Course Outline - 1st Half

- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response, Matrix Exponential (2)
- Transfer Functions & Eigenvalues
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

Introduction

Last meeting we developed a way to represent the equations that describe (model) dynamic systems.

The result was a set of matrix equations of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\underbrace{\mathbf{x}(t)}_{\text{states}} + \mathbf{B}\underbrace{\mathbf{u}(t)}_{\text{inputs}}$$

$$\underbrace{\mathbf{y}(t)}_{\text{outputs}} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Introduction

In this meeting we will derive the solution to this set of differential equations.

The result of the derivation will be an expression for the states as a function of time.

LTI Solution - I

How do we solve for the states as a function of time? Remember,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

represents a set of coupled differential equations. In general, we must solve all of the equations simultaneously to obtain a solution.

LTI Solution - II

Consider a scalar example ($N = 1$):

$$\dot{x}(t) = ax(t) + bu(t)$$

How do we solve this differential equation?

Step 1. Solve the homogeneous equation ($u(t)=0$).

Assume a general solution:

LTI Solution - III

Step 2. Solve for a particular solution

Rearrange the diffeq & multiply by e^{-at}

$$e^{-at}\left[\dot{x}(t) - ax(t)\right] = \frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}\left[bu(t)\right]$$

Integrating this equation from t_0 to t:

$$e^{-at}x(t) = e^{-at_0}x(t_0) + \int_{t_0}^{t} e^{-a\tau}bu(\tau)d\tau$$

In-Class Assignment

Obtain a solution to the following (scalar) state equation:

$$\dot{x} = -2x + 3u$$

with initial condition: $x(0) = 1$

and input: $u(t) = \begin{cases} 0 & t < 0 \\ 2 & t \geq 0 \end{cases}$

In-Class Assignment

LTI Solution - IV

We will use the same procedure to solve the state-space matrix equations.

Step 1. Start with the homogeneous equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

Assume a general solution of the form:

$$\underset{[N\times 1]}{\mathbf{x}(t)} = \underset{[N\times N]}{e^{\mathbf{A}t}}\underset{[N\times 1]}{\mathbf{c}}$$

$\mathbf{c}$ is a vector of unknown constant coefficients

$e^{\mathbf{A}t}$ is called the <u>Matrix Exponential</u>

LTI Solution - V

The matrix exponential is defined as:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\left(\frac{t^2}{2!}\right) + \mathbf{A}^3\left(\frac{t^3}{3!}\right) + \dots$$

We can substitute this result into the homogeneous differential equation…

$$\frac{d}{dt}\left(e^{\mathbf{A}t}\right) = \mathbf{A} + \mathbf{A}^2 t + \mathbf{A}^3\left(\frac{t^2}{2}\right) + \dots =$$

…and see that it truly is a general solution!

LTI Solution - VI

The matrix exponential has several properties that are similar to the scalar exponential function:

Scalar	Matrix
$e^{0} = 1$	$e^{\mathbf{A}0} = \mathbf{I}$
$e^{a(t_1+t_2)} = e^{at_1} e^{at_2}$	$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1} e^{\mathbf{A}t_2}$
$\frac{1}{e^{at}} = e^{-at}$	$\left(e^{\mathbf{A}t}\right)^{-1} = e^{-\mathbf{A}t}$

LTI Solution - VII

Continuing with the solution, we have

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c}$$

What is the constant vector $\mathbf{c}$? Assume we know the state at an initial time

$$\mathbf{x}(t_0) = \mathbf{x}_0 = e^{\mathbf{A}t_0}\mathbf{c}$$

Using the properties from the previous slide, we can solve for the constant vector: $$\mathbf{c} = e^{-\mathbf{A}t_0}\mathbf{x}_0$$

LTI Solution - VIII

Substituting into the assumed solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c} =$$

The homogeneous solution to the state-space Initial Value Problem is:

$$\underbrace{\mathbf{x}(t)}_{\substack{\text{State}\\\text{Response}\\\text{at time } t}} = \underbrace{e^{\mathbf{A}(t-t_0)}}_{\substack{\text{State}\\\text{Transition}\\\text{Matrix}}} \underbrace{\mathbf{x}(t_0)}_{\substack{\text{Initial}\\\text{State}}} = \Phi(t, t_0)\mathbf{x}(t_0)$$

LTI Solution - IX

The State Transition Matrix defines how the state evolves from the IC's:

$$\mathbf{\Phi}\left(t, t_0\right) = e^{\mathbf{A}\left(t - t_0\right)}$$

Note that the state transition matrix is always a function of the difference between the present time t and a previous time t_0. For LTI systems, we can always set this initial time equal to 0 (Not true for LTV systems).

