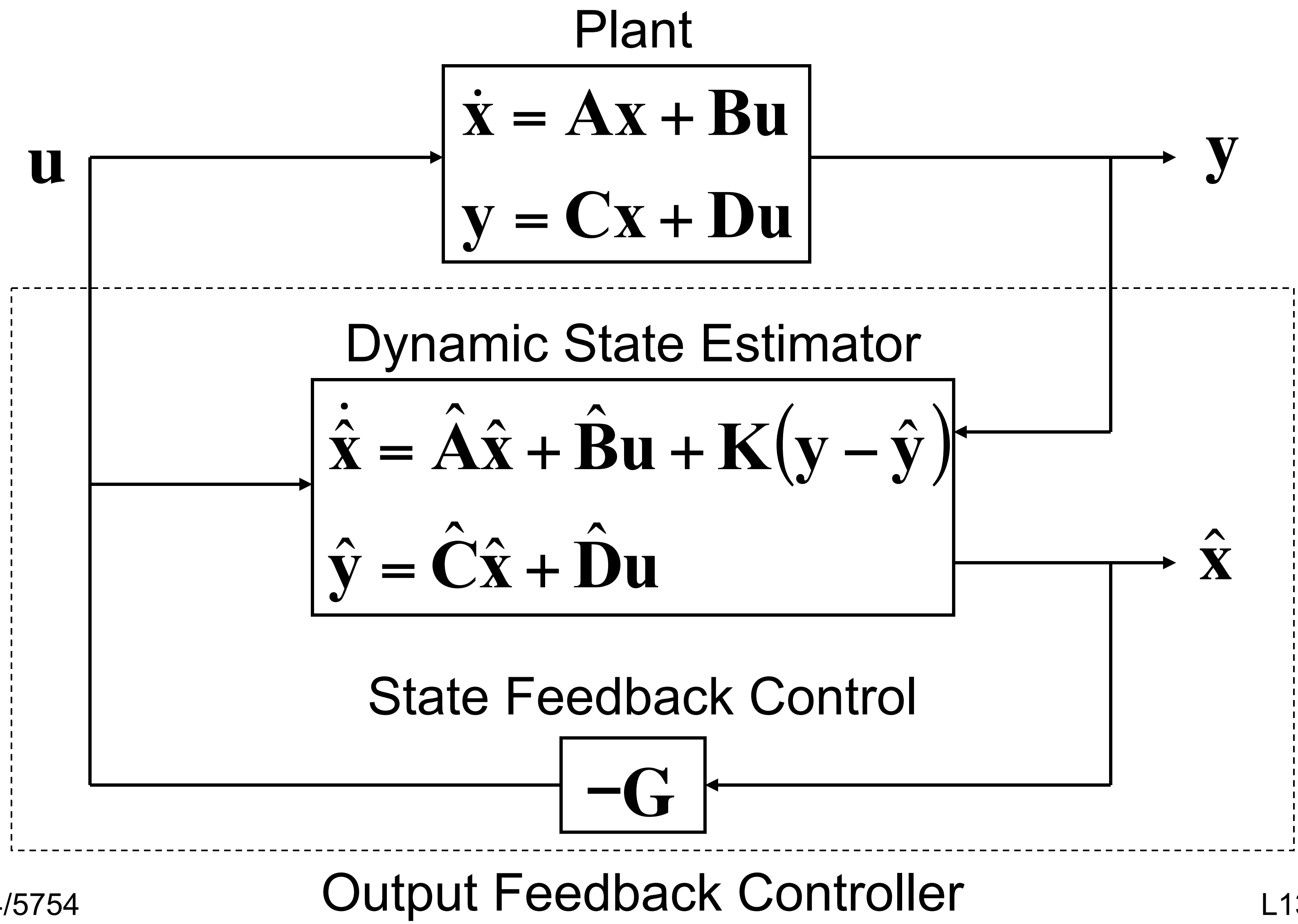
Advanced Control Engineering

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

State Estimation - I

Output feedback w/Luenberger observer



State Estimation - II

An output feedback controller has <u>two</u> parts

- Dynamic state estimator (observer)
- Full state feedback control

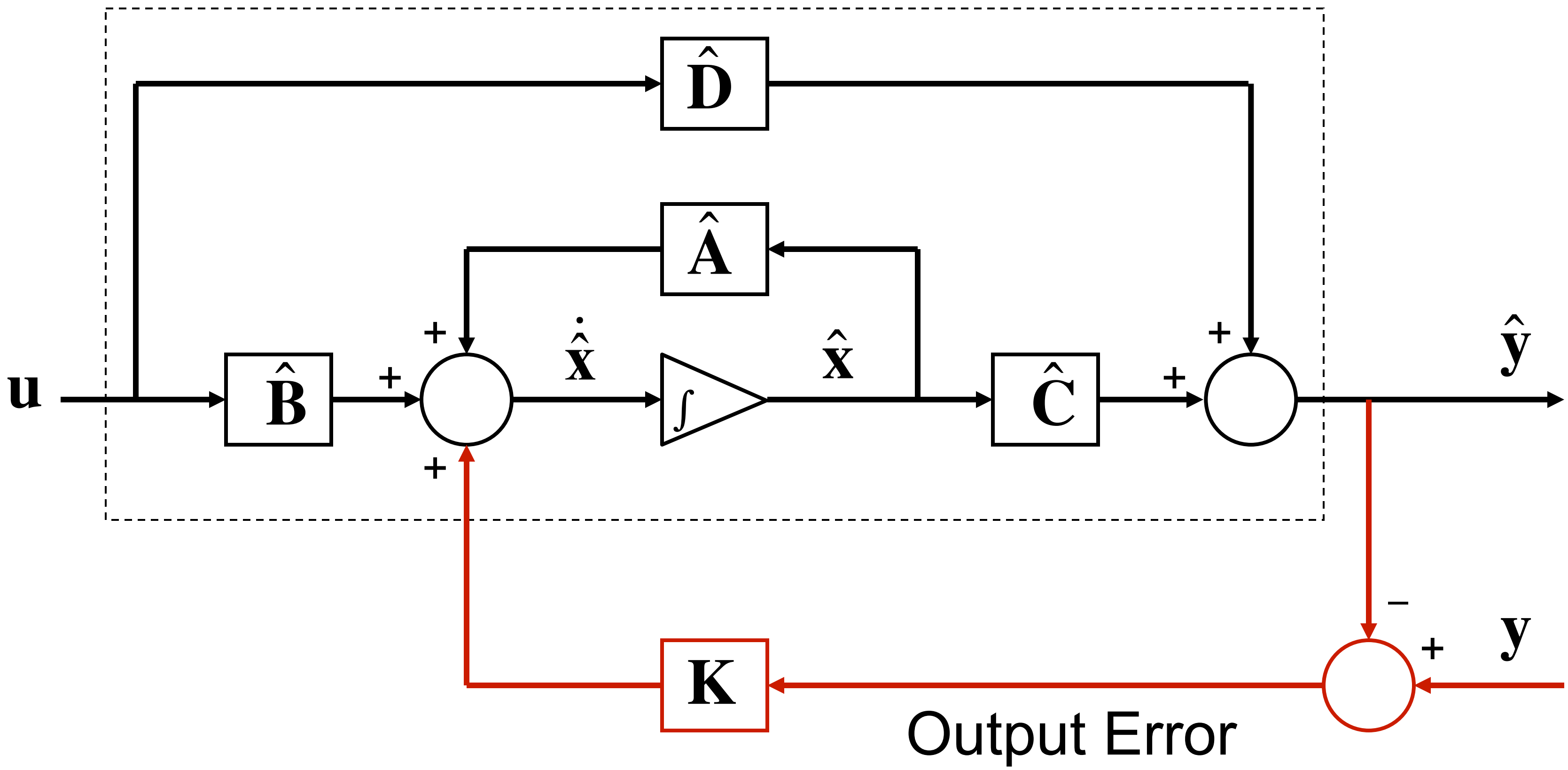
Let's take a closer look at the dynamic observer equations.

$$\underset{[N\times1]}{\dot{\hat{\mathbf{x}}}} = \underset{[N\times N]}{\hat{\mathbf{A}}}\underset{[N\times1]}{\hat{\mathbf{x}}} + \underset{[N\times M]}{\hat{\mathbf{B}}}\underset{[M\times1]}{\mathbf{u}} + \underset{[N\times P]}{\mathbf{K}}\overbrace{\left(\underset{[P\times1]}{\mathbf{y}} - \underset{[P\times1]}{\hat{\mathbf{y}}}\right)}^{\text{Output Error}}$$

$$\underset{[P\times1]}{\hat{\mathbf{y}}} = \underset{[P\times N]}{\hat{\mathbf{C}}}\underset{[N\times1]}{\hat{\mathbf{x}}} + \underset{[P\times M]}{\hat{\mathbf{D}}}\underset{[M\times1]}{\mathbf{u}}$$

State Estimation - III

The observer feedback term is an $N\times 1$ vector that is added to the derivative of the estimated state vector.



State Estimation - IV

We could NOT use this approach (i.e. add a vector to the state derivative) when we looked at full state feedback control, because it is equivalent to creating a new input signal.

We CAN do this for the state estimator because the observer is part of the controller that we are constructing, and is typically implemented on a digital signal processor.

State Estimation - V

We learned that if we assume a perfect plant model, then the observer design reduces to choosing the poles of the state estimation error dynamics

