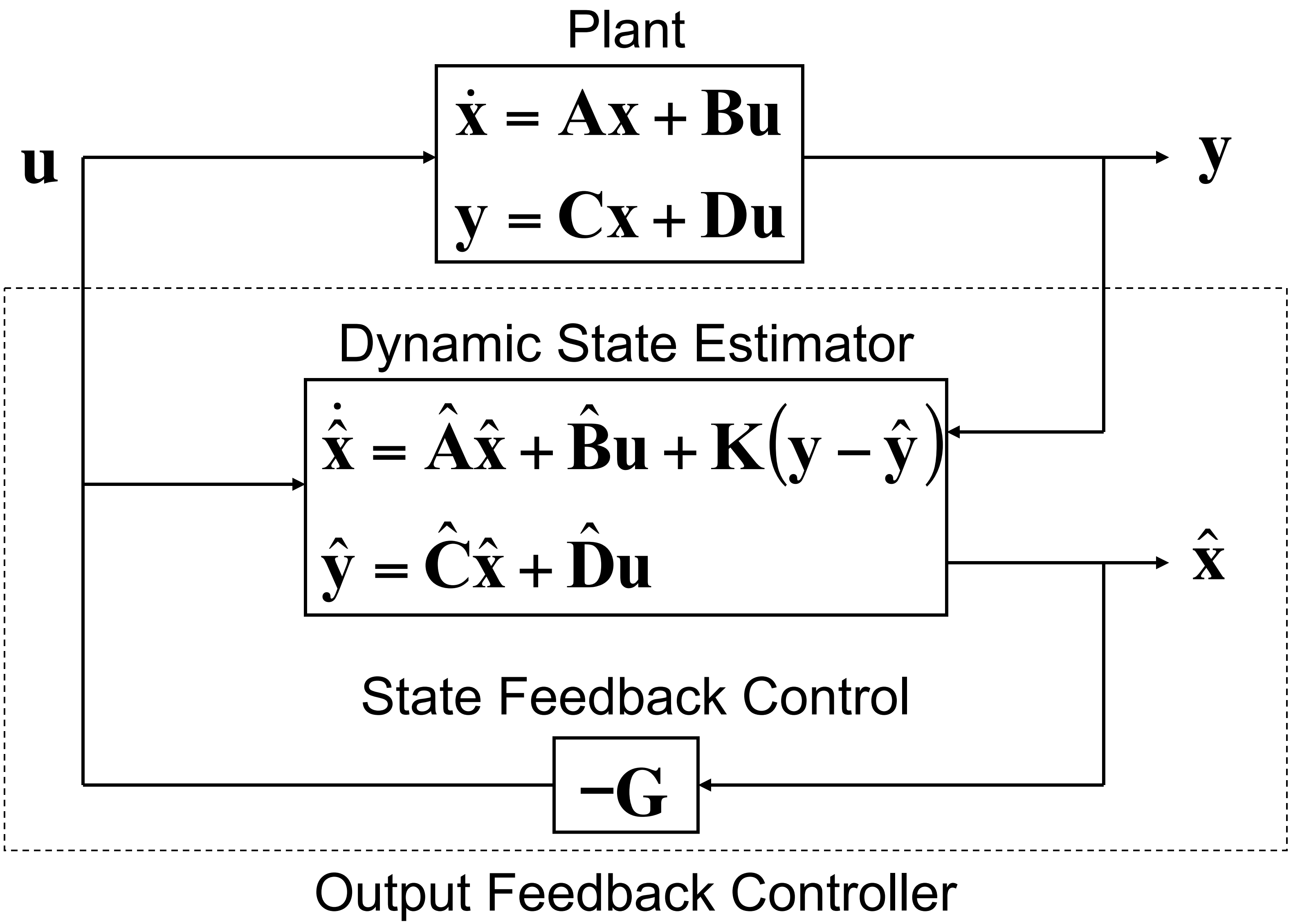
Advanced Control Engineering

- Full State Feedback for MIMO
- Stabilizability
- State Estimation & Output Feedback
- Observability & Duality
- Exogenous Inputs, Integral Control
- Optimal Control (LQR/LQG)
- Robustness & Sensitivity
- Kalman Filtering
- Introduction to Discrete Time

Output Feedback - I



Output Feedback - II

Assuming we have perfect knowledge of the plant, the augmented system dynamics are

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BG} \\ \mathbf{KC} & (\mathbf{A}-\mathbf{KC}-\mathbf{BG}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & -\mathbf{DG} \\ \mathbf{0} & \mathbf{C}-\mathbf{DG} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

and the augmented error dynamics are

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BG} & \mathbf{BG} \\ \mathbf{0} & \mathbf{A}-\mathbf{KC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

In-Class Assignment - I

Write the state-space equations for the augmented system dynamics and the augmented error dynamics associated with the following system

$$\dot{x} = -5x + u$$

$$y = 2x$$

where

$$G = 1$$

$$K = 1$$

In-Class Assignment - II

What are the closed-loop eigenvalues from the:

- Augmented system dynamics
- Augmented error dynamics

What are the eigenvalues of $\mathbf{A-BG}$?

What are the eigenvalues of $\mathbf{A-KC}$?

Observability & Duality - I

For full-state feedback, we could <u>not</u> always place the closed-loop poles of the system wherever we wanted to.

The poles could be placed arbitrarily iff the system was controllable, i.e.
$\mathrm{rank}(\mathbf{Q}) = N$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{N-1}\mathbf{B} \end{bmatrix}$$

Observability & Duality - II

The observer design problem is the "dual" of the controller design problem.

The poles of an observer can be placed arbitrarily iff the system is <u>observable</u> or the <u>observability matrix</u> has full rank,

$$\operatorname{rank}(\mathbf{O}) = N$$

$$\underset{[N\times NP]}{\mathbf{O}} = \left[\underset{[N\times P]}{\mathbf{C}^T} \quad \underset{[N\times P]}{\mathbf{A}^T\mathbf{C}^T} \quad \underset{[N\times P]}{\left(\mathbf{A}^T\right)^2\mathbf{C}^T} \quad \cdots \quad \underset{[N\times P]}{\left(\mathbf{A}^T\right)^{N-1}\mathbf{C}^T} \right]$$

Compensator Design - I

The end result of the observer design procedure is a dynamic compensator that transforms the plant output $\mathbf{y}$ into a control signal $\mathbf{u}$.

The gain and phase characteristics of the compensator can be analyzed in the frequency domain just like any compensator you might have designed using classical techniques.

