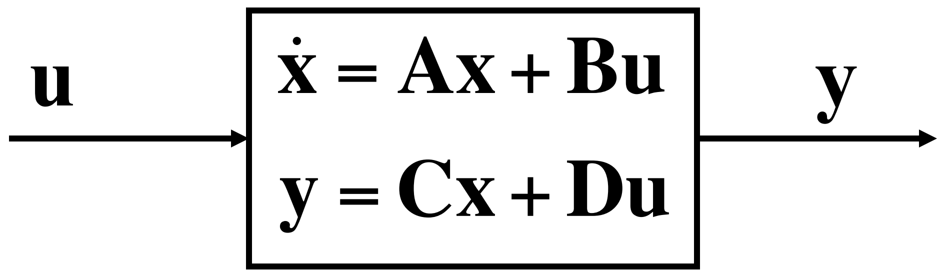
Advanced Control Engineering

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

Introduction - I

Up to this point, we have considered open-loop state-space systems:



and the time response to initial conditions $\mathbf{x}_0$ and arbitrary inputs $\mathbf{u}$.

Introduction - II

We have also considered closed-loop state-space systems with full-state feedback :

$$\mathbf{u} = -\mathbf{G}\mathbf{x} \quad \Rightarrow \quad \boxed{\begin{aligned} \dot{\mathbf{x}} &= [\mathbf{A} - \mathbf{B}\mathbf{G}]\mathbf{x} \\ \mathbf{y} &= [\mathbf{C} - \mathbf{D}\mathbf{G}]\mathbf{x} \end{aligned}} \xrightarrow{\mathbf{y}}$$

The control objective was to “shape” the time response evolving from some initial condition $\mathbf{x}_0$.

Introduction - III

For most practical examples, we must design for disturbances and/or reference inputs. Here are three specific examples:

- Reference input: A robotic arm that must track a desired motion profile
- Process noise: A vibration isolation system that must reject disturbances
- Measurement noise: A feedback control system that has a "noisy" sensor

Introduction - IV

Disturbances and reference inputs are called exogenous variables.

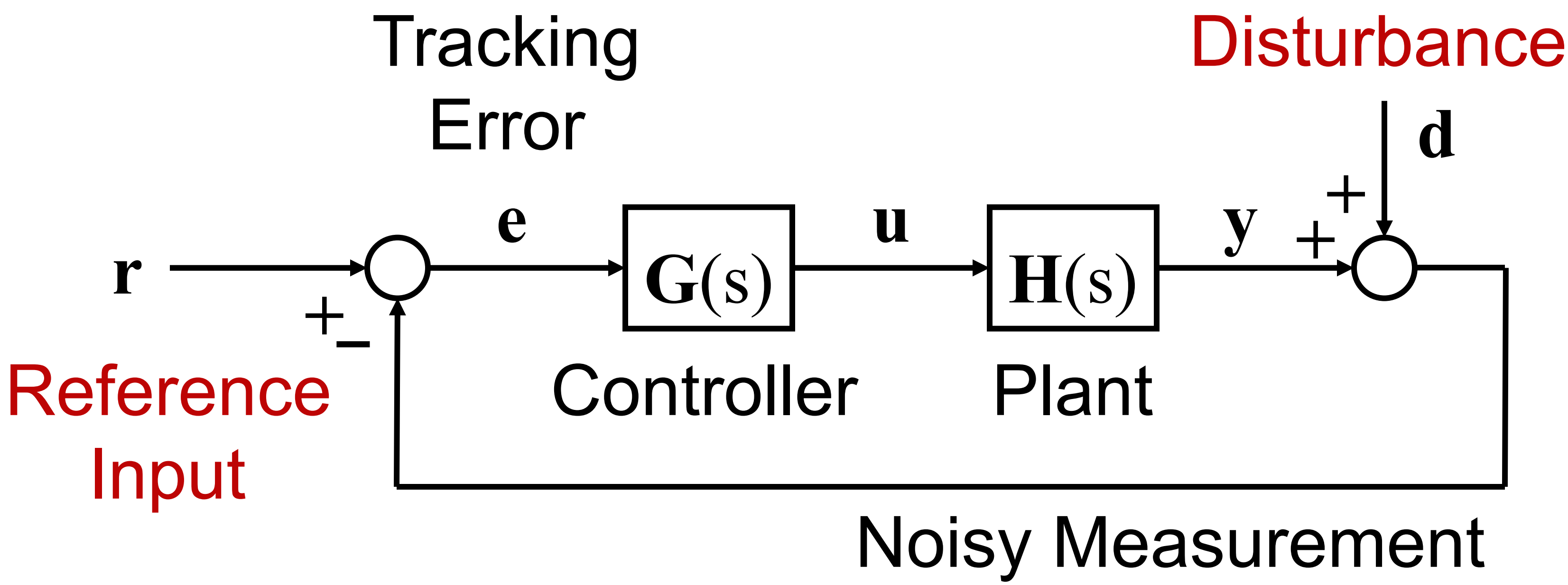
The existence of disturbances and reference inputs typically lead to the following control specifications:

– Minimize the system response to external disturbances.