$$\dot{\mathbf{e}}(t) = [\mathbf{A} - \mathbf{K}\mathbf{C}]\mathbf{e}(t)$$

where $\mathbf{K}$ is the observer feedback gain matrix and $\mathbf{e}(t)$ is the state estimation error. This design allows us to control the rate of convergence of the error states.

State Estimation - VI

How do we compute the gains of the observer feedback matrix $\mathbf{K}$?

The poles of the observer state matrix $[\mathbf{A}-\mathbf{KC}]$ are the solutions of the characteristic equation:

$$\left|s\mathbf{I}-\mathbf{A}+\mathbf{KC}\right| = s^{N} + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0 = 0$$

The desired characteristic equation is:

$$s^{N} + \overline{a}_{N-1}s^{N-1} + \cdots + \overline{a}_1 s + \overline{a}_0 = 0$$

State Estimation - VII

Now we need to take advantage of the fact that the characteristic polynomial of a matrix is equivalent to the characteristic polynomial of its transpose, therefore

$$\left|\left(s\mathbf{I}-\mathbf{A}+\mathbf{KC}\right)^T\right| = \left|s\mathbf{I}-\mathbf{A}^T+\left(\mathbf{KC}\right)^T\right| = \left|s\mathbf{I}-\mathbf{A}^T+\mathbf{C}^T\mathbf{K}^T\right|$$

Compare this to the pole placement problem for full state feedback:

State Estimation - VIII

Comparing the two problems directly:

$$\left| s\mathbf{I} - \mathbf{A}^T + \mathbf{C}^T\mathbf{K}^T \right| = 0 \quad \Leftrightarrow \quad \text{Observer}$$

$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{G} \right| = 0 \quad \Leftrightarrow \quad \text{Full State Feedback}$$

The problem of designing an observer is mathematically equivalent to designing a full state feedback controller. We can use the same algorithms that we have already learned for pole placement!

