Advanced Control Engineering

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

Kalman-Bucy Filter - I

For this course, we are interested in the problem where the plant dynamics are finite-dimensional and LTI:

$$\underset{[N\times 1]}{\dot{\mathbf{x}}} = \underset{[N\times N]}{\mathbf{A}}\underset{[N\times 1]}{\mathbf{x}} + \underset{[N\times M]}{\mathbf{B}}\underset{[M\times 1]}{\mathbf{u}} + \underset{[N\times L]}{\mathbf{F}}\underset{[L\times 1]}{\mathbf{w}}$$

but now the process noise $\mathbf{w}$ is a vector stochastic process that is assumed to have <u>known</u> statistical properties.

Kalman-Bucy Filter - II

In particular, the process noise $\mathbf{w}$ is assumed to be stationary continuous-time white Gaussian noise.

This is a mathematical simplification since continuous white noise does not exist in nature because it requires infinite energy!

The process noise must be zero-mean:

$$\overline{\mathbf{w}} = E\left\langle \mathbf{w}(t) \right\rangle = \mathbf{0} \qquad \forall \ t$$

Kalman-Bucy Filter - III

We also require that the process noise be uncorrelated and that the positive definite symmetric covariance matrix is known to be:

$$\mathbf{C}_{\mathbf{w}}(\tau) = E\left\langle \mathbf{w}(t)\mathbf{w}^T(t-\tau)\right\rangle = \underset{[L\times L]}{Q}\,\delta(\tau) \quad \forall\ t$$

Where $\delta(\tau)$ is the dirac delta function (i.e. the unit impulse at $\tau = 0$).

Kalman-Bucy Filter - IV

What if the system is excited by a colored or non-white input $\mathbf{w}_C$ such as a correlated Gaussian noise?

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\mathbf{w}_C$$

We can use the autocorrelation or power spectral density (PSD) functions from measured data to develop a "shaping filter" that *is* excited by a white Gaussian noise signal.

Kalman-Bucy Filter - V

The shaping filter can be given by:

$$\dot{\mathbf{x}}_{SF} = \mathbf{A}_{SF}\mathbf{x}_{SF} + \mathbf{B}_{SF}\mathbf{w}$$

$$\mathbf{w}_C = \mathbf{C}_{SF}\mathbf{x}_{SF}$$

White Gaussian Noise

When we augment the original system with the shaping filter, the result is a system which is excited by white Gaussian noise:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_{SF} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{F}\mathbf{C}_{SF} \\ \mathbf{0} & \mathbf{A}_{SF} \end{bmatrix}}_{\text{Augmented } \mathbf{A}} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_{SF} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}}_{\text{Augmented } \mathbf{B}} \mathbf{u} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{SF} \end{bmatrix}}_{\text{Augmented } \mathbf{F}} \mathbf{w}$$

Kalman-Bucy Filter - VI

Like the state equations, the output equations are also assumed to be LTI:

$$\underset{[P\times 1]}{\mathbf{y}} = \underset{[P\times N]}{\mathbf{C}}\underset{[N\times 1]}{\mathbf{x}} + \underset{[P\times M]}{\mathbf{D}}\underset{[M\times 1]}{\mathbf{u}} + \underset{[P\times 1]}{\boldsymbol{\theta}}$$

and the measurement noise $\boldsymbol{\theta}$ is also a vector stochastic process that is assumed to have <u>known</u> statistical properties.

Kalman-Bucy Filter - VII

The measurement noise $\boldsymbol{\theta}$ is assumed to be stationary continuous-time white Gaussian zero-mean noise :

$$\overline{\boldsymbol{\theta}} = E\left\langle \boldsymbol{\theta}(t) \right\rangle = \mathbf{0} \qquad \forall\ t$$

The measurement noise must also be uncorrelated and its positive definite symmetric covariance matrix is:

$$\mathbf{C}_{\theta}(\tau) = E\left\langle \boldsymbol{\theta}(t)\boldsymbol{\theta}^T(t-\tau) \right\rangle = \underset{[P\times P]}{\boldsymbol{R}}\,\delta(\tau) \quad \forall\ t$$

Kalman-Bucy Filter - VIII

Finally, we assume that the process noise is uncorrelated with the measurement noise:

$$E\left\langle \mathbf{w}(t)\boldsymbol{\theta}^T(t-\tau)\right\rangle = \underset{[L\times P]}{\mathbf{0}} \qquad \forall\ t,\tau$$

The covariance matrix between the process noise and the measurement noise is zero everywhere for uncorrelated zero-mean signals.

