Advanced Control Engineering

Virginia Tech

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

LQR Solution - I

In the last meeting, we introduced the following quadratic cost function:

$$V(\mathbf{G}) = \int_0^\infty \mathbf{x}^T(\tau)\mathbf{Q}\mathbf{x}(\tau) + \mathbf{u}^T(\tau)\mathbf{R}\mathbf{u}(\tau)d\tau$$

as a means of computing the control gains $\mathbf{G}$ without having to explicitly specify the closed-loop poles.

The gains $\mathbf{G}$ are computed to minimize the cost function given the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$.

LQR Solution - II

How do we find the gains $\mathbf{G}$ that minimize the quadratic cost function?

The answer to this question is a lengthy and complicated proof that involves, among other things, the use of the controllability grammian.

This is also one of the most significant results in modern (state-space) control!

LQR Solution - III

Evaluating the cost function V along system trajectories defined by the closed-loop state equations (plus a whole lot of math) gives the following result for the optimal state-feedback gains:

$$-\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) + \mathbf{Q}$$

$$\mathbf{G}(t) = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t)$$

This celebrated result is known as the continuous Matrix Riccati Equation.

LQR Solution - IV

When our LTI system is either

– Asymptotically stable or

– Controllable and observable

Then a unique constant optimal gain matrix $\mathbf{G}$ exists as the solution of the following Algebraic Riccati Equation:

$$\mathbf{0} = \mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q}$$

$$\mathbf{G} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

LQR Solution - V

Fortunately, the solution to the Algebraic Riccati Equation is readily available using the `lqr` function from the Matlab Control System toolbox:

```
[G,P,E] = lqr(A,B,Q,R);
```

Outputs:
G = State feedback gain matrix
P = Riccati solution
E = Closed-loop eigenvalues

Inputs:
A = State matrix
B = Input matrix
Q = State weighting matrix
R = Input weighting matrix

Weighting Matrix Selection - I

All we have left is the question of how to choose the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$.

In the last meeting we briefly saw how these matrices can *trade off* the cost of the states versus the cost of the control effort.

Making $\mathbf{Q}$ "big" compared to $\mathbf{R}$ will minimize the state response at the expense of large control effort.

Making $\mathbf{R}$ "big" compared to $\mathbf{Q}$ will minimize the required control effort at the expense of large state response.

Weighting Matrix Selection - II

We are required to choose both weighting matrices to be <u>square</u>, <u>real</u>, <u>symmetric</u>, and <u>positive definite</u> (PD).

There are several equivalent definitions of "positive definiteness" for a matrix.

Definition #1: A real symmetric matrix $\mathbf{Q}$ is PD if all of the eigenvalues are positive real: $\lambda_i(\mathbf{Q}) > 0$.

Note that real symmetric matrices always have only real eigenvalues!

Weighting Matrix Selection - III

Definition #2: A real symmetric matrix $\mathbf{Q}$ is PD if, for all non-zero vectors $\mathbf{z}$, the following quadratic form is positive:

$$\mathbf{z}^T \mathbf{Q} \mathbf{z} > 0 \qquad \forall \mathbf{z} \neq \mathbf{0}$$

This is the most common definition for positive definiteness of a matrix.

What does this say about the LQR cost function integral?

Weighting Matrix Selection - IV

Example: Evaluate the positive definiteness of a general 2×2 real symmetric matrix:

$$\mathbf{Q} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

The eigenvalues of this matrix $\mathbf{Q}$ are the roots of the characteristic equation:

$$\begin{vmatrix} \lambda - a & -b \\ -b & \lambda - c \end{vmatrix} = (\lambda - a)(\lambda - c) - b^2 = \lambda^2 - (a+c)\lambda - \left(b^2 - ac\right) = 0$$

Weighting Matrix Selection - V

Solving for the roots:

$$\lambda = \tfrac{1}{2}\left[\left(a+c\right) \pm \sqrt{\left(a+c\right)^2 + 4\left(b^2 - ac\right)}\right]$$

$$= \tfrac{1}{2}\left[\left(a+c\right) \pm \sqrt{\left(a-c\right)^2 + 4b^2}\right]$$

The last equation proves that both roots are always real. Why?