State Estimation - IX

The only change is that we have to use $\mathbf{A}^T$ for the "state matrix" and $\mathbf{C}^T$ for the "input matrix".

In MATLAB, we would use:

```
>> K = place(A',C',des_obs_poles)'
```

Where **`des_obs_poles`** are the desired poles for the $[\mathbf{A}-\mathbf{KC}]$ matrix.

Observer Design Example - I

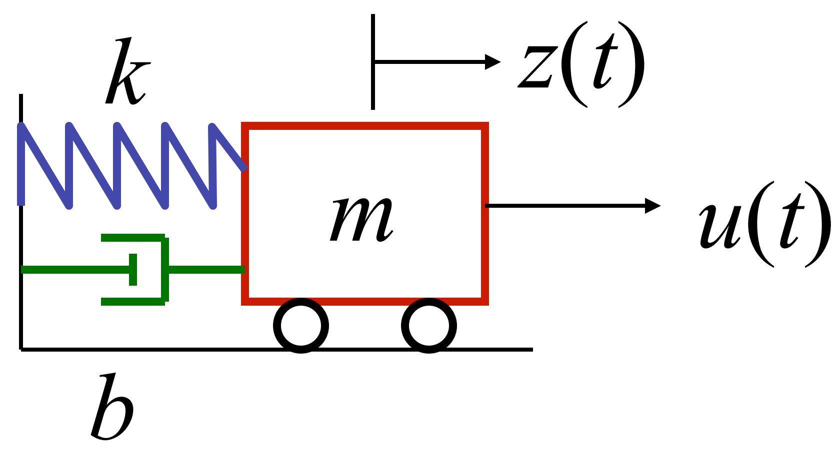
Consider again the damped mass-spring oscillator system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x_1(t) = z(t)$$

$$x_2(t) = \dot{z}(t)$$



Observer Design Example - II

Problem: Compute the observer gains such that the characteristic equation of $[\mathbf{A}-\mathbf{KC}]$ is equal to

$$|s\mathbf{I}-\mathbf{A}+\mathbf{KC}| = \underbrace{s^2+2\zeta_d\omega_d s+\omega_d^2}_{\text{Desired Observer CE}}$$

Solution: Using the brute force approach (L8), we simply write the CE in terms of the observer gains and equate polynomial coefficients.

Observer Design Example - III

Writing the CE in terms of the observer gains:

$$\left|s\mathbf{I}-\mathbf{A}+\mathbf{KC}\right|=\left|s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}0 & 1\\ -\omega^2 & -2\zeta\omega\end{bmatrix}+\begin{bmatrix}k_1\\ k_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right|$$

$$\left|s\mathbf{I}-\mathbf{A}+\mathbf{KC}\right|=\begin{vmatrix}s+k_1 & -1\\ \omega^2+k_2 & s+2\zeta\omega\end{vmatrix}$$

$$\left|s\mathbf{I}-\mathbf{A}+\mathbf{KC}\right|=\left(s+k_1\right)\left(s+2\zeta\omega\right)+\left(\omega^2+k_2\right)$$

$$\left|s\mathbf{I}-\mathbf{A}+\mathbf{KC}\right|=s^2+\left(2\zeta\omega+k_1\right)s+\left(\omega^2+2\zeta\omega k_1+k_2\right)$$

Observer Design Example - IV

Equating this polynomial with the desired polynomial gives us two equations in two unknowns:

$$s^1: \quad 2\zeta\omega + k_1 = 2\zeta_d\omega_d$$

$$s^0: \quad \omega^2 + 2\zeta\omega k_1 + k_2 = \omega_d^2$$

which can be solved for the two gains:

Observer Design Example - V

Now let's apply some numeric values to the problem:

Actual

$$\omega = 2\pi(3\ Hz) = 18.85\,\tfrac{rad}{s}$$

$$\zeta = 0.01$$

Desired

$$\omega_d = 2\pi(6\ Hz) = 37.7\,\tfrac{rad}{s}$$

$$\zeta_d = 0.05$$

The observer gains are:

$$k_1 = 2(0.05 * 37.7 - 0.01 * 18.85) = 3.39$$

$$k_2 = 37.7^2 - 18.85^2 - 2 * 0.01 * 18.85 * 3.39 = 1064.6$$

Observer Design Example - VI

```
%  --define the open-loop plant
w = 2*pi*3;
z = 0.01;
m = 1;

A = [0 1; -w*w -2*z*w];
B = [0; 1/m];
C = [1 0];
D = [0];

%  --define the observer poles
wd = 2*pi*6;
zd = 0.05;
des_obs_poles = [-zd*wd + j*wd*sqrt(1-zd*zd) ...
    -zd*wd - j*wd*sqrt(1-zd*zd) ];

%  --compute the observer gain matrix
K = place(A',C',des_obs_poles)'
```

