Course Outline - 1st Half

- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response, Matrix Exponential
- Transfer Functions & Eigenvalues (2)
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

Dynamic System Models

Transfer Function Matrices

Eigenvalues of the State Matrix

Matlab Representation

Introduction

Last meeting we discussed the a variety of approaches to solving the state equations.

The Laplace transform approach gave us a relationship between the state matrix $\mathbf{A}$ and the state transition matrix $e^{\mathbf{A}t}$,

$$L^{-1}\left\{\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\right\}=e^{\mathbf{A}t}$$

Aside from giving us a method to compute $e^{\mathbf{A}t}$, this relationship gives us the ability to relate the <u>properties of $\mathbf{A}$</u> to the <u>dynamics</u> of the system.

Transfer Functions - I

What about the output equations?

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Notice that the output equations are only a function of the states and the input, therefore, if we know the states and the input as functions of time we can always find the outputs from the expression for $\mathbf{y}(t)$.

Transfer Functions - II

Combining the Laplace representation of the state dynamics with the output equations will yield the input-output transfer functions of the system.

Remember that we discussed single-input-single-output (SISO) transfer functions in L1

$$\frac{y(s)}{u(s)} = H(s) = \left(\frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \right) \quad n \geq m$$

Transfer Functions - III

In this course we will deal with multi-input-multi-output (MIMO) systems.

The state-space formulation naturally leads to systems described by <u>transfer function matrices</u>, where each entry of the matrix is a <u>SISO</u> transfer function

$$\begin{bmatrix} y_1(s) \\ \vdots \\ y_P(s) \end{bmatrix} = \begin{bmatrix} H_{11}(s) & \cdots & H_{1M}(s) \\ \vdots & \ddots & \vdots \\ H_{P1}(s) & \cdots & H_{PM}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ \vdots \\ u_M(s) \end{bmatrix}$$

Transfer Functions - IV

How do we find the matrix $\mathbf{H}(s)$?

Start by taking the Laplace transform of the state and output equations:

$$s\mathbf{x}(s)-\mathbf{x}(0)=\mathbf{A}\mathbf{x}(s)+\mathbf{B}\mathbf{u}(s)$$

$$\mathbf{y}(s)=\mathbf{C}\mathbf{x}(s)+\mathbf{D}\mathbf{u}(s)$$

Next, solve for $\mathbf{x}(s)$ in terms of $\mathbf{u}(s)$ assuming zero initial conditions

Transfer Functions - V

Substitute the expression for $\mathbf{x}(s)$ into the Laplace transformed output equation

$$\mathbf{y}(s)=\mathbf{C}\left[\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}\mathbf{u}(s)\right]+\mathbf{D}\mathbf{u}(s)$$

and factor the input term

The bracketed term above is the matrix of transfer functions

Transfer Functions - VI

Using our definition of the matrix inverse

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \frac{adj[s\mathbf{I}-\mathbf{A}]}{|s\mathbf{I}-\mathbf{A}|} = \frac{\text{Adjoint}}{\text{Determinant}}$$

We can rewrite the transfer function matrix as

$$\underset{[P\times M]}{\mathbf{H}(s)} = \underset{[P\times N]}{\mathbf{C}} \left[\frac{\underset{[N\times N]}{adj[s\mathbf{I}-\mathbf{A}]}}{\underset{\substack{N^{th}\text{ order scalar} \\ \text{polynomial in } s}}{|s\mathbf{I}-\mathbf{A}|}} \right] \underset{[N\times M]}{\mathbf{B}} + \underset{[P\times M]}{\mathbf{D}}$$

Transfer Functions - VII

This analysis tells us that every transfer function element in $\mathbf{H}(s)$ has the same denominator.

This denominator is simply the determinant of $[s\mathbf{I} - \mathbf{A}]$.

The poles of the system are the solutions of the equation:

In-Class Assignment

Find the matrix of transfer functions $\mathbf{H}(s)$ for the system:

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$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

In-Class Assignment

Canonical Forms - I

Let's go back to our general SISO transfer function representation, but w.l.o.g., we can assume $a_n = 1$

$$H(s) = \frac{y(s)}{u(s)} = \left(\frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \right) \quad n > m$$

We can rewrite this transfer function as

$$H(s) = \frac{y(s)}{z(s)} \frac{z(s)}{u(s)} = \left(\frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \right) \quad n > m$$

Canonical Forms - II

Now define

$$\frac{z(s)}{u(s)} = \left(\frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \right)$$

We can rewrite this expression

$$\left\{ s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \right\} z(s) = u(s)$$

and transform back into the time domain

Canonical Forms - III

The following choice of state variables

$$\mathbf{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix}=\begin{bmatrix} z(t) \\ \dot{z}(t) \\ \ddot{z}(t) \\ \vdots \\ \frac{d^{n-1}z(t)}{dt^{n-1}} \end{bmatrix}$$

Leads to the following state equations

Canonical Forms - IV

Based on the previous definition, we must have the following output

$$y(s) = \left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0\right) z(s)$$

Applying the inverse Laplace Transform

$$y(t) = b_m \frac{d^m z(t)}{dt^m} + b_{m-1} \frac{d^{m-1} z(t)}{dt^{m-1}} + \cdots + b_1 \frac{dz(t)}{dt} + b_0 z(t)$$

Substituting the state variable definitions

Canonical Forms - V

Putting all these results into matrix form, the state-space representation is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

Denominator Coefficients

$$\begin{bmatrix} y(t) \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

Numerator Coefficients

Canonical Forms - VI

Notice that the last row of the $\mathbf{A}$ matrix contains the denominator coefficients and the $\mathbf{C}$ matrix contains the numerator coefficients

This transfer function representation is called a companion form, and only works when $n > m$

TF2SS and SS2TF Conversion - I

Start with an example transfer function

$$\frac{y(s)}{u(s)} = \left(\frac{6s+4}{s^2+2s+10} \right)$$

First write the state-space representation in companion form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 4 & 6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

TF2SS and SS2TF Conversion - II

Now lets transform the state-space representation back to a transfer function

We know the answer will be the scalar transfer function:

$$H(s) = \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}$$

TF2SS and SS2TF Conversion - III

Substituting the **A**, **B**, **C**, and **D** matrices into the matrix expression for H

$$H(s) = \begin{bmatrix} 4 & 6 \end{bmatrix} \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \end{bmatrix} \begin{bmatrix} s & -1 \\ 10 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

TF2SS and SS2TF Conversion - IV

The inverse matrix is

$$\begin{bmatrix} s & -1 \\ 10 & s+2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s+2 & 1 \\ -10 & s \end{bmatrix}}{s(s+2)-(-1)(10)} = \frac{\begin{bmatrix} s+2 & 1 \\ -10 & s \end{bmatrix}}{s^2+2s+10}$$

Substituting this result back into the transfer function we get

$$H(s) = \frac{\begin{bmatrix} 4 & 6 \end{bmatrix}\begin{bmatrix} s+2 & 1 \\ -10 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2+2s+10} = \frac{\begin{bmatrix} 4 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix}}{s^2+2s+10}$$

TF2SS and SS2TF Conversion - V

The final expanded result is

$$H(s) = \left(\frac{6s + 4}{s^2 + 2s + 10} \right)$$

Which is identical to our original transfer function

TF2SS and SS2TF Conversion - VI

How do we perform these transformations in Matlab?

We first have to look at how transfer functions and state-space representations are stored in Matlab.

The most efficient method is to use the object-oriented tools in the Control System Toolbox

TF2SS and SS2TF Conversion - VII

Let's say we have the following continuous time transfer function

$$H(s) = \left(\frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \right)$$

In Matlab, we first need to store the numerator and denominator coefficients in arrays starting with the highest power of s

```
num = [b3, b2, b1, b0];
den = [a4, a3, a2, a1, a0];
```

TF2SS and SS2TF Conversion - VIII

To store these polynomial coefficients as a transfer function object, use the `tf()` function

```
tf_obj = tf(num, den);
```

To extract the numerator and denominator coefficients from a transfer function object use the `tfdata()` function

```
[num, den] = tfdata(tf_obj,'v');
```

TF2SS and SS2TF Conversion - IX

Now let's say we have the following state-space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

To store these matrices as a state-space object, use the `ss()` function

```
ss_obj = ss(A, B, C, D);
```

TF2SS and SS2TF Conversion - X

To extract the state-space matrices from a state-space object use the `ssdata()` function

```
[A, B, C, D] = ssdata(ss_obj);
```

TF2SS and SS2TF Conversion - XI

Finally, to convert between a transfer function representation and a state-space representation use either:

```
[A, B, C, D] = tf2ss(num,den);
```

or

```
[num, den] = ss2tf(A, B, C, D);
```

TF2SS and SS2TF Conversion - XII

Matlab/Control System Toolbox has a built-in “viewer” to analyze the time and frequency responses of LTI models:

```
ltiview(tf_obj);

ltiview(ss_obj);

ltiview(tf_obj,ss_obj);
```

Eigenvalues of the State Matrix

Recall our discussion of the eigenvalue problem in the first week of the course. The eigenvalues of the $\mathbf{A}$ matrix are the solutions of the equation

Comparing this with the equation from L6/S10 we see that this expression is equivalent to the expression for the poles of the system.

In-Class Assignment

Using the previous ICA example (L6/S11), show that the eigenvalues of $\mathbf{A}$ are equivalent to the poles of the transfer functions.

Eigenvalues of the State Matrix

This result says that the eigenvalues of the state matrix $\mathbf{A}$ are equivalent to the poles of the system.

Remember that the poles tell us a great deal about the system dynamics:

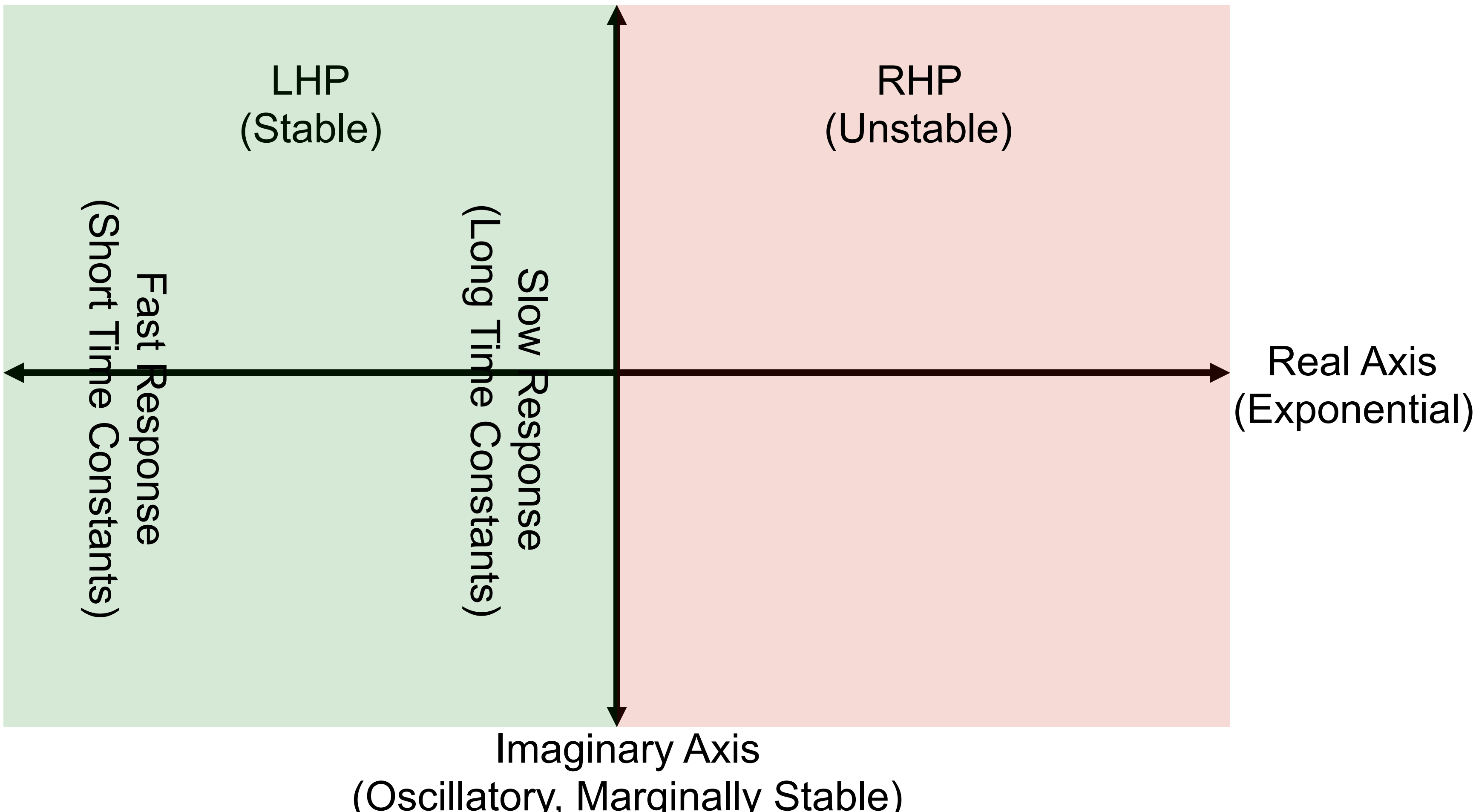
– Will the system be stable or unstable?

– Will the system oscillate?

– How long will it take to reach steady-state?

Eigenvalues of the State Matrix

Remember that the poles are located in the complex s-plane



Eigenvalues of the State Matrix

How hard is it to find the eigenvalues of a matrix? Again our friends at the MathWorks have simplified life for us.

The MATLAB funtion `eig` solves for the eigenvalues of a matrix.

```
lambda = eig(A)
```

Will return a column vector `lambda` of eigenvalues of `A`.

Remember, in general, the eigenvalues of the state matrix will be <u>complex</u>.

Eigenvalues of the State Matrix

The MATLAB funtion `eig` can also be used to solve for the eigenvectors.

```
[V,lambda] = eig(A)
```

Will return a matrix `V` of eigenvectors stored in columns, and a diagonal matrix `lambda` of eigenvalues of `A`.

In general, the eigenvalues and eigenvectors will be <u>complex</u>.

Eigenvalues of the State Matrix

Therefore we now know that:

1. The poles of a system can be determined directly from the eigenvalues of $\mathbf{A}$.
2. Stability is determined by the location of the poles. The response character is also strongly influenced by the location of the system poles.

Summary

- State-space models lead to a matrix of transfer functions. All of the transfer functions will have the same denominator but their numerators will generally differ.
- The eigenvalues of the state matrix A are equivalent to the poles of the system.