MATH/CS 5466 · NUMERICAL ANALYSIS

Practice Examination 1

Instructions:

1. Time limit: **2 uninterrupted hours**.
2. There are three questions worth a total of 100 points, plus a 5-point bonus.
3. You *may not* use any outside resources, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

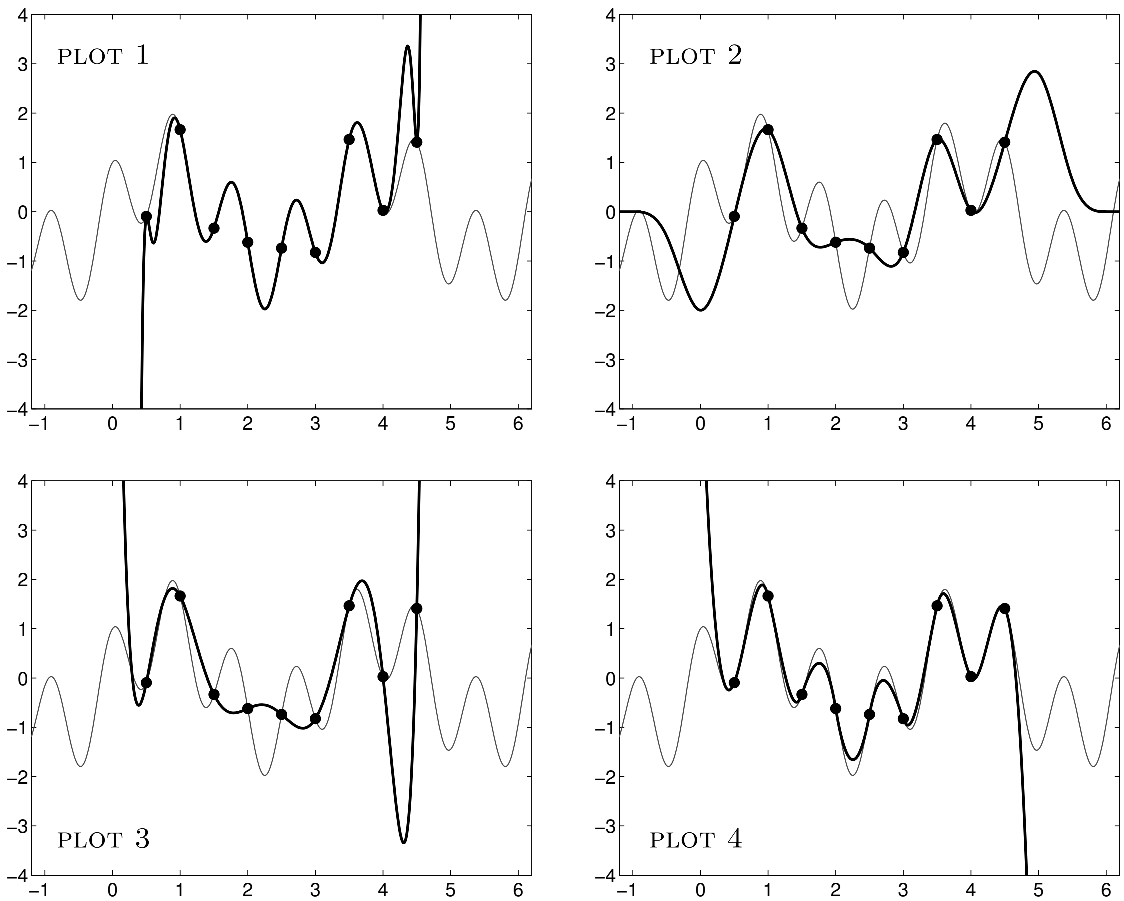
1. [24 points: 6 points per part]

   The following plots show several varieties of interpolants (thick lines) to some mystery function $f(x)$ (thin line) at the points marked with black dots ($\bullet$).

   Match up each plot with the following four interpolation methods:

   (a) standard polynomial interpolation;
   (b) Hermite polynomial interpolation;
   (c) piecewise cubic Hermite interpolation;
   (d) natural cubic spline interpolation.

   Be sure to justify your answers with convincing evidence.



2. [24 points: 8 points per part]

   Rational functions provide useful alternatives to standard polynomial approximations.

   (a) Suppose you wish to construct a rational function of the form

$$R(x) = \frac{\alpha + \beta x}{1 + \gamma x}$$

   that interpolates given data (x_1, f_1), (x_2, f_2), and (x_3, f_3), at *distinct* points x_1, x_2, and x_3. That is, we seek α, β, and γ such that

$$R(x_j) = f_j, \qquad j = 1, 2, 3.$$

   Show how you can determine α, β, and γ by setting up a 3×3 linear system, $\mathbf{Ac} = \mathbf{b}$, for the unknown vector

$$\mathbf{c} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

   (Just write the system down: you do not need to solve it.)