The first equation shows that both roots are positive if:

$$\begin{matrix} a > 0 \\ c > 0 \end{matrix} \quad \text{AND} \quad \left(ac - b^2\right) = \det\left(\mathbf{Q}\right) = \prod \lambda_i > 0$$

Weighting Matrix Selection - VI

From Definition #2, we have:

$$\begin{bmatrix} z_1 & z_2 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = az_1^2 + 2bz_1z_2 + cz_2^2$$

$$= \left(z_1^2\sqrt{a} + z_2^2\sqrt{c}\right)^2 + 2\left(b - \sqrt{ac}\right)z_1z_2$$

Positive definiteness is more difficult to demonstrate here, but the results from Definition #1 confirm that the two definitions are equivalent.

Weighting Matrix Selection - VII

The most direct selection of PD weighting matrices are diagonal:

$$\underset{[N\times N]}{\mathbf{Q}} = \begin{bmatrix} q_1 & & \cdots & 0 \\ & q_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & q_N \end{bmatrix} \qquad \underset{[M\times M]}{\mathbf{R}} = \begin{bmatrix} r_1 & & \cdots & 0 \\ & r_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & r_M \end{bmatrix}$$

This choice is naturally symmetric and will be positive definite if <u>all the diagonal elements are positive</u>.

Weighting Matrix Selection - VIII

Note that q_n is associated with state x_n, and r_m is associated with control u_m.

Care must be taken to account for signal magnitude differences. One solution is to normalize the weights with some peak estimate of the state ($\bar{x}_n$) and control ($\bar{u}_m$) signals:

$$\underset{[N\times N]}{\mathbf{Q}} = \begin{bmatrix} \frac{q_1}{\bar{x}_1} & & \cdots & 0 \\ & \frac{q_2}{\bar{x}_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{q_N}{\bar{x}_N} \end{bmatrix} \qquad \underset{[M\times M]}{\mathbf{R}} = \begin{bmatrix} \frac{r_1}{\bar{u}_1} & & \cdots & 0 \\ & \frac{r_2}{\bar{u}_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{r_M}{\bar{u}_M} \end{bmatrix}$$

Weighting Matrix Selection - IX

Another common simplification is to choose:

$$\underset{[N\times N]}{\mathbf{Q}} = \underset{[1\times 1]}{q}\,\underset{[N\times P]}{\mathbf{C}^T}\,\underset{[P\times N]}{\mathbf{C}}$$

$$\underset{[M\times M]}{\mathbf{R}} = \underset{[1\times 1]}{r}\,\underset{[M\times M]}{\mathbf{I}}$$

Now we only have two scalar constants to choose: q and r.

What is the physical meaning of this particular choice of the cost function?

Weighting Matrix Selection - X

Substituting these choices into the cost function we have (assuming $\mathbf{D} = 0$):

$$V(\mathbf{G}) = \int_0^\infty q \underbrace{\mathbf{x}^T(\tau)\mathbf{C}^T}_{\mathbf{y}^T(\tau)} \underbrace{\mathbf{C}\mathbf{x}(\tau)}_{\mathbf{y}(\tau)} + r\mathbf{u}^T(\tau)\mathbf{u}(\tau)\,d\tau$$

This can be further simplified by rewriting the integrand as:

$$V(\mathbf{G}) = r\int_0^\infty \left(\tfrac{q}{r}\right) \underbrace{\mathbf{y}^T(\tau)\mathbf{y}(\tau)}_{\text{Output Inner Product}} + \underbrace{\mathbf{u}^T(\tau)\mathbf{u}(\tau)}_{\text{Input Inner Product}}\,d\tau$$

We only need to turn the (q/r) knob.