Kalman-Bucy Filter - IX

We have already seen that the Kalman-Bucy state estimator has the same form as the Luenberger observer:

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} + \mathbf{K}\left(\mathbf{y} - \hat{\mathbf{y}}\right)$$

$$\hat{\mathbf{y}} = \hat{\mathbf{C}}\hat{\mathbf{x}} + \hat{\mathbf{D}}\mathbf{u}$$

but now we want to find an "optimal" estimate of the state.

What do we mean by "optimal"?

Kalman-Bucy Filter - X

Under the Gaussian noise assumptions, there are many equivalent "optimal" criterion that lead to the same "optimal" state estimate.

The most popular method is to require the state error to have zero-mean:

$$E\langle \tilde{\mathbf{x}} \rangle = \mathbf{0} \quad \text{where} \quad \tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$$

Kalman-Bucy Filter - XI

This is equivalently expressed with the following quadratic cost function:

$$J = E\left\langle \sum_{i=1}^{N} \tilde{x}_i^2(t) \right\rangle = E\left\langle \tilde{\mathbf{x}}^T \tilde{\mathbf{x}} \right\rangle = trace\left[E\left\langle \tilde{\mathbf{x}}\tilde{\mathbf{x}}^T \right\rangle \right]$$

The *trace* is the sum of the main diagonal elements:

$$\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T = \begin{bmatrix} \tilde{x}_1^2 & \tilde{x}_1\tilde{x}_2 & \cdots & \tilde{x}_1\tilde{x}_N \\ \tilde{x}_2\tilde{x}_1 & \tilde{x}_2^2 & & \\ \vdots & & \ddots & \vdots \\ \tilde{x}_N\tilde{x}_1 & & \cdots & \tilde{x}_N^2 \end{bmatrix}$$

Kalman-Bucy Filter - XII

The matrix we are computing the *trace* of to obtain the cost is called the state estimation error covariance matrix:

$$E\left\langle \tilde{\mathbf{x}}\tilde{\mathbf{x}}^T \right\rangle = E\left\langle \left(\mathbf{x}-\hat{\mathbf{x}}\right)\left(\mathbf{x}-\hat{\mathbf{x}}\right)^T \right\rangle = \underset{[N\times N]}{\mathbf{P}}$$

The matrix $\mathbf{P}$ is the solution of the Lyapunov Matrix Equation (LME):

$$[\mathbf{A}-\mathbf{KC}]\mathbf{P}+\mathbf{P}[\mathbf{A}-\mathbf{KC}]^T+\mathbf{FQF}^T+\mathbf{KRK}^T=\mathbf{0}$$

$$\mathbf{P}=\mathbf{P}^T\geq 0$$

Kalman-Bucy Filter - XIII

To summarize the problem statement, we assume the following:

$\overline{\mathbf{w}} = \mathbf{0}$	Zero-mean process noise
$\overline{\boldsymbol{\theta}} = \mathbf{0}$	Zero-mean measurement noise
$E\left\langle \mathbf{w}\mathbf{w}^T \right\rangle = \boldsymbol{Q}$	PD symmetric process covariance
$E\left\langle \boldsymbol{\theta}\boldsymbol{\theta}^T \right\rangle = \boldsymbol{R}$	PD symmetric measurement covariance
$E\left\langle \mathbf{w}\boldsymbol{\theta}^T \right\rangle = \mathbf{0}$	Uncorrelated noise
$(\mathbf{A},\mathbf{B})$	Stabilizable (or controllable)
$(\mathbf{A},\mathbf{C})$	Detectable (or observable)

Kalman-Bucy Filter - XIV

We want to solve for the LQG optimal gain matrix $\mathbf{K}$ such that the following quadratic cost function is minimized:

$$J = trace\left[\mathbf{P}\right]$$

Subject to the algebraic constraints of the Lyapunov Matrix Equation:

$$\left[\mathbf{A}-\mathbf{KC}\right]\mathbf{P}+\mathbf{P}\left[\mathbf{A}-\mathbf{KC}\right]^T+\mathbf{FQF}^T+\mathbf{KRK}^T=\mathbf{0}$$

$$\mathbf{P}=\mathbf{P}^T\geq 0$$

Kalman-Bucy Filter - XV

Derivation of the LQG solution yields another Matrix Riccati Equation:

$$-\dot{\mathbf{P}}(t) = \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T + \mathbf{F}\mathbf{Q}\mathbf{F}^T - \mathbf{P}(t)\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\mathbf{P}(t)$$

