Course Outline - 1st Half

- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response, Matrix Exponential
- Transfer Functions & Eigenvalues
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

Second Order Poles - I

Let’s investigate the following second-order system:

$$H(s) = \left(\frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2} \right)$$

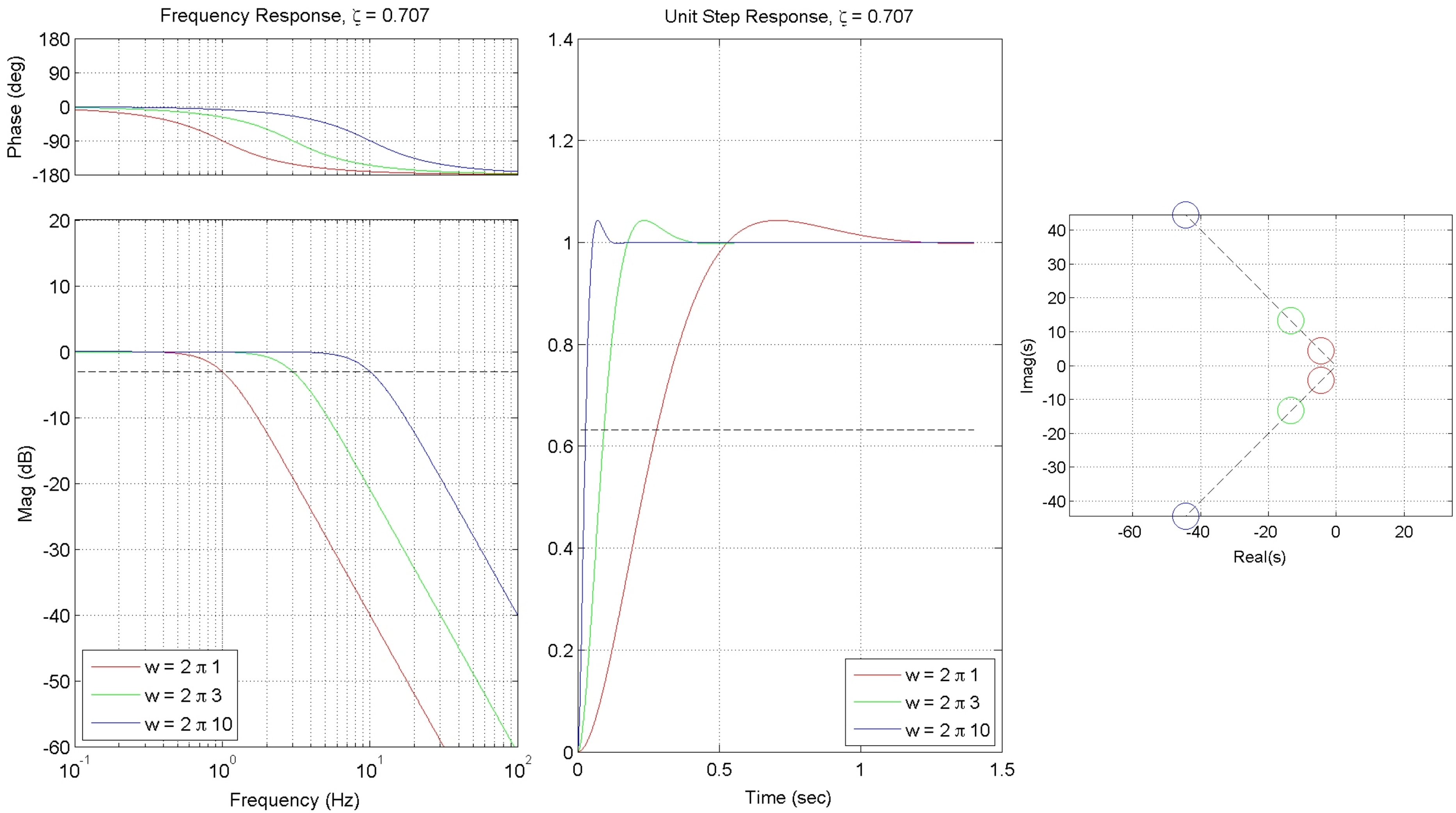
Plot the pole locations, frequency response, and step response for:

$$\xi = \{0.3 \ \ \text{and} \ \ 0.707\}$$

$$\omega = 2\pi f \quad \Rightarrow \quad f = \{1, \ \ 3, \ \ \text{and} \ \ 10\} \quad (Hz)$$

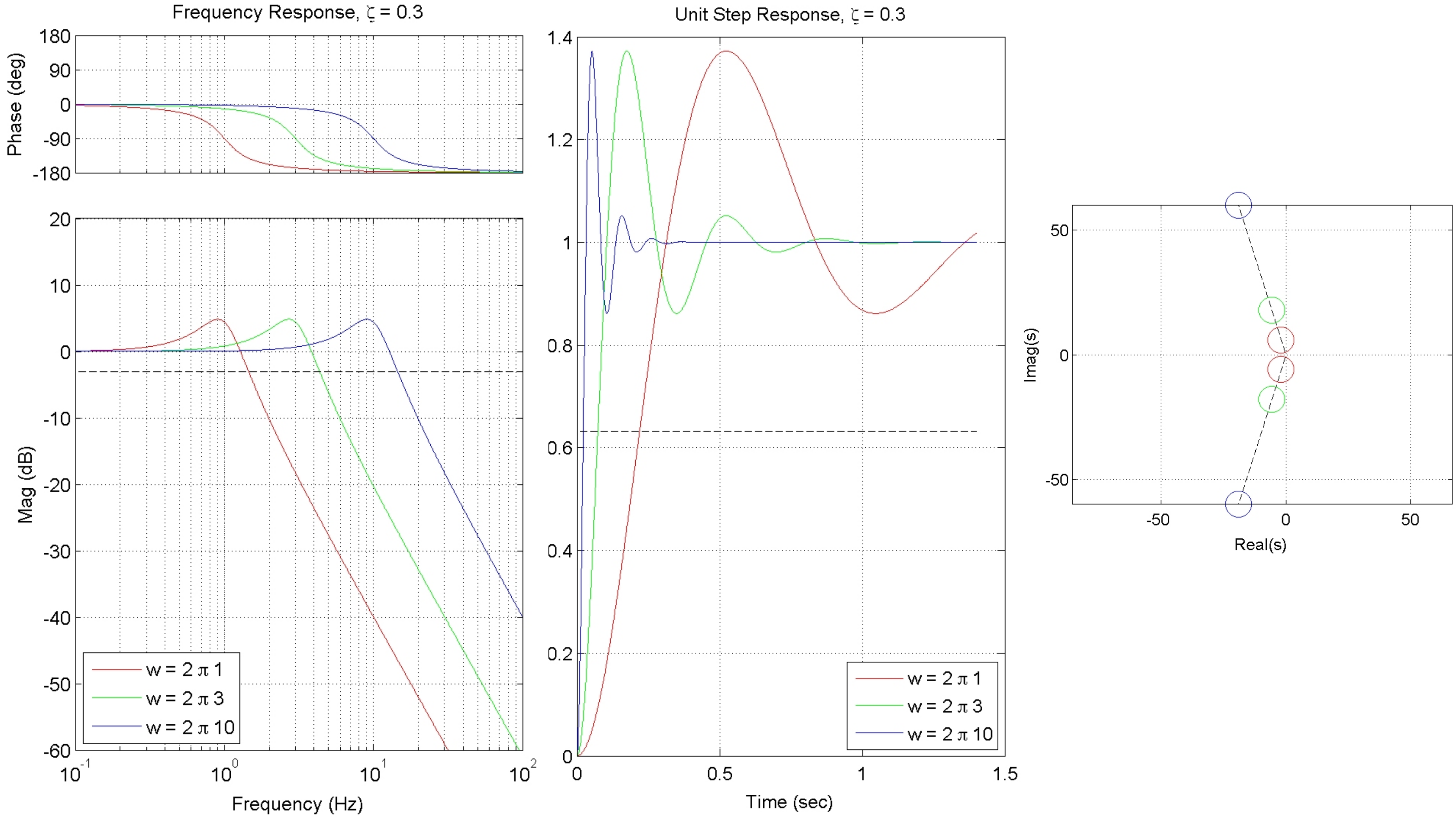
Second Order Poles - II

Critical damping case:



Second Order Poles - III

Lightly damped case:



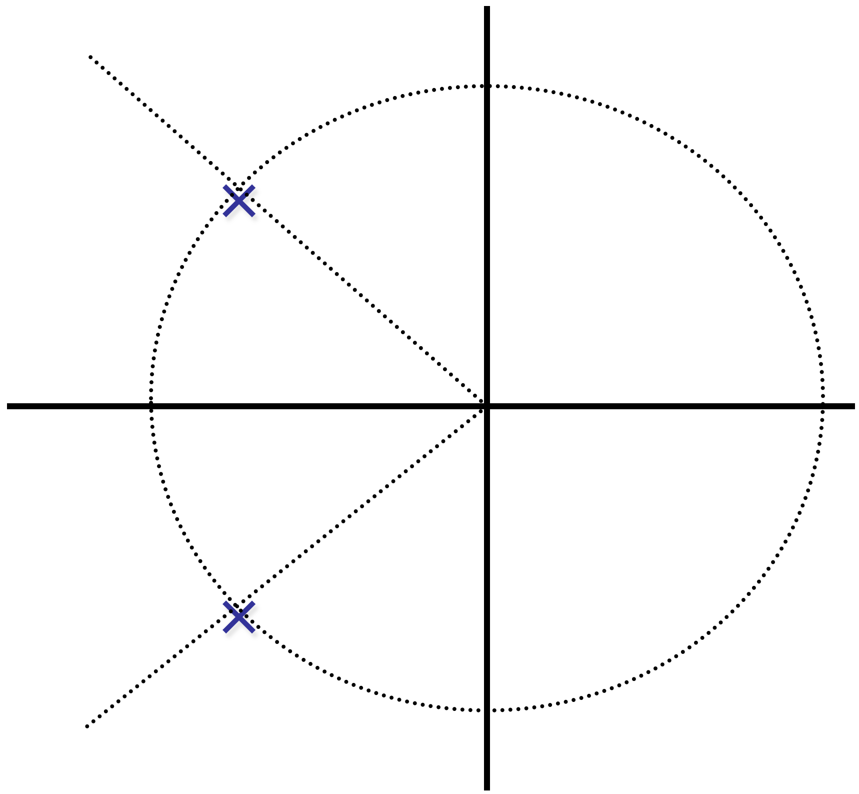
Second Order Poles - IV

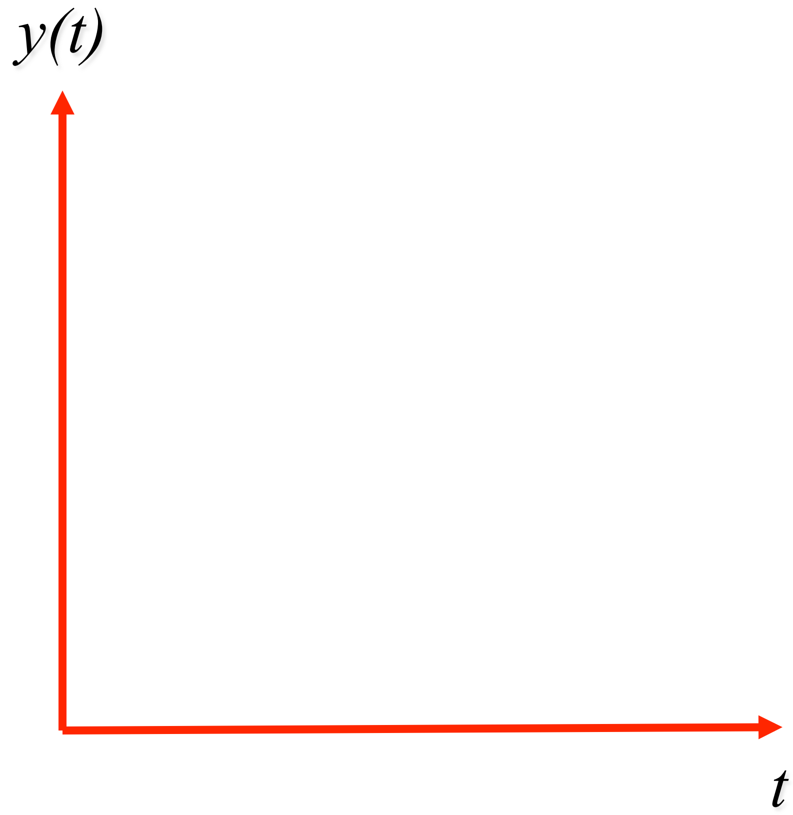
Second-order Observations:

- Increasing the pole frequency at constant damping ratio increases the *bandwidth* AND decreases the *time constant*
- Poles close to the imaginary axis have a damped oscillatory response AND a longer time constant
- Poles far from the imaginary axis have an exponential response AND a short time constant

In-Class Assignment

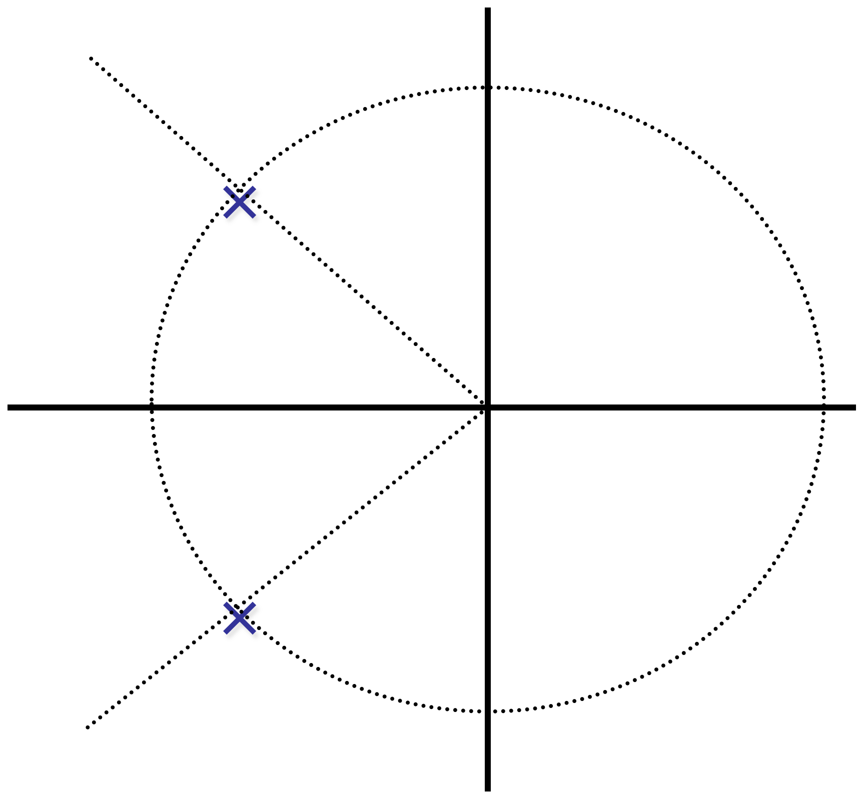
A second-order system has the pole locations shown in the figure, sketch the step response of the system.





