MATH/CS 5466 · NUMERICAL ANALYSIS

Problem Set 1

Posted Friday 29 January 2016. Due Monday 8 February 2016 (5pm).
Students should complete any 5 problems (total of 100 points).
Students are welcome to attempt more problems if they wish, but they will not count for extra points..

1. [20 points]

   This problem addresses the $\xi = \xi(x)$ term that appears in the formula

   $$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

   given in Theorem 1.3 in the course notes, and Section 2.2.2 of Gautschi's book.

   (a) Write down the linear interpolant $p_1(x)$ for the function $f(x) = x^3$ at the interpolation points $x_0 = 0$ and $x_1 = b$. Show that $\xi(x)$ takes the unique value $\xi(x) = (x+b)/3$.

   (b) Write down the linear interpolant $p_1(x)$ for the function $f(x) = 1/x$ at the interpolation points $x_0 = 1$ and $x_1 = 2$. Explicitly write down the function $\xi(x)$ for this case, and find the extreme values $\min_{1 \le x \le 2} \xi(x)$ and $\max_{1 \le x \le 2} \xi(x)$.

   [Süli and Mayers, Gautschi]

2. [20 points]

   Recall that for $\mathbf{A} \in \mathbb{C}^{n \times n}$, the linear system $\mathbf{Ac} = \mathbf{f}$ has a unique solution for any $\mathbf{f}$ provided $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{0}\}$, where $\mathrm{Ker}(\mathbf{A})$ denotes the kernel (null space) of $\mathbf{A}$.

   If the kernel of $\mathbf{A}$ is larger, i.e., if there is a nonzero vector $\mathbf{z} \in \mathrm{Ker}(\mathbf{A})$, then there are two possibilities:

   - If $\mathbf{f} \notin \mathrm{Ran}(\mathbf{A})$, then there is *no solution* $\mathbf{c}$ to the linear system $\mathbf{Ac} = \mathbf{f}$.
   - If $\mathbf{f} \in \mathrm{Ran}(\mathbf{A})$, then there are *infinitely many solutions* to the linear system $\mathbf{Ac} = \mathbf{f}$. In particular, if $\widehat{\mathbf{c}}$ satisfies $\mathbf{A}\widehat{\mathbf{c}} = \mathbf{f}$, then any $\mathbf{c}$ of the form $\mathbf{c} = \widehat{\mathbf{c}} + \gamma\mathbf{z}$ is also a solution, where γ is an arbitrary constant.

   With these facts in mind, please answer the following questions.

   (a) Suppose we wish to construct a polynomial $p_5 \in \mathcal{P}_5$ that interpolates a function $f \in \mathbb{C}^2[-1,1]$ in the following (somewhat unusual) manner: $p_5(-1) = f(-1)$; $p_5'(-1) = f'(-1)$; $p_5(0) = f(0)$; $p_5''(0) = f''(0)$; $p_5(1) = f(1)$; $p_5'(1) = f'(1)$. Write down a linear system to determine the coefficients $c_0, \ldots, c_5$ for p in the monomial basis: $p_5(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5$.

   (b) What is the kernel of the matrix $\mathbf{A}$ constructed in part (a)?
   (You may use the MATLAB command `null(A,'r')`.)
   What does your answer imply about the existence and uniqueness of the interpolant p_5?

   (c) Consider the data: $f(-1) = -1$, $f'(-1) = 0$, $f(0) = 1$, $f''(0) = -2$, $f(1) = 3$, $f'(1) = 4$. Show that there are infinitely many choices for the polynomial p_5 that interpolate this data. Plot six of them. (Superimpose all on the same plot.)

3. [20 points]

   The *Hermite interpolant* $h_n \in \mathcal{P}_{2n+1}$ of $f \in C^1[a,b]$ at the points $\{x_j\}_{j=0}^n$ can be written in the form

   $$h_n(x) = \sum_{j=0}^{n} \Big(A_j(x) f(x_j) + B_j(x) f'(x_j) \Big),$$

where the functions A_j and B_j generalize the Lagrange basis functions:

$$A_j(x) = \big(1 - 2\,\ell_j'(x_j)(x - x_j)\big)\,\ell_j^2(x)$$
$$B_j(x) = (x - x_j)\,\ell_j^2(x),$$

with $\ell_j(x) = \prod_{k=0,k\neq j}^{n}(x - x_k)/(x_j - x_k)$.