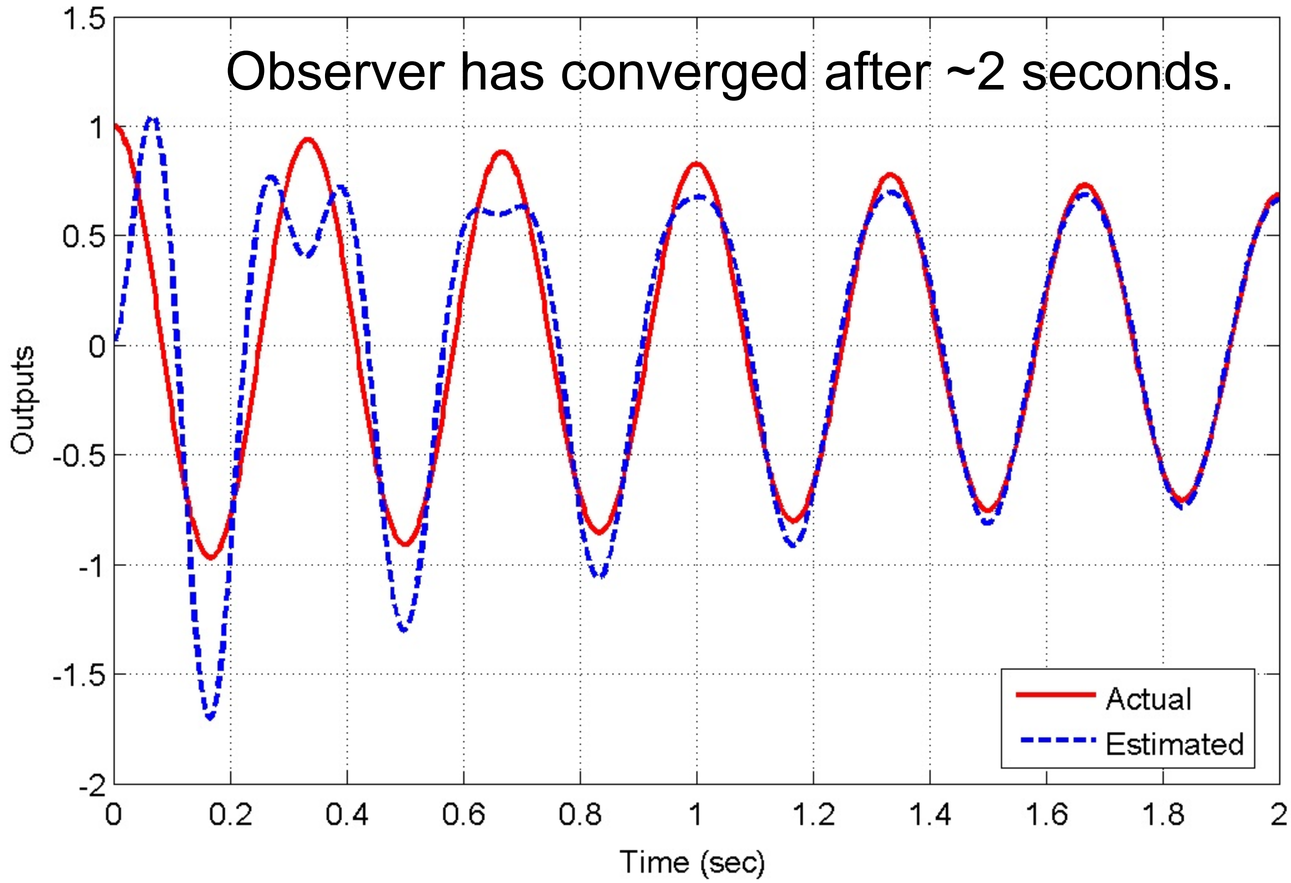
```
K =
     3.3929e+00
     1.0646e+03
```

Observer Design Example - VII

The response of the plant & observer due to the following initial conditions is:

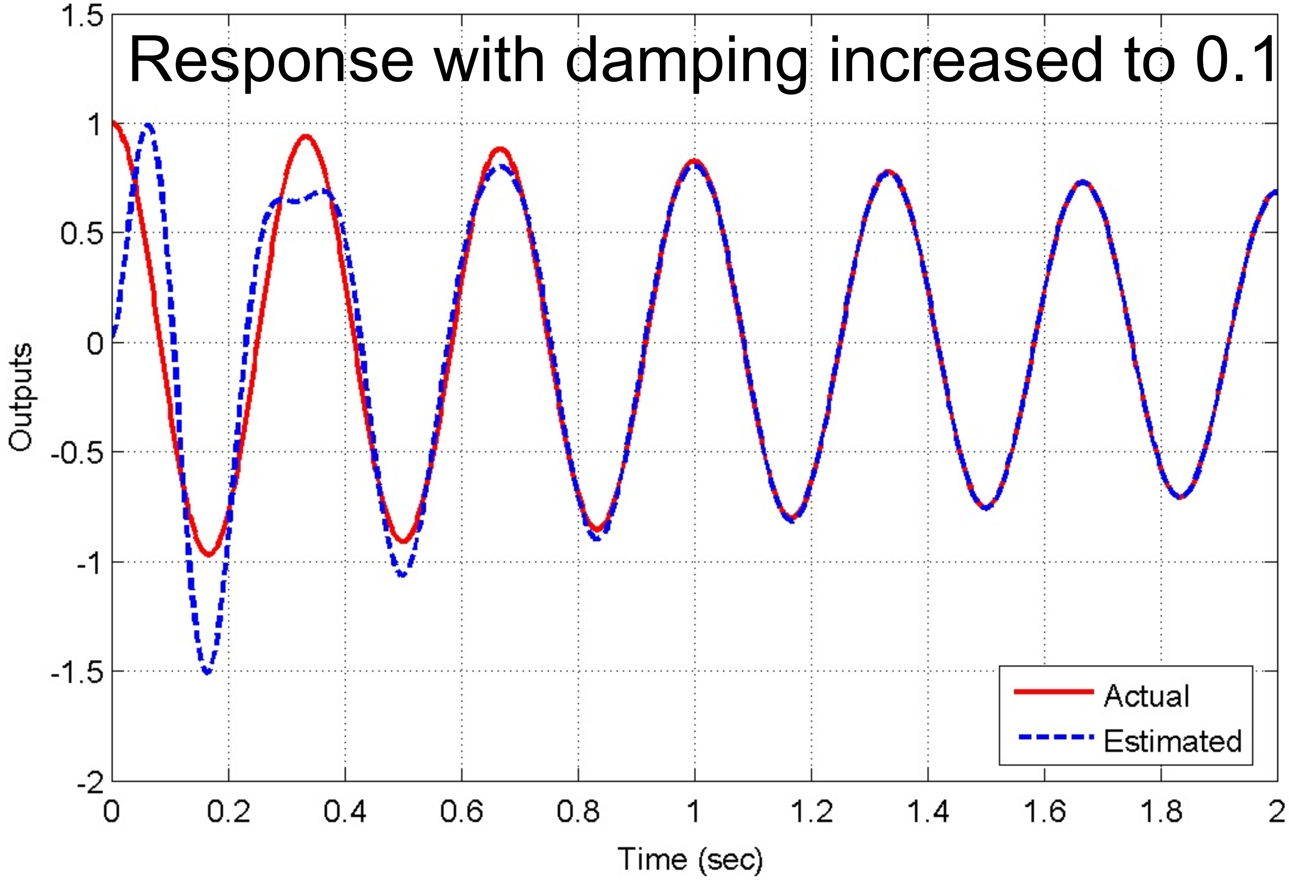
$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



