ME 5554 / AOE5754 / ECE5754

Applied Linear Systems

HW2

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1. Place the following nonlinear ODE into first-order form:

$$M\ddot{x}(t) + B\dot{x}(t)^2 + Kx(t) = u_1(t) + G\theta(t)$$

$$G\dot{x}(t) + L\dot{\theta}(t) = u_2(t) - R\theta(t)x(t)$$

Define: $x_1(t) = x(t)$

$x_2(t) = \dot{x}(t) = \dot{x}_1(t)$

$x_3(t) = \theta(t)$

$y(t) = \theta(t)$

So: $M\dot{x}_2(t) + Bx_2(t)^2 + Kx_1 = u_1(t) + Gx_3(t)$

$Gx_2(t) + L\dot{x}_3(t) = u_2(t) - Rx_3(t)x_1(t)$

The resulting nonlinear state equation plus output equation then are:

$\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = -\frac{K}{M}x_1(t) - \frac{B}{M}x_2(t)^2 + \frac{G}{M}x_3(t) + \frac{1}{M}u_1(t)$

$\dot{x}_3(t) = -\frac{R}{L}x_1(t)x_3(t) - \frac{G}{L}x_2(t) + \frac{1}{L}u_2(t)$

$y(t) = x_3(t)$

Assume that at initial condition t = t$_o$: $\tilde{x}_1(t) = \tilde{x}(t) = x_0(t)$

$\tilde{x}_2(t) = \dot{\tilde{x}}(t) = x_0$

$\tilde{x}_3(t) = \tilde{\theta}(t) = \theta_0(t)$

$\tilde{u}_1(t) = -Kx_0(t) - Bx_0(t)^2 + G\theta_0(t)$

$\tilde{u}_2(t) = R\theta_0(t)x_0(t) + Gx_o + L\theta_0$

We have: $\dot{\tilde{x}}_1(t) = x_0$

$\dot{\tilde{x}}_2(t) = 0$

$\dot{\tilde{x}}_3(t) = \theta_0$

with: $\tilde{x}_2(t) = x_0$

$$-\frac{K}{M}\tilde{x}_0(t) - \frac{B}{M}\tilde{x}_0(t)^2 + \frac{G}{M}\tilde{\theta}_0(t) + \frac{1}{M}[Kx_0(t) + Bx_0(t)^2 - G\theta_0(t)] = 0$$

$$\tilde{x}_3(t) = -\frac{R}{L}\tilde{x}_1(t)\tilde{x}_3(t) - \frac{G}{L}\tilde{x}_2(t) + \frac{1}{L}\tilde{u}_2(t).$$

$$\Leftrightarrow\ \tilde{x}_3(t) = -\frac{R}{L}x_0(t)\theta_0(t) - \frac{G}{L}x_0(t) + \frac{1}{L}[R\theta_0(t)x_0(t) + Gx_o + L\theta_0]$$

$$\Leftrightarrow\ \tilde{x}_3(t) = \theta_0$$

It follows directly that deviation variables are specified by:

$$x_\delta(t) = \begin{bmatrix} x(t) - \tilde{x}(t) \\ \dot{x}(t) - \dot{\tilde{x}}(t) \\ \theta(t) - \tilde{\theta}(t) \end{bmatrix}$$

$$u(t) = \begin{bmatrix} u_1(t) + Kx_0(t) + Bx_0(t)^2 - G\theta_0(t) \\ u_2(t) - R\theta_0(t)x_0(t) - Gx_o - L\theta_0 \end{bmatrix}$$

$$y_\delta(t) = \theta(t) - \tilde{\theta}(t)$$

With:

$$f(x,u) = \begin{bmatrix} f_1(x_1,x_2,x_3,u_1,u_2) \\ f_2(x_1,x_2,x_3,u_1,u_2) \\ f_3(x_1,x_2,x_3,u_1,u_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ -\frac{K}{M}x_1(t) - \frac{B}{M}x_2(t)^2 + \frac{G}{M}x_3(t) + \frac{1}{M}u_1(t) \\ -\frac{R}{L}x_1(t)x_3(t) - \frac{G}{L}x_2(t) + \frac{1}{L}u_2(t) \end{bmatrix}$$