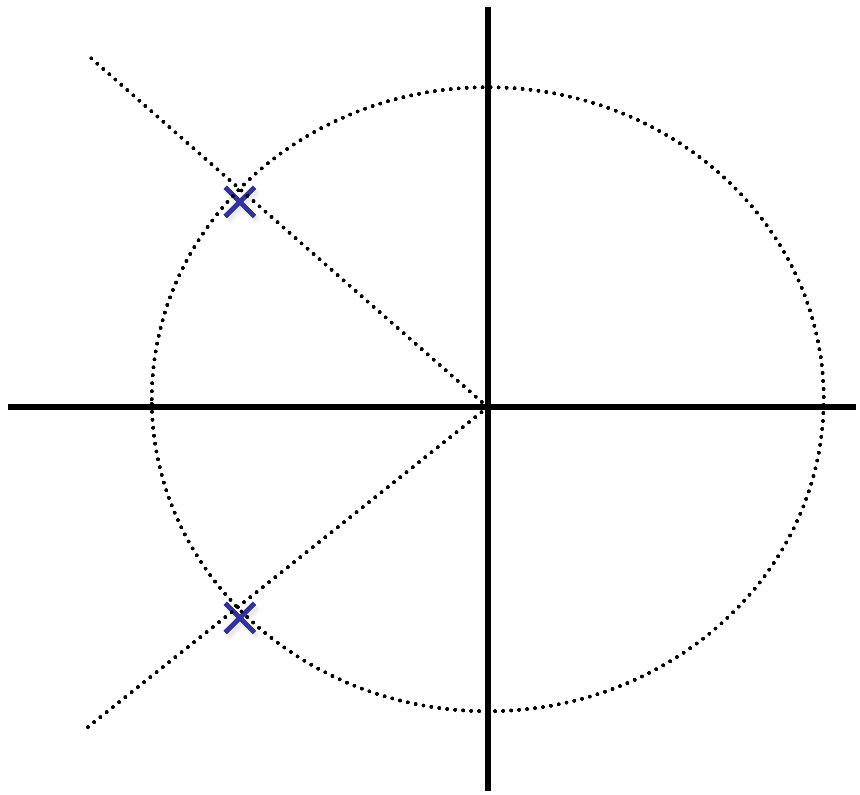
In-Class Assignment

Where would you move the poles if you wanted to increase the <u>damping</u> of the response?



In-Class Assignment

Where would you move the poles if you wanted to increase the speed (natural frequency) of the response?



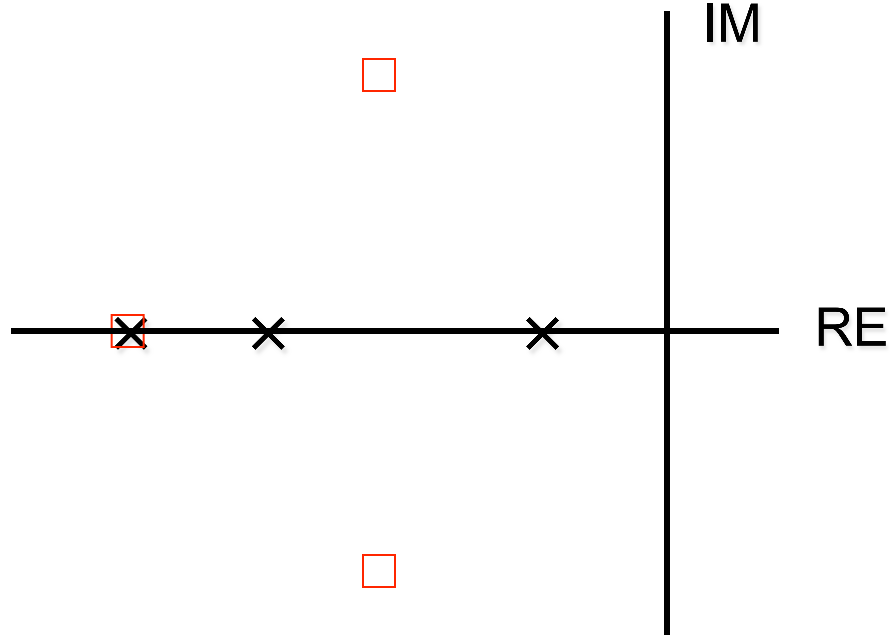
Virginia Tech

Intro to FSF - I

Pole placement is a feedback control design technique which enables, under certain restrictions, the arbitrary placement of the closed-loop system poles.

× - open-loop pole

□ desired closed-loop pole



Introduction to FSF - II

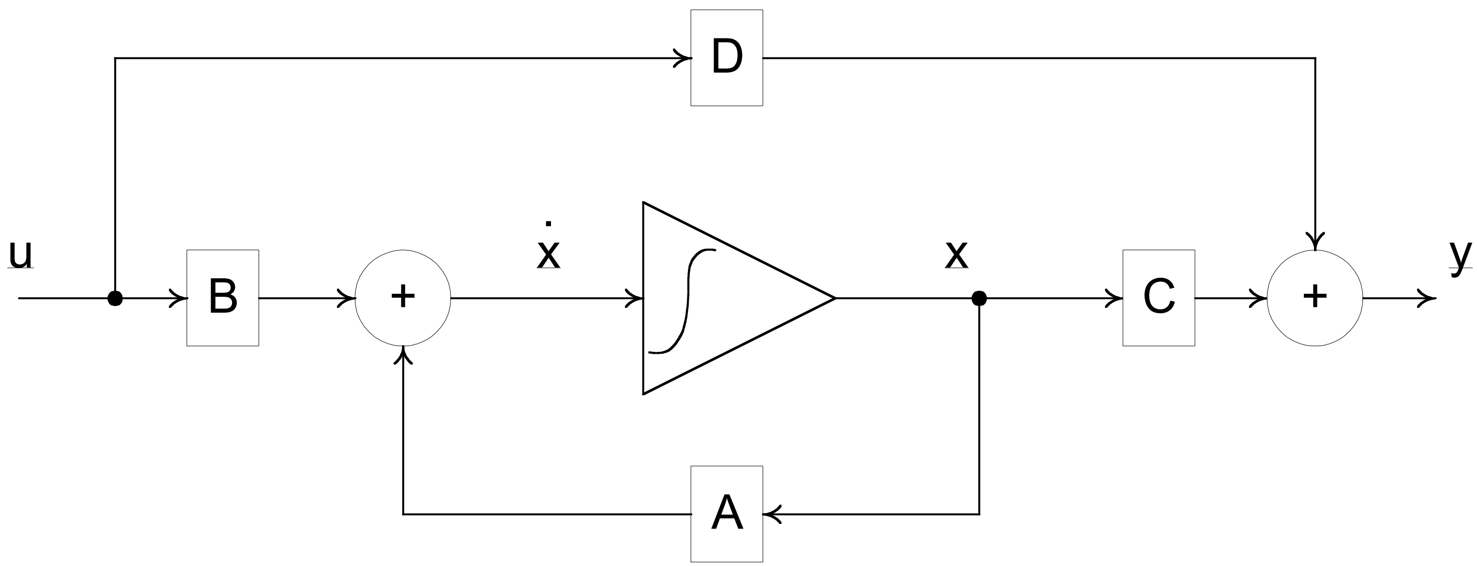
We know that the state transition matrix, eigenvalues, eigenvectors, and therefore the system response is dependent on the $\mathbf{A}$ matrix

