Course Outline - 1st Half

- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response, Matrix Exponential
- Transfer Functions & Eigenvalues
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

Frequency Response Functions (FRF's)

- Review of Classical SISO Theory
- Stability from the FRF
- Harmonic Response
- Impulse Response and Convolution

FRF - Review - I

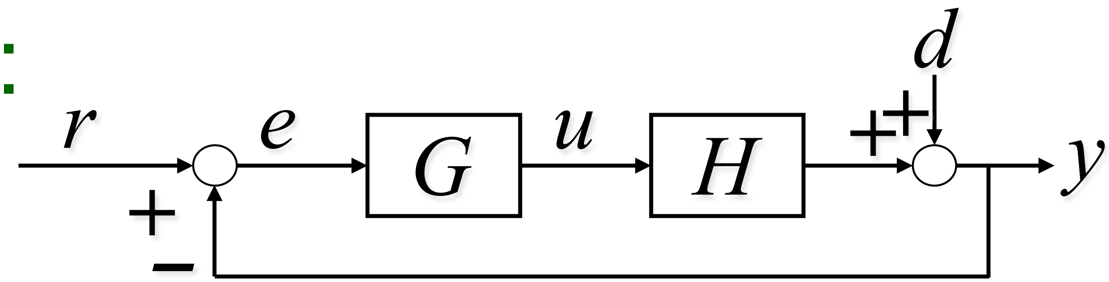
So far we have talked about the time domain solutions of the state equations.

Oftentimes it will be important to relate the frequency domain response of the state equations to the time domain.

A key element to understanding state-space control systems is to be able to relate the frequency response to the time response.

FRF - Review - II

SISO Example:



Consider the proportional control system

$$G(s) = 100$$

for the plant

$$H(s) = \frac{1}{s(s+1)(s+100)}$$

Note the three real open-loop poles:

$$s = -1\left(\frac{\text{rad}}{\text{sec}}\right) \quad \Rightarrow \quad \left(\frac{1\ \text{rad}}{\text{sec}}\right)\left(\frac{1\ \text{cycle}}{2\pi\ \text{rad}}\right) = 0.159(\text{Hz})$$

$$s = -100\left(\frac{\text{rad}}{\text{sec}}\right) \quad \Rightarrow \quad \left(\frac{100\ \text{rad}}{\text{sec}}\right)\left(\frac{1\ \text{cycle}}{2\pi\ \text{rad}}\right) = 15.9(\text{Hz})$$

FRF - Review - III

Remember that $s = \sigma + j\omega$ is <u>any</u> complex number in the complex plane and has units of (rad/sec).

Note that $GH(s)$ is also a complex number:

$$GH(s) = \frac{100}{s(s+1)(s+100)} = \frac{100}{s^3 + 101s^2 + 100s}$$

$$GH(\sigma + j\omega) = \frac{100}{(\sigma + j\omega)^3 + 101(\sigma + j\omega)^2 + 100(\sigma + j\omega)}$$

FRF - Review - IV

The complex terms can be expanded:

$$(\sigma + j\omega)^2 = (\sigma + j\omega)(\sigma + j\omega) = (\sigma^2 - \omega^2) + j(2\sigma\omega)$$

$$(\sigma + j\omega)^3 = (\sigma + j\omega)\left((\sigma^2 - \omega^2) + j(2\sigma\omega)\right)$$

$$= (\sigma^3 - \sigma\omega^2) + j(2\sigma^2\omega) + j(\omega\sigma^2 - \omega^3) - (2\sigma\omega^2)$$

$$= (\sigma^3 - 3\sigma\omega^2) + j(3\omega\sigma^2 - \omega^3)$$

Substituting these results back into GH:

$$GH(\sigma + j\omega) = \frac{100}{\left[\sigma^3 - 3\sigma\omega^2 + 100\sigma + 101(\sigma^2 - \omega^2)\right] + j\left[100\omega + 202\sigma\omega + 3\omega\sigma^2 - \omega^3\right]}$$

FRF - Review - V

Define:

$$\alpha = \left[\sigma^3 - 3\sigma\omega^2 + 100\sigma + 101\left(\sigma^2 - \omega^2\right)\right]$$

$$\beta = \left[100\omega + 202\sigma\omega + 3\omega\sigma^2 - \omega^3\right]$$

Now we can simplify the expression for the complex transfer function:

$$GH\left(\sigma + j\omega\right) =$$

FRF - Review - VI

The "Frequency Response" is evaluated only along the imaginary axis ($s = j\omega$).

For our example problem:

$$\sigma = 0 \quad \Rightarrow \quad \begin{cases} \\ \\ \end{cases}$$

$$GH(j\omega) = \underbrace{\left(\frac{100\alpha}{\alpha^2 + \beta^2}\right)}_{RE\{GH\}} - j\underbrace{\left(\frac{100\beta}{\alpha^2 + \beta^2}\right)}_{IM\{GH\}}$$

FRF - Review - VII

You should be familiar with frequency response plots (Bode, Nyquist, Nichols) from your undergraduate controls course.