LQR Example - I

For a first order system, the solution is:

$$x(t) = e^{(a-bg)t} x(0)$$

$$u(t) = -gx(t) = -ge^{(a-bg)t} x(0)$$

Substituting this solution into the cost function:

$$V(g) = \int_0^\infty qe^{2(a-bg)t} x^2(0) + rg^2 e^{2(a-bg)t} x^2(0) d\tau$$

$$= r\left(\tfrac{q}{r} + g^2\right) x^2(0) \int_0^\infty e^{2(a-bg)t} d\tau$$

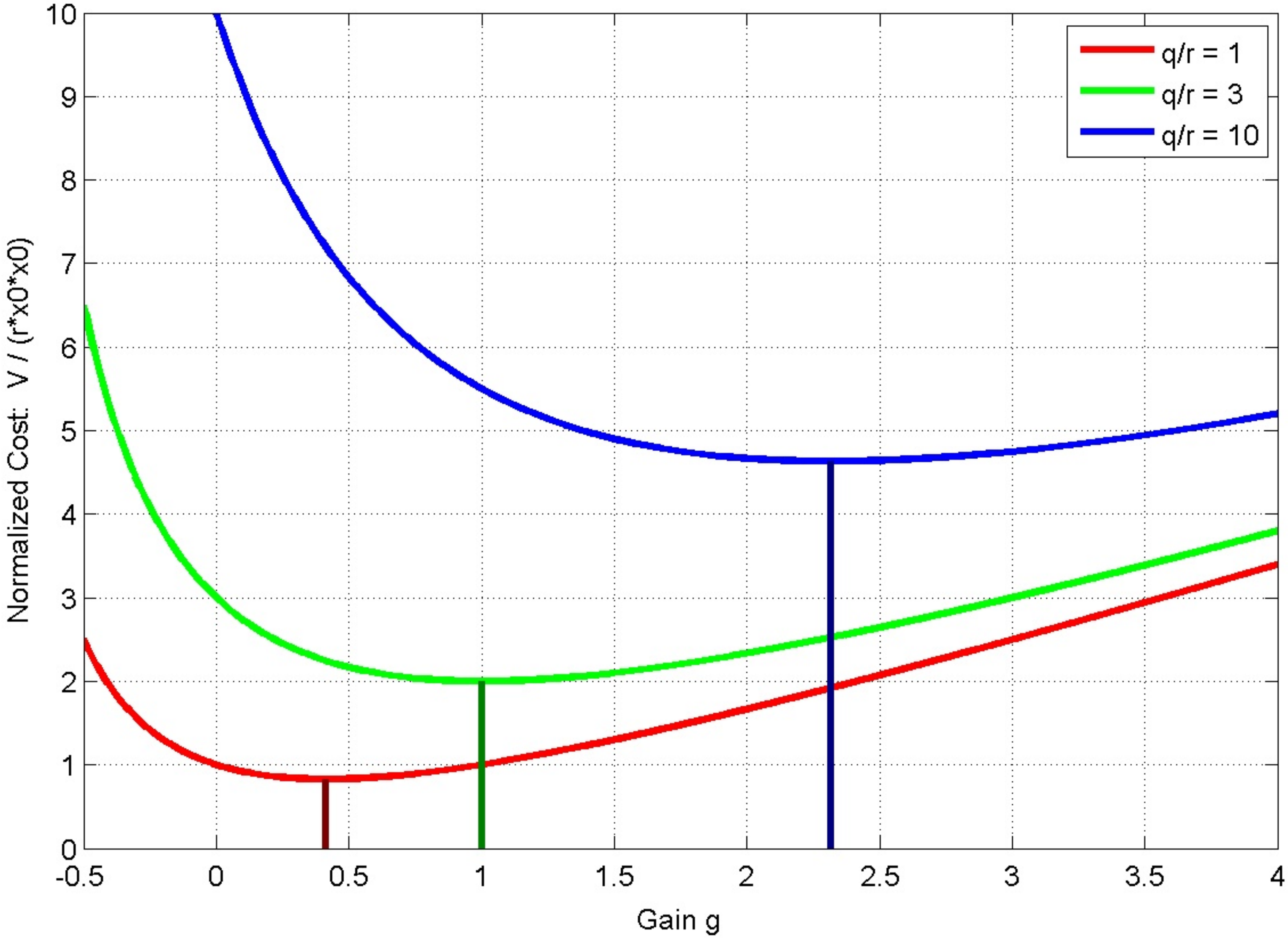
LQR Example - II

Assuming that $a-bg$ represents a stable system, the integral can be evaluated:

$$V(g) = -rx^2(0)\frac{\left(\frac{q}{r}+g^2\right)}{2(a-bg)}$$

Changing the ratio (q/r) affects the optimal gain of the cost function.