(a) Verify that

$$A_j(x_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k, \end{cases} \qquad A_j'(x_k) = 0, \qquad B_j(x_k) = 0, \qquad B_j'(x_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases}$$

(b) The above expression for the Hermite interpolating polynomial mimics the *Lagrange form* of the the standard interpolating polynomial. Devise a scheme for constructing Hermite interpolants that generalizes the *Newton form*. What are your new Newton-like basis functions for $\mathcal{P}_{2n+1}$?

4. [20 points]

The one-dimensional interpolation scheme studied in class can be adapted to higher dimensions. For example, suppose we are given a scalar-valued function $f(x, y)$, such as

$$f(x, y) = \mathrm{e}^x \sin y,$$

and wish to construct a function of the form

$$p(x, y) = c_0 + c_1 x + c_2 y + c_3 xy + c_4 x^2 + c_5 y^2$$

that interpolates $f(x, y)$ at (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5).

(a) Set up a linear system $\mathbf{Ac} = \mathbf{f}$ to determine the coefficients $c_0, \ldots, c_5$.

(b) Write a MATLAB code to determine $\mathbf{c}$ when $f(x, y) = \mathrm{e}^x \sin y$ and the (x_j, y_j) pairs take the values listed in the following table.

j	0	1	2	3	4	5
x_j	0	0	1	1	2	2
y_j	0	2	0	2	1	3

Report your value for $\mathbf{c}$.

(c) Plot your model function $p(x, y)$ over $x \in [-1, 3]$, $y \in [-1, 3]$ using MATLAB's `surf` command. Compare this plot to the similar plot for $f(x, y)$, which can be obtained in the following manner.

```
f = inline('exp(x).*sin(y)','x','y');
[xx,yy] = meshgrid(linspace(-1,3,25),linspace(-1,3,25));
zz = f(xx,yy);
figure(1), clf
surf(xx,yy,zz)
```

Please submit plots of both $p(x, y)$ and $f(x, y)$.

5. [20 points]

Suppose the complex-valued function $f(z)$ of the variable $z \in \mathbb{C}$ is analytic in a region D of the complex plane whose boundary C is a simple closed contour. Furthermore, suppose the interpolation points $x_0, \ldots, x_n$ $(n \geq 1)$ and the point x all lie in D.

(a) Let $p_n \in \mathcal{P}_n$ denote the polynomial that interpolates f at $x_0, \ldots, x_n$.
For any $x \in D$, confirm the identity

$$f(x) - p_n(x) \quad = \quad \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{z-x} \prod_{j=0}^{n} \frac{x - x_j}{z - x_j} \,\mathrm{d}z$$

by computing the integral on the right. (Hint: Consider the poles of the integrand, and use the Cauchy integral formula.)

For the rest of the problem, suppose that the real number x and the interpolation points $x_0, \ldots, x_n$ all lie in the real interval $[a, b]$, and define, for constant $K > 0$,

$$D = \{z \in \mathbb{C} : |z - t| < K \text{ for some } t \in [a, b]\}.$$

(b) Plot (or draw) the boundary C of D for $[a, b] = [-1, 1]$ and $K = 1$.

(c) Show that the length of the contour C is $2(b - a) + 2\pi K$, and that the integral formula in (a) leads to the bound

$$|f(x) - p_n(x)| < \frac{(b - a + \pi K)M}{\pi K} \Big(\frac{b-a}{K}\Big)^{n+1},$$

where M is such that $|f(z)| \le M$ on C.

(d) Deduce that if f is analytic on D for some $K > |b - a|$, then the sequence $\{p_n\}$ converges to f uniformly on $[a, b]$ as $n \to \infty$.

(e) Show that the requirements for the conclusion in (d) *are not satisfied* by Runge's function, $f(x) = 1/(1 + x^2)$ over $[a, b] = [-5, 5]$. For what values of α are the conditions satisfied by this f over $[a, b] = [-\alpha, \alpha]$?