In that course you learned that the closed-loop stability could be determined by looking at two parameters extracted from the open-loop (GH) frequency response:

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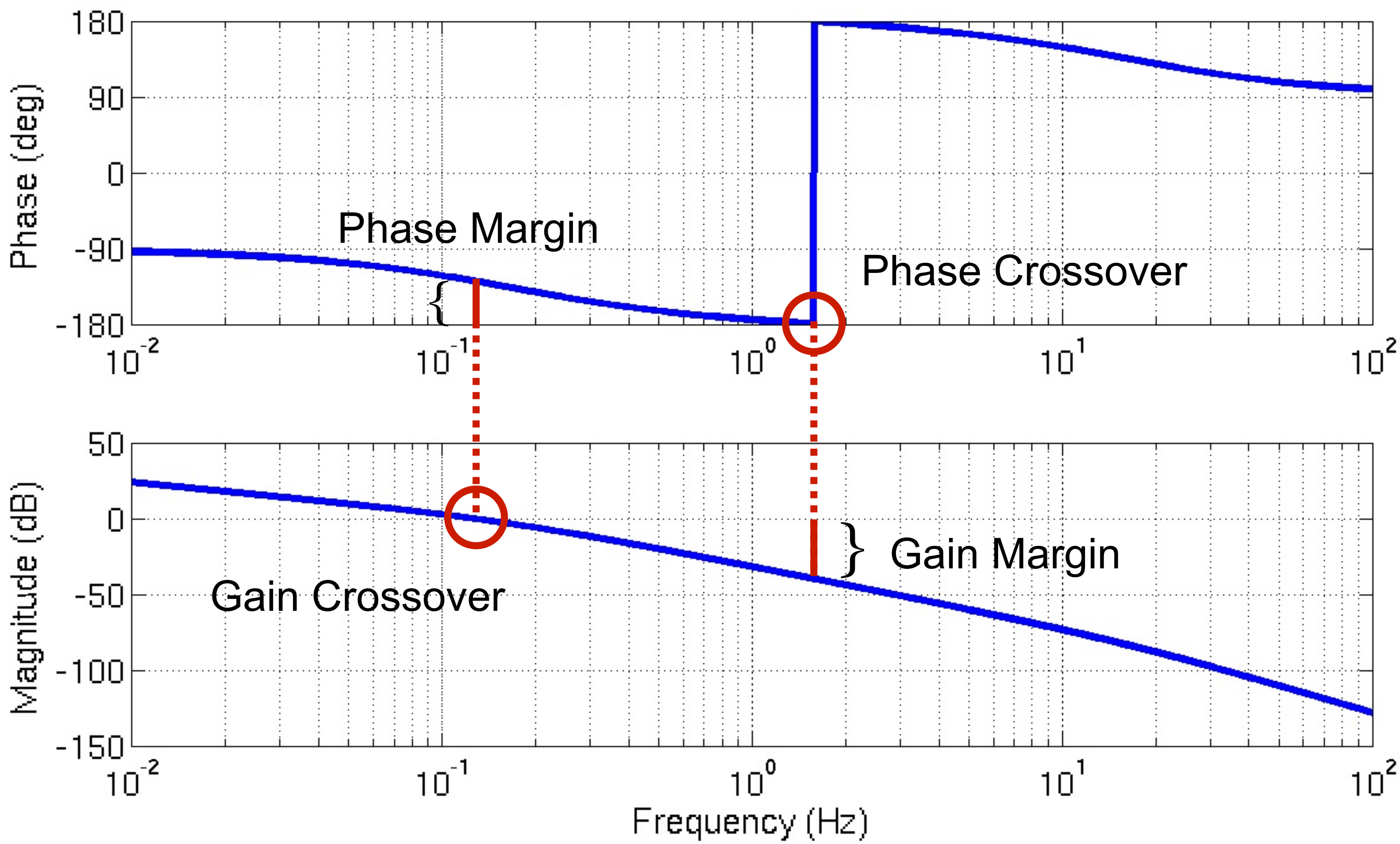
FRF - Review - VIII

Magnitude vs. frequency and Phase vs. frequency is called a “Bode” plot.



FRF - Review - IX

The closed-loop system is stable if

The phase margin (PM) is positive:

$$\angle GH > -180^\circ \quad \text{when} \quad |GH| = 1$$

How much more can the phase drop down?

(Good design practice requires PM > 30º)

The gain margin (GM) is positive:

$$|GH| < 1 \quad \text{when} \quad \angle GH = -180^\circ$$

How much more can we increase the gain?

(Good design practice requires GM > 6dB)

FRF - State-space - I

Next, we will investigate some relationships between the frequency domain and the time domain

- We will show that the steady-state time response to harmonic excitations is directly related to the frequency response.
- We will also show that the impulse response of the system is directly related to the frequency response.