Which is very similar to what we saw for the LQR problem:

$$-\dot{\mathbf{P}}(t) = \mathbf{A}^T\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{Q} - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t)$$

Kalman-Bucy Filter - XVI

The LQG solution also simplifies for an asymptotically stable LTI system, and is derived from the Algebraic Riccati Equation. This equation must first be solved for the steady-state error covariance matrix $\mathbf{P}$:

$$\mathbf{0} = \mathbf{AP} + \mathbf{PA}^T + \mathbf{FQF}^T - \mathbf{PC}^T\mathbf{R}^{-1}\mathbf{CP}$$

Then the optimal Kalman gain matrix is:

$$\mathbf{K} = \mathbf{PC}^T\mathbf{R}^{-1}$$

Kalman-Bucy Filter - XVII

Matlab computes the solution to the Kalman optimal gain matrix with the `kalman` function:

```
sys = ss(A, [B F], C, D);
[Kest,K,P] = kalman(sys,Q,R);
```

where:

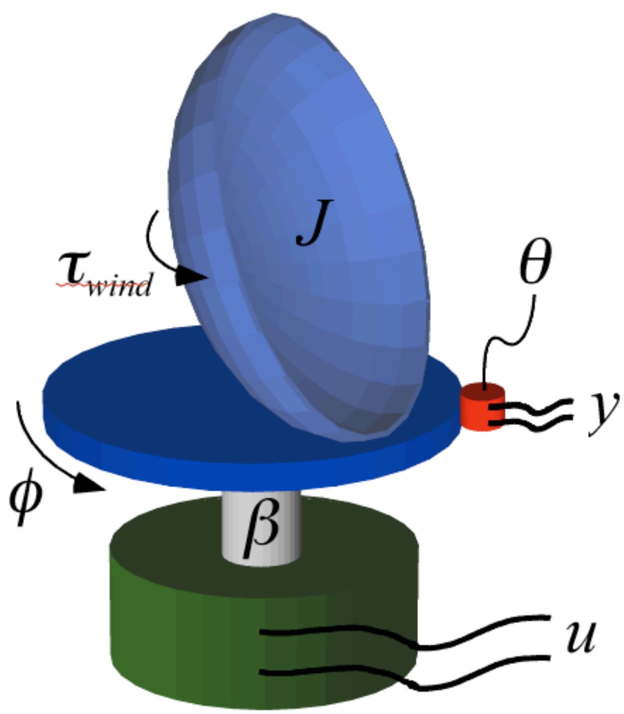
`Kest` = Kalman estimator state-space object

`K` = Optimal Kalman estimator gains

`P` = Steady-state error covariance matrix

Kalman Example - I

Let's look at the problem of regulating the angular position of a radar antenna in the presence of a wind disturbance τ_{wind} as well as sensor noise θ.



$$\underbrace{J\ddot{\phi}}_{\text{Rotational Inertia}} = -\underbrace{\beta\dot{\phi}}_{\text{Rotational Damping}} + \underbrace{\alpha u}_{\text{Control Torque}} + \underbrace{\tau_{wind}}_{\text{Disturbance Torque}}$$

$$\begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\left(\frac{\beta}{J}\right) \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \left(\frac{\alpha}{J}\right) \end{bmatrix}}_{\mathbf{B}} u + \underbrace{\begin{bmatrix} 0 \\ \left(\frac{1}{J}\right) \end{bmatrix}}_{\mathbf{F}} \tau_{wind}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} + \theta$$

Kalman Example - II

Assume that $J = \alpha = \beta = 1$, then this system is controllable and observable.

