MATH/CS 5466 NUMERICAL ANALYSIS
Homework 1

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Problem 1. This problem addresses the $\xi = \xi(x)$ term that appear in the formula:

$$f(x) - p_n(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!} \sum_{j=0}^{n} (x - x_j)$$

1. Write down the linear interpolant $p_1(x)$ for the function $f(x) = x^3$ at the interpolant points $x_0 = 0$ and $x_1 = b$:

   We have: $\quad f(x_0) = 0^3 = 0 \qquad f(x_1) = b^3$

$$\begin{aligned} p_1(x) = p_0(x) + c_1 q_1(x_1) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \\ &= 0^3 + \frac{b^3 - 0^3}{b - 0}(x - 0) \\ &= b^2 x \end{aligned}$$

   According to Theorem 1.3 in course notes we have:

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

   Applied above Theorem for $n = 1$ we have:

$$f(x) - p_1(x) = \frac{f^{(2)}(\xi(x))}{2!} \prod_{j=0}^{1} (x - x_j)$$

$$\begin{aligned} &\Leftrightarrow x^3 - b^2 x = \frac{6(\xi(x))}{2}(x - x_0)(x - x_1) \\ &\Leftrightarrow x(x - b)(x + b) = 3(\xi(x))x(x - b) \\ &\Leftrightarrow (x + b) = 3(\xi(x)) \\ &\Leftrightarrow \xi(x) = \frac{(x + b)}{3} \end{aligned}$$

2. Write down the linear interpolant $p_1(x)$ for the function $f(x) = 1/x$ at the interpolation points $x_0 = 1$ and $x_1 = 2$:

   We have: $\quad f(x_0) = \frac{1}{1} = 1 \qquad f(x_1) = \frac{1}{2}$

$$\begin{aligned} p_1(x) = p_0(x) + c_1 q_1(x_1) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \\ &= 1 + \frac{\frac{1}{2} - 1}{2 - 1}(x - 1) \\ &= -\frac{x}{2} + \frac{3}{2} \end{aligned}$$

   Applied above Theorem for $n = 1$ with $f^{(2)}(\frac{1}{x}) = \frac{2}{x^3}$ we have:

$$\frac{1}{x} - \left(-\frac{x}{2} + \frac{3}{2}\right) = \frac{\frac{2}{(\xi(x))^3}}{2}(x-1)(x-2)$$

$$\Leftrightarrow \xi^3(x)(x^2 - 3x + 2) = 2x(x-1)(x-2)$$

$$\Leftrightarrow \xi^3(x) = 2x$$

$$\Leftrightarrow \xi(x) = \sqrt[3]{2x} = (2x)^{\frac{1}{3}}$$

We have: $\xi'(x) = \frac{2}{3}(2x)^{-\frac{2}{3}} \Rightarrow \begin{cases} \xi'(x) = 0 & x = 0 \\ \xi'(x) > 0 & \forall x \neq 0 \end{cases}$

$\Rightarrow \xi(x)$ is increasing in [1 2] so:

$$min_{1\leq x\leq 2}\xi(x) = \xi(1) = (2.1)^{\frac{1}{3}} = 1.2599$$

$$max_{1\leq x\leq 2}\xi(x) = \xi(2) = (2.2)^{\frac{1}{3}} = 1.5874$$

Problem 2. Recall for $\mathbf{A} \in \mathbb{C}^{n\times n}$, the linear system $\mathbf{Ac} = \mathbf{f}$ has a unique solution for any $\mathbf{f}$ provided $\mathrm{Ker}(\mathbf{A}) = \{0\}$, where $\mathrm{Ker}(\mathbf{A})$ denotes the kernel (null space) of $\mathbf{A}$.