It stands to reason that if we can manipulate the $\mathbf{A}$ matrix, we should be able to change (control) the dynamics of the system

Intro to FSF - III

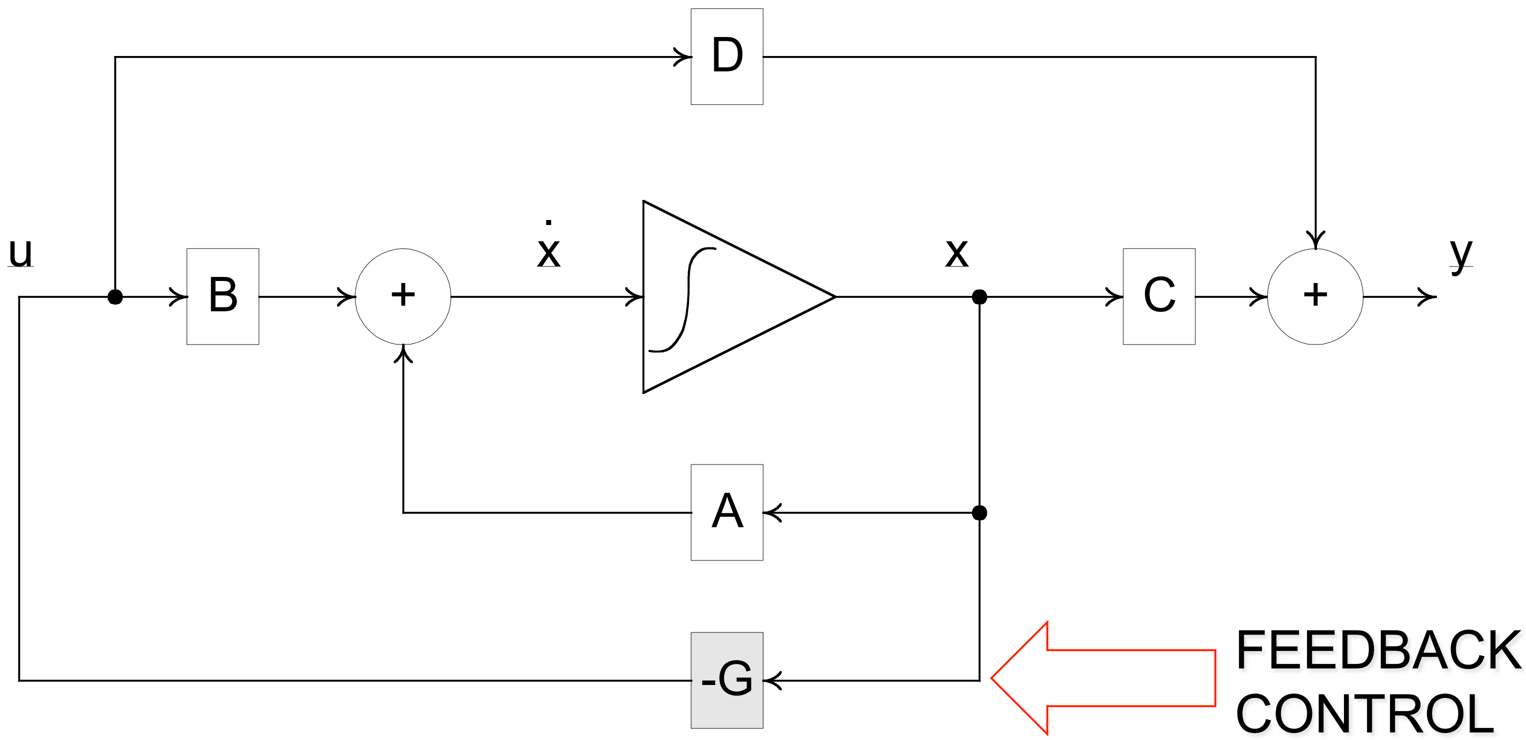
How can we manipulate the **A** matrix?

First consider a block diagram of our state-space system:



Intro to FSF - IV

What if we could access <u>all of the states</u> to use for feedback?



Intro to FSF - V

If we have all of the states available for feedback, then let the feedback control law be a matrix of static (i.e. constant) gains multiplied by our state vector:

$$\underset{[M\times 1]}{\mathbf{u}(t)} = -\underset{[M\times N]}{\mathbf{G}}\,\underset{[N\times 1]}{\mathbf{x}(t)}$$

How does this control law change the system dynamics?

Intro to FSF - VI

Substitute into the state equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\left(-\mathbf{G}\mathbf{x}(t)\right)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\left(-\mathbf{G}\mathbf{x}(t)\right)$$

After combining terms we see that the closed-loop state matrix is a function of the control parameters:

$$\dot{\mathbf{x}}(t) = \underbrace{\left[\mathbf{A} - \mathbf{B}\mathbf{G}\right]}_{\substack{\text{Closed-Loop} \\ \text{State Matrix}}}\mathbf{x}(t)$$

Intro to FSF - VII

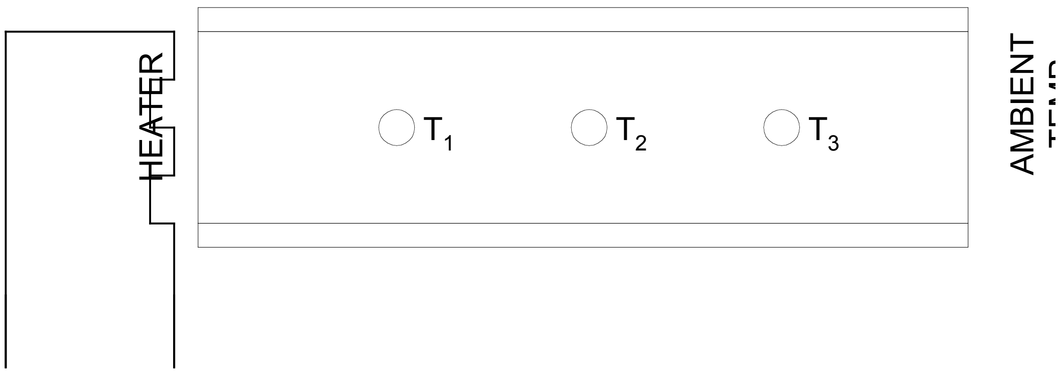
Now we have a set of knobs to twiddle that will change the state matrix and, therefore, the system dynamics.

The question is:

"How do we choose the gain matrix $\mathbf{G}$?"

