

## LECTURE 36: Linear Multistep Methods: Zero Stability

### 5.6 Linear multistep methods: zero stability

Does consistency imply convergence for linear multistep methods? This is always the case for one-step methods, as proved in section 5.3, but the example at the end of the last lecture suggests the issue is less straightforward for multistep methods. By understanding the subtleties, we will come to appreciate one of the most significant themes in numerical analysis: *stability* of discretizations.

We are interested in the behavior of linear multistep methods as  $h \rightarrow 0$ . In this limit, the right hand side of the formula for the generic multistep method,

$$\sum_{j=0}^m \alpha_j x_{k+j} = h \sum_{j=0}^m \beta_j f(t_{k+j}, x_{k+j}),$$

makes a negligible contribution. This motivates our consideration of the trivial model problem  $x'(t) = 0$  with  $x(0) = 0$ . Does the linear multistep method recover the exact solution,  $x(t) = 0$ ?

When  $x'(t) = 0$ , clearly we have  $f_{k+j} = 0$  for all  $j$ . The condition  $\alpha_m \neq 0$  allows us to write

$$x_m = -\frac{(\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{m-1} x_{m-1})}{\alpha_m}$$

Hence if the method is started with exact data

$$x_0 = x_1 = \cdots = x_{m-1} = 0,$$

then

$$x_m = -\frac{(\alpha_0 \cdot 0 + \alpha_1 \cdot 0 + \cdots + \alpha_{m-1} \cdot 0)}{\alpha_m} = 0,$$

and this pattern will continue:  $x_{m+1} = 0, x_{m+2} = 0, \dots$ . Any linear multistep method with exact starting data produces the exact solution for this special problem, regardless of the time-step.

Of course, for more complicated problems it is unusual to have exact starting values  $x_1, x_2, \dots, x_{m-1}$ ; typically, these values are only approximate, obtained from some high-order one-step ODE solver or from an asymptotic expansion of the solution that is accurate in a neighborhood of  $t_0$ . To discover how multistep methods behave, we must first understand how these errors in the initial data pollute future iterations of the linear multistep method.

**Definition 5.6.** Suppose the initial value problem  $x'(t) = f(t, x)$ ,  $x(t_0) = x_0$  satisfies the requirements of Picard's Theorem over the

interval  $[t_0, t_{\text{final}}]$ . For an  $m$ -step linear multistep method, consider two sequences of starting values for a fixed time-step  $h$

$$\{x_0, x_1, \dots, x_{m-1}\} \quad \text{and} \quad \{\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{m-1}\},$$

that generate the approximate solutions  $\{x_j\}_{j=0}^n$  and  $\{\hat{x}_j\}_{j=0}^n$ , where  $t_n = t_{\text{final}}$ . The multistep method is *zero-stable* for this initial value problem if for sufficiently small  $h$  there exists some constant  $M$  (independent of  $h$ ) such that

$$|x_k - \hat{x}_k| \leq M \max_{0 \leq j \leq m-1} |x_j - \hat{x}_j|$$

for all  $k$  with  $t_0 \leq t_k \leq t_{\text{final}}$ . More plainly, a method is zero-stable for a particular problem if errors in the starting values are not magnified in an unbounded fashion.

Proving zero-stability directly from this definition would be a chore. Fortunately, there is an easy way to check zero stability all at once *for all sufficiently 'nice' differential equations*. To begin with, consider a particular example.

**Example 5.11** (A novel second order method). The truncation error formulas from the last lecture can be used to derive a variety of linear multistep methods that satisfy a given order of truncation error. You can use those conditions to verify that the explicit two-step method

$$(5.4) \quad x_{k+2} = 2x_k - x_{k+1} + h\left(\frac{1}{2}f_k + \frac{5}{2}f_{k+1}\right)$$

is second order accurate. Now we will test the zero-stability of this algorithm on the trivial model problem,  $x'(t) = 0$  with  $x(0) = 0$ . Since  $f(t, x) = 0$  in this case, the method reduces to

$$x_{k+2} = 2x_k - x_{k+1}.$$

As seen above, this method produces the exact solution if given exact initial data,  $x_0 = x_1 = 0$ . But what if  $x_0 = 0$  but  $x_1 = \varepsilon$  for some small  $\varepsilon > 0$ ? This method produces the iterates