If the kernel of $\mathbf{A}$ is larger, i.e., if there is a nonzero vector $\mathbf{z} \in \mathrm{Ker}(\mathbf{A})$, then there are two possibilities:

- If $\mathbf{f} \notin \mathrm{Ran}(\mathbf{A})$ then there is *no solution* $\mathbf{c}$ to the linear system $\mathbf{Ac} = \mathbf{f}$.
- If $\mathbf{f} \in \mathrm{Ran}(\mathbf{A})$, then there are *infinitely many solutions* to the linear system $\mathbf{Ac} = \mathbf{f}$. In particular, if $\hat{\mathbf{c}}$ satisfies $\mathbf{A}\hat{\mathbf{c}} = \mathbf{f}$, then there any $\mathbf{c}$ of the form $\mathbf{c} = \hat{\mathbf{c}} + \gamma\mathbf{z}$ is also a solution, where γ is an arbitrary constant.

1. Suppose we wish to construct a polynomial $p_5 \in \mathcal{P}_5$ that interpolates a function $f \in \mathbb{C}^2[-1,1]$ in the following manner: $p_5(-1) = f(-1)$; $p_5'(-1) = f'(-1)$; $p_5(0) = f(0)$; $p"_5(0) = f"(0)$; $p_5(1) = f(1)$; $p_5'(1) = f'(1)$.

   Write down a linear system to determine the coefficients $c_0, ..., c_5$ for p in the monomial basis: $p_5(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5$.

   As defined above, we have: $x_0 = -1; x_1 = 0; x_2 = 1$ and:

   $$p_5'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4$$

   $$p"_5(x) = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3$$

   (1). $p_5(x_0) = c_0 + c_1x_0 + c_2x_0^2 + c_3x_0^3 + c_4x_0^4 + c_5x_0^5 = f(x_0)$

   $$\Leftrightarrow c_0 + c_1(-1) + c_2(-1)^2 + c_3(-1)^3 + c_4(-1)^4 + c_5(-1)5 = f(-1)$$

$$\Leftrightarrow c_0 - c_1 + c_2 - c_3 + c_4 - c_5 = f(-1)$$

(2). $p_5'(x_0) = c_1 + 2c_2x_0 + 3c_3x_0^2 + 4c_4x_0^3 + 5c_5x_0^4 = f'(x_0)$

$$\Leftrightarrow c_1 + 2c_2(-1) + 3c_3(-1)^2 + 4c_4(-1)^3 + 5c_5(-1)^4 = f'(-1)$$

$$\Leftrightarrow c_1 - 2c_2 + 3c_3 - 4c_4 + 5c_5 = f'(-1)$$

(3). $p_5(x_1) = c_0 + c_1x_1 + c_2x_1^2 + c_3x_1^3 + c_4x_1^4 + c_5x_1^5 = f(x_1)$

$$\Leftrightarrow c_0 + c_1.0 + c_2.0^2 + c_3.0^3 + c_4.0^4 + c_5.0^5 = f(0)$$

$$\Leftrightarrow c_0 = f(0)$$

(4). $p"_5(x_1) = 2c_2 + 6c_3x_1 + 12c_4x_1^2 + 20c_5x_1^3 = f"(x_1)$

$$\Leftrightarrow 2c_2 + 6c_3.0 + 12c_4.0^2 + 20c_5.0^3 = f"(0)$$

$$\Leftrightarrow 2c_2 = f"(0)$$

(5). $p_5(x_2) = c_0 + c_1x_2 + c_2x_2^2 + c_3x_2^3 + c_4x_2^4 + c_5x_2^5 = f(x_2)$

$$\Leftrightarrow c_0 + c_1.1 + c_2.1^2 + c_3.1^3 + c_4.1^4 + c_5.1^5 = f(1)$$

$$\Leftrightarrow c_0 + c_1 + c_2 + c_3 + c_4 + c_5 = f(0)$$

(6). $p_5'(x_2) = c_1 + 2c_2x_2 + 3c_3x_2^2 + 4c_4x_2^3 + 5c_5x_2^4 = f'(x_2)$

$$\Leftrightarrow c_1 + 2c_2.1 + 3c_3.1^2 + 4c_4.1^3 + 5c_5.1^4 = f'(1)$$

$$\Leftrightarrow c_1 + 2c_2 + 3c_3 + 4c_4 + 5c_5 = f'(1)$$

From 6 equations above $((1),(2),(3),(4),(5),(6))$ we form a linear system to determine coefficient c:

$$\mathbf{Ac} = \mathbf{f} \Leftrightarrow \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} f(-1) \\ f'(-1) \\ f(0) \\ f"(0) \\ f(1) \\ f'(1) \end{bmatrix}$$

2. What is the kernel of the matrix **A** constructed in part (a)?

```
% Kernel of constructed matrix A
A = [1,-1,1,-1,1,-1; 0,1,-2,3,-4,5; 1,0,0,0,0,0; 0,0,2,0,0,0; ..
     1,1,1,1,1,1; 0,1,2,3,4,5];
Z = null(A,'r');
```

```
>> Z

Z =     0
        1
        0
       -2
        0
        1
```

$Z \neq 0 \rightarrow$ so we have two possibilities:

• If $\mathbf{f} \notin \text{Ran}(\mathbf{A})$ then there is *no solution* $\mathbf{c}$ to the linear system $\mathbf{Ac} = \mathbf{f}$.

• If $\mathbf{f} \in \text{Ran}(\mathbf{A})$, then there are *infinitely many solutions* to the linear system $\mathbf{Ac} = \mathbf{f}$. In particular, if $\hat{\mathbf{c}}$ satisfies $\mathbf{A}\hat{\mathbf{c}} = \mathbf{f}$, then there any $\mathbf{c}$ of the form $\mathbf{c} = \hat{\mathbf{c}} + \gamma\mathbf{z}$ is also a solution, where γ is an arbitrary constant.

3. Consider the data: $f(-1) = -1$, $f'(-1) = 0$, $f(0) = 1$, $f"(0) = -2$, $f(1) = 3$, $f'(1) = 4$. We have linear system:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}$$

Solve for c_j we have: $\begin{cases} c_0 = 1 \\ c_2 = -1 \\ c_4 = 1 \\ c_1 + c_3 + c_5 = 2 \\ c_1 + 3c_3 + 5c_5 = 2 \end{cases}$

$\Rightarrow$ Two equations with 3 unknown (c_1, c_3, c_5) so we can have infinitely many choices for the polynomial $p_5(x)$.

We can randomly choose values of c_1 to get the polynomial p_5:

For example here I chose 6 values of c_1: $c_1 = \{-3, -2, -1, 1, 2, 3\}$ and we have 6 different polynomials of $p_5(x)$:

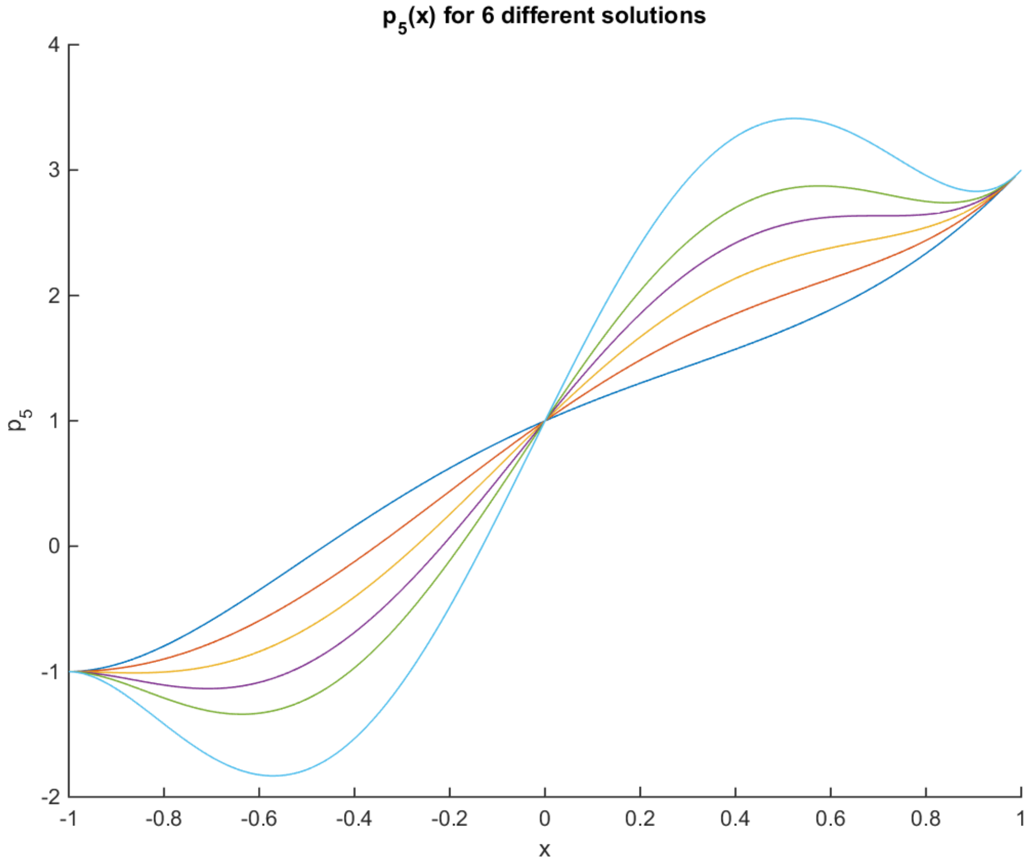

Figure 1: Polynomial of p_5 with 6 different solutions

Problem 3. The *Hermite interpolant* $h_n \in \mathcal{P}_{2n+1}$ of $f \in C^1[a,b]$ at the points $\{x_j\}_{j=0}^n$ can be written in the form:

$$h_n(x) = \sum_{j=0}^{n} (A_j(x)f(x_j) + B_j(x)f'(x_j))$$

where the functions A_j and B_j generalize the Lagrange basis functions:

$$A_j(x) = (1 - 2l_j'(x_j)(x - x_j))l_j^2(x)$$
$$B_j(x) = (x - x_j)l_j^2(x)$$

with $l_j = \prod_{k=0,k\neq j}^{n}(x - x_k)/(x_j - x_k)$

1. Verify that:

$$A_j(x_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \qquad A'_j(x_k) = 0 \qquad B_j(x_k) = 0 \qquad B'_j(x_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

As Lagrange basis function:

- if $k \neq j$ then $l_j(x_k) = 0 \Rightarrow \begin{cases} A_j(x_k) = 0 \\ B_j(x_k) = 0 \end{cases}$
- if $k = j$ then $\begin{cases} l_j(x_k) = 1 \\ x_k - x_j = xj - x_j = 0 \end{cases} \Rightarrow \begin{cases} A_j(x_j) = 1 \\ B_j(x_j) = 0 \end{cases}$

as a result: $\quad A_j(x_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$ and $\quad B_j(x_k) = 0 \quad \forall j, k$

$\mathbf{A}'_j(x) = -2l'_j(x_j)l_j^2(x) + 2[1 - 2l'_j(x_j)(x - x_j)]l'_j(x)l_j(x)$

- if $k \neq j \Rightarrow l_j(x_k) = 0$:

$$\Rightarrow A'_j(x_k) = -2l'_j(x_j).0^2 + 2[1 - 2l'_j(x_j)(x_k - x_j)]l'_j(x_k).0 = 0$$