Compensator Design - II

To model the compensator in the Laplace domain, we combine the full-state feedback control signal

$$\mathbf{u} = -\mathbf{G}\hat{\mathbf{x}}$$

with the observer state equations

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} + \mathbf{K}\left(\mathbf{y} - \hat{\mathbf{C}}\hat{\mathbf{x}} - \hat{\mathbf{D}}\mathbf{u}\right)$$

To give us the following state equation

Compensator Design - III

The complete set of state equations for the Compensator are:

$$\dot{\hat{\mathbf{x}}} = \left[\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{G} - \mathbf{K}\hat{\mathbf{C}} + \mathbf{K}\hat{\mathbf{D}}\mathbf{G}\right]\hat{\mathbf{x}} + \mathbf{K}\mathbf{y}$$

$$\mathbf{u} = -\mathbf{G}\hat{\mathbf{x}} + \mathbf{0}\mathbf{y}$$

Compensator Design - IV

The transfer function matrix for the compensator is obtained by transforming the state equations into the Laplace domain

$$s\hat{\mathbf{x}}(s)=\left[\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{G}-\mathbf{K}\hat{\mathbf{C}}+\mathbf{K}\hat{\mathbf{D}}\mathbf{G}\right]\hat{\mathbf{x}}(s)+\mathbf{K}\mathbf{y}(s)$$

$$\mathbf{u}(s)=-\mathbf{G}\hat{\mathbf{x}}(s)$$

Solving for the Laplace transform of the estimated states

$$\hat{\mathbf{x}}(s)=\left[s\mathbf{I}-\hat{\mathbf{A}}+\hat{\mathbf{B}}\mathbf{G}+\mathbf{K}\hat{\mathbf{C}}-\mathbf{K}\hat{\mathbf{D}}\mathbf{G}\right]^{-1}\mathbf{K}\mathbf{y}(s)$$

Compensator Design - V

Substituting into the output expression gives us the transfer function matrix of the compensator

$$\mathbf{u}(s) = -\mathbf{G}\left[s\mathbf{I} - \hat{\mathbf{A}} + \hat{\mathbf{B}}\mathbf{G} + \mathbf{K}\hat{\mathbf{C}} - \mathbf{K}\hat{\mathbf{D}}\mathbf{G}\right]^{-1}\mathbf{K}\mathbf{y}(s)$$

Note:

- The poles of the compensator are the eigenvalues of $[\mathbf{A} - \mathbf{BG} - \mathbf{KC} + \mathbf{KDG}]$
- The compensator is the same order as the plant.

Compensator Design - VI

Why do we care about the compensator transfer function matrix?

- It allows us to check the magnitude and phase of the controller
- It allows us to compute the gain and phase margin of the plant in series with the compensator
- It allows us to assess the robustness of our design (more later)

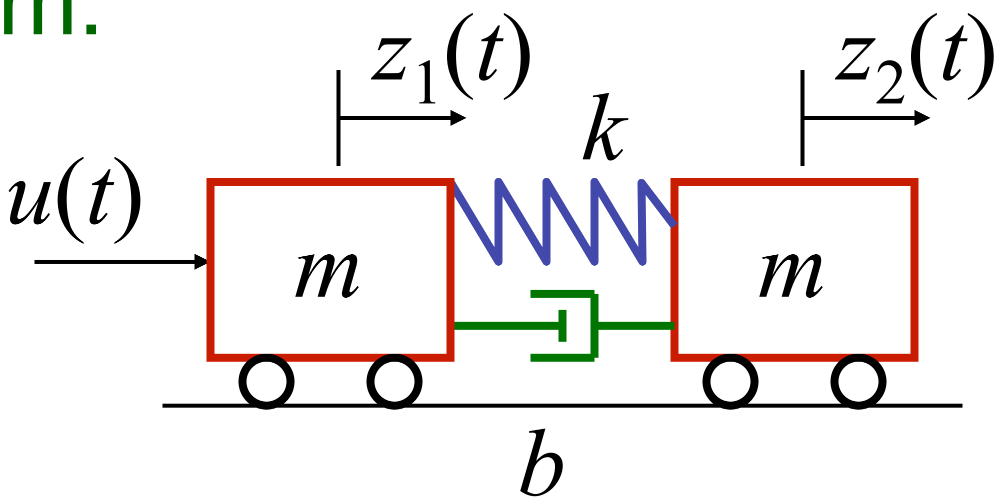
Compensator Design - VII

The compensator design procedure is:

1. Check Observability & Controllability
2. Choose poles of $[\mathbf{A}-\mathbf{BG}]$ and compute gain matrix $\mathbf{G}$
3. Choose poles of $[\mathbf{A}-\mathbf{KC}]$ and compute gain matrix $\mathbf{K}$
4. Simulate closed-loop system
5. Iterate on pole locations if necessary

Design Example - I

For the following mass-spring-mass system:



The equations of motion are:

$$m\ddot{z}_1 = u + k(z_2 - z_1) + b(\dot{z}_2 - \dot{z}_1)$$

$$m\ddot{z}_2 = k(z_1 - z_2) + b(\dot{z}_1 - \dot{z}_2)$$

Design Example - II

The state space equations are:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k}{m}\right) & \left(\frac{k}{m}\right) & -\left(\frac{b}{m}\right) & \left(\frac{b}{m}\right) \\ \left(\frac{k}{m}\right) & -\left(\frac{k}{m}\right) & \left(\frac{b}{m}\right) & -\left(\frac{b}{m}\right) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \left(\frac{1}{m}\right) \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}$$

Design Example - III

Note that for this particular problem, we could have written our equations in standard second order vibration form:

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{\text{Mass Matrix}}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} b & -b \\ -b & b \end{bmatrix}}_{\text{Damping Matrix}}\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\text{Stiffness Matrix}}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

The general form is given by:

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{B}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{F}u$$

This is NOT the same as our state-space representation!

Design Example - IV

We can construct a valid state-space state equation from the standard vibrational form with the following:

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathbf{M}^{-1}\mathbf{F} \end{bmatrix} u$$

A close check with the state-space result on S16 will confirm that this is an identical result.

Design Example - V

Using the following parameters:

$$m = 1 \text{ kg}, \quad k = 10 \text{ N/m}, \quad b = 0.05 \text{ Ns/m}$$

The open-loop poles (from $\mathbf{A}$) are:

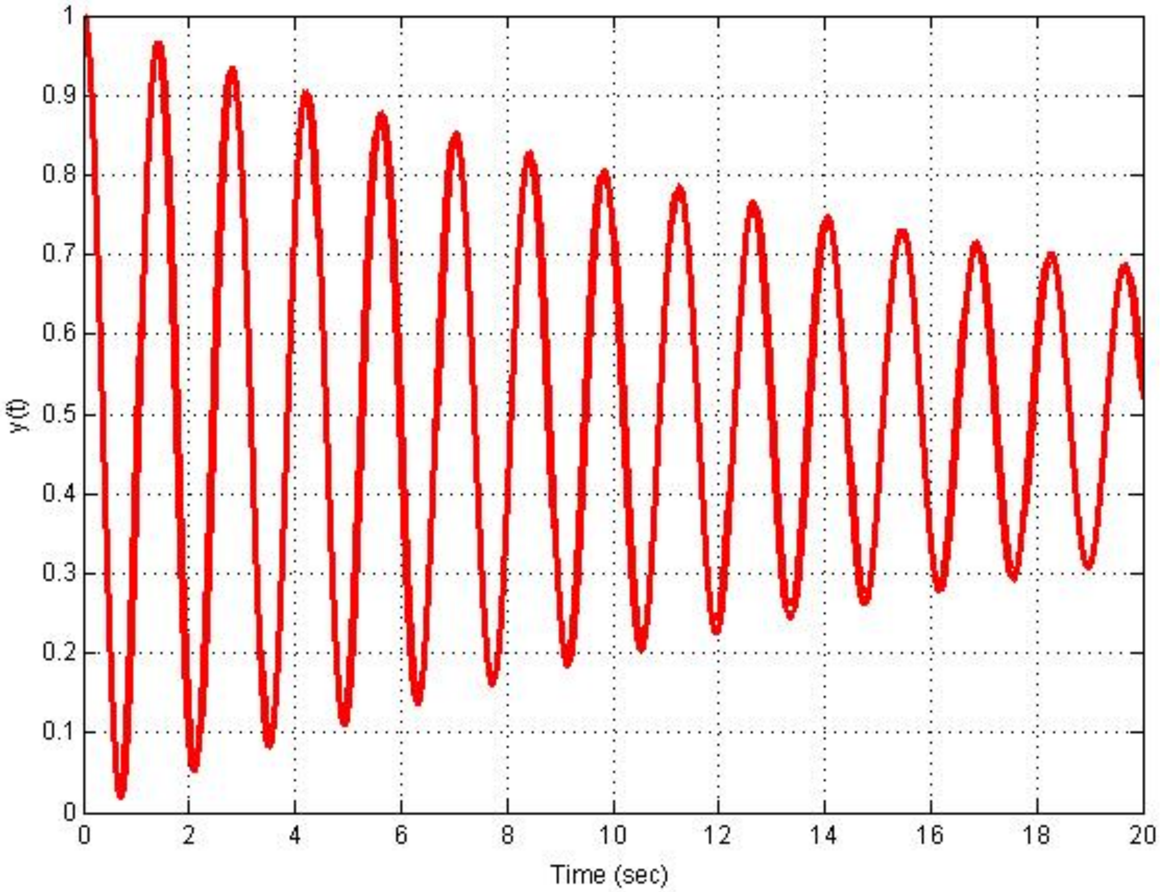
$$\lambda_{OL} = \{0, 0, -0.05 \pm j4.472\}$$

What should we expect to see in the time response based on these poles?

Design Example - VI

The open-loop initial condition response is lightly damped and has a long settling time (>20 sec).

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



Design Example - VII

Now check controllability & observability

```
>> Q = ctrb(A,B)
Q =
         0    1.0000   -0.0500   -9.9950
         0         0    0.0500    9.9950
    1.0000   -0.0500   -9.9950    1.9995
         0    0.0500    9.9950   -1.9995
>> rank(Q)
ans =
     4
>> O = obsv(A,C)' % also:  O = ctrb(A',C')
O =
         0         0   10.0000   -1.0000
    1.0000         0  -10.0000    1.0000
         0         0    0.0500    9.9950
         0    1.0000   -0.0500   -9.9950
>> rank(O)
ans =
     4
```

Design Example - VIII

Since this system is controllable and observable, let's place the poles such that the system settling time is less than 10 seconds.

To accomplish this we must move the complex poles further into the LHP. Move the rigid body modes such that their damping ratio is 0.707 (critically damped).

Design Example - IX

If $\tau = -1$, then the time constant is 1 second.

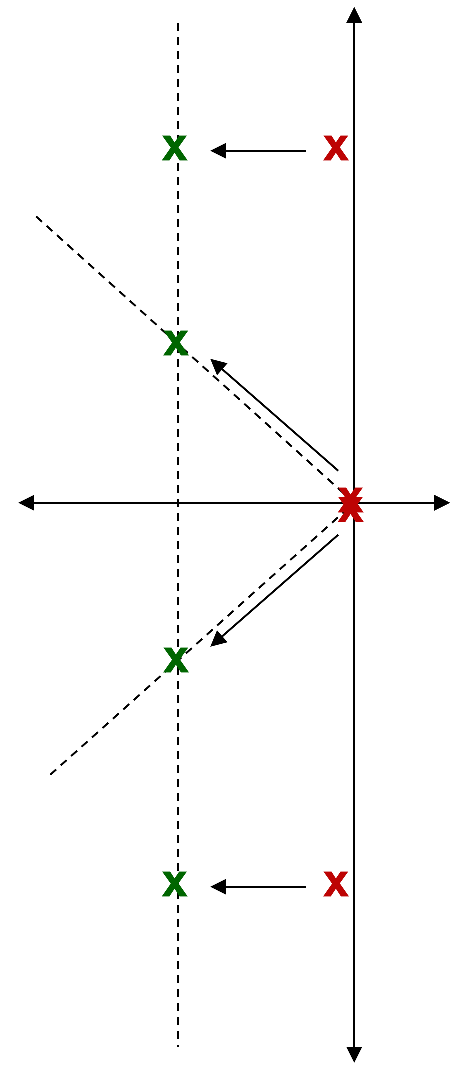
The system will reach 95% of the final value in 3 time constants or ~3 seconds.

Therefore the desired closed-loop poles are chosen to be:

$$\lambda_{CL} = \{-1 \pm j, -1 \pm j4.472\}$$

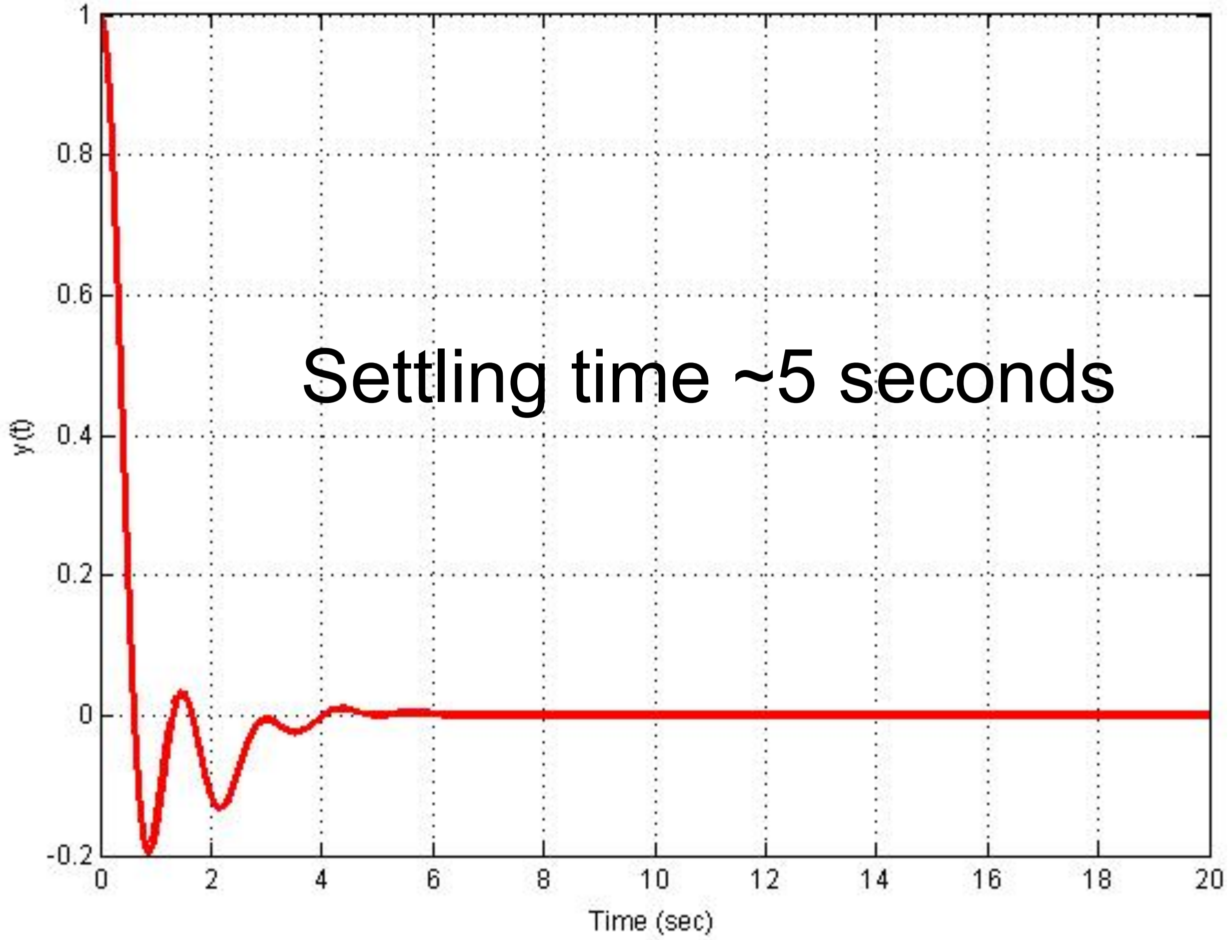
For these pole values:

```
>> G = place(A,B,[-1+j,-1-j,-1+4.472*j,-1-4.472*j])
G =
    6.7698   -2.5701    3.9000    0.6788
```



