Advanced Control Engineering

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

Intro. To Optimal Control - I

The state-space design methodology we have developed allows us to choose the location of the closed-loop poles through observer design and full-state feedback.

The pole locations are typically chosen to satisfy design criteria such as settling time, peak effort, peak response, etc.

Intro. To Optimal Control - II

Compared to classical LTI feedback control design methods such as loop-shaping, the state-space method has several advantages:

- Generally easier to place closed-loop poles than to shape the loop transfer function
- Relatively straightforward means to handle multiple inputs and multiple outputs

Intro. To Optimal Control - III

We have traded one problem (How do we shape the loop TF?) for a new problem (Where do we place the poles?).

We have already seen that it is not easy to translate all design specifications directly to pole locations.

Even if we have a feasible solution, i.e. one that satisfies all the design specs, do we ever really know if it is the BEST solution?

Intro. To Optimal Control - IV

To answer this question, we have to define what we mean by "best".

This is the motivation for the development of "Optimal Controls".

Optimal Control enables the design of compensators that are the BEST possible w.r.t. some cost function.

Intro. To Optimal Control - V

Notice that now we are trading the problem of pole placement for a new set of problems:

- How do we choose the cost function?
- Once we have a cost function, how do we find the optimal state-feedback control gains?

Optimization Basics - I

Let's start by reviewing the general optimization problem.

Given some cost function $V(\mathbf{p})$, where $\mathbf{p}$ is a matrix of design parameters, the generic unconstrained optimization problem is mathematically given by:

Find the Optimal set of design parameters

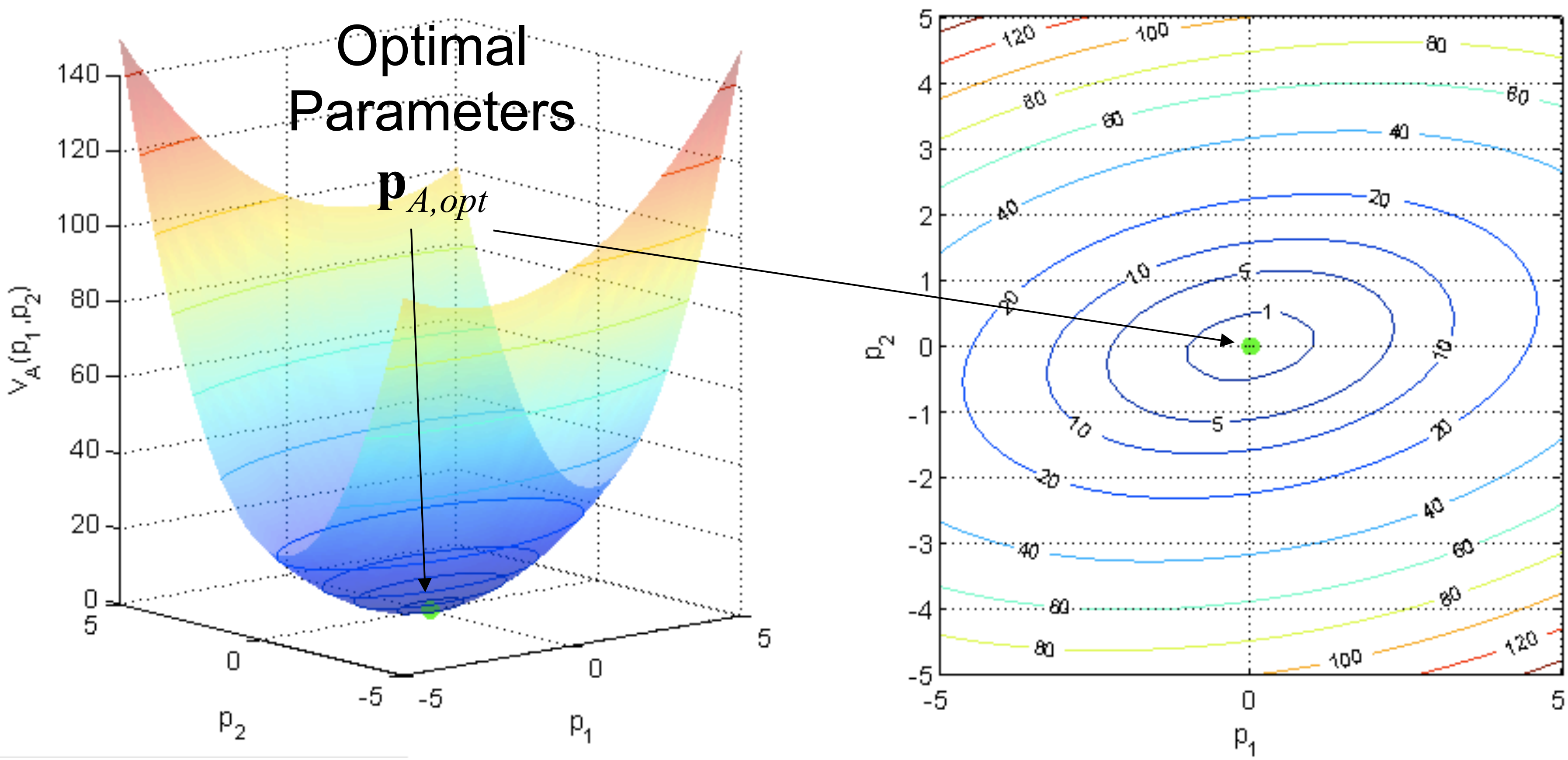
$$\boxed{\mathbf{p}_{opt}} = \min_{\mathbf{p}} \boxed{V(\mathbf{p})}$$

