Course Outline - 1st Half

- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response & Matrix Exponential
- Transfer Functions & Eigenvalues
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

Review of Matrix Analysis

Topics:

- Vector & Matrix Operations
- Types of Matrices
- The Eigenvalue Problem

Matrix Analysis Review - I

A matrix is a compact method of writing a coupled set of linear equations:

$$
\begin{aligned}
x_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1m}y_m \\
x_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2m}y_m \\
&\vdots \\
x_n &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nm}y_m
\end{aligned}
$$

$$a_{ij} \quad \begin{cases} i = \text{Row Index} \\ j = \text{Column Index} \end{cases}$$

In vector/matrix form, these equations are:

$$
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NM} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} \Leftrightarrow \underset{[N\times 1]}{\mathbf{x}} = \underset{[N\times M]}{\mathbf{A}}\,\underset{[M\times 1]}{\mathbf{y}}
$$

Matrix Analysis Review - II

Standard Notational Convention (typical)

Scalars are represented by lower-case italic variables in Times or Symbol

$$u \quad y \quad x \quad r \quad e \quad d \quad a \quad \omega \quad \lambda$$

Vectors are represented by lower-case bold-face variables in Times or Symbol

$$\mathbf{u} \quad \mathbf{y} \quad \mathbf{x} \quad \mathbf{r} \quad \mathbf{e} \quad \mathbf{d} \quad \mathbf{a}$$

Matrices are represented by upper-case bold-face variables in Times or Symbol

$$\mathbf{A} \quad \mathbf{B} \quad \mathbf{C} \quad \mathbf{D}$$

Matrix Analysis Review - III

The elements of a vector/matrix can be

- Real numbers
- Complex numbers
- Time Varying
- Functions
- etc…

We will work with both real matrices (state-space models) and complex matrices (matrix of transfer functions)

Matrix Operations - I

The fundamental matrix operations are:

Matrix addition:

$$\underset{[N\times M]}{\mathbf{C}} = \underset{[N\times M]}{\mathbf{A}} + \underset{[N\times M]}{\mathbf{B}}$$

Matrix multiplication:

$$\underset{[N\times M]}{\mathbf{C}} = \underset{[N\times P]}{\mathbf{A}}\underset{[P\times M]}{\mathbf{B}}$$

Matrix division requires the use of a <u>matrix inverse</u> (more on this later)

Matrix Operations - II

Matrix addition is an element-by-element operation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NM} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1M} \\ b_{21} & b_{22} & \cdots & b_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{NM} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1M}+b_{1M} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2M}+b_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}+b_{N1} & a_{N2}+b_{N2} & \cdots & a_{NM}+b_{NM} \end{bmatrix}$$

In compact (indicial) notation:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \qquad \Leftrightarrow$$

Matrix Operations - III

Matrix multiplication is the result of dot products between rows and columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

In compact (indicial) notation:

$$\mathbf{C} = \mathbf{AB} \quad \Leftrightarrow \quad c_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}$$

Types of Matrices - I

The <u>identity matrix</u> has 1's along the diagonal and zeros everywhere else:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A <u>symmetric matrix</u> is one in which the transpose of the matrix is equal to the original matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Types of Matrices - II

A <u>square matrix</u> is one in which $N = M$

A square matrix $\mathbf{A}$ has an <u>inverse</u> if a matrix can be found such that:

$$\mathbf{BA} = \mathbf{I}$$

The matrix $\mathbf{B}$ is the <u>inverse</u> of $\mathbf{A}$

$$\mathbf{B} = \mathbf{A}^{-1} \quad \Rightarrow$$

Only square matrices can have an inverse, but not all square matrices will have an inverse!

