LECTURE 13: *Equioscillation, Part 1*

2.2 *Oscillation Theorem*

The previous example hints that the points at which the error $f - p_*$ attains its maximum magnitude play a central role in the theory of minimax approximation. The Theorem of de la Vallée Poussin is a first step toward such a result. We include its proof to give a flavor of how such results are established.

The proof is adapted from Section 8.3 of Süli and Mayers, *An Introduction to Numerical Analysis* (Cambridge, 2003).

Theorem 2.1 (de la Vallée Poussin's Theorem)**.**
Let $f \in C[a,b]$ and suppose $r \in \mathcal{P}_n$ is some polynomial for which there exist $n+2$ points $\{x_j\}_{j=0}^{n+1}$ with $a \le x_0 < x_1 < \cdots < x_{n+1} \le b$ at which the error $f(x) - r(x)$ oscillates signs, i.e.,

$$\mathrm{sgn}(f(x_j) - r(x_j)) = -\mathrm{sgn}(f(x_{j+1}) - r(x_{j+1}))$$

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$

for $j = 0, \ldots, n$. Then

$$\min_{p \in \mathcal{P}_n} \|f - p\|_\infty \ge \min_{0 \le j \le n+1} |f(x_j) - r(x_j)|. \tag{2.2}$$

Before proving this result, look at Figure 2.2 for an illustration of the theorem. Suppose we wish to approximate $f(x) = e^x$ with some quintic polynomial, $r \in \mathcal{P}_5$ (i.e., $n = 5$). *This polynomial is not necessarily the minimax approximation to f over the interval* $[0,1]$. However, in the figure it is clear that for this r, we can find $n+2 = 7$ points at which the sign of the error $f(x) - r(x)$ oscillates. The red curve shows the error for the optimal minimax polynomial p_* (whose computation is discussed below). This is the point of de la Vallée Poussin's theorem: *Since the error $f(x) - r(x)$ oscillates sign $n+2$ times, the minimax error $\pm\|f - p_*\|_\infty$ exceeds $|f(x_j) - r(x_j)|$ at one of the points x_j that give the oscillating sign.* In other words, de la Vallée Poussin's theorem gives a nice mechanism for developing *lower bounds* on $\|f - p_*\|_\infty$.

These $n+2$ points are by no means unique: we have a continuum of choices available. However, taking the extrema of $f - r$ will give the the best bounds in the theorem.

Proof. Suppose we have $n+2$ ordered points, $\{x_j\}_{j=0}^{n+1} \subset [a,b]$, such that $f(x_j) - r(x_j)$ alternates sign at consecutive points, and let p_* denote the minimax polynomial,

$$\|f - p_*\|_\infty = \min_{p \in \mathcal{P}_n} \|f - p\|_\infty.$$

We will prove the result by contradiction. Thus suppose

$$\|f - p_*\|_\infty < |f(x_j) - r(x_j)|, \qquad \text{for all } j = 0, \ldots, n+1. \tag{2.3}$$

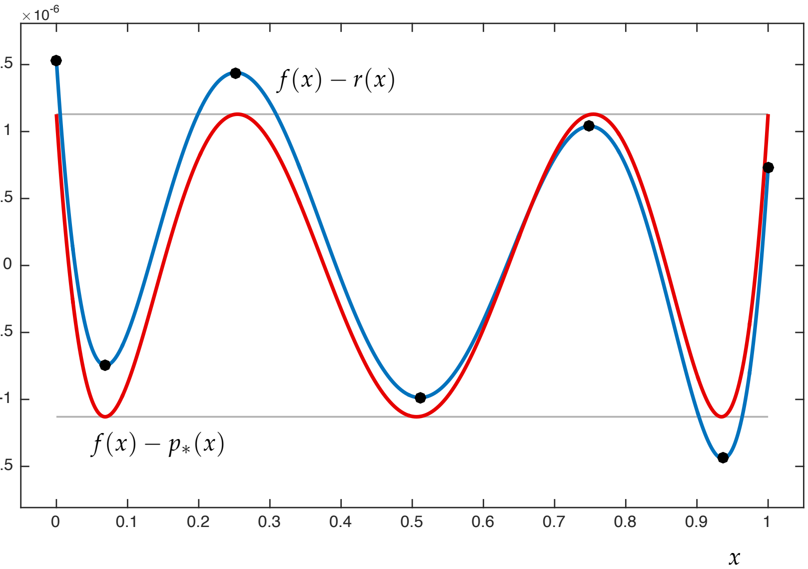

Figure 2.2: Illustration of de la Vallée Poussin's theorem for $f(x) = e^x$ and $n = 5$. Some polynomial $r \in \mathcal{P}_5$ gives an error $f - r$ for which we can identify $n + 2 = 7$ points x_j, $j = 0, \ldots, n+1$ (black dots) at which $f(x_j) - r(x_j)$ oscillates sign. The minimum value of $|f(x_j) - r(x_j)|$ gives a *lower bound* the maximum error $\|f - p_*\|_\infty$ of the optimal approximation $p_* \in \mathcal{P}_5$.