Partial differentiation yields:

$$\frac{\partial f}{\partial x}(x,u) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{K}{M} & -\frac{2B}{M} & \frac{G}{M} \\ -\frac{R}{L}x_3 & -\frac{G}{L} & -\frac{R}{L}x_1 \end{bmatrix}$$

$$\frac{\partial f}{\partial u}(x,u) = \begin{bmatrix} 0 \\ \frac{1}{M} \\ \frac{1}{L} \end{bmatrix}$$

$$\frac{\partial h}{\partial x}(x,u) = [1 \quad 0 \quad 0]$$

$$\frac{\partial h}{\partial u}(x,u) = [0]$$

Evaluating at the nominal trajectory gives:

$$A(t) = \frac{\partial f}{\partial x}[\tilde{x}(t), \tilde{u}(t)] = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{K}{M} & -\frac{2B}{M} & \frac{G}{M} \\ -\frac{R}{L}\theta_0 & -\frac{G}{L} & -\frac{R}{L}x_0 \end{bmatrix}$$

$$B(t) = \frac{\partial f}{\partial u}[\tilde{x}(t), \tilde{u}(t)] = \begin{bmatrix} 0 \\ \frac{1}{M} \\ \frac{1}{L} \end{bmatrix}$$

$$C(t) = \frac{\partial h}{\partial x}[\tilde{x}(t), \tilde{u}(t)] = [1 \quad 0 \quad 0]$$

$$D(t) = \frac{\partial h}{\partial u}[\tilde{x}(t), \tilde{u}(t)] = [0]$$

2. Given the following State Equation and initial condition:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad \text{with } x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2a. Use the Matlab function EXPM() to compute a numerical representation for the Matrix Exponential.

```
A = [0 -5; 1 -2];    % state space matrix
t = linspace(0,5,10);     % time vector
phi = zeros(2,2,10);     % create temperate matrix
for i = 1:10
    phi(:,:,i) = expm(A*t(i));
end
```

Result:

```
phi(:,:,1) =  1      0

              0      1

phi(:,:,2) =  0.5117   -1.2855

              0.2571   -0.0025

phi(:,:,3) =  -0.0687    -0.6545

               0.1309   -0.3305
```

```
phi(:,:,4) =  -0.2034    0.0900
              -0.0180   -0.1674
phi(:,:,5) =  -0.0809    0.2613
              -0.0523    0.0236
phi(:,:,6) =  0.0258    0.1034
              -0.0207    0.0671
phi(:,:,7) =  0.0398   -0.0334
              0.0067    0.0264
phi(:,:,8) =  0.0118   -0.0510
              0.0102   -0.0086
phi(:,:,9) =  -0.0071   -0.0150
              0.0030   -0.0131
phi(:,:,10) = -0.0075    0.0092
              -0.0018   -0.0038
```

2b. Use the Symbolic Toolbox with the EXPM() function in Matlab to generate a symbolic representation for the Matrix Exponential:

```
syms a b c d t;
A = [a b;c d];

PhiB = expm(A*t);
```

2c. Numerically evaluate the symbolic representation from 2b using the same time vector from 2a:

```
A(1,1) = 0; A(1,2) = -5;
A(2,1) = 1; A(2,2) = -2;    % numerical elements in A
PB = expm(A*t);             % re-calculate
T = linspace(0,5,10);       % time vector

p11 = PB(1,1);
p12 = PB(1,2);
```

```
p21 = PB(2,1);
p22 = PB(2,2);

for i = 1:10
    PB1(i) = double(subs(p11,{t},T(i)));
    PB2(i) = double(subs(p12,{t},T(i)));
    PB3(i) = double(subs(p21,{t},T(i)));
    PB4(i) = double(subs(p22,{t},T(i)));
end
```

Result:

```
>> PB11

PB11 =

    1.0000    0.5117   -0.0687   -0.2034   -0.0809    0.0258
0.0398    0.0118   -0.0071   -0.0075


>> PB12

PB12 =

         0   -1.2855   -0.6545    0.0900    0.2613    0.1034
-0.0334   -0.0510   -0.0150    0.0092


>> PB21

PB21 =

         0     0.2571     0.1309    -0.0180    -0.0523    -
0.0207    0.0067    0.0102    0.0030   -0.0018


>> PB22

PB22 =

    1.0000   -0.0025   -0.3305   -0.1674    0.0236    0.0671
0.0264   -0.0086   -0.0131   -0.0038
```

2d. Generate an analytic representation for the Matrix Exponential using Laplace Transforms:

We have: $A = \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix}$

Taking Laplace transform for an analytic representation: $L^{-1}\{[sI - A]^{-1}\}$

$$[sI - A] = \begin{bmatrix} s & 5 \\ -1 & s+2 \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} s & 5 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{detA}\begin{bmatrix} s+2 & -5 \\ 1 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+2}{s^2+2s+5} & \frac{-5}{s^2+2s+5} \\ \frac{1}{s^2+2s+5} & \frac{s}{s^2+2s+5} \end{bmatrix}$$