Types of Matrices - III

A general 2×2 matrix has the following inverse:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} =$$

Verify this:

Types of Matrices - IV

The Determinant of a matrix $\mathbf{A}$ is a scalar function of the matrix components

For a 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Rightarrow \qquad |\mathbf{A}| =$$

For a 3×3 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \Rightarrow \qquad |\mathbf{A}| =$$

Types of Matrices - V

A general expression for the matrix inverse is:

$$\mathbf{A}^{-1} = \frac{adj(\mathbf{A})}{|\mathbf{A}|} = \frac{\text{Adjoint of } \mathbf{A}}{\text{Determinant of } \mathbf{A}}$$

The inverse of $\mathbf{A}$ does not exist if $|\mathbf{A}| = 0$

$\mathbf{A}$ is <u>singular</u> if its inverse does not exist

$\mathbf{A}$ is <u>non-singular</u> if its inverse exists

Types of Matrices - VI

Find the inverse of:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

3x3 Matrices

Given a general 3x3 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|\mathbf{A}| =$$

3x3 Matrices

Note that you can compute the determinant using <u>any</u> row or column (Hint: choose wisely!)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

$$= -b(di - fg) + e(ai - cg) - h(af - cd)$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

3x3 Matrices

The Inverse requires the Adjoint which is the transpose of the Cofactor matrix

$$\operatorname{adj}(\mathbf{A}) = [\operatorname{cof}(\mathbf{A})]^T$$

$$\operatorname{cof}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

The Eigenvalue Problem - I

One of the most important problems in matrix analysis is the Eigenvalue Problem (EVP)

Given an $N \times N$ matrix $\mathbf{A}$

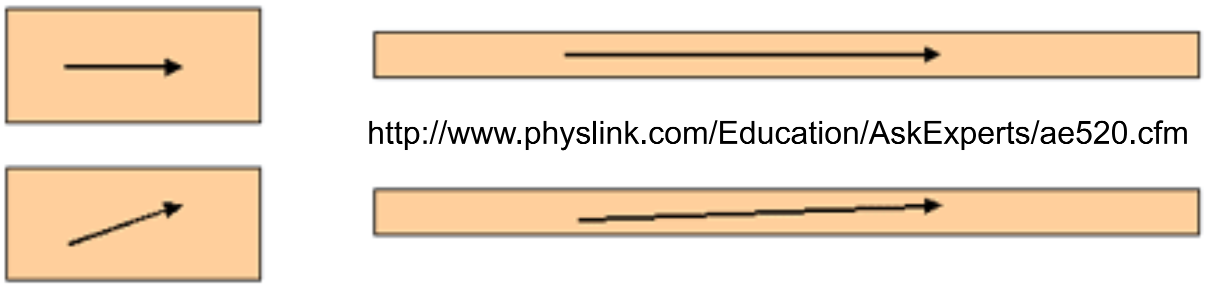
Find the scalar(s) λ and vector(s) $\mathbf{v}$ that satisfy:

$$\underset{[N\times N]}{\mathbf{A}}\;\underset{[N\times 1]}{\mathbf{v}} = \underset{[1\times 1]}{\lambda}\;\underset{[N\times 1]}{\mathbf{v}}$$

Consider $\mathbf{A}$ as a transformation matrix and $\mathbf{v}$ as a vector in a certain direction

The Eigenvalue Problem - II

Geometric Explanation: Draw an arrow on an elastic band. Stretching the band is a transformation. Some directions are preserved, but other directions are changed.