LTI Solution - X

The state transition matrix has the following useful properties:

$$\Phi(t_0, t_0) = \mathbf{I}$$

$$\Phi(t_2, t_1)\Phi(t_1, t_0) = \Phi(t_2, t_0) \quad \forall\ t_0, t_1, t_2$$

$\Phi(t, t_0)$ is nonsingular $\forall\ t, t_0$

$$\Phi^{-1}(t, t_0) = \Phi(t_0, t) \quad \forall\ t, t_0$$

$\forall$ means "for all"

LTI Solution - XI

Step 2. Now find a particular solution.

Assume the following form:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c}(t)$$

Substitute into the original state equation

$$\dot{\mathbf{x}}(t) = \underbrace{e^{\mathbf{A}t}\dot{\mathbf{c}}(t) + \mathbf{A}e^{\mathbf{A}t}\mathbf{c}(t)}_{\text{From Chain Rule}} = \mathbf{A}e^{\mathbf{A}t}\mathbf{c}(t) + \mathbf{B}\mathbf{u}(t)$$

And solve for

LTI Solution - XII

Integrate this equation from t_0 to t:

$$d\mathbf{c} = e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\int_{t_0}^{t} d\mathbf{c} = \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{c}(t) = \mathbf{c}(t_0) + \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{c}(t) = e^{-\mathbf{A}t_0}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Substitute into the assumed solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left[e^{-\mathbf{A}t_0}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau\right]$$

LTI Solution - XIII

Rearranging terms yields:

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)}_{\text{Response due to Initial Conditions}} + \underbrace{\int_{t_0}^{t} \left[e^{\mathbf{A}(t-\tau)}\mathbf{B}\right]\mathbf{u}(\tau)\,d\tau}_{\text{Forced Response}}$$

The total response at time t is a summation (superposition) of the IC response and the forced response.

The forced response is a convolution integral where $e^{\mathbf{A}t}\mathbf{B}$ is the system <u>impulse response</u>

LTI Solution - Alternative Forms - I

There is an alternative form which is also useful in solving for the state response. Consider the substitution:

$$T = t - \tau \quad \Leftrightarrow \quad \tau = t - T$$

Then the original result becomes:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \left[e^{\mathbf{A}T}\mathbf{B}\right]\mathbf{u}(t-T)\,dT$$

LTI Solution - Alternative Forms - III

The variable T is a dummy variable, so we can replace it by τ, now we have

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \left[e^{\mathbf{A}\tau}\mathbf{B}\right]\mathbf{u}(t-\tau)d\tau$$

where the original form was

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \left[e^{\mathbf{A}(t-\tau)}\mathbf{B}\right]\mathbf{u}(\tau)d\tau$$

Either expression can be used to compute the state response.

LTI Solution - Example - I

Let the state equations be:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The matrix exponential for this system is:

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-3t} & \frac{1}{7}\left(e^{-3t} - e^{-10t}\right) \\ 0 & e^{-10t} \end{bmatrix}$$

We will discuss how to obtain $e^{\mathbf{A}t}$ later.

