Advanced Control Engineering

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

Dynamic Compensator Design - I

In L15, we incorporated exogenous inputs into the state-space framework. Exogenous inputs were classified into two categories:

- **References**: signals we want the *outputs* to track
- **Disturbances**: signals we want to reject because they corrupt the performance

In both cases we want to eliminate steady-state error to these inputs.

Dynamic Compensator Design - II

There were two distinct kinds of disturbance inputs:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\mathbf{w}$$

Process Noise

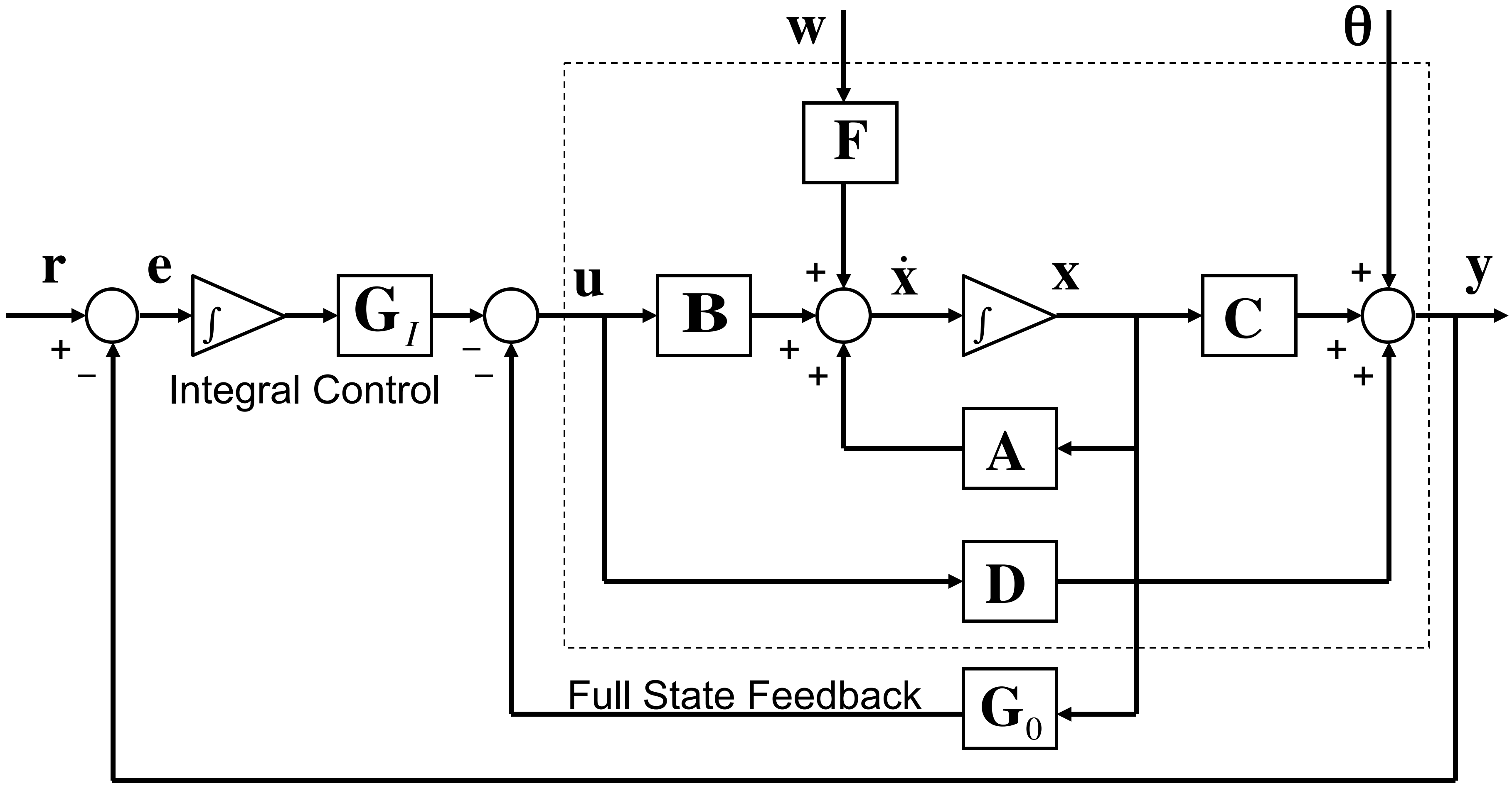
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \boldsymbol{\theta}$$

Sensor Noise

The reference inputs were incorporated through integral control of the tracking error vector. Higher-order controllers such as PID are possible, but integral control allows for a simpler design.

Dynamic Compensator Design - III

The overall block diagram is:



Dynamic Compensator Design - IV

As mentioned in the previous meeting, this controller structure assumed that we had access to the actual system states (full state vector!).

Now we will develop the compensator using a linear observer to estimate the states for state feedback.

Dynamic Compensator Design - V

The augmented system equations for reference tracking are (L15/S18):

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_I \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_I \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{D} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix} ?$$

The observer equations were derived in L13. We will use the following form of the equations from L13/S25:

$$\dot{\hat{\mathbf{x}}} = [\mathbf{KC}]\mathbf{x} + \left[\hat{\mathbf{A}} - \mathbf{K}\hat{\mathbf{C}}\right]\hat{\mathbf{x}} + \left[\hat{\mathbf{B}} + \mathbf{K}\left(\mathbf{D} - \hat{\mathbf{D}}\right)\right]\mathbf{u}$$

Dynamic Compensator Design - VI

Combining these two equations we get:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_I \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{KC} & \mathbf{0} & \hat{\mathbf{A}} - \mathbf{K}\hat{\mathbf{C}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_I \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{D} \\ \hat{\mathbf{B}} + \mathbf{K}\left(\mathbf{D} - \hat{\mathbf{D}}\right) \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix} \boldsymbol{\theta}$$

Note that the observer gains $\mathbf{K}$ are designed using the original plant $(\mathbf{A},\mathbf{C})$ and NOT the augmented plant. So $\mathbf{K}$ is always an $N \times P$ matrix.