[Süli and Mayers, Problem 6.11]

6. [20 points]

The standard Lagrange interpolation formula for the polynomial $p_n \in \mathcal{P}_n$ that interpolates $f \in C[a, b]$ at the distinct points $\{x_j\}$,

$$p_n(x) = \sum_{j=0}^{n} \ell_j(x) f(x_j), \qquad \text{where} \quad \ell_j(x) = \prod_{k=0, k\ne j}^{n} \frac{(x - x_k)}{(x_j - x_k)},$$

requires $O(n^2)$ floating point operations to evaluate for each point x. In this exercise, we construct an alternative Lagrange interpolation formula, known as the *barycentric interpolant*, that can evaluated more efficiently and also has superior numerical stability.

Let $w(x) = \displaystyle\prod_{k=0}^{n} (x - x_k)$ and define the *barycentric weight* as

$$\beta_j = \frac{1}{\prod_{k=0, k\ne j}^{n} (x_j - x_k)}, \qquad j = 0, \ldots, n.$$

(a) Show that the Lagrange form for p_n can be rewritten as

$$p_n(x) = w(x) \sum_{j=0}^{n} \frac{\beta_j}{x - x_j} f(x_j).$$

(b) Verify that

$$1 = w(x) \sum_{j=0}^{n} \frac{\beta_j}{x - x_j}.$$

(Hint: This follows from part (a) with a special choice of f.)

(c) Dividing the result of part (a) by the result of part (b) yields the *barycentric interpolation formula*

$$p_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{\beta_j}{x - x_j} f(x_j)}{\displaystyle\sum_{j=0}^{n} \frac{\beta_j}{x - x_j}}.$$

Assuming the β_j values are already known, how many floating point operations are required to evaluate $p_n(x)$ for some point x?

(d) Suppose $[a, b] = [0, 1]$ and $x_j = j/n$ for $j = 0, \ldots, n$. Derive a simple formula for β_j in terms of j and n. For which j values is β_j largest (in absolute value)? (These terms will be favored in the formula in part (c).)

[Berrut and Trefethen]

7. [20 points]

As mentioned in class, the Weierstrass Approximation Theorem states that for any $f \in C[a, b]$ and any $\varepsilon > 0$, there exists some polynomial (of unspecified degree) such that $\max_{x\in[a,b]} |f(x) - p(x)| < \varepsilon$. The most common proof of this fact is *constructive*: one can use for the approximating polynomial the *Bernstein polynomial* of appropriate degree. When $[a, b] = [0, 1]$, the degree-n Bernstein polynomial is defined as

$$B_n(x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1 - x)^{n-k},$$

where $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$ can be obtained in MATLAB via the `nchoosek` command.

Remarkably, it turns out that for any $f \in C[a, b]$, we have $\max_{x\in[a,b]} |f(x) - B_n(x)| \to 0$ as $n \to \infty$. In this exercise you shall explore the rate at which this convergence occurs.

(a) Confirm that $B_n(x) \to f(x)$ for $x \in [0, 1]$ and $f(x) = \sin(3\pi x)$ by producing a MATLAB plot that compares $f(x)$ to $B_n(x)$ on $x \in [0, 1]$ for $n = 5, 10, 20$. (Please label the plot clearly!)

(b) Describe how to modify the definition of B_n so as to work for a general interval $[a, b] \neq [0, 1]$.

(c) Let $f(x) = \mathrm{e}^x$ and $[a, b] = [-1, 1]$. Write MATLAB code to compute $B_n(x)$ as well as the polynomial $p_n(x)$ that interpolates f at the Chebyshev points

$$x_k = \cos(k\pi/n), \qquad k = 0, \ldots, n.$$

(You may use the monomial, Newton, or Lagrange basis.) Turn in a `semilogy` plot that compares $\max_{x\in[-1,1]} |f(x) - B_n(x)|$ with $\max_{x\in[-1,1]} |f(x) - p_n(x)|$ for $n = 1, \ldots, 40$. (For purposes of this problem, you may ignore any warnings issued by `nchoosek` for large n.)

(d) Repeat part (c) with $f(x) = x^2 - 1$ and $[a, b] = [-1, 1]$.