ME-5554 Applied Linear System
Homework 4

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NE2.2 - For the following systems described by the given state equations, solve for the State Feedback Control gains that place the closed loop poles at $[-3+j, -3-j]$:

The desired characteristic equation is define by (1):

$$|sI - A + bg| = (s+3-j)(s+3+j) = s^2 + 6s + 10$$

We have:

1. $$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t)$$

The closed-loop pole locations (CPL) is defined by:

$$[CPL] = \begin{bmatrix} sI - A + bg \end{bmatrix} = s. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix}$$

$$[CPL] = \begin{bmatrix} s+3 & 0 \\ 0 & s+4 \end{bmatrix} + \begin{bmatrix} g_1 & g_2 \\ g_1 & g_2 \end{bmatrix} = \begin{bmatrix} s+3+g_1 & g_2 \\ g_1 & s+4+g_2 \end{bmatrix}$$

The determinant of [CPL] is defined:

$$\begin{aligned} det[CPL] = |sI - A + bg| &= (s+3+g_1)(s+4+g_2) - g_1.g_2 \\ &= s^2 + (g_1+g_2+7)s + (4g_1+3g_2+12) \end{aligned}$$

From (1) we have: $\begin{cases} g_1 + g_2 + 7 = 6 \\ 4g_1 + 3g_2 + 12 = 10 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The control gain is defined: $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

The State Feedback control is: $u(t) = -g.x(t)$

$$\Leftrightarrow u(t) = - \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -x_1(t) + 2x_2(t)$$

2. $$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t)$$

The closed-loop pole locations (CPL) is defined by:

$$[CPL] = \begin{bmatrix} sI - A + bg \end{bmatrix} = s. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix}$$

$$[CPL] = \begin{bmatrix} s & -1 \\ 3 & s+2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g_1 & g_2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 3+g_1 & s+2+g_2 \end{bmatrix}$$

The determinant of [CPL] is defined:

$$det[CPL] = |sI - A + bg| = s(s+2+g_2) - (3+g_1)(-1) = s^2 + (g_2+2)s + (g_1+3)$$

From (1) we have: $\begin{cases} g_2 + 2 = 6 \\ g_1 + 3 = 10 \end{cases} \Leftrightarrow \begin{cases} g_1 = 7 \\ g_2 = 4 \end{cases}$

The control gain is defined: $\Rightarrow \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

The State Feedback control is: $u(t) = -g.x(t)$

$$\Leftrightarrow u(t) = -\begin{bmatrix} 7 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -7x_1(t) - 4x_2(t)$$

3. $$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -12 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t)$$

The closed-loop pole locations (CPL) is defined by:

$$[CPL] = \begin{bmatrix} sI - A + bg \end{bmatrix} = s. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & -12 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix}$$

$$[CPL] = \begin{bmatrix} s & 2 \\ -1 & s+12 \end{bmatrix} + \begin{bmatrix} g_1 & g_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s+g_1 & g_2+2 \\ -1 & s+12 \end{bmatrix}$$

The determinant of [CPL] is defined:

$$\begin{aligned} det[CPL] = |sI - A + bg| &= (s+g_1)(s+12) - (-1)(g_2+2) \\ &= s^2 + (g_1+12)s + (12g_1 + g_2 + 2) \end{aligned}$$

From (1) we have: $\begin{cases} g_1 + 12 = 6 \\ 12g_1 + g_2 + 2 = 10 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ 8 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ -12 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 8 \end{bmatrix} = \begin{bmatrix} -6 \\ 80 \end{bmatrix}$$

The control gain is defined: $\Rightarrow \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 80 \end{bmatrix}$

The State Feedback control is: $u(t) = -g.x(t)$

$$\Leftrightarrow u(t) = -\begin{bmatrix} -6 & 80 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 6x_1(t) - 80x_2(t)$$

4. $$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 7 & 8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [9]u(t)$$

The closed-loop pole locations (CPL) is defined by:

$$[CPL] = \begin{bmatrix} sI - A + bg \end{bmatrix} = s. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix}$$

$$[CPL] = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix} + \begin{bmatrix} 5g_1 & 5g_2 \\ 6g_1 & 6g_2 \end{bmatrix} = \begin{bmatrix} s+5g_1-1 & 5g_2-2 \\ 6g_1-3 & s+6g_2-4 \end{bmatrix}$$

The determinant of [CPL] is defined:

$$\begin{aligned} det[CPL] = |sI - A + bg| &= (s+5g_1-1)(s+6g_2-4) - (6g_1-3).(5g_2-2) \\ &= s^2 + (5g_1+6g_2-5)s + (-8g_1+9g_2-2) \end{aligned}$$

From (1) we have: $\begin{cases} 5g_1 + 6g_2 - 5 = 6 \\ -8g_1 + 9g_2 - 2 = 10 \end{cases} \Leftrightarrow \begin{bmatrix} 5 & 6 \\ -8 & 9 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -8 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 12 \end{bmatrix} = \frac{1}{93} \begin{bmatrix} 9 & -6 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} 11 \\ 12 \end{bmatrix} = \frac{1}{93} \begin{bmatrix} 42 \\ 148 \end{bmatrix} = \begin{bmatrix} \frac{42}{93} \\ \frac{148}{93} \end{bmatrix}$$

The control gain is defined: $\Rightarrow \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \frac{42}{93} \\ \frac{148}{93} \end{bmatrix}$

The State Feedback control is: $u(t) = -g.x(t)$

$$\Leftrightarrow u(t) = -\begin{bmatrix} \frac{42}{93} & \frac{148}{93} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -\frac{42}{93}x_1(t) - \frac{148}{93}x_2(t)$$