LQR Example - III

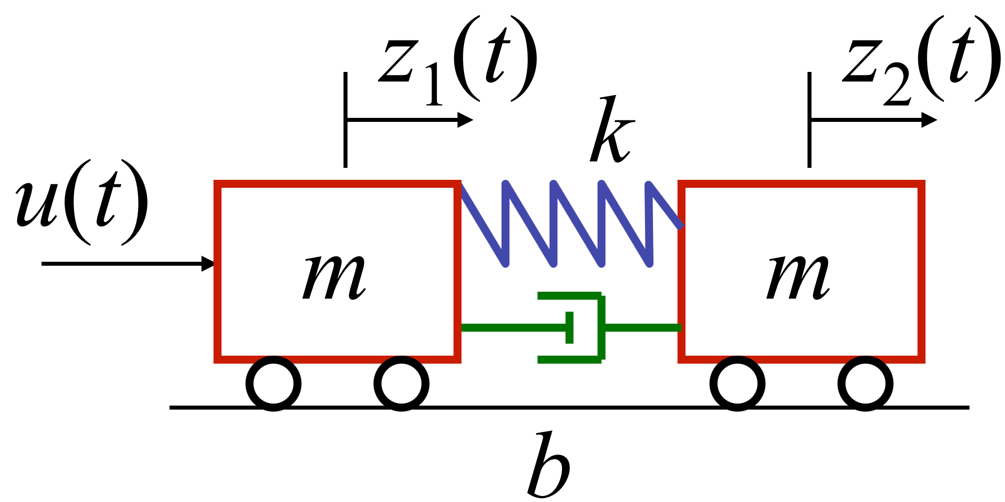


LQR Example - IV

Consider our 4th order damped mass-spring-mass system from L14:

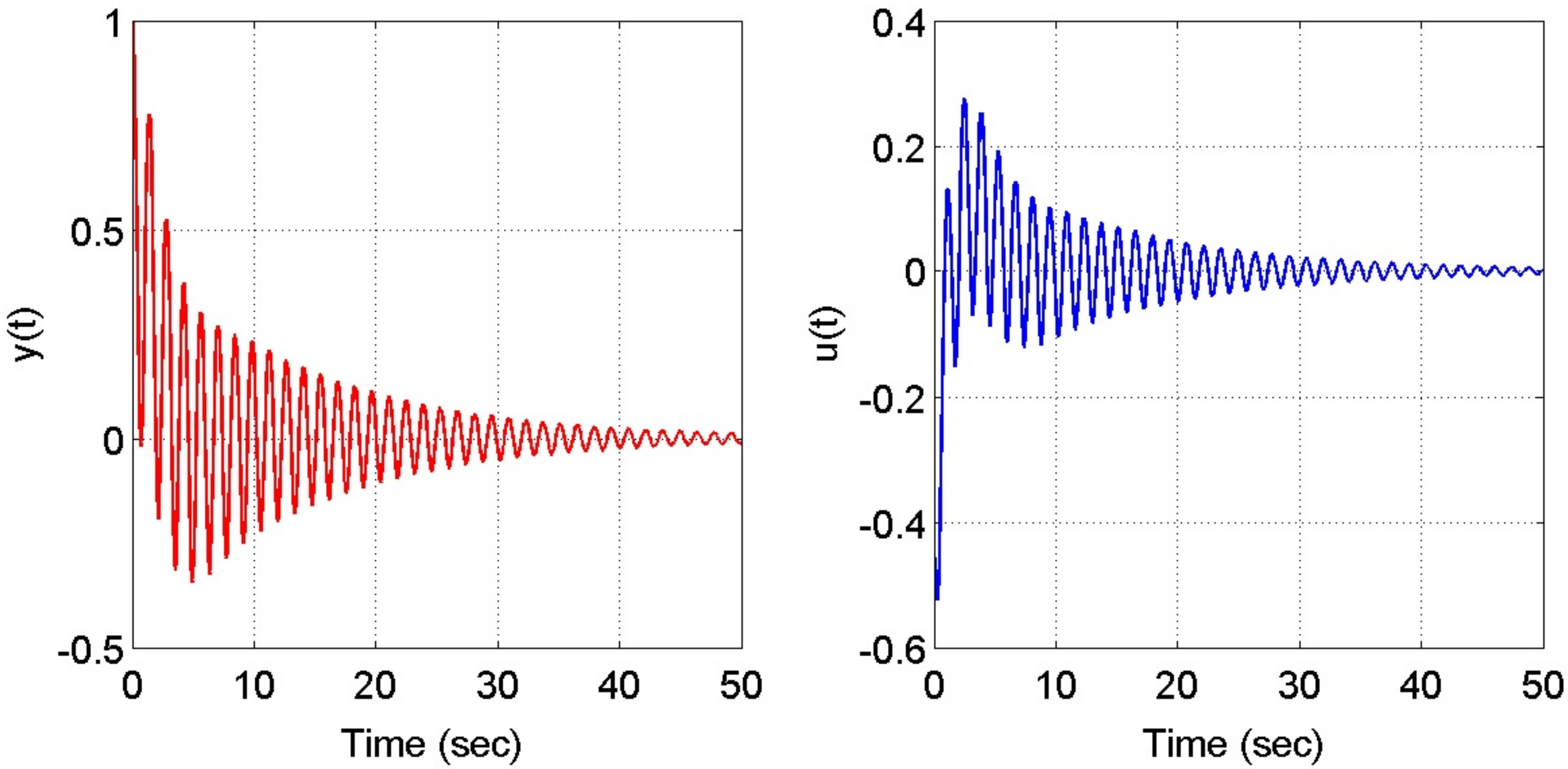
$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k}{m}\right) & \left(\frac{k}{m}\right) & -\left(\frac{b}{m}\right) & \left(\frac{b}{m}\right) \\ \left(\frac{k}{m}\right) & -\left(\frac{k}{m}\right) & \left(\frac{b}{m}\right) & -\left(\frac{b}{m}\right) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \left(\frac{1}{m}\right) \\ 0 \end{bmatrix} u \qquad \mathbf{z}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}$$



