1+k_2 & d_2k_2-d_1k_1 \\ d_2k_2-d_1k_1 & d_1^2k_1+d_2^2k_2 \end{bmatrix}\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix}=\begin{bmatrix} 1 \\ d_2 \end{bmatrix}F(t)$$

And the output equation is:

$$y(t)=x(t)+d_2\theta(t)$$

FSF Design Example - IV

Compute the state-space model of the system. First choose a set of state variables

$$\mathbf{z}(t) = \begin{bmatrix} x(t) \\ \theta(t) \\ \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix}$$

Substitute into the equations of motion

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \dot{z}_3(t) \\ \dot{z}_4(t) \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & d_2 k_2 - d_1 k_1 \\ d_2 k_2 - d_1 k_1 & d_1^2 k_1 + d_2^2 k_2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ d_2 \end{bmatrix} F(t)$$

$$y(t) = z_1(t) + d_2 z_2(t)$$

FSF Design Example - V

Rewrite the first-order equations:

$$\left.\begin{aligned}\dot{z}_1(t) &= z_3(t)\\ \dot{z}_2(t) &= z_4(t)\end{aligned}\right\}\ \text{Identity equations}$$

$$\dot{z}_3(t) = -\left(\frac{k_1+k_2}{m}\right)z_1(t) + \left(\frac{d_1k_1-d_2k_2}{m}\right)z_2(t) + \left(\frac{1}{m}\right)F(t)$$

$$\dot{z}_4(t) = \left(\frac{d_1k_1-d_2k_2}{J}\right)z_1(t) - \left(\frac{d_1^2k_1+d_2^2k_2}{J}\right)z_2(t) + \left(\frac{d_2}{J}\right)F(t)$$

Two second-order equations result in <u>four</u> first-order equations

FSF Design Example - VI

Place the equations in state-space matrix form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k_1+k_2}{m}\right) & \left(\frac{d_1k_1-d_2k_2}{m}\right) & 0 & 0 \\ \left(\frac{d_1k_1-d_2k_2}{J}\right) & -\left(\frac{d_1^2k_1+d_2^2k_2}{J}\right) & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \left(\frac{1}{m}\right) \\ \left(\frac{d_2}{J}\right) \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & d_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

FSF Design Example - VII

Define the following system parameters

$$m = 10 \qquad k_1 = 1000 \qquad d_1 = 0.402$$

$$J = 40 \qquad k_2 = 1000 \qquad d_2 = 0.668$$

The Open-Loop poles are at:

$$\lambda = \left\{ +4j, \quad -4j, \quad +14j, \quad -14j \right\}$$

FSF Design Example - VIII

Transform the design requirements into 'acceptable' regions of the *s*-plane.

Which of the design requirements can be related directly to the location of the closed-loop poles?

Take the specifications one at a time.

FSF Design Example - IX

Specification 1: Minimize the settling time of the output (5% of initial value) to an initial condition of $\theta_0 = 100$ micro-radians

FSF Design Example - X

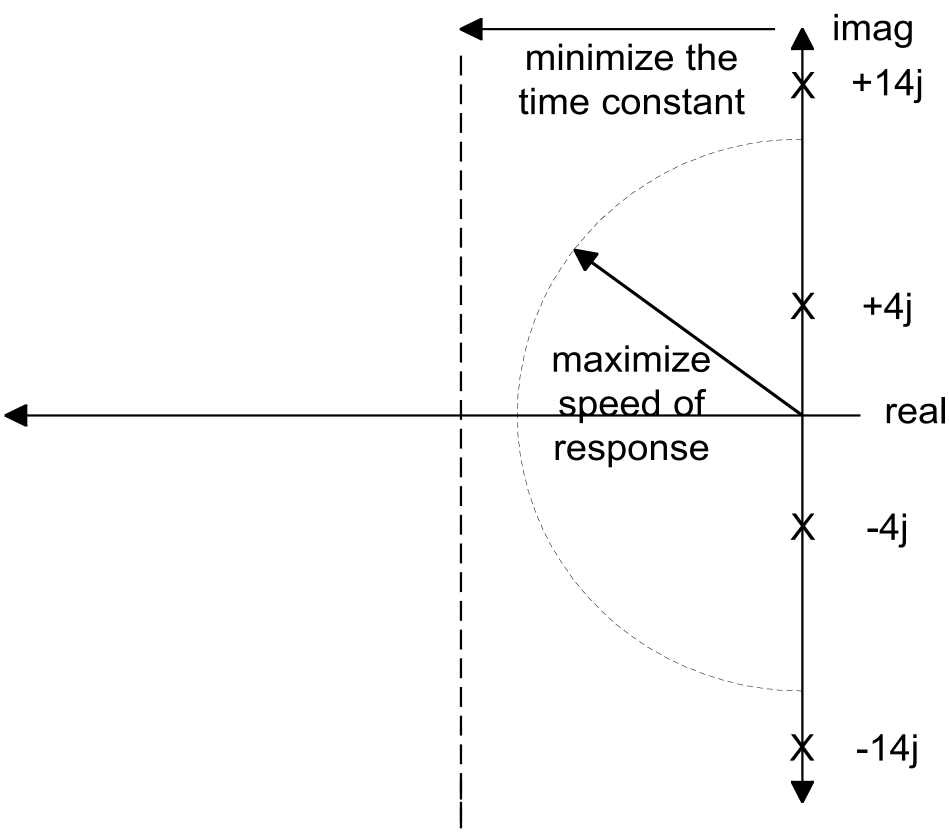
Specification 2: Peak output no greater than 100 microns (1 micron = 1e-6 meters)