– Track the reference inputs by minimizing the steady-state error between the outputs and the reference outputs.

Introduction - V

We have already studied the effects and accommodation of exogenous variables in classical feedback control:



Remember <u>Sensitivity</u> & <u>Complementary Sensitivity</u>?

Modeling Disturbances - I

Process noise and Measurement (or Sensor) noise are similar to each other because they are both disturbance inputs and can easily be confused.

Process noise affects both the states AND the outputs

Measurement or Sensor noise only affects the outputs but NOT the states.

Modeling Disturbances - II

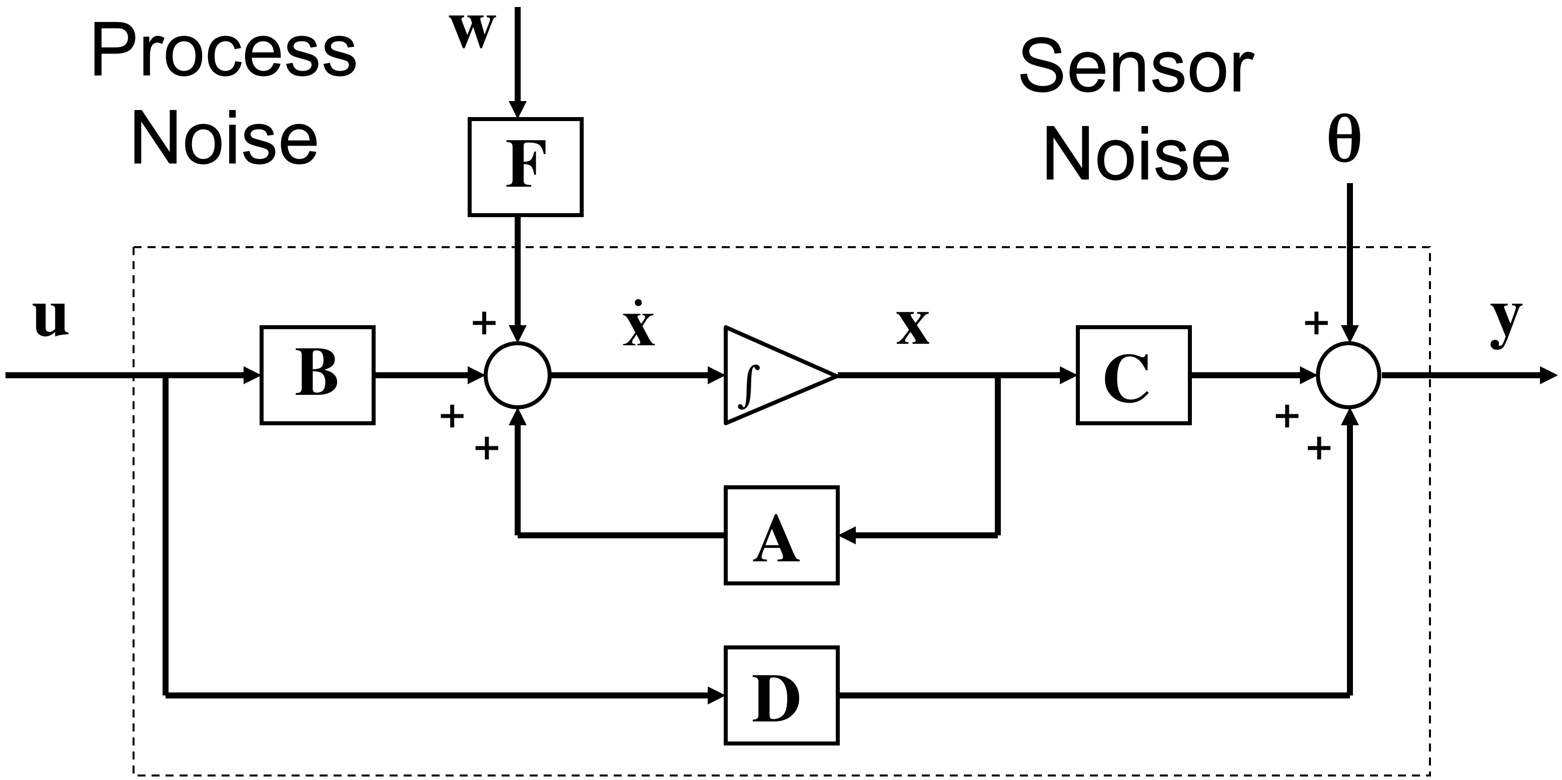
To see this distinction more clearly, we can see how the two disturbances are incorporated into our state-space equations:

$$\underset{[N\times 1]}{\dot{\mathbf{x}}} = \underset{[N\times N]}{\mathbf{A}}\,\underset{[N\times 1]}{\mathbf{x}} + \underset{[N\times M]}{\mathbf{B}}\,\underset{[M\times 1]}{\mathbf{u}} + \underbrace{\underset{[N\times L]}{\mathbf{F}}\,\underset{[L\times 1]}{\mathbf{w}}}_{\text{Process Noise}}$$

$$\underset{[P\times 1]}{\mathbf{y}} = \underset{[P\times N]}{\mathbf{C}}\,\underset{[N\times 1]}{\mathbf{x}} + \underset{[P\times M]}{\mathbf{D}}\,\underset{[M\times 1]}{\mathbf{u}} + \underbrace{\underset{[P\times 1]}{\boldsymbol{\theta}}}_{\text{Sensor Noise}}$$

Modeling Disturbances - III

In block diagram form we have:



Notice that there is no direct path for θ to get into the states $\mathbf{x}$.

Modeling Disturbances - IV

In general, the disturbance inputs which make up the elements of $\mathbf{w}$ and θ are either deterministic or stochastic and are typically considered uncertain.

Deterministic examples: constant, step, ramp, sinusoidal, periodic…

Stochastic examples: random, white noise, colored noise, Markov sequence…

Modeling Disturbances - V

In order to properly compensate for these disturbances, some information is usually required to characterize the disturbance:

Deterministic: What is the form, period, or frequency?

Stochastic: What are the statistical properties?

Modeling Disturbances - VI

Disturbance rejection and/or accommodation techniques are fairly well developed.

For more information see:

– Robust Servomechanism Design
– Internal Model Principle
– Feedforward Control
– Adaptive Control

Modeling References - I

Reference input tracking in a state-space system is similar to the classical feedback control system.

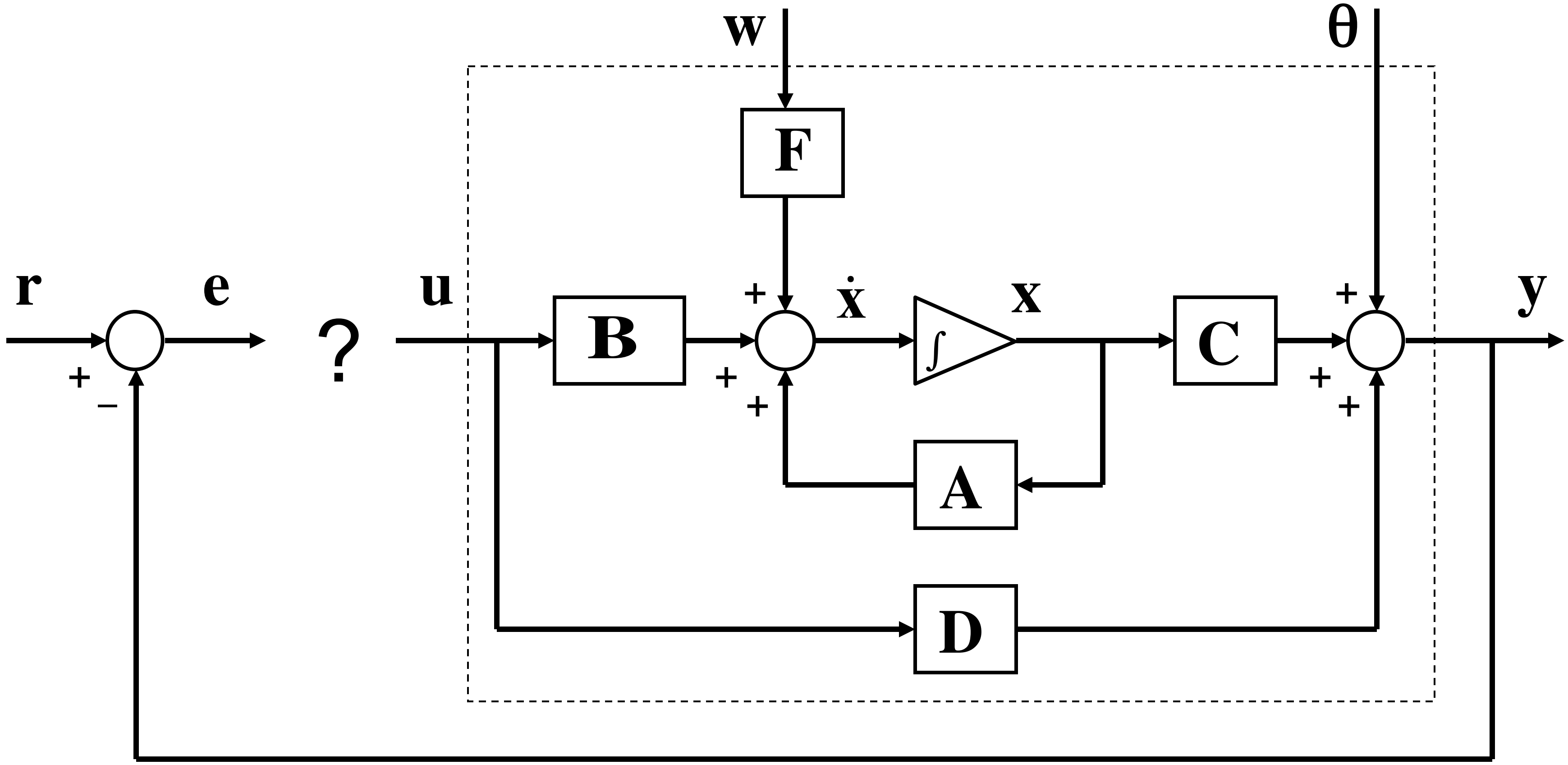
The <u>tracking error</u> vector is defined as:

$$\mathbf{e}(t) = \mathbf{r}(t) - \mathbf{y}(t)$$

Non-zero error indicates that the output $\mathbf{y}$ is not tracking the reference $\mathbf{r}$.

Modeling References - II

The reference tracking block diagram is



Modeling References - III

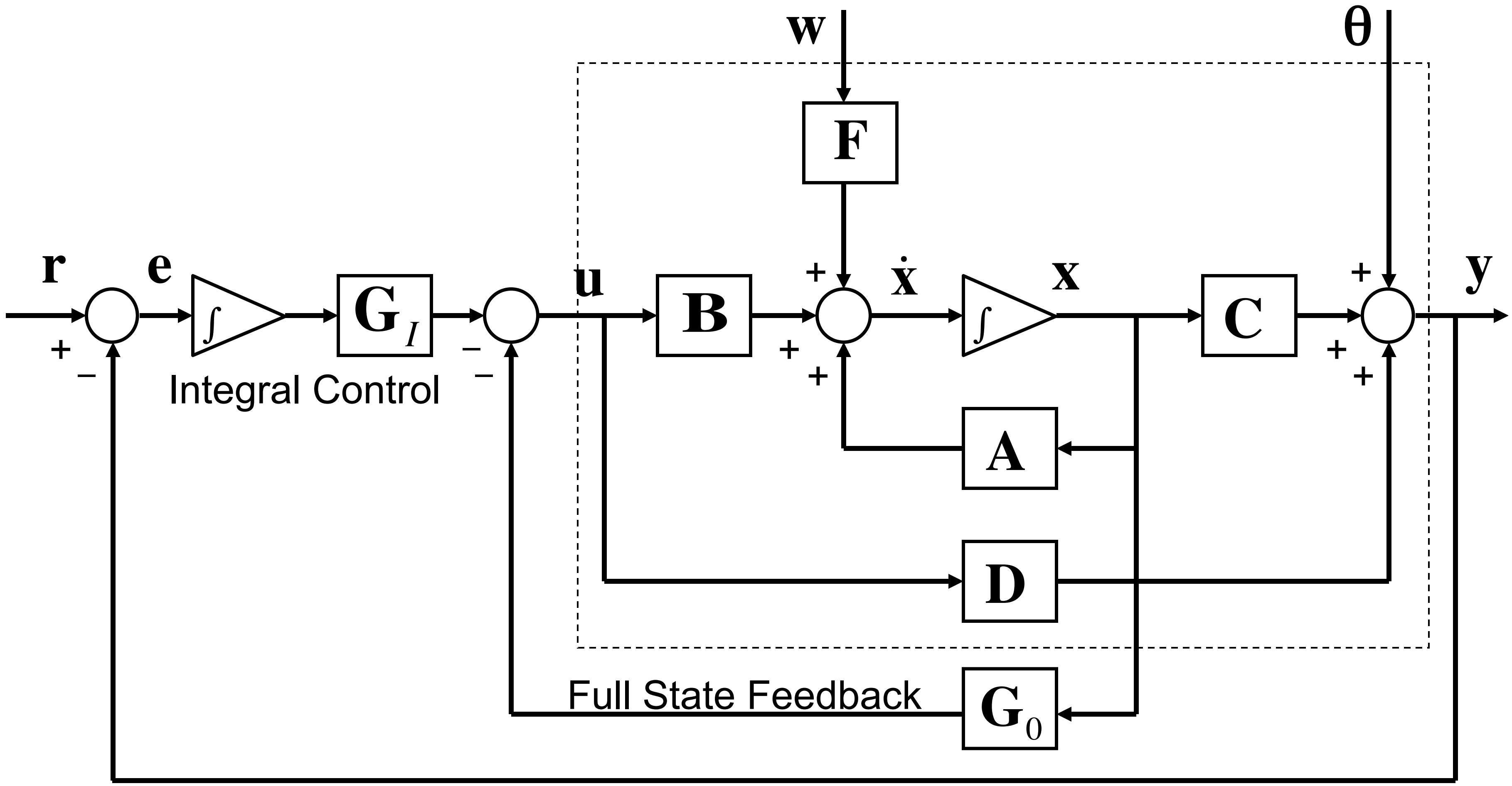
How do we connect the P-dimensional tracking error vector $\mathbf{e}$ to the M-dimensional control vector $\mathbf{u}$?

