Course Outline - 1st Half

Virginia Tech

- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response, Matrix Exponential (2)
- Transfer Functions & Eigenvalues
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

LTI Response Solutions

- Laplace Transform Method
- Method of Diagonalization
- Numerical Methods

Introduction

In the last meeting we solved the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

using a matrix exponential for the state transition matrix.

$$\Phi(t, t_0) = e^{\mathbf{A}(t - t_0)}$$

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} [\Phi(t, \tau)\mathbf{B}]\mathbf{u}(\tau)\, d\tau$$

Introduction

In this meeting we will derive alternative means of solving for the states as a function of time.

In doing this, we will develop a means of computing the state transition matrix of an LTI system.

Review of Laplace Transforms

Laplace Transform Definition:

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$L^{-1}\{F(s)\} = f(t) = \left(\tfrac{1}{2\pi i}\right)\int_{-i\infty}^{+i\infty} e^{sx} F(s)\,ds \qquad t > 0$$

Common Transform Pairs:

$$L\{\tfrac{d}{dt} f(t)\} = sF(s) - f(0)$$

$$L\left\{\int_{-\infty}^{t} f(\tau)\,d\tau\right\} = \frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} f(\tau)\,d\tau$$

$$L\{\delta(t)\} = 1$$

$$L\{\mu(t)\} = \tfrac{1}{s}$$

$$L\{t\} = \tfrac{1}{s^2}$$

$$L\{e^{-at}\} = \frac{1}{s+a}$$

$$L\{\sin(\omega t)\} = \frac{\omega^2}{s^2+\omega^2}$$

$$L\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

Laplace Solution - I

Apply the Laplace Transform to our State Space system:

$$L\{\dot{\mathbf{x}}(t)\} = L\{\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\}$$

$$s\mathbf{x}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s)$$

Rearrange terms and solve for $\mathbf{x}(s)$

Laplace Solution - II

Examine the Laplace solution:

$$\mathbf{x}(s) = \underbrace{[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0)}_{\text{Response due to Initial Conditions}} + \underbrace{[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{u}(s)}_{\text{Forced Response}}$$

How do we get the time response?

Take the **inverse** Laplace Transform of both sides:

Laplace Solution - III

This solution must be the same as the previous solution we derived, therefore, looking only at the IC term:

$$L^{-1}\left\{\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{x}(0)\right\}=e^{\mathbf{A}t}\mathbf{x}(0)$$

Since the initial condition vector is a constant, we must have

$$L^{-1}\left\{\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\right\}=e^{\mathbf{A}t}$$

Laplace Solution - IV

Continuing with the solution, the last term in the Laplace solution is

$$L^{-1}\left\{\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{Bu}(s)\right\}=?$$

We can apply the <u>Convolution Integral</u>

$$L^{-1}\left\{f(s)g(s)\right\}=\int_0^t f(t-\tau)g(\tau)d\tau$$

Therefore

Laplace Solution - V

The solution using Laplace Transforms is identical to the solution we obtained through direct integration

$$L^{-1}\left\{\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{x}(0)+\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}\mathbf{u}(s)\right\}$$

$$= e^{\mathbf{A}t}\mathbf{x}(0)+\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

More importantly, the Laplace solution gives us a direct relationship between the <u>state matrix</u> $\mathbf{A}$ and the <u>matrix exponential</u>

Laplace Transform of $e^{\mathbf{A}t}$ - I

Now we know that the matrix exponential is related to the state matrix $\mathbf{A}$ through the inverse Laplace Transform:

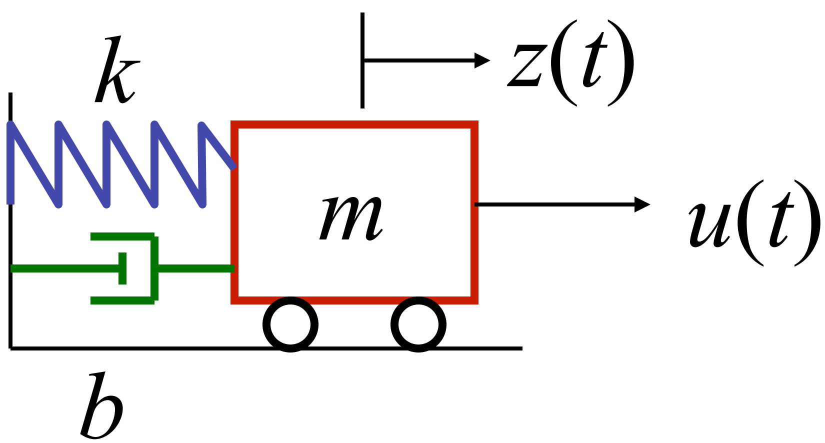
$$L^{-1}\left\{\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\right\}=e^{\mathbf{A}t}$$

How can we use this information?

Laplace Transform of $e^{\mathbf{A}t}$ - II

Example: The model of a damped mass-spring oscillator is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$



States:

$$x_1(t) = z(t)$$

$$x_2(t) = \dot{z}(t)$$

Parameters:

$$\omega^2 = \left(\tfrac{k}{m}\right)$$

$$2\zeta\omega = \left(\tfrac{b}{m}\right)$$

Laplace Transform of $e^{\mathbf{A}t}$ - III

Let's compute $e^{\mathbf{A}t}$ for this system using the Laplace Transform.

First, find $[s\mathbf{I} - \mathbf{A}]$

$$[s\mathbf{I}-\mathbf{A}] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} = \begin{bmatrix} s & -1 \\ \omega^2 & s+2\zeta\omega \end{bmatrix}$$

Next, compute the inverse

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \begin{bmatrix} s & -1 \\ \omega^2 & s+2\zeta\omega \end{bmatrix}^{-1} =$$

Laplace Transform of $e^{\mathbf{A}t}$ - IV

The final result is:

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \begin{bmatrix} \left(\dfrac{s+2\zeta\omega}{s^2+2\zeta\omega s+\omega^2}\right) & \left(\dfrac{1}{s^2+2\zeta\omega s+\omega^2}\right) \\ \left(\dfrac{-\omega^2}{s^2+2\zeta\omega s+\omega^2}\right) & \left(\dfrac{s}{s^2+2\zeta\omega s+\omega^2}\right) \end{bmatrix}$$

Note that $[s\mathbf{I} - \mathbf{A}]^{-1}$ is an $N \times N$ matrix of transfer functions in the complex Laplace variable s, where N is the number of states (2 in this case!).

Laplace Transform of $e^{\mathbf{A}t}$ - V

The matrix $e^{\mathbf{A}t}$ is found by applying the inverse Laplace Transform to $[s\mathbf{I} - \mathbf{A}]^{-1}$.

This operation is performed element by element in the matrix:

$$L^{-1}\left\{\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\right\} = \begin{bmatrix} L^{-1}\left\{\dfrac{s+2\zeta\omega}{s^2+2\zeta\omega s+\omega^2}\right\} & L^{-1}\left\{\dfrac{1}{s^2+2\zeta\omega s+\omega^2}\right\} \\ L^{-1}\left\{\dfrac{-\omega^2}{s^2+2\zeta\omega s+\omega^2}\right\} & L^{-1}\left\{\dfrac{s}{s^2+2\zeta\omega s+\omega^2}\right\} \end{bmatrix}$$

This is the tedious part of the process…

Laplace Transform of $e^{\mathbf{A}t}$ - VI

From a table of Laplace Transform pairs:

$$L^{-1}\left\{\frac{1}{s^2+2\zeta\omega s+\omega^2}\right\} = \left(\frac{1}{\omega_d}\right)e^{-\zeta\omega t}\sin(\omega_d t)$$

We can also look up the following:

$$L^{-1}\left(\frac{s}{s^2+2\zeta\omega s+\omega^2}\right) = -\left(\frac{\omega}{\omega_d}\right)e^{-\zeta\omega t}\sin(\omega_d t-\phi)$$

$$\phi = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega}\right)$$

where

$$\omega_d = \omega\sqrt{1-\zeta^2} \qquad \zeta<1$$

Laplace Transform of $e^{\mathbf{A}t}$ - VII

This result provides the solution for the second column in the matrix.

$$\begin{bmatrix} L^{-1}\left\{\dfrac{s+2\zeta\omega}{s^2+2\zeta\omega s+\omega^2}\right\} & L^{-1}\left\{\dfrac{1}{s^2+2\zeta\omega s+\omega^2}\right\} \\ L^{-1}\left\{\dfrac{-\omega^2}{s^2+2\zeta\omega s+\omega^2}\right\} & L^{-1}\left\{\dfrac{s}{s^2+2\zeta\omega s+\omega^2}\right\} \end{bmatrix}$$

The first column will follow the same procedure.