Given this Cost Function

Optimization Basics - II

Consider the example cost function:

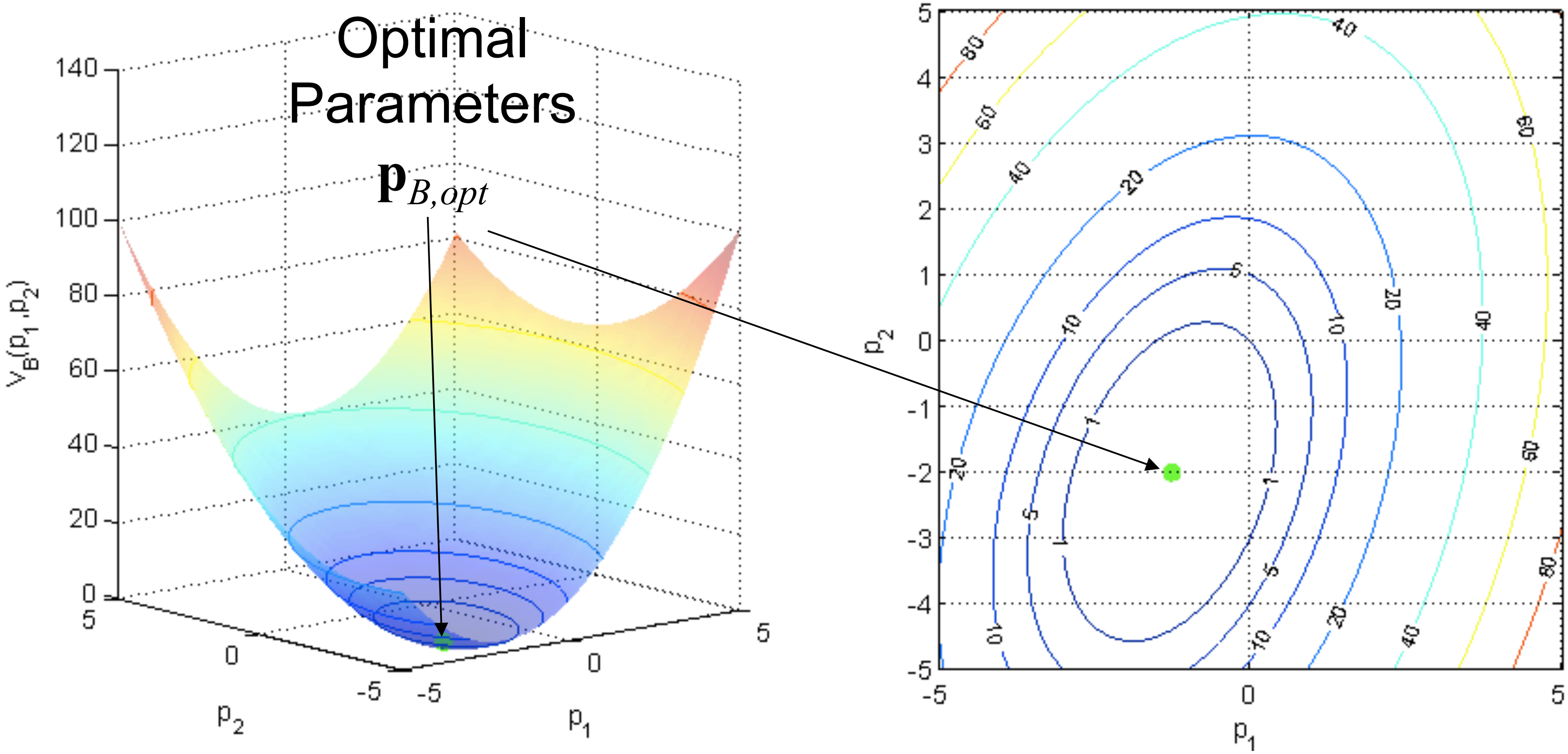
$$V_A(\mathbf{p}) = p_1^2 - p_1 p_2 + 4 p_2^2$$