FSF Design Example - XI

Specification 3: Peak control force no greater than 5 N

FSF Design Example - XII

Specification 1: Minimize settling time



Moving the real part into the left-half plane will decrease the time constant.

Moving the pole further from the origin will increase the speed of response.

Result: Specification 1 is directly related to the pole location in the *s*-plane

FSF Design Example - XIII

Specification 2: The peak output is related to the maximum over time of this matrix multiplication.

$$\left|y(t)\right|_{peak} = \left|\mathbf{C}\mathbf{x}(t)\right|_{peak} = \left|\mathbf{C}e^{(\mathbf{A}-\mathbf{B}\mathbf{G})t}\mathbf{x}(0)\right|_{peak} < 100 \text{ microns}$$

<u>Not easily related to the s-plane pole locations.</u>

The peak could be found by taking the time derivative of this expression and computing the maximum response. Again, not easy to relate to the s-plane pole locations.

FSF Design Example - XIV

Specification 3:

$$\left|u(t)\right|_{peak} = \left|-\mathbf{G}\mathbf{x}(t)\right|_{peak} = \left|-\mathbf{G}e^{(\mathbf{A}-\mathbf{B}\mathbf{G})t}\mathbf{x}(0)\right|_{peak} < 5\ \mathrm{N}$$

The peak output is again related to the maximum response over time of this expression. <u>Not easy to relate to the *s*-plane pole locations</u>.

We do know that the further into the LHP we place the closed-loop poles, the larger the gains will become.