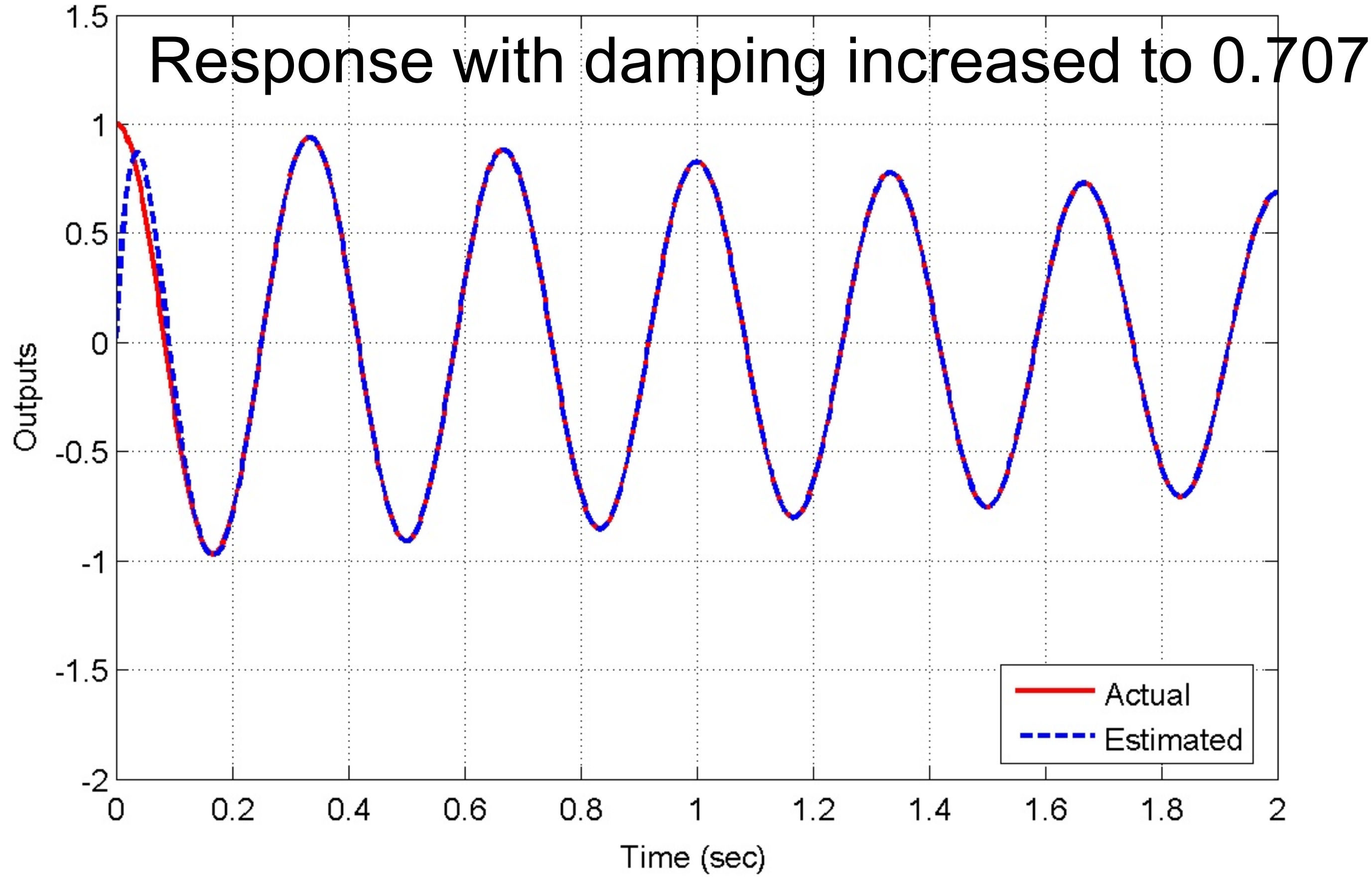
Observer Design Example - VIII

Changing the poles will change the convergence rate of the estimate.