Ignoring the disturbances, we can design a state-feedback controller to place the closed-loop poles at critically damped locations: $\{-2+2j, -2-2j\}$.

$$\mathbf{G} = \begin{bmatrix} 8 & 3 \end{bmatrix} \quad \Rightarrow \quad u = -\mathbf{G}\hat{\mathbf{x}} = -8\hat{\phi} - 3\dot{\hat{\phi}}$$

Kalman Example - III

Before we design the Kalman filter, we need to know something about the noise inputs. Assume both inputs are Gaussian (white), stationary, and:

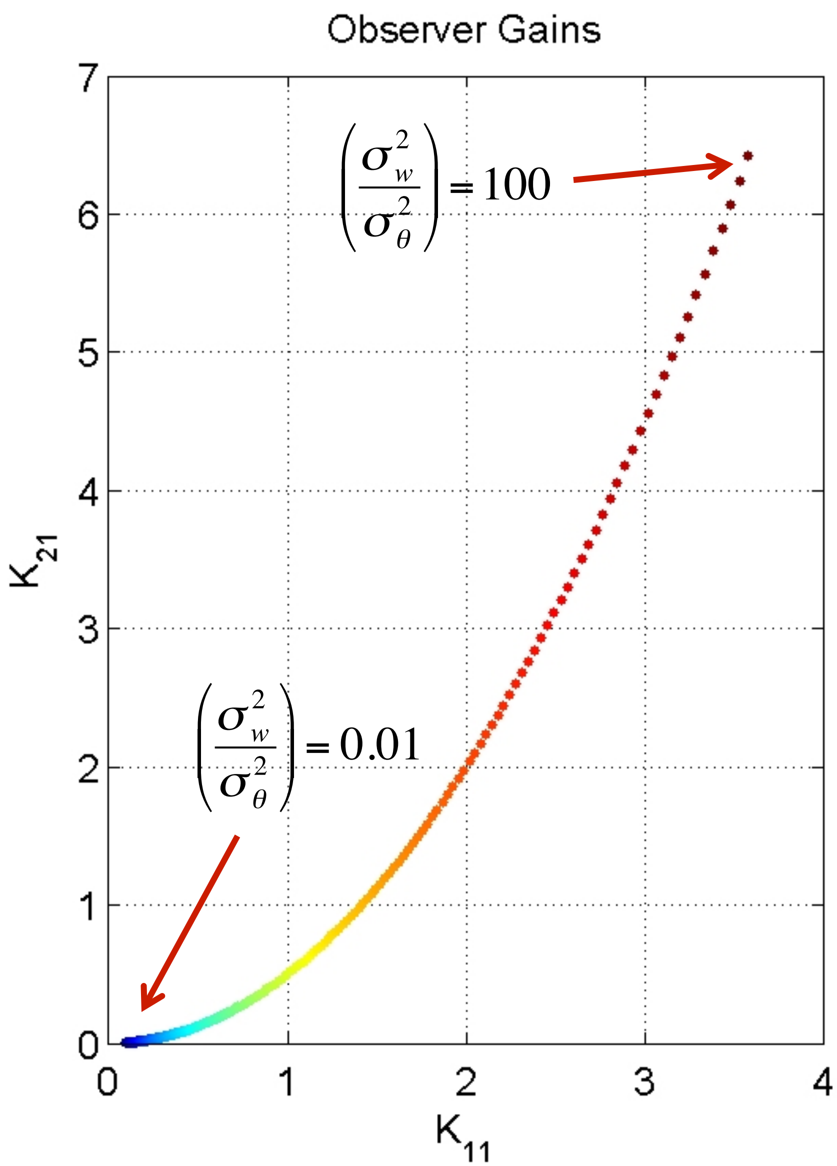
$E\langle \tau_{wind} \rangle = 0$	Zero-mean process noise
$E\langle \tau_{wind}^2 \rangle = \sigma_w^2$	Process noise covariance
$E\langle \theta \rangle = 0$	Zero-mean measurement noise
$E\langle \theta^2 \rangle = \sigma_\theta^2$	Measurement noise covariance
$E\langle \tau_{wind} \theta \rangle = 0$	Noises are uncorrelated