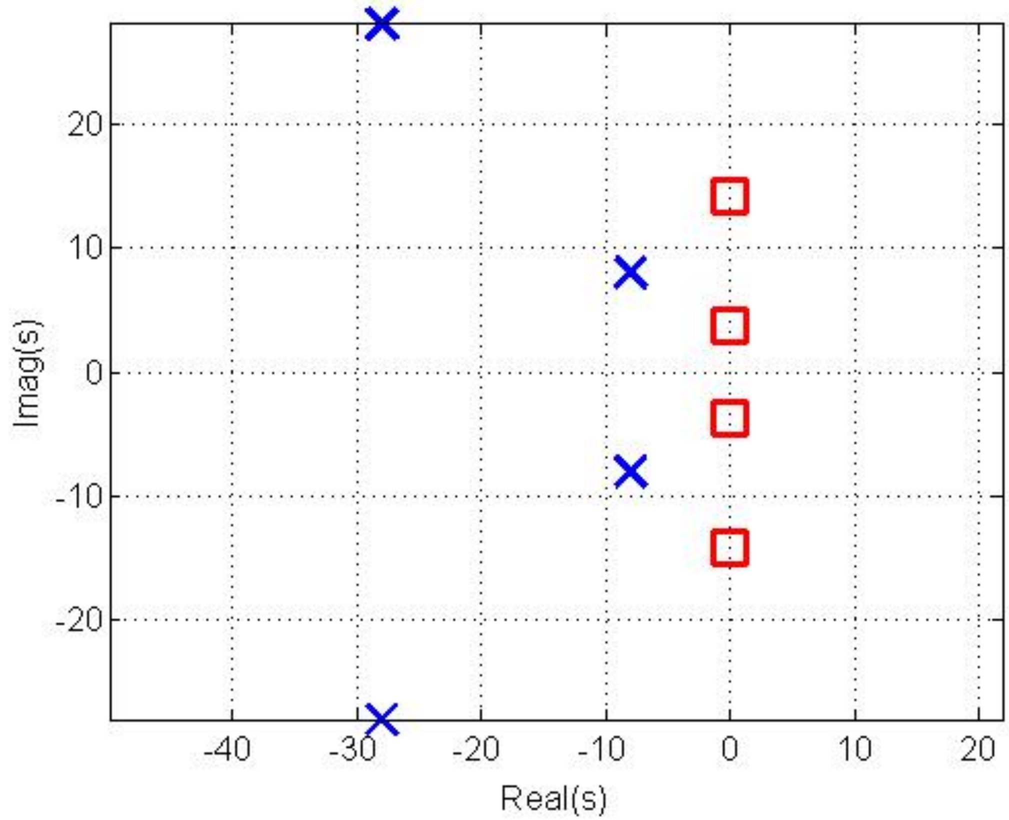
FSF Design Example - XV

Summary of results:

- Specification 1 can be translated into 'acceptable' regions of the s-plane.
- Specification 2 is difficult to translate into regions of the s-plane.
- Specification 3 is also difficult, but we know that the gains will increase the more we change the characteristic equation.

FSF Design Example - XVI

Iteration 1: Let's try to increase the speed of response by a factor of 2 and damp both poles with a damping factor of 0.707

