Course Outline - 1st Half

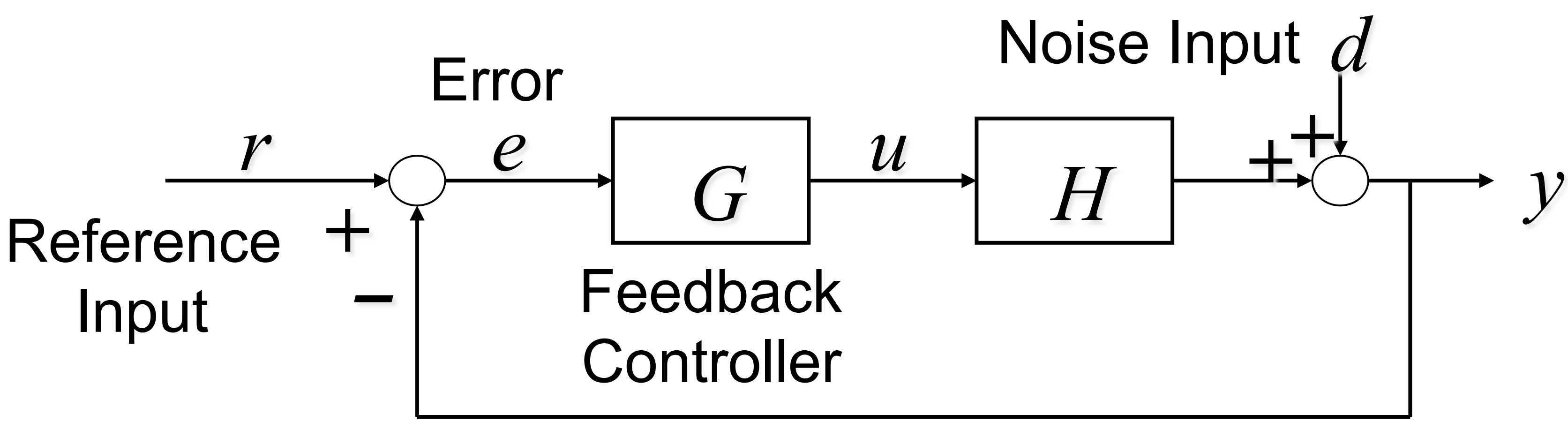
- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response, Matrix Exponential
- Transfer Functions & Eigenvalues
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

Course Summary - I

1. For the SISO open-loop transfer function GH, the closed-loop is characterized by:

$$T(s) = \left(\frac{G(s)H(s)}{1+G(s)H(s)} \right) \Rightarrow$$ Complementary Sensitivity (Tracking Performance)

$$S(s) = \left(\frac{1}{1+G(s)H(s)} \right) \Rightarrow$$ Sensitivity Function (Disturbance Rejection)



Course Summary - II

2. The SISO Characteristic Equation (CE) is

$$1+G(s)H(s)=0$$

3. Linear time invariant (LTI) systems can be modeled with first order differential equations of the form

$$\left.\underset{[N\times 1]}{\dot{\mathbf{x}}} = \underset{[N\times N]}{\mathbf{A}}\,\underset{[N\times 1]}{\mathbf{x}} + \underset{[N\times M]}{\mathbf{B}}\,\underset{[M\times 1]}{\mathbf{u}}\right\}\ \text{State Equations}$$

$$\left.\underset{[P\times 1]}{\mathbf{y}} = \underset{[P\times N]}{\mathbf{C}}\,\underset{[N\times 1]}{\mathbf{x}} + \underset{[P\times M]}{\mathbf{D}}\,\underset{[M\times 1]}{\mathbf{u}}\right\}\ \text{Output Equations}$$

Course Summary - III

4. State-space <u>realizations</u> of a dynamic system are <u>not unique</u>, but the <u>input-output relationships are invariant</u>. The input-output transfer functions are expressed by the matrix

$$\mathbf{H}(s) = \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}$$

Course Summary - IV

- Nonlinear state-space systems may be linearized about some operating point using the Jacobian matrices