Intro to FSF - VIII

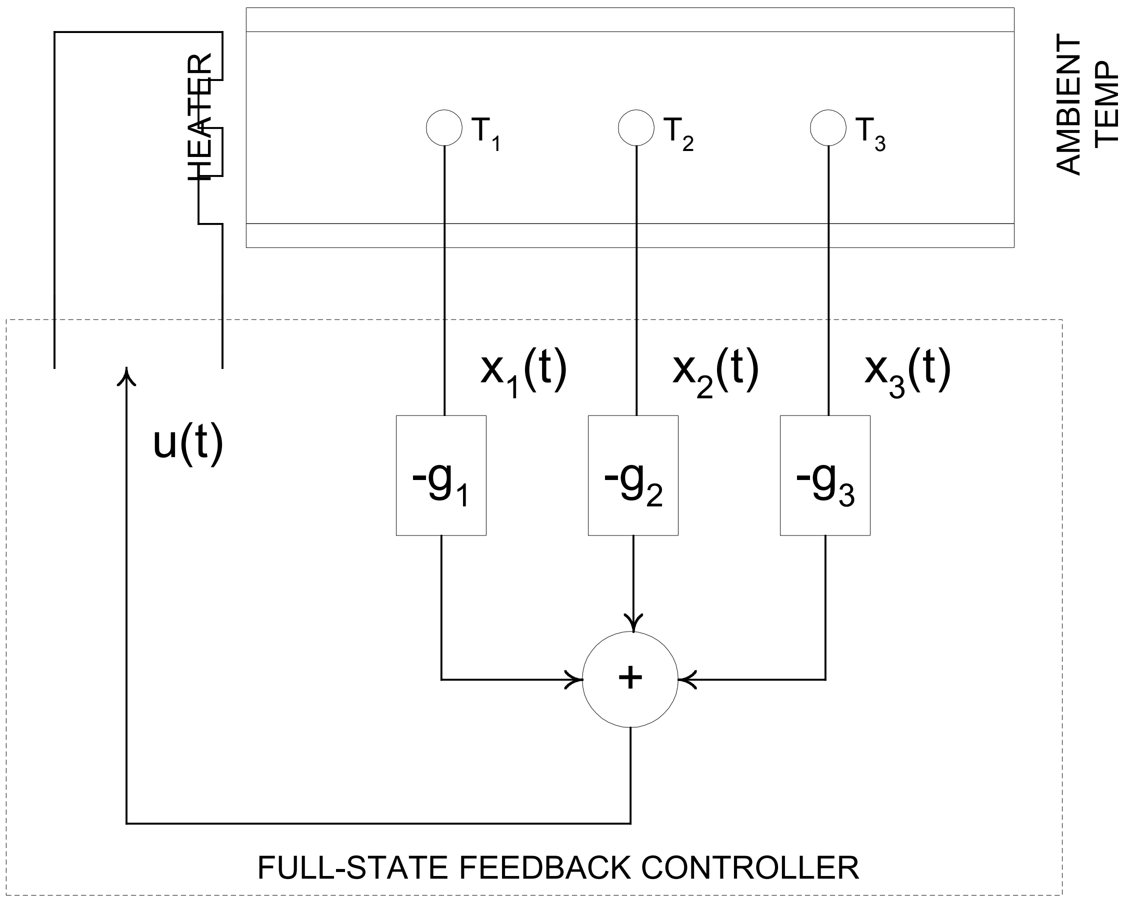
What does it mean to have to feedback all of the states? Consider the heat conduction example:



$$u(t) = -\mathbf{G}\mathbf{x}(t) = -\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$= -g_1 x_1(t) - g_2 x_2(t) - g_3 x_3(t)$$

Intro to FSF - IX



Brute force pole placement - I

Can we choose a gain matrix that places the closed-loop poles at the desired locations?

First, let's start with a single-input state-space model.

$$\underset{[N\times 1]}{\dot{\mathbf{x}}(t)} = \underset{[N\times N]}{\mathbf{A}}\,\underset{[N\times 1]}{\mathbf{x}(t)} + \underset{[N\times 1]}{\mathbf{b}}\,\underset{[1\times 1]}{u(t)}$$

The full-state feedback control law is

$$\underset{[1\times 1]}{u(t)} = -\underset{[1\times N]}{\mathbf{g}}\,\underset{[N\times 1]}{\mathbf{x}(t)}$$

Brute force pole placement - II

Combining the two equations yields

$$\underset{[N\times 1]}{\dot{\mathbf{x}}(t)} = \underset{[N\times N]}{\mathbf{A}}\,\underset{[N\times 1]}{\mathbf{x}(t)} - \underbrace{\underset{[N\times 1]}{\mathbf{b}}\;\underset{[1\times N]}{\mathbf{g}}}_{[N\times N]}\,\underset{[N\times 1]}{\mathbf{x}(t)}$$

Denote the <u>closed-loop state matrix</u> as

$$\mathbf{A}_c = \left[\mathbf{A} - \mathbf{bg}\right]$$

Brute force pole placement - III

Step 1: Choose a desired set of closed-loop poles based on performance requirements

Step 2: Determine the associated closed-loop characteristic equation

Step 3: Choose the gain vector $\mathbf{g}$ such that the characteristic equation of the closed-loop state matrix is equal to the desired characteristic equation

$$\left|s\mathbf{I}-\mathbf{A}_c\right| = \left|s\mathbf{I}-\mathbf{A}+\mathbf{bg}\right| = \underbrace{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}_{\text{Desired CL characteristic equation}}$$

Brute force pole placement - IV

Expanding the determinant

$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{bg} \right|$$

yields a polynomial in which the coefficients are functions of the control gains

$$s^N + \bar{a}_{N-1}(\mathbf{A},\mathbf{b},\mathbf{g})s^{N-1} + \cdots + \bar{a}_1(\mathbf{A},\mathbf{b},\mathbf{g})s + \bar{a}_0(\mathbf{A},\mathbf{b},\mathbf{g})$$

Brute force pole placement - V

Setting this polynomial equal to the desired polynomial will yield a set of N equations in N unknowns which can be solved for the unknown gains in $\mathbf{g}$.

$$
\begin{aligned}
a_{N-1} &= \bar{a}_{N-1}(\mathbf{A},\mathbf{b},\mathbf{g}) \\
\vdots \quad & \qquad \vdots \\
a_1 &= \bar{a}_1(\mathbf{A},\mathbf{b},\mathbf{g}) \\
\underbrace{a_0}_{\text{From desired CL CE}} &= \underbrace{\bar{a}_0(\mathbf{A},\mathbf{b},\mathbf{g})}_{\text{From closed-loop state matrix } \mathbf{A}_c}
\end{aligned}
$$

Regulator Problem - I

Notice that the closed-loop state-space system with full state feedback has no inputs!

$$\dot{\mathbf{x}}(t) = [\mathbf{A} - \mathbf{bg}]\mathbf{x}(t)$$

The closed-loop system will only evolve if there are nonzero initial conditions.

Assuming asymptotic stability, what will the closed-loop states evolve to?

$$\dot{\mathbf{x}}(t) = \mathbf{0} \quad \Rightarrow \quad \mathbf{x}(t) \to \mathbf{0} = \text{Origin of State - Space}$$

Regulator Problem - II

This means that our closed-loop system (assuming asymptotic stability) is designed to drive the states to the origin from any initial condition.

Whenever the control objective is to drive states or outputs to constant desired values, this is called the <u>Regulator Problem</u>.