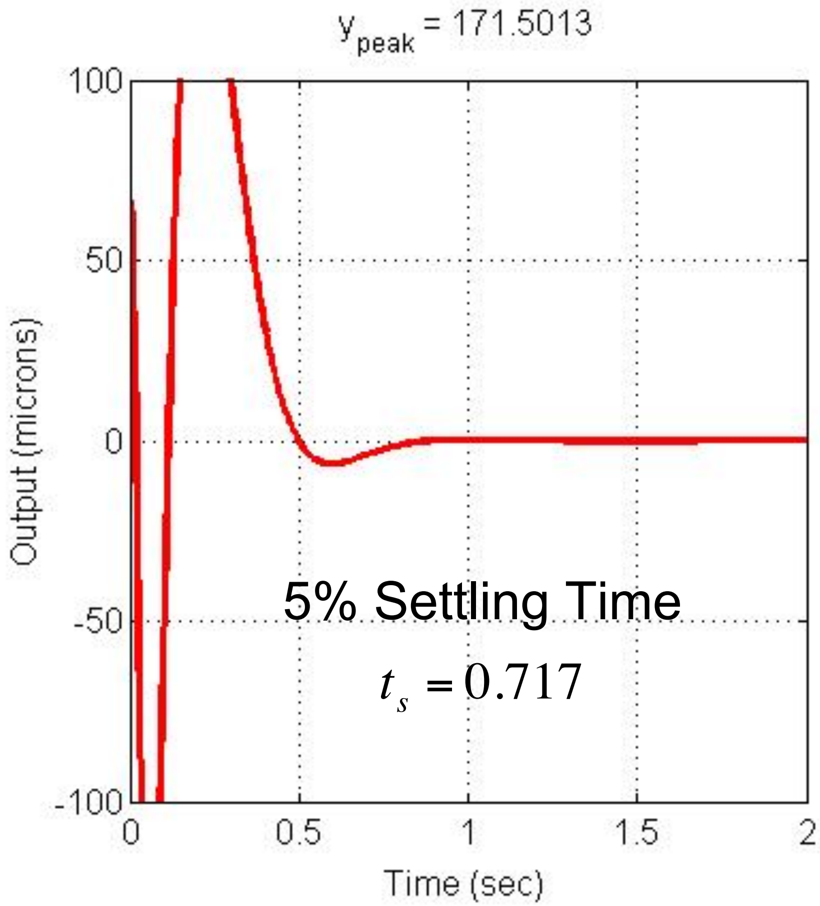


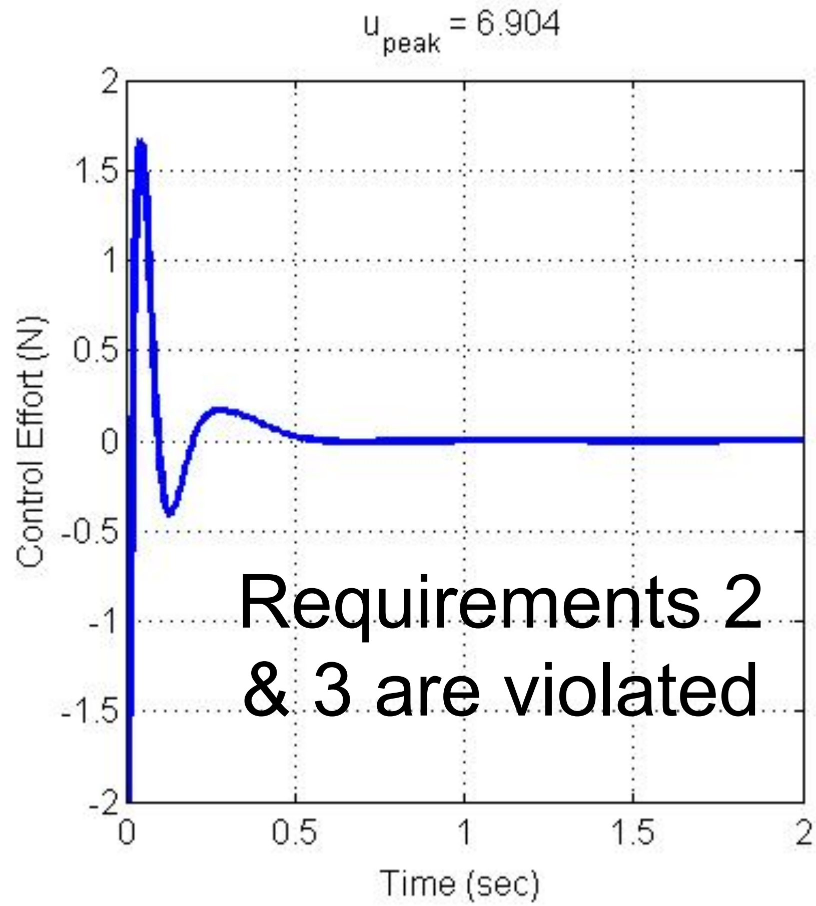
Desired Poles

$$\left\{-8 \pm 8j, -28 \pm 28j\right\}$$