Also note that neither $\mathbf{r}$, $\mathbf{w}$, or $\boldsymbol{\theta}$ directly affect the estimated states.

Dynamic Compensator Design - VII

The feedback control is a linear combination of full-state feedback of the integrated error state and the estimated states:

$$\mathbf{u}(t) = -\mathbf{G}_0\hat{\mathbf{x}}(t) - \mathbf{G}_I\mathbf{x}_I(t) = -\begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_I \end{bmatrix}\begin{bmatrix} \hat{\mathbf{x}}(t) \\ \mathbf{x}_I(t) \end{bmatrix}$$

Remember that these gains are computed using full-state feedback on the augmented plant from S6.

Dynamic Compensator Design - VIII

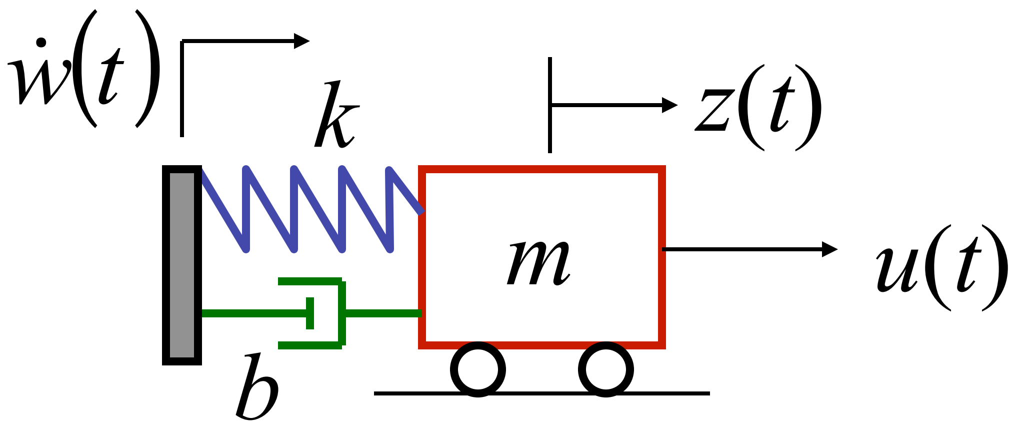
The closed-loop state equations with integral control and state estimation are given by:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_I \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{G}_I & -\mathbf{B}\mathbf{G}_0 \\ -\mathbf{C} & \mathbf{D}\mathbf{G}_I & \mathbf{D}\mathbf{G}_0 \\ \mathbf{K}\mathbf{C} & -\left(\hat{\mathbf{B}}+\mathbf{K}\left(\mathbf{D}-\hat{\mathbf{D}}\right)\right)\mathbf{G}_I & \left(\hat{\mathbf{A}}-\mathbf{K}\hat{\mathbf{C}}\right)-\left(\hat{\mathbf{B}}+\mathbf{K}\left(\mathbf{D}-\hat{\mathbf{D}}\right)\right)\mathbf{G}_0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_I \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix} \theta$$

The equations seem to be growing, but they are simply additional layers on the concept of full-state feedback with a linear observer.

Design Example - I

Let's return to the damped mass-spring oscillator from L15.



The augmented equations were:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_I(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ -\omega^2 & -2\zeta\omega & \omega^2 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_I(t)\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{m}\\ 0\\ 0\end{bmatrix}u(t)+\begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}r(t)+\begin{bmatrix}0\\ 2\zeta\omega\\ 1\\ 0\end{bmatrix}\dot{w}(t)$$

Design Example - II

The control design specifications were:

- Minimize the settling time to a step input r
- Zero steady-state error to a 1 cm step input r and a 1 mm step disturbance at the base
- Control force no greater than 10 N

The full-state feedback design resulted in the following poles and gains:

$$\lambda_{CL} = \{-48,\quad -48 \pm j48,\quad 0\}$$

$$\mathbf{G} = \begin{bmatrix} 9116 & 143.9 & 0 & -221184 \end{bmatrix}$$

Design Example - III

The observer design requires choosing a set of observer poles and computing the observer gains **K**. We must use the original plant for this design!

The original plant from L15/S23 was:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\omega^2 & -2\zeta\omega & \omega^2 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