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{u}) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x},\mathbf{u})\end{aligned} \quad \Rightarrow \quad \begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}(\mathbf{x}_0,\mathbf{u}_0)\mathbf{x} + \mathbf{B}(\mathbf{x}_0,\mathbf{u}_0)\mathbf{u} \\ \mathbf{y} &= \mathbf{C}(\mathbf{x}_0,\mathbf{u}_0)\mathbf{x} + \mathbf{D}(\mathbf{x}_0,\mathbf{u}_0)\mathbf{u}\end{aligned}$$

- The N Eigenvalues (λ) and the corresponding Eigenvectors ($\mathbf{v}$) are solutions of $\underset{[N\times N]}{\mathbf{A}}\ \underset{[N\times 1]}{\mathbf{v}} = \underset{[1\times 1]}{\lambda}\ \underset{[N\times 1]}{\mathbf{v}}$

$$(\mathbf{A} - \lambda_n \mathbf{I})\mathbf{v}_n = 0 \qquad n = 1,\ldots,N$$

Course Summary - V

7. The time-domain solution of the State-Space differential equation is

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)}_{\text{Response due to Initial Conditions}} + \underbrace{\int_{t_0}^{t}\left[e^{\mathbf{A}(t-\tau)}\mathbf{B}\right]\mathbf{u}(\tau)\,d\tau}_{\text{Forced Response}}$$

8. A key part of this solution is the <u>state transition matrix</u>, or <u>matrix exponential</u> which is related to the matrix $\mathbf{A}$ through the inverse Laplace transform

$$\Phi(t,t_0) = e^{\mathbf{A}(t-t_0)} = L^{-1}\left\{\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\right\}$$

Course Summary - VI

9. Transfer functions can be transformed into a state-space representation through one of the <u>canonical forms</u>.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$[y(t)] = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + [0]u(t)$$

Course Summary - VII

10. The poles of the system are the eigenvalues of the characteristic equation

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$$

which are also the N roots of

$$\left|s\mathbf{I} - \mathbf{A}\right| = s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0 = 0$$

$$\left(s - s_1\right)\left(s - s_2\right)\cdots\left(s - s_N\right) = \prod_{i=1}^{N}\left(s - s_i\right) = 0$$

Course Summary - VIII

11. The pole locations in the complex s-plane provide information about the stability and time response of the open-loop system

Asymptotically Stable	$\Rightarrow$	$\mathrm{RE}(\lambda_n) < 0 \quad \forall n$
Marginally Stable	$\Rightarrow$	$\mathrm{RE}(\lambda_n) = 0$ for at least one n
Unstable	$\Rightarrow$	$\mathrm{RE}(\lambda_n) > 0$ for at least one n
Oscillatory	$\Rightarrow$	Complex poles near IM-axis

Course Summary - IX

12. The steady-state response of an asymptotically-stable LTI system is related to the magnitude and phase of $\mathbf{H}(j\omega)$.
    - The steady-state response amplitude is the amplitude of the input multiplied by $|\mathbf{H}(j\omega)|$.
    - The phase shift between the input and the output is equal to the phase of $\mathbf{H}(j\omega)$.

Course Summary - X

13. The impulse response function is related to the transfer function

$$h_{pm}(t) = L^{-1}\left\{H_{pm}(s)\right\} \qquad \Leftrightarrow \qquad H_{pm}(s) = L\left\{h_{pm}(t)\right\}$$

14. Once the impulse response is known for an LTI system, the response to <u>any arbitrary input</u> can be determined from the convolution integral:

$$y_p(t) = \sum_{m=1}^{M}\left(\int_0^t h_{pm}(t-\tau)u_m(\tau)d\tau\right)$$

Course Summary - XI

15. For an LTI state-space system, the closed-loop poles can be "placed" using full-state feedback:

$$\underset{[M\times 1]}{\mathbf{u}(t)} = -\underset{[M\times N]}{\mathbf{G}}\underset{[N\times 1]}{\mathbf{x}(t)}$$

and the resulting closed-loop system does not have any inputs:

$$\dot{\mathbf{x}}(t) = \underbrace{\left[\mathbf{A}-\mathbf{B}\mathbf{G}\right]}_{\text{Closed-Loop State Matrix}}\mathbf{x}(t)$$