   (b) Describe a situation in which your matrix $\mathbf{A}$ in part (a) is singular.
   (You do not need to find all possible examples – one will suffice – but you must recall that x_1, x_2, and x_3 are distinct. Do not use the trivial example of $x_1 = x_2 = x_3$.)

   What does this singularity imply about the existence and/or uniqueness of the interpolant $R(x)$ in this situation?

   (c) The (1,1) Padé approximation to $f \in C^2(-1, 1)$ at $x = 0$ takes the form

$$R_{1,1}(x) = \frac{\alpha + \beta x}{1 + \gamma x},$$

   where the constants α, β, and γ are determined such that

$$\begin{aligned} R_{1,1}(0) &= f(0) \\ R'_{1,1}(0) &= f'(0) \\ R''_{1,1}(0) &= f''(0). \end{aligned}$$

   Show that the (1,1) Padé approximation to $f(x) = \mathrm{e}^{\lambda x}$ (with $\lambda \neq 0$) is given by

$$R_{1,1}(x) = \frac{1 + \frac{1}{2}\lambda x}{1 - \frac{1}{2}\lambda x}.$$

3. [28 points: 8 points for (a); 10 points each for (b), (c) per part]

   Suppose $f \in C^\infty[a,b]$, i.e., f is a continuous function on $[a,b]$, and all of its derivatives exist and are also continuous on $[a,b]$. Let $p_n \in \mathcal{P}_n$ denote the polynomial of degree-n (or less) that interpolates f at the points $x_0, \ldots, x_n$,

$$a \le x_0 < x_1 < \cdots < x_n \le b.$$

   (a) State (without proof) the standard formula for the error $f(x) - p_n(x)$ for $x \in [a,b]$.

   (b) Suppose there exist constants $C, M > 0$, independent of n, such that for all $n \ge 0$,

$$\max_{\xi \in [a,b]} |f^{(n)}(\xi)| \;\le\; CM^n.$$

   Show that as $n \to \infty$,

$$\max_{x \in [a,b]} |f(x) - p_n(x)| \to 0$$

   for *any* choice for the (distinct) interpolation points $x_0, \ldots, x_n \in [a,b]$.

   (c) Now consider uniformly spaced interpolation points on $[a,b] = [0,1]$, i.e., suppose that for each $n \ge 0$, $p_n \in \mathcal{P}_n$ interpolates f at

$$x_j = j/n, \qquad j = 0, \ldots, n.$$

   Show that the absolute value of the product term in the answer to (a) cannot exceed

$$\frac{n!}{n^{n+1}}.$$

   Based on this analysis, suggest an upper bound on

$$\max_{\xi \in [a,b]} |f^{(n)}(\xi)|$$

   that is larger than the bound on the magnitude of the derivative given in part (b), but still ensures that

$$\max_{x \in [a,b]} |f(x) - p_n(x)| \to 0$$

   as $n \to 0$ provided the interpolation occurs at uniformly spaced points.

4. [24 points: 4 points each for (a), (b), (c); 12 points for (d)]

   Recall that degree-n Chebyshev polynomial $T_n(x)$ is defined by

$$T_n(x) = \cos(n \cos^{-1}(x))$$

for $x \in [-1, 1]$, or, equivalently, via the recurrence

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n = 1, 2, \ldots,$$

where $T_0(x) = 1$ and $T_1(x) = x$.

   (a) Determine all roots of $T_n(x)$ in $[-1, 1]$.

   (b) Determine the extrema of $T_n(x)$ over $[-1, 1]$, i.e., find all $x \in [-1, 1]$ for which $|T_n(x)|$ is maximal.

   (c) Describe, in terms of T_{n+1}, the polynomial $p_n \in \mathcal{P}_n$ that minimizes

$$\max_{x\in[-1,1]} |x^{n+1} - p_n(x)|.$$

   Be sure to justify your answer.

   (d) Now consider the function

$$f(x) = \sum_{k=0}^{n+1} a_k x^k$$

   for constants $a_0, \ldots, a_{n+1}$ with $a_{n+1} \neq 0$. Find the polynomial $q_n \in \mathcal{P}_n$ that minimizes

$$\max_{x\in[-1,1]} |f(x) - q_n(x)|.$$

[Süli & Mayers]