

ME 5514 VIBRATION MECHANICS SYSTEMS
Homework 2

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Consider the dx element of the given cantilever beam we have: V and M are the shear force and bending moments respectively.

- Summing the force in Y -direction for the dm elements: $dm\bar{Y} = \sum F_y \Leftrightarrow \rho A \frac{\partial^2 Y}{\partial t^2} = -\frac{\partial V}{\partial X}$
with: ρ is the density (mass/volume): $\rho = 2.7 \text{ g/cm}^3$

A is the cross-sectional area: $A = 0.32 * 2.45 = 0.784 \text{ cm}^2$

The bending moment M is defined by: $M = EI \frac{\partial^2 Y}{\partial X^2}$ with I is moment inertial of cross-sectional

$$I = \frac{bh^3}{12} = \frac{2.45 * 10^{-2} * (3.2 * 10^{-3})^3}{12} = 6.69 \times 10^{-11} \text{ m}^4$$

The shear force V is calculated from M : $V = \frac{\partial M}{\partial X}$

Because EI is constant, we come up with: $\frac{\partial^2 Y}{\partial t^2} + \frac{EI}{\rho A} \frac{\partial^4 Y}{\partial X^4} = 0$ (1)

Assume the vibration solution is: $Y(X, t) = \bar{Y}(X) \sin \omega t$

(1) become: $\frac{d^4 \bar{Y}}{dX^4} - \beta^4 \bar{Y} = 0$ with $\beta^4 = \frac{\rho A \omega^2}{EI}$

Solution: $\bar{Y}(X) = a \cosh \beta X + b \sinh \beta X + c \cos \beta X + d \sin \beta X$ (2)

1. Assume the tip mas can be modeled as a point mass \Rightarrow the boundary conditions are:

- End displacement (on left): $Y(0, t) = 0 \Rightarrow \bar{Y}(0) = 0 \Leftrightarrow \bar{Y}(0) = a + c = 0$ (3)

- End slope (on left): $Y'(0, t) = 0 \Rightarrow \bar{Y}'(0) = 0$

with $\bar{Y}'(X, t) = \beta a \sinh \beta X + \beta b \cosh \beta X - \beta c \sin \beta X + \beta d \cos \beta X$

$\Rightarrow \bar{Y}'(0, t) = \beta b + \beta d = 0 \Leftrightarrow b + d = 0$ (4)

- End moment (on right): $M(L, t) = EI \frac{\partial^2 Y}{\partial X^2}(L, t) = 0 \Leftrightarrow \bar{Y}''(L, t) = 0$

with $\bar{Y}''(X, t) = \beta^2 a \cosh \beta X + \beta^2 b \sinh \beta X - \beta^2 c \cos \beta X - \beta^2 d \sin \beta X$

$\Rightarrow \bar{Y}''(L, t) = \beta^2 (a \cosh \beta L + b \sinh \beta L - c \cos \beta L - d \sin \beta L) = 0$

$\Leftrightarrow a \cosh \beta L + b \sinh \beta L - c \cos \beta L - d \sin \beta L = 0$ (because $\beta \neq 0$)

$\Leftrightarrow \frac{a}{b} = -\frac{\sinh \beta L + \sin \beta L}{\cosh \beta L + \cos \beta L}$ (5)

- Shear force V at right end: $V(L, t) = \bar{M} Y'''(L, t) \Leftrightarrow \bar{Y}'''(L, t) = -\frac{\bar{M} L}{m} \beta^4 \bar{Y}(L)$

with: \bar{M} is the tip mass: $\bar{M} = (1.13 * \pi * 1.15^2 + 1.17 * \pi * 0.45^2) * 2.7 = 14.69 \text{ g}$

m - mass of beam: $m = V\rho = 0.32 * 2.45 * 46 * 2.7 = 97.4 \text{ g}$

$\bar{Y}'''(X, t) = \beta^3 a \sinh \beta X + \beta^3 b \cosh \beta X + \beta^3 c \sin \beta X - \beta^3 d \cos \beta X$

$\Rightarrow \bar{Y}'''(L, t) = \beta^3 (a \sinh \beta L + b \cosh \beta L + c \sin \beta L - d \cos \beta L) = -\frac{\bar{M} L}{m} \beta^4 \bar{Y}(L)$