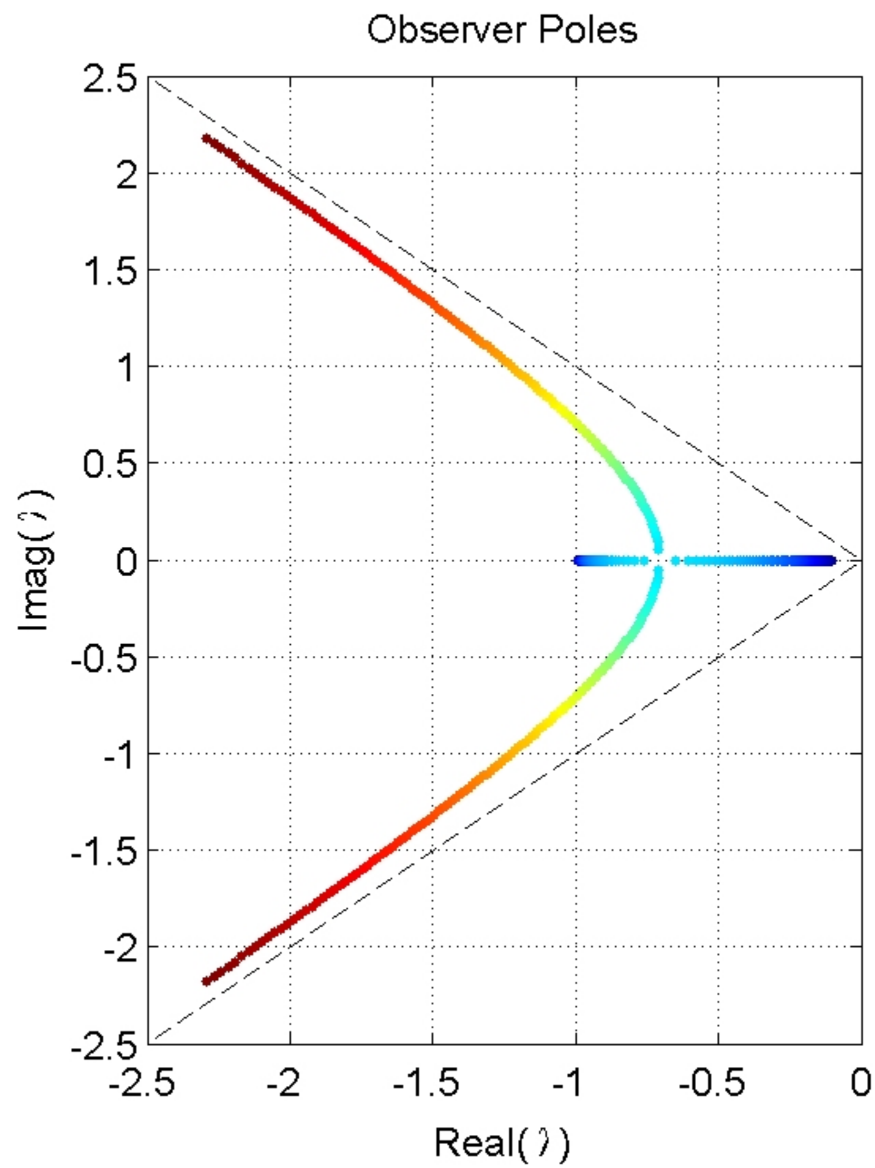
Kalman Example - IV

We can compute the Kalman optimal state estimation gains $\mathbf{K}$ as well as the Kalman filter poles for any value of the process and measurement noise covariances.

Let's plot the Kalman gains and the corresponding poles as a function of the ratio of covariances: $\left(\dfrac{\sigma_w^2}{\sigma_\theta^2}\right)$

Kalman Example - V





Kalman Example - VI

When the process noise is much larger than the sensor noise, the Kalman gains are large and the observer poles asymptotically approach critical damping (45° line).

When the sensor noise is much larger than the process noise, the Kalman gains are small and the observer poles are on the real axis.