Optimization Basics - III

Consider a 2[nd] example cost function:

$$V_B(\mathbf{p}) = 2p_1^2 - p_1 p_2 + p_2^2 + 3p_1 + 3p_2 + 1$$



Optimization Basics - IV

Important observations:

- Both cost functions are quadratic in both parameters

- Both cost functions have a <u>unique global</u> minimum (optimal solution)

- Both cost functions are ellipsoidal

Optimization Basics - V

For linear combinations of two quadratic cost functions:

$$V(\mathbf{p}) = \alpha V_A(\mathbf{p}) + \beta V_B(\mathbf{p})$$

Then $V(\mathbf{p})$:

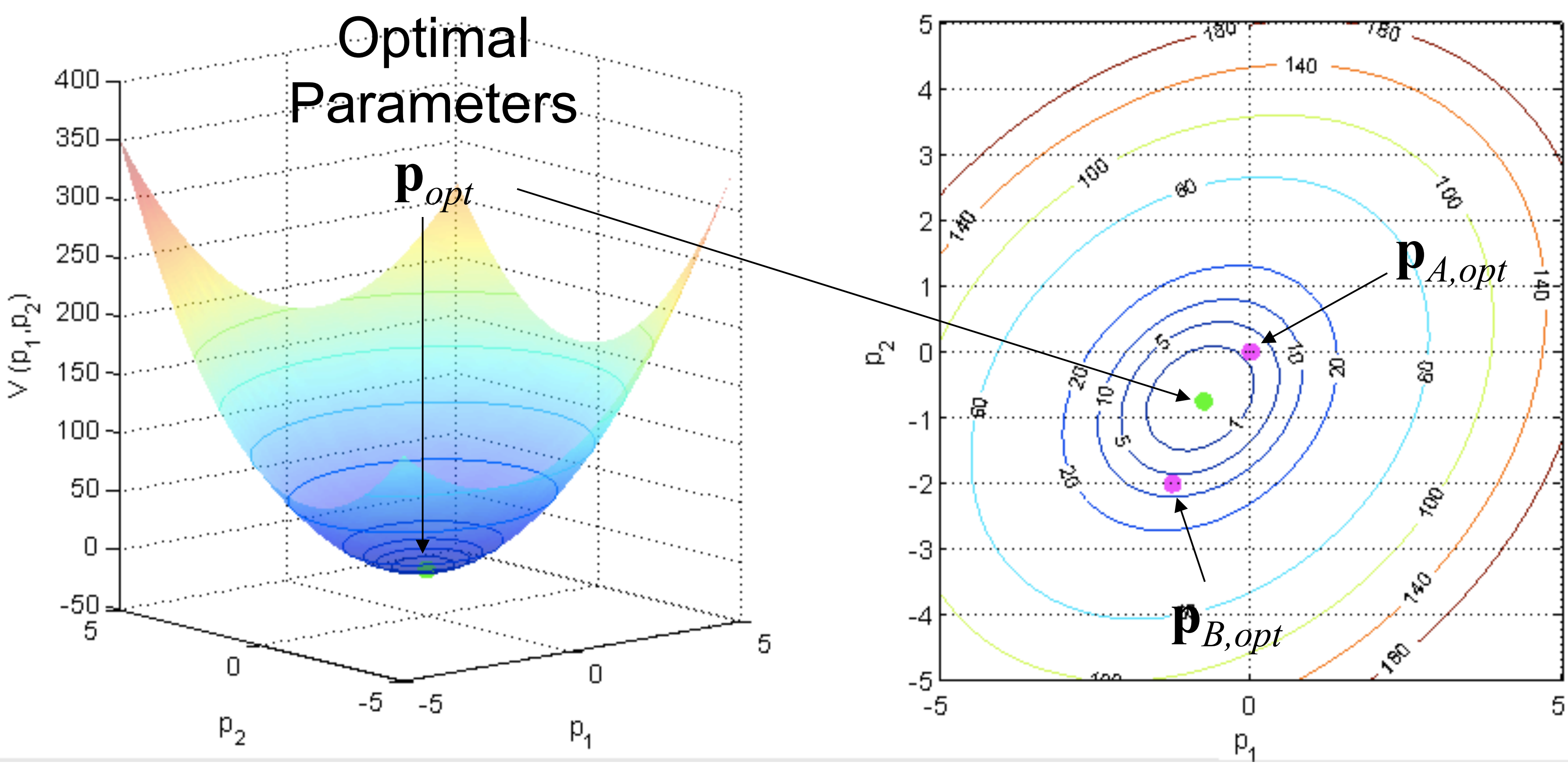
- Is quadratic in both parameters
- Has only a single optimum
- Is an ellipsoidal surface

The fundamental shape of the surface has not changed.

Optimization Basics - VI

Consider the cost function:

$$V(\mathbf{p}) = V_A(\mathbf{p}) + 2V_B(\mathbf{p})$$



Optimization Basics - VII

The optimal solution for $V(\mathbf{p})$ lies somewhere in *between* the optimal solutions for $V_A(\mathbf{p})$ and $V_B(\mathbf{p})$.

At the extreme values:

$$\alpha >> \beta \qquad V(\mathbf{p}) \approx \alpha V_A(\mathbf{p})$$

$$\beta >> \alpha \qquad V(\mathbf{p}) \approx \beta V_B(\mathbf{p})$$

the optimal solutions converge to the optimal solutions for $V_A(\mathbf{p})$ or $V_B(\mathbf{p})$.

Introduction to LQR - I

What cost function should we choose for our control problem?