- if $k = j \Rightarrow A'_j(x_j) = -2l'_j(x_j)l_j^2(x_j) + 2[1 - 2l'_j(x_j)(x_j - x_j)]l'_j(x_j)l_j(x_j)$

$$= -2l'_j(x_j) + 2l'_j(x_j) \qquad \text{(because } l_j(x_j) = 1\text{)}$$

$$= 0$$

$$\Rightarrow A'_j(x) = 0 \quad \forall j, k$$

$\mathbf{B}'_j(x) = l_j^2(x) + 2(x - x_j)l'_j(x)l_j(x)$

- if $k \neq j \Rightarrow l_j(x_k) = 1 \Rightarrow B'_j(x) = 1^2 + 2(x_j - x_j).l'_j(x_j).1 = 1$
- if $k \neq j \Rightarrow l_j(x_k) = 0 \Rightarrow B'_j(x) = 0^2 + 2(x_k - x_j)l'_j(x_k).0 = 0$

$$\Rightarrow B'_j(x_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

2. The above expression for the Hermite interpolating polynomial mimics the *Lagrange form* of the standard interpolating polynomial. Devise a scheme for constructing Hermite interpolants the generalizes the *Newton form.* What are your Newton-like basis function for $\mathcal{P}_{2n+1}$?

First consider:

$\lim_{x \to x_0} f[x_0, x] = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$

This motivates the notation: $f[x_0, x_0] = f'(x_0)$

Using the identity: $f[x_0, x_1, ..., x_k] = \dfrac{1}{k!} f^{(k)}(\xi)$, $\xi \in (x_0, x_k)$

we have: $f[x_0, ..., x_0] = \dfrac{1}{k!} f^{(k)}(x_0)$

We can form:

$$p(x) = \sum_{j=0}^{n} f[x_0, .., x_j] \prod_{i=0}^{j-1} (x - x_i)$$

Problem 4. The one-dimensional interpolation scheme studied in class can be adapted to higher dimensions. For example, suppose we are given a scalar-valued function $f(x, y)$, such as:

$$f(x, y) = e^x siny$$

and wish to construct a function of the form:

$p(x, y) = c_0 + c_1 x + c_2 y + c_3 xy + c_4 x^2 + c_5 y^2$ that interpolates $f(x, y)$ at $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)$.

1. Set up a linear system $\mathbf{Ac} = \mathbf{f}$ to determine the coefficients $c_0, ..., c_5$.

   Based on interpolate points we form 6 different equations:

   $p(x_0, y_0) = c_0 + c_1 x_0 + c_2 y_0 + c_3 x_0 y_0 + c_4 x_0^2 + c_5 y_0^2 = f(x_0, y_0) = e^{x_0} \sin y_0$

   $p(x_1, y_1) = c_0 + c_1 x_1 + c_2 y_1 + c_3 x_1 y_1 + c_4 x_1^2 + c_5 y_1^2 = f(x_1, y_1) = e^{x_1} \sin y_1$

   $p(x_2, y_2) = c_0 + c_1 x_2 + c_2 y_2 + c_3 x_2 y_2 + c_4 x_2^2 + c_5 y_2^2 = f(x_2, y_2) = e^{x_2} \sin y_2$

   $p(x_3, y_3) = c_0 + c_1 x_3 + c_2 y_3 + c_3 x_3 y_3 + c_4 x_3^2 + c_5 y_3^2 = f(x_3, y_3) = e^{x_3} \sin y_3$

   $p(x_4, y_4) = c_0 + c_1 x_4 + c_2 y_4 + c_3 x_4 y_4 + c_4 x_4^2 + c_5 y_4^2 = f(x_4, y_4) = e^{x_4} \sin y_4$

   $p(x_5, y_5) = c_0 + c_1 x_5 + c_2 y_5 + c_3 x_5 y_5 + c_4 x_5^2 + c_5 y_5^2 = f(x_5, y_5) = e^{x_5} \sin y_5$