FSF Design Example - XVII

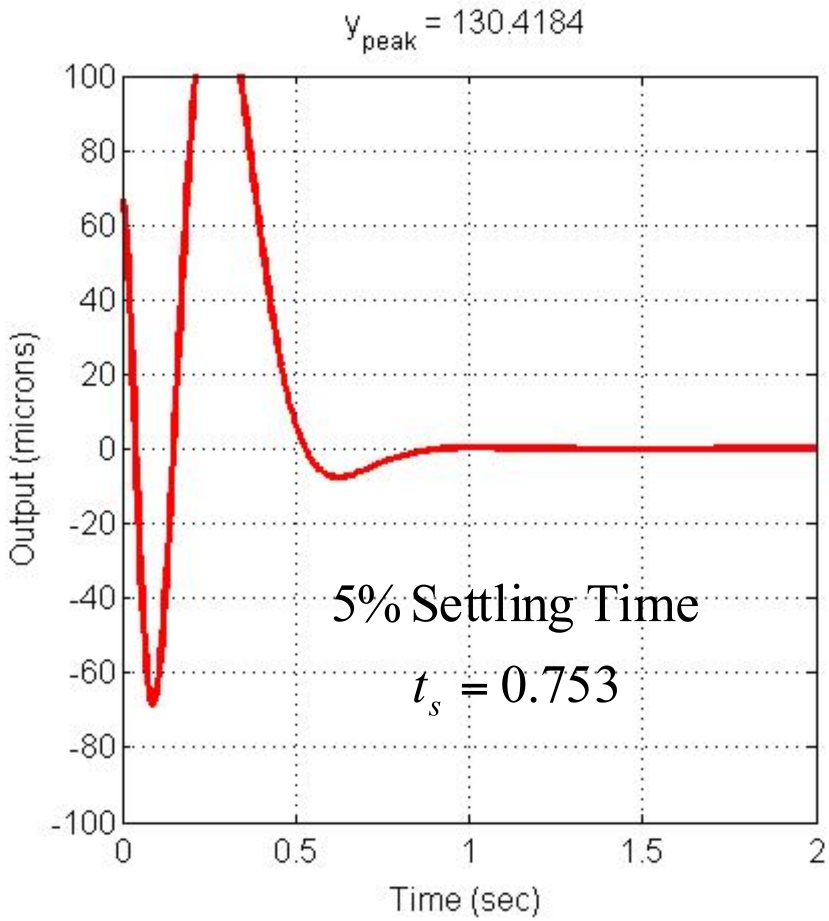
Iteration 1: Solve for the response using MATLAB

