LECTURE 6: *Interpolating Derivatives*

1.8 Hermite Interpolation and Generalizations

Example 1.1 demonstrated that polynomial interpolants to $\sin(x)$ attain arbitrary accuracy for $x \in [-5,5]$ as the polynomial degree increases, even if the interpolation points are taken exclusively from $[-1,1]$. In fact, as $n \to \infty$ interpolants based on data from $[-1,1]$ will converge to $\sin(x)$ *for all* $x \in \mathbb{R}$. More precisely, for any $x \in \mathbb{R}$ and any $\varepsilon > 0$, there exists some positive integer N such that $|\sin(x) - p_n(x)| < \varepsilon$ for all $n \geq N$, where p_n interpolates $\sin(x)$ at $n+1$ uniformly-spaced interpolation points in $[-1,1]$.

In fact, this is not as surprising as it might first appear. The Taylor series expansion uses derivative information at a single point to produce a polynomial approximation of f that is accurate at nearby points. In fact, the interpolation error bound derived in the previous lecture bears close resemblance to the remainder term in the Taylor series. If $f \in C^{(n+1)}[a,b]$, then expanding f at $x_0 \in (a,b)$, we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

This is the *Lagrange form* of the error.

for some $\xi \in [x, x_0]$ that depends on x. The first sum is simply a degree n polynomial in x; from the final term – the Taylor remainder – we obtain the bound

$$\max_{x\in[a,b]} \left| f(x) - \Big(\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \Big) \right| \leq \left(\max_{\xi\in[a,b]} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \right) \left(\max_{x\in[a,b]} |x-x_0|^{n+1} \right),$$

which should certainly remind you of the interpolation error formula in Theorem 1.3.

One can view polynomial interpolation and Taylor series as two extreme approaches to approximating f: one uses global information, but only about f; the other uses only local information, but requires extensive knowledge of the derivatives of f. In this section we shall discuss an alternative based on the best features of each of these ideas: use global information about both f and its derivatives.

1.8.1 Hermite interpolation

In cases where the polynomial interpolants of the previous sections incurred large errors for some $x \in [a,b]$, one typically observes that the slope of the interpolant differs markedly from that of f at some of the interpolation points $\{x_j\}$. (Recall Runge's example in Figure 1.8.) Why not then *force the interpolant to match both f and f' at the interpolation points*?

Often the underlying application provides a motivation for such derivative matching. For example, if the interpolant approximates the position of a particle moving in space, we might wish the interpolant to match not only position, but also velocity. *Hermite interpolation* is the general procedure for constructing such interpolants.

Typically the position of a particle is given in terms of a second-order differential equation (in classical mechanics, arising from Newton's second law, $F = ma$). To solve this second-order ODE, one usually writes it as a system of first-order equations whose numerical solution we will study later in the semester. One component of the system is position, the other is velocity, and so one automatically obtains values for both f (position) and f' (velocity) simultaneously.

> Given $f \in C^1[a,b]$ and $n+1$ points $\{x_j\}_{j=0}^n$ satisfying
>
> $$a \le x_0 < x_1 < \cdots < x_n \le b,$$
>
> determine some $h_n \in \mathcal{P}_{2n+1}$ such that
>
> $$h_n(x_j) = f(x_j), \qquad h_n'(x_j) = f'(x_j) \quad \text{for } j = 0, \ldots, n.$$

Note that h must generally be a polynomial of degree $2n+1$ to have sufficiently many degrees of freedom to satisfy the $2n+2$ constraints. We begin by addressing the existence and uniqueness of this interpolant.

Existence is best addressed by explicitly constructing the desired polynomial. We adopt a variation of the *Lagrange approach* used in Section 1.5. We seek degree-$(2n+1)$ polynomials $\{A_k\}_{k=0}^n$ and $\{B_k\}_{k=0}^n$ such that

$$A_k(x_j) = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases} \qquad A_k'(x_j) = 0 \text{ for } j = 0, \ldots, n;$$

$$B_k(x_j) = 0 \text{ for } j = 0, \ldots, n, \qquad B_k'(x_j) = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}.$$

These polynomials would yield a basis for $\mathcal{P}_{2n+1}$ in which h_n has a simple expansion:

$$h_n(x) = \sum_{k=0}^n f(x_k) A_k(x) + \sum_{k=0}^n f'(x_k) B_k(x). \tag{1.29}$$

To show how we can construct the polynomials A_k and B_k, we recall the Lagrange basis polynomials used for the standard interpolation problem,

$$\ell_k(x) = \prod_{j=0, j\neq k}^n \frac{(x - x_j)}{(x_k - x_j)}.$$

Consider the definitions

$$A_k(x) := \big(1 - 2(x - x_k)\ell_k'(x_k)\big)\ell_k^2(x),$$
$$B_k(x) := (x - x_k)\ell_k^2(x).$$