Design Example - IV

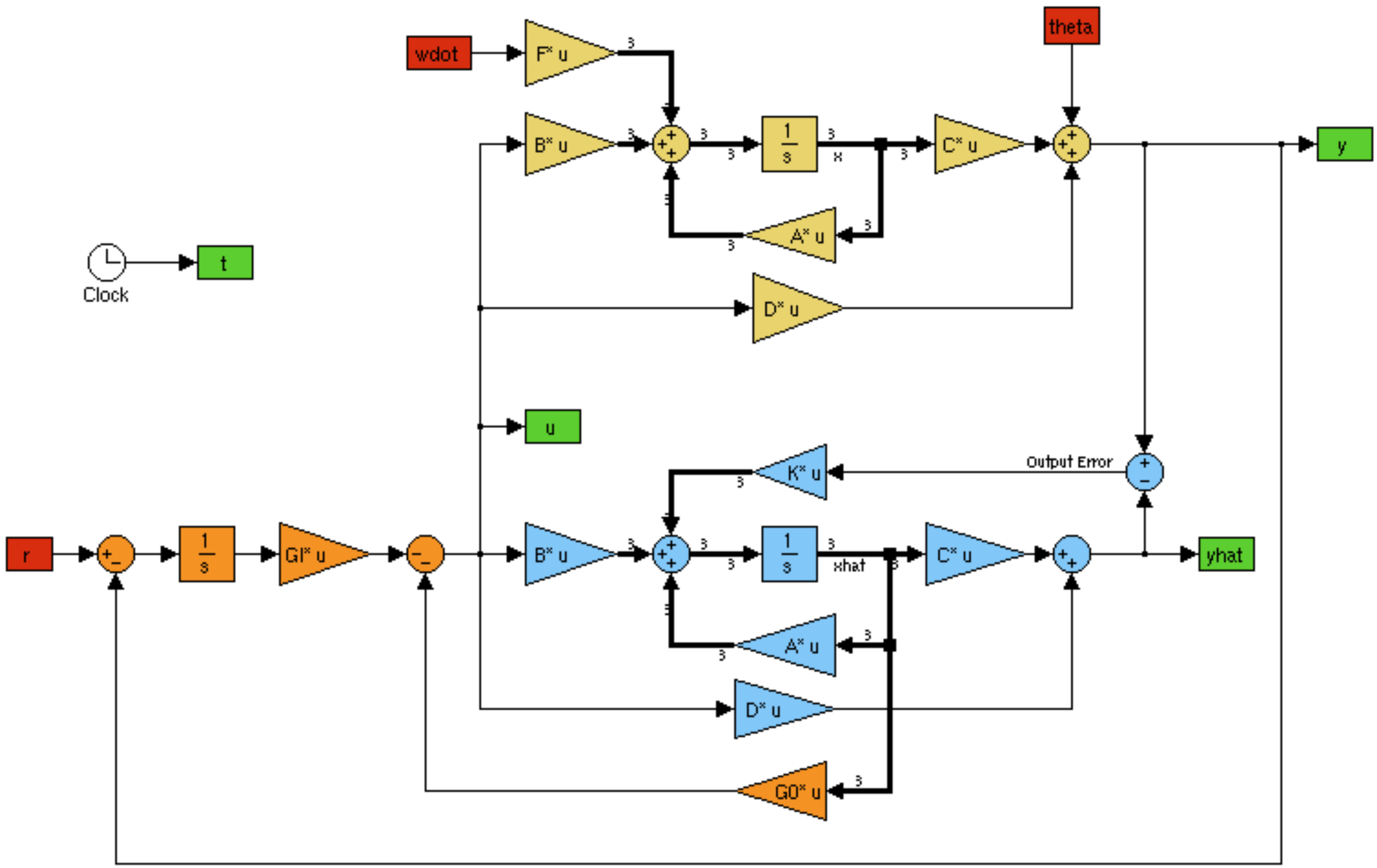
For this problem, we need to place three observer poles. Let's use the "simple" method we explored in L14.

$$\lambda_{OBS} = \alpha \times \{-48, \quad -48 \pm j48\}$$

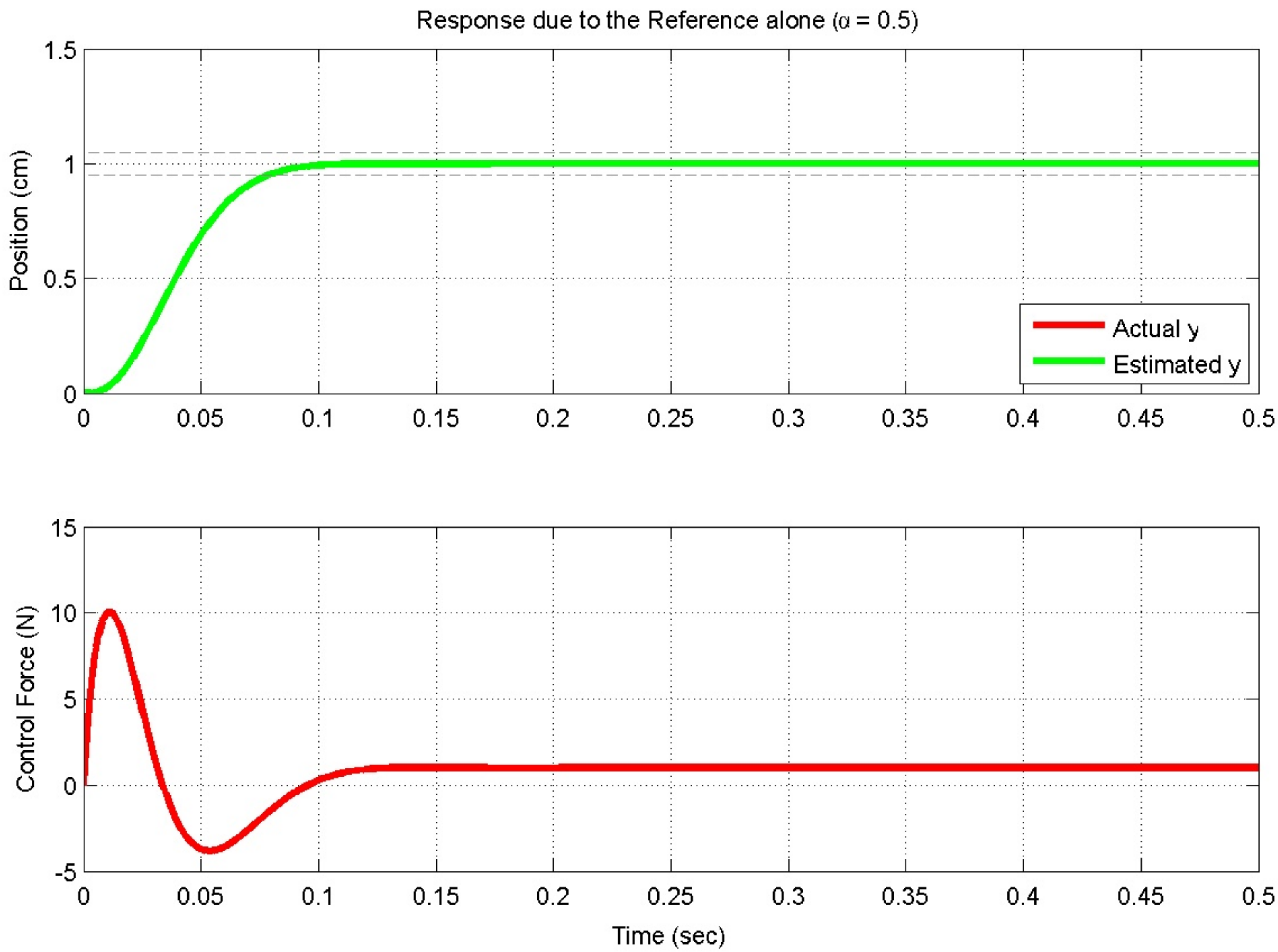
Select $\alpha = 0.5$ then use the `place` function to compute the gains:

```
>> K = place(A',C',obspoles)'
K =
   1.0e+03 *
    0.0719
    2.1968
    0.2765
```