$$\begin{aligned} x_2 &= 2x_0 - x_1 = 2 \cdot 0 - \varepsilon = -\varepsilon \\ x_3 &= 2x_1 - x_2 = 2(\varepsilon) - (-\varepsilon) = 3\varepsilon \\ x_4 &= 2x_2 - x_3 = 2(-\varepsilon) - 3\varepsilon = -5\varepsilon \\ x_5 &= 2x_3 - x_4 = 2(3\varepsilon) - (-5\varepsilon) = 11\varepsilon \\ x_6 &= 2x_4 - x_5 = 2(-5\varepsilon) - (11\varepsilon) = -21\varepsilon \\ x_7 &= 2x_5 - x_6 = 2(11\varepsilon) - (-21\varepsilon) = 43\varepsilon \\ x_8 &= 2x_6 - x_7 = 2(-21\varepsilon) - (43\varepsilon) = 85\varepsilon. \end{aligned}$$

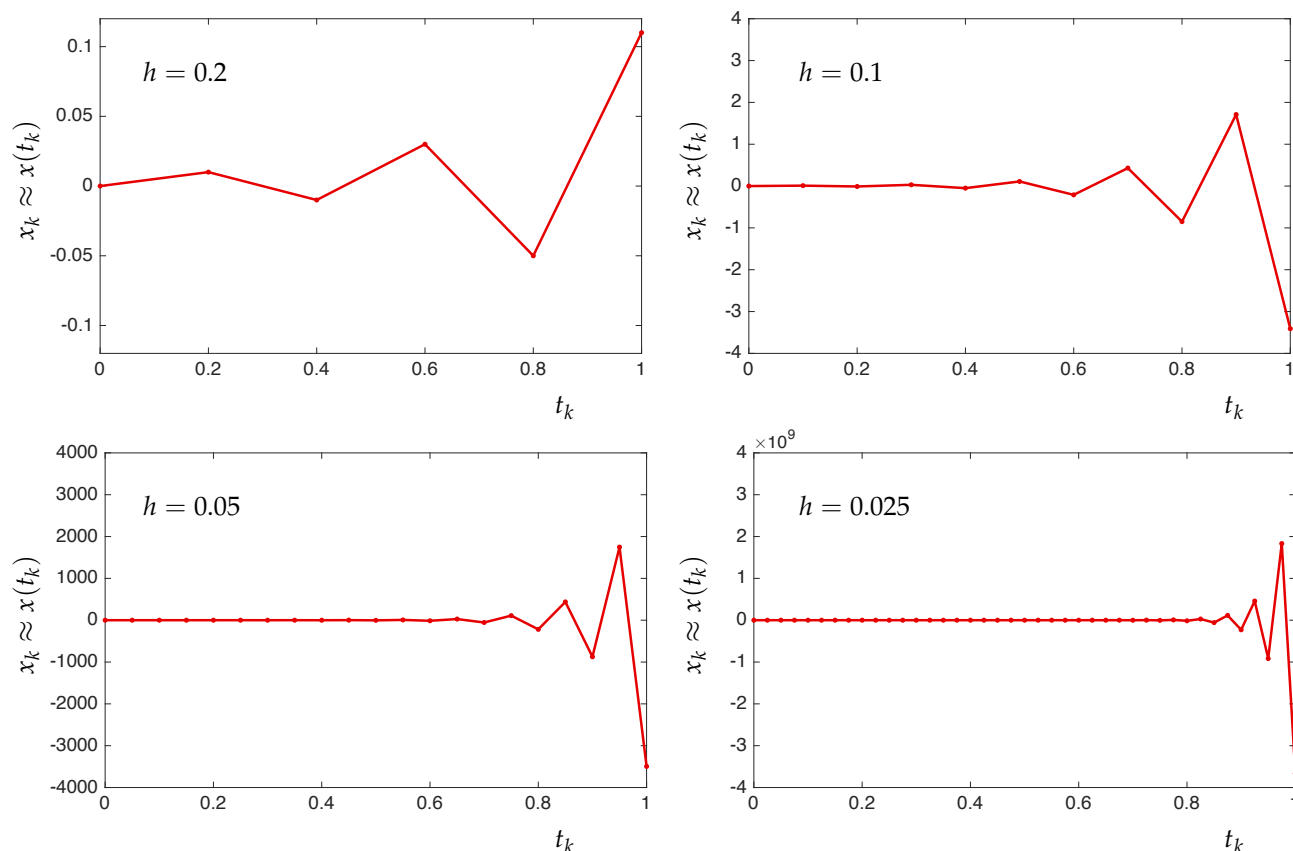


Figure 5.5: Approximate solutions to  $x'(t) = 0$  using the novel second-order method (5.4) with initial data  $x_0 = 0$  and  $x_1 = 0.01$ . As  $h$  gets smaller, the method produces approximations to the true solution  $x(t) = 0$  of exponentially degrading quality! (Note how the scale of the vertical axis grows from plot to plot.)

In just seven steps, the error has been multiplied 85-fold. The error is roughly doubling at each step, and before long the approximate ‘solution’ is complete garbage. Figure 5.5 shows this instability, plotting  $x_k$  for four different values of  $h$  and  $\varepsilon = 0.01$ .

This examples illustrates another quirk. When applied to this particular model problem, the linear multistep method reduces to  $\sum_{j=0}^m \alpha_j x_{k+j} = 0$ , and thus never incorporates the time-step,  $h$ . Hence *the error at some fixed time  $t_{\text{final}} = hk$  gets worse as  $h$  gets smaller and  $k$  grows accordingly!* Figure 5.6 puts all four of solutions from Figure 5.5 together in one plot, dramatically illustrating how the solutions degrade as  $h$  gets smaller!

Though this method has second-order local (truncation) error, it blows up if fed incorrect initial data for  $x_1$ . Decreasing  $h$  can magnify this effect, even if, for example, the error in  $x_1$  is proportional to  $h$ . We can draw a larger lesson from this simple problem: For linear multistep methods, consistency (i.e.,  $T_k \rightarrow 0$  as  $h \rightarrow 0$ ) is *not sufficient* to ensure convergence.

Let us analyze our unfortunate method a little more carefully. Setting the starting values  $x_0$  and  $x_1$  aside for the moment, we want to

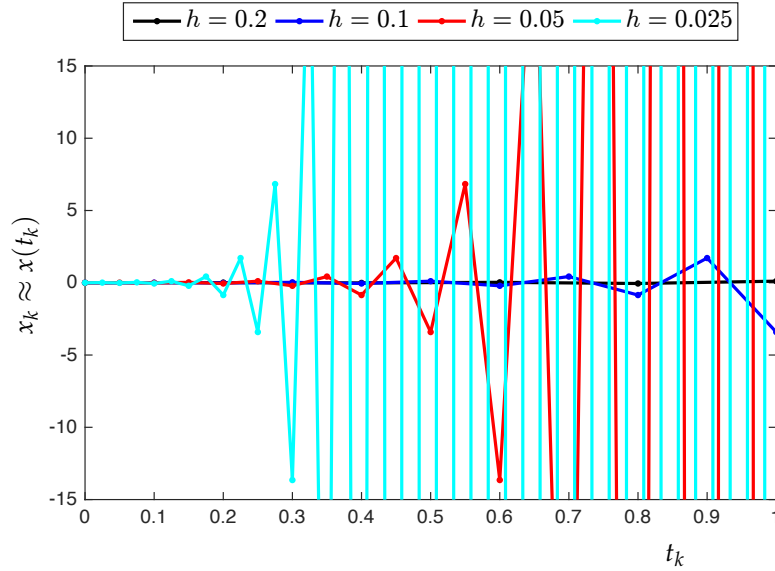


Figure 5.6: All four solutions from Figure 5.5 plotted together, to illustrate how the approximate solutions from the method (5.4) to  $x'(t) = 0$  with  $x_0 = 0$  and  $x_1 = \varepsilon$  degrade as  $h \rightarrow 0$ .

find all sequences  $\{x_j\}_{j=0}^{\infty}$  that satisfy the linear, constant-coefficient recurrence relation

$$x_{k+2} = 2x_k - x_{k+1}.$$

Since the  $x_k$  values grew exponentially in the example above, assume that this recurrence has a solution of the form  $x_k = \gamma^k$  for all  $k = 0, 1, \dots$ , where  $\gamma$  is some number that we will try to determine. Plug this *ansatz* for  $x_k$  into the recurrence relation to see if you can make it work as a solution:

$$\gamma^{k+2} = 2\gamma^k - \gamma^{k+1}.$$

Divide this equation through by  $\gamma^k$  to obtain the quadratic equation

$$\gamma^2 = 2 - \gamma.$$

If  $\gamma$  solves this quadratic, then the putative solution  $x_k = \gamma^k$  indeed satisfies the difference equation. Since

$$\gamma^2 + \gamma - 2 = (\gamma + 2)(\gamma - 1)$$

the roots of this quadratic are simply  $\gamma = -2$  and  $\gamma = 1$ . Thus we expect solutions of the form  $x_k = (-2)^k$  and the less interesting  $x_k = 1^k = 1$ .

If  $x_k = \gamma_1^k$  and  $x_k = \gamma_2^k$  are both solutions of the recurrence, then  $x_k = A\gamma_1^k + B\gamma_2^k$  is also a solution, for any real numbers  $A$  and  $B$ . To see this, note that

$$\gamma_1^2 + \gamma_1 - 2 = \gamma_2^2 + \gamma_2 - 2 = 0,$$

and so

$$A\gamma_1^k(\gamma_1^2 + \gamma_1 - 2) = B\gamma_2^k(\gamma_2^2 + \gamma_2 - 2) = 0.$$

Rearranging this equation,

$$A\gamma_1^{k+2} + B\gamma_1^{k+2} = 2(A\gamma_1^k + B\gamma_1^k) - (A\gamma_1^{k+1} + B\gamma_1^{k+1}),$$

which implies that  $x_k = A\gamma_1^k + B\gamma_2^k$  is a solution to the recurrence.

In fact, this is the general form of a solution to our recurrence. For any starting values  $x_0$  and  $x_1$ , one can determine the associated constants  $A$  and  $B$ . For example, with  $\gamma_1 = -2$  and  $\gamma_2 = 1$ , the initial conditions  $x_0 = 0$  and  $x_1 = \varepsilon$  require that

$$\begin{aligned} A + B &= 0 \\ -2A + B &= \varepsilon, \end{aligned}$$

which implies

$$A = -\varepsilon/3, \quad B = \varepsilon/3.$$

Indeed, the solution

$$(5.5) \quad x_k = \frac{\varepsilon}{3} - \frac{\varepsilon}{3}(-2)^k$$

generates the iterates  $x_0 = 0, x_1 = \varepsilon, x_2 = -\varepsilon, x_3 = 3\varepsilon, x_4 = -5\varepsilon, \dots$  computed previously. Notice that (5.5) reveals *exponential* growth with  $k$ : this growth overwhelms algebraic improvements in the estimate  $x_1$  that might occur as we reduce  $h$ . For example, if  $\varepsilon = x_1 - x(t_0 + h) = ch^p$  for some constant  $c$  and  $p \geq 1$ , then  $x_k = ch^p(1 - (-2)^k)/3$  still grows exponentially in  $k$ .

### 5.6.1 The Root Condition

The real trouble with the previous method was that the formula for  $x_k$  involves the term  $(-2)^k$ . Since  $|-2| > 1$ , this component of  $x_k$  grows exponentially in  $k$ . This term is simply an artifact of the finite difference equation, and has nothing to do with the underlying differential equation. As  $k$  increases, this  $(-2)^k$  term swamps the other term in the solution. It is called a *parasitic solution*.

Let us review how we determined the general form of the solution. We assumed a solution of the form  $x_k = \gamma^k$ , then plugged this solution into the recurrence  $x_{k+2} = 2x_k - x_{k+1}$ . The possible values for  $\gamma$  were roots of the equation  $\gamma^2 = 2 - \gamma$ .

The process we just applied to one specific linear multistep method can readily be extended to the general case

$$\sum_{j=0}^m \alpha_j x_{k+j} = h \sum_{j=0}^m \beta_j f(t_{k+j}, x_{k+j}).$$

For the differential equation  $x'(t) = 0$ , the method reduces to

$$\sum_{j=0}^m \alpha_j x_{k+j} = 0.$$

Substituting  $x_k = \gamma^k$  yields

$$\sum_{j=0}^m \alpha_j \gamma^{k+j} = 0.$$

Canceling  $\gamma^k$ ,

$$\sum_{j=0}^m \alpha_j \gamma^j = 0.$$

**Definition 5.7.** The *characteristic polynomial* of an  $m$ -step linear multistep method is the degree- $m$  polynomial

$$\rho(z) = \sum_{j=0}^m \alpha_j z^j.$$

For  $x_k = \gamma^k$  to be a solution to the above recurrence,  $\gamma$  must be a root of the characteristic polynomial,  $\rho(\gamma) = 0$ . Since the characteristic polynomial has degree  $m$ , it will have  $m$  roots. If these roots are distinct, call them  $\gamma_1, \gamma_2, \dots, \gamma_m$ , the general form of the solution of

$$\sum_{j=0}^m \alpha_j x_{k+j} = 0$$

is

$$x_k = c_1 \gamma_1^k + c_2 \gamma_2^k + \dots + c_m \gamma_m^k.$$

for constants  $c_1, \dots, c_m$  that are determined from the starting values  $x_0, \dots, x_m$ .

To avoid parasitic solutions to a linear multistep method, all the roots of the characteristic polynomial should be located within the unit disk in the complex plane, i.e.,  $|\gamma_j| \leq 1$  for all  $j = 1 \dots, m$ . Thus, for the simple differential equation  $x'(t) = 0$ , we have found a way to describe zero stability: Initial errors will not be magnified if the characteristic polynomial has all its roots in the unit disk; any roots on the unit disk should be simple (i.e., not multiple). What is remarkable is that this criterion actually characterizes zero stability not just for  $x'(t) = 0$ , but for *all* well-behaved differential equations! This was discovered in the late 1950s by Germund Dahlquist.

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**Theorem 5.4.** A linear multistep method is zero-stable for any ‘well-behaved’ initial value problem provided it satisfies the *root condition*:

- all roots of  $\rho(\gamma) = 0$  lie in the unit disk, i.e.,  $|\gamma| \leq 1$ ;
  - any roots on the unit circle ( $|\gamma| = 1$ ) are simple (i.e., not multiple).
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If some root, say  $\gamma_1$  is repeated  $p$  times, then instead of contributing the term  $c_1 \gamma_1^k$  to the general solution, it will contribute a term of the form  $c_{1,1} \gamma_1^k + c_{1,2} k \gamma_1^k + \dots + c_{1,p} k^{p-1} \gamma_1^k$ .

Following on from the previous marginal note, we note that if some root, say  $\gamma_1$ , is a repeated root on the unit circle,  $|\gamma_1| = 1$ , then the general solution will have terms like  $k \gamma_1^k$ , so  $|k \gamma_1^k| = k |\gamma_1|^k = k$ . While this term will not grow exponentially in  $k$ , it does grow algebraically, and errors will still grow enough as  $h \rightarrow 0$  to violate zero stability.

An excellent discussion of this theoretical material is given in Süli and Mayers, *Numerical Analysis: An Introduction*, Cambridge University Press, 2003; these notes follow, in part, their exposition. See also E. Hairer, S. P. Nørsett, and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, 2nd ed., Springer-Verlag, 1993.

One can see now where the term *zero-stability* comes from: it is necessary and sufficient for the stability definition to hold for the differential equation  $x'(t) = 0$ . In recognition of the discoverer of this key result, zero-stability is sometimes called *Dahlquist stability*. (Another synonymous term is *root stability*.) In addition to making this beautiful characterization, Dahlquist also answered the question about the conditions necessary for a multistep method to be convergent.

**Theorem 5.5** (Dahlquist Equivalence Theorem).

Suppose an  $m$ -step linear multistep method applied to a ‘well-behaved’ initial value problem on  $[t_0, t_{\text{final}}]$  with consistent starting values,

$$x_k \rightarrow x(t_k) \quad \text{for } t_k = t_0 + hk, k = 0, \dots, m-1$$

as  $h \rightarrow 0$ . This method is convergent, i.e.,

$$x_{\lceil (t-t_0)/h \rceil} \rightarrow x(t) \quad \text{for all } t \in [t_0, t_{\text{final}}].$$

as  $h \rightarrow 0$  if and only if the method is *consistent* and *zero-stable*.

If the exact solution is sufficiently smooth,  $x(t) \in C^{p+1}[t_0, t_{\text{final}}]$  and the multistep method is order- $p$  accurate ( $T_k = \mathcal{O}(h^p)$ ), then

$$x(t_k) - x_k = \mathcal{O}(h^p)$$

for all  $t_k \in [t_0, t_{\text{final}}]$ .

Dahlquist also characterized the maximal order of convergence for a zero-stable  $m$ -step multistep method.

**Theorem 5.6** (First Dahlquist Stability Barrier).

A zero-stable  $m$ -step linear multistep method has truncation error no better than

- $\mathcal{O}(h^{m+1})$  if  $m$  is odd
- $\mathcal{O}(h^m)$  if  $m$  is even.

**Example 5.12** (A method on the brink of stability). We close this lecture with an example of a method that, we might figuratively say, is ‘on the brink of stability.’ That is, the method is zero-stable, but it stretches that definition to its limit. Consider the method

$$(5.6) \quad x_{k+2} = x_k + 2hf_{k+1},$$

which has  $\mathcal{O}(h^2)$  truncation error. The characteristic polynomial is  $z^2 - 1 = (z+1)(z-1)$ , which has the two roots  $\gamma_1 = -1$  and

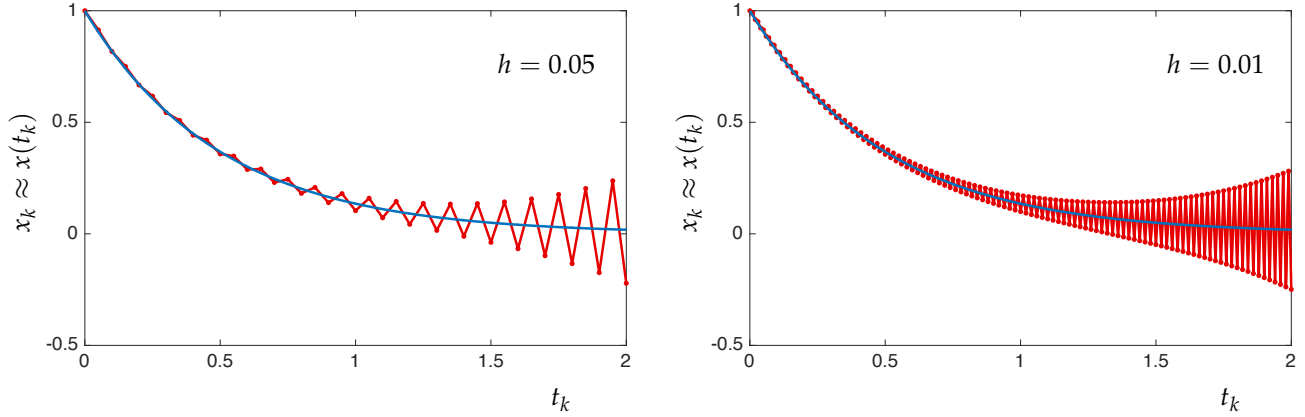


Figure 5.7: The second-order zero-stable method (5.6) applied to  $x'(t) = -2x(t)$ , started with initial condition  $x_0 = 1$  and a 10% error in  $x_1$ , i.e.,  $x_1 = 1.01e^{-2h}$  (shown in red). The method is ‘on the brink of stability’: the systematic error in  $x_1$  prevents the method from converging, but the deviation from the true solution  $x(t) = e^{-2t}$  (in blue) is bounded. This would not be the case if the method violated the root condition.

$\gamma_2 = 1$ . These are distinct roots on the unit circle, so the method is zero-stable.

Apply this method to the model problem  $x'(t) = \lambda x$  with  $x(0) = 1$ . Substituting  $f(t_k, x_k) = \lambda x_k$  into the method gives

$$x_{k+2} = x_k + 2\lambda h x_{k+1}.$$

For a fixed  $\lambda$  and  $h$ , this is just another recurrence relation like we have considered above. It has solutions of the form  $\gamma^k$ , where  $\gamma$  is a root of the polynomial

$$\gamma^2 - 2\lambda h \gamma - 1 = 0.$$

In fact, those roots are simply

$$\gamma = \lambda h \pm \sqrt{\lambda^2 h^2 + 1}.$$

Since  $\sqrt{\lambda^2 h^2 + 1} \geq 1$  for any  $h > 0$  and  $\lambda \neq 0$ , at least one of the roots  $\gamma$  will always be greater than one in modulus, thus leading to a solution  $x_k$  that grows exponentially with  $k$ . Of course, the exact solution to this equation is  $x(t) = e^{\lambda t}$ , so if  $\lambda < 0$ , then we have  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . The numerical approximation will generally diverge, giving the qualitatively opposite behavior!

How is this possible for a zero-stable method? The key is that here, unlike our previous zero-unstable method, the exponential growth rate depends upon the time-step  $h$ . Zero stability only requires that on a fixed finite time interval  $t \in [t_0, t_{\text{final}}]$ , the amount by which errors in the initial data are magnified be *bounded*.

Figure 5.7 shows what this means. Set  $\lambda = -2$  and  $[t_0, t_{\text{final}}] = [0, 2]$ . Start the method with  $x_0 = 1$  and  $x_1 = 1.01 e^{-2h}$ . That is, the second data point has an initial error of 1%. The plot on the left shows the solution for  $h = 0.05$ , while the plot on the right uses  $h = 0.01$ . In both cases, the solution oscillates wildly across the true



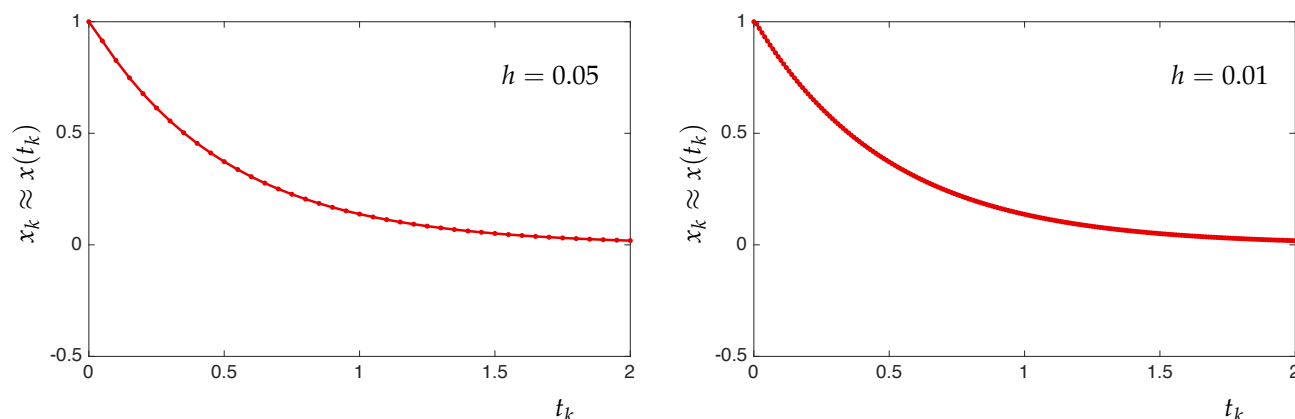


Figure 5.8: Second-order Adams–Bashforth method applied to  $x'(t) = -2x(t)$ , started with initial condition  $x_0 = 1$  and a 10% error in  $x_1$ , i.e.,  $x_1 = 1.01e^{-2h}$ . Despite the systematic error, the method still behaves like the true solution,  $x(t) = e^{-2t}$ .

solution, and the amplitude of these oscillations grows with  $t$ . As we reduce the step-size, the solution remains equally bad. (If the method were not zero-stable, we would expect the error to magnify as  $h$  shrinks.)

The solution does not blow up, but nor does it converge as  $h \rightarrow 0$ . So does this example contradict the Dahlquist Equivalence Theorem? No! The hypotheses for that theorem require consistent starting values. In this case, that means  $x_1 \rightarrow x(t_0 + h)$  as  $h \rightarrow 0$ . (We assume that  $x_0 = x(t_0)$  is exact.) In the example shown above, we have kept fixed  $x_1$  to have a 1% error as  $h \rightarrow 0$ , so it is not consistent.

Not all linear multistep methods behave as badly as this one in the presence of imprecise starting data. Recall the second-order Adams–Bashforth method from the previous lecture.

$$x_{k+2} - x_{k+1} = \frac{h}{2}(3f_{k+1} - f_k).$$

This method is zero stable, as  $\rho(z) = z^2 - z = z(z - 1)$ . Figure 5.8 repeats the exercise of Figure 5.7, with the same errors in  $x_1$ , but with the second-order Adams–Bashforth method. Though the initial value error will throw off the solution slightly, we recover the correct qualitative behavior.

Judging from the different manner in which our two second-order methods handle this simple problem, it appears that there is still more to understand about linear multistep methods. This is the subject of the next lecture.