LTI Solution - Example - II

Find the state response to the following step input

$$u(t) = \begin{cases} 0 & t < 0 \\ 5 & t \geq 0 \end{cases}$$

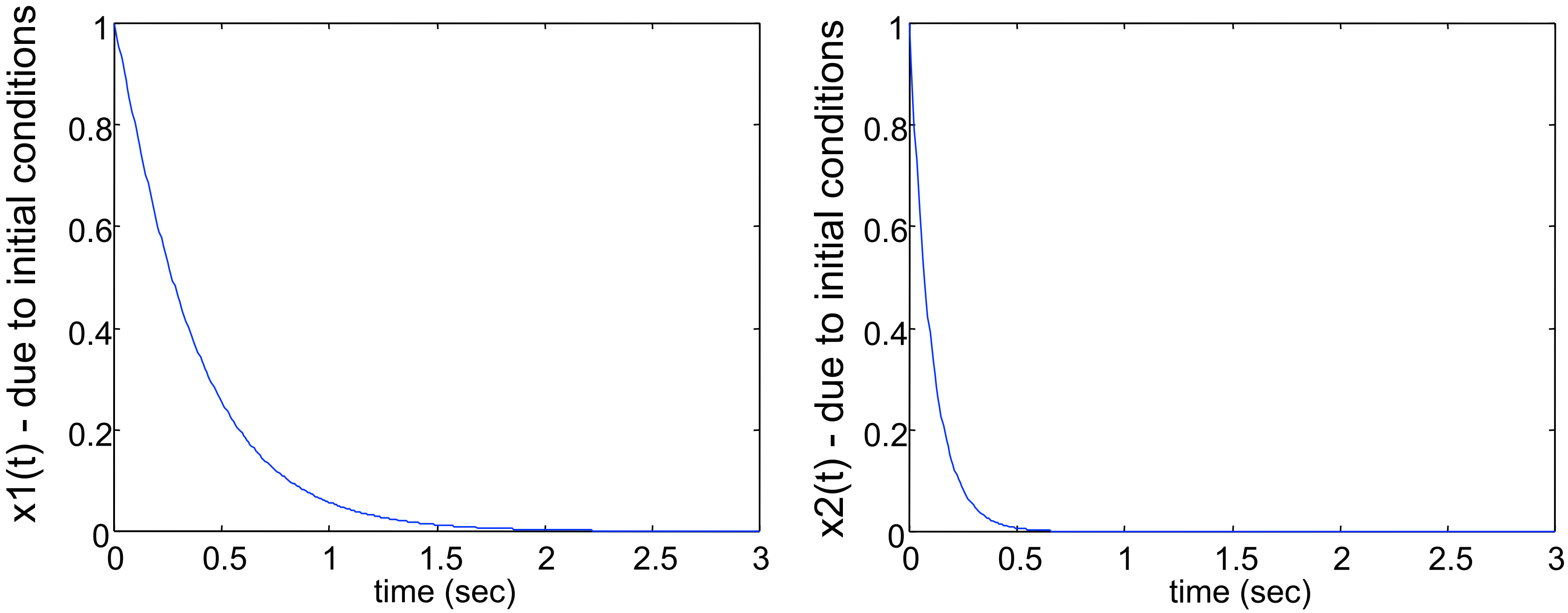
with the following initial conditions

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

LTI Solution - Example - III

First compute the response due to initial conditions:

$$e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} e^{-3t} & \frac{1}{7}\left(e^{-3t} - e^{-10t}\right) \\ 0 & e^{-10t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{7}e^{-3t} - \frac{1}{7}e^{-10t} \\ e^{-10t} \end{bmatrix}$$



LTI Solution - Example - IV

Next compute the forced response using the second form of the convolution integral:

$$\int_{t_0}^{t} \left[e^{\mathbf{A}\tau} \mathbf{B} \right] \mathbf{u}(t-\tau) d\tau$$

This will simplify the integration since

$$\mathbf{u}(t-\tau) = \begin{cases} 0 & (t-\tau) < 0 \\ 5 & (t-\tau) \geq 0 \end{cases}$$

Never Possible! Why?

LTI Solution - Example - V

$$\int_{t_0}^{t}\left[e^{\mathbf{A}\tau}\mathbf{B}\right]\mathbf{u}(t-\tau)d\tau = \int_0^t \begin{bmatrix} e^{-3\tau} & \frac{1}{7}\left(e^{-3\tau}-e^{-10\tau}\right) \\ 0 & e^{-10\tau} \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}5d\tau$$

$$= \int_0^t \begin{bmatrix} \frac{5}{7}\left(e^{-3\tau}-e^{-10\tau}\right) \\ 5e^{-10\tau} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} -\frac{5}{21}e^{-3\tau}+\frac{1}{14}e^{-10\tau} \\ -\frac{1}{2}e^{-10\tau} \end{bmatrix}\Bigg|_0^t$$

$$= \begin{bmatrix} -\frac{5}{21}e^{-3t}+\frac{1}{14}e^{-10t}+\frac{1}{6} \\ \frac{1}{2}\left(1-e^{-10\tau}\right) \end{bmatrix}$$

Verify that the forced response is 0 at $t = 0$

LTI Solution - Example - VI

The complete solution is the combination of the initial condition response and the forced response