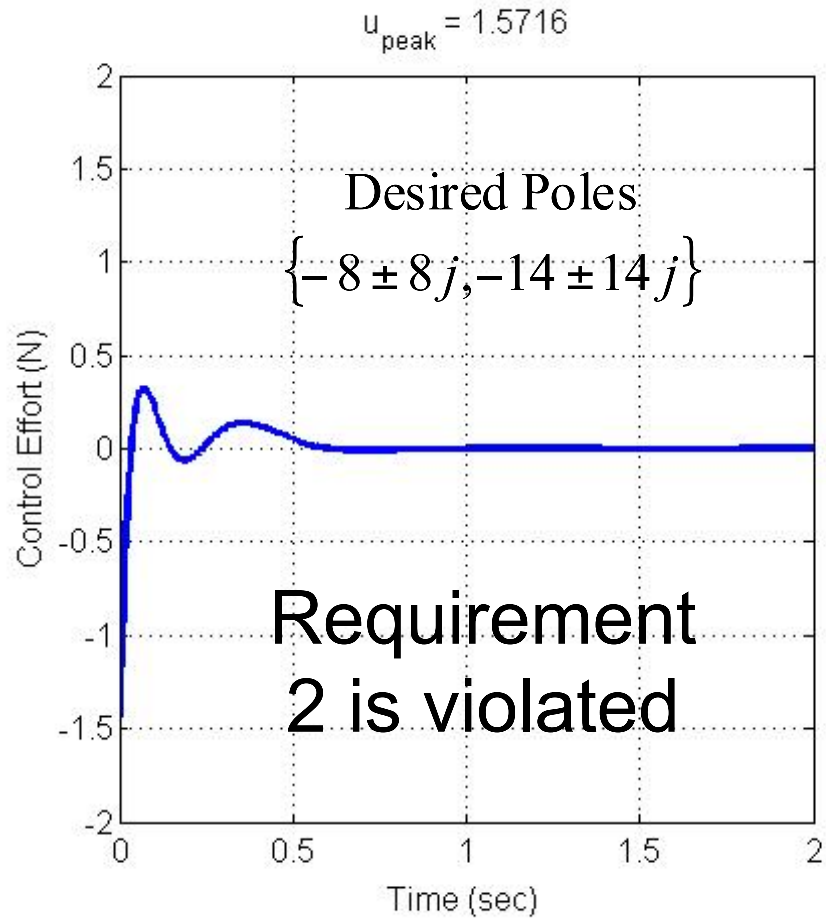




FSF Design Example - XVIII

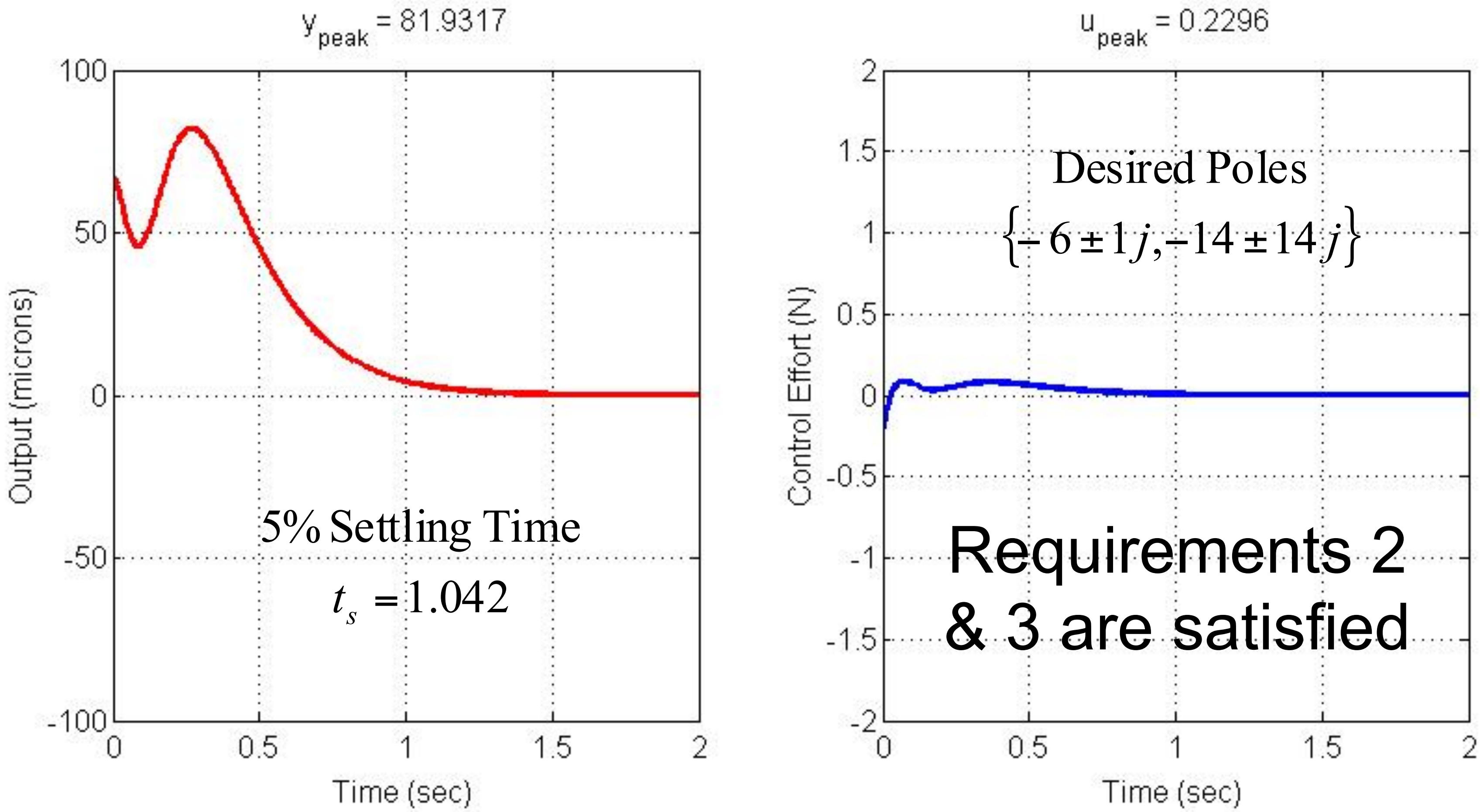
Iteration 2: Reduce the speed of response of the second set of poles