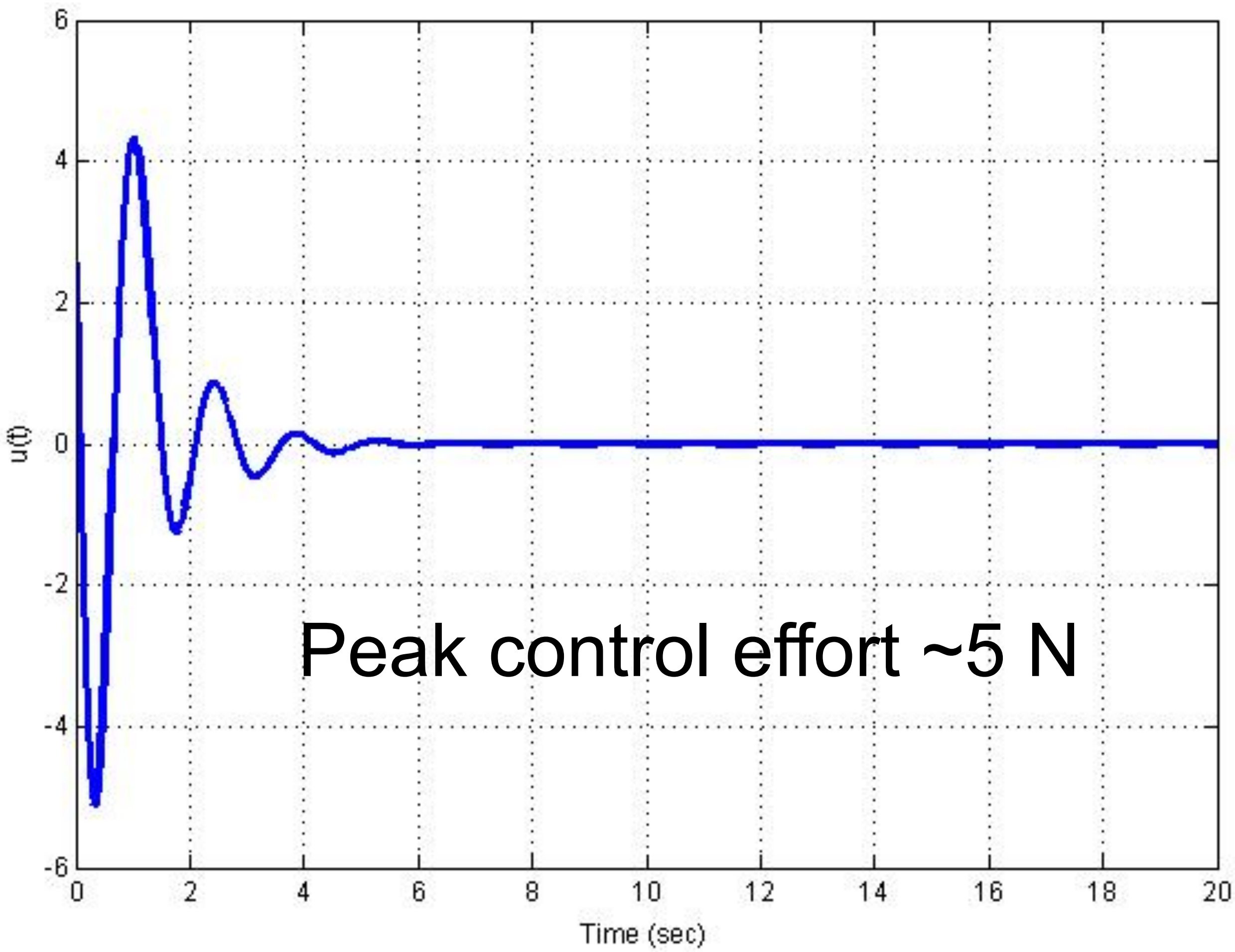
Design Example - X

The closed-loop IC response using full state feedback with the exact states is:



Design Example - XI

The control effort using full state feedback is:



Design Example - XII

Now we can design an observer to estimate the states.

For this example, let's choose observer poles that are scalar multiples of the poles of $[\mathbf{A}-\mathbf{BG}]$

$$\lambda_{\mathbf{A}-\mathbf{KC}} = \alpha \times \lambda_{\mathbf{A}-\mathbf{BG}}$$