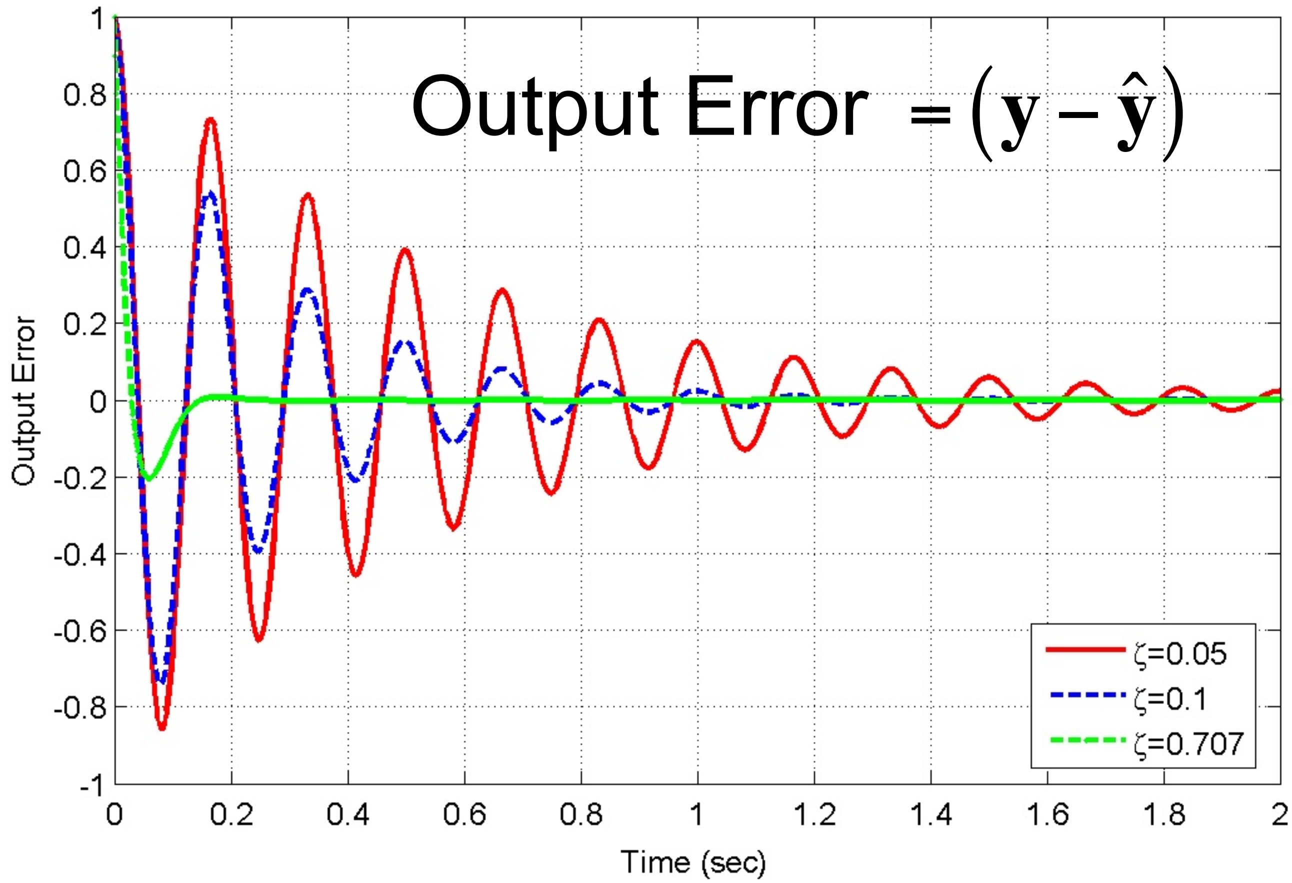
Observer Design Example - IX

Increasing the damping yet again yields even faster convergence.



Observer Design Example - X

The output error shows that increasing damping speeds up the convergence.



Augmented State Equations - I

State estimates are generated by the observer, which enables full state feedback using the estimates instead of the actual states.

We now have a two-step procedure for a complete compensator design

- Design the full-order observer
- Design the full-state feedback controller

Augmented State Equations - II

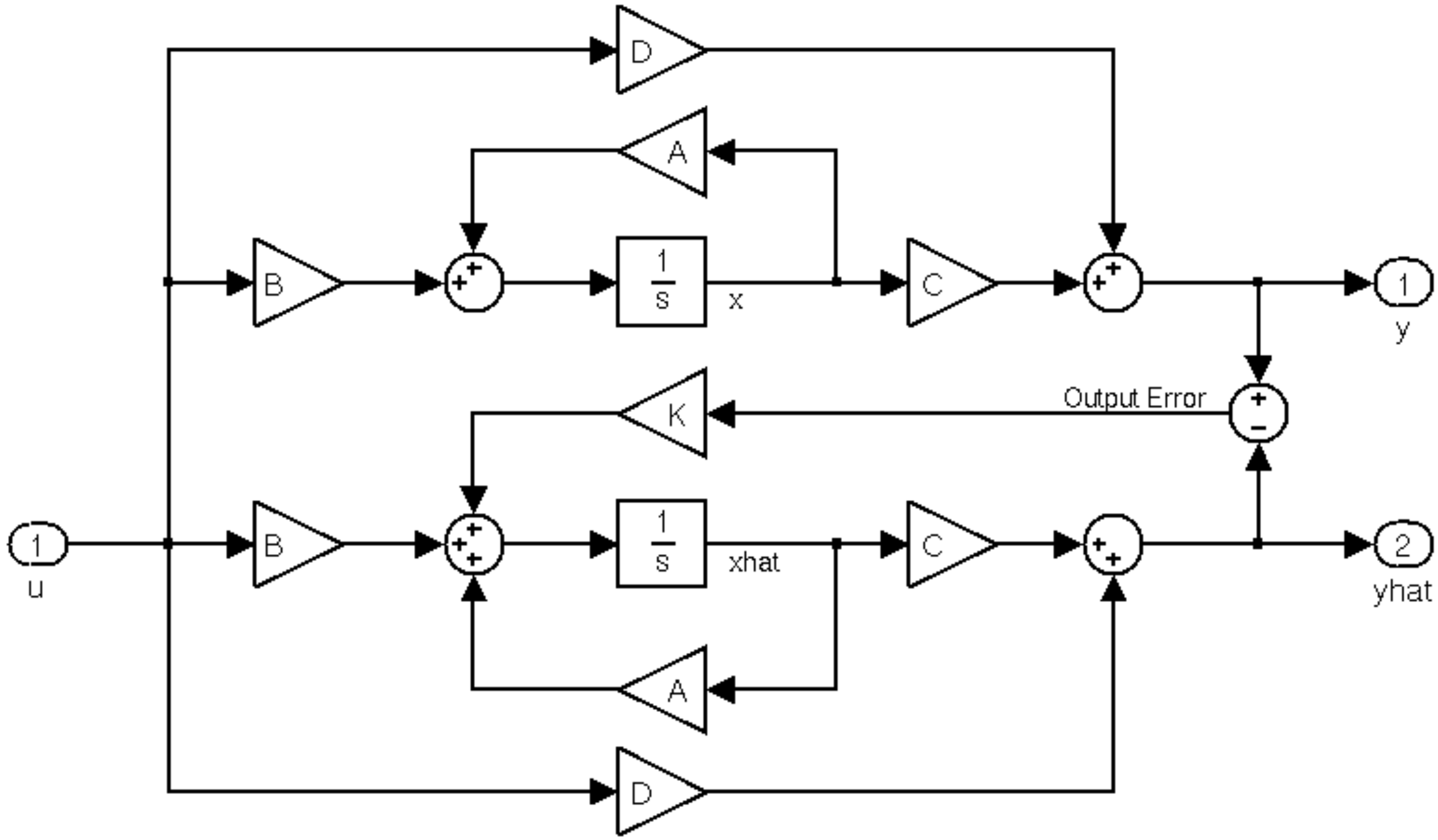
There are two related questions to ask.