LQR Example - V

Setting $q/r = 1$ yields the following:



$$\int_0^\infty y^2(\tau)\,d\tau = 1.16$$

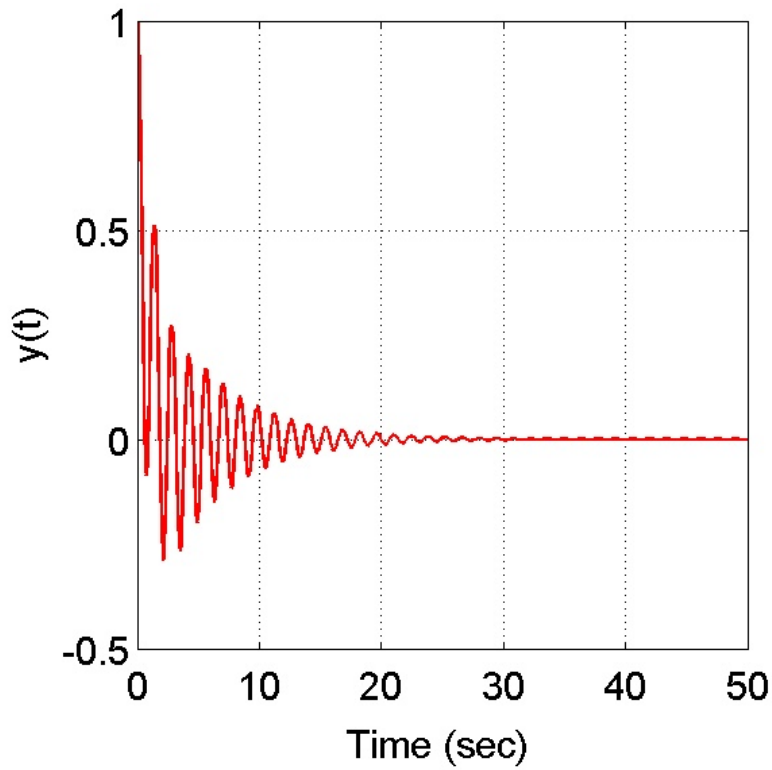
$t_{\text{settling}} = 30.2$ sec.

