Advanced Control Engineering

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

Uncertainty - I

Up to this point, we have assumed that an exact dynamic model of the plant ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$) is available.

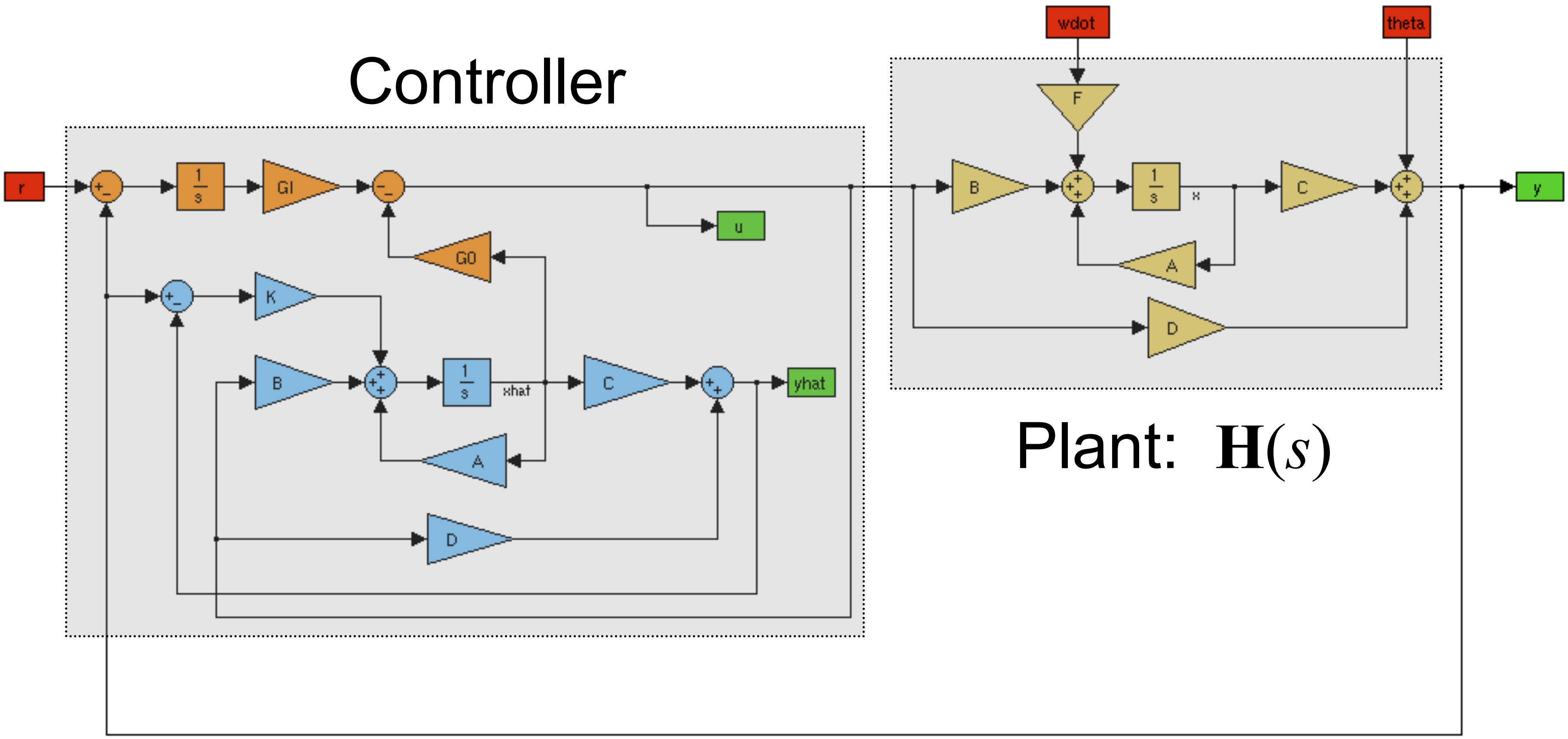
What if the model is not the same as the plant?

