ME 5554 / AOE5754 / ECE5754

Applied Linear Systems

HW1

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1. Determine the eigenvalues and corresponding eigenvectors of matrix:

$$A = \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & -d & c \end{bmatrix}$$

***** Eigenvalue of A is a number λ with: $A - \lambda I = 0$

$$A - \lambda I = \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & -d & c \end{bmatrix} - \lambda . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a-\lambda & b & 0 & 0 \\ -b & a-\lambda & 0 & 0 \\ 0 & 0 & c-\lambda & d \\ 0 & 0 & -d & c-\lambda \end{bmatrix}$$

Determinant of $(A - \lambda I)$:

$$det(A - \lambda I) = (a-\lambda) . \begin{bmatrix} a-\lambda & 0 & 0 \\ 0 & c-\lambda & d \\ 0 & -d & c-\lambda \end{bmatrix} - b . \begin{bmatrix} -b & 0 & 0 \\ 0 & c-\lambda & d \\ 0 & -d & c-\lambda \end{bmatrix}$$

$$+0 . \begin{bmatrix} -b & a & 0 \\ 0 & 0 & d \\ 0 & 0 & c-\lambda \end{bmatrix} + 0 . \begin{bmatrix} a-\lambda & b & 0 \\ 0 & 0 & c-\lambda \\ 0 & 0 & -d \end{bmatrix}$$

$$= (a-\lambda)^2 . (c-\lambda)^2 - (a-\lambda)^2 . d . (-d) - b(-b) . (c-\lambda)^2$$

$$+b(-b) . d . (-d)$$

$$= (a-\lambda)^2 . (c-\lambda)^2 + (a-\lambda)^2 . d^2 + b^2 . (c-\lambda)^2 + b^2 . d^2$$

$$= [(a-\lambda)^2 + b^2] . [(c-\lambda)^2 + d^2]$$

Assume that $det(A - \lambda I) = 0$, we have:

$$[(a-\lambda)^2 + b^2] . [(c-\lambda)^2 + d^2] = 0$$

So we have 4 results of λ: $\left[\begin{array}{l} \lambda = a + bi \\ \lambda = a - bi \\ \lambda = c + di \\ \lambda = c - di \end{array}\right.$

***** Eigenvector v of A is a 4x1 matrix: $v = [v_1 \quad v_2 \quad v_3 \quad v_4]^T$

and: $(A - \lambda I)v = 0$

- For $\lambda_1 = a + bi$:

$$A - \lambda_1 I = \begin{bmatrix} -bi & b & 0 & 0 \\ -b & -bi & 0 & 0 \\ 0 & 0 & c-a-bi & d \\ 0 & 0 & -d & c-a-bi \end{bmatrix}$$

$(A - \lambda_1 I)v_1 = 0$ so:

$$\begin{bmatrix} -bi & b & 0 & 0 \\ -b & -bi & 0 & 0 \\ 0 & 0 & c-a-bi & d \\ 0 & 0 & -d & c-a-bi \end{bmatrix} . \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} (-bi).v_{11} + b.v_{12} = 0 \\ -b.v_{11} - (bi).v_{12} = 0 \\ (c-a-bi).v_{13} + d.v_{14} = 0 \\ -d.v_{13} + (c-a-bi).v_{14} = 0 \end{cases}$$

$$\Rightarrow v_1 = \alpha_1 . \begin{bmatrix} 1 \\ -i \\ -d \\ c-a-bi \end{bmatrix} \qquad \text{for any } \alpha_1 \neq 0$$

- For $\lambda_2 = a - bi$:

$$A - \lambda_2 I = \begin{bmatrix} bi & b & 0 & 0 \\ -b & bi & 0 & 0 \\ 0 & 0 & c-a+bi & d \\ 0 & 0 & -d & c-a+bi \end{bmatrix}$$

$(A - \lambda_2 I)v_2 = 0$ so:

$$\begin{bmatrix} bi & b & 0 & 0 \\ -b & bi & 0 & 0 \\ 0 & 0 & c-a+bi & d \\ 0 & 0 & -d & c-a+bi \end{bmatrix} . \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} bi.v_{21} + b.v_{22} = 0 \\ -b.v_{21} + bi.v_{22} = 0 \\ (c-a+bi).v_{23} + d.v_{24} = 0 \\ -d.v_{23} + (c-a+bi).v_{24} = 0 \end{cases}$$

$$\Rightarrow v_2 = \alpha_2.\begin{bmatrix} 1 \\ i \\ -d \\ c-a+bi \end{bmatrix} \qquad \text{for any } \alpha_2 \neq 0$$

- For $\lambda_3 = c + di$:

$$A - \lambda_3 I = \begin{bmatrix} a-c-di & b & 0 & 0 \\ -b & a-c-di & 0 & 0 \\ 0 & 0 & -di & d \\ 0 & 0 & -d & -di \end{bmatrix}$$

$(A - \lambda_3 I)v_3 = 0$ so:

$$\begin{bmatrix} a-c-di & b & 0 & 0 \\ -b & a-c-di & 0 & 0 \\ 0 & 0 & -di & d \\ 0 & 0 & -d & -di \end{bmatrix}.\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \\ v_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} (a-c-di).v_{31} + b.v_{32} = 0 \\ -b.v_{31} + (a-c-di).v_{32} = 0 \\ -di.v_{33} + d.v_{34} = 0 \\ -d.v_{33} - di.v_{34} = 0 \end{cases}$$

$$\Rightarrow v_3 = \alpha_3.\begin{bmatrix} -b \\ a-c-di \\ -i \\ 1 \end{bmatrix} \qquad \text{for any } \alpha_3 \neq 0$$

- For $\lambda_4 = c - di$:

$$A - \lambda_4 I = \begin{bmatrix} a-c+di & b & 0 & 0 \\ -b & a-c+di & 0 & 0 \\ 0 & 0 & di & d \\ 0 & 0 & -d & di \end{bmatrix}$$

$(A-\lambda_4 I)v_4 = 0$ so:

$$\begin{bmatrix} a-c+di & b & 0 & 0 \\ -b & a-c+di & 0 & 0 \\ 0 & 0 & di & d \\ 0 & 0 & -d & di \end{bmatrix}.\begin{bmatrix} v_{41} \\ v_{42} \\ v_{43} \\ v_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} (a-c+di).v_{41} + b.v_{42} = 0 \\ -b.v_{41} + (a-c+di).v_{42} = 0 \\ di.v_{43} + d.v_{44} = 0 \\ -d.v_{43} + di.v_{44} = 0 \end{cases}$$

$$\Rightarrow v_4 = \alpha_4.\begin{bmatrix} -b \\ a-c+di \\ i \\ 1 \end{bmatrix} \qquad \text{for any } \alpha_4 \neq 0$$

2. Compute the determinant of matrix A:

$$A = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}$$

***** Compute determinant by using the rule of Sarrus:

$$\det(A) = 0.0.(-a_2) + 0.(-a_1).0 + (-a_0).1.1 - (-a_0).0.0 - 0.1.(-a_2) - 0.(-a_1).1$$

$$= -a_0$$

***** Compute determinant using the first column:

$$\det(A) = 0.\begin{bmatrix} 0 & -a_1 \\ 1 & -a_2 \end{bmatrix} - 1.\begin{bmatrix} 0 & -a_0 \\ 1 & -a_2 \end{bmatrix} + 0.\begin{bmatrix} 0 & -a_0 \\ 0 & -a_1 \end{bmatrix}$$

$$= 0.(0.(-a_2) - 1.(-a_1)) - 1.\big(0.(-a_2) - 1.(-a_0)\big) + 0.(0.(-a_1) - 0.(-a_0))$$

$$= -a_0$$

NE1.2 Define the state variables and derive the coefficient matrices

a. $\dot{y}(t) + 2y(t) = u(t)$

Define: $X_1(t) = y(t)$

so: $\dot{X}_1(t) = \dot{y}(t)$

We have: $\dot{X}_1(t) + 2X_1(t) = u(t) \quad \Leftrightarrow \dot{X}_1(t) = -2X_1(t) + u(t)$

and $y(t) = X_1(t)$

So: State matrix A: A = -2

Input matrix B: B = 1

Output matrix: C = 1

Direct Transmission matrix D: D = 0

b. $\ddot{y}(t) + 3\dot{y}(t) + 10y(t) = u(t)$

Define: $X_1 = y(t)$

$X_2 = \dot{y}(t) = \dot{X}_1$

$U = u(t)$

so: $\ddot{y}(t) = \dot{X}_2$

We have: $\dot{X}_1 = X_2$

$\dot{X}_2 + 3X_2 + 10X_1 = U$

$\Leftrightarrow \dot{X}_2 = -10X_1 - 3X_2 + U$

So: $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$

and: $y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

Finally: State matrix A is: $A = \begin{bmatrix} 0 & 1 \\ -10 & -3 \end{bmatrix}$

Input matrix B is: $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Output matrix C is: $C = [1 \quad 0]$

Direct Transmission matrix: D = 0

c. $\dddot{y}(t) + 2\ddot{y}(t) + 3\dot{y}(t) + 5y(t) = u(t)$

Define: $X_1 = y(t)$

$X_2 = \dot{y}(t) = \dot{X}_1$

$X_3 = \ddot{y}(t) = \dot{X}_2$

$U = u(t)$

so: $\dddot{y}(t) = \dot{X}_3$

We have: $\dot{X}_1 = X_2$

$\dot{X}_2 = X_3$

$\dot{X}_3 + 2X_3 + 3X_2 + 5X_1 = U$

$\Leftrightarrow \dot{X}_3 = -5X_1 - 3X_2 - 2X_3 + U$

So: $$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

and: $$y = [1 \quad 0 \quad 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Finally: State matrix A is: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -2 \end{bmatrix}$

Input matrix B is: $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Output matrix C is: $C = [1 \quad 0 \quad 0]$

Direct transmission matrix: D = 0

d. $\ddot{y}_1(t) + 5y_1(t) - 10[y_2(t) - y_1(t)] = u_1(t)$ (1)

$2\ddot{y}_2(t) + \dot{y}_2(t) + 10[y_2(t) - y_1(t)] = u_2(t)$ (2)

Define: $X_1 = y_1(t)$

$X_2 = \dot{y}_1(t) = \dot{X}_1$

$X_3 = y_2(t)$

$X_4 = \dot{y}_2(t) = \dot{X}_3$

$U_1 = u_1(t)$

$U_2 = u_2(t)$

so: $\ddot{y}_1(t) = \dot{X}_2$

$\ddot{y}_2(t) = \dot{X}_4$

We have: $\dot{X}_1 = X_2$

$\dot{X}_3 = X_4$

(1) $\dot{X}_2 + 5X_1 - 10(X_3 - X_1) = U_1$

$\Leftrightarrow \dot{X}_2 = -15X_1 + 10X_3 + U_1$

(2) $\dot{X}_4 + X_4 + 10(X_3 - X_1) = U_2$

$\Leftrightarrow \dot{X}_4 = 10X_1 - 10X_3 - X_4 + U_2$

So:
$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -15 & 0 & 10 & 0 \\ 0 & 0 & 1 & 0 \\ 10 & 0 & -10 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

Finally: State matrix A is: $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -15 & 0 & 10 & 0 \\ 0 & 0 & 1 & 0 \\ 10 & 0 & -10 & -1 \end{bmatrix}$

Input matrix B is: $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

Output matrix C is: $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Direct Transmission matrix D is: D = 0