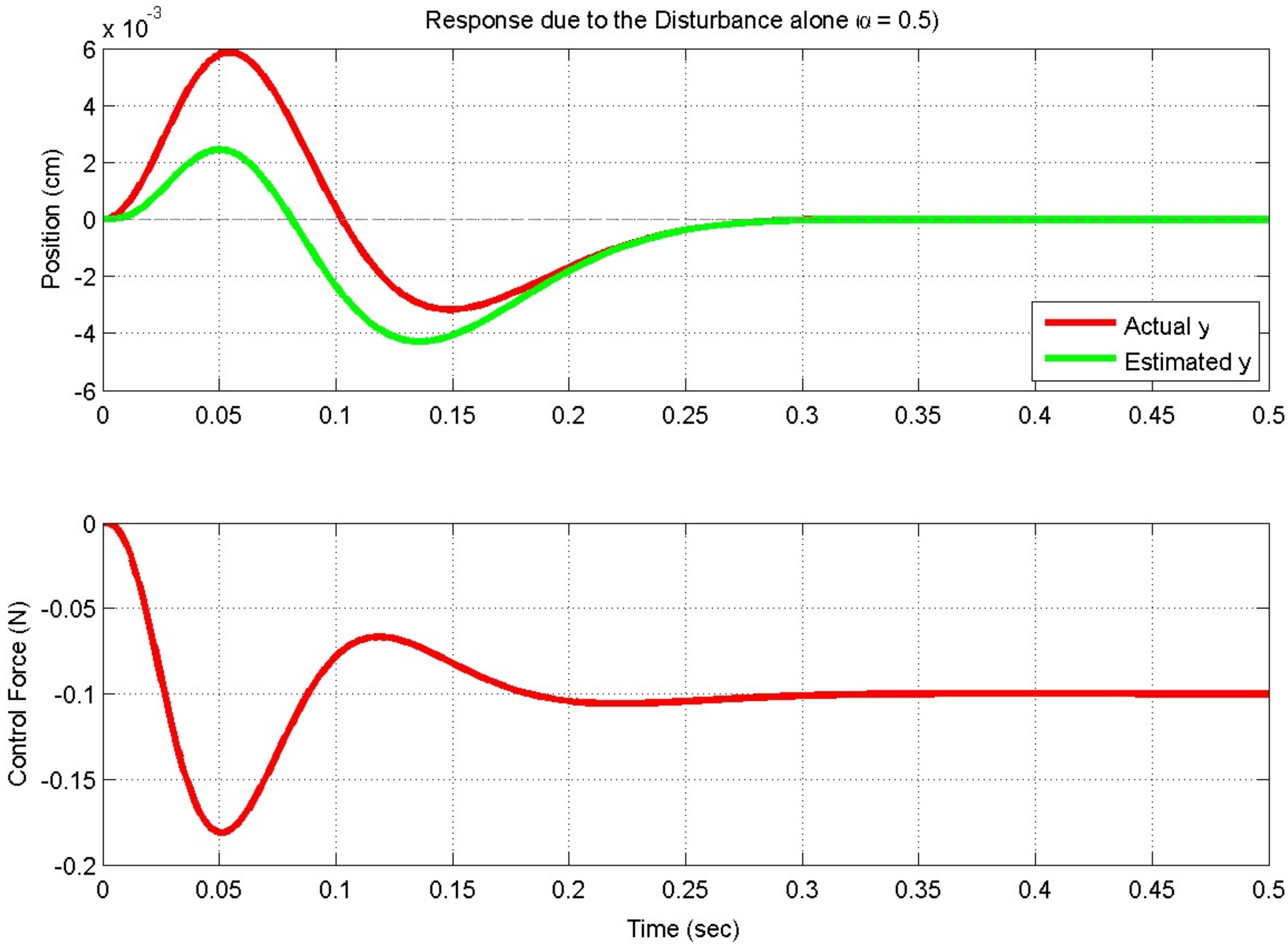
Design Example - V



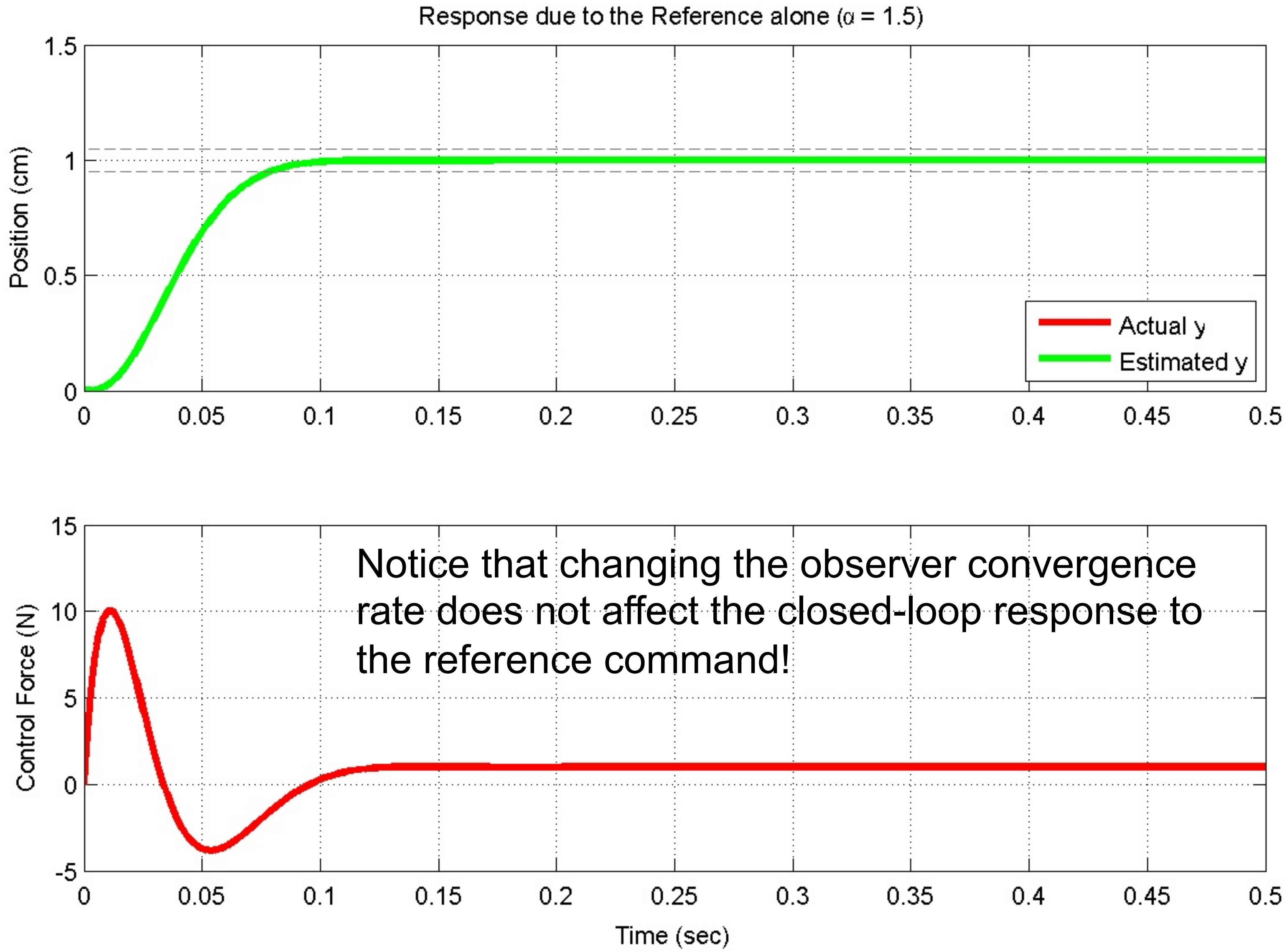
Design Example - VI



Design Example - VII

