Course Outline - 1st Half

- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response, Matrix Exponential
- Transfer Functions & Eigenvalues
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

Review - I

A system is controllable if and only if…

Algebraic Test:

- The Controllability matrix has full rank.

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{N-1}\mathbf{B} \end{bmatrix}$$

Integral Test:

- The Controllability grammian is non-singular.

$$\mathbf{P}(t_1, t_0) = \int_{t_0}^{t_1} e^{\mathbf{A}(t_1 - \tau)} \mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T (t_1 - \tau)} d\boldsymbol{\tau}$$

Review - II

These "tests" are known as <u>necessary and sufficient conditions</u> because they are part of an "if and only if" statement.

From wikipedia.org:

A *necessary* condition is one that must be satisfied for the result to happen. Breathing is *necessary* to stay alive; if you did not breathe, you would not stay alive. Breathing is not *sufficient* to stay alive, for if you did nothing but breathe, you could still die.

A *sufficient* condition is one that, if it is satisfied, the result is certain to happen. Jumping is *sufficient* to leave the ground, since the act of jumping causes you to leave the ground. Jumping is not *necessary* to leave the ground however, since one could step onto a ladder and leave the ground in a way which isn't jumping.

Review - III

In order for our LTI system to be completely controllable:

- it is <u>necessary</u> that either of these two conditions be true, and
- it is <u>sufficient</u> that either of these are the only conditions that are required

Review - IV

The mathematical statement:

$$\underbrace{(\text{A is true})}_{\substack{\text{System is} \\ \text{controllable}}} \quad \underbrace{\Leftrightarrow}_{\substack{\text{If and only if} \\ \text{(or IFF)}}} \quad \underbrace{(\text{B is true})}_{\substack{\text{Controllability} \\ \text{matrix is full rank}}}$$

Is equivalent to saying that the following are <u>both</u> true statements

$$(\text{A is true}) \quad \underbrace{\Rightarrow}_{\text{Implies That}} \quad (\text{B is true})$$

$$(\text{B is true}) \quad \underbrace{\Rightarrow}_{\text{Implies That}} \quad (\text{A is true})$$

Review - V

We can show how our two controllability "tests" are equivalent by going through part of a proof that the controllability grammian must be non-singular for complete controllability.

We will NOT be proving the "only if" condition here; however, a complete proof would require it.

Algebraic Conditions - I

Let's start with the following equation (Note that the t_0 has been dropped)

$$\underset{[N\times N]}{\mathbf{P}(t_1)}\underset{[N\times 1]}{\mathbf{v}} = \underset{[N\times 1]}{\mathbf{0}}$$

Either $\mathbf{P}(t_1)$ is singular (not invertible) or it is non-singular (invertible).

If it is non-singular, then the only solution to this equation is:

$$\mathbf{P}^{-1}(t_1)\mathbf{P}(t_1)\mathbf{v} = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{v} = \mathbf{0}$$

Algebraic Conditions - II

Now assume $\mathbf{P}(t_1)$ is singular, then $\mathbf{P}(t_1)$ has a non-trivial null-space, i.e. there are non-zero solutions $\mathbf{v} \neq \mathbf{0}$ such that

$$\mathbf{P}\left(t_1\right)\mathbf{v} = \mathbf{0}$$

The set of all such vectors is the null space. We can pre-multiply this equation by $\mathbf{v}^T$ and the result is still valid, but now we have a scalar

$$\underset{[1\times N]}{\mathbf{v}^T}\,\underset{[N\times N]}{\mathbf{P}\left(t_1\right)}\,\underset{[N\times 1]}{\mathbf{v}} = \underset{[1\times 1]}{0}$$

Algebraic Conditions - III

Now substitute the definition of the controllability grammian

$$\mathbf{v}^T \left\{ \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)} \mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t_1-\tau)} d\boldsymbol{\tau} \right\} \mathbf{v} = 0$$

And rewrite the expression

$$\int_0^{t_1} \left(\underset{[1\times N]}{\mathbf{v}^T} \underset{[N\times N]}{e^{\mathbf{A}(t_1-\tau)}} \underset{[N\times M]}{\mathbf{B}} \right) \left(\underset{[M\times N]}{\mathbf{B}^T} \underset{[N\times N]}{e^{\mathbf{A}^T(t_1-\tau)}} \underset{[N\times 1]}{\mathbf{v}} \right) d\boldsymbol{\tau} = 0$$

Remember: N = # of states, and M = # of inputs

Algebraic Conditions - IV

By defining the following vector quantity

$$\underset{[M\times 1]}{\mathbf{z}(\tau)} = \underset{[M\times N]}{\mathbf{B}^T} \underset{[N\times N]}{e^{\mathbf{A}^T(t_1-\tau)}} \underset{[N\times 1]}{\mathbf{v}}$$

With the fact that the transpose of a product is the product of transposes in reverse order, we can simplify to:

$$\int_0^{t_1} \left(\underset{[1\times N]}{\mathbf{v}^T} \underset{[N\times N]}{e^{\mathbf{A}(t_1-\tau)}} \underset{[N\times M]}{\mathbf{B}} \right) \left(\underset{[M\times N]}{\mathbf{B}^T} \underset{[N\times N]}{e^{\mathbf{A}^T(t_1-\tau)}} \underset{[N\times 1]}{\mathbf{v}} \right) d\tau = 0$$

$$\Rightarrow \qquad \int_0^{t_1} \underset{[1\times M]}{\mathbf{z}^T(\tau)} \underset{[M\times 1]}{\mathbf{z}(\tau)}\, d\tau = 0$$

Algebraic Conditions - V

The integrand:

$$\underset{[1\times M]}{\mathbf{z}^T(\tau)}\ \underset{[M\times 1]}{\mathbf{z}(\tau)}$$

is a <u>positive definite quadratic form</u>, which is also known as an <u>inner product</u> between the two $\mathbf{z}$ vectors.

This scalar result is commonly used to represent norms (measuring length).

Algebraic Conditions - VI

From the definition of $\mathbf{z}$, each element is a general time-domain function made up of terms from the matrix exponential

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ \vdots \\ z_M(t) \end{bmatrix}$$

By expanding the integrand, we can clearly see the <u>positive definite</u> nature

$$\mathbf{z}^T(\tau)\mathbf{z}(\tau) = z_1^2(\tau) + z_2^2(\tau) + \cdots + z_M^2(\tau) \geq 0$$

Algebraic Conditions - VII

To recap so far, our assumption of a singular $\mathbf{P}(t_1)$ has led to the following integral requirement

$$\int_0^{t_1} \mathbf{z}^T(\tau)\mathbf{z}(\tau)d\tau = 0$$

Because the integrand is positive definite, the integral (area under the curve) can only be zero if $\mathbf{z}(t) = \mathbf{0}$ for all values of t.

Algebraic Conditions - VIII

If $\mathbf{P}(t_1)$ is singular, then there exists a non-zero vector $\mathbf{v}$ such that

$$\mathbf{z}(t) = \mathbf{B}^T e^{\mathbf{A}^T t}\mathbf{v} = \mathbf{0} \qquad \text{for} \qquad 0 \le t \le t_1$$

If $\mathbf{z}(t)$ is zero, then all derivatives of $\mathbf{z}(t)$ must be zero as well

$$\frac{d\mathbf{z}(t)}{dt} = \mathbf{B}^T \mathbf{A}^T e^{\mathbf{A}^T t}\mathbf{v} = 0$$

$$\frac{d^2\mathbf{z}(t)}{dt^2} = \mathbf{B}^T \left(\mathbf{A}^T\right)^2 e^{\mathbf{A}^T t}\mathbf{v} = 0$$

$$\vdots$$

$$\frac{d^{N-1}\mathbf{z}(t)}{dt^{N-1}} = \mathbf{B}^T \left(\mathbf{A}^T\right)^{N-1} e^{\mathbf{A}^T t}\mathbf{v} = 0$$

Note: We only need the first N-1 derivatives!

Algebraic Conditions - IX

Collecting all these equations into a matrix equation gives us a result that looks somewhat familiar

$$\begin{bmatrix} \mathbf{B}^T \\ \mathbf{B}^T \mathbf{A}^T \\ \mathbf{B}^T \left(\mathbf{A}^T\right)^2 \\ \vdots \\ \mathbf{B}^T \left(\mathbf{A}^T\right)^{N-1} \end{bmatrix} e^{\mathbf{A}^T t} \mathbf{v} = \mathbf{Q}^T e^{\mathbf{A}^T t} \mathbf{v} = 0$$

The large matrix on the left is just the transpose of the controllability matrix $\mathbf{Q}$

Algebraic Conditions - X

Looking at the matrix exponential product with $\mathbf{v}$ we have a time-varying vector result

$$\underset{[N\times N]}{e^{\mathbf{A}^T t}}\underset{[N\times 1]}{\mathbf{v}} = \begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_N(t) \end{bmatrix}$$

We can also write the $\mathbf{Q}^T$ matrix as a set of N column vectors

$$\underset{[NM\times N]}{\mathbf{Q}^T} = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_N \end{bmatrix}$$

Algebraic Conditions - XI

The matrix expression can be written as a linear combination of the columns

$$\mathbf{Q}^T e^{\mathbf{A}^T t}\mathbf{v} = \mathbf{q}_1\alpha_1(t) + \mathbf{q}_2\alpha_2(t) + \cdots + \mathbf{q}_N\alpha_N(t) = \mathbf{0}$$

This linear combination of N columns can only be zero if the columns are <u>dependent</u> which means that the rank of $\mathbf{Q}^T$, and therefore $\mathbf{Q}$, must be less than N

$$\mathrm{rank}\left(\mathbf{Q}^T\right) = \mathrm{rank}(\mathbf{Q}) < N$$

Algebraic Conditions - XII

We have now shown that if $\mathbf{P}(t_1)$ is singular, then the controllability matrix $\mathbf{Q}$ cannot have full rank.

Working backwards, we see that if the controllability matrix $\mathbf{Q}$ has full rank, then the only possible vector $\mathbf{v}$ that satisfies the equations is $\mathbf{v} = 0$, which then implies that $\mathbf{P}(t_1)$ must be non-singular.

Summary

The following controllability conditions are equivalent and interchangeable:

- An LTI system is completely controllable iff the controllability grammian $\mathbf{P}(t_1)$ is non-singular

- An LTI system is completely controllable iff the controllability matrix $\mathbf{Q}$ has full rank

Summary

An algebraic controllability condition was derived that enabled us to check whether a system was controllable

The result is that a system is controllable if and only if the *controllability matrix*

$$\underset{[N\times NM]}{\mathbf{Q}} = \begin{bmatrix} \underset{[N\times M]}{\mathbf{B}} & \underset{[N\times M]}{\mathbf{AB}} & \underset{[N\times M]}{\mathbf{A}^2\mathbf{B}} & \cdots & \underset{[N\times M]}{\mathbf{A}^{N-1}\mathbf{B}} \end{bmatrix}$$

has rank = N.