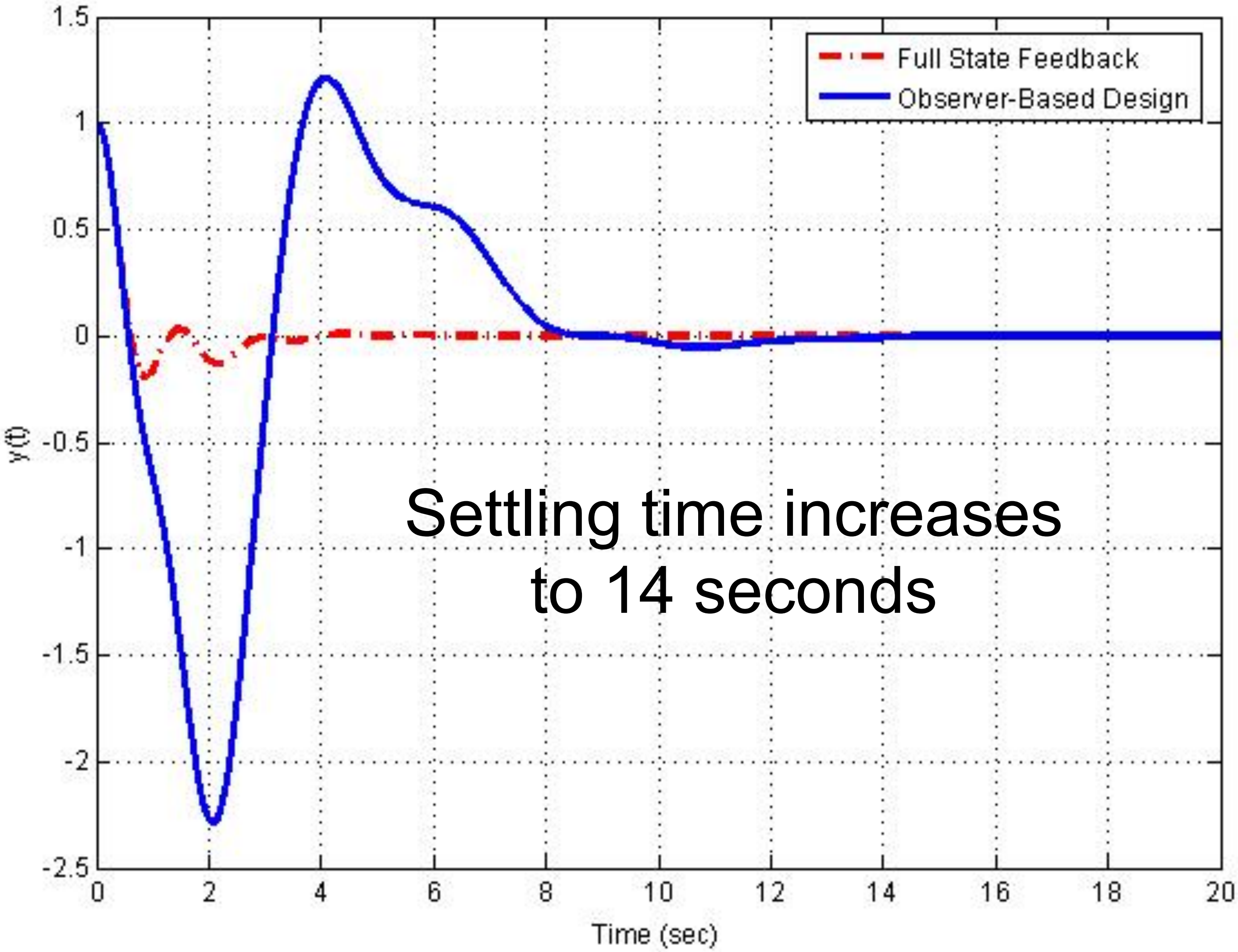
$$\alpha \ll 1 \quad \Rightarrow \quad \text{observer is "slower" than the CL poles}$$

$$\alpha \approx 1 \quad \Rightarrow \quad \text{observer is approximately same speed}$$

$$\alpha \gg 1 \quad \Rightarrow \quad \text{observer is "faster" than the CL poles}$$