Regulator Problem - III

With no external disturbance, the closed-loop response becomes

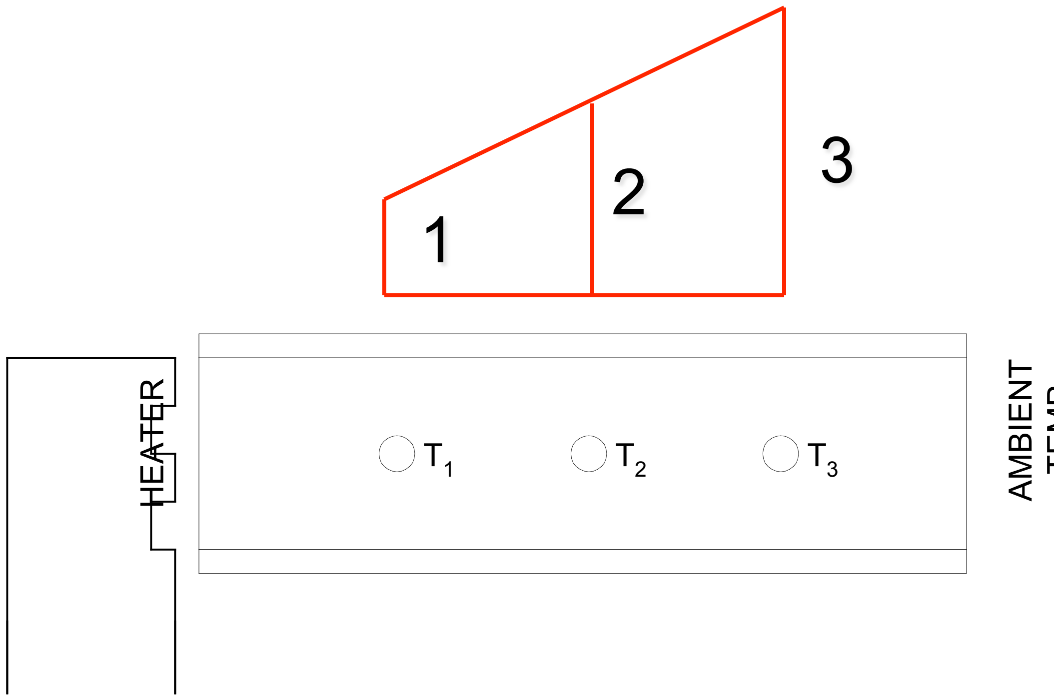
$$\mathbf{x}(t) = \underbrace{e^{(\mathbf{A}-\mathbf{bg})t}}_{\text{CL State Transition Matrix}} \mathbf{x}(0) = e^{\mathbf{A}_c t}\mathbf{x}(0)$$

and the control input is

$$\mathbf{u}(t) = -\mathbf{g}\mathbf{x}(t) = -\mathbf{g}e^{(\mathbf{A}-\mathbf{bg})t}\mathbf{x}(0)$$

Brute force pole placement - I

Let's return to our heat conduction example. Assume that the rod has an initial temperature profile that is linear from location 1 to location 3:

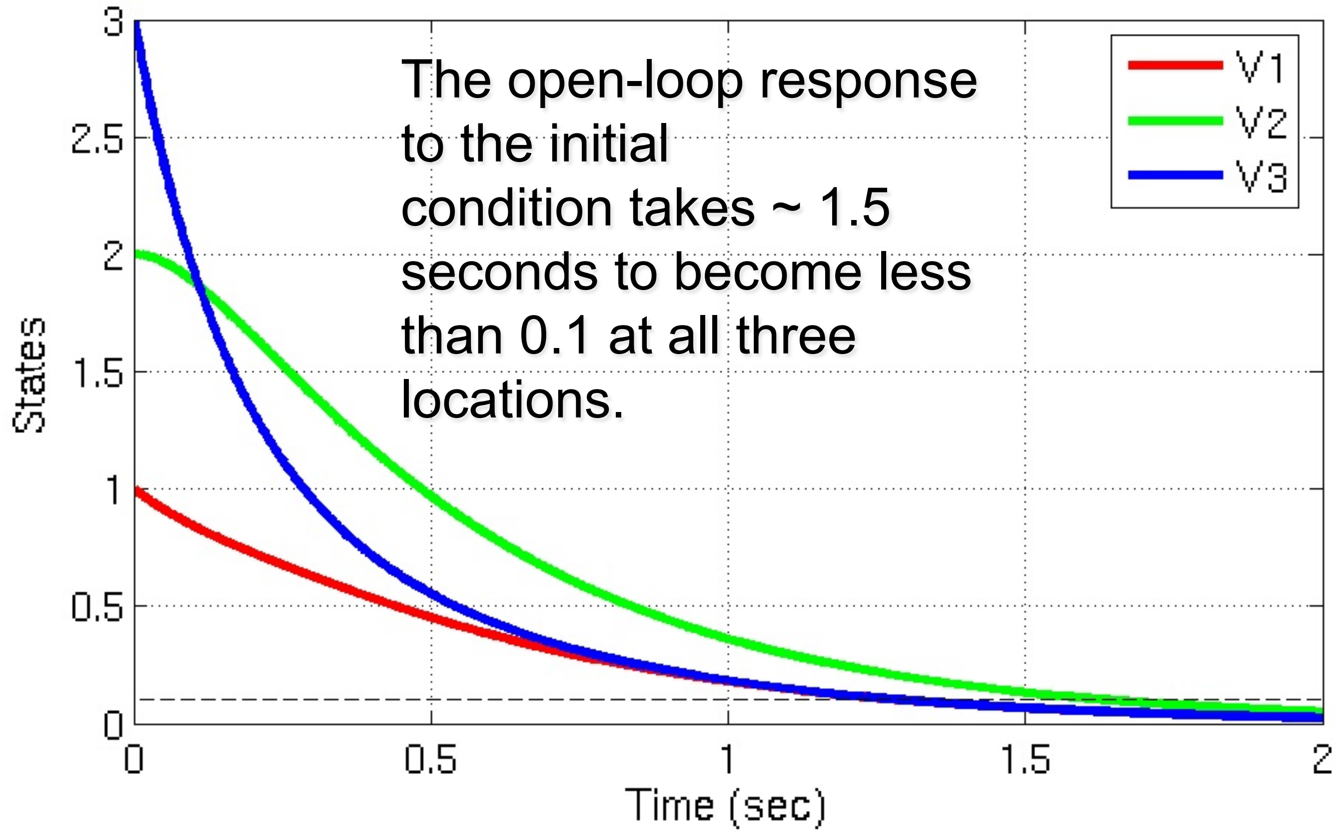


$$\mathbf{v}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Brute force pole placement - II

The state equations are on slide L7/S28

For this example, use $RC = 0.5$, and only consider the heater input ($u = 0$).



Brute force pole placement - III

To determine the closed-loop pole locations, form

$$[s\mathbf{I}-\mathbf{A}+\mathbf{bg}] = \begin{bmatrix} s+6 & -2 & 0 \\ -2 & s+4 & -2 \\ 0 & -2 & s+6 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

and find the determinant

$$|s\mathbf{I}-\mathbf{A}+\mathbf{bg}| =$$

$$s^3 + \underbrace{(4g_1+16)}_{\bar{a}_2} s^2 + \underbrace{(40g_1+8g_2+76)}_{\bar{a}_1} s + \underbrace{(80g_1+48g_2+16g_3+96)}_{\bar{a}_0}$$

Brute force pole placement - IV

Now we have a set of three equations and three unknowns. We can set these up as a matrix expression

$$\begin{bmatrix} 4 & 0 & 0 \\ 48 & 8 & 0 \\ 80 & 48 & 16 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \bar{a}_2 - 16 \\ \bar{a}_1 - 76 \\ \bar{a}_0 - 96 \end{bmatrix}$$

This set of equations <u>always</u> has a solution as long as the matrix on the LHS has an inverse.

Brute force pole placement - V

<u>For this example</u>:

- We can choose **ANY COEFFICIENTS THAT WE WANT**, and solve for the associated control gains in $\mathbf{g}$.

- Thus, we can **PLACE THE POLES ANYWHERE WE WANT**.