Note that since $\ell_k \in \mathcal{P}_n$, we have $A_k, B_k \in \mathcal{P}_{2n+1}$. Figure 1.12 shows these Hermite basis polynomials and their derivatives for $n = 5$ using

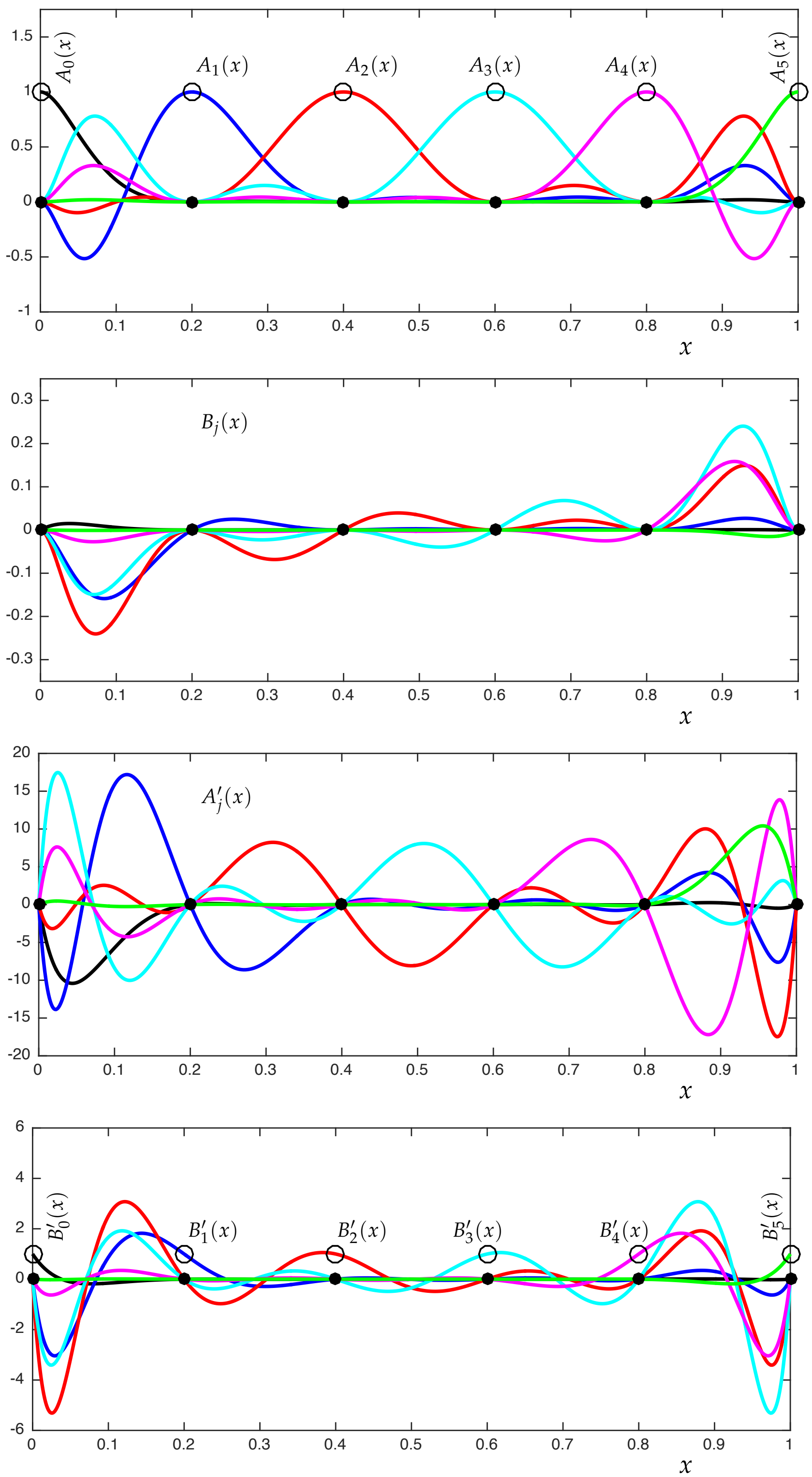

Figure 1.12: The Hermite basis polynomials for $n = 5$ on the interval $[a, b] = [0, 1]$ with $x_j = j/5$ (black dots).