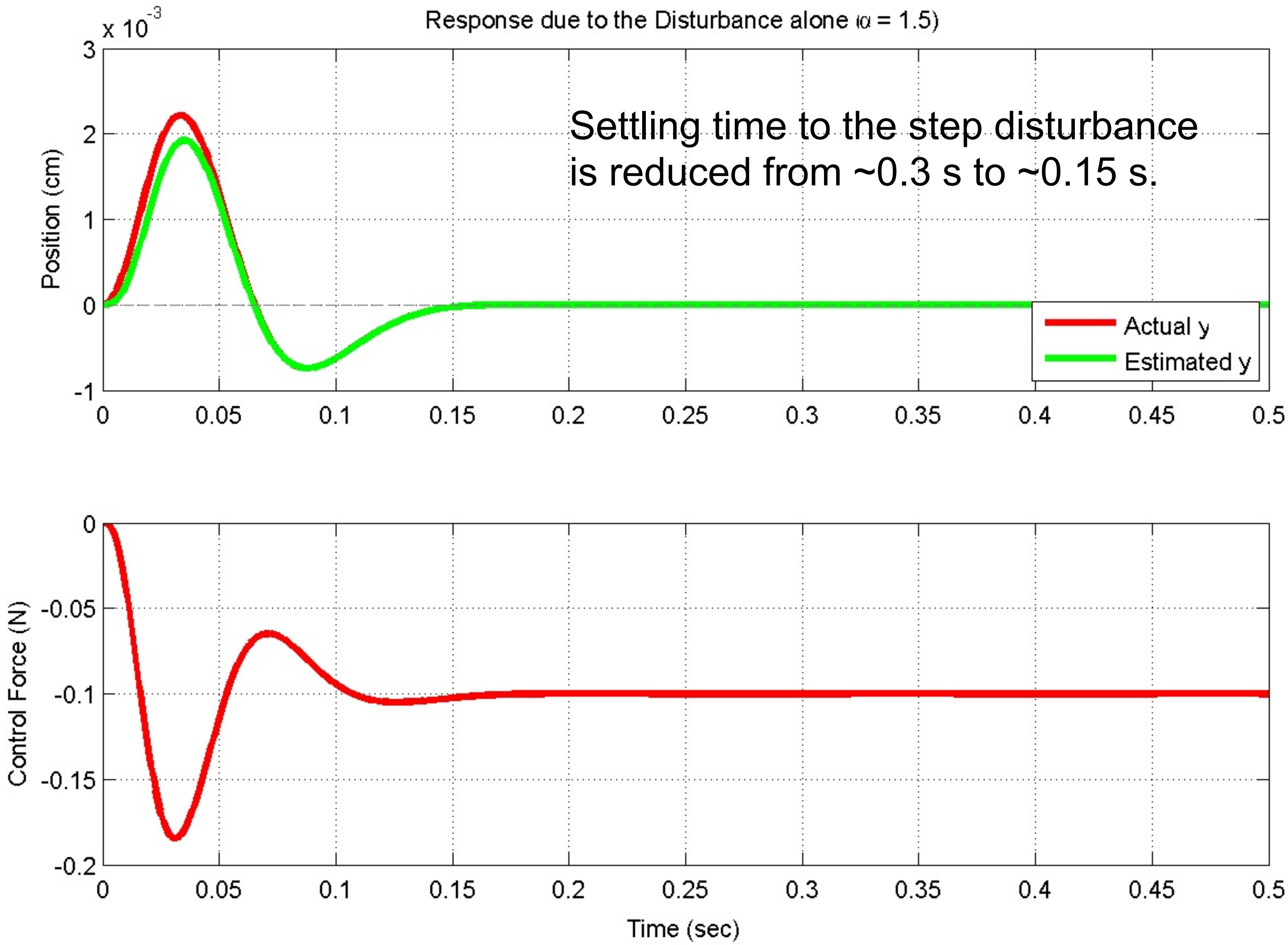


Design Example - VIII



Notice that changing the observer convergence rate does not affect the closed-loop response to the reference command!