Laplace Transform of $e^{\mathbf{A}t}$ - VIII

Notice that after Laplace transorming each element of the $[s\mathbf{I} - \mathbf{A}]^{-1}$ matrix, the result is a matrix of time-domain functions:

$$L^{-1}\left\{\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\right\} = \Phi\left(t, 0\right) = e^{\mathbf{A}t}$$

Again, this time-domain result is called the State Transition Matrix

Method of Diagonalization - I

We can also compute the matrix exponential using the eigenvalues ($\mathbf{\Lambda}$) and eigenvectors ($\mathbf{V}$) of the $\mathbf{A}$ matrix.

First define the following:

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_N \\ \vdots & & \vdots \end{bmatrix}$$

then

$$\underbrace{\mathbf{AV} = \mathbf{V}\Lambda}_{\text{Eigenvalue Problem}} \quad \Rightarrow \quad \begin{aligned} \mathbf{A} &= \mathbf{V}\Lambda\mathbf{V}^{-1} \\ \Lambda &= \mathbf{V}^{-1}\mathbf{AV} \end{aligned}$$

Method of Diagonalization - II

Use this factorization to compute $\mathbf{A}^2$

$$\mathbf{A}^2 = \underbrace{\mathbf{V}\Lambda\mathbf{V}^{-1}}_{\mathbf{A}}\underbrace{\mathbf{V}\Lambda\mathbf{V}^{-1}}_{\mathbf{A}} =$$

Generalizing for any power $\mathbf{A}^k$

$$\mathbf{A}^k = \underbrace{\mathbf{V}\Lambda\mathbf{V}^{-1}}_{\mathbf{A}}\underbrace{\mathbf{V}\Lambda\mathbf{V}^{-1}}_{\mathbf{A}}\cdots\underbrace{\mathbf{V}\Lambda\mathbf{V}^{-1}}_{\mathbf{A}} =$$

Method of Diagonalization - III

Now substitute this result into the series expansion definition

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\left(\frac{t^2}{2!}\right) + \mathbf{A}^3\left(\frac{t^3}{3!}\right) + \dots$$

$$= \mathbf{V}\mathbf{V}^{-1} + \mathbf{V}\Lambda\mathbf{V}^{-1}t + \mathbf{V}\Lambda^2\mathbf{V}^{-1}\left(\frac{t^2}{2!}\right) + \mathbf{V}\Lambda^3\mathbf{V}^{-1}\left(\frac{t^3}{3!}\right) + \dots$$

$$=$$

Method of Diagonalization - IV

Use the EVP example from L2/S25:

$$\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \quad \mathbf{V}^{-1} = \tfrac{1}{3}\begin{bmatrix} -2 & 5 \\ 1 & -1 \end{bmatrix}$$

Then from the result on the previous slide we have:

$$e^{\mathbf{A}t} = \mathbf{V}e^{\Lambda t}\mathbf{V}^{-1} = \tfrac{1}{3}\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -1 \end{bmatrix}$$

$$= \tfrac{1}{3}\begin{bmatrix} 5e^{2t} - 2e^{-t} & 5e^{-t} - 5e^{2t} \\ 2e^{2t} - 2e^{-t} & 5e^{-t} - 2e^{2t} \end{bmatrix}$$

Method of Diagonalization - V

Remember from a previous meeting that state space representations are not unique and we can transform from one state representation to another?

