MATH/CS 5466 · NUMERICAL ANALYSIS

Problem Set 3

Posted Friday 26 February 2016. Due Friday 4 March 2016 (5pm).
Due to Spring Break, late problem sets are due on Monday 14 March 2016 (5pm).
Students should complete all 4 problems (total of 100 points).

1. [25 points]

   Consider approximations to $f(x)\sqrt{x}$ for $x \in [0, 1]$.

   (a) Construct the interpolant $p_1 \in \mathcal{P}_1$ to $f(x) = \sqrt{x}$ using the interpolation points $x_0 = 1/3$ and $x_1 = 1/3$ and produce a plot of the error $\sqrt{x} - p_1(x)$. Identify three points (call them $\widetilde{x}_0$, $\widetilde{x}_1$, and $\widetilde{x}_2$ in this case) at which the error $f(x) - p_1(x)$ oscillates sign. Apply de la Valée Poussin's theorem to obtain a lower bound on the error $\|f - p_*\|_\infty$ in the minimax approximation $p_* \in \mathcal{P}_1$ to f on $[0, 1]$.

   (b) For the same interpolant p_1 at $x_0 = 1/3$ and $x_1 = 2/3$, Problem 3 of Problem Set 2 gives a different lower bound on $\|f - p_*\|_\infty$:

   $$\frac{\|f - p_1\|_\infty}{1 + \|\Pi_1\|_\infty} \le \|f - p_*\|_\infty.$$

   Compute (by hand) the left-hand side.

   Note that, unlike Problem 2 of Problem Set 3, $\|\Pi_n\|_\infty \ne 1$, since the interpolation points are not at the ends of the interval $[0, 1]$. You will need to compute

   $$\|\Pi_1\|_\infty = \max_{x\in[0,1]} |\ell_0(x)| + |\ell_1(x)|.$$

   (c) Now compute $p_* \in \mathcal{P}_1$ that best approximates $\sqrt{x}$ in the minimax sense, and report the error. Is this error consistent with the bounds you obtained in part (a) ?
   [Kincaid & Cheney]

2. [25 points]

   (a) Find the line $P_* \in \mathcal{P}_1$ that gives a least squares approximation to $f(x) = \sqrt{x}$ on $x \in [0, 1]$, and report the error $\|f - P_*\|_2$.

   (b) For a general interval $[a, b]$, prove that for all $f \in C[a, b]$,

   $$\min_{p\in\mathcal{P}_n} \|f - p\|_2 \le \sqrt{b - a} \min_{p\in\mathcal{P}_n} \|f - p\|_\infty.$$

   Confirm that your solutions to Problems 1(c) and 2(a) are consistent with this bound.

   (c) Let $[a, b]$ be any fixed interval. Given any $\varepsilon > 0$, show that there exists some $f \in C[a, b]$ such that $\|f\|_2 \le \varepsilon$, while $\|f\|_\infty \ge 1/\varepsilon$. (This implies that there exists *no* constant M such that $\|f\|_\infty \le M\|f\|_2$ for all $f \in C[a, b]$.) [Süli and Mayers, problem 8.1]

3. [25 points]

Suppose we have a function $f \in C(a,b)$ but $f(x)$ is unbounded as $x \to a$. Then for any polynomial $p \in \mathcal{P}_n$, $\|f-p\|_\infty$ would be infinite over $[a,b]$, and hence there exists no minimax approximation. The last problem hints that the situation could possibly be better for least squares approximation. This problem, adapted from Gautchi, explores an example where this is the case.

The *shifted Legendre polynomials* for the interval $x \in [0,1]$ are defined as $\pi_0(x) = 1$, $\pi_1(x) = 2x-1$, and then, for $n \geq 1$, via the recurrence

$$\pi_{n+1} = \frac{2n+1}{n+1}(2x-1)\pi_n(x) - \frac{n}{n+1}\pi_{n-1}(x).$$

You may use without proof that these polynomials are *orthogonal* with respect to the inner product (and norm)

$$(f,g) = \int_0^1 f(x)g(x)\,\mathrm{d}x, \qquad \|f\|_2 = (f,f)^{1/2}.$$

By longstanding convention these polynomials are constructed such that $\pi_n(1) = 1$, so they are not *orthonormal*. You may thus also use without proof that

$$(\pi_j, \pi_j) = \frac{1}{2j+1}.$$

(a) Work out explicit formulas for $\pi_2(x)$ and $\pi_3(x)$; simplify as much as possible.

We wish to approximate the *unbounded* function $f(x) = \log(1/x)$ (log denotes the natural logarithm).

We will use these orthogonal polynomials as a basis for least squares approximation with degree-n polynomials, writing the optimal polynomial $P_{*,n} \in \mathcal{P}$ in the form

$$P_{*,n}(x) = \sum_{j=0}^{n} c_j \pi_j(x).$$

(b) Use the fact that

$$\int_0^1 \pi_j \log(1/x)\,\mathrm{d}x = \begin{cases} 1, & j = 0 \\ \dfrac{(-1)^j}{j(j+1)}, & j \geq 1, \end{cases}$$

to determine formulas for the coefficients $c_0, \ldots, c_n$ in the best approximation.

(c) Prove that $\|f - P_{*,n}\| = \dfrac{1}{n+1}$. (Hint: use induction.)
This implies that $P_{*,n}$ converges to f as $n \to \infty$, in the 2-norm.

(d) Create a plot comparing f to the first four least squares approximations: $P_{*,0}$, $P_{*,1}$, $P_{*,2}$, $P_{*,3}$.

4. [25 points]

The *conjugate gradient algorithm* is the most important iterative method for solving the linear system of equations $\mathbf{Ac} = \mathbf{b}$ when the matrix $\mathbf{A} \in \mathbb{R}^{N\times N}$ is large, sparse, and symmetric positive definite.

Since $\mathbf{A}$ is symmetric positive definite, its eigenvalues are real and positive; write them as

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

One can show that at step k, the conjugate gradient algorithm constructs an iterate $\mathbf{c}_k$ such that

$$\frac{\|\mathbf{c} - \mathbf{c}_k\|_{\mathbf{A}}}{\|\mathbf{b}\|_{\mathbf{A}}} \leq \min_{\substack{p\in\mathcal{P}_k\\ p(0)=1}} \max_{x\in[\lambda_1,\lambda_N]} |p(x)|.$$

(Here $\|\cdot\|_{\mathbf{A}}$ defines the *energy norm* induced by $\mathbf{A}$: $\|\mathbf{z}\|_{\mathbf{A}} = (\mathbf{z}^T\mathbf{A}\mathbf{z})^{1/2}$.)

(a) Derive the solution to the problem

$$\min_{\substack{p\in\mathcal{P}_k\\ p(0)=1}} \max_{x\in[\lambda_1,\lambda_N]} |p(x)|.$$

in terms of Chebyshev polynomials (with appropriate change-of-variables and scaling).

(b) Write a MATLAB routine `cheby.m` such that `Tkx = cheby(x,k)` evaluates $T_k(x)$.

(c) Plot the minimizing polynomial in part (a) for $k = 3$ and $k = 5$ for
  (i) $\lambda_1 = 1$, $\lambda_N = 4$;
  (ii) $\lambda_1 = 1$, $\lambda_N = 16$;
  (iii) $\lambda_1 = 1/4$, $\lambda_N = 4$.

(d) Use your answer from part (a) to derive the classic error bound for conjugate gradients:

$$\min_{\substack{p\in\mathcal{P}_k\\ p(0)=1}} \max_{x\in[\lambda_1,\lambda_N]} |p(x)| \leq 2\left(\frac{\sqrt{\lambda_N/\lambda_1}-1}{\sqrt{\lambda_N/\lambda_1}+1}\right)^k.$$

What kind of eigenvalue intervals $[\lambda_1, \lambda_N]$ lead the conjugate gradient algorithm to converge quickly? Slowly?