- $A_0, \ldots, A_5$: $A_j(x_j) = 1$ (black circles).
- $B_0, \ldots, B_5$: $B_j(x_k) = 0$ for all j, k.
- $A_0', \ldots, A_5'$: $A_j'(x_k) = 0$ for all j, k.
- $B_0', \ldots, B_5'$: $B_j'(x_j) = 1$ (black circles).

uniformly spaced points on $[0, 1]$. It is a straightforward exercise to verify that these A_k and B_k, and their first derivatives, obtain the specified values at $\{x_j\}_{j=0}^n$.

These interpolation conditions at the points $\{x_j\}$ ensure that the $2n + 2$ polynomials $\{A_k, B_k\}_{k=0}^n$, each of degree $2n + 1$, form a basis for $\mathcal{P}_{2n+1}$, and thus we can always write h_n via the formula (1.29).

Figure 1.13 compares the standard polynomial interpolant $p_n \in \mathcal{P}_n$ to the Hermite interpolant $h_n \in \mathcal{P}_{2n+1}$ and the standard interpolant of the same degree, $p_{2n+1} \in \mathcal{P}_{2n+1}$ for the example $f(x) = \sin(20x) + e^{5x/2}$ using uniformly spaced points on $[0, 1]$ with $n = 5$. Note the distinction between h_n and p_{2n+1}, which are both polynomials of the same degree.

Here are a couple of basic results whose proofs follow the same techniques as the analogous proofs for the standard interpolation problem.

Theorem 1.7. The Hermite interpolant $h_n \in \mathcal{P}_{2n+1}$ is unique.

The uniqueness result hinges on the fact that we interpolate f and f' both at all interpolation points. If we vary the number of derivatives interpolated at each data point, we open the possibility of non-unique interpolants.

Theorem 1.8. Suppose $f \in C^{2n+2}[x_0, x_n]$ and let $h_n \in \mathcal{P}_{2n+1}$ such that $h_n(x_j) = f(x_j)$ and $h_n'(x_j) = f'(x_j)$ for $j = 0, \ldots, n$. Then for any $x \in [x_0, x_n]$, there exists some $\xi \in [x_0, x_n]$ such that

$$f(x) - h_n(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{j=0}^{n} (x - x_j)^2.$$

The proof of this latter result is directly analogous to the standard polynomial interpolation error in Theorem 1.3. Think about how you would prove this result for yourself.

Hint: the proof has some resemblance to our proof of Theorem 1.6. Invoke Rolle's theorem to get n roots of a certain function, then use the derivative interpolation to get another $n + 1$ roots.

1.8.2 Hermite–Birkhoff interpolation

Of course, one need not stop at interpolating f and f'. Perhaps your application has more general requirements, where you want to interpolate higher derivatives, too, or have the number of derivatives interpolated differ at each interpolation point. Such general polynomials are called *Hermite–Birkhoff interpolants*, and you already have the tools at your disposal to compute them. Simply formulate the problem as a linear system and find the desired coefficients, but beware that in some situations, *there may be infinitely many polynomials that satisfy the interpolation conditions*. For these problems, it is generally simplest to work with the monomial basis, though one could design Newton- or Lagrange-inspired bases for particular situations.

For example, suppose you seek an interpolant that is particularly accurate in the vicinity of one of the interpolation points, and so you wish to interpolate higher derivatives at that point: a hybrid between an interpolating polynomial and a Taylor expansion.

