Course Outline - 2nd Half

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

Review - I

We know that an LTI state-space system

$$\underset{[N\times 1]}{\dot{\mathbf{x}}} = \underset{[N\times N]}{\mathbf{A}}\,\underset{[N\times 1]}{\mathbf{x}} + \underset{[N\times M]}{\mathbf{B}}\,\underset{[M\times 1]}{\mathbf{u}}$$

$$\underset{[P\times 1]}{\mathbf{y}} = \underset{[P\times N]}{\mathbf{C}}\,\underset{[N\times 1]}{\mathbf{x}} + \underset{[P\times M]}{\mathbf{D}}\,\underset{[M\times 1]}{\mathbf{u}}$$

has a unique set of N open-loop poles, or eigenvalues, which are the roots of

$$|s\mathbf{I}-\mathbf{A}| = 0 \quad \Rightarrow \quad s_{OL} = \{s_1, s_2, \ldots, s_N\}$$

Review - II

We also learned that this state-space system is completely controllable iff the Controllability Grammian is non-singular, invertible, or full rank:

$$\underset{[N\times N]}{\mathbf{P}\left(t_1,t_0\right)} = \int_{t_0}^{t_1} \underset{[N\times N]}{e^{\mathbf{A}\left(t_1-\tau\right)}} \underset{[N\times M]}{\mathbf{B}} \underset{[M\times N]}{\mathbf{B}^T} \underset{[N\times N]}{e^{\mathbf{A}^T\left(t_1-\tau\right)}} \, d\tau$$

This matrix result depends on $\mathbf{A}$, $\mathbf{B}$, and the initial time t_0 and final time t_1 and is generally difficult to work with.

Review - III

A simpler but equivalent result is that the state-space system is completely controllable iff the Controllability Matrix has full rank (which is N).

$$\underset{[N\times NM]}{\mathbf{Q}} = \left[\underset{[N\times M]}{\mathbf{B}} \quad \underset{[N\times M]}{\mathbf{AB}} \quad \underset{[N\times M]}{\mathbf{A}^2\mathbf{B}} \quad \cdots \quad \underset{[N\times M]}{\mathbf{A}^{N-1}\mathbf{B}}\right]$$

For a single-input system, $\mathbf{Q}$ is square.

For a multi-input system, $\mathbf{Q}$ has more columns than rows.

Review - IV

We learned that we can change the state-space system dynamics using full state feedback (assuming we have all of the states available!).

The Full State Feedback control law is a matrix of static (i.e. fixed) gains multiplied by the state vector:

Review - V

The closed-loop state-space system is:

$$\underset{[N\times 1]}{\dot{\mathbf{x}}} = \left[\underset{[N\times N]}{\mathbf{A}} - \underset{[N\times M]}{\mathbf{B}}\underset{[M\times N]}{\mathbf{G}}\right]\underset{[N\times 1]}{\mathbf{x}}$$

$$\underset{[P\times 1]}{\mathbf{y}} = \left[\underset{[P\times N]}{\mathbf{C}} - \underset{[P\times M]}{\mathbf{D}}\underset{[M\times N]}{\mathbf{G}}\right]\underset{[N\times 1]}{\mathbf{x}}$$

We can see that the closed-loop state matrix $[\mathbf{A}-\mathbf{BG}]$ is a function of the control gains in the feedback matrix $\mathbf{G}$.

Review - VI

To determine the control gains, we first need to select a set of desired closed-loop poles based on performance requirements

$$s_{CL} = \left\{ s_1, s_2, \ldots, s_N \right\}$$

We also saw that it is sometimes clear and sometimes difficult to know exactly where to place the closed-loop poles.

Review - VII

From the set of desired closed-loop poles, it is easy to determine the coefficients of the desired closed-loop characteristic equation

$$(s - s_1)(s - s_2)\cdots(s - s_N) = 0$$

$$s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0 = 0$$

Remember that ALL coefficients a_n MUST always be REAL, even if some of the poles are complex.

Review - VIII

For example, let's say we have the following poles:

$$s_1 = -2 \qquad\qquad s_2 = -1 + j3$$

The characteristic equation is:

$$(s+2)(s+1-j3) = s^2 + (3-j3)s + (2-j6) = 0$$

This does NOT represent a real physical system because the CE coefficients are complex.