We can borrow from the classical solutions such as PID. In particular, integral control is very useful for many reference tracking problems so we will explore that in detail.

This analysis is not limited to integral control only!

Modeling References - IV

The block diagram for integral control is:



Modeling References - V

Notice that the control law includes both integral control on the tracking error as well as full state feedback:

$$\mathbf{u}(t) = \underbrace{-\mathbf{G}_0\mathbf{x}(t)}_{\substack{\text{State}\\ \text{Feedback}}} - \underbrace{\mathbf{G}_I\mathbf{x}_I(t)}_{\substack{\text{Integral}\\ \text{Control}}}$$

where $\mathbf{x}_I$ satisfies the differential equation:

Modeling References - VI

Because the integral control action introduces new dynamics, we must augment the state-space system in order to analyze the dynamics or simulate the response.

$$\underbrace{\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_I \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_I \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{D} \end{bmatrix} \mathbf{u}}_{\text{Open-Loop Augmented State Equations}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix} \boldsymbol{\theta}}_{\text{Exogenous Inputs}}$$

Modeling References - VII

Using the augmented state vector, the control signal is defined to be:

$$\mathbf{u}(t) =$$

We can now use our standard tools for checking controllability and placing the closed-loop poles, but we have to use the augmented system matrices!

Modeling References - VIII

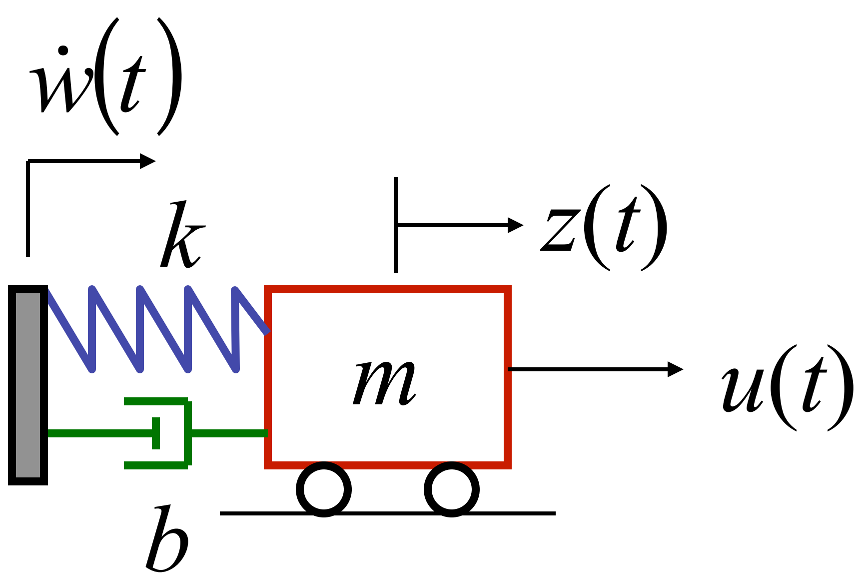
Substituting the control signal into the augmented system equations, the closed-loop system is:

$$\underbrace{\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_I \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{G}_0 & -\mathbf{B}\mathbf{G}_I \\ -\mathbf{C} + \mathbf{D}\mathbf{G}_0 & \mathbf{D}\mathbf{G}_I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_I \end{bmatrix}}_{\text{Closed-Loop Augmented State Equations}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix} \boldsymbol{\theta}}_{\text{Exogenous Inputs}}$$

This is a new result! After closing the loop, we still have inputs to our system in the form of exogenous inputs.

Tracking Control Example - I

Let's return to the damped mass-spring oscillator, except now we will include a base excitation velocity.



States:

$$x_1(t) = z(t)$$

$$x_2(t) = \dot{z}(t)$$

Parameters:

$$\omega^2 = \left(\frac{k}{m}\right)$$

$$2\zeta\omega = \left(\frac{b}{m}\right)$$

Tracking Control Example - II

The first-order differential equations are:

$$\dot{x}_1 = \dot{z} = x_2$$

$$\dot{x}_2 = \ddot{z} = 2\zeta\omega\left(\dot{w} - \dot{z}\right) + \omega^2\left(w - z\right) + \left(\tfrac{1}{m}\right)u$$

We need one additional state to provide the base displacement:

$$x_3 = w \quad \Rightarrow \quad \dot{x}_3 = \dot{w}$$

Tracking Control Example - III

The resulting state-space system with mass position as the output is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\omega^2 & -2\zeta\omega & \omega^2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 2\zeta\omega \\ 1 \end{bmatrix} \dot{w}(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Tracking Control Example - IV

For the following parameters:

$$m = 1\ (kg) \qquad k = 100\ \left(\tfrac{N}{M}\right) \qquad b = 0.1\ \left(\tfrac{Ns}{M}\right)$$

Consider the design specifications:

- Minimize the settling time to a step input r
- Zero steady-state error to a 1 cm step input r and a 1 mm step disturbance at the base
- Control force no greater than 10 N

Tracking Control Example - V

Using the integral control results from S20, the augmented state equations are given by:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_I(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & -2\zeta\omega & \omega^2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_I(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t) + \begin{bmatrix} 0 \\ 2\zeta\omega \\ 1 \\ 0 \end{bmatrix} \dot{w}(t)$$

The feedback control law is:

$$u(t) = -\begin{bmatrix} G_{01} & G_{02} & G_{03} & G_I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_I(t) \end{bmatrix}$$

Tracking Control Example - VI

Substituting the control law into the augmented state matrices yields:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_I(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 - \frac{1}{m}G_{01} & -2\zeta\omega - \frac{1}{m}G_{02} & \omega^2 - \frac{1}{m}G_{03} & -\frac{1}{m}G_I \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_I(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2\zeta\omega \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{w}(t) \end{bmatrix}$$

We can augment the output equations to directly provide access to the control force

$$\mathbf{y}(t) = \begin{bmatrix} z(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -G_{01} & -G_{02} & -G_{03} & -G_I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_I(t) \end{bmatrix}$$

Tracking Control Example - VII

Due to the simple structure of the augmented $\mathbf{A}$ matrix, it is easy to solve for the characteristic equation:

$$\begin{vmatrix} s & -1 & 0 & 0 \\ \omega^2 + \frac{G_{01}}{m} & s + 2\zeta\omega + \frac{G_{02}}{m} & -\omega^2 + \frac{G_{03}}{m} & \frac{G_I}{m} \\ 0 & 0 & s & 0 \\ 1 & 0 & 0 & s \end{vmatrix} = s^4 + \left(2\zeta\omega + \tfrac{G_{02}}{m}\right)s^3 + \left(\omega^2 + \tfrac{G_{01}}{m}\right)s^2 - \left(\tfrac{G_I}{m}\right)s$$

Given a desired CE, the gains are:

$$\begin{aligned} a_3 &= 2\zeta\omega + \tfrac{G_{02}}{m} & &\Rightarrow & G_{02} &= m\left(a_3 - 2\zeta\omega\right) \\ a_2 &= \omega^2 + \tfrac{G_{01}}{m} & &\Rightarrow & G_{01} &= m\left(a_2 - \omega^2\right) \\ a_1 &= -\left(\tfrac{G_I}{m}\right) & &\Rightarrow & G_I &= -ma_1 \end{aligned}$$

Tracking Control Example - VIII

Notice that G_{03} is arbitrary so we will choose it to be zero. Also notice that one of the poles must be at the origin.

Even though this system is not controllable, we can still choose the following closed-loop poles:

$$\lambda_{CL} = \{-20, \quad -20 \pm j20\}$$

The feedback gains are:

$$\mathbf{G} = \begin{bmatrix} 1500 & 59.9 & 0 & -16000 \end{bmatrix}$$

Tracking Control Example - IX

The desired mass position is:

$$r = 1\ (cm)\left(\tfrac{1m}{100cm}\right) = 0.01\ (m)$$

the base disturbance displacement is

$$\dot{w} = 0 \qquad w(0) = 1\ (mm)\left(\tfrac{1m}{1000mm}\right) = 0.001\ (m)$$

then in steady-state, the total spring deflection will be

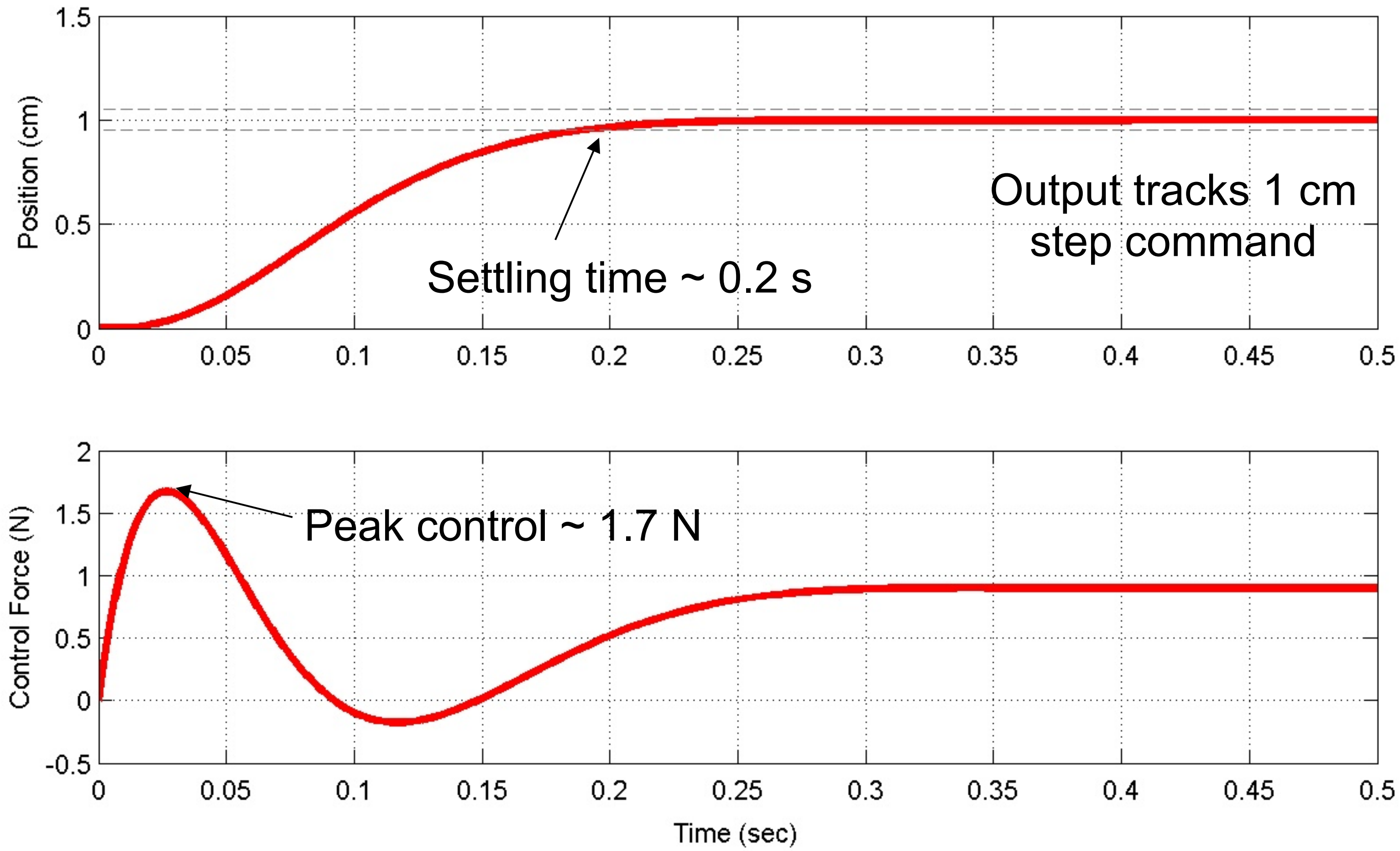
$$(z_{SS} - w_{SS}) = 0.01\ (m) - 0.001\ (m) = 0.009\ (m) = 9\ (mm)$$

The steady state control force should be:

$$u_{SS} = k(z_{SS} - w_{SS}) = 100\,\tfrac{N}{m} \times 0.009m = 0.9N$$

Tracking Control Example - X

The closed-loop response is:



Tracking Control Example - XI

We are not close to violating the authority limit so let’s move the poles further to the left.

The authority limit is reached for the following poles:

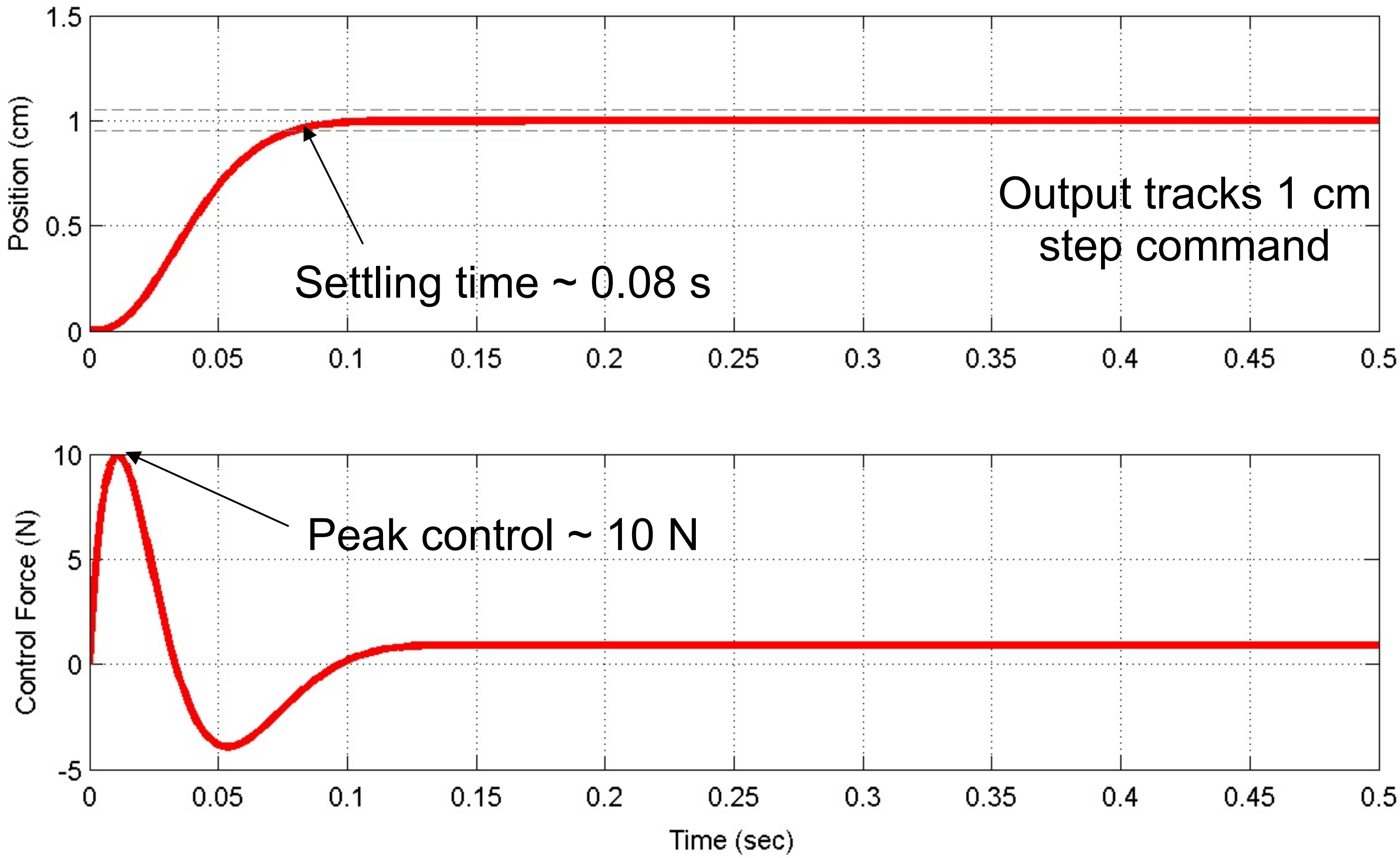
$$\lambda_{CL} = \{-48, \quad -48 \pm j48\}$$

The corresponding gains are:

$$\mathbf{G} = \begin{bmatrix} 9116 & 143.9 & 0 & -221184 \end{bmatrix}$$

Tracking Control Example - XII

The closed-loop response is:



Summary

Reference inputs and external disturbances can be incorporated into the state-space model using additional input matrices.

Steady-state error can be eliminated using integral control.

Dynamics introduced by the tracking controller must be included as augmented states.