As the left hand side is the maximum difference of $f - p_*$ over all $x \in [a, b]$, that difference can be no larger at $x_j \in [a, b]$, and so:

$$|f(x_j) - p_*(x_j)| < |f(x_j) - r(x_j)|, \qquad \text{for all } j = 0, \ldots, n+1. \tag{2.4}$$

Now consider

$$p_*(x) - r(x) = (f(x) - r(x)) - (f(x) - p_*(x)),$$

which is a degree n polynomial, since $p_*, r \in \mathcal{P}_n$. Equation (2.4) states that $f(x_j) - r(x_j)$ always has larger magnitude than $f(x_j) - p_*(x_j)$. Thus, regardless of the sign of $f(x_j) - p_*(x_j)$, the magnitude $|f(x_j) - p_*(x_j)|$ will never be large enough to overcome $|f(x_j) - r(x_j)|$, and hence

$$\operatorname{sgn}(p_*(x_j) - r(x_j)) = \operatorname{sgn}(f(x_j) - r(x_j)).$$

We know from the hypothesis that $f(x) - r(x)$ must change sign at least $n + 1$ times (at least once in each interval (x_j, x_{j+1}) for $j = 0, \ldots, n$), and thus the degree-n polynomial $p_* - r$ must do the same. But $n + 1$ sign changes implies $n + 1$ roots; the only degree-n polynomial with $n + 1$ roots is the zero polynomial, i.e., $p_* = r$. However, this contradicts the strict inequality in equation (2.3). Hence, there must be at least one j for which

$$\|f - p_*\|_\infty \geq |f(x_j) - r(x_j)|,$$

thus yielding the theorem. ■

Now suppose we can find some degree-n polynomial, call it $\tilde{r} \in \mathcal{P}_n$, and $n + 2$ points $x_0 < \cdots < x_n$ in $[a, b]$ such that not only does the sign of $f - \widetilde{r}$ oscillate, but the error takes its extremal values at these points. That is,

$$|f(x_j) - \widetilde{r}(x_j)| = \|f - \widetilde{r}\|_\infty, \quad j = 0, \ldots, n+1,$$

and

$$f(x_j) - \widetilde{r}(x_j) = -\big(f(x_{j+1}) - \widetilde{r}(x_{j+1})\big), \quad j = 0, \ldots, n.$$

The red curve in Figure 2.2 shows an error function that satisfies these requirements.

Now apply de la Valée Poussin's theorem to this special polynomial $\widetilde{r}$. Equation (2.2) gives

$$\min_{p \in \mathcal{P}_n} \|f - p_*\| \geq \min_{0 \leq j \leq n+1} |f(x_j) - \widetilde{r}(x_j)|.$$

On the other hand, we have presumed that

$$|f(x_j) - \widetilde{r}(x_j)| = \|f - \widetilde{r}\|_\infty$$

for all $j = 0, \ldots, n+1$. Thus,

$$\min_{p \in \mathcal{P}_n} \|f - p_*\| \geq \|f - \widetilde{r}\|_\infty.$$

Since $\widetilde{r} \in \mathcal{P}_n$, it follows that

$$\min_{p \in \mathcal{P}_n} \|f - p_*\| = \|f - \widetilde{r}\|_\infty,$$

and this *equioscillating* $\widetilde{r}$ must be an optimal approximation to f.

The question remains: Does such a polynomial with equioscillating error always exist? The following theorem ensures it does.

For a direct proof, see Section 8.3 of Süli and Mayers. Another excellent resource is G. W. Stewart, *Afternotes Goes to Graduate School*, SIAM, 1998; see Stewart's Lecture 3.

Theorem 2.2 (Oscillation Theorem)**.** Suppose $f \in C[a,b]$. Then $p_* \in \mathcal{P}_n$ is a minimax approximation to f from $\mathcal{P}_n$ on $[a,b]$ *if and only if* there exist $n+2$ points $x_0 < x_1 < \cdots < x_{n+1}$ such that

$$|f(x_j) - p_*(x_j)| = \|f - p_*\|_\infty, \quad j = 0, \ldots, n+1$$

and the sign of the error oscillates at these points:

$$f(x_j) - p_*(x_j) = -\big(f(x_{j+1}) - p_*(x_{j+1})\big), \quad j = 0, \ldots, n.$$

Note that this result is *if and only if*: the oscillation property exactly *characterizes* the minimax approximation. We have proved one direction already by appeal to de la Vallée Poussin's theorem. The proof of the other direction is rather more involved.