**MATH/CS 5466 · NUMERICAL ANALYSIS**

Practice Examination 2

Instructions:

1. Time limit: **2 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
3. You *may not* use any outside resources, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   *Show all your work to maximize partial credit.*
5. Print your name on the line below:

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1. [26 points: (a), (b), (c) = 5 points each; (d) = 8 points; (e) = 3 points]

   (a) Suppose we wish to approximate the integral $\int_{-1}^{1} f(x)\,\mathrm{d}x$ with the quadrature rule

   $$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx Af(0) + Bf'(0).$$

   How should $A$ and $B$ be selected to make this rule exact for all linear polynomials?

   (b) Suppose we improve this rule to

   $$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx Af(0) + Bf'(0) + Cf''(0).$$

   How should $A$, $B$, and $C$ be selected to make this rule exact for all quadratic polynomials?

   (c) With three evaluations of $f$, Simpson's rule is exact for all cubic polynomials. Is the same true of your method in (b)?

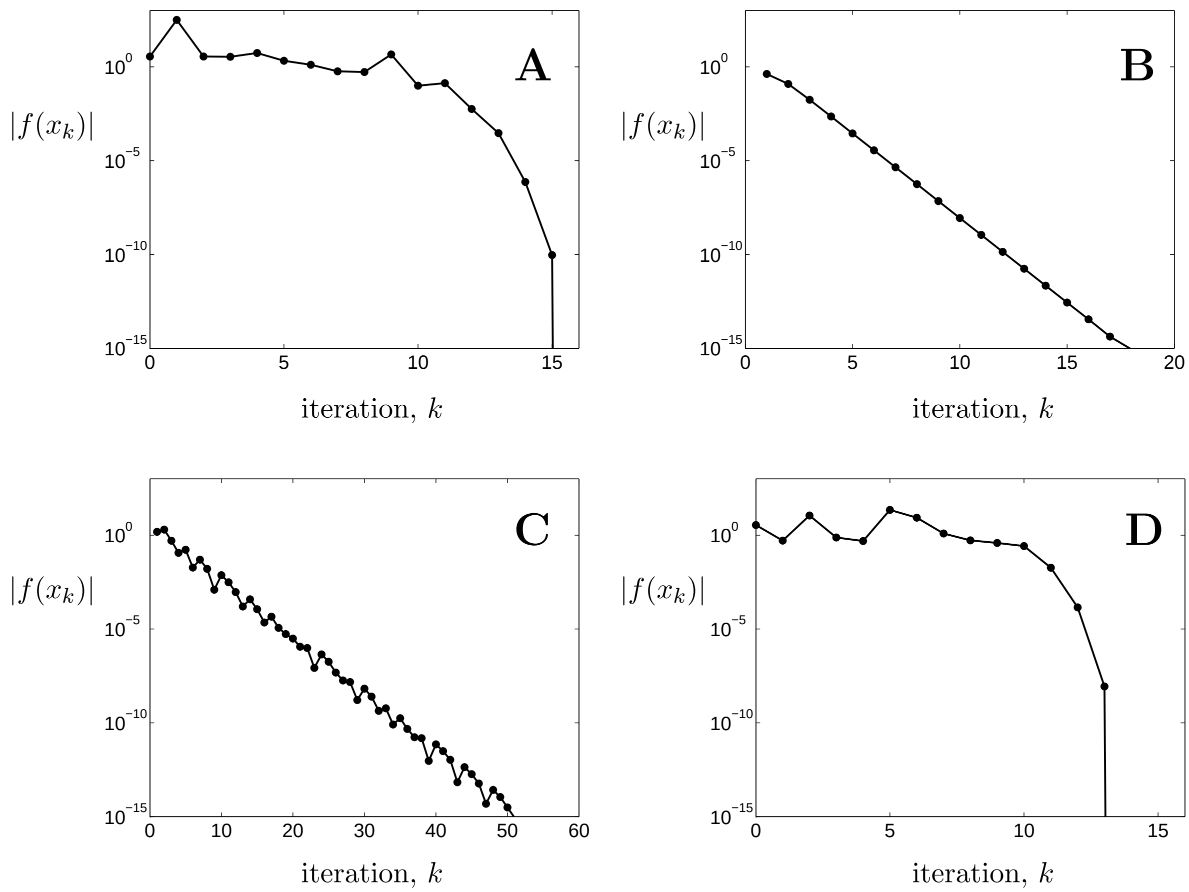
   (d) Assuming $f$ is sufficiently differentiable, use the Taylor expansion with remainder term to derive a bound on