Design Example - IX



Design Example - X

These results demonstrate that the convergence rate of the observer has very little effect on the response to a step reference input.

The observer convergence rate has a significant impact on the response to disturbance inputs.

Why does this happen?

State Error Response - I

To answer this question, we can look at the block diagram on S14.

Our control system now consists of an observer-based design that utilizes the plant output to estimate the states for full-state feedback.

The plant output is also used to implement integral control to eliminate steady-state error to exogenous inputs.

State Error Response - II

Why doesn't the observer convergence rate affect the response to a reference input? Consider the case in which the initial state error is zero.

The reference input produces a control signal **u** which is passed to both the plant and the observer simultaneously.

The reference input affects the plant and observer equally and produces no additional state error.

State Error Response - III

Why does the observer convergence rate affect the response to a disturbance input? Again, consider the case in which the initial state error is zero.

The disturbance affects only the plant, therefore it produces error in the estimated states. The estimator error is eliminated by feeding back the residual output error. Thus the observer convergence rate affects how fast the observer can respond to error induced by the disturbance.

State Error Response - IV

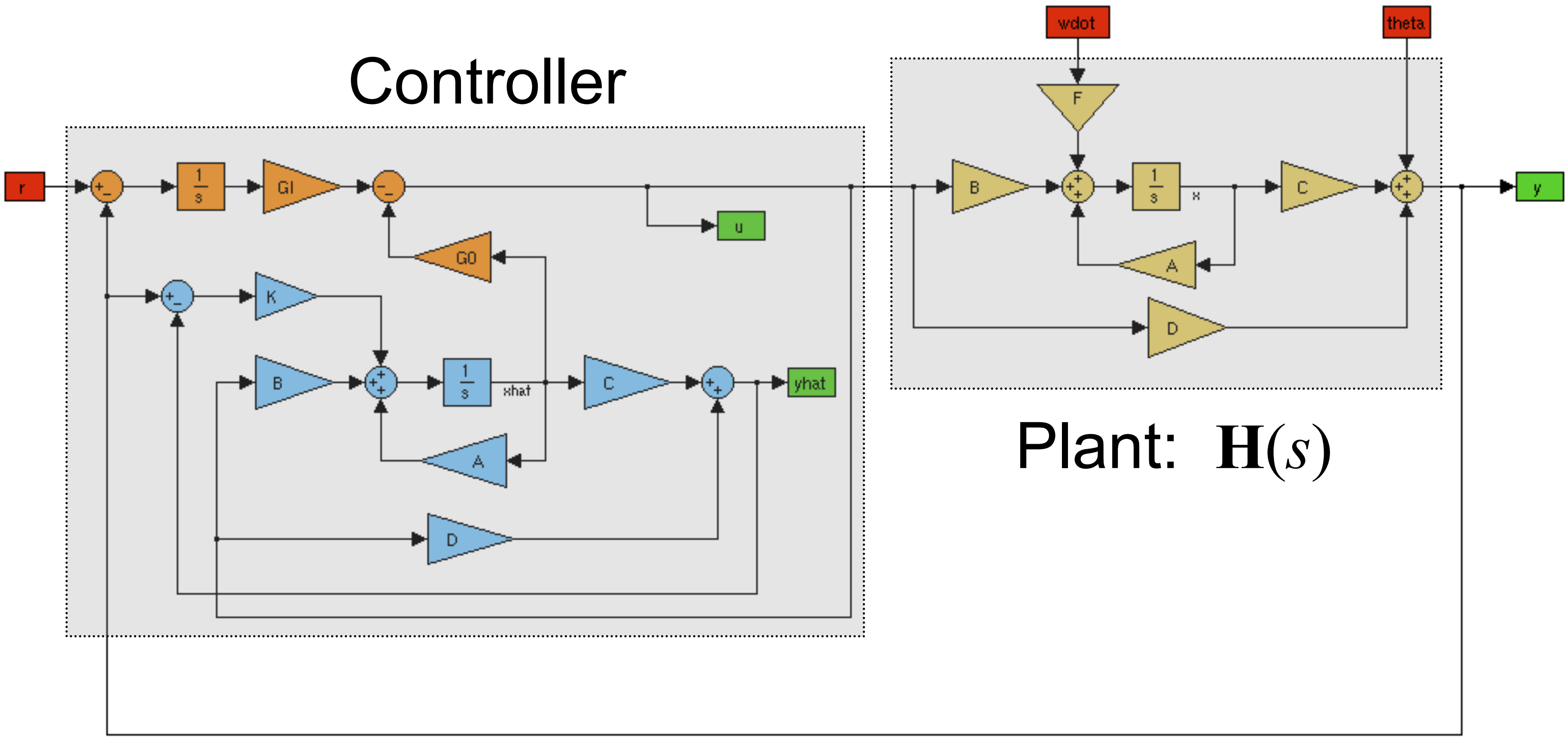
Result:

The observer convergence rate affects the response to initial estimation error and the response to disturbance inputs.

The observer convergence rate does not affect the response to reference inputs because reference inputs do not induce any estimation error.

Loop Analysis - I

Let's rearrange the diagram on S14 to obtain a more familiar feedback loop structure. Notice that the controller has two inputs: $\mathbf{r}$ and $\mathbf{y}$.



Loop Analysis - II

Let's analyze the closed-loop system using block diagram techniques. We can write the compensator output as:

$$\mathbf{u}(s) = \mathbf{K}_{ur}(s)\mathbf{r}(s) + \mathbf{K}_{uy}(s)\mathbf{y}(s)$$

Where $\mathbf{K}_{ur}$ and $\mathbf{K}_{uy}$ are appropriately dimensioned transfer function matrices.