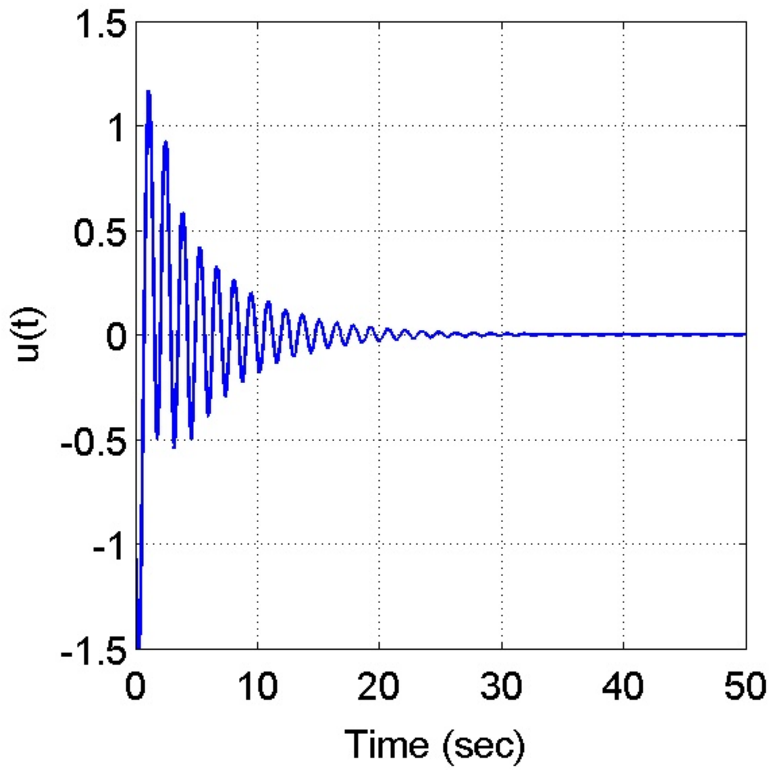
$$\int_0^\infty u^2(\tau)\,d\tau = 0.301$$

$$\left|u(t)\right|_{\text{peak}} = 0.523$$

LQR Example - VI

Setting $q/r = 10$ yields the following:





$$\int_0^\infty y^2(\tau)d\tau = 0.528$$

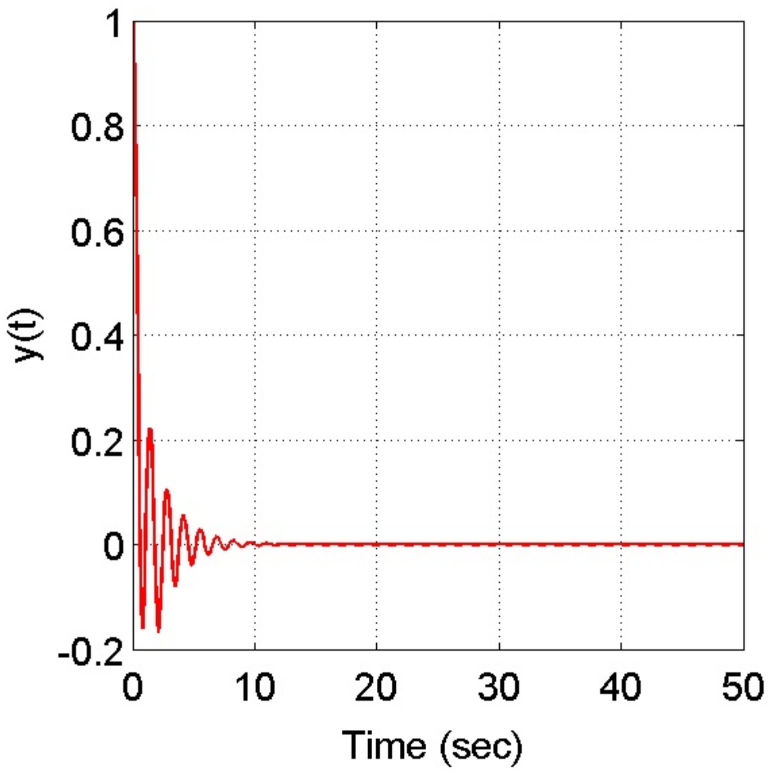
$$t_{\text{settling}} = 11.96 \text{ sec.}$$