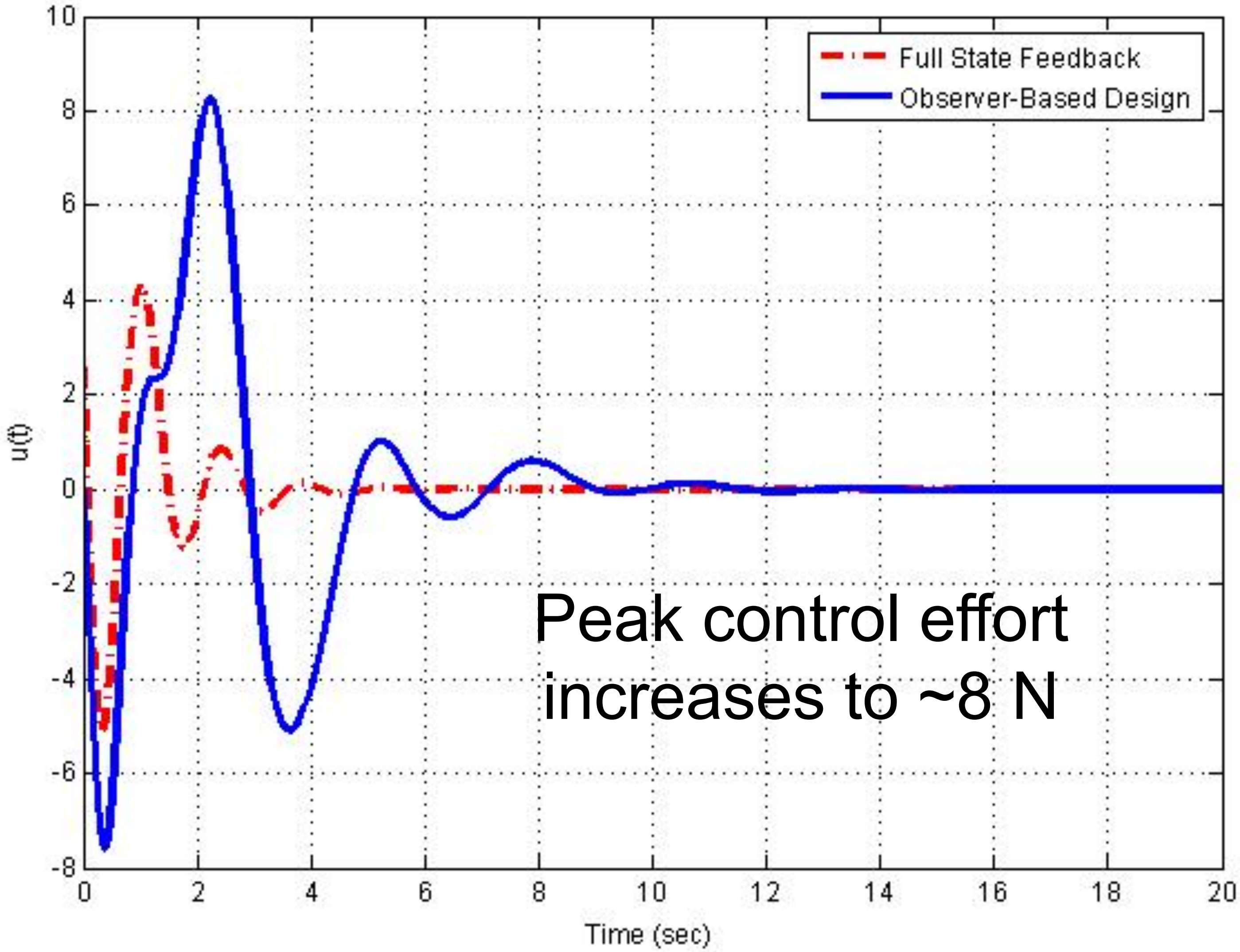
Design Example - XIII

Observer-based Design: $\alpha = 0.5$



Design Example - XIV

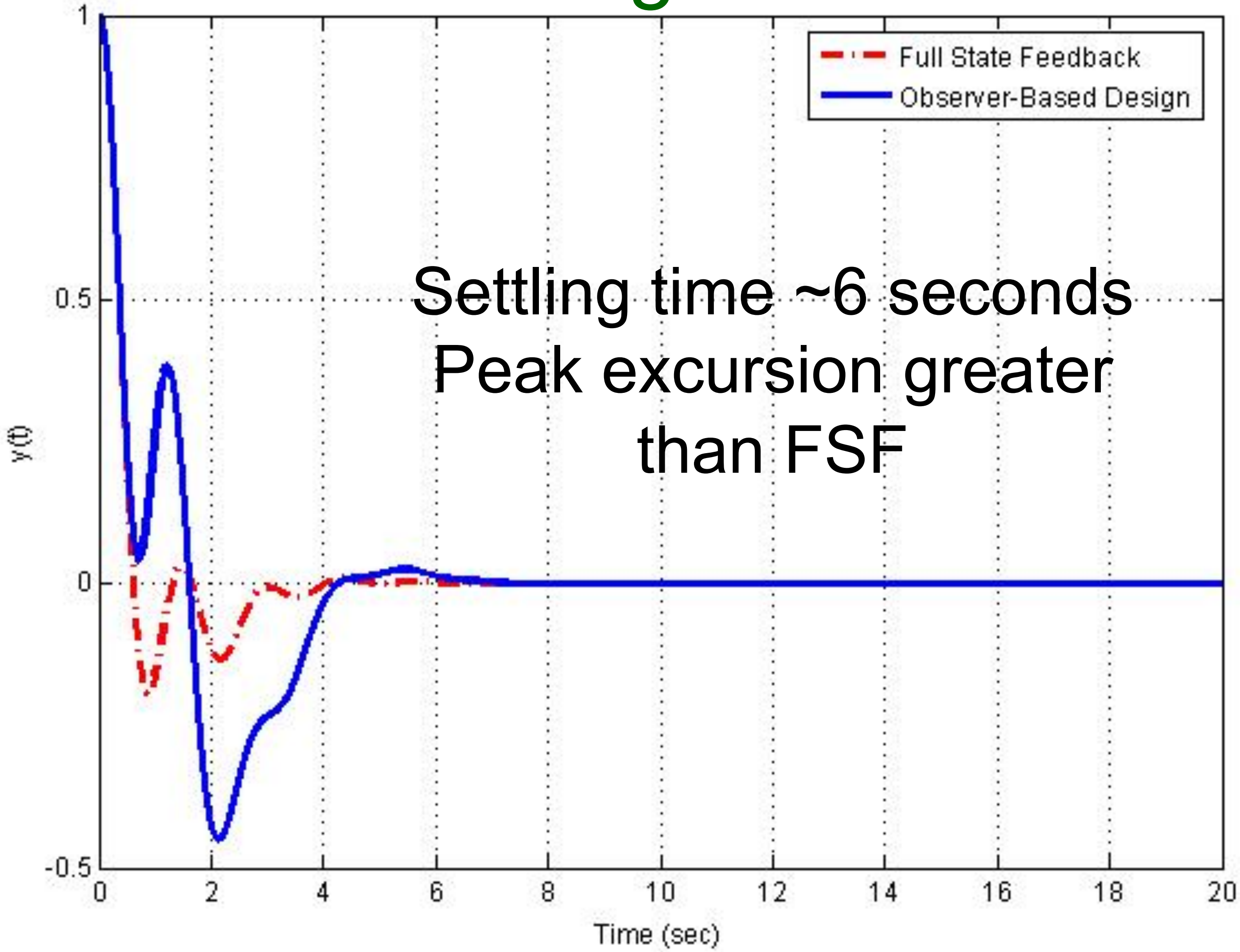
Observer-based Design: $\alpha = 0.5$



Peak control effort increases to ~8 N

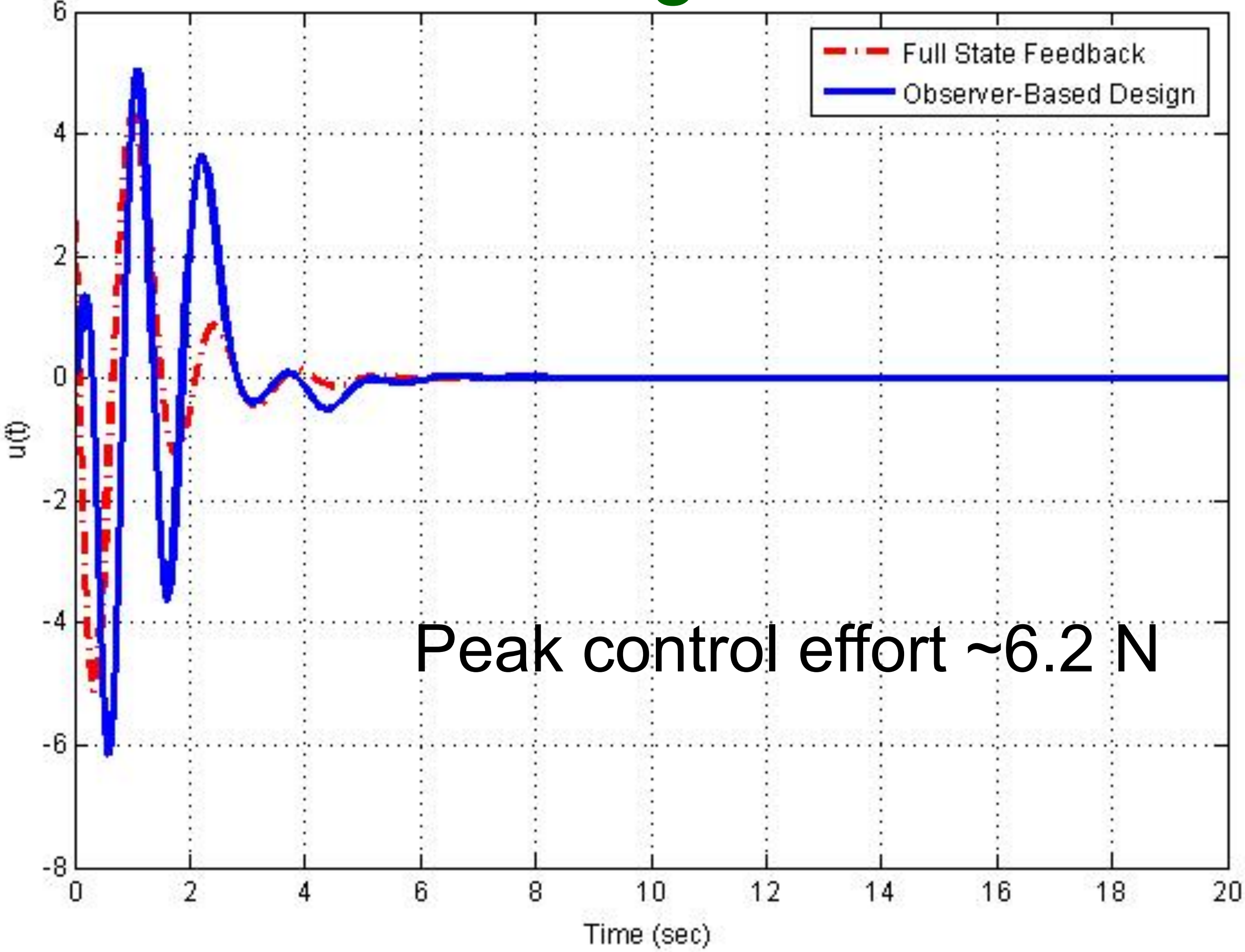
Design Example - XV

Observer-based Design: $\alpha = 1.5$



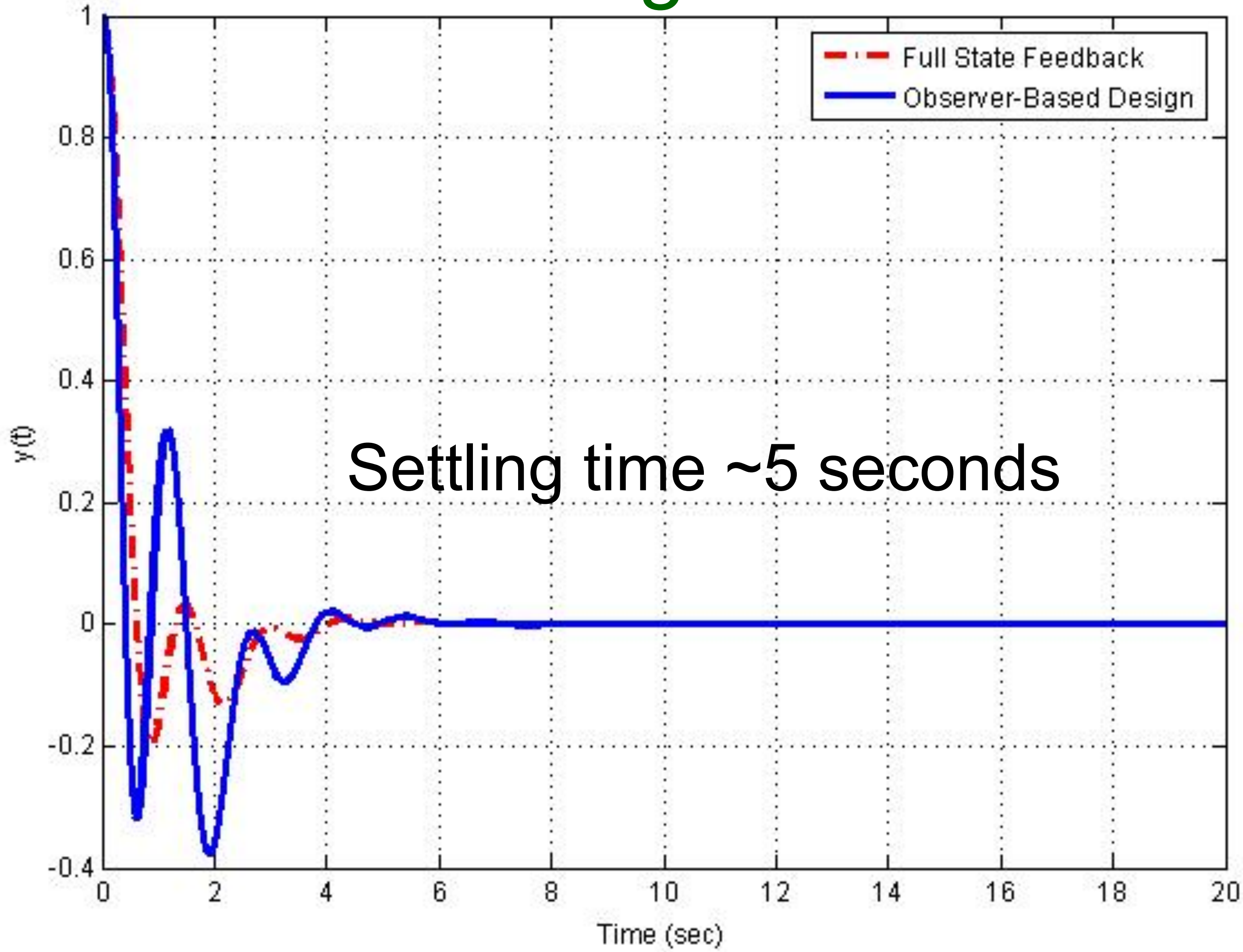
Design Example - XVI

Observer-based Design: $\alpha = 1.5$



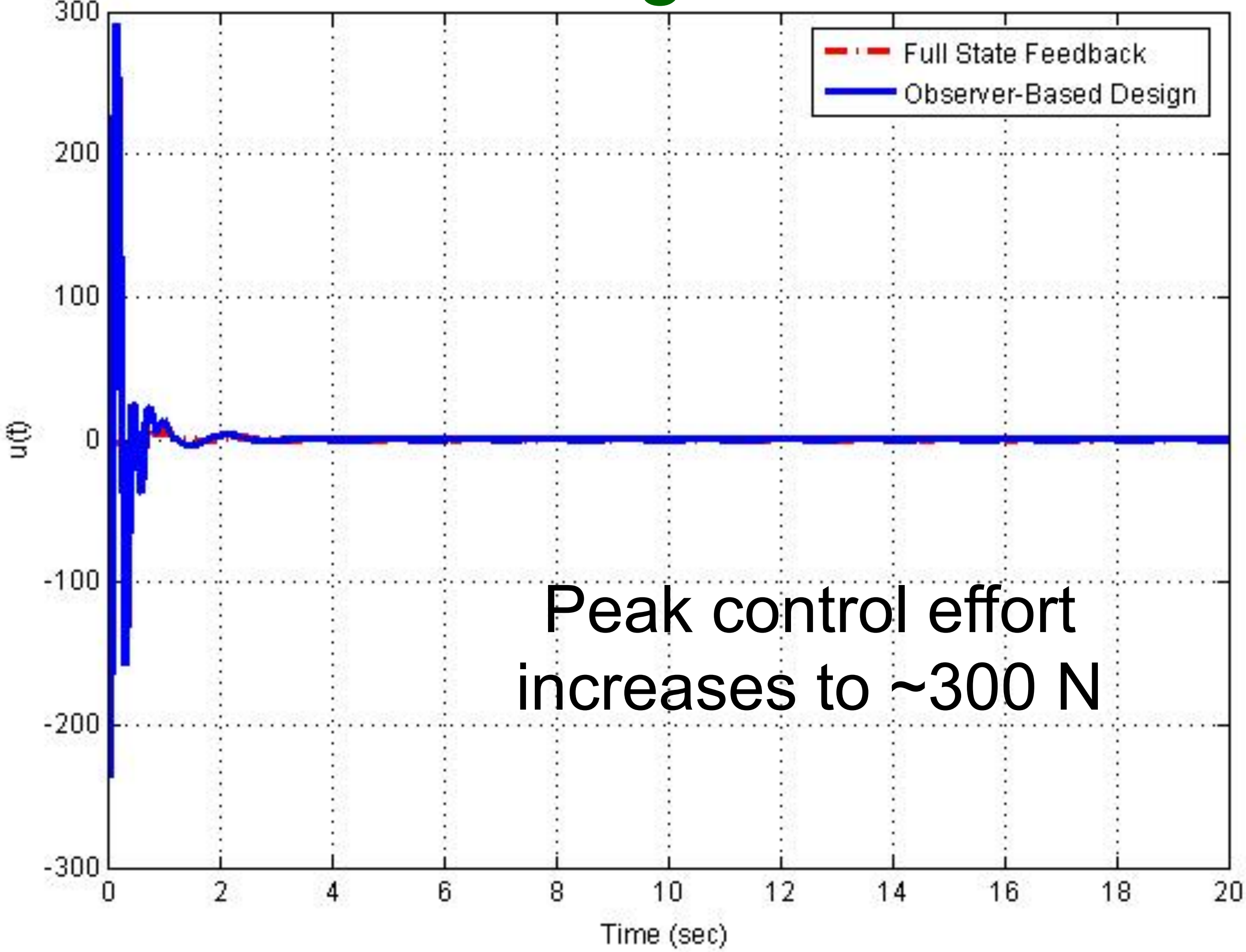
Design Example - XVII

Observer-based Design: $\alpha = 5$



Design Example - XVIII

Observer-based Design: $\alpha = 5$



Design Example - XIX

Design Summary:

	Settling Time	Peak u
FSF	5	5
Obs $\alpha = 0.5$	14	8
Obs $\alpha = 1.5$	6	6.2
Obs $\alpha = 5$	5	300