What if we choose the eigenvector matrix as our state transformation?

$$\mathbf{x} = \mathbf{V}\hat{\mathbf{x}} \qquad \hat{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x}$$

Method of Diagonalization - VI

The transformed state equations are:

$$\left[\mathbf{V}^{-1}\mathbf{V}\right]\dot{\hat{\mathbf{x}}} = \left[\mathbf{V}^{-1}\mathbf{A}\mathbf{V}\right]\hat{\mathbf{x}} + \left[\mathbf{V}^{-1}\mathbf{B}\right]\mathbf{u}$$

$$\mathbf{y} = \left[\mathbf{C}\mathbf{V}\right]\hat{\mathbf{x}} + \mathbf{D}\mathbf{u}$$

From our previous definition, the transformed $\mathbf{A}$ matrix is now diagonal!

Method of Diagonalization - VII

A diagonal state matrix means that the differential equations are <u>decoupled</u>

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \vdots \\ \dot{\hat{x}}_N \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_N \end{bmatrix} + \left[\mathbf{V}^{-1} \mathbf{B} \right] \mathbf{u}$$

These particular transformed states are called <u>Modal States</u> or <u>Modal Coordinates</u>

Method of Diagonalization - VIII

The method of diagonalization works well when the eigenvalues are real as in the previous example.

The method can still be used for <u>complex</u> eigenvalues as well as <u>repeated</u> eigenvalues; however, the result is more involved.

Method of Diagonalization - IX

Here is an interesting fact about transformations that we may use later.

For any invertible transformation $\mathbf{T}$

$$\mathbf{A}_1 = \mathbf{T}^{-1}\mathbf{A}_2\mathbf{T}$$

The matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ have the exact same eigenvalues but not the same eigenvectors. Mathematicians would call $\mathbf{A}_1$ and $\mathbf{A}_2$ "similar" and $\mathbf{T}$ is called a <u>similarity transform</u>.

Numerical Calculation of $e^{\mathbf{A}t}$ - I

The solution to the state equation

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} [\Phi(t, \tau)\mathbf{B}]\mathbf{u}(\tau)d\tau$$

tells us that we need to know 4 items to compute the state response $\mathbf{x}(t)$

1.
2.
3.
4.

Numerical Calculation of $e^{\mathbf{A}t}$ - II

How do we numerically compute the state transition matrix?

Closed-form solutions using $[s\mathbf{I} - \mathbf{A}]^{-1}$ or the diagonalization method are possible by hand ($N = 2$) or by using symbolic math packages ($N = 5$ or 6).

In general, the state transition matrix must be computed numerically for an arbitrary number of states.

Numerical Calculation of $e^{\mathbf{A}t}$ - IV

Remember the definition of $e^{\mathbf{A}t}$

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\left(\frac{t^2}{2!}\right) + \mathbf{A}^3\left(\frac{t^3}{3!}\right) + \ldots$$

Is this a good way to compute the state transition matrix?

Try it using the state matrix from our damped oscillator example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}$$

Numerical Calculation of $e^{\mathbf{A}t}$ - V

Use only the first three terms

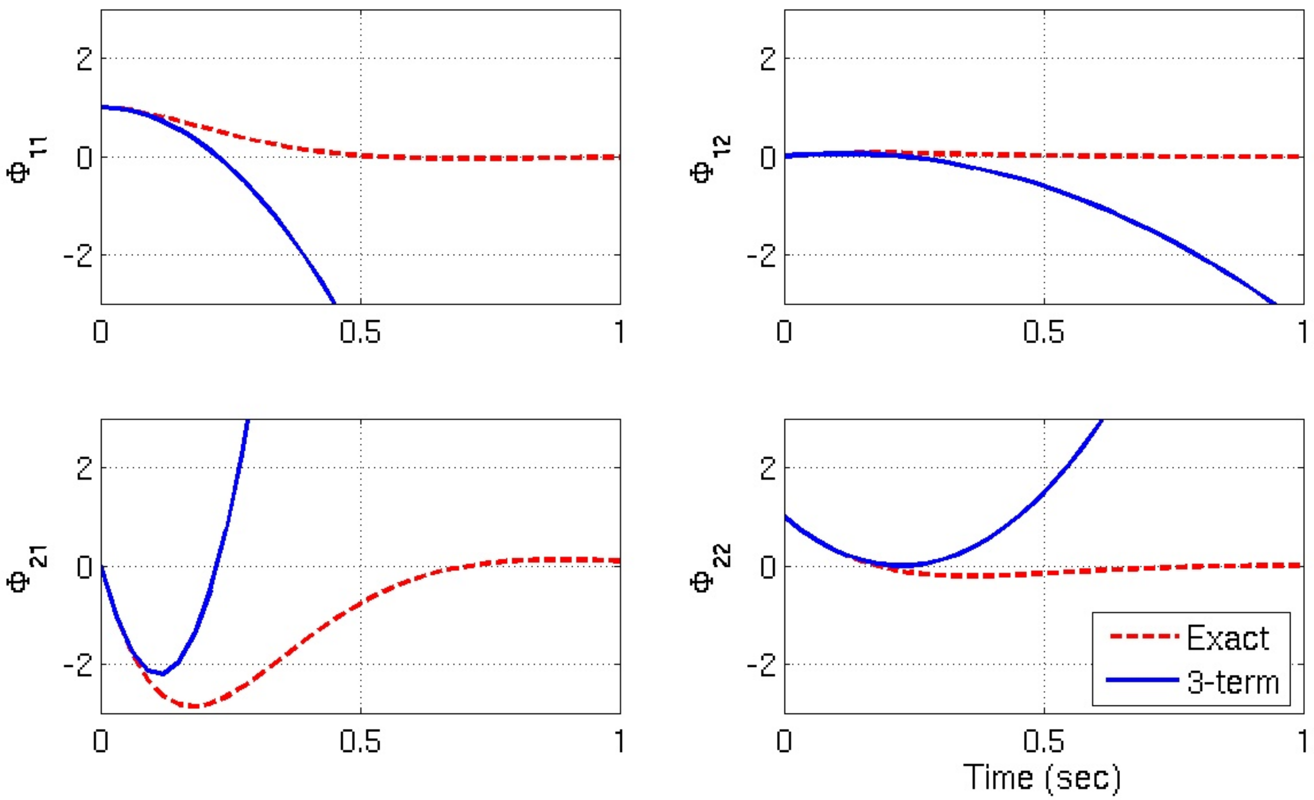
$$e^{\mathbf{A}t} \approx \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\left(\frac{t^2}{2}\right)$$

The result is

$$e^{\mathbf{A}t} \approx \begin{bmatrix} 1-\tfrac{1}{2}(\omega t)^2 & t(1-\zeta\omega t) \\ -\omega^2 t(1-\zeta\omega t) & 1-\omega t\left(2\zeta-\omega t\left(2\zeta^2-\tfrac{1}{2}\right)\right) \end{bmatrix}$$

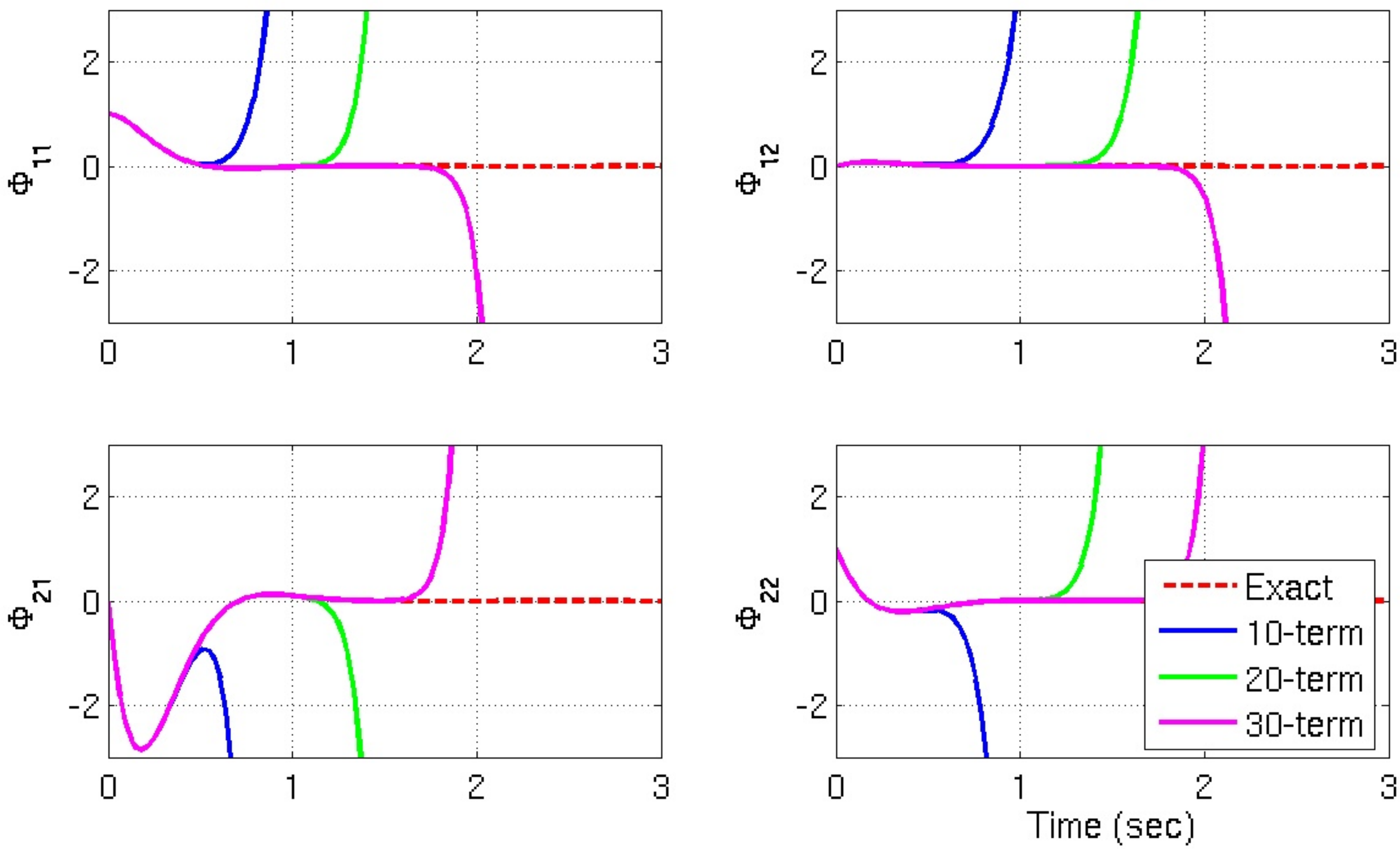
Numerical Calculation of $e^{\mathbf{A}t}$ - VI

Now let's numerically compare the three-term solution to the exact solution for a damping ratio of 0.707.



Numerical Calculation of $e^{\mathbf{A}t}$ - VII

What happens when the number of terms in the approximation increases?



Numerical Calculation of $e^{\mathbf{A}t}$ - VIII

Patel, Laub, and Van Dooren have surveyed **nineteen** "dubious" computational techniques for the matrix exponential. Their conclusion:

- Methods based on Padé approximations are generally attractive.
- Methods based on Taylor series are generally unattractive.

(*Numerical Linear Algebra Techniques for Systems and Control*, IEEE Press, 1994)

Numerical Calculation of $e^{\mathbf{A}t}$ - IX

The folks at the MathWorks have carefully studied this problem and provide a numerically robust Padé approximation in Matlab.

`EXPM(A)` is the matrix exponential of matrix `A`. EXPM is computed using a scaling and squaring algorithm with a Pade approximation.

Numerical Calculation of $e^{\mathbf{A}t}$ - X

Here is example Matlab code to compute the state transition matrix for a given 2x2 $\mathbf{A}$ matrix over an array of times:

```
t = linspace(0,tmax,N);
PHI = zeros(2,2,N);
for i = 1:N,
   PHI(:,:,i) = expm(A * t(i));
end
```

Summary

Three methods of computing the state transition matrix were discussed:

1. Laplace Transform Methods

   Analytical techniques OK for low order systems ($N = 2, 3$)

2. Diagonalization Method

   Analytical technique OK for low order systems with real eigenvalues

3. Numerical Techniques

   Will work for any order system