FRF - State-space - II

We have already shown that state equations are transformed to transfer function matrices through the expression

$$\mathbf{H}(s) = \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}$$

which can also be written as

$$\mathbf{H}(s) = \mathbf{C}\left[\frac{adj\left[s\mathbf{I} - \mathbf{A}\right]}{\left|s\mathbf{I} - \mathbf{A}\right|}\right]\mathbf{B} + \mathbf{D}$$

FRF - State-space - III

This illustrates that the denominator of every transfer function in $\mathbf{H}$ is equal to $|s\mathbf{I} - \mathbf{A}|$. Therefore the characteristic equation of the system is equal to

$$\left| s\mathbf{I} - \mathbf{A} \right| = 0$$

which is equivalent to the equation for finding the eigenvalues of $\mathbf{A}$.

Therefore, the eigenvalues of $\mathbf{A}$ are equivalent to the poles of any input-output transfer function in $\mathbf{H}$.

FRF - State-space - IV

The characteristic equation (CE) of the system is a polynomial of the form

$$\left|s\mathbf{I}-\mathbf{A}\right| = s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0 = 0$$

Where the coefficients a_n are functions of the elements of $\mathbf{A}$.

There are N roots of this equation:

$$\left(s-s_1\right)\left(s-s_2\right)\cdots\left(s-s_N\right) = \prod_{i=1}^{N}\left(s-s_i\right) = 0$$

FRF - State-space - V

The stability of the LTI system is determined by the roots of the CE

- Asymptotically stable: The real part of every root is less than zero. ($\mathbf{A}$ is known as a Hurwitz matrix, system is BIBO stable)
- Marginally stable: At least one root has a real part which is equal to zero.
- Unstable: At least one root has a real part which is greater than zero.

Harmonic Response - I

One of the most useful aspects of the frequency response of an LTI system is determining the steady-state response to harmonic (sinusoidal) inputs.

In general, the input-output relationship of an LTI system is expressed as a matrix expression

$$\underset{[P\times 1]}{\mathbf{y}(s)} = \underset{[P\times M]}{\mathbf{H}(s)}\,\underset{[M\times 1]}{\mathbf{u}(s)}$$

Harmonic Response - II

Let's analyze the response of our state-space system to a harmonic excitation

We need the following assumptions:

- $\mathbf{H}(s)$ represents the transfer function matrix of an <u>asymptotically stable</u> (Hurwitz) LTI system.
- Only one of the M inputs is active ($u_m(t)$), all other inputs are zero. Superposition allows the study of individual solutions.

Harmonic Response - III

We can mathematically select only the m^{th} actuator as follows

$$\mathbf{u}(t)=\begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \\ \vdots \\ u_M(t) \end{bmatrix}=\begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} U\sin(\omega t)$$

Harmonic Response - IV

Use Laplace transforms to find the output $\mathbf{y}(t)$ to this harmonic input.

First, transform $u_m(t)$ into the Laplace domain:

$$L\{U \sin(\omega t)\} = U\left(\frac{\omega}{s^2 + \omega^2}\right) =$$

Harmonic Response - V

Substituting this result into the matrix expression:

$$\mathbf{y}(s)=\begin{bmatrix} H_{11}(s) & \cdots & H_{1m}(s) & \cdots & H_{1M}(s) \\ \vdots & & \vdots & & \vdots \\ H_{p1}(s) & \cdots & H_{pm}(s) & & H_{pM}(s) \\ \vdots & & \vdots & & \vdots \\ H_{P1}(s) & \cdots & H_{Pm}(s) & \cdots & H_{PM}(s) \end{bmatrix}\begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} U\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$y_p(s)=H_{pm}(s)\times U\left(\frac{\omega}{(s-j\omega)(s+j\omega)}\right)$$

Harmonic Response - VI

The time response $\mathbf{y}(t)$ is found by taking the inverse Laplace transform.

Remember that all elements in this resultant vector have the same denominators given by:

$$\underbrace{(s-s_1)(s-s_2)\cdots(s-s_N)}_{\substack{\text{System poles} \\ \text{(eigenvalues of } \mathbf{A}\text{)}}}\underbrace{(s-j\omega)(s+j\omega)}_{\substack{\text{Terms due to} \\ \text{forcing function}}}$$

Harmonic Response - VII

Ignore the numerators for the moment and set up a partial-fraction expansion for the p^{th} element of $\mathbf{y}(s)$:

$$UH_{pm}(s)\frac{\omega}{s^2+\omega^2} = \frac{Q_1}{s+j\omega} + \frac{Q_2}{s-j\omega} + \frac{R_1}{s-s_1} + \frac{R_2}{s-s_2} + \cdots \frac{R_N}{s-s_N}$$

Transforming back into the time domain:

$$L^{-1}\left\{UH_{pm}(s)\frac{\omega}{s^2+\omega^2}\right\} = \underbrace{Q_1 e^{-j\omega t} + Q_2 e^{j\omega t}}_{\text{These are the steady-state terms}} + \underbrace{R_1 e^{s_1 t} + R_2 e^{s_2 t} + \cdots R_N e^{s_N t}}_{\text{These terms will approach zero because the system is asymptotically stable}}$$

For the steady-state response, we only need to solve for Q_1 and Q_2.