Review - IX

For physical systems, complex poles MUST exist in conjugate pairs:

$$s_1 = -1 - j3 \qquad s_2 = -1 + j3$$

The characteristic equation is:

$$(s + 1 + j3)(s + 1 - j3) = s^2 + 2s + 10 = 0$$

Which does represent a physical system because the coefficients are all real.

Review - X

Given a set of desired closed-loop pole locations, we have studied one procedure (Brute Force) for computing the unknown control gains that place the poles at the desired locations.

For the <u>Brute Force</u> approach, we first construct an analytic expression for the closed-loop characteristic equation

Review - XI

This process will generate a polynomial where the CE coefficients are functions of the control gains and the parameters in **A** and **B**:

$$s^N + \bar{a}_{N-1}(\mathbf{A},\mathbf{B},\mathbf{G})s^{N-1} + \cdots + \bar{a}_1(\mathbf{A},\mathbf{B},\mathbf{G})s + \bar{a}_0(\mathbf{A},\mathbf{B},\mathbf{G}) = 0$$

These functions are <u>linear</u> functions of the control gains when the system has only a <u>single input</u>.

Review - XII

The last step in the Brute Force approach was to equate the two sets of polynomial coefficients which always gives us a set of N equations.

$$\begin{aligned}
a_{N-1} &= \bar{a}_{N-1}(\mathbf{A},\mathbf{B},\mathbf{G}) \\
\vdots \quad & \qquad \vdots \\
a_1 &= \bar{a}_1(\mathbf{A},\mathbf{B},\mathbf{G}) \\
\underbrace{a_0}_{\substack{\text{From} \\ \text{desired} \\ \text{CL CE}}} &= \underbrace{\bar{a}_0(\mathbf{A},\mathbf{B},\mathbf{G})}_{\substack{\text{From closed-loop} \\ \text{state matrix}}}
\end{aligned}$$

Review - XIII

We only studied the restricted brute force case of a single input system which led to a set of N linear equations in the N unknown control gains.

$$
\begin{aligned}
a_{N-1} &= \bar{a}_{N-1}(\mathbf{A},\mathbf{b},\mathbf{g}) \\
&\vdots \\
a_1 &= \bar{a}_1(\mathbf{A},\mathbf{b},\mathbf{g}) \\
\underbrace{a_0}_{\text{From desired CL CE}} &= \underbrace{\bar{a}_0(\mathbf{A},\mathbf{b},\mathbf{g})}_{\text{From closed-loop state matrix}}
\end{aligned}
$$

$$\underset{[N\times 1]}{\mathbf{b}} \qquad \underset{[1\times N]}{\mathbf{g}}$$

$$
\begin{bmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix}
\begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix}
=
\begin{bmatrix} * \\ \vdots \\ * \end{bmatrix}
$$

Review - XIV

We have not yet learned:

- How to place the closed-loop poles for multiple input systems (next!)
- What do we do if we don't have all the states available for feedback (State Estimation/Observer)
- How do we handle the real-world control problems that have measurement noise and reference inputs (Exogenous Inputs)

Multi-Input Pole Placement - I

Let's start with a simple 2^{nd} order multi-input system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

With the following standard controller definition:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The closed-loop system is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left[\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Multi-Input Pole Placement - II

With a little help from Mathematica…

```
In[1]:= A = {{a11, a12}, {a21, a22}};

In[2]:= B = {{b11, b12}, {b21, b22}};

In[3]:= G = {{g11, g12}, {g21, g22}};

In[4]:= Ac = A - B . G;

In[5]:= MatrixForm[Ac]

Out[5]//MatrixForm=
```

$$\begin{pmatrix} a_{11} - b_{11} g_{11} - b_{12} g_{21} & a_{12} - b_{11} g_{12} - b_{12} g_{22} \\ a_{21} - b_{21} g_{11} - b_{22} g_{21} & a_{22} - b_{21} g_{12} - b_{22} g_{22} \end{pmatrix}$$

```
In[6]:= Collect[FullSimplify[Det[{{s, 0}, {0, s}} - Ac]], s]
```