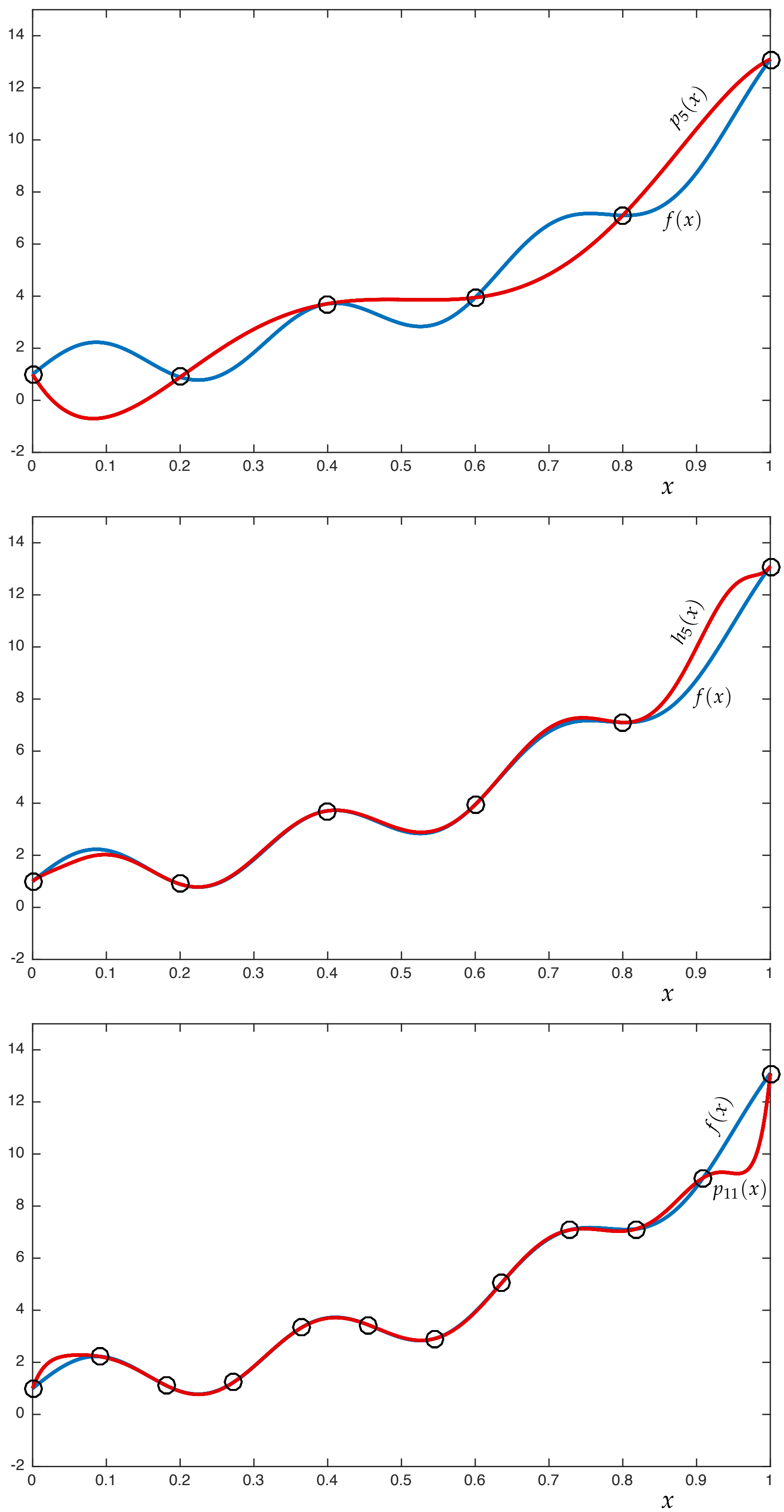

Figure 1.13: Interpolation of $f(x) = \sin(20x) + e^{5x/2}$ at uniformly spaced points for $x \in [0, 1]$. Top plot: the standard polynomial interpolant $p_5 \in \mathcal{P}_5$. Middle plot: the Hermite interpolant $h_5 \in \mathcal{P}_{11}$. Bottom plot: the standard interpolant $p_{11} \in \mathcal{P}_{11}$.

Though the last two plots show polynomials of the same degree, notice how the interpolants differ. (At first glance it appears the Hermite interpolation condition fails at the rightmost point in the middle plot; zoom in to see that the slope of the interpolant indeed matches $f'(1)$.)

1.8.3 *Hermite–Fejér interpolation*

Another Hermite-like scheme initially sounds like a bad idea: make the interpolant have *zero slope* at all the interpolation points.

> Given $f \in C^1[a,b]$ and $n+1$ points $\{x_j\}_{j=0}^n$ satisfying
>
> $$a \le x_0 < x_1 < \cdots < x_n \le b,$$
>
> determine some $h_n \in \mathcal{P}_{2n+1}$ such that
>
> $$h_n(x_j) = f(x_j), \qquad h_n'(x_j) = 0 \quad \text{for } j = 0, \ldots, n.$$

That is, explicitly construct h_n such that its derivatives in general *do not match those of f*. This method, called *Hermite–Fejér interpolation*, turns out to be remarkably effective, even better than standard Hermite interpolation in certain circumstances. In fact, Fejér proved that if we choose the interpolation points $\{x_j\}$ in the right way, h_n is guaranteed to converge to f uniformly as $n \to \infty$.

Theorem 1.9. For each $n \ge 1$, let h_n be the Hermite–Fejér interpolant of $f \in C[a,b]$ at the Chebyshev interpolation points

$$x_j = \frac{a+b}{2} + \Big(\frac{b-a}{2}\Big)\cos\Big(\frac{(2j+1)\pi}{2n+2}\Big), \quad j = 0, \ldots, n.$$

Then $h_n(x)$ converges uniformly to f on $[a,b]$.

For a proof of Theorem 1.9, see page 57 of I. P. Natanson, *Constructive Function Theory*, vol. 3 (Ungar, 1965).