In general, we want to choose a cost function that:

- Has a unique global minimum
- Is positive definite (or strictly positive)
- Has practical use for a wide variety of control problems

Introduction to LQR - II

Consider the regulator problem using full-state feedback:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$ State Equations

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t)$$ Full-State Feedback Control

$$\mathbf{x}(t) = e^{(A-BG)t}\mathbf{x}(0)$$ Closed-Loop State Response

$$\mathbf{u}(t) = -\mathbf{G}e^{(A-BG)t}\mathbf{x}(0)$$ Control Response

Introduction to LQR - III

Now consider the following cost function for the optimal control problem:

$$V(\mathbf{G}) = \int_{t_0}^{t_F} \underbrace{\mathbf{x}^T(\tau)\mathbf{Q}\mathbf{x}(\tau)}_{\substack{\text{Quadratic term} \\ \text{related to the} \\ \text{state response}}} + \underbrace{\mathbf{u}^T(\tau)\mathbf{R}\mathbf{u}(\tau)}_{\substack{\text{Quadratic term} \\ \text{related to control} \\ \text{effort}}} d\tau$$

The knobs that we can tweak in the design process are:

$\mathbf{Q}$ = The State weighting matrix

$\mathbf{R}$ = The Control Effort weighting matrix

Introduction to LQR - IV

How do we visualize the meaning of the quadratic forms $\mathbf{x}^T\mathbf{Q}\mathbf{x}$ and $\mathbf{u}^T\mathbf{R}\mathbf{u}$?

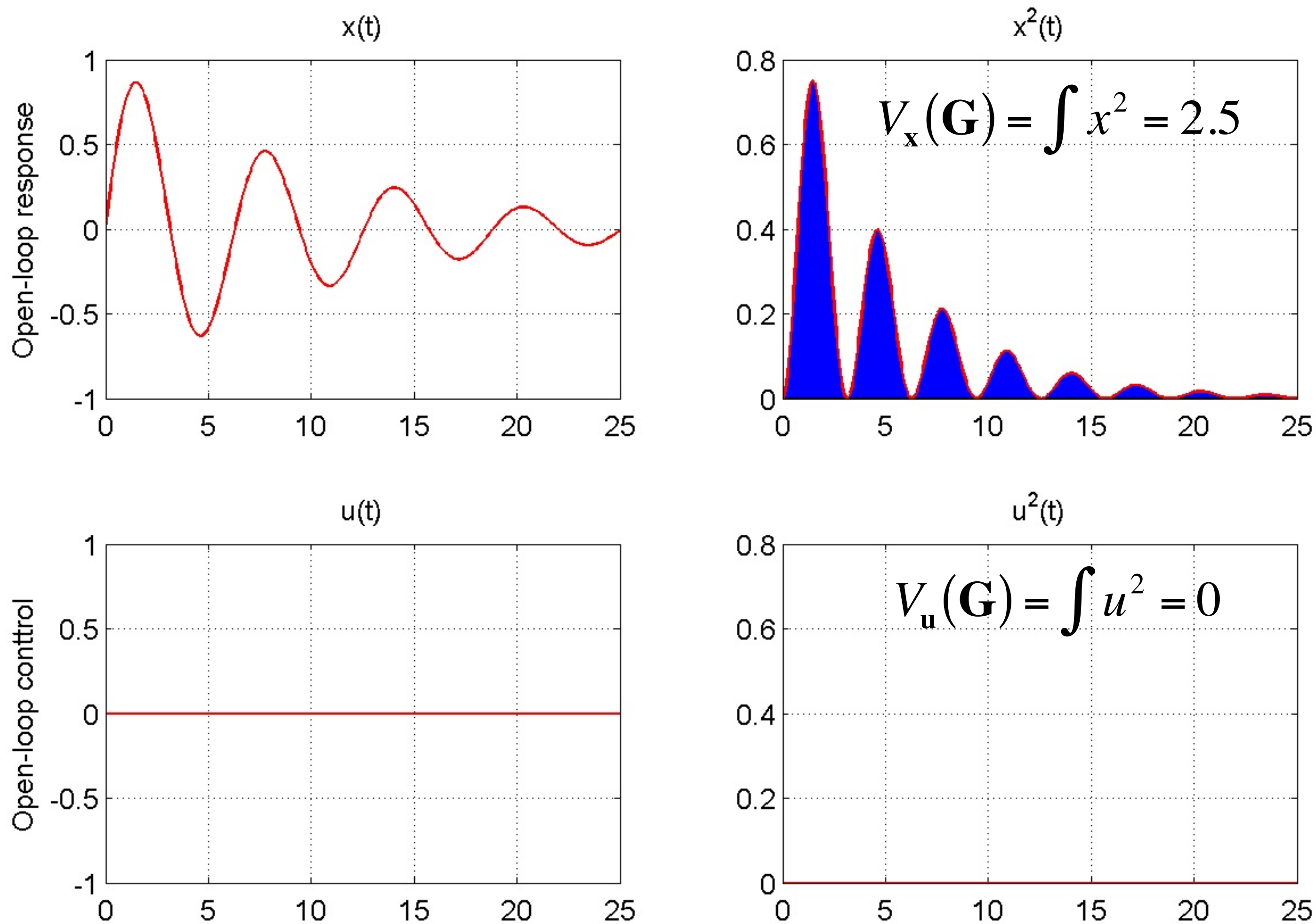
Let's evaluate each term separately for an open-loop and a closed-loop response of a second-order system:

$$\underset{[1\times 1]}{V_{\mathbf{x}}(\mathbf{G})} = \int_{t_0}^{t_F} \underset{[1\times N]}{\mathbf{x}^T(\tau)} \underset{[N\times N]}{\mathbf{Q}} \underset{[N\times 1]}{\mathbf{x}(\tau)} d\tau$$

$$\underset{[1\times 1]}{V_{\mathbf{u}}(\mathbf{G})} = \int_{t_0}^{t_F} \underset{[1\times M]}{\mathbf{u}^T(\tau)} \underset{[M\times M]}{\mathbf{R}} \underset{[M\times 1]}{\mathbf{u}(\tau)} d\tau$$

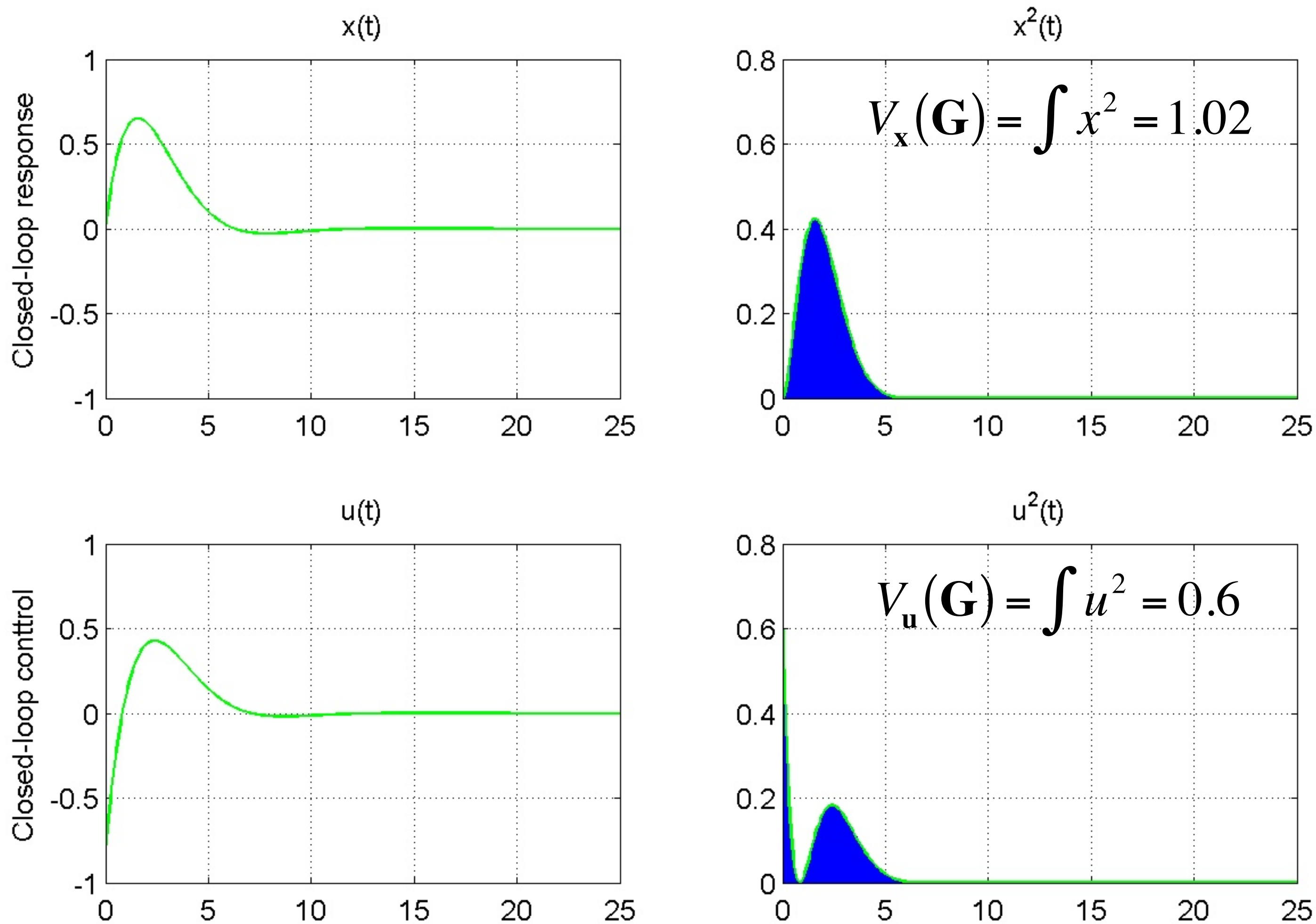
Introduction to LQR - V

Open-Loop evaluation of cost:



Introduction to LQR - VI

Closed-Loop evaluation of cost:



Introduction to LQR - VII

The terms in this cost function have physical meaning.

$$V(\mathbf{G}) = \int_{t_0}^{t_F} \overbrace{\mathbf{x}^T(\tau)\mathbf{Q}\mathbf{x}(\tau)}^{\text{"Size" of State Response}} + \underbrace{\mathbf{u}^T(\tau)\mathbf{R}\mathbf{u}(\tau)}_{\text{"Size" of Control Effort}} d\tau$$

Varying the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ allows us to trade off the “size” of the state response with the “size” of the control effort.

Introduction to LQR - VIII

