Course Outline - 1st Half

- Review Classical Feedback Control
- Review Vector/Matrix Theory
- State-Space Representations
- LTI Response, Matrix Exponential
- Transfer Functions & Eigenvalues
- Frequency-Domain Analysis
- Harmonic & Impulse Responses
- Pole Placement
- Controllability

Intro to Controllability - I

The concepts of *controllability* and *observability* are important to state-space design techniques.

Controllability is related to whether or not we can arbitrarily place the closed-loop poles using full-state feedback.

Observability is related to our ability to design state observers or estimators for output feedback (more later)

Intro to Controllability - II

Consider the following state-space system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

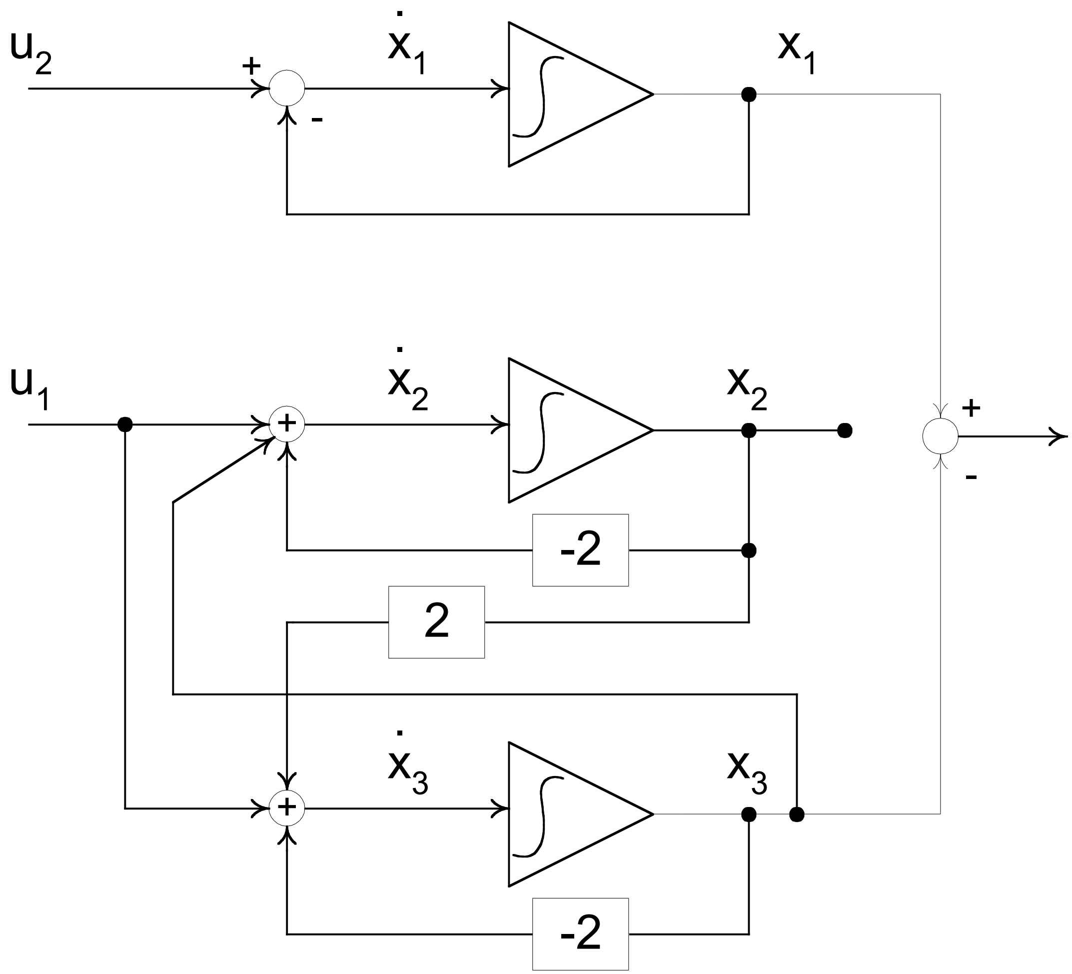
$$y(t) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Which states are controllable?

Which states are observable?

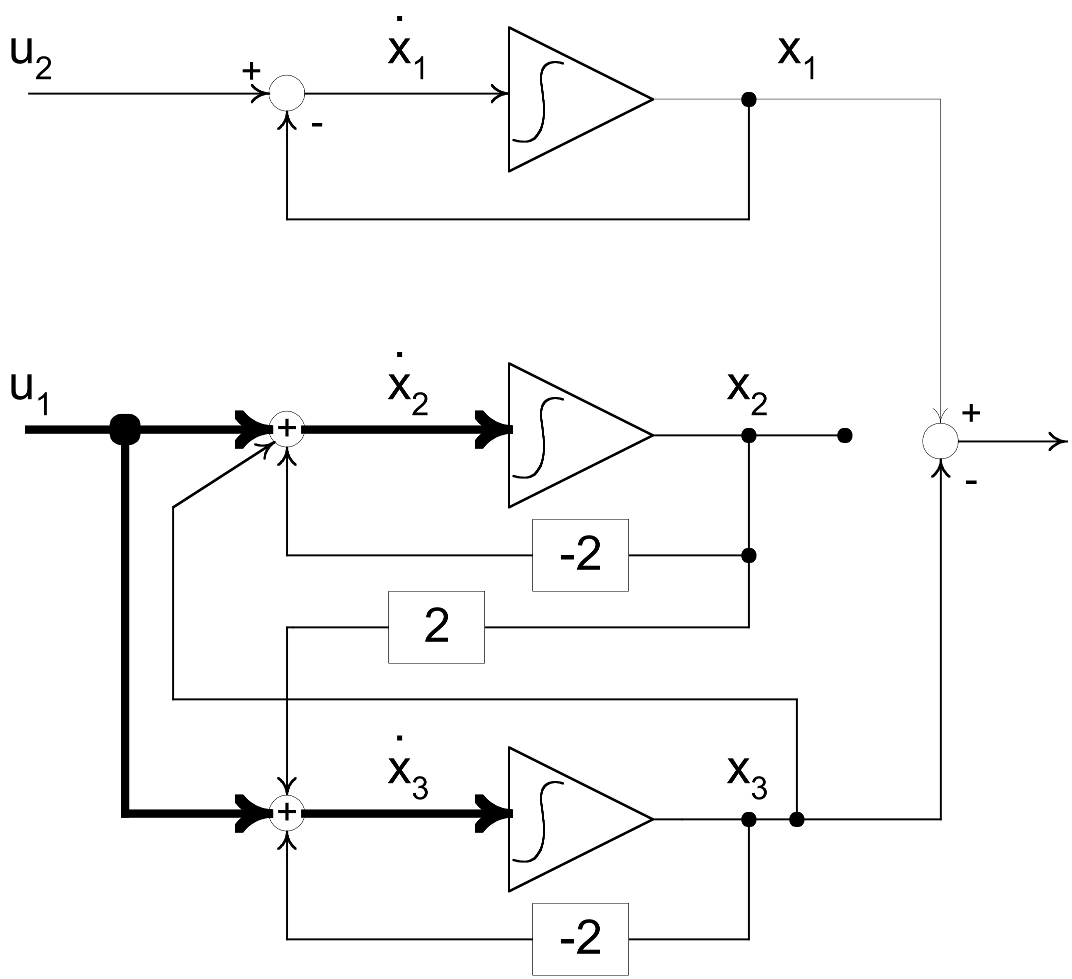
Intro to Controllability - III

First, let's draw a block diagram



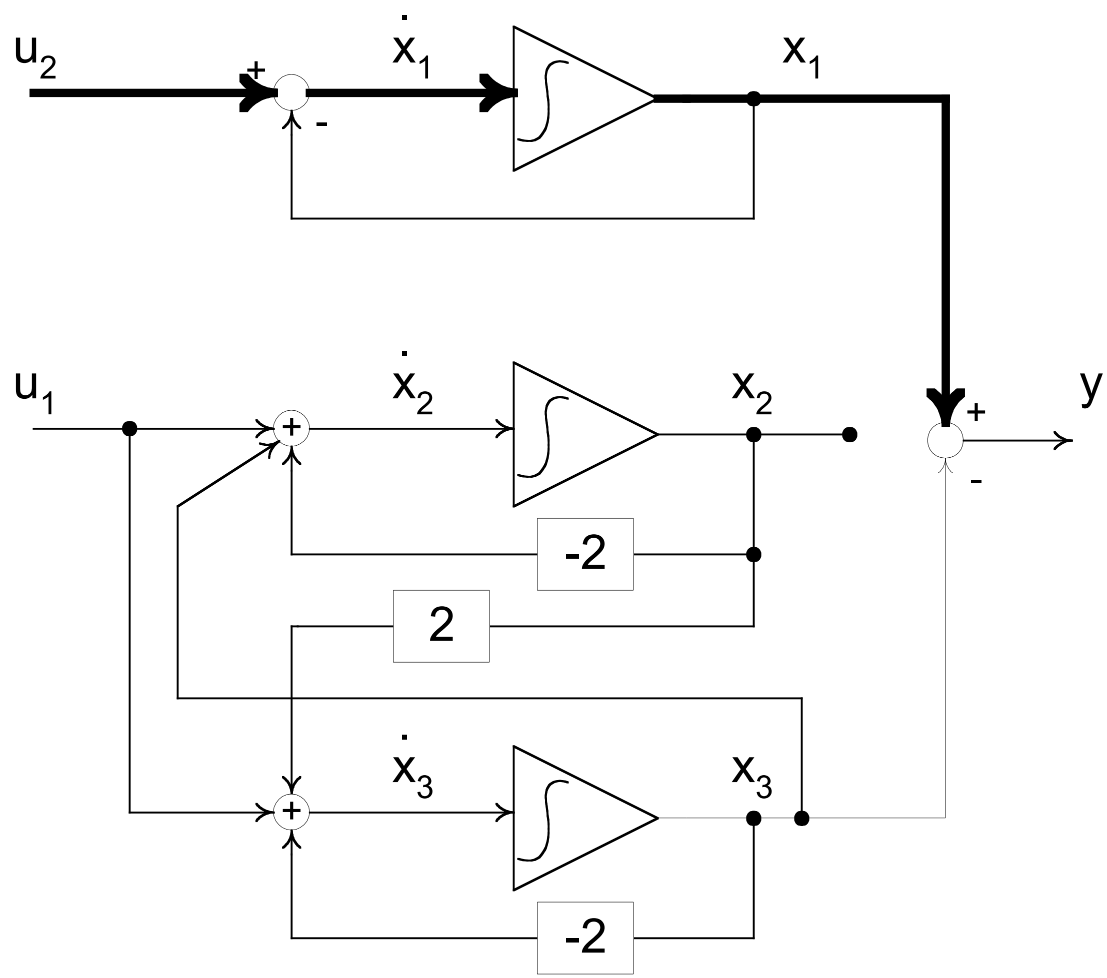
Intro to Controllability - IV

Can x_2 and x_3 be affected by u_1?



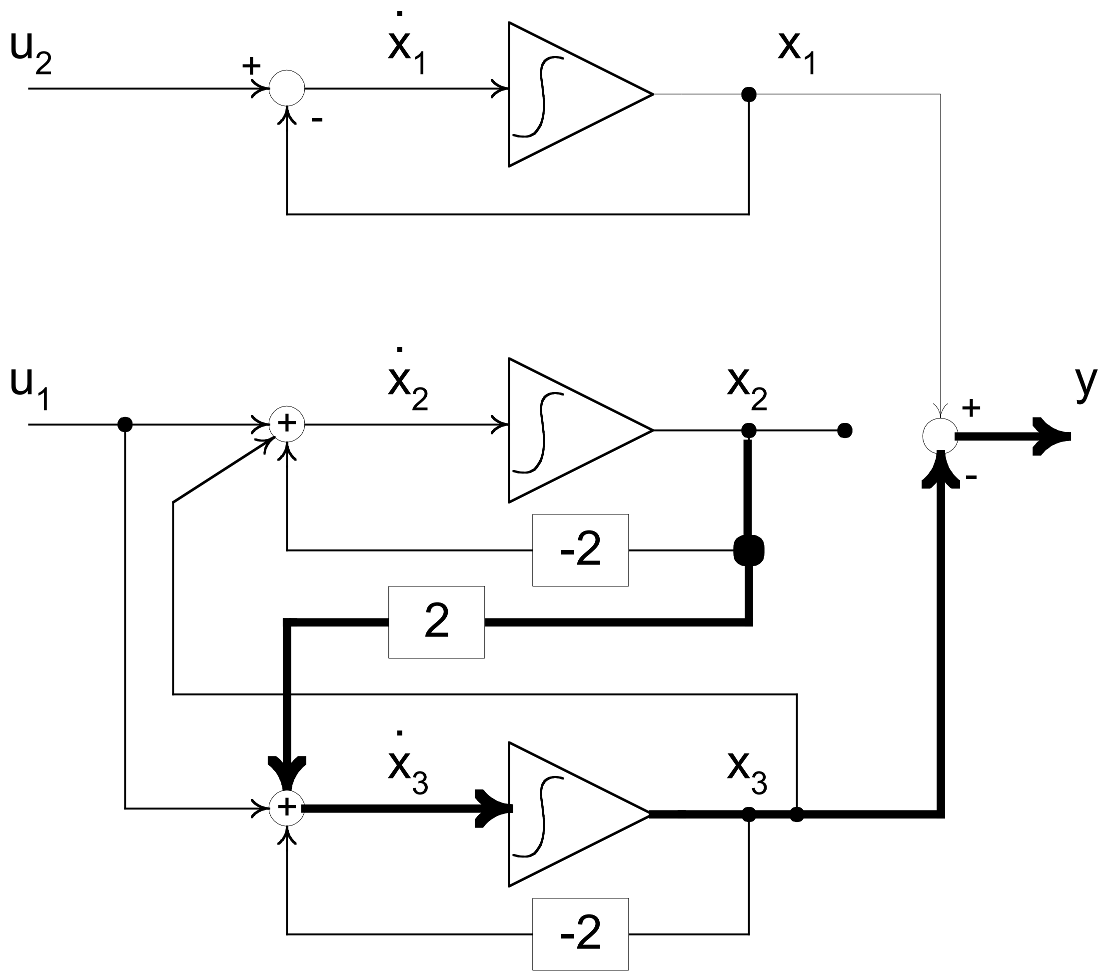
Intro to Controllability - V

Can x_2 and x_3 be affected by u_2?



Intro to Controllability - VI

Can x_2 be "seen" in y?



Intro to Controllability - VII

Some results (there are others…):

1. x_2 and x_3 are <u>controllable</u> from u_1.

2. x_2 and x_3 are <u>not controllable</u> from u_2.

3. x_2 is <u>observable</u> from y.

Intro to Controllability - VIII

Consider the response to the input

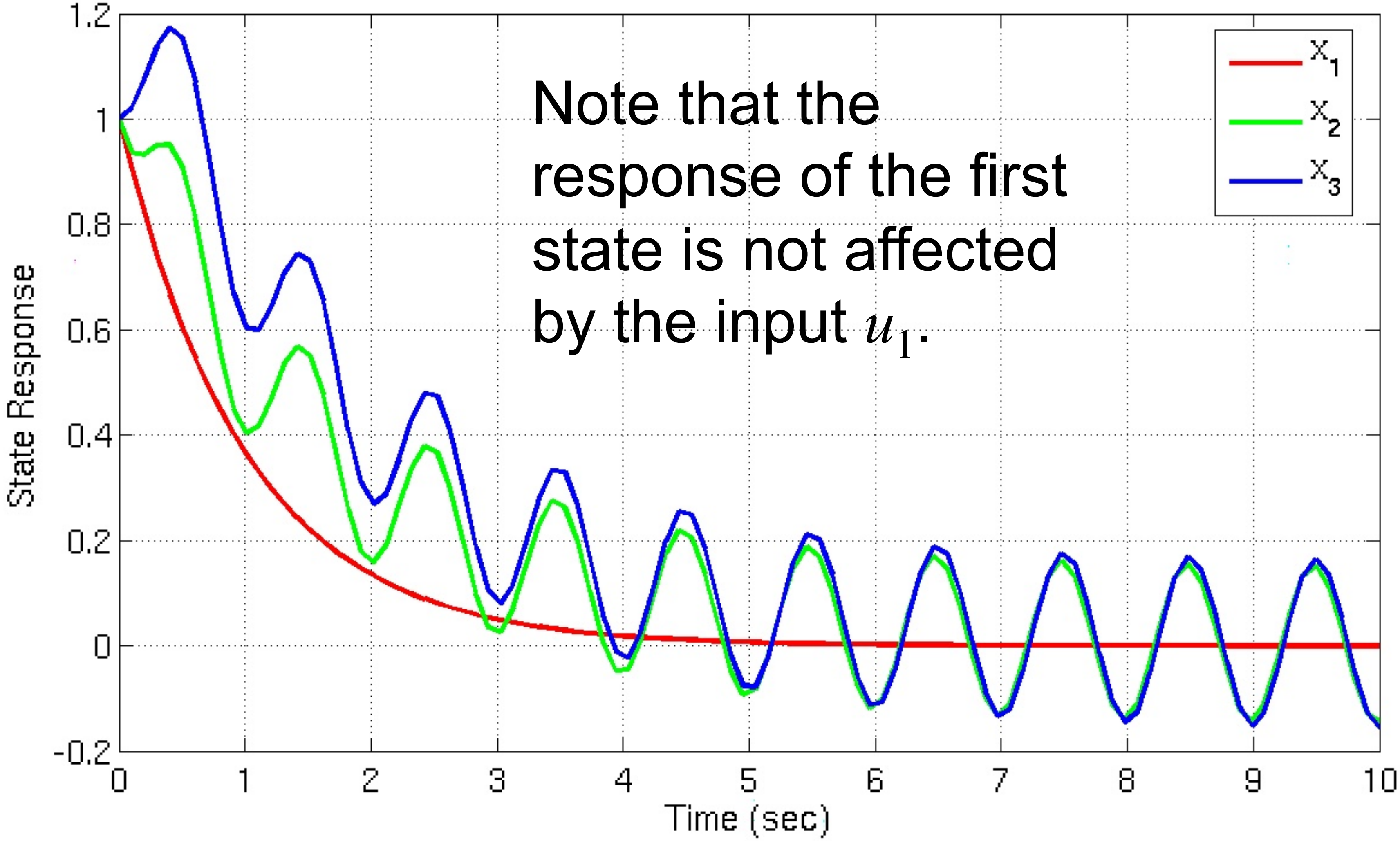
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sin t \\ 0 \end{bmatrix}$$

with the non-zero initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

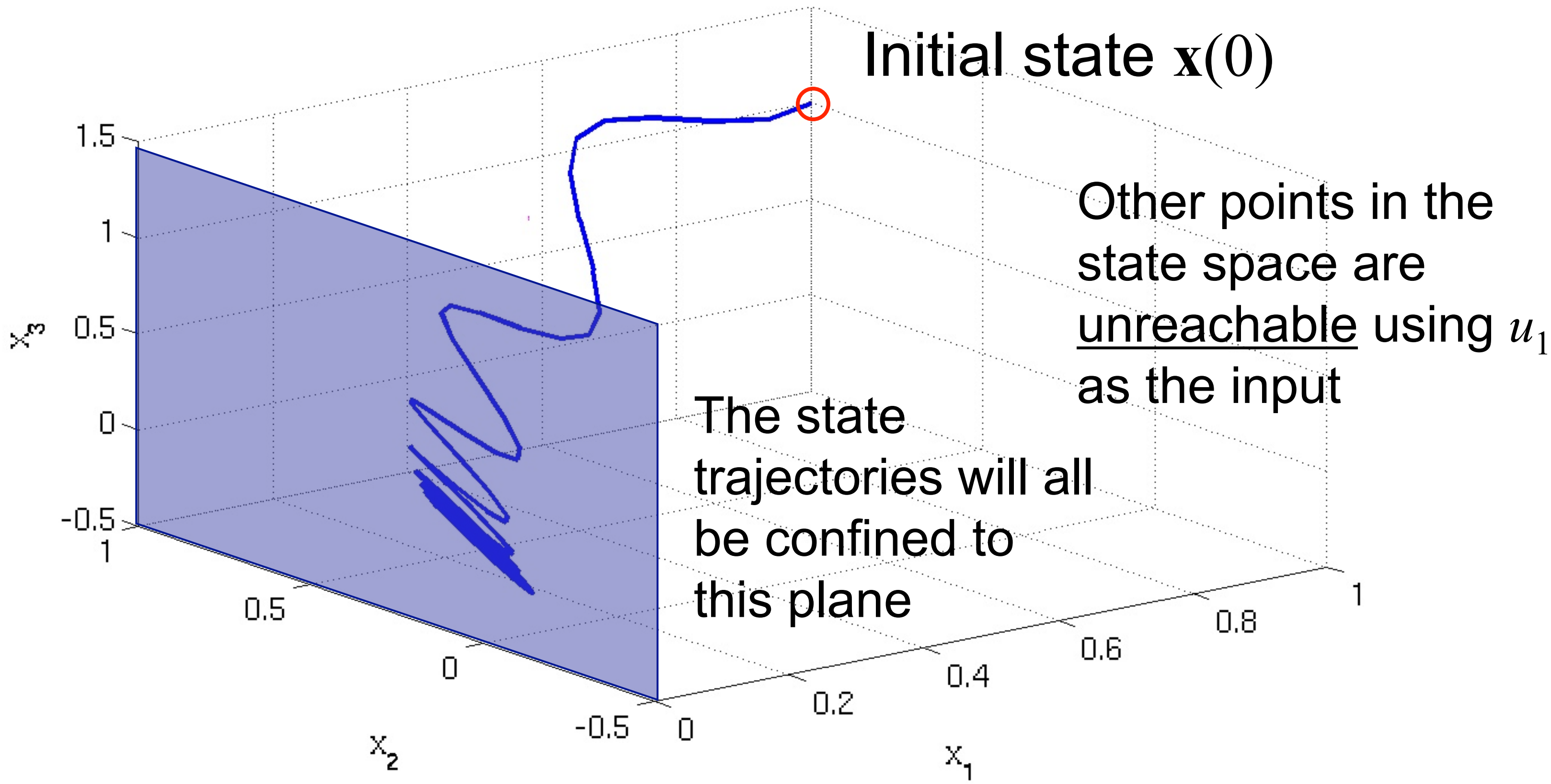
Intro to Controllability - IX

The state vector time-response is:



Intro to Controllability - X

We can plot the same data as a state trajectory in three dimensions:



Controllability Conditions - I

Formal definition of controllability for LTI systems:

The linear differential system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is said to be completely controllable if and only if it can be transferred from any initial state $\mathbf{x}_0$ at any initial time t_0 to any terminal state $\mathbf{x}(t_1) = \mathbf{x}_1$ within a finite time t_1 - t_0

Controllability Conditions - II

Equivalent definition: A system is controllable if and only if the column vectors of the controllability matrix

$$\underset{[N\times NM]}{\mathbf{Q}} = \left[\underset{[N\times M]}{\mathbf{B}} \quad \underset{[N\times M]}{\mathbf{AB}} \quad \underset{[N\times M]}{\mathbf{A}^2\mathbf{B}} \quad \cdots \quad \underset{[N\times M]}{\mathbf{A}^{N-1}\mathbf{B}} \right]$$

span the N-dimensional space. This is equivalent to $\mathbf{Q}$ having rank N.

The rank of a matrix is the number of linearly independent columns or rows.

In-Class Assignment - I

Consider the following example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Is the system controllable?

First, form the controllability matrix:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} =$$

In-Class Assignment - II

Now try the following modified example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

First compute the controllability matrix:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} =$$

In-Class Assignment - III

What is the rank of the matrix $\mathbf{Q}$?

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

Controllability Conditions - III

How do we check the rank for larger matrices or for matrices where the rank is not so obvious?

A <u>square matrix</u> whose rank is less than the number of columns (or rows) will have a zero determinant.

In Matlab, use `det(Q)` to get the determinant. A zero determinant indicates that the rank is less than N.

Controllability Conditions - IV

The number of linearly independent columns or rows can be computed for any matrix (i.e. square or nonsquare) using the `rank(Q)` function in Matlab

```
» Q = [1 0 1;1 2 3;3 -2 1];
» rank(Q)
ans =
      2
```

Example 2

Given the system

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ a \end{bmatrix}$$

Use the rank of the controllability matrix to determine the values of a that result in an uncontrollable system.

Example 2

First construct the controllability matrix

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} & \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & a-2 \\ a & -a \end{bmatrix}$$

Since $\mathbf{Q}$ is a square matrix, we can check the determinant to evaluate the rank (not true for multiple inputs!)

$$|\mathbf{Q}| = (1)(-a) - (a)(a-2) = -a^2 + a = -a(a-1)$$

Result: $\mathrm{rank}(\mathbf{Q}) < 2$ when $a = \{0,1\}$

Integral Conditions - II

A corollary of the controllability statement on L10/S12 is the following:

If an LTI system is completely controllable, then we should be able to find a control $\mathbf{u}(t)$ that transfers the state from any initial state $\mathbf{x}_0(t_0)$ to any terminal state $\mathbf{x}(t_1)$

Let's try to find such a control.

Integral Conditions - III

From our study of the state transition matrix we know the response at t_1 is

$$\mathbf{x}(t_1) = e^{\mathbf{A}(t_1 - t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Now define a matrix called the <u>Controllability Grammian</u>

$$\underset{[N\times N]}{\mathbf{P}(t_1, t_0)} = \int_{t_0}^{t_1} \underset{[N\times N]}{e^{\mathbf{A}(t_1-\tau)}}\;\underset{[N\times M]}{\mathbf{B}}\;\underset{[M\times N]}{\mathbf{B}^T}\;\underset{[N\times N]}{e^{\mathbf{A}^T(t_1-\tau)}}\,d\tau$$

Integral Conditions - IV

It is not obvious, but the input

$$\mathbf{u}(t) = \mathbf{B}^T e^{\mathbf{A}^T (t_1 - t)} \mathbf{P}^{-1}(t_1, t_0)\left[\mathbf{x}(t_1) - e^{\mathbf{A}(t_1 - t_0)}\mathbf{x}(t_0)\right]$$

will transfer the system from the initial state $\mathbf{x}(t_0)$ to the final state $\mathbf{x}(t_1)$.

How do we show this is the answer?

Integral Conditions - V

Substitute this result into the expression for the final state response

$$\mathbf{x}(t_1) = e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\underbrace{\mathbf{B}^T e^{\mathbf{A}^T(t_1-\tau)}\mathbf{P}^{-1}(t_1,t_0)\left[\mathbf{x}(t_1) - e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0)\right]}_{\mathbf{u}(\tau)}d\tau$$

$$\mathbf{x}(t_1) = e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0) + \underbrace{\int_{t_0}^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t_1-\tau)}d\tau}_{\mathbf{P}(t_1,t_0)} \times \mathbf{P}^{-1}(t_1,t_0)\left[\mathbf{x}(t_1) - e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0)\right]$$

$$\mathbf{x}(t_1) = e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0) + \underbrace{\mathbf{P}(t_1,t_0)\mathbf{P}^{-1}(t_1,t_0)}_{\mathbf{I}}\left[\mathbf{x}(t_1) - e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0)\right]$$

$$\mathbf{x}(t_1) = \cancel{e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0)} + \mathbf{x}(t_1) - \cancel{e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0)} = \mathbf{x}(t_1)$$

Integral Conditions - VI

The input that transforms the states from the initial state to the final state depends on the existence of the inverse of the controllability grammian

$$\mathbf{u}(t) = \mathbf{B}^T e^{\mathbf{A}^T(t_1 - t)} \boxed{\mathbf{P}^{-1}(t_1, t_0)} \left[\mathbf{x}(t_1) - e^{\mathbf{A}(t_1 - t_0)} \mathbf{x}(t_0)\right]$$

An equivalent requirement is that an LTI system is completely controllable if and only if the controllability grammian is non-singular.

Integral Conditions - VII

Let's return to the example on L10/S14:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For this system, the state transition matrix is:

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

Integral Conditions - VIII

The controllability grammian is

$$\mathbf{P}(t_1,t_0)=\int_{t_0}^{t_1}\begin{bmatrix}1 & 1-e^{-(t_1-\tau)}\\ 0 & e^{-(t_1-\tau)}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 1-e^{-(t_1-\tau)} & e^{-(t_1-\tau)}\end{bmatrix}d\tau$$

After simplification

$$\mathbf{P}(t_1,t_0)=\int_{t_0}^{t_1}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}d\tau=\begin{bmatrix}(t_1-t_0) & 0\\ 0 & 0\end{bmatrix}$$

The controllability grammian is singular therefore this system is uncontrollable.

Integral Conditions - IX

Now recall the example on L10/S15 where only the $\mathbf{B}$ vector was changed.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state transition matrix does not depend on $\mathbf{B}$ so it is still

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

Integral Conditions - X

Now the controllability grammian is

$$\mathbf{P}(t_1,t_0)=\int_{t_0}^{t_1}\begin{bmatrix}1 & 1-e^{-(t_1-\tau)}\\ 0 & e^{-(t_1-\tau)}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1-e^{-(t_1-\tau)} & e^{-(t_1-\tau)}\end{bmatrix}d\tau$$

Setting $t_0 = 0$ (for convenience) we have

$$\mathbf{P}(t_1,0)=\int_{0}^{t_1}\begin{bmatrix}\left(1-e^{-(t_1-\tau)}\right)^2 & \left(1-e^{-(t_1-\tau)}\right)e^{-(t_1-\tau)}\\ \left(1-e^{-(t_1-\tau)}\right)e^{-(t_1-\tau)} & \left(e^{-(t_1-\tau)}\right)^2\end{bmatrix}d\tau$$

$$=\begin{bmatrix}-\frac{3}{2}+e^{-2t_1}\left(t_1e^{2t_1}+2e^{t_1}-\frac{1}{2}\right) & \frac{1}{2}+e^{-2t_1}\left(\frac{1}{2}-e^{t_1}\right)\\ \frac{1}{2}+e^{-2t_1}\left(\frac{1}{2}-e^{t_1}\right) & \frac{1}{2}-\frac{1}{2}e^{-2t_1}\end{bmatrix}$$

Integral Conditions - XI

The determinant of the controllability grammian is

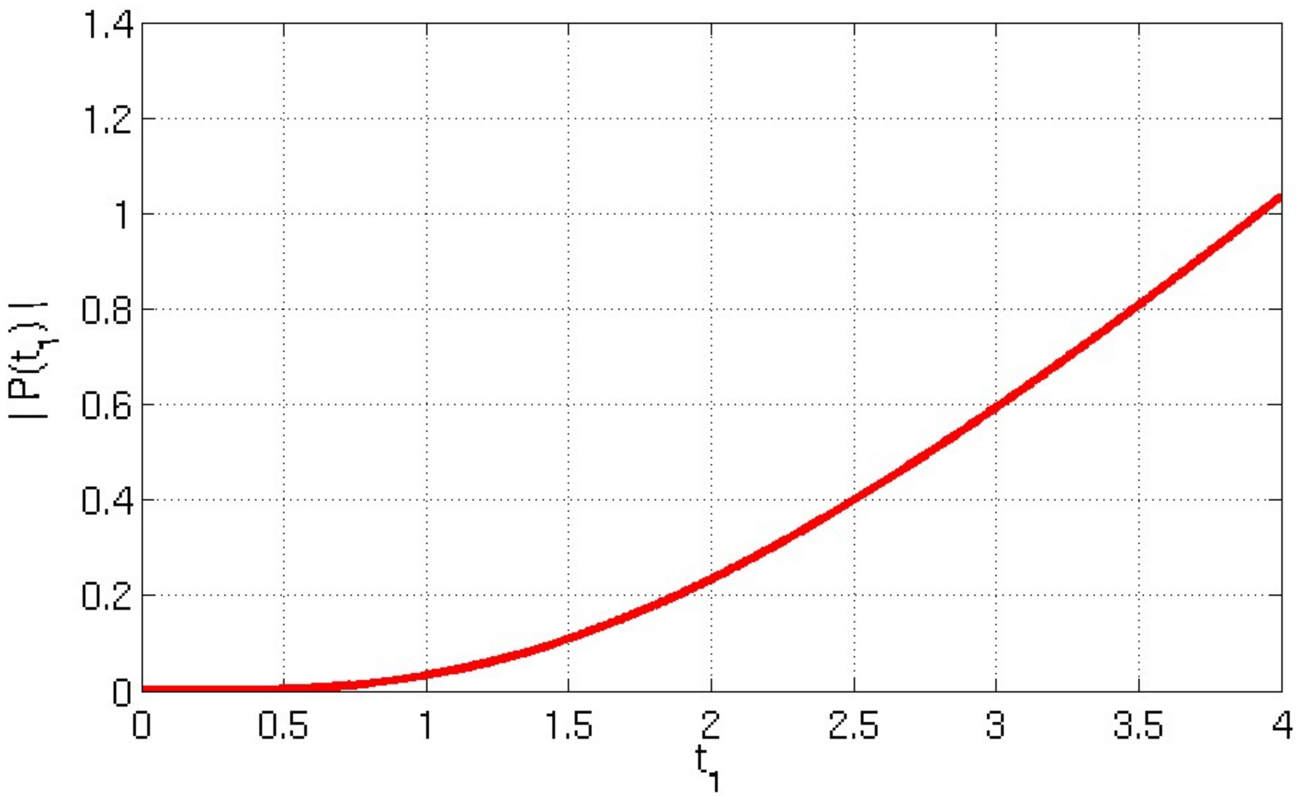
$$\left|\mathbf{P}(t_1, 0)\right| = -1 + 2e^{-t_1} + \tfrac{1}{2}t_1 - \tfrac{1}{2}t_1 e^{-2t_1} - e^{-2t_1}$$

Is this determinant equal to zero for all final times $t_1 > 0$?

$$\left|\mathbf{P}(t_1, 0)\right| = -1 + \underbrace{\tfrac{1}{2}t_1}_{\substack{\to\infty \\ \text{as} \\ t_1\to\infty}} + \underbrace{2e^{-t_1} - \tfrac{1}{2}t_1 e^{-2t_1} - e^{-2t_1}}_{\substack{\to 0 \\ \text{as} \\ t_1\to\infty}}$$

Integral Conditions - XII

$|\mathbf{P}(t_1,0)|$ is never zero for $t_1 > 0$, therefore the system is always controllable.



Summary - I

We have looked at several physical interpretations of controllability:

- No path through a block diagram

  Good visual aid, but can be difficult for large systems

- Unreachable locations in the state space

  The state vector can pass through unreachable locations, but we cannot control our system to go there

Summary - II

We have also looked at several mathematical tests for controllability:

- Controllability matrix must be full rank.

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{N-1}\mathbf{B} \end{bmatrix}$$

- Controllability grammian must be non-singular.

$$\mathbf{P}(t_1, t_0) = \int_{t_0}^{t_1} e^{\mathbf{A}(t_1-\tau)} \mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t_1-\tau)} d\boldsymbol{\tau}$$

Note that both tests require $\mathbf{A}$ <u>and</u> $\mathbf{B}$! Are these tests related?