Assuming $\mathbf{w} = \mathbf{0}$, the plant TF is:

$$\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s)$$

Loop Analysis - III

Combining these two equations and solving for $\mathbf{u}(s)$ results in:

$$\mathbf{u}(s) = \left[\mathbf{I} - \mathbf{K}_{uy}(s)\mathbf{H}(s)\right]^{-1}\mathbf{K}_{ur}(s)\mathbf{r}(s)$$

Using this result with the definition from the previous slide, the closed-loop TF between $\mathbf{y}$ and $\mathbf{r}$ is:

$$\mathbf{y}(s) = \underbrace{\mathbf{H}(s)\left[\mathbf{I} - \mathbf{K}_{uy}(s)\mathbf{H}(s)\right]^{-1}\mathbf{K}_{ur}(s)}_{\text{Closed-Loop Transfer Function Matrix}}\mathbf{r}(s)$$

Loop Analysis - IV

If all of the transfer function matrices are SISO, then the previous result reduces to a more familiar form:

$$y(s) = \left(\frac{H(s) K_{ur}(s)}{1 - K_{uy}(s) H(s)} \right) r(s)$$

For a conventional SISO feedback loop with a controller $G(s)$:

$$y(s) = \left(\frac{G(s) H(s)}{1 + G(s) H(s)} \right) r(s)$$

Loop Analysis - V

The TF analysis gives us two important results.

- Closed-loop stability is only a function of the compensator TF $K_{uy}(s)$, because it is in the denominator.
- Our sign convention for state variable feedback is actually positive feedback. We have to remember this sign change when using classical tools.

Loop Analysis - VI

The transfer function that determines the stability is the loop transfer function, $K_{\mathrm{uy}}(\mathrm{s})H(\mathrm{s})$. We can rewrite the TF in the style of negative feedback as:

$$y(s) = \left(\frac{H(s)K_{ur}(s)}{1 + \left(-K_{uy}(s)H(s)\right)} \right) r(s)$$

Examining the loop $-K_{\mathrm{uy}}(\mathrm{s})H(\mathrm{s})$, we can apply all the results of classical control theory to our state variable compensator design.

Loop Analysis - VII

Let's apply these results to two different kinds of analyses:

- Determine the type of compensator that is produced by state variable feedback
- Determine the characteristics of a SISO loop TF such as bandwidth, gain crossover, phase margin, etc.

Loop Analysis - VIII

The state equations of the compensator with state variable feedback, a linear observer, and integral control can be derived from the block diagram on S14

$$\begin{bmatrix} \dot{\mathbf{x}}_I \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{K}\hat{\mathbf{D}}\mathbf{G}_I - \hat{\mathbf{B}}\mathbf{G}_I & \hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{G}_0 - \mathbf{K}\hat{\mathbf{C}} + \mathbf{K}\hat{\mathbf{D}}\mathbf{G}_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_I \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{r} + \begin{bmatrix} -\mathbf{I} \\ \mathbf{K} \end{bmatrix} \mathbf{y}$$

$$\mathbf{u} = \begin{bmatrix} -\mathbf{G}_I & -\mathbf{G}_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_I \\ \hat{\mathbf{x}} \end{bmatrix}$$

As expected, this system has two inputs $\mathbf{y}$ and $\mathbf{r}$, and one output $\mathbf{u}$.