Brute force pole placement - VI

Assume that we want to place the poles at

$s_1 = -8$ originally -8

$s_2 = -6 + j6$ originally -6

$s_3 = -6 - j6$ originally -2

therefore the desired characteristic equation is

$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{bg} \right| = (s+8)(s+6-j6)(s+6+j6)$$

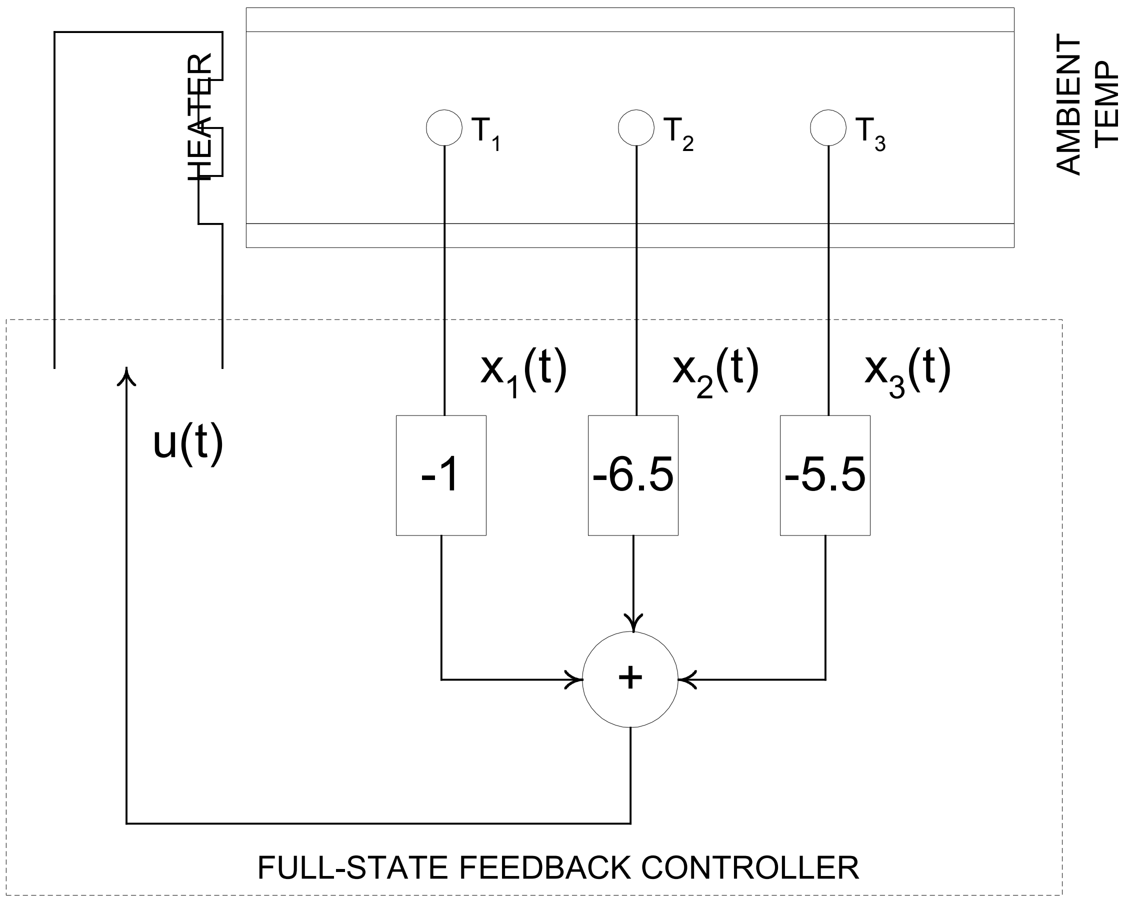
$$= s^3 + \underbrace{20}_{\bar{a}_2} s^2 + \underbrace{168}_{\bar{a}_1} s + \underbrace{576}_{\bar{a}_0}$$

Brute force pole placement - VII

The gains are computed from the matrix expression

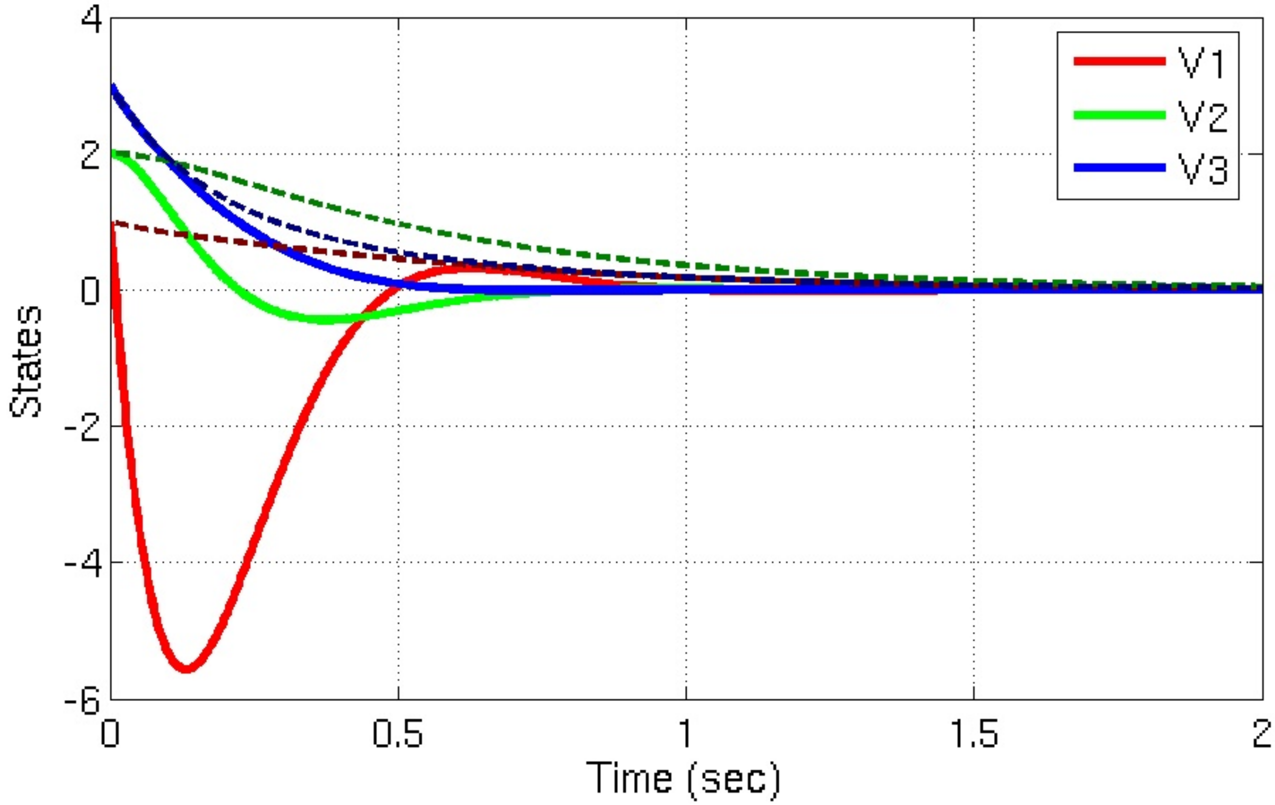
$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 48 & 8 & 0 \\ 80 & 48 & 16 \end{bmatrix}^{-1} \begin{bmatrix} 20-16 \\ 168-76 \\ 576-96 \end{bmatrix} = \begin{bmatrix} 1 \\ 6.5 \\ 5.5 \end{bmatrix}$$