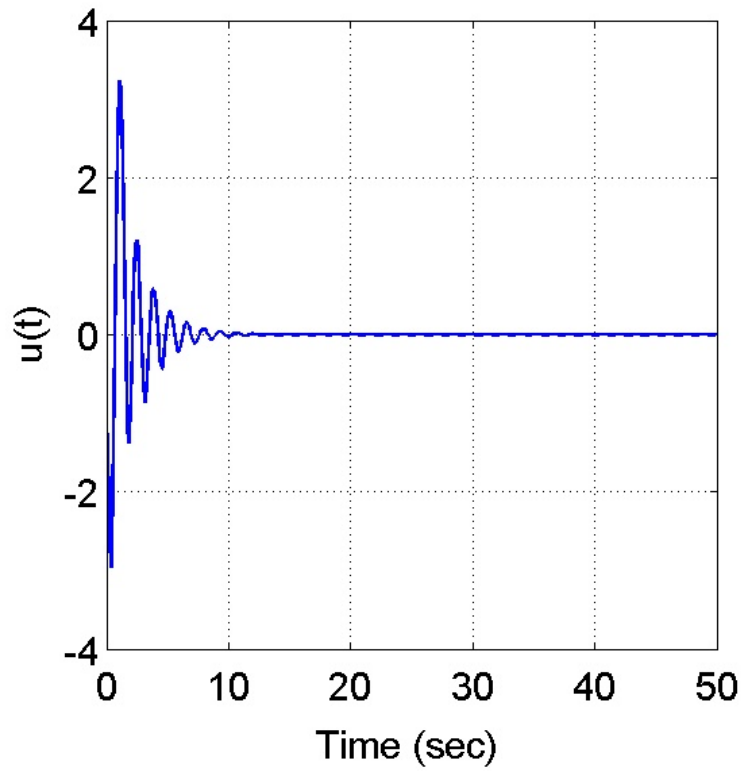
$$\int_0^\infty u^2(\tau)d\tau = 2.52$$

$$|u(t)|_{\text{peak}} = 1.49$$

LQR Example - VII

Setting $q/r = 100$ yields the following:





$$\int_0^\infty y^2(\tau)d\tau = 0.303$$

$$t_{\text{settling}} = 4.25 \text{ sec.}$$

$$\int_0^\infty u^2(\tau)d\tau = 8.99$$

$$|u(t)|_{\text{peak}} = 3.23$$

LQR Example - VIII

Changing the ratio (q/r) allows us to trade off the integral of the output squared versus the integral of the control squared:

q/r	$\int_0^\infty y^2(\tau)d\tau$	$\int_0^\infty u^2(\tau)d\tau$
1	1.16	0.301
10	0.528	2.52
100	0.303	8.99

Decreasing Increasing

LQR Example - IX

LQR can also be used to trade off other specifications such as settling time and peak control effort:

q/r	$t_{settling}$	$\lvert u \rvert_{peak}$
1	30.2	0.523
10	11.96	1.49
100	4.25	3.23

However, these design specs are not explicitly part of the cost function so there are no guarantees!