Harmonic Response - VIII

To compute the steady-state terms:

$$Q_1 = U(s + j\omega) H_{pm}(s) \frac{\omega}{s^2 + \omega^2}\bigg|_{s=-j\omega} = -\left(\frac{U}{2j}\right) H_{pm}(-j\omega)$$

$$Q_2 = U(s - j\omega) H_{pm}(s) \frac{\omega}{s^2 + \omega^2}\bigg|_{s=j\omega} = \left(\frac{U}{2j}\right) H_{pm}(j\omega)$$

The terms $H_{pm}(j\omega)$ and $H_{pm}(-j\omega)$ are complex-valued functions of the form

$$H_{pm}(j\omega) = \left|H_{pm}(j\omega)\right| e^{j\angle H_{pm}(j\omega)}$$

$$H_{pm}(-j\omega) = \left|H_{pm}(j\omega)\right| e^{-j\angle H_{pm}(j\omega)}$$

Harmonic Response - IX

The steady-state response of output p due to input m is the following

$$y_p(t) = U\left|H_{pm}(j\omega)\right| \underbrace{\left(\frac{e^{j\left(\omega t + \angle H_{pm}(j\omega)\right)} - e^{-j\left(\omega t + \angle H_{pm}(j\omega)\right)}}{2j} \right)}_{\text{This term is just a sine wave!}}$$

The simplified steady-state response is

$$y_p(t) =$$

Harmonic Response - Summary

The steady-state response of an asymptotically-stable LTI system to a sinusoidal excitation

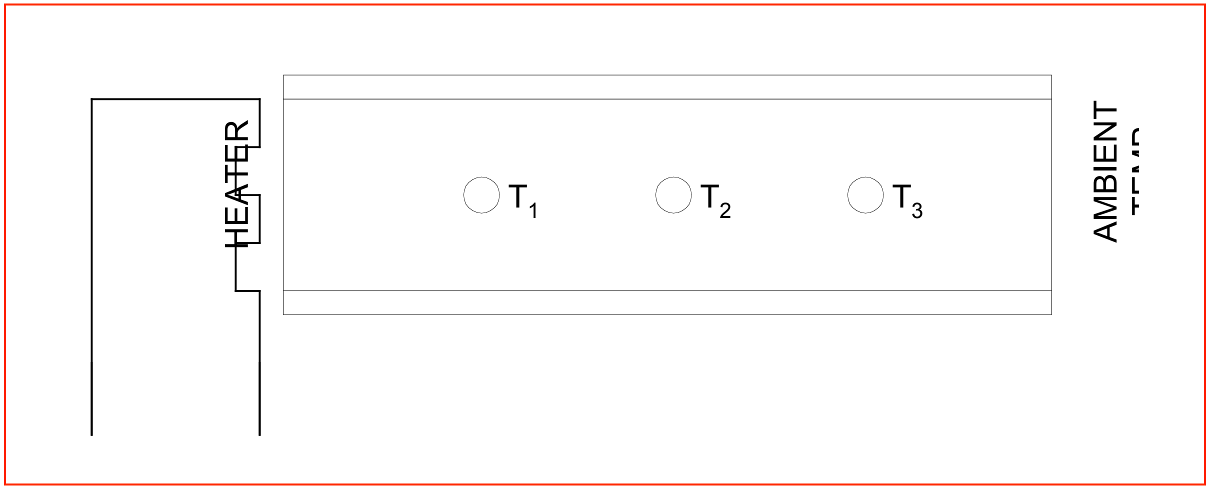
$$\mathbf{u}(t) = \begin{bmatrix} \cdots & 0 & \underset{m^{th}}{1} & 0 & \cdots \end{bmatrix}^T U \sin(\omega t)$$

is a sine wave that is amplified (or attenuated) by $|H_{pm}(j\omega)|$ and phase-shifted by $\angle H_{pm}(j\omega)$

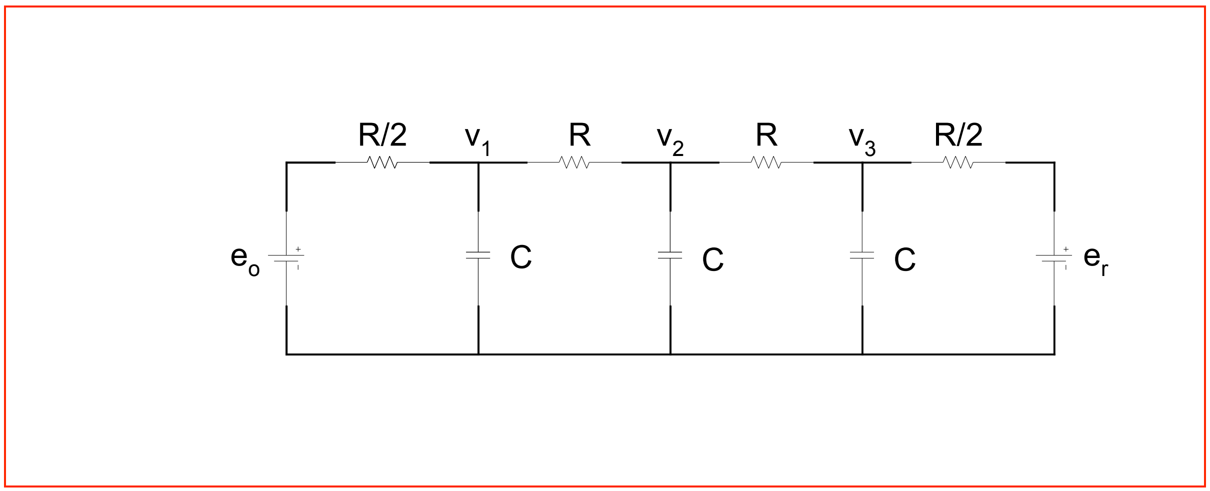
$$y_p(t) = U \left| H_{pm}(j\omega) \right| \sin\left(\omega t + \angle H_{pm}(j\omega) \right)$$

Harmonic Excitation - Example - I

Example - Heat Conduction in a Rod



Thermal Model



Analogous Lumped-Parameter Electrical Model (Voltage = Temp)

Harmonic Excitation - Example - II

The state equations are

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \frac{1}{RC}\begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \frac{1}{RC}\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} e_o \\ e_r \end{bmatrix}$$

and the output equations are

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

For this example, assume $RC = 0.1 \text{ sec.}$

Harmonic Excitation - Example - III

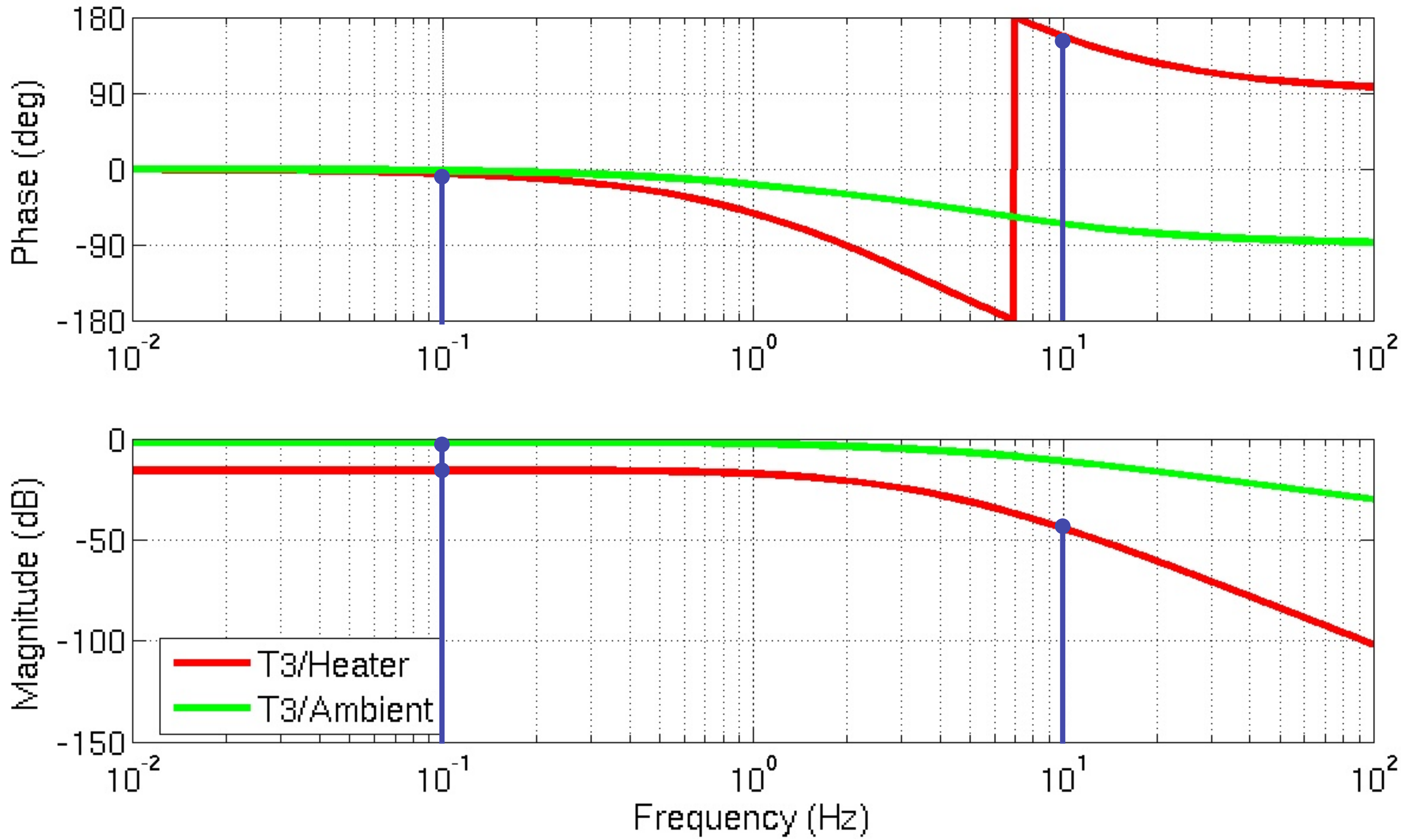
The matrix of transfer functions is found from

$$\mathbf{H}(s) = \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}$$

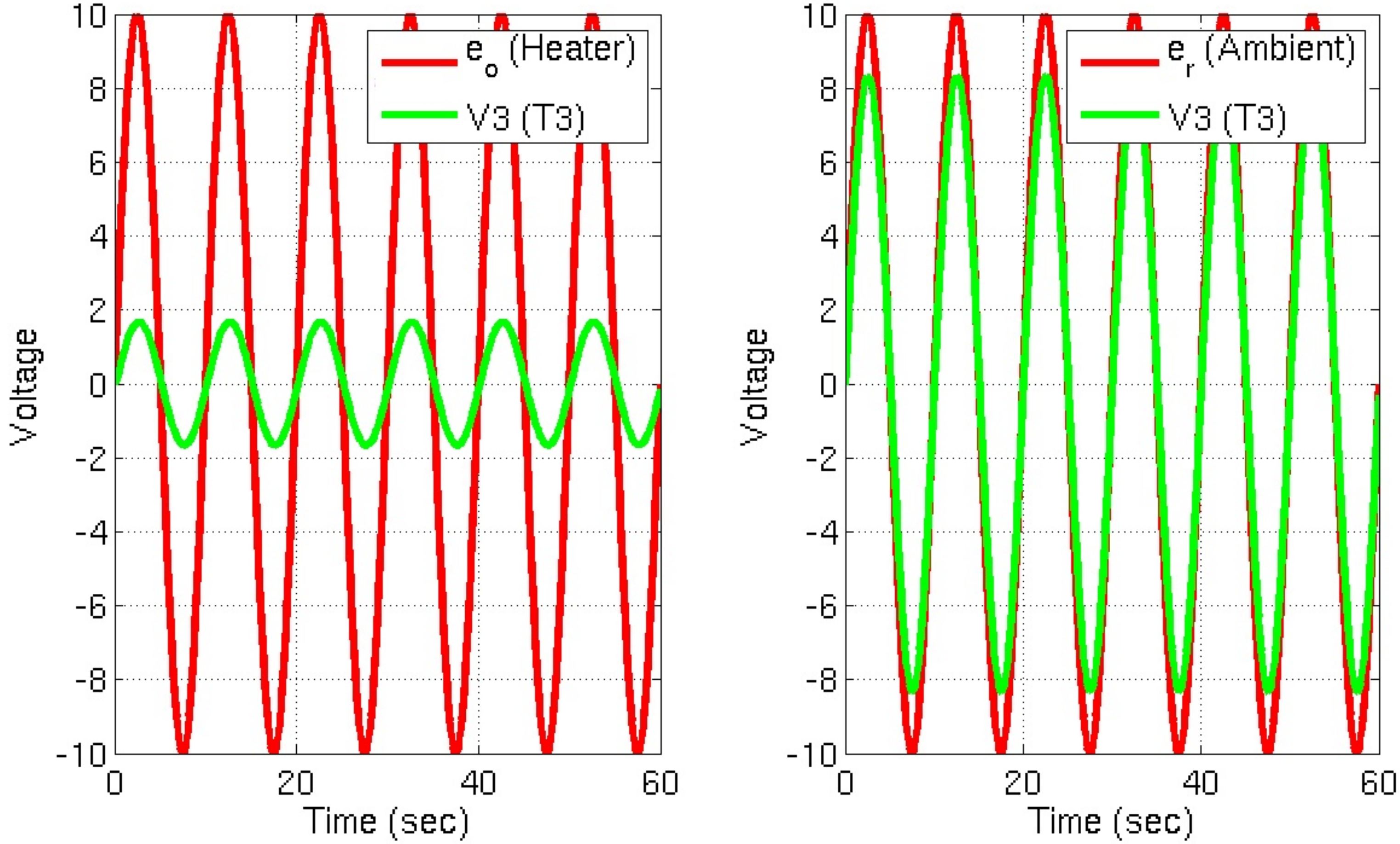
$$\underset{[1\times 2]}{\mathbf{H}(s)} = \left[\underbrace{\left(\frac{2000}{s^3+80s^2+1900s+12000}\right)}_{\frac{v_3}{e_o} \Rightarrow \frac{T_3}{\text{Heater}}} \quad \underbrace{\left(\frac{20\left(s^2+50s+500\right)}{s^3+80s^2+1900s+12000}\right)}_{\frac{v_3}{e_r} \Rightarrow \frac{T_3}{\text{Ambient}}} \right]$$

Harmonic Excitation - Example - IV

Evaluate the FRF's at 0.1 Hz and 10 Hz

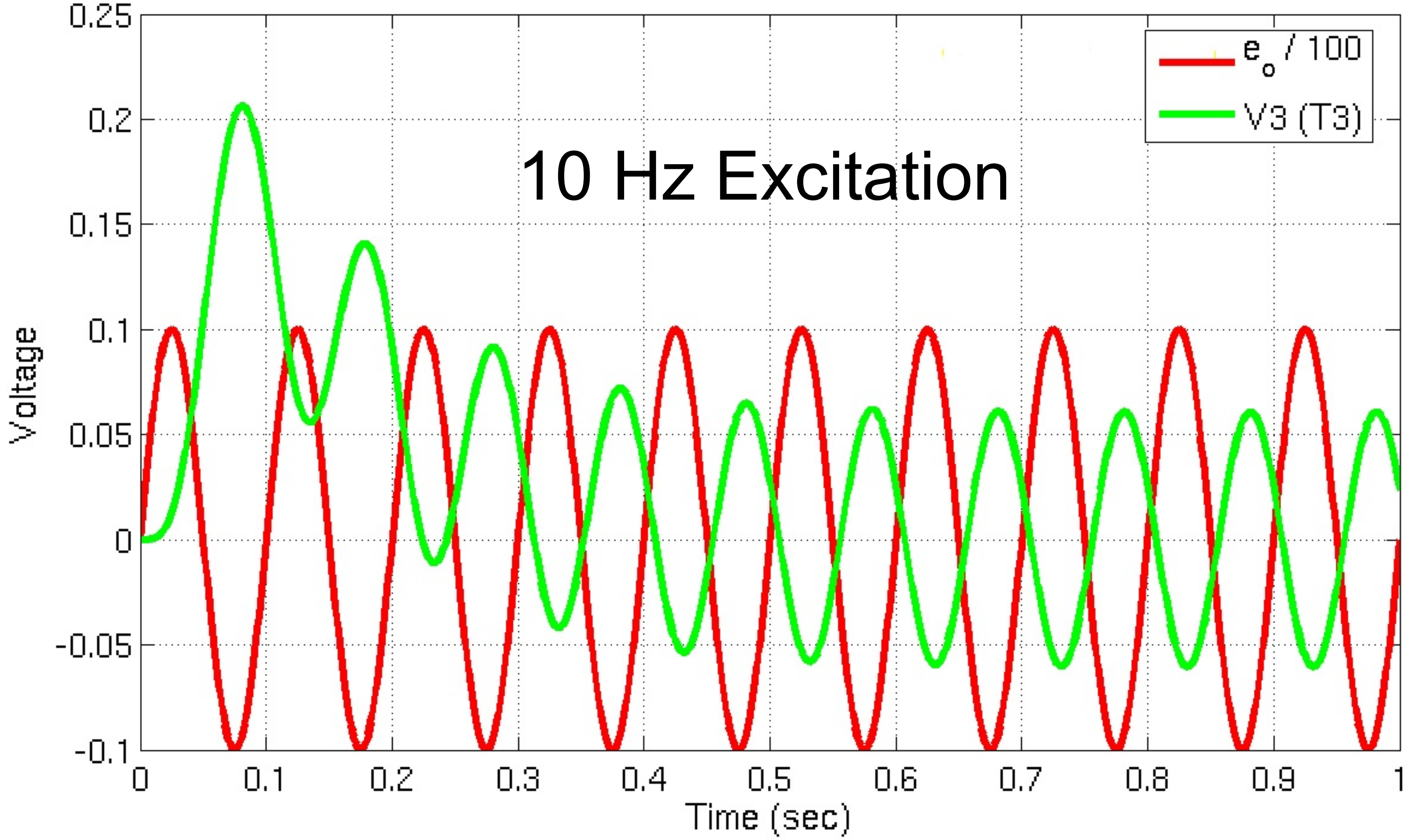


Harmonic Excitation - Example - V



- T3 is in-phase with 0.1 Hz excitation
- T3 is more sensitive to Ambient excitation