So:

$$L^{-1}\left\{\frac{s+2}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{s+2}{(s+1)^2+2^2}\right\} = L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2} + \frac{1}{(s+1)^2+2^2}\right\}$$

$$= e^{-t}.\left(cos2t + \frac{1}{2}sin2t\right)$$

$$L^{-1}\left\{\frac{-5}{s^2+2s+5}\right\} = L^{-1}\left\{-\frac{5}{2}\frac{2}{(s+1)^2+2^2}\right\} = -\frac{5}{2}.e^{-t}.sin2t$$

$$L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{1}{2}\frac{2}{(s+1)^2+2^2}\right\} = \frac{1}{2}.e^{-t}.sin2t$$

$$L^{-1}\left\{\frac{s}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}\right\} = e^{-t}.\left(cos2t - \frac{1}{2}sin2t\right)$$

Final result:

$$L^{-1}\{[sI - A]^{-1}\} = \begin{bmatrix} e^{-t}.\left(cos2t + \frac{1}{2}sin2t\right) & -\frac{5}{2}.e^{-t}.sin2t \\ \frac{1}{2}.e^{-t}.sin2t & e^{-t}.\left(cos2t - \frac{1}{2}sin2t\right) \end{bmatrix}$$

Test with Matlab:

```
A = [0 -5; 1 -2];
syms s;
R = s*eye(2) - A;
RI = inv(R);
PD = ilaplace(RI);
```

Result:

```
>> PD
PD =
[ exp(-t)*(cos(2*t) + sin(2*t)/2),         -(5*sin(2*t)*exp(-t))/2]
[             (sin(2*t)*exp(-t))/2, exp(-t)*(cos(2*t) - sin(2*t)/2)]
```

2e. Numerically evaluate the symbolic representation from 2d using the same time vector from 2a:

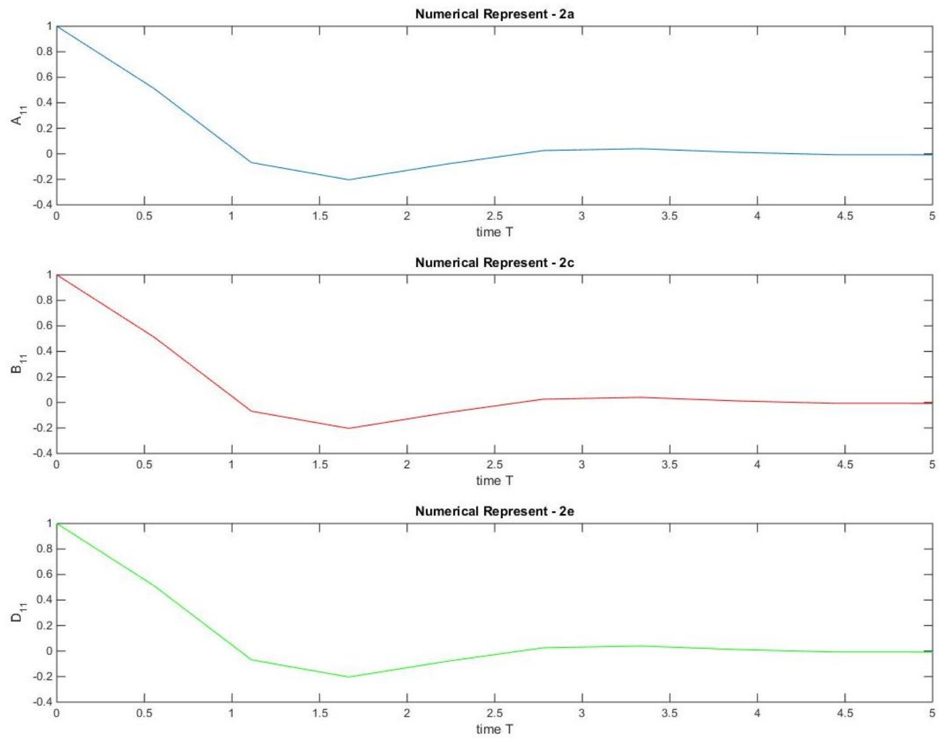
```
T = linspace(0,5,10);    % time vector
for i = 1:10
            PD11(i) = double(subs(PD(1,1),{t},T(i)));
            PD12(i) = double(subs(PD(1,2),{t},T(i)));
            PD21(i) = double(subs(PD(2,1),{t},T(i)));
      PD22(i) = double(subs(PD(2,2),{t},T(i)));
end
```