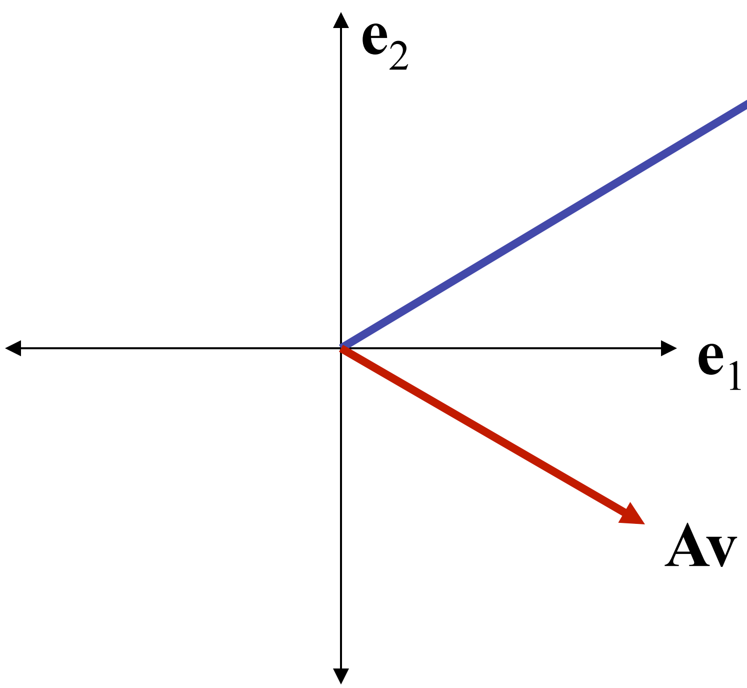


The stretching of an elastic band

The preserved direction is called an <u>Eigenvector</u> of the transformation and the associated scalar amount by which it has been stretched is called an <u>Eigenvalue</u>.

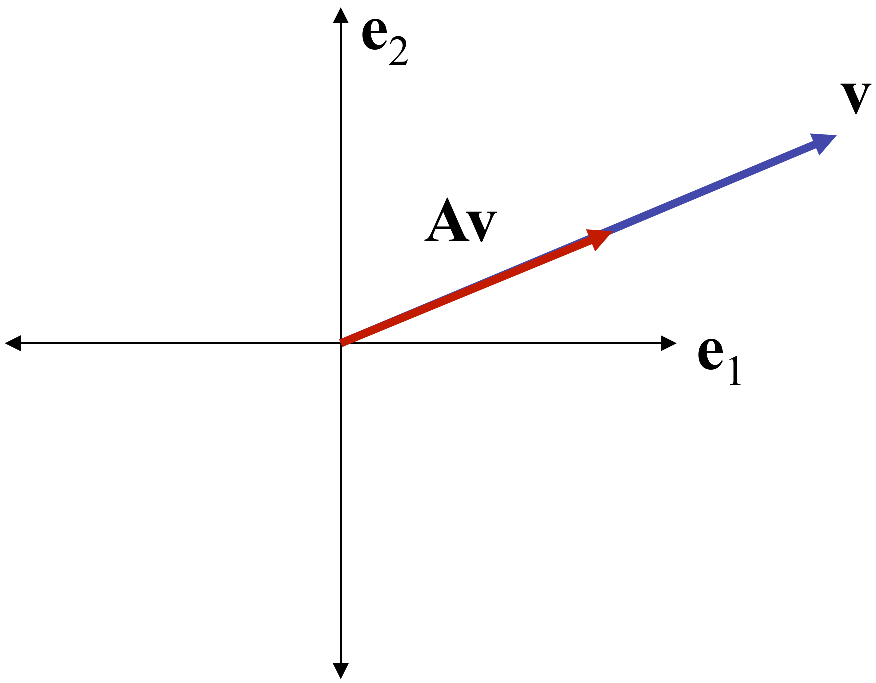
The Eigenvalue Problem - III

Consider a two-dimensional space ($N = 2$) spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. For real-valued symmetric transformation matrices $\mathbf{A}$:



The transformed vector $\mathbf{Av}$ does not lie in the same direction as $\mathbf{v}$, therefore $\mathbf{v}$ is not a solution to the EVP

The Eigenvalue Problem - IV



In this case, the transformed vector $\mathbf{Av}$ does lie in the same direction as $\mathbf{v}$, therefore $\mathbf{v}$ is a solution to the EVP.

The Eigenvalue Problem - V

How do we solve the EVP?