LQR Example - X

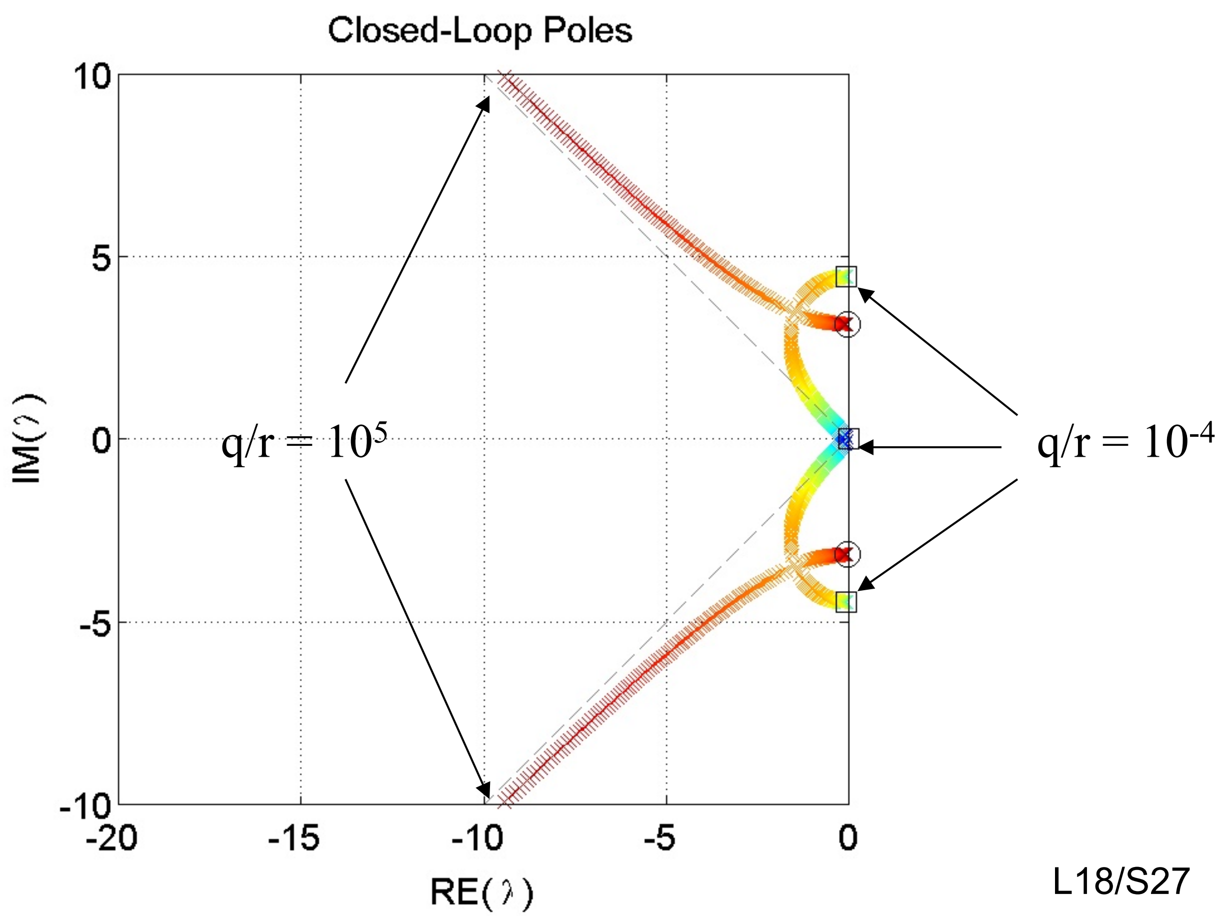
What is LQR doing to the closed-loop system poles?

Remember in pole placement how we started with the desired closed-loop poles then solved for the gains $\mathbf{G}$.

For LQR control, we choose weighting matrices $\mathbf{Q}$ and $\mathbf{R}$, then solve for the gains $\mathbf{G}$. To find the closed-loop poles, we must solve for the roots of $[\mathbf{A}-\mathbf{BG}]$.

LQR Example - XI

For the mass-spring-mass example, the following is a plot of the LQR closed-loop pole locus as q/r is varied.



O = Open-Loop Zeros
□ = Open-Loop Poles

LQR Example - XII

This plot is very much like a root locus from classical control theory.

The locus shows that as q/r is increased, LQR is placing a pair of poles near the open-loop zeros, and one pair of poles are asymptotically approaching critical damping (0.707).

Summary

Linear Quadratic Regulator (LQR) control is an optimal control technique that allows us to design full-state feedback controllers without specifying the closed-loop poles.

The technique allows us to trade off the response of the output versus the response of the control effort.

- The technique explicitly accounts for the integral of the response and the control effort.
- Other design specifications can also be analyzed (e.g. settling time and peak control effort) but there are no guarantees regarding their convergence properties.