Harmonic Excitation - Example - VI



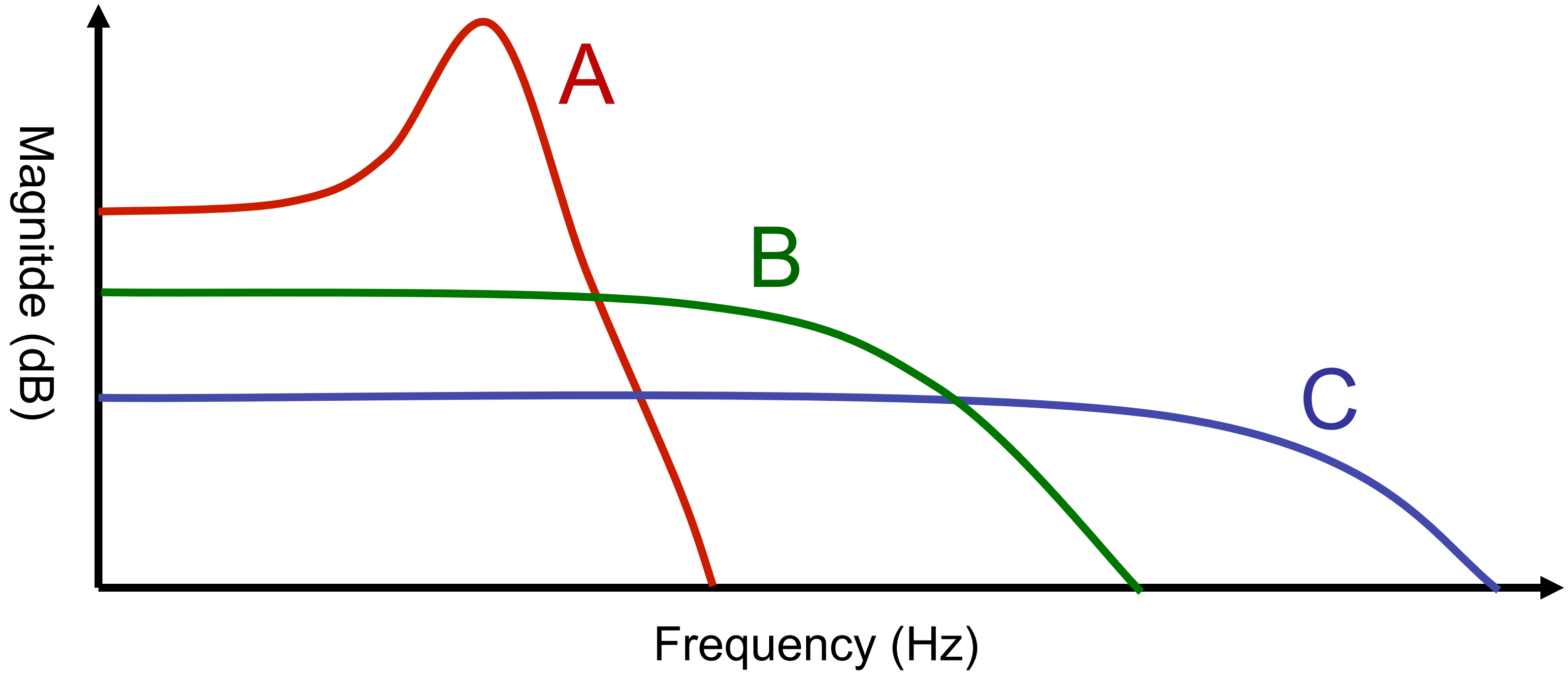
- Transient dies out within ~5 cycles
- T3 is out of phase with Heater excitation

Harmonic Excitation - Summary

The steady-state response of an asymptotically-stable system to harmonic excitation is directly related to the magnitude and phase of the frequency response $\mathbf{H}(j\omega)$

Because we are assuming that the system is linear, the frequency response can be used to determine the steady-state response to multiple harmonic inputs and any periodic excitation.

In-Class Assignment



1. Which system has the fastest time constant?
2. Which system has the highest bandwidth?
3. Which system will exhibit the greatest overshoot to a step input?
4. Which system will have the largest steady-state response to a step input?

Impulse Response - I

How do we compute the impulse response?

$$\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s)$$

If the m^{th} input is a Dirac delta impulse at time $t = 0$

$$L\{\delta(t)\} = 1 \quad \Rightarrow \quad \mathbf{u}(s) = \begin{bmatrix} u_1(s) \\ \vdots \\ u_m(s) \\ \vdots \\ u_M(s) \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

Impulse Response - II

The Laplace transform of the p^{th} output is

$$\begin{bmatrix} \vdots \\ y_p(s) \\ \vdots \end{bmatrix} = \begin{bmatrix} H_{11}(s) & \cdots & \cdots & \cdots & H_{1M}(s) \\ \vdots & \vdots & H_{pm}(s) & \vdots & \vdots \\ H_{P1}(s) & \cdots & \cdots & \cdots & H_{PM}(s) \end{bmatrix} \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

$$y_p(s) = H_{pm}(s)$$

and the time response associated with an impulse excitation is

$$L^{-1}\left\{y_p(s)\right\} = L^{-1}\left\{H_{pm}(s)\right\} = h_{pm}(t)$$

Impulse Response - III

The term $h_{pm}(t)$ is the <u>impulse response function</u> between the p^{th} output and the m^{th} input.

Why is this so important?

If we know the impulse response functions, then we can solve for the response to <u>any arbitrary input</u> $u_m(t)$ through the convolution integral

Impulse Response - IV

The convolution integral (or superposition integral) for an LTI system is directly related to the impulse response function.

$$y_p(t) = \int_0^t h_{pm}(t-\tau) u_m(\tau) d\tau$$

Impulse Response - Summary

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The inverse Laplace transform of $\mathbf{H}(s)$ is equivalent to the matrix of impulse response functions.

The response to any arbitrary input can be computed once you know the impulse response functions.

Does this mean that if we know the frequency response function $\mathbf{H}(j\omega)$ we can compute the impulse response?