   $$\Big|\int_{-1}^{1} f(x)\,\mathrm{d}x - \Big(Af(0) + Bf'(0) + Cf''(0)\Big)\Big|.$$

   (e) Why is the method you derived in (b) less famous than Simpson's rule?

2. [20 points: 5 points per part]

   The bisection, *regula falsi*, Newton, and secant methods are used to find the root of some particular function $f \in C(\mathbb{R})$. Graphs showing the convergence behavior obtained for each method are shown below, but the order is scrambled. Based on typical behavior of these algorithms, match up each plot with the appropriate method. *Justify your answers.*



3. [28 points: (a) = 5 points; (b) = 10 points; (c) = 13 points]

   Consider a Gaussian quadrature rule

$$I_n(f) = \sum_{j=0}^{n} w_j f(x_j)$$

   that approximates the integral

$$\int_a^b f(x)w(x)\,\mathrm{d}x$$

   for some weight function $w$ that is continuous and positive-valued on $(a, b)$.

   (a) Describe how the nodes $\{x_j\}$ relate to orthogonal polynomials. (No proof is necessary.)

   (b) Suppose $f$ is a polynomial. For what degrees of polynomial will the quadrature rule be exact? Applying this result for special choices of $f$, conclude that

$$\sum_{j=0}^{n} w_j = \int_a^b w(x)\,\mathrm{d}x$$

   and

$$w_j > 0, \qquad \text{for all } j = 0, \ldots, n.$$

   (c) Let $f$ be a continuous function on $[a, b]$. The Weierstrass approximation theorem states that for any $\varepsilon > 0$, there exists some polynomial $p$ such that

$$\max_{x\in[a,b]} |f(x) - p(x)| < \varepsilon.$$

   Use this result, along with those of part (b), to show that

$$\lim_{n\to\infty} I_n(f) \;=\; \int_a^b f(x)w(x)\,\mathrm{d}x.$$

   (If you were unable to prove the results in part (b), please take them as given and attempt this part of the problem anyway.)

4. [26 points: (a), (b), (c), (d), (f) = 4 points each; (e) = 6 points]
Consider a grid of points $t_0, t_1, t_2, \ldots$, where $t_k = t_0 + hk$ for some fixed $h > 0$.

(a) Construct the linear polynomial $p(t)$ that interpolates the data points $(t_k, x_k)$ and $(t_{k+1}, x_{k+1})$.

(b) Compute $p'(t)$.
To obtain a numerical method for the initial value problem

$$x'(t) = f(t, x), \quad x(t_0) = x_0,$$

set $p'(t_{k+1}) = f_{k+1}$, where $x_k \approx x(t_k)$ and $f_{k+1} = f(t_{k+1}, x_{k+1})$.
Write down this method in terms of only $x_k$, $x_{k+1}$, $h$, and $f_{k+1}$.
(Hint: you should obtain the backward Euler method.)

(c) Now construct the quadratic polynomial $q(t)$ that interpolates $(t_k, x_k)$, $(t_{k+1}, x_{k+1})$, and $(t_{k+2}, x_{k+2})$.

(d) Following the model of part (b), one can obtain a numerical method for $x'(t) = f(t, x)$ by setting $q'(t_{k+2}) = f_{k+2}$. Write down this method in terms of only $x_k$, $x_{k+1}$, $x_{k+2}$, $h$, and $f_{k+2}$.

(e) Is the method in part (d) zero-stable?

(f) Describe computational challenges to implementing the methods in part (b) and (d).
Be as complete as possible.