Out[6]=

$$\begin{aligned} &s^2 + a_{11} a_{22} - a_{22} b_{11} g_{11} + a_{21} b_{11} g_{12} - a_{11} b_{21} g_{12} - a_{22} b_{12} g_{21} + \\ &b_{12} b_{21} g_{12} g_{21} - b_{11} b_{22} g_{12} g_{21} + a_{12} (-a_{21} + b_{21} g_{11} + b_{22} g_{21}) + \\ &a_{21} b_{12} g_{22} - a_{11} b_{22} g_{22} - b_{12} b_{21} g_{11} g_{22} + b_{11} b_{22} g_{11} g_{22} + \\ &s (-a_{11} - a_{22} + b_{11} g_{11} + b_{21} g_{12} + b_{12} g_{21} + b_{22} g_{22}) \end{aligned}$$

Notice that the closed-loop CE coefficients are non-linear functions of the unknown feedback control gains.

Multi-Input Pole Placement - III

For this simple 2^{nd} order example, we will get two equations (where do they come from?), but we have four unknown control gains to solve for!

For the general state-feedback control design problem, we will always have:

- Only N equations
- and $N \times M$ unknown control gains

This is known as an under-determined problem when $M > 1$.

Multi-Input Pole Placement - IV

This is great news for you, the control designer! It means that you have more flexibility in your final design.

You can place the poles where you want (assuming complete controllability) and still have some design freedom.

Of course the price of flexibility is that you now have to supply additional constraint equations.

Multi-Input Pole Placement - V

What are some useful or practical design constraints?

Example 1: Select the gain matrix $\mathbf{G}$ that has minimum norm.

- This tends to minimize the absolute values of the individual gains in $\mathbf{G}$ which in turn tends to minimize the control signal magnitude and the distance that the open-loop poles move
- Requires function minimization

Multi-Input Pole Placement - VI

Example 2: Select key elements of the $\mathbf{G}$ matrix to be zero.

- If you zero out a column of $\mathbf{G}$, then you will not require that state in the feedback loop

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} 0 & g_{12} \\ 0 & g_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} g_{12} \\ g_{22} \end{bmatrix} x_2$$

- You might want to make each control signal depend on a specific set of states

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} 0 & g_{12} \\ g_{21} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} g_{12} x_2 \\ g_{21} x_1 \end{bmatrix}$$

Multi-Input Pole Placement - VII

Example 3: The `place` function in the Matlab Control System Toolbox:

```
[G] = place(A,B,clpoles);
```

returns a gain matrix **G** that minimizes the sensitivity of the closed-loop pole locations to uncertainties in **A** and **B**

- Excellent choice since **A** and **B** are almost always approximations to the actual system
- Requires function minimization

Multi-Input Pole Placement - VIII

Numerical Example: Given the following controllable state-space system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 & 0 \\ 1 & -5 & 4 & -3 \\ -2 & -4 & -4 & 2 \\ 0 & 3 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 3 & 0 \\ -2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The open-loop poles from Matlab are:

$$s_{OL} = \{-4.4482 \pm 5.5054i, \quad -0.3747, \quad -3.7288\}$$

Multi-Input Pole Placement - IX

First use the `place` function to place the following closed-loop poles

$$s_{CL} = \{-6 \pm 6i, \quad -2, \quad -7\}$$

```
>> clpoles = [-6+j*6, -6-j*6, -2, -7];
>> G = place(A,B,clpoles)
G =
   -0.8285    1.1602   -0.7339    0.0967
   -1.5141    0.7035   -0.0115    0.8980
>> Ac = A - B*G;
>> eig(Ac)
ans =
  -6.0000 + 6.0000i
  -6.0000 - 6.0000i
  -2.0000
  -7.0000
```