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad \Rightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

Case I: $\mathbf{v} = \mathbf{0}$ is a solution to the equation, but it is the boring (trivial) solution

Case II: Assume $(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})^{-1}$ exists

$$(\mathbf{A} - \lambda\mathbf{I})^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})^{-1}\mathbf{0} \quad \Rightarrow \quad \mathbf{v} = \mathbf{0}$$

Again we have the boring solution

The Eigenvalue Problem - VI

Case III: Assume $(\mathbf{A}-\boldsymbol{\lambda}\mathbf{I})^{-1}$ does not exist

This is equivalent to assuming that the determinant of $(\mathbf{A}-\boldsymbol{\lambda}\mathbf{I})$ is equal to zero:

$$\left|\mathbf{A}-\lambda\mathbf{I}\right| = 0$$

This equation yields an N^{th}-order polynomial in λ

$$\lambda^N + c_{N-1}\lambda^{N-1} + \cdots + c_1\lambda + c_0 = 0$$

The solutions (roots) are the N eigenvalues of $\mathbf{A}$

The Eigenvalue Problem - VII

Each eigenvalue is denoted by:

$$\lambda_n \qquad n = 1, \ldots, N$$

The eigenvalues are typically ordered such that:

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{N-1} \le \lambda_N$$

Each eigenvalue has an associated <u>eigenvector</u> that is the solution of

$$\left(\mathbf{A} - \lambda_n \mathbf{I}\right)\mathbf{v}_n = \mathbf{0} \qquad n = 1, \ldots, N$$

Eigenvalue Example - I

Find the eigenvalues of the matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

First construct $(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})$:

$$\left(\mathbf{A} - \lambda\mathbf{I}\right) =$$

Then compute the determinant:

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| =$$

NOTE:
Can also
Construct
$(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A})$
Why?

Eigenvalue Example - II

Using the quadratic equation solution:

$$a\lambda^2 + b\lambda + c = 0 \quad \Rightarrow \quad \lambda =$$

The eigenvalues are:

$$\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 2 \end{matrix} \quad \Rightarrow \quad (\lambda + 1)(\lambda - 2) = 0$$

NOTE: The eigenvalues of a real-valued matrix <u>can</u> be complex. This will often happen in control analysis.

Eigenvalue Example - III

What are the corresponding eigenvectors?

Substitute λ_1 back into the EVP equation:

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = 0 \quad \Rightarrow \quad \begin{bmatrix} 4-(-1) & -5 \\ 2 & -3-(-1) \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} 5v_{11} - 5v_{21} &= 0 \\ 2v_{11} - 2v_{21} &= 0 \end{aligned}$$

The only useful information we have is:

Eigenvalue Example - IV

Now try the second eigenvalue…

Substitute λ_2 back into the EVP equation:

$$\left(\mathbf{A}-\lambda_2\mathbf{I}\right)\mathbf{v}_2=0 \quad \Rightarrow \quad \begin{bmatrix} 4-(2) & -5 \\ 2 & -3-(2) \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} 2v_{12}-5v_{22}&=0 \\ 2v_{12}-5v_{22}&=0 \end{aligned}$$

The only useful information we have is:

Eigenvalue Example - V

The eigenvectors are not uniquely determined!

We are free to choose any vector that satisfies the constraints just derived, i.e. any vector parallel to these vectors are eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for any } \alpha_1 \neq 0$$

$$\mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \alpha_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \text{for any } \alpha_2 \neq 0$$

MATLAB normalizes each eigenvector such that: $\mathbf{v}_1^T \mathbf{v}_1 = 1 \quad \text{and} \quad \mathbf{v}_2^T \mathbf{v}_2 = 1$

Matrix Analysis - Summary

How does this relate to controls?

- All of the topics we will study in this course will require matrix analysis.
- Derivation of the design techniques will require a thorough understanding of matrix operations.
- The eigenvalue problem will resurface when we begin discussing stability and performance of continuous-time feedback control systems.