LECTURE 8: *Piecewise interpolation*

1.10 *Piecewise polynomial interpolation*

WE HAVE SEEN, through Runge's example, that high degree polynomial interpolation can lead to large errors when the $(n+1)$st derivative of f is large in magnitude. In other cases, the interpolant converges to f, but the polynomial degree must be fairly high to deliver an approximation of acceptable accuracy throughout $[a, b]$. Beyond theoretical convergence questions, high-degree polynomials can be delicate to work with, even when using a stable implementation (the Lagrange basis, in its barycentric form). Many practical approximation problems are better solved by a simpler 'piecewise' alternative: instead of approximating f with one high-degree interpolating polynomial over a large interval $[a, b]$, patch together many low-degree polynomials that each interpolate f on some subinterval of $[a, b]$.

1.10.1 *Piecewise linear interpolation*

The simplest piecewise polynomial interpolation uses linear polynomials to interpolate between adjacent data points. Informally, the idea is to 'connect the dots.' Given $n+1$ data points $\{(x_j, f_j)\}_{j=0}^n$, we need to construct n linear polynomials $\{s_j\}_{j=1}^n$ such that

$$s_j(x_{j-1}) = f_{j-1}, \qquad \text{and} \qquad s_j(x_j) = f_j$$

for each $j = 1, \ldots, n$. It is simple to write down a formula for these polynomials,

Note that all the s_j's are *linear* polynomials. Unlike our earlier notation, the subscript j *does not* reflect the polynomial degree.

$$s_j(x) = f_j - \frac{(x_j - x)}{(x_j - x_{j-1})}(f_j - f_{j-1}).$$

Each s_j is valid on $x \in [x_{j-1}, x_j]$, and the interpolant $S(x)$ is defined as $S(x) = s_j(x)$ for $x \in [x_{j-1}, x_j]$.

To analyze the error, we can apply the interpolation bound developed in the last lecture. If we let Δ denote the largest space between interpolation points,

$$\Delta := \max_{j=1,\ldots,n} |x_j - x_{j-1}|,$$

then the standard interpolation error bound gives

$$\max_{x\in[x_0,x_n]} |f(x) - S(x)| \le \max_{x\in[x_0,x_n]} \frac{|f''(x)|}{2} \Delta^2.$$

In particular, this proves convergence as $\Delta \to 0$ provided $f \in C^2[x_0, x_n]$.

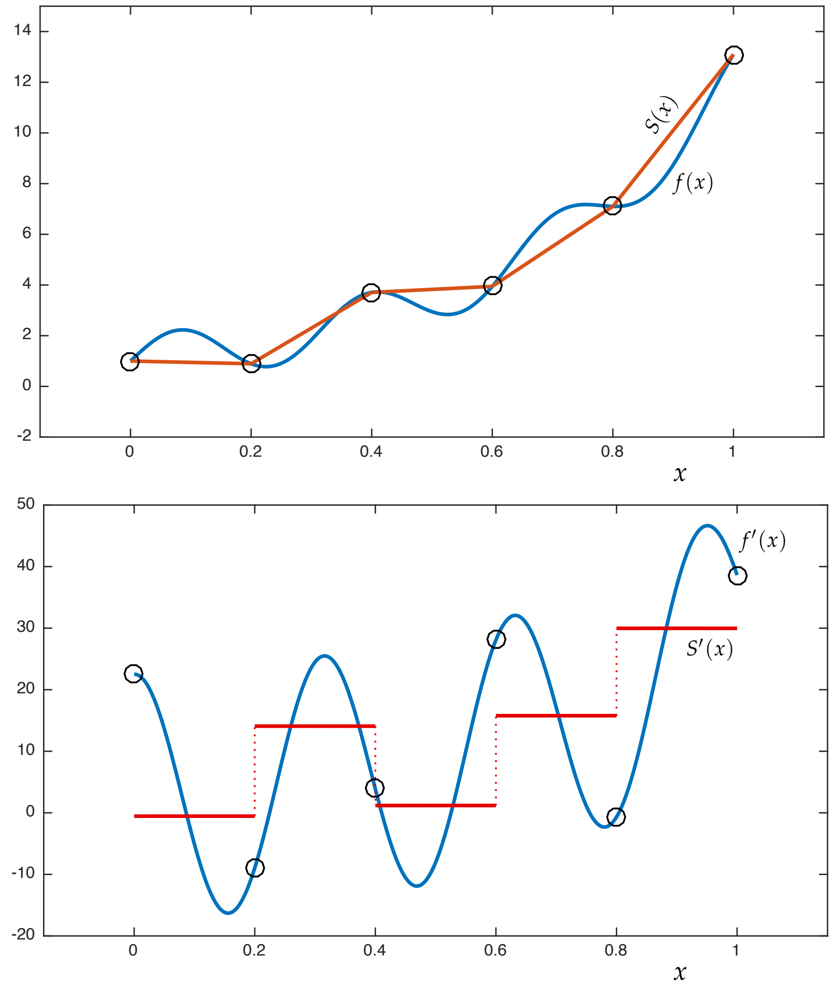

Figure 1.18: Piecewise linear interpolant to $f(x) = \sin(20x) + e^{5x/2}$ at $n = 5$ uniformly spaced points (top), and the derivative of this interpolant (bottom). Notice that the interpolant is continuous, but its derivative has jump discontinuities.

What could go wrong with this simple approach? The primary difficulty is that the interpolant is *continuous*, but generally not *continuously differentiable*. Still, these functions are easy to construct and cheap to evaluate, and can be very useful despite their simplicity.

1.10.2 *Piecewise cubic Hermite interpolation*

To remove the discontinuities in the first derivative of the piecewise linear interpolant, we begin by modeling our data with cubic polynomials over each interval $[x_j, x_{j+1}]$. Each such cubic has four free parameters (since $\mathcal{P}_3$ is a vector space of dimension 4); we require these polynomials to interpolate both f and its first derivative:

$$\begin{aligned} s_j(x_{j-1}) &= f(x_{j-1}), & j &= 1,\ldots,n; \\ s_j(x_j) &= f(x_j), & j &= 1,\ldots,n; \\ s_j'(x_{j-1}) &= f'(x_{j-1}), & j &= 1,\ldots,n; \\ s_j'(x_j) &= f'(x_j), & j &= 1,\ldots,n. \end{aligned}$$

To satisfy these conditions, take s_j to be the Hermite interpolant to the data $(x_{j-1}, f(x_{j-1}), f'(x_{j-1}))$ and $(x_j, f(x_j), f'(x_j))$. The resulting function, $S(x) := s_j(x)$ for $x \in [x_{j-1}, x_j]$, will always have a continuous derivative, $S \in C^1[x_0, x_n]$, but generally $S \notin C^2[x_0, x_n]$ due to discontinuities in the second derivative at the interpolation points.

In many applications, we lack specific values for $S'(x_j) = f'(x_j)$; we simply want the function $S(x)$ to be as *smooth* as possible. That motivates our next topic: splines.

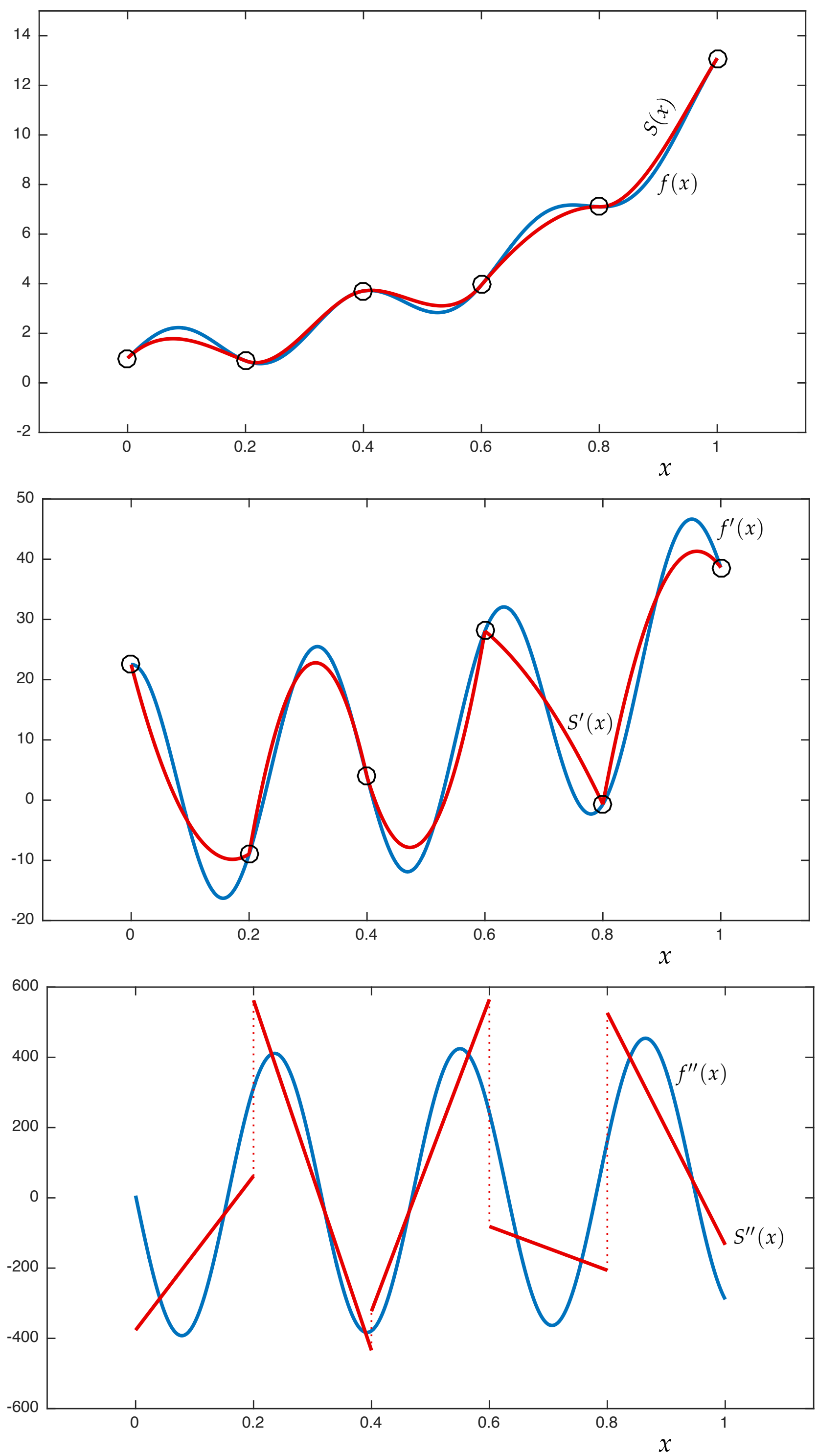

Figure 1.19: Piecewise cubic Hermite interpolant to $f(x) = \sin(20x) + e^{5x/2}$ at $n = 5$ uniformly spaced points (top), and the derivative of this interpolant (middle). Now both the interpolant and its derivative are continuous, and the derivative interpolates f'. However, the second derivative of the interpolant now has jump discontinuities (bottom).