Multi-Input Pole Placement - X

Next, let's force the control solution to feedback only x_2 and x_4

$$\mathbf{G} = \begin{bmatrix} 0 & g_{12} & 0 & g_{14} \\ 0 & g_{22} & 0 & g_{24} \end{bmatrix}$$

```
In[1]:= A = {{-1, -2, -1, 0}, {1, -5, 4, -3}, {-2, -4, -4, 2}, {0, 3, -2, -3}};

In[2]:= B = {{1, -2}, {3, 0}, {-2, 4}, {0, 1}};

In[3]:= G = {{0, g12, 0, g14}, {0, g22, 0, g24}};

In[4]:= Ac = A - B.G

Out[4]= {{-1, -2 - g12 + 2 g22, -1, -g14 + 2 g24}, {1, -5 - 3 g12, 4, -3 - 3 g14},
  {-2, -4 + 2 g12 - 4 g22, -4, 2 + 2 g14 - 4 g24}, {0, 3 - g22, -2, -3 - g24}}

In[5]:= Collect[Det[s IdentityMatrix[4] - Ac], s]

Out[5]= 70 + s^4 - 20 g12 + 44 g14 + 100 g22 + 52 g14 g22 - 38 g24 - 52 g12 g24 + s^3 (13 + 3 g12 + g24) +
  s (218 + 20 g12 + 76 g14 + 79 g22 + 16 g14 g22 + 31 g24 - 16 g12 g24) +
  s^2 (88 + 17 g12 + 13 g14 + 11 g22 - 3 g14 g22 + 2 g24 + 3 g12 g24)
```

Multi-Input Pole Placement - XI

Using the same desired closed loop poles as before, the characteristic equation is:

$$s^4 + 21s^3 + 194s^2 + 816s + 1008 = 0$$

Equating the polynomial coefficients:

$$s^3: \qquad 21 = 13 + 3g_{12} + g_{24}$$

$$s^2: \qquad 194 = 88 + 17g_{12} + 13g_{14} + 11g_{22} + 2g_{24} - 3\left(g_{14}g_{22} - g_{12}g_{24}\right)$$

$$s^1: \qquad 816 = 218 + 20g_{12} + 76g_{14} + 79g_{22} + 31g_{24} + 16\left(g_{14}g_{22} - g_{12}g_{24}\right)$$

$$s^0: \qquad 1008 = 70 - 20g_{12} + 44g_{14} + 100g_{22} - 38g_{24} + 52\left(g_{14}g_{22} - g_{12}g_{24}\right)$$

Multi-Input Pole Placement - XII

In matrix form, these four equations are:

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 1 \\ 17 & 13 & 11 & 2 \\ 20 & 76 & 79 & 31 \\ -20 & 44 & 100 & -38 \end{bmatrix}\begin{bmatrix} g_{12} \\ g_{14} \\ g_{22} \\ g_{24} \end{bmatrix} = \begin{bmatrix} 21-13 \\ 194-88 \\ 816-218 \\ 1008-70 \end{bmatrix}}_{\text{Linear Part}} + \underbrace{\begin{bmatrix} 0 \\ 3 \\ -16 \\ -52 \end{bmatrix}\left(g_{14}g_{22} - g_{12}g_{24}\right)}_{\text{Non-Linear Part}}$$

This non-linear set of equations does not have a closed-form solution. We can find the solution using an iterative search or an optimization code.

Multi-Input Pole Placement - XIII

A Matlab check indicates that the iterative solution does place the poles at the desired closed-loop locations:

```
>> G = [0, g12, 0, g14; 0, g22, 0, g24]
G =
         0    3.3147         0   -0.1512
         0    6.5338         0   -1.9440
>> Ac = A - B*G;
>> eig(Ac)
ans =
  -6.0000 + 6.0000i
  -6.0000 - 6.0000i
  -7.0000
  -2.0000
```

Multi-Input Pole Placement - XIV

For this numerical example, two different constraint approaches led to two different feedback solutions, both of which placed the closed-loop poles at the exact same desired locations.

$$\mathbf{G} = \begin{bmatrix} -0.8285 & 1.1602 & -0.7339 & 0.0967 \\ -1.5141 & 0.7035 & -0.0115 & 0.8980 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 0 & 3.3147 & 0 & -0.1512 \\ 0 & 6.5338 & 0 & -1.9440 \end{bmatrix}$$

Summary

State-space systems with more than one input will provide more design freedom to the designer in placing the closed-loop poles using full state feedback.

The designer may choose from a large number of additional constraints in order to determine a unique set of feedback gains.