   We have the linear system:

$$\mathbf{Ac} = \mathbf{f} \Leftrightarrow \begin{bmatrix} 1 & x_0 & y_0 & x_0 y_0 & x_0^2 & y_0^2 \\ 1 & x_1 & y_1 & x_1 y_1 & x_1^2 & y_1^2 \\ 1 & x_2 & y_0 2 & x_2 y_2 & x_2^2 & y_2^2 \\ 1 & x_3 & y_3 & x_3 y_3 & x_3^2 & y_3^2 \\ 1 & x_4 & y_4 & x_4 y_4 & x_4^2 & y_4^2 \\ 1 & x_5 & y_5 & x_5 y_5 & x_5^2 & y_5^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} e^{x_0} \sin y_0 \\ e^{x_1} \sin y_1 \\ e^{x_2} \sin y_2 \\ e^{x_3} \sin y_3 \\ e^{x_4} \sin y_4 \\ e^{x_5} \sin y_5 \end{bmatrix}$$

2. Write a MATLAB code to determine $\mathbf{c}$ when $f\{x, y\} = e^x \sin y$ and the (x_j, y_j)

pairs take the values listed in the following table.

j	0	1	2	3	4	5
x_j	0	0	1	1	2	2
y_j	0	2	0	2	1	3

Based on the provided interpolant points we can rewrite above function $\mathbf{Ac} = \mathbf{f}$ as follow:

$$\mathbf{Ac} = \mathbf{f} \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 4 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 & 1 & 4 \\ 1 & 2 & 1 & 2 & 4 & 1 \\ 1 & 2 & 3 & 6 & 4 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.9093 \\ 0 \\ 2.472 \\ 6.2177 \\ 1.043 \end{bmatrix}$$

```
% Determine c based on interpolant points
A = [1,0,0,0,0,0; 1,0,2,0,0,4; 1,1,0,0,1,0; 1,1,2,2,1,4; ...
     1,2,1,2,4,1; 1,2,3,6,4,9];
B = [0;0.9093;0;2.472;6.2177;1.043];
c = linsolve(A,B);
```

```
>> c

c =

    0
    -0.9490
    5.0593
    0.7813
    0.9490
    -2.3023
```

3. Plot your model function $p(x, y)$ over $x \in [-1, 3], y \in [-1, 3]$ using `MATLAB's` `surf` command.

Compare this plot to the similar plot for $f(x, y)$, which can be obtained in the following manner.

```
% Plot from function f(x,y)
f = inline('exp(x).*sin(y)','x','y');
[xx,yy] = meshgrid(linspace(-1,3,25),linspace(-1,3,25));
zz = f(xx,yy);
figure(1), clf
surf(xx,yy,zz);
title('f(x,y)'); print('f_xy','-dpng');
%Plot from funtion p(x,y)
p = @(x,y) c(1,1) + c(2,1)*x + c(3,1)*y + c(4,1)*x.*y ...
     + c(5,1)*x.^2 + c(6,1)*y.^2;
pp = p(xx,yy);
figure;
surf(xx,yy,pp);
title('p(x,y)'); print('p_xy','-dpng');
```