Does the closed-loop performance change?

Can our compensator destabilize the plant?

Uncertainty - II

Which parts of the closed-loop system might be a source of uncertainty?



Uncertainty - III

Let' s review the principal functions of a fixed-gain controller:

1. Maintainance of stability
2. Proper response to command inputs
3. Reduction of disturbance perturbations

The ability to do (1) with uncertainty is called <u>stability robustness</u>.

The ability to do (2) and (3) with uncertainty is called <u>performance robustness</u>.

Uncertainty - IV

Robustness analysis is the analysis of the stability and performance of the closed-loop system in the presence of uncertainty.

A control system is said to be robust if the stability and/or performance is insensitive to uncertainties in the system.

Uncertainty - V

There are two general categories of uncertainties in a control system:

1. Uncertain exogenous inputs

   Exogenous inputs are usually handled by augmented control system design:

   – Integral control

   – Linear Quadratic Gaussian (LQG) control

   – Disturbance accommodation with Robust Servomechanism, Internal Model Control, etc.

2. Uncertain plant dynamics

Uncertainty - VI

The most significant form of uncertainty is associated with the plant model.

- The physics of the plant may only be approximately understood
- Only an empirical data-based estimate of the plant is available
- The exact dynamics may be known but too complicated to include in the control system design (i.e. infinite dimensional systems governed by PDE's)
- The plant may change slowly with time

Uncertainty - VII

Plant model uncertainties may be classified into two categories:

Parametric Uncertainties: The structure of the model is known, but one or more parameters are unknown, unmeasurable, or slowly time varying.

Structural Uncertainties: The structure of the model is unknown and the parameters are empirically obtained, or the structure is intentionally simplified (e.g. finite-mode truncation).

Robustness - I

Robust Control is probably the most heavily researched topic in control theory over the last 25 years.

In this course, we will only provide a very brief introduction and demonstrate how parametric uncertainty can affect the stability and performance of a feedback control system.