Course Summary - XII

16. For a Single-Input state-space system, we can use the "brute force" method of pole placement which equates coefficients of the desired and actual closed-loop CE's:

$$\left|s\mathbf{I}-\mathbf{A}_c\right| = \underbrace{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}_{\text{Desired CL characteristic equation}}$$

$$\left|s\mathbf{I}-\mathbf{A}+\mathbf{bg}\right| = \underbrace{s^N + \bar{a}_{N-1}(\mathbf{A},\mathbf{b},\mathbf{g})s^{N-1} + \cdots + \bar{a}_1(\mathbf{A},\mathbf{b},\mathbf{g})s + \bar{a}_0(\mathbf{A},\mathbf{b},\mathbf{g})}_{\text{Actual CL characteristic equation}}$$

Course Goals

Remember the original course goals from the first meeting (L1):

- Represent multi-input-multi-output (MIMO) dynamic systems using <u>state-space</u> models
- Solve for the time response of a linear dynamic system and relate the response to the <u>state-space</u> system description
- Design linear feedback control systems using frequency domain, state estimation, and pole placement techniques

Midterm Rules

- Thursday October 15, 11:00-12:15
- You may NOT use: laptop computers, notes, books
- You may bring 1 sheet of 8.5”x11” paper with anything you want printed or written on it (both sides)
- Any required Laplace transforms will be provided
- Exam will cover material up to and including full state feedback for SISO

Fair game for the Midterm Exam

- ALGEBRA!!!!!
- Transfer functions
- Basic matrix operations
- Eigenvalues & eigenvectors
- State-space representations
- State transition matrix
- Impulse response
- Time domain response
- Harmonic response
- Poles, zeros, and stability

Transfer Functions

You should know how to:

- Convert higher-order differential equations into Laplace-domain transfer functions (L1)
- Convert a state-space model into a Laplace domain transfer function matrix (L6)

Basic Matrix Operations

You should know how to:

- Add and multiply matrices & vectors (L2)
- Compute the inverse of a matrix (up to 3x3) by hand (L2)
- Compute the determinant of a matrix (up to 3x3) by hand (L2)
- Determine whether a matrix is singular or non-singular (L2)

Eigenvalues & Eigenvectors

You should know how to:

- Define the eigenvalue problem for any square matrix (L2)
- Solve the eigenvalue problem (up to 3x3) by hand for the eigenvalues and corresponding eigenvectors (L2)

State-Space Representations

You should know how to:

- Write one or more higher-order diffeq's in first-order form (L3)
- Place first-order diffeq's into state-space matrix form (L3)
- Write the output equations in state-space matrix form (L3)
- Place a SISO transfer function into state-space canonical form (L6)

State-Space Representations

You should know how to:

- Understand how state-space representations are not unique, but the input/output relationships are invariant (L3)
- Transform a state-space model with a non-singular transformation (L3)
- Understand the difference between LTI, LTV, and NL systems (L3)

State Transition Matrix

You should know how to:

- Use the series definition of the matrix exponential (L4)
- Use basic properties of the matrix exponential (state transition matrix) (L4)
- Compute the state transition matrix using Laplace transforms (L5)
- Use eigenvectors to compute the state transition matrix (L5)

Impulse Response

You should know how to:

- Understand the relationship between an impulse response and a transfer function (L7)
- Understand how to use an impulse response to compute the time response to an arbitrary excitation

Time Domain Response

You should know how to:

- Solve for a forced response and/or the IC response using the state transition matrix (L4)
- Solve for a forced response using the convolution integral (L7)
- Understand that the total response is a summation of multiple modes (L9)

Harmonic Response

You should know how to:

- Compute the magnitude and phase of a Laplace domain transfer function at a given frequency (L7)
- Compute the steady-state time response to a harmonic excitation (L7)
- Understand superposition

Poles, Zeros, and Stability

You should know how to:

- Obtain the SISO closed-loop characteristic equation (L1)
- Solve the quadratic equation
- Determine the poles of a state-space model (L6, L7)
- Understand the difference between asymptotic stability, marginal stability, and instability (L7)

Poles, Zeros, and Stability

You should know how to:

- Understand the concepts of bandwidth and time constant and how they are related (L7)
- Understand how the time response is related to pole locations (L8)

Pole Placement

You should know how to:

- Place the poles for a single-input state-space system using the "brute force" method. (L8)