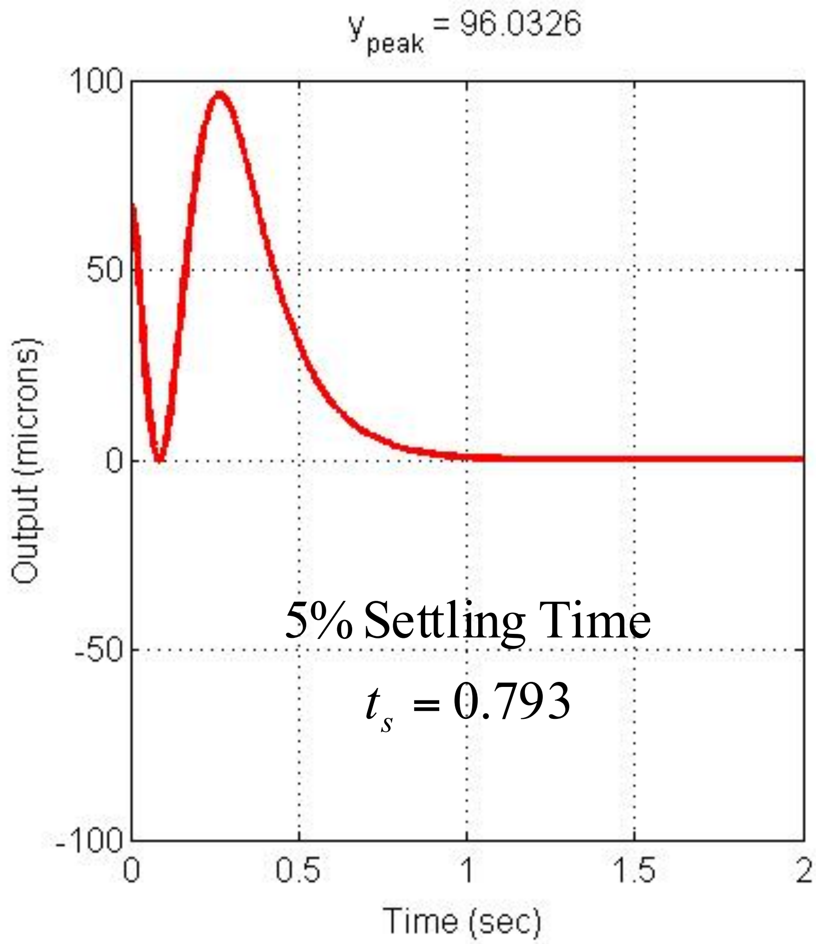
FSF Design Example - XIX

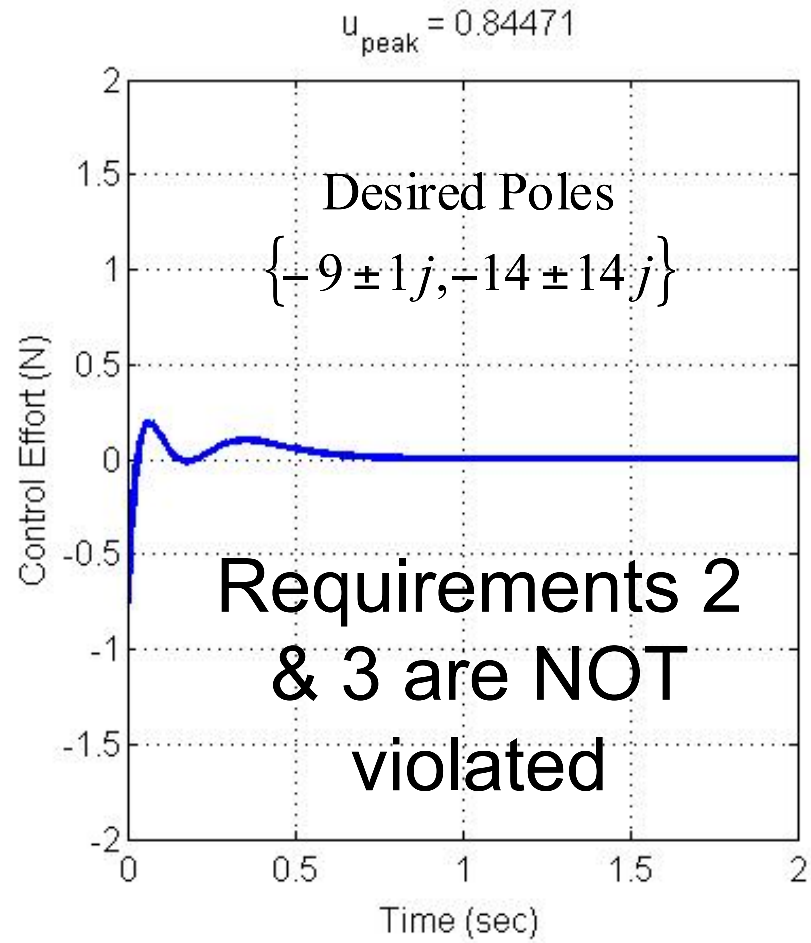
Iteration 3: Reduce the speed of the first set of poles and increase the damping



FSF Design Example - XX

Iteration 4: Increase the speed of the dominant pole and reduce the damping





FSF Design Example - XXI

A table summarizing the iterations is shown below

Pole Locations	Settling Time (sec)	Peak Output (microns)	Peak Force (N)	Meet Specs?
-8±j8, -28±j28	0.717	171.5	6.9	No
-8±j8, -14±j14	0.753	130.4	1.57	No
-6±j1, -14±j14	1.042	81.9	0.229	Yes
-9±j1, -14±j14	0.793	96.0	0.845	Yes

The fourth design yields a set of gains that achieves the specifications and results in a 0.793 second settling time.

FSF Design Example - XXII

Not all of the requirements limited the design in the iterations that we performed.

The constraint on the peak output was the important constraint for the pole locations that we chose.

Design Summary

- Transforming the specifications into 'acceptable' regions of the s-plane helped us choose the location of the closed-loop poles…but not all of the specifications could be directly related to pole locations.
- Several iterations were required to obtain a design that met the specifications.
- We are not guaranteed that we have the 'best' design possible, simply one that meets the specified requirements (more on this later….)