Question 1: How can we build a simulation model of the plant and observer systems?

Question 2: Can we guarantee that the observer design is decoupled from the full state feedback design? Can state feedback change the observer?

Augmented State Equations - III

For Question #1, we could build the following simulation in Simulink:



Augmented State Equations - IV

Or we could combine the state and output equations for the plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

with the state and output equations for the observer

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} + \mathbf{K}\left(\mathbf{y} - \hat{\mathbf{y}}\right)$$

$$\hat{\mathbf{y}} = \hat{\mathbf{C}}\hat{\mathbf{x}} + \hat{\mathbf{D}}\mathbf{u}$$

Augmented State Equations - V

Substituting the output equations

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\hat{\mathbf{y}} = \hat{\mathbf{C}}\hat{\mathbf{x}} + \hat{\mathbf{D}}\mathbf{u}$$

into the observer state equation:

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} + \mathbf{K}\left(\left(\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\right) - \left(\hat{\mathbf{C}}\hat{\mathbf{x}} + \hat{\mathbf{D}}\mathbf{u}\right)\right)$$

$$\dot{\hat{\mathbf{x}}} = \mathbf{K}\mathbf{C}\mathbf{x} + \left[\hat{\mathbf{A}} - \mathbf{K}\hat{\mathbf{C}}\right]\hat{\mathbf{x}} + \left[\hat{\mathbf{B}} + \mathbf{K}\left(\mathbf{D} - \hat{\mathbf{D}}\right)\right]\mathbf{u}$$

Augmented State Equations - VI

Combining these results, we can form an <u>augmented</u> state-space system

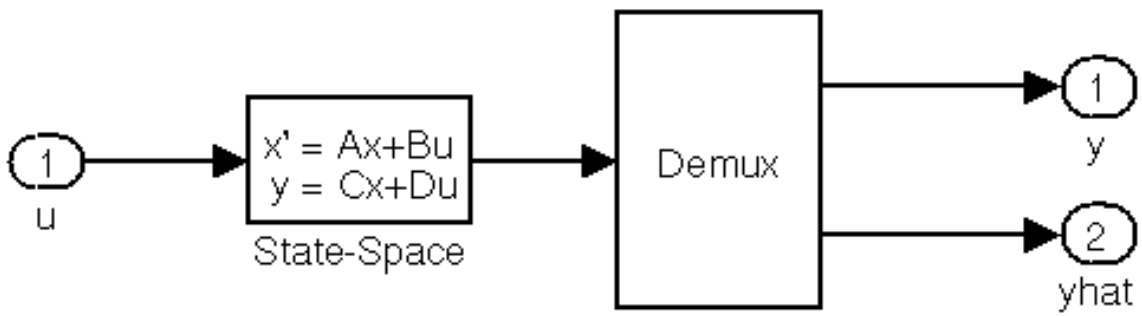
$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{KC} & \hat{\mathbf{A}} - \mathbf{K}\hat{\mathbf{C}} \end{bmatrix}}_{\bar{\mathbf{A}}} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B} \\ \hat{\mathbf{B}} + \mathbf{K}\left(\mathbf{D} - \hat{\mathbf{D}}\right) \end{bmatrix}}_{\bar{\mathbf{B}}} \mathbf{u}$$

$$\begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{C}} \end{bmatrix}}_{\bar{\mathbf{C}}} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{D} \\ \hat{\mathbf{D}} \end{bmatrix}}_{\bar{\mathbf{D}}} \mathbf{u}$$

This augmented system has twice as many states and outputs as the plant or observer!

Augmented State Equations - VII

Using the augmented state-space system, we could now build a simpler Simulink (or LSIM) model as follows:



Where **A**, **B**, **C**, and **D** are defined on the previous slide, and a DEMUX block could be used to separate the outputs.