When $|\mathbf{Q}| << |\mathbf{R}|$, then the cost function will minimize the amount of control effort needed (gains will be small).

When $|\mathbf{R}| << |\mathbf{Q}|$, then the cost function will minimize the state response without regard to the amount of control effort that we need (gains will be large).

Introduction to LQR - IX

When **Q** and **R** are chosen to be positive definite symmetric matrices, then the problem of determining state-feedback control gains **G** that minimize the following cost function:

$$V(\mathbf{G}) = \int_{t_0}^{t_F} \mathbf{x}^T(\tau)\mathbf{Q}\mathbf{x}(\tau) + \mathbf{u}^T(\tau)\mathbf{R}\mathbf{u}(\tau)\,d\tau$$

is called the **L**inear **Q**uadratic **R**egulator problem or the Linear Optimal Regulator problem.

Summary

Linear Quadratic control is a method for designing control laws that are optimal with respect to the cost function:

$$V(\mathbf{G}) = \int_{t_0}^{t_F} \mathbf{x}^T(\tau)\mathbf{Q}\mathbf{x}(\tau) + \mathbf{u}^T(\tau)\mathbf{R}\mathbf{u}(\tau)d\tau$$

The weighting matrices allow us to trade off state response for control effort.

How do we compute the optimal gains?