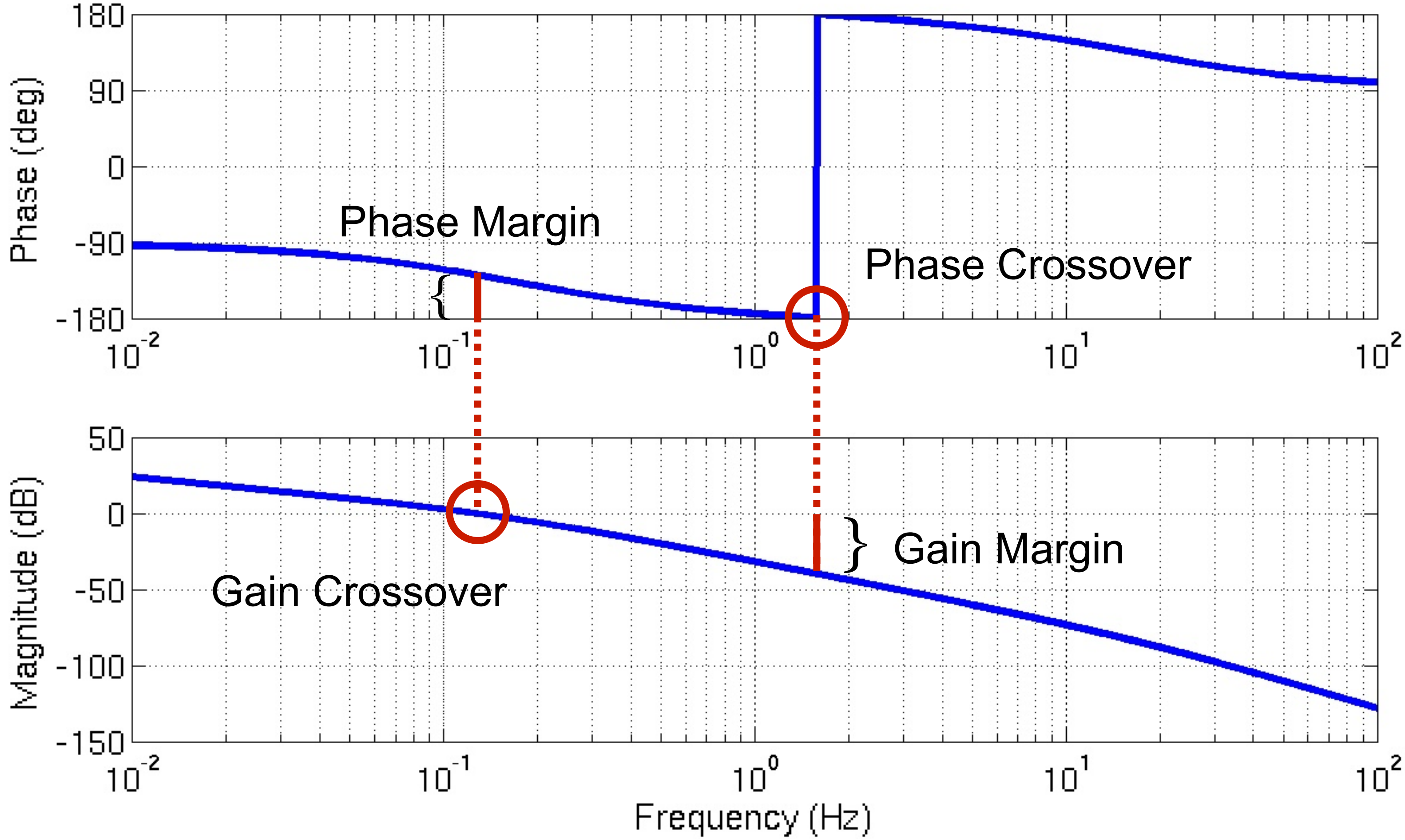
Robustness - II

Can we analyze the robustness of a MIMO state-space system using SISO techniques from classical control?

One classical analysis method is to evaluate stability margins such as the gain margin and the phase margin. This is a frequency domain technique using either a Bode diagram, a Nyquist plot, or a Nichols chart.

Robustness - III

Here is an example SISO Bode Diagram showing the gain and phase margins.



Robustness - IV

It is not easy to extend these frequency domain stability margins to the multivariable case (i.e. state-space) because there are multiple transfer functions ($\dim\{\mathbf{H}(s)\} = [P \times M]$).

There are a number of techniques which use frequency domain gain and phase margins to analyze the robustness of multivariable systems. In fact, there is even a multivariable Nyquist stability criterion.

Robustness - V

Another technique from classical control is to evaluate the root locus generated by varying one uncertain parameter.

This technique easily extends to the multivariable case except that it has one significant drawback: <u>You can only vary one parameter at a time</u>!

Robustness - VI

For example, in the mass-spring-mass example from L14 & L18, we could assume that the stiffness was known to an accuracy of ±50%. We could model the stiffness with a nominal value k_0 plus an additive perturbation:

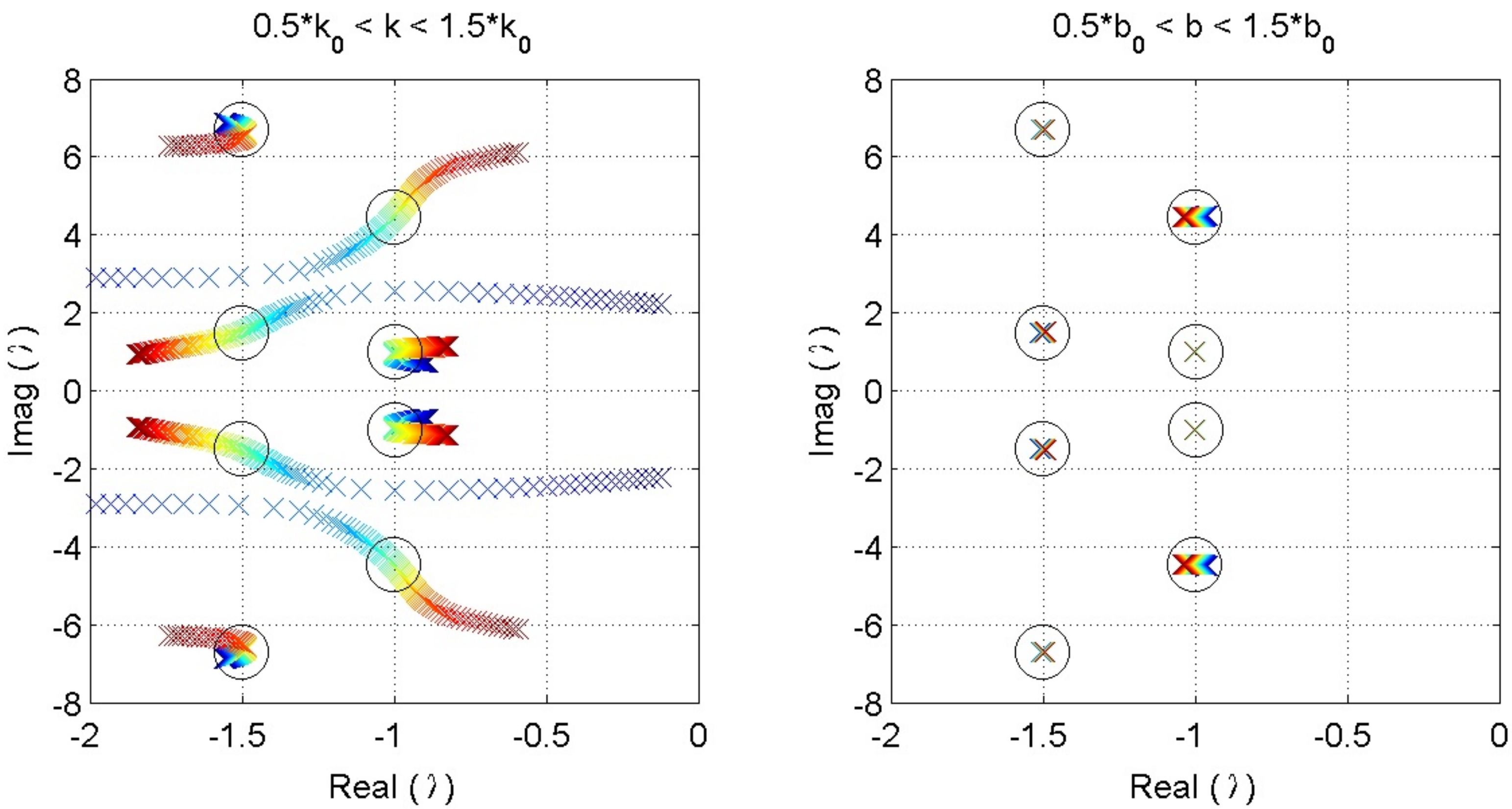
$$k = k_0 + \tilde{k} \qquad -0.5k_0 < \tilde{k} < 0.5k_0$$

or a multiplicative perturbation:

$$k = k_0(1+\delta) \qquad -0.5 < \delta < 0.5$$

Robustness - VII

Here are two root locus plots when the stiffness and damping are varied ±50% in our mass-spring-mass example.



Robustness - VIII

For any feedback control design, we start with a nominal model of the plant, but in reality the actual plant is uncertain. For parametric uncertainty:

Actual Plant
$$\begin{cases} \dot{\mathbf{x}} = \left[\mathbf{A}_0 + \tilde{\mathbf{A}}\right]\mathbf{x} + \left[\mathbf{B}_0 + \tilde{\mathbf{B}}\right]\mathbf{u} \\ \mathbf{y} = \left[\mathbf{C}_0 + \tilde{\mathbf{C}}\right]\mathbf{x} + \left[\mathbf{D}_0 + \tilde{\mathbf{D}}\right]\mathbf{u} \end{cases}$$

Perturbation

Nominal Plant

Robustness - IX

We then use the nominal model:

$$\hat{\mathbf{A}} = \mathbf{A}_0 \qquad \hat{\mathbf{B}} = \mathbf{B}_0 \qquad \hat{\mathbf{C}} = \mathbf{C}_0 \qquad \hat{\mathbf{D}} = \mathbf{D}_0$$

to design a state-feedback controller and the dynamic observer feedback gains.

For example:

```
G = place(Ahat,Bhat,cl_poles);

K = place(Ahat',Chat',obs_poles)';
```

Robustness - X

The augmented closed-loop system is:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \underbrace{\begin{bmatrix} \left(\mathbf{A}_0 + \tilde{\mathbf{A}}\right) & -\left(\mathbf{B}_0 + \tilde{\mathbf{B}}\right)\mathbf{G} \\ \mathbf{K}\left(\mathbf{C}_0 + \tilde{\mathbf{C}}\right) & \left(\hat{\mathbf{A}} - \mathbf{K}\hat{\mathbf{C}} - \hat{\mathbf{B}}\mathbf{G} - \mathbf{K}\left(\left(\mathbf{D}_0 + \tilde{\mathbf{D}}\right) - \hat{\mathbf{D}}\right)\mathbf{G}\right) \end{bmatrix}}_{\text{Closed-loop Augmented } \mathbf{A} \text{ matrix}} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

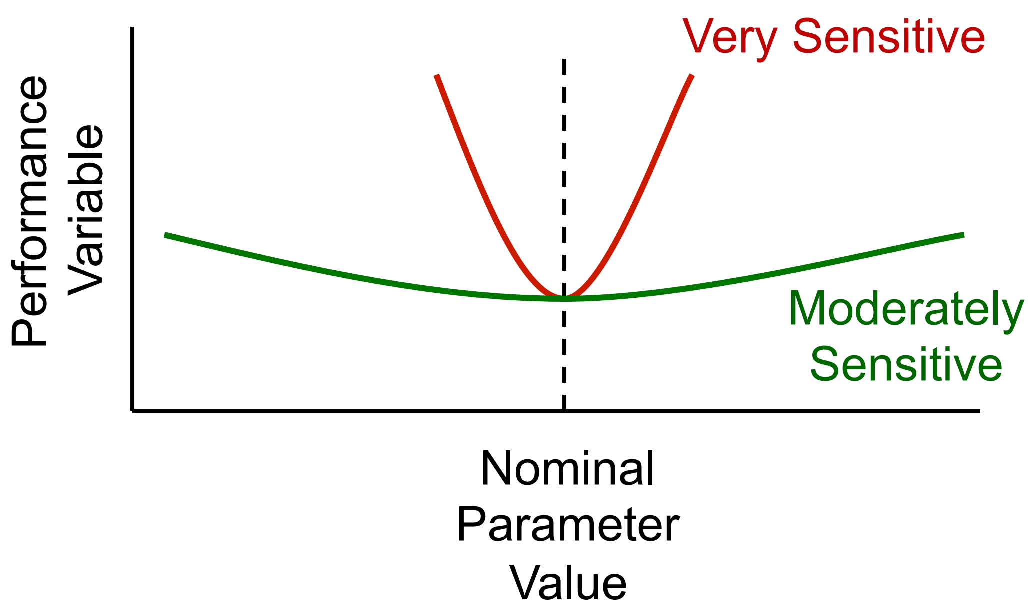
$$\begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{C}_0 + \tilde{\mathbf{C}}\right) & -\left(\mathbf{D}_0 + \tilde{\mathbf{D}}\right)\mathbf{G} \\ \mathbf{0} & \hat{\mathbf{C}} - \hat{\mathbf{D}}\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

A root locus can be generated by plotting the eigenvalues of the closed-loop $\mathbf{A}$ matrix while varying <u>one</u> uncertain parameter or perturbation.

Robustness - XI

Instead of the closed-loop eigenvalues, we could plot some other performance variable such as <u>settling time</u> or <u>peak control effort</u> as a function of the uncertain parameter.

Unfortunately, we can still only plot the variation due to a single parameter.



SVD Analysis - I

Conservative bounds for acceptable additive perturbations can be obtained using singular value analysis.

Singular values are analogous to eigenvalues for complex or real matrices that are not necessarily square. Singular Value Decompositions (SVD) are extremely powerful and show up in many branches of engineering.

SVD Analysis - II

We can express any general complex matrix $\mathbf{H}$ in terms of its Singular Value Decomposition (SVD):

$$\underset{[P\times M]}{\mathbf{H}} = \underset{[P\times P]}{\mathbf{U}}\underset{[P\times M]}{\Sigma}\underset{[M\times M]}{\mathbf{V}^{H}}$$

$\mathbf{U}$ is a complex matrix whose columns are the left singular vectors of $\mathbf{H}$

$\mathbf{V}$ is a complex matrix whose columns are the right singular vectors of $\mathbf{H}$

$\mathbf{\Sigma}$ is a diagonal matrix of singular values

SVD Analysis - III

Each of the SVD components have very interesting properties, but for MIMO robustness analysis, we are only interested in the singular values:

$$\underset{[P\times M]}{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & & & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ & & \sigma_J & & & \\ \vdots & & & 0 & & \\ 0 & \cdots & & & 0 & 0 \end{bmatrix} \quad \Leftrightarrow \quad rank(\mathbf{H}) = J$$

$$\bar{\sigma} \equiv \max\{\sigma_1, \ldots, \sigma_J\}$$

$$\underline{\sigma} \equiv \min\{\sigma_1, \ldots, \sigma_J\}$$

SVD Analysis - IV

How does the SVD relate to our robust feedback control problem?

We can always write any state-space system in the frequency domain as a matrix of transfer functions:

$$\mathbf{H}(j\omega) = \mathbf{C}\left[j\omega\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}$$

SVD Analysis - V

The effects of any additive perturbations to the state-space system can be expressed as follows:

$$\mathbf{H}(j\omega) = \mathbf{H}_0(j\omega) + \tilde{\mathbf{H}}(j\omega)$$

Where $\mathbf{H}_0$ is the transfer function matrix associated with the nominal state-space system, and $\tilde{\mathbf{H}}$ is the associated variation of the transfer function matrix.

SVD Analysis - VI

At any selected frequency, we can compute the SVD of the nominal transfer function matrix:

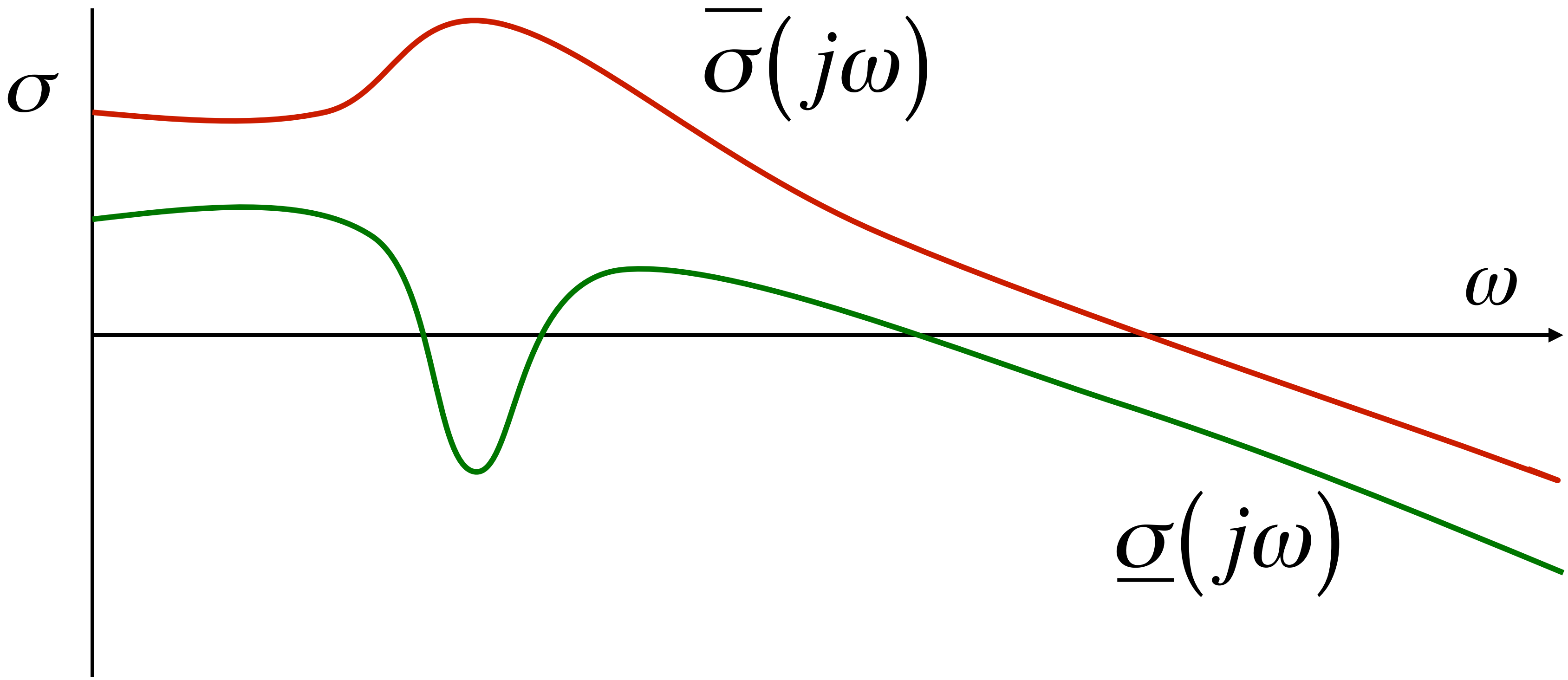
$$\mathbf{H}_0(j\omega) = \mathbf{U}(j\omega)\boldsymbol{\Sigma}(j\omega)\mathbf{V}^H(j\omega)$$

And in particular, extract the max and min singular values:

$$\overline{\sigma}(j\omega) \text{ and } \underline{\sigma}(j\omega)$$

SVD Analysis - VI

We can then plot the min and max singular values over a range of frequencies:



SVD Analysis - VI

The singular value plot can provide valuable information about the properties of the MIMO system.

In particular:

- It quantifies the "gain band" of the plant at each frequency.
- It is a natural generalization of the information in the classical Bode magnitude plot for SISO plants

SVD Analysis - VI

And finally, if the nominal system is stable, then the perturbation will not destabilize the system as long as the following criterion is met:

$$\overline{\sigma}\left[\tilde{\mathbf{H}}(j\omega)\right] < \underline{\sigma}\left[\mathbf{I}+\mathbf{H}_0(j\omega)\right]$$

This is a conservative bound so if it fails at any frequency, then stability is not guaranteed, but it is still possible.

Summary

Model error, or plant uncertainty, can affect the performance of feedback control systems.

- It can change the performance
- It can cause closed-loop instability

Most classical techniques for analyzing robustness have MIMO counterparts, but are not as easy to use.