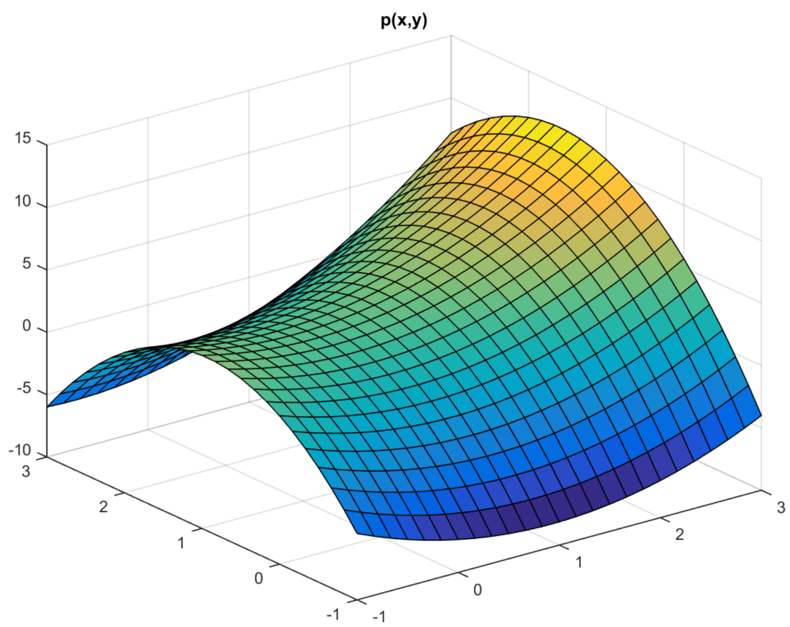

Figure 2: $p(x, y)$ plot with **surf** over $x \in [-1, 3], y \in [-1, 3]$

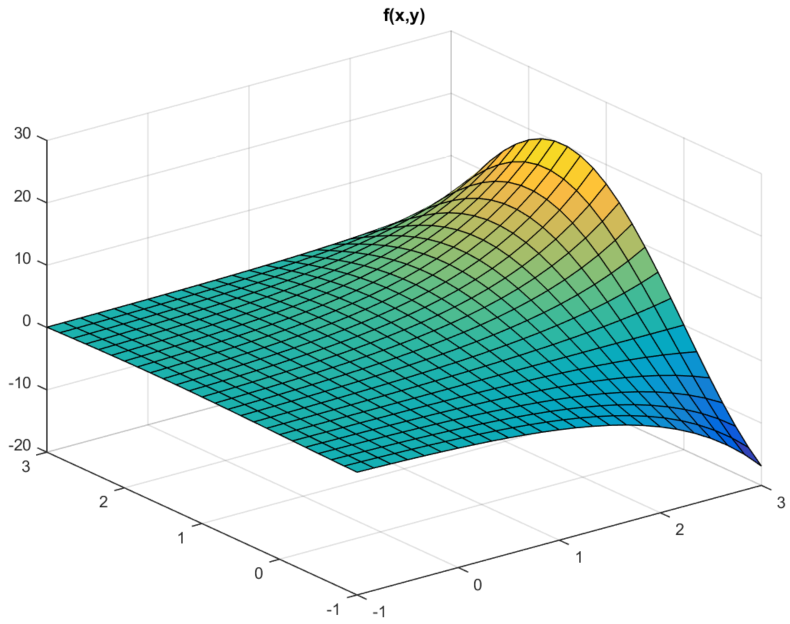

Figure 3: $f(x,y)$ plot with provided function

Problem 6. The standard Lagrange interpolation formula for the polynomial $p_n \in \mathcal{P}_n$ that interpolates $f \in C[a,b]$ at the distinct points $\{x_j\}$:

$$p_n(x) = \sum_{j=0}^{n} l_j(x) f(x_j), \quad where \;\; l_j = \prod_{k=0,k\neq j}^{n} \frac{(x-x_k)}{(x_j-x_k)}$$

require $O(n^2)$ floating point operations to evaluate for each point x. In this exercise, we construct an alternative Lagrange interpolation formula, known as the *barycentric interpolant*, that can evaluated more efficiently and also has superior numerical stability.

Let

$$w(x) = \prod_{k=0}^{n} (x-x_k)$$

and define the *barycentric weight* as: $\qquad \beta_j = \dfrac{1}{\prod_{k=0,k\neq j}^{n}(x-x_j)}, \; j = 0, ..., n$