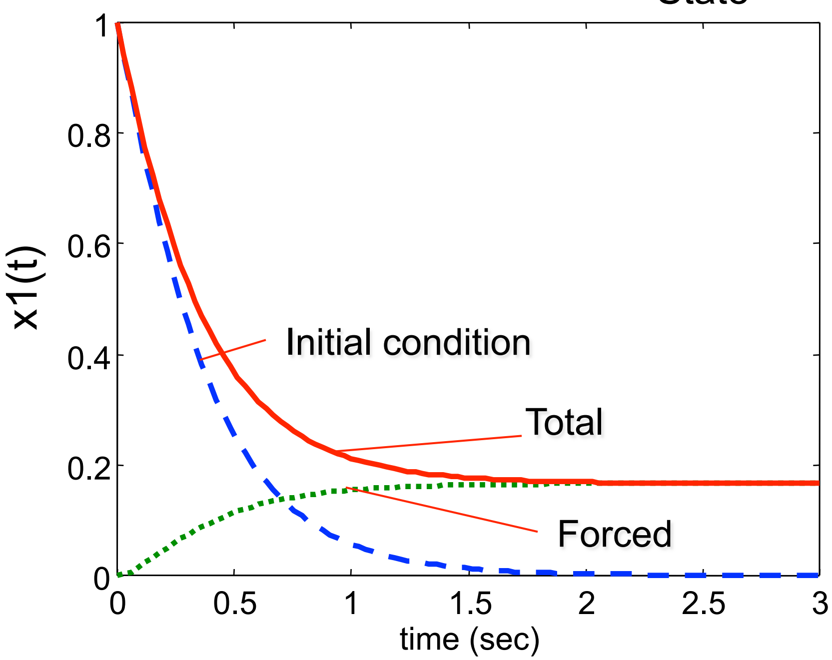
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{8}{7}e^{-3t} - \frac{1}{7}e^{-10t} \\ e^{-10t} \end{bmatrix}}_{\text{Response due to Initial Conditions}} + \underbrace{\begin{bmatrix} -\frac{5}{21}e^{-3t} + \frac{1}{14}e^{-10t} + \frac{1}{6} \\ \frac{1}{2}\left(1 - e^{-10\tau}\right) \end{bmatrix}}_{\text{Forced Response}}$$

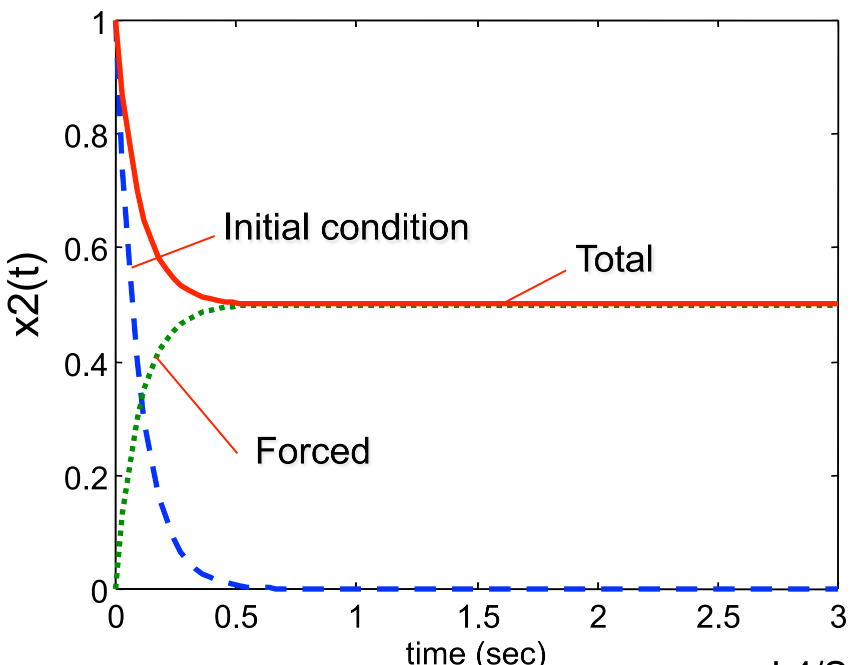
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{6} + \frac{19}{21}e^{-3t} - \frac{1}{14}e^{-10t} \\ \frac{1}{2} + \frac{1}{2}e^{-10t} \end{bmatrix}$$

LTI Solution - Example - VII

The plots clearly show the Transient and the Steady-State Response

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}}_{\text{Steady State}} + \underbrace{\begin{bmatrix} \frac{19}{21} e^{-3t} - \frac{1}{14} e^{-10t} \\ \frac{1}{2} e^{-10t} \end{bmatrix}}_{\text{Transient Response}}$$





Summary

The solution of the state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is given by

$$\mathbf{x}(t) = \Phi(t,t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} [\Phi(t,\tau)\mathbf{B}]\mathbf{u}(\tau)d\tau$$