Conclusion: Increasing observer conv. rate comes at the expense of increased control effort.

Design Example - XX

The compensator transfer function we derived on S12 is:

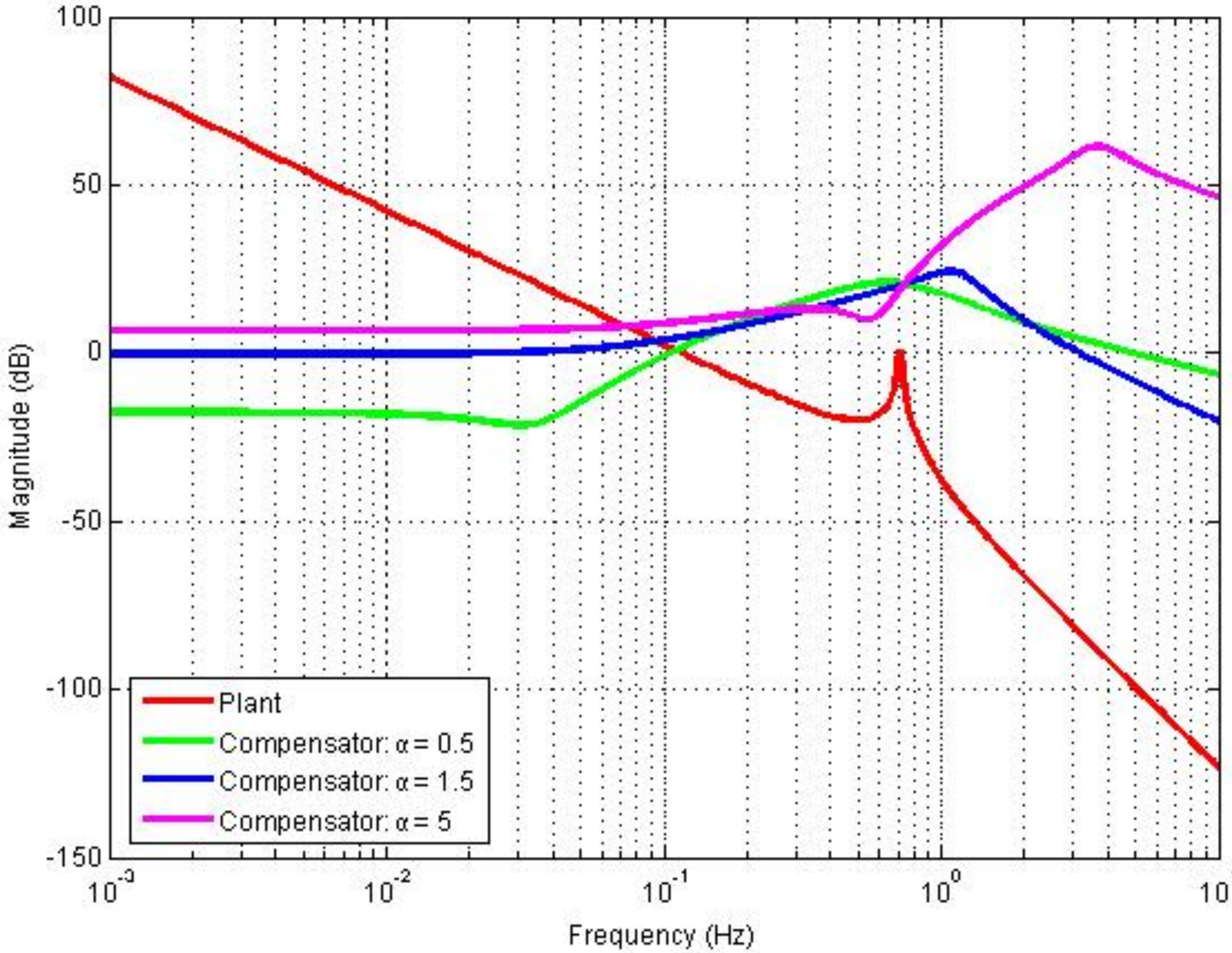
$$\mathbf{y} \longrightarrow \boxed{-\mathbf{G}\left[s\mathbf{I}-\hat{\mathbf{A}}+\hat{\mathbf{B}}\mathbf{G}+\mathbf{K}\hat{\mathbf{C}}-\mathbf{K}\hat{\mathbf{D}}\mathbf{G}\right]^{-1}\mathbf{K}} \longrightarrow \mathbf{u}$$

We can compute the frequency response like any other transfer function by substituting: $s = j\omega$

$$\mathbf{H}_C(j\omega) = -\mathbf{G}\left[j\omega\mathbf{I}-\hat{\mathbf{A}}+\hat{\mathbf{B}}\mathbf{G}+\mathbf{K}\hat{\mathbf{C}}-\mathbf{K}\hat{\mathbf{D}}\mathbf{G}\right]^{-1}\mathbf{K}$$

Design Example - XXI

What does increasing the observer convergence rate do to the compensator?



Design Example - XXII

Compensator Analysis (from the frequency domain magnitude plot)

- Compensator is similar to a lag-lead filter with additional high frequency roll-off
- Average gain of compensator increases as the observer rate of convergence increases
- High-frequency gain increases significantly as scaling parameter α increases from 1.5 to 5.0

Summary

Increasing the convergence rate of the observer can recover the performance of the full-state feedback controller.

It is generally considered good design practice to place the observer poles to the left of the closed-loop poles.

Increasing the convergence rate of the observer typically increases the control effort, compensator gain, and compensator bandwidth.

Use caution when increasing the observer bandwidth!