1. Based of above definitions, the Lagrange form for p_n can be rewritten as:

$$p_n(x) = \sum_{j=0}^{n} l_j(x) f(x_j) = \sum_{j=0}^{n} \left(\prod_{k=0,k\neq j}^{n} \frac{(x-x_k)}{(x_j-x_k)} f(x_j) \right)$$

$$= \sum_{j=0}^{n} \left(\prod_{k=0,k\neq j}^{n} (x - x_k) \prod_{k=0,k\neq j}^{n} \frac{1}{(x_j - x_k)} f(x_j) \right)$$

$$= \sum_{j=0}^{n} \left(\left(\prod_{k=0}^{n} (x - x_k) \right) \frac{1}{x - x_j} \beta_j f(x_j) \right)$$

$$= \sum_{j=0}^{n} w(x) \frac{\beta_j}{(x - x_j)} f(x_j)$$

$$= w(x) \sum_{j=0}^{n} \frac{\beta_j}{(x - x_j)} f(x_j)$$

2. Verify that:

$$1 = w(x) \sum_{j=0}^{n} \frac{\beta_j}{x - x_j}$$

Belong to above expression we have: $l_j(x) = w(x).\dfrac{\beta_j}{x - x_j}$

$$\Rightarrow \sum_{j=0}^{n} l_j(x) = \sum_{j=0}^{n} w(x) \frac{\beta_j}{x - x_j} = w(x) \sum_{j=0}^{n} \frac{\beta_j}{x - x_j}$$

Besides, based on definition of Lagrange basis function:

$$\begin{cases} l_j(x_k) = 0 & j \neq k \\ l_j(x_k) = 1 & j = k \end{cases} \Rightarrow \sum_{j=0}^{n} l_j(x) = 1$$

From 2 above equations we can affirm that:

$$1 = w(x) \sum_{j=0}^{n} \frac{\beta_j}{x - x_j}$$

Or by choosing $f(x) = 1$ we also have the same result.

3. Dividing the result of part (a) by the result of part (b) yields the *barycentric interpolation formular*:

$$p_n(x) = \frac{\sum_{j=0}^{n} \dfrac{\beta_j}{x - x_j} f(x_j)}{\sum_{j=0}^{n} \dfrac{\beta_j}{x - x_j}}$$

Assuming the β_j values are already known, in order to evaluate p_5 we will need $O(n)$ floating point operations:

For the numerator each j value we have $3(n+1)$ operations ($2(n+1)$ for 2 times of $(x - x_j)$ and one time for $f(x_j)$).

For the denominator we need n operations

$\Rightarrow$ in total we need $4n + 3$ operations $= O(n)$.

4. Suppose $[a, b] = [0, 1]$ and $x_j = j/n$ for $j = 0, ..., n$. Derive a simple formula for β_j in terms of j and n:

   Replaced x_j into definition of the *barycentric weight* we have:

   $$\beta_j = \frac{1}{\prod_{k=0,k\neq j}^{n}(\frac{j}{n} - \frac{k}{n})} = \frac{1}{\frac{1}{n^n}\prod_{k=0,k\neq j}^{n}(j-k)} = \frac{n^n}{\prod_{k=0,k\neq j}^{n}(j-k)} = \frac{n^n}{j!(n-j)!}$$

   β_j get the largest value when $g(j) = j!(n-j)!$ hit the smallest value. We have:
   Assume that n is a even number, we'll make a comparison between $g(\frac{n}{2})$ and $g(\frac{n}{2}+1)$:

   $$\frac{g(\frac{n}{2})}{g(\frac{n}{2}+1)} = \frac{\frac{n}{2}!\frac{n}{2}!}{(\frac{n}{2}-1)!(\frac{n}{2}+1)!} = \frac{\frac{n}{2}}{\frac{n}{2}+1} < 1$$

   $\Rightarrow \beta_j$ largest at $j = \frac{n}{2}$: $\beta_j = \frac{n^n}{\frac{n}{2}!\frac{n}{2}!} = \frac{4n^n}{(n!)^2}$

   In case n is a odd number, follow the same step, we could proof that the highest value of β_j at $ceil(\frac{n}{2})$ or $floor(\frac{n}{2})$.