Kalman Example - VII

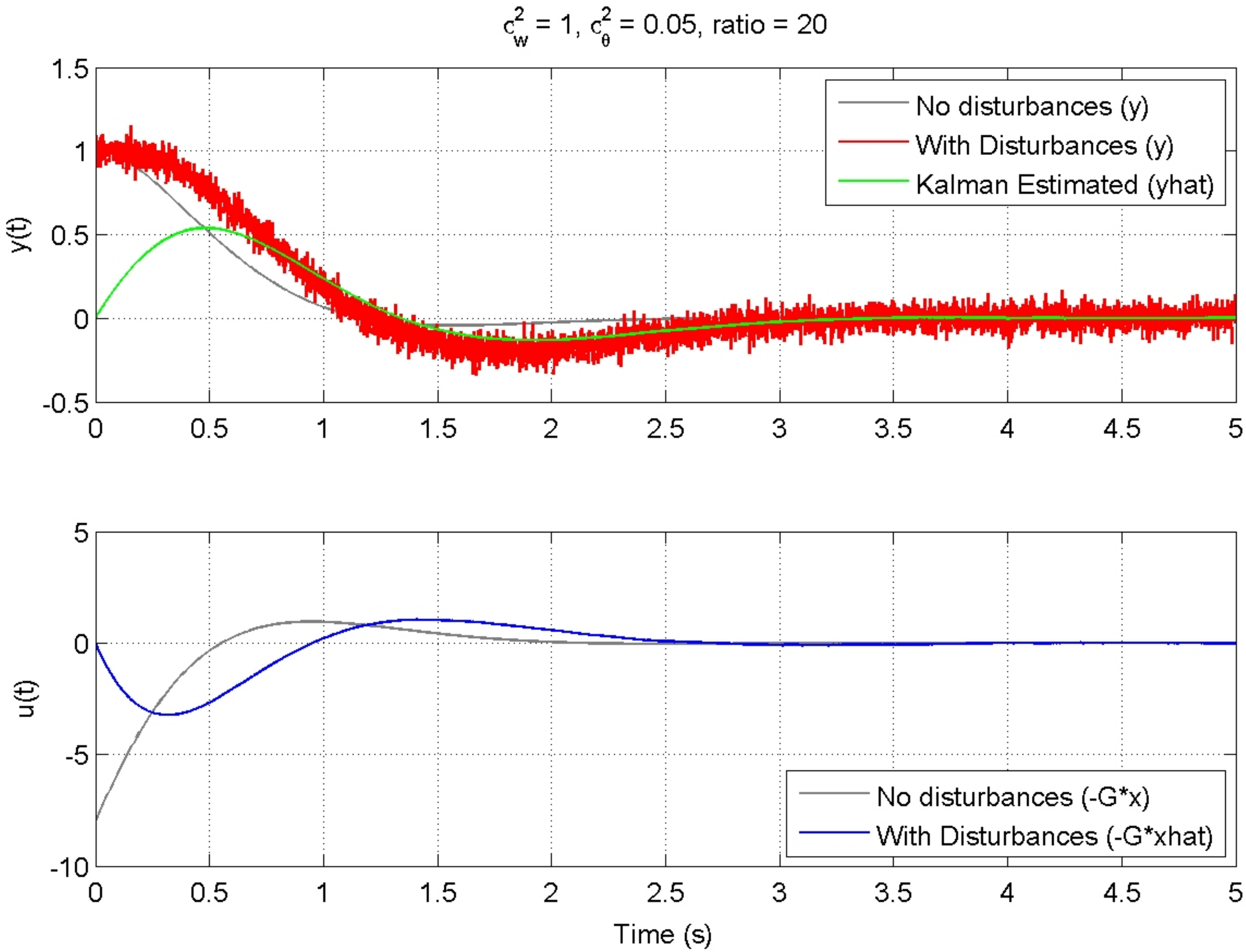
To simulate the time response, we need to construct an augmented system:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BG} \\ \mathbf{KC} & (\mathbf{A}-\mathbf{KC}-\mathbf{BG}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \begin{bmatrix} w \\ \theta \end{bmatrix}$$