Results:

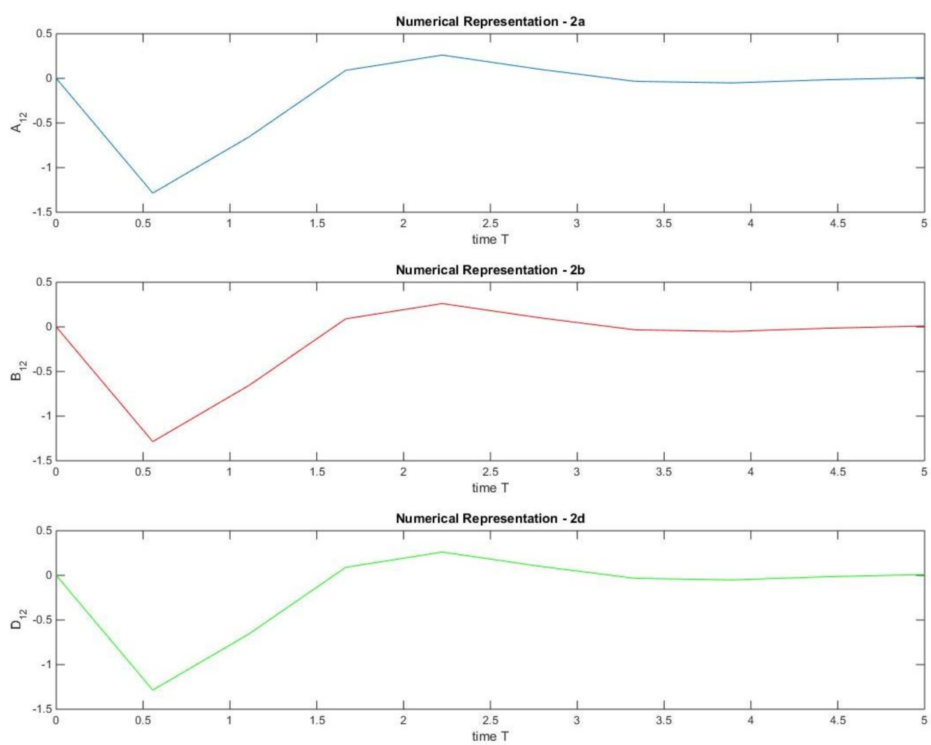
```
>> PD11

PD11 =

    1.0000    0.5117   -0.0687   -0.2034   -0.0809    0.0258    0.0398
0.0118   -0.0071   -0.0075


>> PD12

PD12 =

         0   -1.2855   -0.6545    0.0900    0.2613    0.1034    -
0.0334   -0.0510   -0.0150    0.0092


>> PD21

PD21 =

         0    0.2571    0.1309   -0.0180   -0.0523   -0.0207    0.0067
0.0102    0.0030   -0.0018


>> PD22

PD22 =

    1.0000   -0.0025   -0.3305   -0.1674    0.0236    0.0671    0.0264
-0.0086   -0.0131   -0.0038
```

2f. Use Matlab to generate a plot of all elements in the Matrix Exponential:

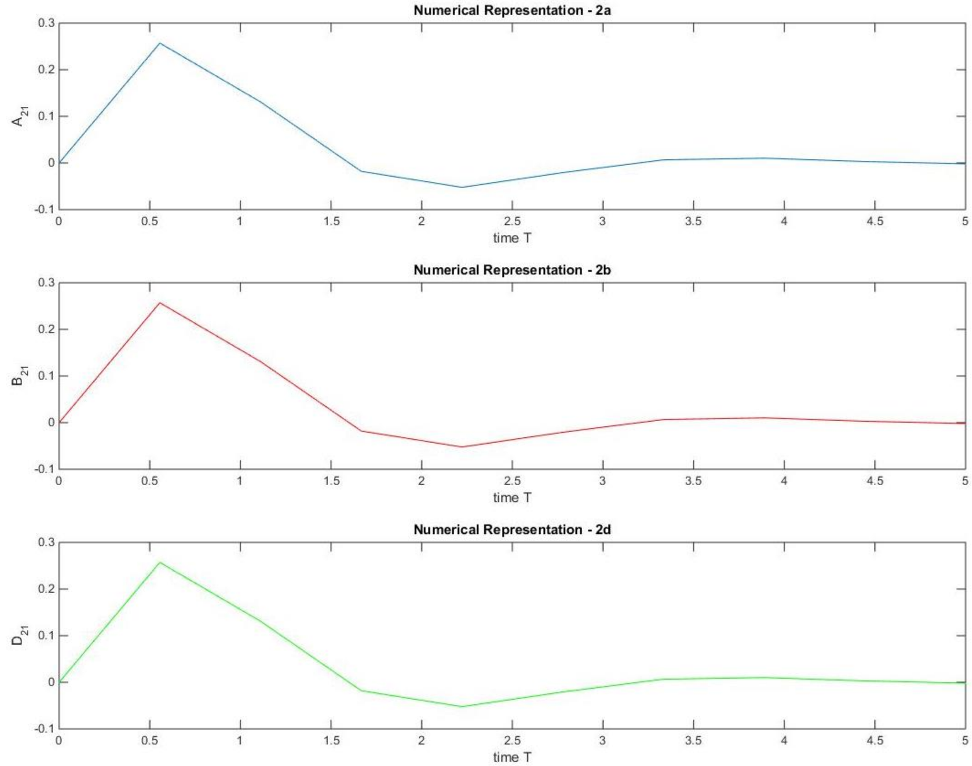
First element: first row + first column



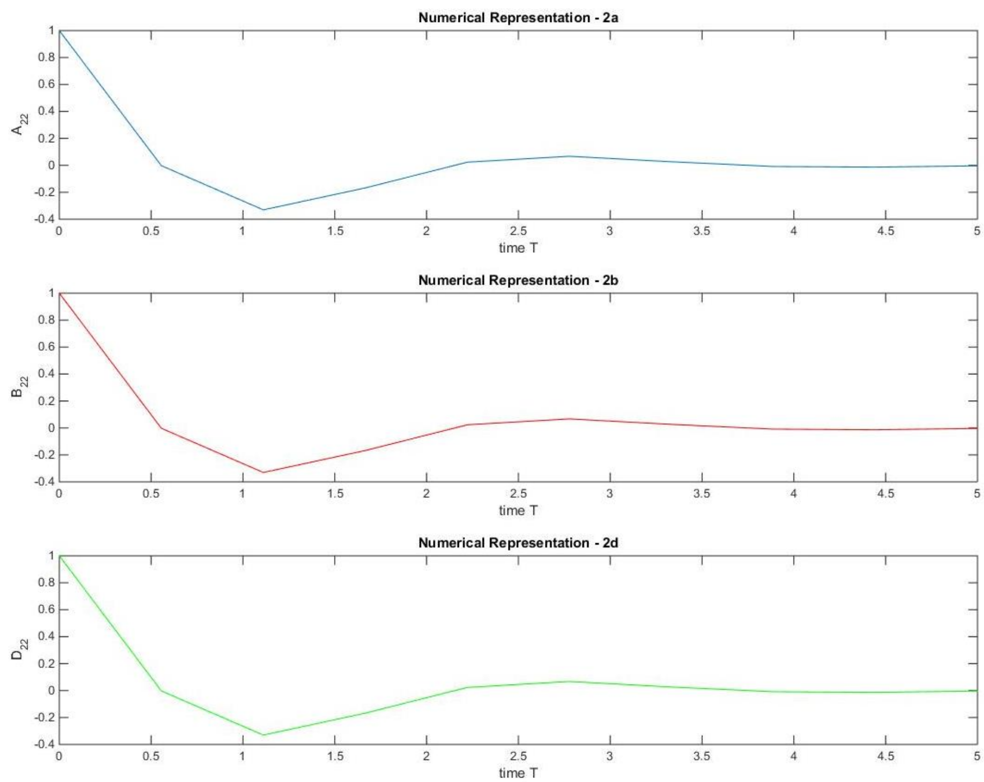
Second element: first row + second column:



Third element: second row + first column:



Forth element: second row + second column:



3. Use any of the result from problem 2 to compute and plot the state trajectory for the Initial Value response of this dynamic system:

- Use the result from problem 2b with assume that input u(t) = 0. We have:

```
A = [0 -5;1 -2];
syms t;
x0 = [1; -1];

PB = expm(A*t);                         % expm(A*t)
T = linspace(0,5,10);                   % time vector

x_t = PB*x0;

for i = 1:10
     x_t11(i) = double(subs(x_t(1,1),{t},T(i)));
     x_t21(i) = double(subs(x_t(2,1),{t},T(i)));
end

plot(T,x_t11,T,x_t21,'r');
xlabel('time T');
ylabel('X');
title('State Trajectory');
legend('x_1','x_2');
```

Result: