MATH/CS 5466 NUMERICAL ANALYSIS
Homework 3

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March 14, 2016

Problem 1. Consider the approximation to $f(x) = \sqrt{x}$ for $x \in [0,1]$:

1. Construct the interpolant $p_1 \in \mathcal{P}_1$ to $f(x) = \sqrt{x}$ using the interpolation points $x_0 = 1/3$ and $x_1 = 1/3$ and produce a plot of the error $\sqrt{x} - p_1(x)$. Identify three points (call them $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2$ in this case) at which the error $f(x) - p_1(x)$ oscillates sign. Apply *de la Valee Poussin*'s theorem to obtain a lower bound on the error $\|f - p_*\|_\infty$ in the minimax approximation $p_* \in \mathcal{P}_1$ to f on $[0,1]$.

Apply Newton basis for the interpolating polynomial, we have:

$$p_1(x) = p_0(x) + c_1 q_1(x)$$

with: $p_0(x) = f(x_0) \Rightarrow p_0(x) = \sqrt{\dfrac{1}{3}}$

$$c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\sqrt{\dfrac{2}{3}} - \sqrt{\dfrac{1}{3}}}{\dfrac{2}{3} - \dfrac{1}{3}} = \sqrt{3}(\sqrt{2} - 1)$$

$$q_1 = x - x_0 = x - \frac{1}{3};$$

$$\Rightarrow p_1(x) = \sqrt{\frac{1}{3}} + \sqrt{3}(\sqrt{2} - 1)\left(x - \frac{1}{3}\right) = \sqrt{3}(\sqrt{2} - 1)x + \frac{2 - \sqrt{2}}{\sqrt{3}}$$
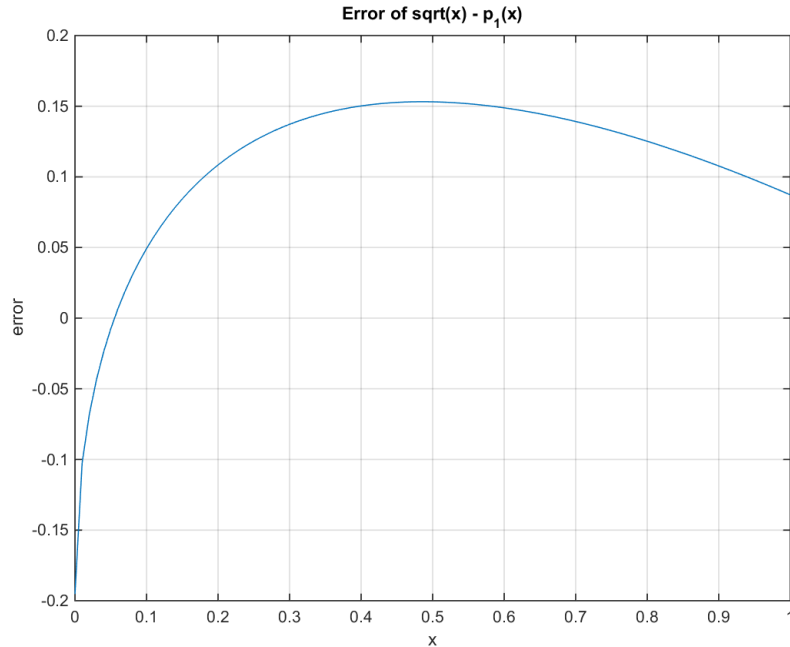
Plot the error $\sqrt{x} - p_1(x)$ over $[0,1]$:

Figure 1: Error of $\sqrt{x} - p_1(x)$ over $[1/3, 2/3]$

Identify 3 points $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2$, at which the error $f(x) - p_1(x)$ oscillates sign:

Base on above figure, we could point out three oscillates points:
$$\tilde{x}_0 = 0.2, \tilde{x}_1 = 0.5, \tilde{x}_2 = 0.8$$

Apply **de la Valee Poussin**'s theorem:

$$\|f - p_*\|_\infty \geq \min_{0 \leq j \leq n+1} |f(x_j) - p_1(x_j)| = \min_{0 \leq x_j \leq 1} \left| \sqrt{x} - (\sqrt{6} - \sqrt{3})x - \frac{2 - \sqrt{2}}{\sqrt{3}} \right|$$

For error function: $e(x) = \sqrt{x} - (\sqrt{6} - \sqrt{3})x - \dfrac{2 - \sqrt{2}}{\sqrt{3}}$

$$e'(x) = \frac{1}{2\sqrt{x}} - (\sqrt{6} - \sqrt{3})$$

so that $e'(x) = 0 \Leftrightarrow x = \dfrac{1}{4(9 - 6\sqrt{2})} = 0.4857$

$$\Rightarrow \min_{0 \leq x_j \leq 1} |e(x)| = \left| \sqrt{x} - (\sqrt{6} - \sqrt{3})x - \frac{2 - \sqrt{2}}{\sqrt{3}} \right|_{x=0.4857} = 0.0103$$

The lower bound on the error: $\|f - p_*\|_\infty \geq 0.0103$

2. For the same interpolant p_1 at $x_0 = 1/3$ and $x_2 = 2/3$, Problem 3 of Problem set 2 gives a different lower bound on $\|f - p_*\|_\infty$:

$$\frac{\|f - p_1\|_\infty}{1 + \|\Pi_1\|_\infty} \leq \|f - p_*\|_\infty$$

Compute (by hand) the left hand side:

We first have: $\|f - p_1\|_\infty = \min_{x \in [0,1]} \left| \sqrt{x} - (\sqrt{6} - \sqrt{3})x - \dfrac{2 - \sqrt{2}}{\sqrt{3}} \right| = 0.0103$

After that: $\|\Pi_1\|_\infty = \max_{x \in [0,1]} |l_0(x)| + |l_1(x)|$

$$l_0(x) = \prod_{k=0, k \neq 0}^{1} \frac{x - x_k}{x_0 - x_k} = \frac{x - x_1}{x_0 - x_1} = \frac{x - \frac{2}{3}}{\frac{1}{3} - \frac{2}{3}} = -3\left(x - \frac{2}{3}\right) = 2 - 3x$$

$$l_1(x) = \prod_{k=0, k \neq 1}^{1} \frac{x - x_k}{x_1 - x_k} = \frac{x - x_0}{x_1 - x_0} = \frac{x - \frac{1}{3}}{\frac{2}{3} - \frac{1}{3}} = 3\left(x - \frac{1}{3}\right) = 3x - 1$$

$$\Rightarrow \|\Pi_1\|_\infty = \max_{x \in [0,1]} |l_0(x)| + |l_1(x)| = \max_{x \in [0,1]} (|2 - 3x| + |3x - 1|) = 3$$

So that: $\dfrac{\|f - p_1\|_\infty}{1 + \|\Pi_1\|_\infty} = \dfrac{0.0103}{1 + 3} = 0.002575$

3. Now compute $p_* \in \mathcal{P}_1$ that best approximates $\sqrt{x}$ in the minimax sense, and report the error. Is this error consistent with the bounds you obtained in part (a)? [Kincaid & Cheney]

For minimax sense: $p_*(x) = \alpha + \beta x \in \mathcal{P}_1$ with $x_0 = 0, x_2 = 1$:

$$\begin{cases} \delta = f(x_0) - (\alpha + \beta x_0) \\ -\delta = f(x_1) - (\alpha + \beta x_1) \\ \delta = f(x_2) - (\alpha + \beta x_2) \\ f'(x_1) - p'_*(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} \delta = 0 - (\alpha + \beta.0) \\ -\delta = \sqrt{x_1} - (\alpha + \beta x_1) \\ \delta = 1 - (\alpha + \beta.1) \\ \dfrac{1}{2\sqrt{x_1}} = \beta \end{cases} \Leftrightarrow \begin{cases} \alpha = \dfrac{1}{8} \\ \beta = 1 \\ \delta = -\dfrac{1}{8} \\ x_1 = \dfrac{1}{4} \end{cases}$$

So we have: $p_* = \dfrac{1}{8} + x$
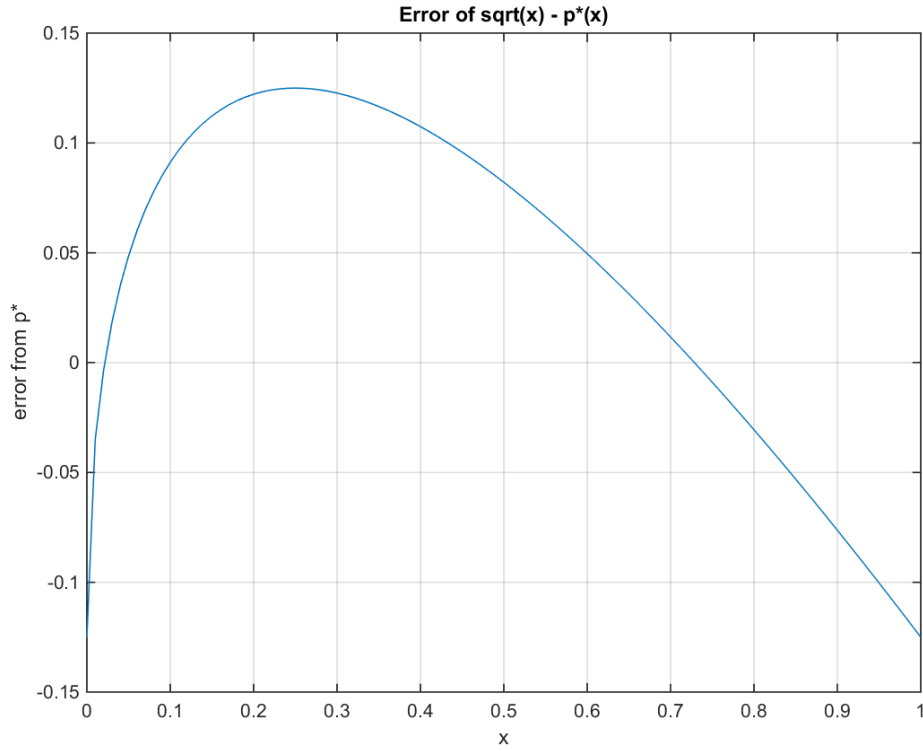
The error is presented in following figure:

Figure 2: Error of p_* approximation

The error is bounded by: $\min \|f - p_*\|_\infty = \dfrac{1}{8}$ at $x = x_1 = \dfrac{1}{4}$

From that we see: $\dfrac{1}{8} = 0.125 >> 0.0103 \Rightarrow$ so the error is consistent with the bounds obtained from part (a).

Problem 2.

1. Find the line $P_* \in \mathcal{P}_1$ that gives a least square approximation to $f(x) = \sqrt{x}$ on $x \in [0, 1]$, and report the error $\|f - P_*\|_2$

   We have the form of P_*: $P_* = P_1 = c_0 \Phi_0 + c_1 \Phi_1$ with $\begin{cases} \Phi_0 = x^0 = 1 \\ \Phi_1 = x^1 = x \end{cases}$

   So: $P_* = c_0 + c_1 x$, so that we form the following equation:

   $$\begin{bmatrix} \langle \Phi_0, \Phi_0 \rangle & \langle \Phi_0, \Phi_1 \rangle \\ \langle \Phi_1, \Phi_0 \rangle & \langle \Phi_1, \Phi_1 \rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \langle f, \Phi_0 \rangle \\ \langle f, \Phi_1 \rangle \end{bmatrix}$$

   with:

   $$\langle \Phi_0, \Phi_0 \rangle = \langle 1, 1 \rangle = \int_0^1 1.1.\mathrm{d}x = x\big|_0^1 = 1$$

   $$\langle \Phi_0, \Phi_1 \rangle = \langle \Phi_1, \Phi_0 \rangle = \langle 1, x \rangle = \int_0^1 1.x.\mathrm{d}x = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$$

   $$\langle \Phi_1, \Phi_1 \rangle = \langle x, x \rangle = \int_0^1 x.x\mathrm{d}x = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

   $$\langle f, \Phi_0 \rangle = \langle x^{\frac{1}{2}}, 1 \rangle = \int_0^1 x^{\frac{1}{2}}.1.\mathrm{d}x = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}$$

   $$\langle f, \Phi_1 \rangle = \langle x^{\frac{1}{2}}, x \rangle = \int_0^1 x^{\frac{1}{2}}.x.\mathrm{d}x = \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\Big|_0^1 = \frac{2}{5}$$

   So: $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{5} \end{bmatrix} \Leftrightarrow \begin{cases} c_0 = \dfrac{4}{15} \\ c_1 = \dfrac{4}{5} \end{cases}$

   $\Rightarrow P_* = \dfrac{4}{15} + \dfrac{4}{5}x$

   The error is presented as bellow (figure 3):

2. For a general interval $[a, b]$, prove that for all $f \in C[a, b]$:

   $$\min_{p \in \mathcal{P}_n} \|f - p\|_2 \leq \sqrt{b - a} \min_{p \in \mathcal{P}_n} \|f - p\|_\infty$$

   We have: $\min\limits_{p \in \mathcal{P}_n} \|f - p\|_2 = \|f - p_*\|_2$ and $\min\limits_{p \in \mathcal{P}_n} \|f - p\|_\infty = \|f - p_*\|_\infty$

   and $|f - p_*| \leq \|f - p_*\|_\infty$

   Based on Example 8.2 {Suli and Mayers} equation:

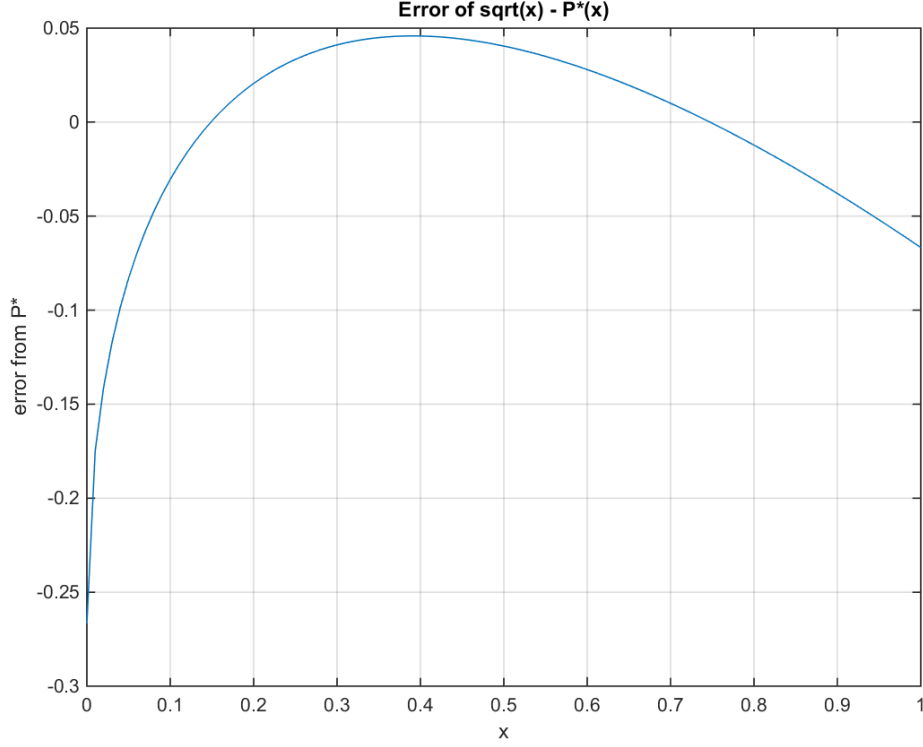 $$\|f\|_2 = \left(\int_a^b w(x)|f(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

Figure 3: Error of P_* approximation

Applied above equation with the **_weight function_** $w(x) = 1$ we have:

$$\|f - p_*\|_2 = \left(\int_a^b w(x)|f(x) - p_*(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq \left(\int_a^b 1.\left(\|f(x) - p_*(x)\|_\infty\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\Leftrightarrow \|f - p_*\|_2 \leq \left(\int_a^b 1.\mathrm{d}x \int_a^b \left(\|f(x) - p_*(x)\|\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\Leftrightarrow \|f - p_*\|_2 \leq \left(x\big|_a^b\right)^{\frac{1}{2}}.\left(\int_a^b \left(\|f(x) - p_*(x)\|_\infty\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\Leftrightarrow \|f - p_*\|_2 \leq \sqrt{b - a}.\|f(x) - p_*(x)\|_\infty$$

$$\Leftrightarrow \min_{p \in \mathcal{P}_n} \|f - p\|_2 \leq \sqrt{b - a} \min_{p \in \mathcal{P}_n} \|f - p\|_\infty$$

For problem 1c and 2a we have $a = 0, b = 1 \Rightarrow$ The lower bound is defined as:

$$\min_{p \in \mathcal{P}_n} \|f - p\|_2 \leq \min_{p \in \mathcal{P}_n} \|f - p\|_\infty$$

i. From part 1c: $\min\limits_{p \in \mathcal{P}_n} \|f - p\|_\infty = 0.125$

ii. From part 2a: $\min\limits_{p \in \mathcal{P}_n} \|f - p\|_2 = 0.0458$

We have: $0.125 > 0.0458 \Rightarrow$ these 2 problems are consistent with the bound.

3. Let $[a, b]$ be any fixed interval. Given any $\epsilon > 0$, show that there exist some $f \in C[a, b]$ such that $\|f\|_2 \le \epsilon$, while $\|f\|_\infty \ge \frac{1}{\epsilon}$.

Define: $f(x) = \dfrac{2\frac{1}{\epsilon}}{\sqrt{1 + k^2(x - c)^2}}$

with: $c = \dfrac{1}{2}(a + b)$ and k is a constant to be determined ($k > 0$).

We first have: $\|f\|_\infty = 2\dfrac{1}{\epsilon} \ge \dfrac{1}{\epsilon}$

After that:

$$(\|f(x)\|_\infty)^2 = \int_a^b |f(x)|^2 \mathrm{d}x = \int_a^b \frac{\frac{4}{\epsilon^2}}{1 + k^2(x - c)^2}\mathrm{d}x$$

$$\Leftrightarrow (\|f(x)\|_\infty)^2 = \frac{4}{k\epsilon^2}\left(tan^{-1}\frac{1}{2}k(b - a) - tan^{-1}\frac{1}{2}k(a - b)\right) \le \frac{4\pi}{k\epsilon^2}$$

$$\Leftrightarrow \|f(x)\|_\infty \le \frac{2\sqrt{\pi}}{\epsilon\sqrt{k}} \qquad (\Rightarrow k \le \pi)$$

Applied above result:

$$\begin{cases} \|f\|_2 \le \sqrt{b - a}\|f\|_\infty \\ \|f\|_2 \le \epsilon \quad (required) \end{cases} \Leftrightarrow \epsilon \le \sqrt{b - a}.\frac{2\sqrt{\pi}}{\epsilon\sqrt{k}} \Leftrightarrow k \le \frac{4\pi(b - a)}{\epsilon^4}$$

With k meet above conditions ($k \le \pi$, and $k \le \dfrac{4\pi(b - a)}{\epsilon^4}$), function $f(x)$ is exist.

Problem 3. Suppose we have a function $f \in C[a, b]$ but $f(x)$ is unbounded as $x \to a$. Then for any polynomial $p \in \mathcal{P}_n, \|f - p\|_\infty$ would be infinite over $[a, b]$, and hence there exists no minimax approximation. The last problem hints that the situation could possibly be better for least squares approximation. This problem, adapted from Gautchi, explores an example where this is the case.

The *shifted Legendre polynomials* for the interval $x \in [0, 1]$ are defined as $\pi_0(x) = 1, \pi_1(x) = 2x - 1$, and then, for $n \ge 1$, via the recurrence:

$$\pi_{n+1} = \frac{2n + 1}{n + 1}(2x - 1)\pi_n(x) - \frac{n}{n + 1}\pi_{n-1}(x)$$

You may use without proof that these polynomial are *orthogonal* with respect to inner product (and norm):

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx, \qquad \|f\|_2 = (f, f)^{1/2}$$

By longstanding convention these polynomials are constructed such that $\pi_n(1) = 1$, so they are not *orthonormal*. You may thus also use without proof that:

$$(\pi_j, \pi_j) = \frac{1}{2j + 1}$$

1. Work out explicit formulas for $\pi_2(x)$ and $\pi_3(x)$; simplify as much as possible.

   - For n=1:
   $$\pi_2(x) = \pi_{1+1}(x) = \frac{2*1+1}{1+1}(2x-1)\pi_1(x) - \frac{1}{1+1}\pi_0(x)$$
   $$= \frac{3}{2}(2x-1)(2x-1) - \frac{1}{2}.1$$
   $$= 6x^2 - 6x + 1$$

   - For n=2:
   $$\pi_3(x) = \pi_{2+1}(x) = \frac{2*2+1}{2+1}(2x-1)\pi_2(x) - \frac{2}{2+1}\pi_1(x)$$
   $$= \frac{5}{3}(2x-1)(6x^2 - 6x + 1) - \frac{2}{3}.(2x-1)$$
   $$= 20x^3 - 30x^2 + 12x - 1$$

   We wish to approximate the *unbounded* function $f(x) = log(1/x)$ (log denotes the natural logarithm). We will use these orthogonal polynomials as a basis for least squares approximation with degree-n polynomials, writing the optimal polynomial $P_{*,n} \in \mathcal{P}$ in the form:

   $$P_{*,n}(x) = \sum_{j=0}^{n} c_j \pi_j(x)$$

2. Use the fact that:

   $$\int_0^1 \pi_j \log(1/x)\mathrm{d}x = \begin{cases} 1 & j = 0 \\ \dfrac{(-1)^j}{j(j+1)} & j \geq 1 \end{cases}$$

   to determine formulas for the coefficient $c_0, ..., c_n$ in the best approximation
   The error based on least square approximation:

   $$\|f - P_{*,n}\|_2 = \left(\int_0^1 (f(x) - P_{*,n}(x))^2 \, \mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_0^1 \left(\log(\frac{1}{x}) - \sum_{j=0}^{n} c_j \pi_j(x)\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

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Define the Error function:

$$E(c_0, c_1, ..., c_n) = \int_0^1 \left(\log(\frac{1}{x}) - \sum_{j=0}^n c_j \pi_j(x) \right)^2 dx$$

$$\Leftrightarrow E(c_0, c_1, ..., c_n) = \|f\|^2 - 2\sum_{k=0}^n c_k \langle \log(\frac{1}{x}), \pi_k \rangle + \sum_{j=0}^n \sum_{k=0}^n c_j c_k \langle \pi_j, \pi_k \rangle$$

Taking derivative of above equation for $c_0, c_1, ..., c_n$ respectively, we have:

$$\sum_{j=0}^n c_j \langle \pi_j, \pi_l \rangle = \langle \log(\frac{1}{x}), \pi_l \rangle \qquad l = 0, ..., n$$

Linear equation for coefficient c_j is formed:

$$\begin{bmatrix} \langle \pi_0, \pi_0 \rangle & \langle \pi_1, \pi_0 \rangle & ... & ... & ... & \langle \pi_n, \pi_0 \rangle \\ \langle \pi_0, \pi_1 \rangle & \langle \pi_1, \pi_1 \rangle & ... & ... & ... & \langle \pi_n, \pi_1 \rangle \\ ... & ... & ... & ... & ... & ... \\ \langle \pi_0, \pi_j \rangle & \langle \pi_1, \pi_j \rangle & ... & ... & ... & \langle \pi_n, \pi_j \rangle \\ ... & ... & ... & ... & ... & ... \\ \langle \pi_0, \pi_n \rangle & \langle \pi_1, \pi_n \rangle & ... & ... & ... & \langle \pi_n, \pi_n \rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ ... \\ c_j \\ ... \\ c_n \end{bmatrix} = \begin{bmatrix} \langle \log(\frac{1}{x}), \pi_0 \rangle \\ \langle \log(\frac{1}{x}), \pi_1 \rangle \\ ... \\ \langle \log(\frac{1}{x}), \pi_j \rangle \\ ... \\ \langle \log(\frac{1}{x}), \pi_n \rangle \end{bmatrix}$$

We can easily see that:

$$\langle \pi_j, \pi_k \rangle = \int_0^1 \pi_j(x) \pi_k(x) dx = 0 \qquad for \ j \neq k$$

$$(\pi_j, \pi_j) = \frac{1}{2j+1} \quad and \quad \int_0^1 \pi_j \log(1/x) dx = \begin{cases} 1 & j = 0 \\ \dfrac{(-1)^j}{j(j+1)} & j \geq 1 \end{cases}$$

Linear equation for coefficient c_j is defined as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & \frac{1}{2j+1} & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2n+1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ ... \\ c_j \\ ... \\ c_n \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ ... \\ \frac{(-1)^j}{j(j+1)} \\ ... \\ \frac{(-1)^n}{n(n+1)} \end{bmatrix}$$

3. Prove that $\|f - P_{*,n}\| = \dfrac{1}{n+1}$ (Hint: use induction)

We have:

$$\|f - P_{*,n}\|^2 = \|f\|^2 - \|P_{*,n}\|^2 = \|f\|^2 - \sum_{k=0}^{n} \frac{\langle f, \pi_k \rangle^2}{\langle \pi_k, \pi_k \rangle} \qquad (*)$$

Assume that $x = e^{-t}$ we have:

$$\|f\|^2 = \int_0^1 [\log(1/x)]^2 \, dx = \int_\infty^0 t^2(-e^{-t}) dx = \int_0^\infty t^2 e^{-t} dx = 2! = 2$$

Utilize the fact given in part (b):

$$\langle \pi_j, f \rangle = \int_0^1 \pi_j \log(1/x) dx = \begin{cases} 1 & j = 0 \\ \dfrac{(-1)^j}{j(j+1)} & j \geq 1 \end{cases} \Rightarrow \langle \pi_j, f \rangle^2 = \begin{cases} 1 & j = 0 \\ \dfrac{1}{j^2(j+1)^2} & j \geq 1 \end{cases}$$

Besides, from question's data: $\langle \pi_j, \pi_j \rangle = \dfrac{1}{2j+1}$

Replaced to (*) we have final result:

$$\|f - P_{*,n}\|^2 = 2 - 1 - \sum_{j=1}^{n} \frac{\frac{1}{j^2(j+1)^2}}{\frac{1}{2j+1}} = 1 - \sum_{j=1}^{n} \frac{2j+1}{j^2(j+1)^2}$$

$$\Leftrightarrow \|f - P_{*,n}\|^2 = 1 - \sum_{j=1}^{n} \frac{(j+1)^2 - j^2}{j^2(j+1)^2} = 1 - \sum_{j=1}^{n} \frac{1}{j^2} + \sum_{j=1}^{n} \frac{1}{(j+1)^2}$$

$$\Leftrightarrow \|f - P_{*,n}\|^2 = 1 - \sum_{j=1}^{n} \frac{1}{j^2} + \sum_{j=2}^{n+1} \frac{1}{j^2} = 1 - 1 + \frac{1}{(n+1)^2} = \frac{1}{(n+1)^2}$$

So that: $\|f - P_{*,n}\| = \dfrac{1}{n+1}$

4. Create a plot comparing f to the first four least squares approximation: $P_{*,0}, P_{*,1}, P_{*,2}, P_{*,3}$.

For $n = 3$ we have the specific linear equation for coefficient c_j:

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1}{3} & 0 & 0 \\ 0 & 0 & \dfrac{1}{5} & 0 \\ 0 & 0 & 0 & \dfrac{1}{7} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\dfrac{1}{2} \\ \dfrac{1}{6} \\ -\dfrac{1}{12} \end{bmatrix} \Leftrightarrow \begin{cases} c_0 = 1 \\ c_1 = -\dfrac{3}{2} \\ c_2 = \dfrac{5}{6} \\ c_3 = -\dfrac{7}{12} \end{cases}
$$

The first four least square approximation is defined:

$$P_{*,0} = c_0\pi_0 = 1.1 = 1$$

$$P_{*,1} = c_0\pi_0 + c_1\pi_1 = 1.1 - \frac{3}{2}(2x - 1) = -3x + \frac{5}{2}$$

$$P_{*,2} = c_0\pi_0 + c_1\pi_1 + c_2\pi_2 = 1.1 - \frac{3}{2}(2x - 1) + \frac{5}{6}(6x^2 - 6x + 1) = 5x^2 - 8x + \frac{10}{3}$$

$$
\begin{aligned}
P_{*,3} &= c_0\pi_0 + c_1\pi_1 + c_2\pi_2 + c_3\pi_3 \\
&= 1.1 - \frac{3}{2}(2x - 1) + \frac{5}{6}(6x^2 - 6x + 1) - \frac{7}{12}(20x^3 - 30x^2 + 12x - 1) \\
&= -\frac{35}{3}x^3 + \frac{45}{2}x^2 - 15x + \frac{47}{12}
\end{aligned}
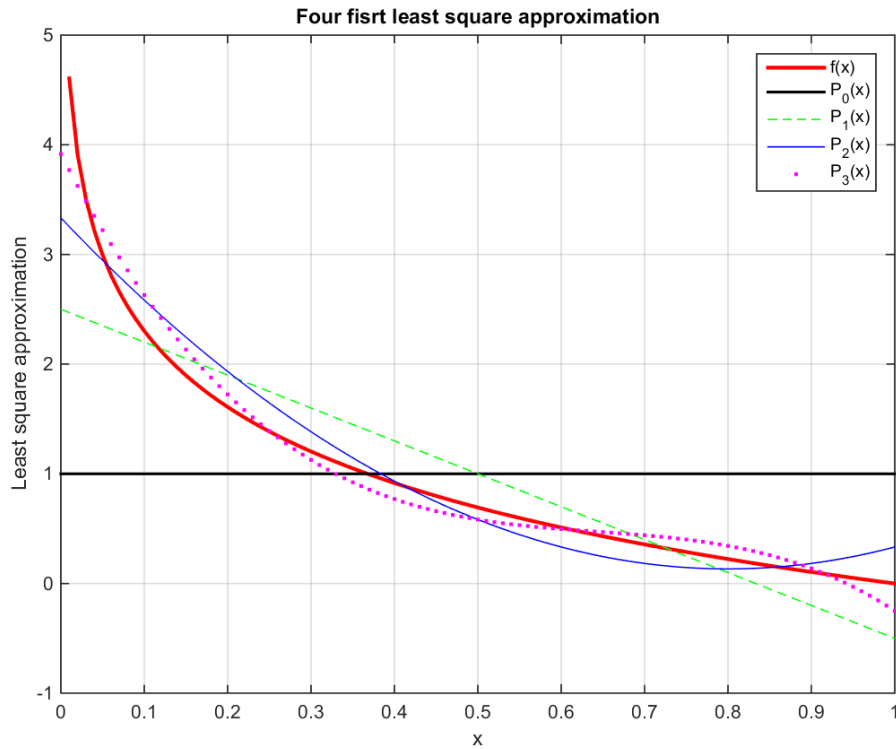$$

Approximations are presented in following plot:

Figure 4: First four least square approximation of $f(x) = \log(\dfrac{1}{x})$

Problem 4. The *conjugate gradient algorithm* is the most important iterative method for solving the linear system of equations $\mathbf{Ac} = \mathbf{b}$ when the matrix $\mathbf{A} \in \mathbb{R}^{NxN}$ is large, sparse, and symmetric positive define. Since $\mathbf{A}$ is symmetric positive definite, its eigenvalues are real and positive; write them as: $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$

One can show that at step k, the conjugate gradient algorithm constructs an iterate $\mathbf{c}_k$ such that:

$$\frac{\|\mathbf{c} - \mathbf{c}_k\|_{\mathbf{A}}}{\|\mathbf{b}\|_{\mathbf{A}}} \leq \min_{p \in \mathcal{P}_k, p(0)=1} \max_{x \in [\lambda_1, \lambda_N]} |p(x)|$$

(Here $\|.\|_{\mathbf{A}}$ defines the *energy norm* induced by $\mathbf{A}$: $\|z\|_{\mathbf{A}} = (z^T \mathbf{A} z)^{1/2}$.)

1. Derive the solution to the problem: $\quad \min\limits_{p \in \mathcal{P}_k, p(0)=1} \max\limits_{x \in [\lambda_1, \lambda_N]} |p(x)|$

   in term of Chebyshev polynomials.

   Chebyshev polynomials is defined by: $T_n(x) = \cos(n \cos^{-1} x)$ for $x \in [-1, 1]$

   $n = 0$: $T_0(x) = \cos(0 \cos^{-1} x) = \cos(0) = 1$

   $n = 1$: $T_1(x) = \cos(1 \cos^{-1} x) = x$

   $n = 2$: $T_2(x) = \cos(2 \cos^{-1} x) = 2 \cos^2(\cos^{-1} x) - 1 = 2x^2 - 1$

   In general, for $n \geq 2$: $T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$

   We also can represent Chebyshev polynomial by:

   $$T_n(x) = \frac{1}{2} \left[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right]$$

   so if $x = x_j = \cos(\frac{n\pi}{n})$ then $T_n(x_j) = (-1)^n$, $j = 0, 1, ...n$

   Define a polynomial q_n: $q_n(\lambda) = \left[T_n \left(\frac{\lambda_N + \lambda_1}{\lambda_N - \lambda_1} \right) \right]^{-1} . T_n \left(\frac{\lambda_N + \lambda_1 - 2\lambda}{\lambda_N - \lambda_1} \right)$

   because of $|T_n(x)| \leq 1$ for all $x \in [-1, 1]$ so:

   $$\max_{\lambda \in [\lambda_1, \lambda_N]} |q_n| = \left| T_n \left(\frac{\lambda_N + \lambda_1}{\lambda_N - \lambda_1} \right) \right|^{-1} = t_*$$

   Let $\mu_j = \frac{\lambda_1 - \lambda_N}{2} x_j + \frac{\lambda_1 + \lambda_N}{2}$ with defined x_j above, so:

   i. $p_n(\mu_j) = (-1)^j t_*$, $j = 0, ..., n$

   ii. $\mu_j \in [\lambda_1, \lambda_N]$

   Assume that there exists a polynomial $p_n \in \mathcal{P}_n$, which is the best approximation for $T_n(x)$ with $p_n(0) = 1$ and: $|p_n(\lambda)| < |t_*|$ for all $\lambda \in [\lambda_1, \lambda_N]$

   it means that: $-|t_*| < p_n(\mu_j) < |t_*|$ with $j = 0, ..., n$

   If $sign(t_*) > 0$ then:
   $$\begin{cases} p_n(\mu_j) - q_n(\mu_j) < 0 & for \ m \ even \\ p_n(\mu_j) - q_n(\mu_j) > 0 & for \ m \ odd \end{cases}$$

If $sign(t_*) < 0$ then:
$$\begin{cases} p_n(\mu_j) - q_n(\mu_j) < 0 & for\ m\ odd \\[2mm] p_n(\mu_j) - q_n(\mu_j) > 0 & for\ m\ even \end{cases}$$

Hence, regardless of the `sign` of t_*, the different $p_n - q_n$ has a zero in every interval (μ_j, μ_{j+1}). There are n such intervals.

But we also have that $p_n(0) - q_n(0) = 1 - 1 = 0$. Since $p_n - q_n$ is a polynomial of degree n, so $p_n \equiv q_n$.

As a result: $\min\limits_{p \in \mathcal{P}_k, p(0)=1} \max\limits_{x \in [\lambda_1, \lambda_N]} |p(x)| = \min\limits_{p \in \mathcal{P}_k, p(0)=1} \max\limits_{x \in [\lambda_1, \lambda_N]} |q_n(x)| = \left| T_n \left(\dfrac{\lambda_N + \lambda_1}{\lambda_N - \lambda_1} \right) \right|^{-1}$

2. Write `MATLAB` routine `cheby.m` such that `Tkx = cheby(x,k)` evaluate $T_k(x)$

```
function  Tkx  =  cheby(x,k)
      Tkx  =  cos(k*acos(x));
end
```

3. Plot the minimizing polynomial in part (a) for $k = 3$ and $k = 5$ for:
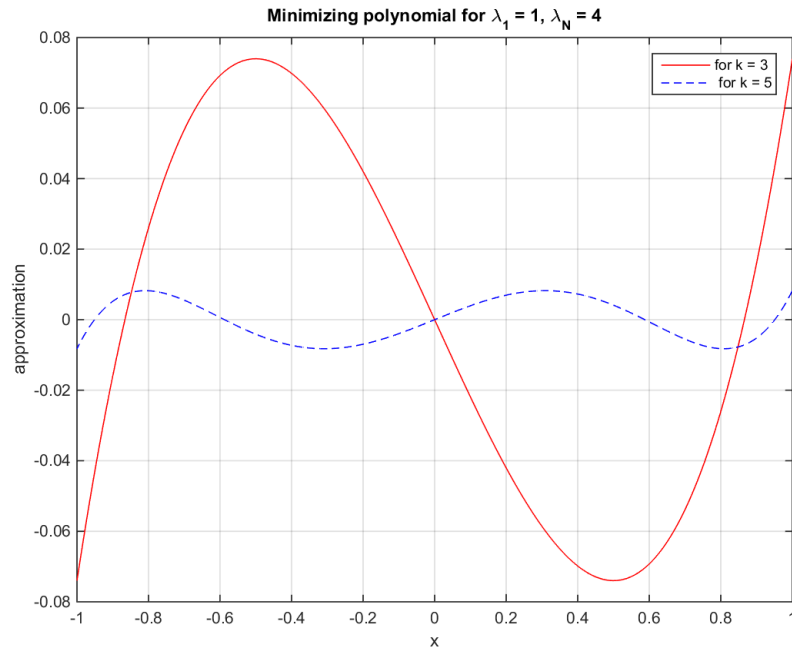
(i) $\lambda_1 = 1, \lambda_N = 4$

Figure 5: Minimizing polynomial for $\lambda_1 = 1, \lambda_N = 4$

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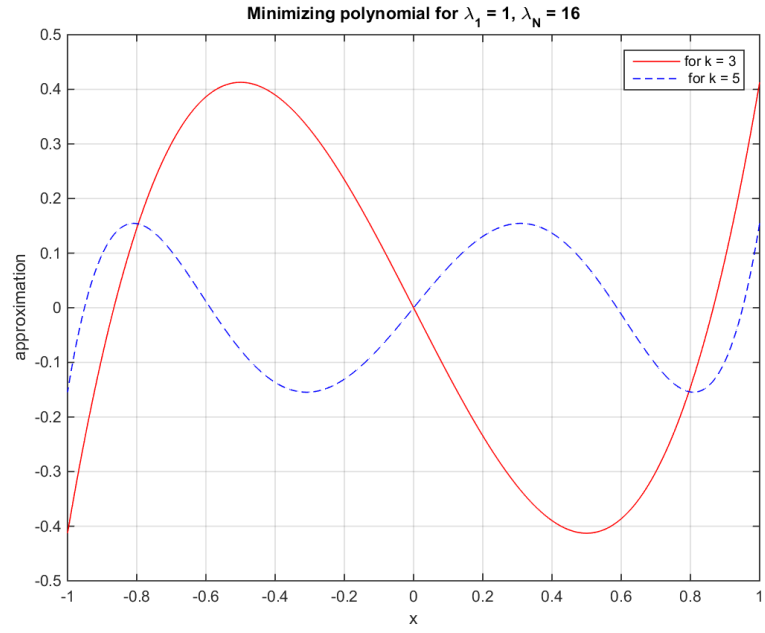(ii) $\lambda_1 = 1, \lambda_N = 16$

Figure 6: Minimizing polynomial for $\lambda_1 = 1, \lambda_N = 4$

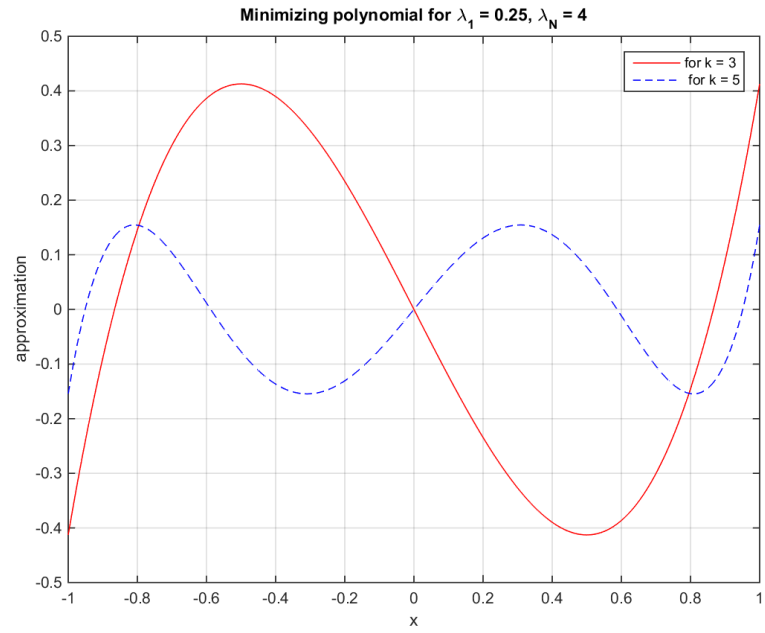(iii) $\lambda_1 = 1/4, \lambda_N = 4$

Figure 7: Minimizing polynomial for $\lambda_1 = 1/4, \lambda_N = 4$

4. Use your answer from part (a) to derive the classic error bound for conjugate gradients:

$$\min_{p \in \mathcal{P}_k, p(0)=1} \max_{x \in [\lambda_1, \lambda_N]} |p(x)| \leq 2 \left(\frac{\sqrt{\lambda_N/\lambda_1} - 1}{\sqrt{\lambda_N/\lambda_1} + 1} \right)^k$$

For $x = \dfrac{\lambda_N + \lambda_1}{\lambda_N - \lambda_1} = \dfrac{\lambda_N/\lambda_1 + 1}{\lambda_N/\lambda_1 - 1}$, replaced into above equation we have:

$$x \pm \sqrt{x^2 - 1} = \frac{\lambda_N/\lambda_1 + 1}{\lambda_N/\lambda_1 - 1} \pm \frac{2\sqrt{\lambda_N/\lambda_1}}{\lambda_N/\lambda_1 - 1} = \frac{\lambda_N/\lambda_1 + 1 \pm 2\sqrt{\lambda_N/\lambda_1}}{\lambda_N/\lambda_1 - 1}$$

$$= \frac{(\sqrt{\lambda_N/\lambda_1} \pm 1)^2}{(\sqrt{\lambda_N/\lambda_1} - 1)(\sqrt{\lambda_N/\lambda_1} + 1)}$$

$$= \frac{\sqrt{\lambda_N/\lambda_1} \pm 1}{\sqrt{\lambda_N/\lambda_1} \mp 1}$$

Thus:

$$T_n\left(\frac{\lambda_N + \lambda_1}{\lambda_N - \lambda_1}\right) = \frac{1}{2}\left[\left(\frac{\sqrt{\lambda_N/\lambda_1} + 1}{\sqrt{\lambda_N/\lambda_1} - 1}\right)^n + \left(\frac{\sqrt{\lambda_N/\lambda_1} - 1}{\sqrt{\lambda_N/\lambda_1} + 1}\right)^n\right]$$

Finally:

$$\left|T_n\left(\frac{\lambda_N + \lambda_1}{\lambda_N - \lambda_1}\right)\right|^{-1} = \frac{2}{\left(\dfrac{\sqrt{\lambda_N/\lambda_1} + 1}{\sqrt{\lambda_N/\lambda_1} - 1}\right)^n + \left(\dfrac{\sqrt{\lambda_N/\lambda_1} - 1}{\sqrt{\lambda_N/\lambda_1} + 1}\right)^n}$$

$$\leq 2\left(\frac{\sqrt{\lambda_N/\lambda_1} - 1}{\sqrt{\lambda_N/\lambda_1} + 1}\right)^n$$

So: $\displaystyle\min_{p \in \mathcal{P}_k, p(0)=1} \max_{x \in [\lambda_1, \lambda_N]} |p(x)| \leq 2\left(\frac{\sqrt{\lambda_N/\lambda_1} - 1}{\sqrt{\lambda_N/\lambda_1} + 1}\right)^k$ **Proved!**

What kind of eigenvalue intervals $[\lambda_1, \lambda_N]$ lead the conjugate gradient algorithm to converge quickly? Slowly?

From above result we have:

i. If $\lambda_N \gg \lambda_1 \Rightarrow \left(\dfrac{\sqrt{\lambda_N/\lambda_1} - 1}{\sqrt{\lambda_N/\lambda_1} + 1}\right)$ is closer to 1 $\Rightarrow$ the upper bound is closer to 2 (get the maximum value) $\Rightarrow$ the conjugate gradient algorithm to converge is slow.

ii. If $\lambda_N \approx \lambda_1 \Rightarrow \left(\dfrac{\sqrt{\lambda_N/\lambda_1} - 1}{\sqrt{\lambda_N/\lambda_1} + 1}\right) \approx 0$ (closer to 0) $\Rightarrow$ the upper bound is closer to 0 (get the minimum value) $\Rightarrow$ the conjugate gradient algorithm to converge is fast.