Brute force pole placement - VIII



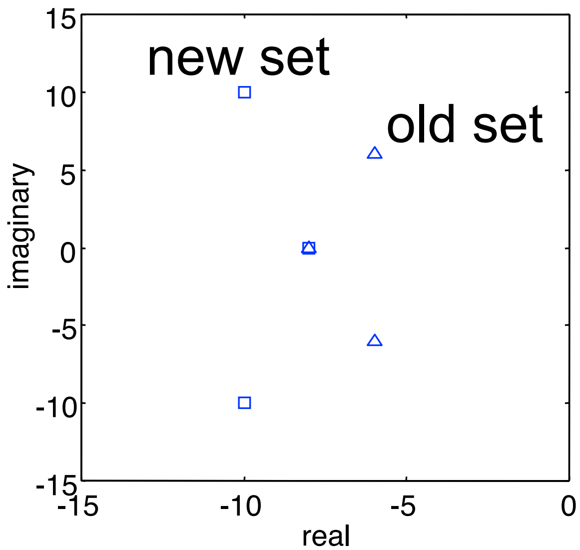
Brute force pole placement - IX

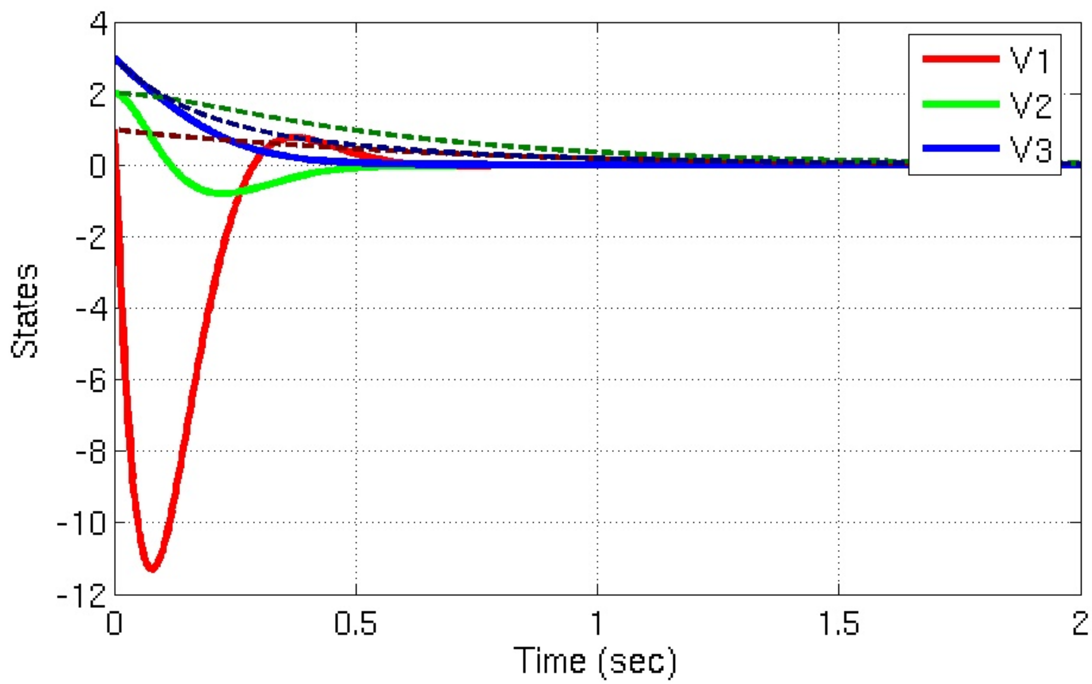
Full-state feedback can speed up the response by a factor of ~2 at the expense of increased overshoot



Brute force pole placement - X

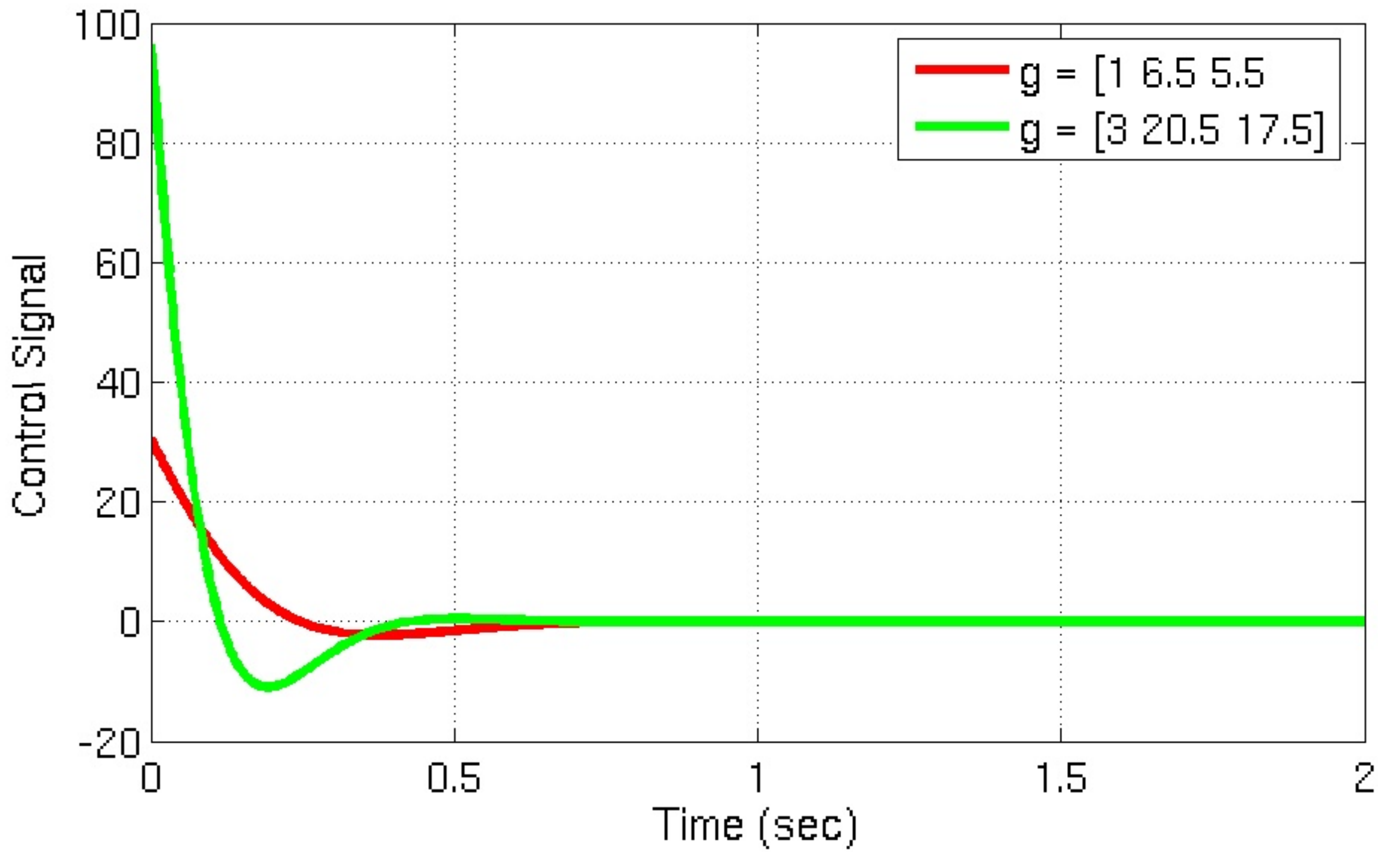
If the speed of response is not satisfactory, then new closed-loop poles can be chosen. With new gains, the speed of response is now ~0.5 seconds.





Brute force pole placement - XI

Fundamental Tradeoff: The penalty for increasing the speed of response is <u>increased control effort</u>.



Summary

Pole placement is a design technique that allows you to choose the location of the poles of the closed-loop state matrix.

Changing the location of the poles will change the time response (settling time, overshoot, etc) to meet the design specifications.

Placement of poles will be constrained by actuator authority.