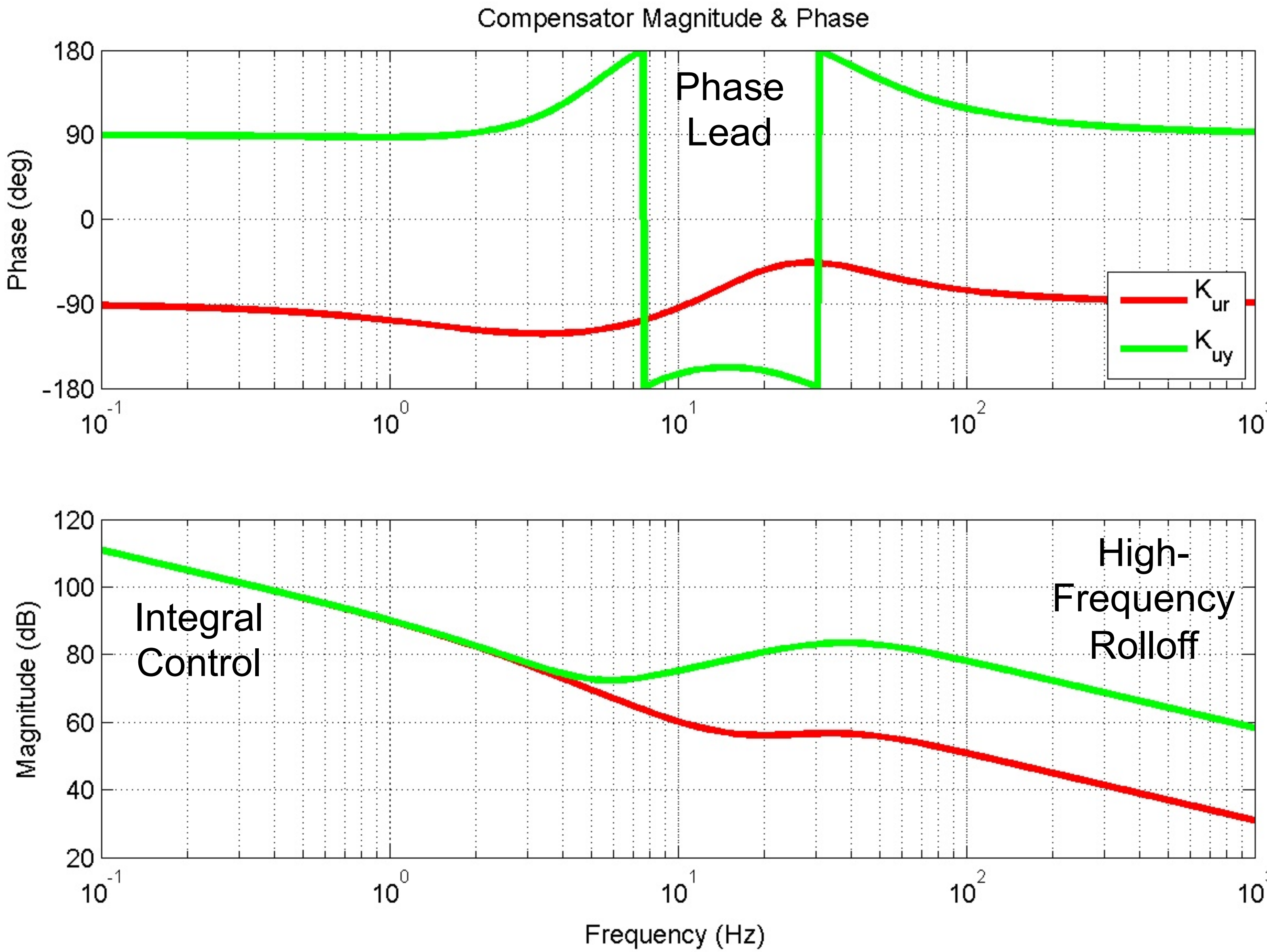
Loop Analysis - IX

Let's now consider the damped spring-mass oscillator example.

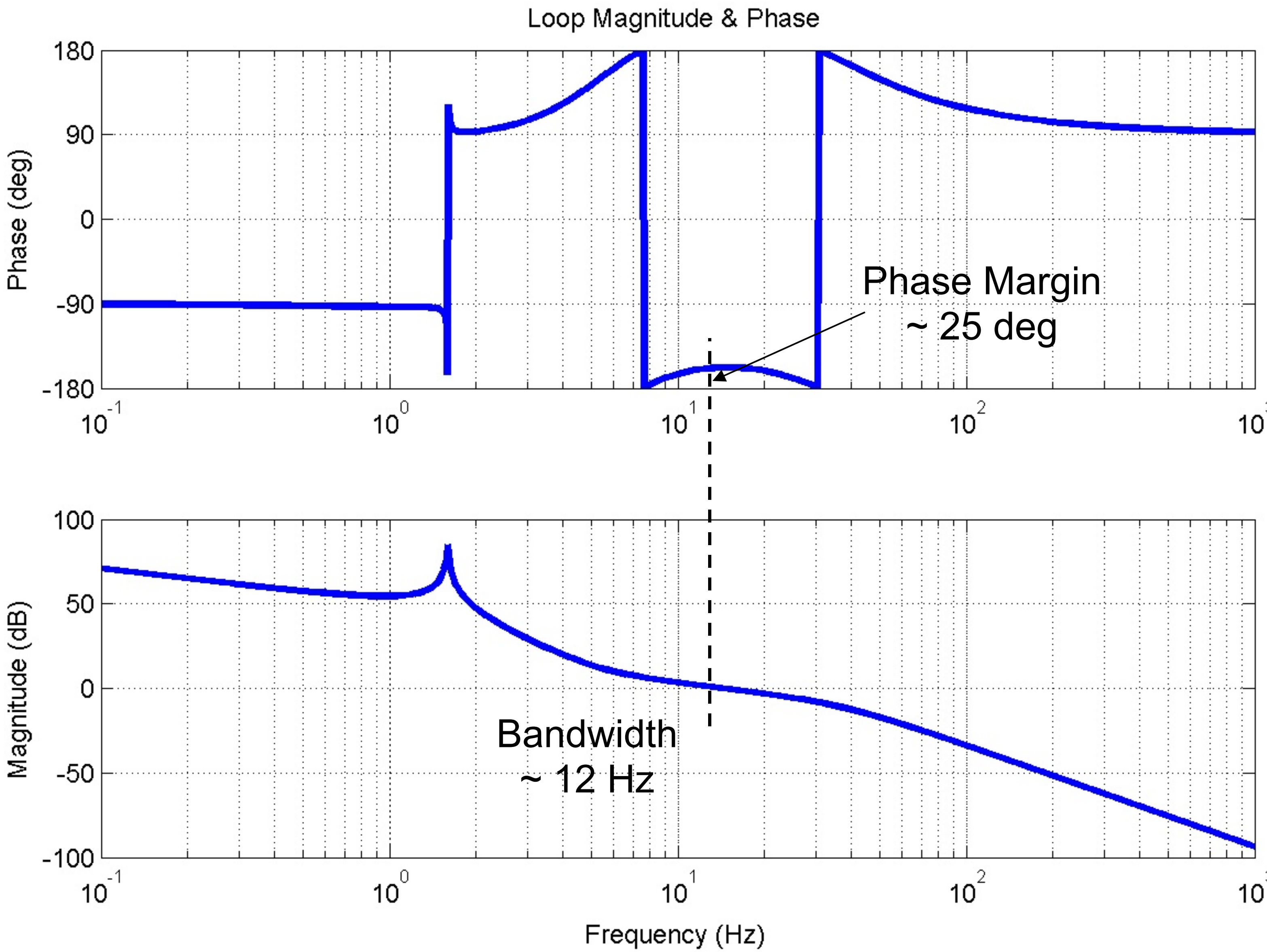
Using the gains from our full-state feedback design and the observer when $\alpha = 1.5$ produces a compensator that includes integral control.

We can plot the frequency response of the compensator and the loop TF.

Loop Analysis - X



Loop Analysis - XI



Loop Analysis - XII

Note that the loop transfer function must represent a stable system because we have chosen the closed-loop poles directly using full-state feedback and an observer.

The primary difference between this technique and classical control designs is that we do not directly affect the compensator during the design process. The compensator is a by-product of the choice of poles.

Summary

Integral control is implemented by incorporating a linear observer into the plant equations augmented with additional integrator states.

The linear observer is designed with the original plant, not the plant augmented with the integrator states.

Summary

Results demonstrate that the convergence rate of the observer does not affect the response to a reference input, but does affect the response to disturbances.

This completes our analysis of full-state feedback and observer design.

The next big question is: What is the “best” way to choose the poles?