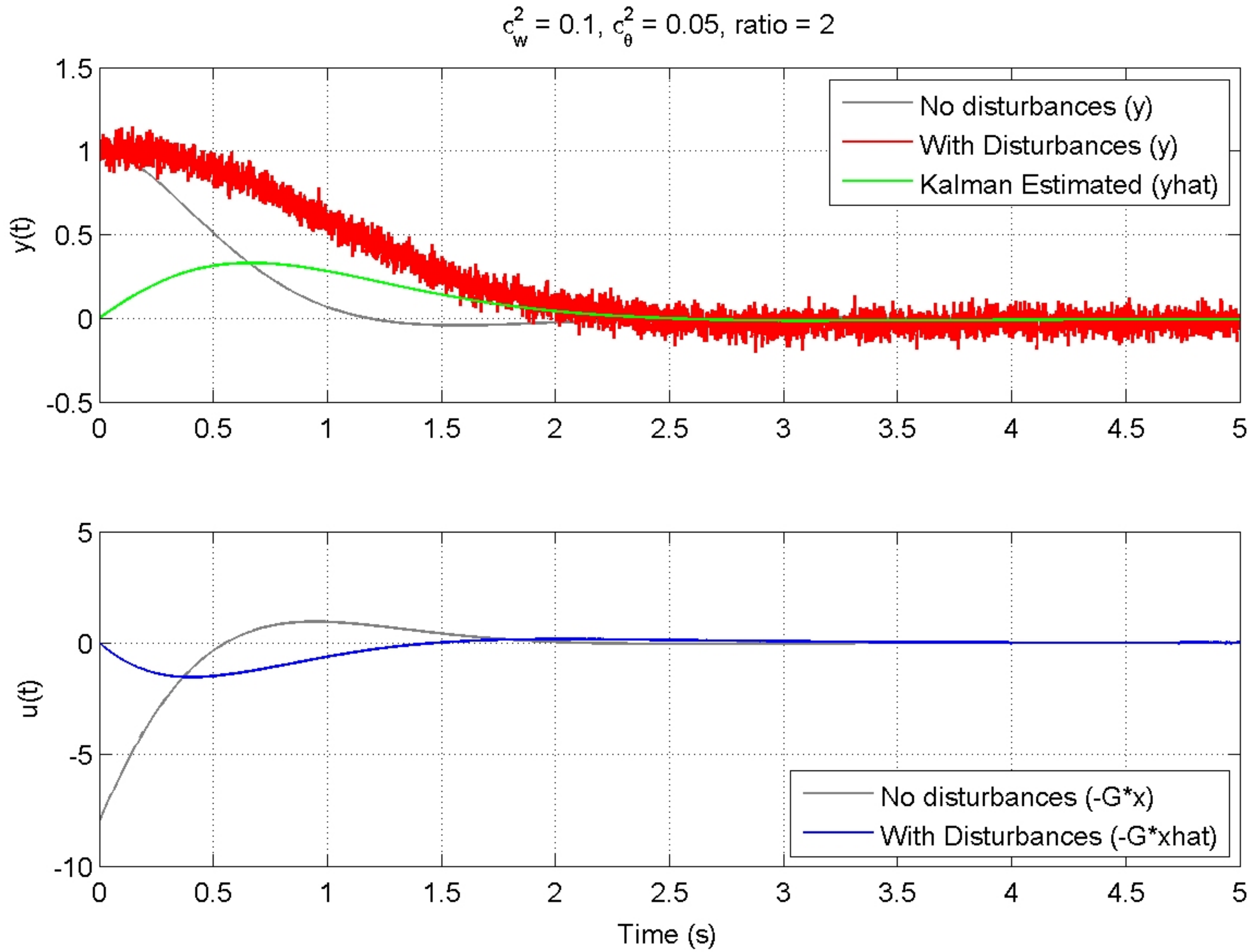
$$\begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & -\mathbf{DG} \\ \mathbf{0} & \mathbf{C}-\mathbf{DG} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \theta \end{bmatrix}$$

Of course this result assumes perfect knowledge of the plant matrices.