Separation Principle - I

This answers Question #1 regarding how to simulate the augmented system.

We can also use these results to answer Question #2 about whether the state feedback is decoupled from the observer design.

Separation Principle - II

Substituting our full state feedback control law with estimated states

$$\mathbf{u} = -\mathbf{G}\hat{\mathbf{x}}$$

into the augmented state space system gives us the following result

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{KC} & \hat{\mathbf{A}} - \mathbf{K}\hat{\mathbf{C}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} - \begin{bmatrix} \mathbf{BG} \\ \hat{\mathbf{B}}\mathbf{G} + \mathbf{K}\left(\mathbf{D} - \hat{\mathbf{D}}\right)\mathbf{G} \end{bmatrix} \hat{\mathbf{x}}$$

$$\begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{C}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} - \begin{bmatrix} \mathbf{DG} \\ \hat{\mathbf{D}}\mathbf{G} \end{bmatrix} \hat{\mathbf{x}}$$

Separation Principle - III

This result simplifies to the following

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BG} \\ \mathbf{KC} & \left(\hat{\mathbf{A}} - \mathbf{K}\hat{\mathbf{C}} - \hat{\mathbf{B}}\mathbf{G} - \mathbf{K}\left(\mathbf{D} - \hat{\mathbf{D}}\right)\mathbf{G}\right) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & -\mathbf{DG} \\ \mathbf{0} & \hat{\mathbf{C}} - \hat{\mathbf{D}}\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

These equations can be used to simulate the closed-loop response to initial conditions.

Separation Principle - IV

Now, to answer Question #2, we could try to determine how the poles of the closed-loop system relate to the poles of the observer and the full-state feedback system.

We first have to rewrite the closed-loop equations in terms of the states $\mathbf{x}$ and the estimation error $\mathbf{e}$.

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$$

Separation Principle - V

Using the results from the previous two slides, we see that

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}$$

$$\dot{\mathbf{e}} = \left(\mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{G}\hat{\mathbf{x}}\right) - \left(\mathbf{K}\mathbf{C}\mathbf{x} + \left(\hat{\mathbf{A}} - \mathbf{K}\hat{\mathbf{C}} - \hat{\mathbf{B}}\mathbf{G} - \mathbf{K}\left(\mathbf{D} - \hat{\mathbf{D}}\right)\mathbf{G}\right)\hat{\mathbf{x}}\right)$$

$$\dot{\mathbf{e}} = \left[\mathbf{A} - \mathbf{K}\mathbf{C}\right]\mathbf{x} - \left[\hat{\mathbf{A}} - \mathbf{K}\hat{\mathbf{C}} + \left(\mathbf{B} - \hat{\mathbf{B}}\right)\mathbf{G} - \mathbf{K}\left(\mathbf{D} - \hat{\mathbf{D}}\right)\mathbf{G}\right]\hat{\mathbf{x}}$$

Assuming we have perfect knowledge of the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ matrices

Separation Principle - VI

And the original closed-loop state equation in terms of $\mathbf{e}$ becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{G}\underbrace{(\mathbf{x} - \mathbf{e})}_{\hat{\mathbf{x}}} = [\mathbf{A} - \mathbf{B}\mathbf{G}]\mathbf{x} + [\mathbf{B}\mathbf{G}]\mathbf{e}$$

In augmented matrix form we have

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Separation Principle - VII

This system is simply a transformed version of the original closed-loop equations

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

Therefore the poles of this representation are identical to the poles of the original representation.

What are the eigenvalues (poles)?

Separation Principle - VIII

Start with the following <u>general</u> block partitioned matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$$

where $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ are appropriately dimensioned matrices. From a CRC handbook we can find the following general result for the determinant:

$$|\mathbf{A}| = \begin{vmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{vmatrix} = |\mathbf{E}|\left|\mathbf{B} - \mathbf{C}\mathbf{E}^{-1}\mathbf{D}\right|$$

Separation Principle - IX

For our particular state matrix, the lower left matrix block is zero ($\mathbf{D} = \mathbf{0}$) so the determinant reduces to

$$|\mathbf{A}| = \begin{vmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{E} \end{vmatrix} = |\mathbf{E}||\mathbf{B}|$$

Whenever either of the off-diagonal block matrices is the zero matrix, the determinant is simply equal to the product of the determinants of the diagonal matrix blocks.

Separation Principle - X

For our block partitioned system:

$$\begin{vmatrix} s\mathbf{I}-\mathbf{A}+\mathbf{BG} & -\mathbf{BG} \\ \mathbf{0} & s\mathbf{I}-\mathbf{A}+\mathbf{KC} \end{vmatrix} = \left|s\mathbf{I}-\mathbf{A}+\mathbf{BG}\right|\left|s\mathbf{I}-\mathbf{A}+\mathbf{KC}\right|$$

Setting this determinant equal to zero gives us the characteristic equation whose roots are the poles of the closed-loop system

$$\underbrace{\left|s\mathbf{I}-\mathbf{A}+\mathbf{BG}\right|}_{\text{Full-State Feedback Poles}}\underbrace{\left|s\mathbf{I}-\mathbf{A}+\mathbf{KC}\right|}_{\text{Observer Poles}}=0$$

Separation Principle - XI

This key result is called the Separation Principle. This result confirms the following facts to answer Question #2:

- We can choose the observer gains and the full-state feedback gains independent of each other
- Our choice for one set of gains will not alter our choice for the other set of gains

Summary

To design the observer feedback gains, we can use the same pole placement algorithms we used for full state feedback except that we have to use the transposes: $\mathbf{A}^{T}$ and $\mathbf{C}^{T}$.

Because of the separation principle, placement of the observer poles is decoupled from the placement of the closed-loop state feedback poles.