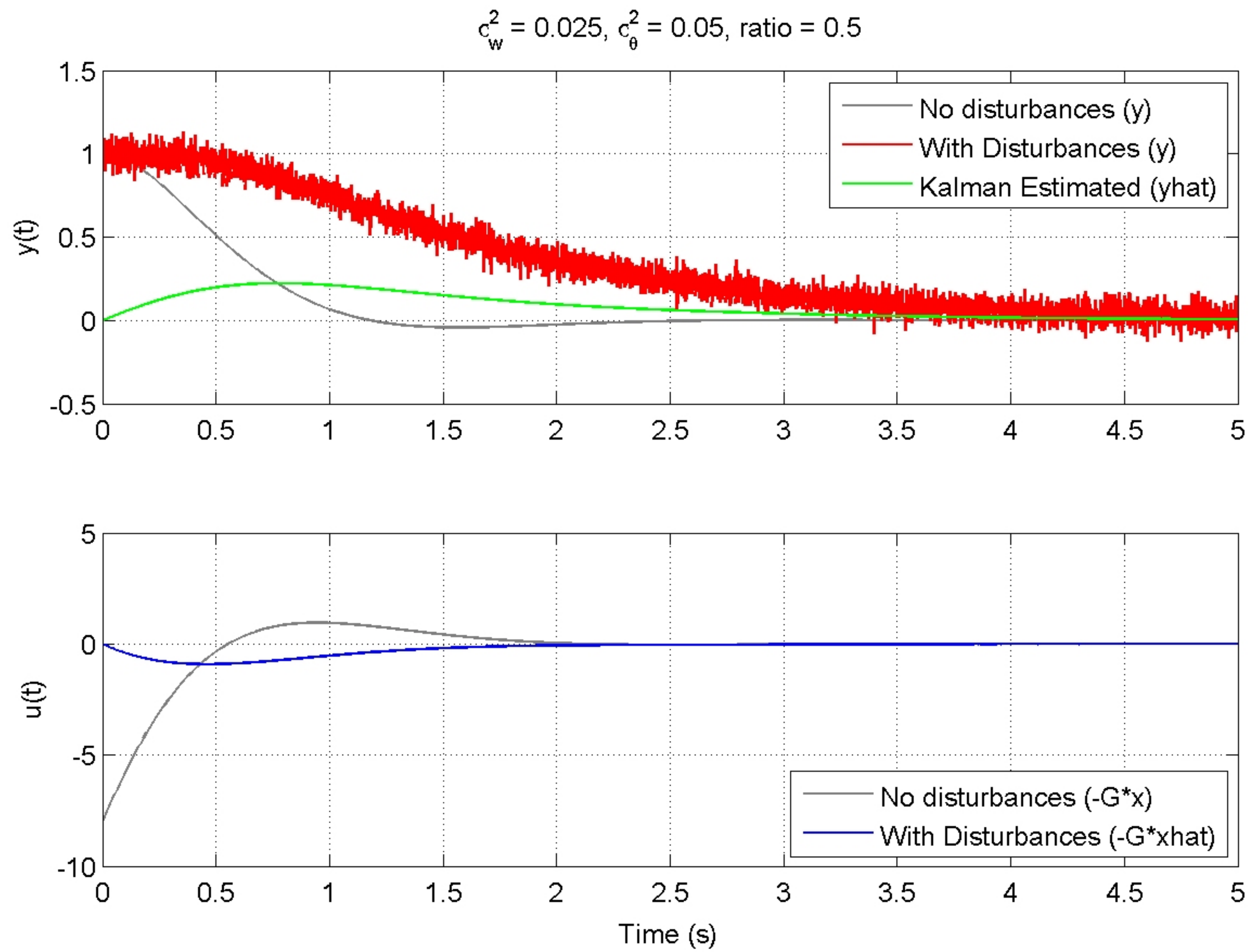
Kalman Example - VIII



Kalman Example - IX



Kalman Example - X



Where to go from here?

The LQG method is not just associated with the Kalman Filtering problem. LQG may also be applied to the control problem when Gaussian exogenous inputs are present.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\mathbf{w}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \theta$$

$$\mathbf{E}\left\langle \begin{bmatrix} \mathbf{w}^T & \theta^T \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \theta \end{bmatrix} \right\rangle = \begin{bmatrix} \mathbf{C}_{\mathbf{w}} & \mathbf{C}_{\theta\mathbf{w}} \\ \mathbf{C}_{\mathbf{w}\theta} & \mathbf{C}_{\theta} \end{bmatrix}$$

$$J(\mathbf{G}) = \int_0^\infty \begin{bmatrix} \mathbf{x}^T & \mathbf{u}^T \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} dt$$

Where to go from here?

The linear Kalman filter has been used very successfully as a state estimator for many smooth nonlinear problems.

In particular, the Kalman filter provides a linearized approximation of the effects of small perturbations to the state of the nonlinear system.

This type of application is called the Extended Kalman Filter (EKF) or the Kalman-Schmidt Filter.

Where to go from here?

The Extended Kalman Filter can be used for estimating not only the states of a dynamic system, but also the uncertain parameters.

For every uncertain parameter, an additional state is introduced and a new augmented state vector is formed. The resulting nonlinear system can be linearized about the nominal or estimated trajectory.

Where to go from here?

For more information see:

Kalman Filtering: Theory and Applications, H. Sorensen, IEEE Press, 1985.

Kalman Filtering: Theory and Practice Using MATLAB, 2nd edition, M. Grewal & A. Andrews, Wiley, 2001.