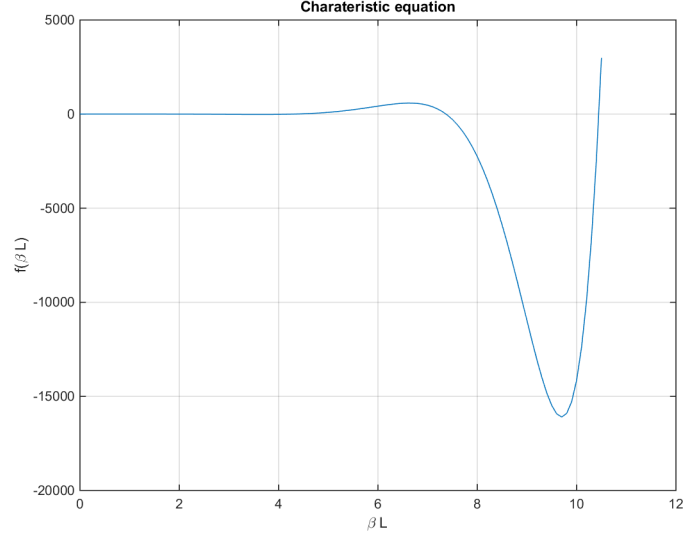
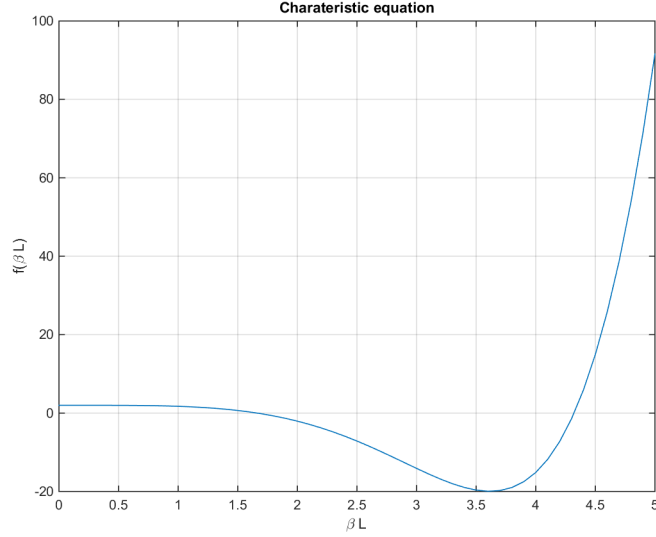
$\Leftrightarrow a \left[(\sinh \beta L - \sin \beta L) + \frac{\bar{M}}{m} \beta L (\cosh \beta L - \cos \beta L) \right] = -b \left[(\cosh \beta L + \cos \beta L) + \frac{\bar{M}}{m} \beta L (\sinh \beta L - \sin \beta L) \right]$

$\Leftrightarrow \frac{a}{b} = -\frac{(\cosh \beta L + \cos \beta L) + \frac{\bar{M}}{m} \beta L (\sinh \beta L - \sin \beta L)}{(\sinh \beta L - \sin \beta L) + \frac{\bar{M}}{m} \beta L (\cosh \beta L - \cos \beta L)}$ (6)

From (5) and (6) we have the characteristic equation:

$$\frac{\sinh \beta L + \sin \beta L}{\cosh \beta L + \cos \beta L} = \frac{(\cosh \beta L + \cos \beta L) + \frac{\bar{M}}{m} \beta L (\sinh \beta L - \sin \beta L)}{(\sinh \beta L - \sin \beta L) + \frac{\bar{M}}{m} \beta L (\cosh \beta L - \cos \beta L)}$$

$$\begin{aligned}
&\Leftrightarrow \sinh^2 \beta L - \sin^2 \beta L + \frac{\bar{M}}{m} \beta L \sinh \beta L \cosh \beta L - \frac{\bar{M}}{m} \beta L \sinh \beta L \cos \beta L + \frac{\bar{M}}{m} \beta L \sin \beta L \cosh \beta L - \\
&\frac{\bar{M}}{m} \beta L \sin \beta L \cos \beta L = \cosh^2 \beta L + 2 \cosh \beta L \cos \beta L + \cos^2 \beta L + \frac{\bar{M}}{m} \beta L \sinh \beta L \cosh \beta L - \frac{\bar{M}}{m} \beta L \cosh \beta L \sin \beta L + \\
&\frac{\bar{M}}{m} \beta L \cos \beta L \sinh \beta L - \frac{\bar{M}}{m} \beta L \cos \beta L \sin \beta L \\
&\Leftrightarrow 2 + 2 \cosh \beta L \cos \beta L + 2 \frac{\bar{M}}{m} \beta L \cos \beta L \sinh \beta L - 2 \frac{\bar{M}}{m} \beta L \cosh \beta L \sin \beta L = 0 \\
&\Leftrightarrow 1 + \cosh \beta L \cos \beta L + \frac{\bar{M}}{m} \beta L \cos \beta L \sinh \beta L - \frac{\bar{M}}{m} \beta L \cosh \beta L \sin \beta L = 0
\end{aligned}$$



The first four nonzero roots of characteristic equation are: $\beta_1 L = 1.665$, $\beta_2 L = 4.322$, $\beta_3 L = 7.37$ and $\beta_4 L = 10.45$

Natural frequencies is defined from computed values of β , we have:

$$\text{since: } \beta^4 = \frac{\rho A \omega^2}{EI} \Rightarrow \omega = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta L)^2 \sqrt{\frac{EI}{\rho A L^4}}$$

$$\text{we first have: } \sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{68.9 \bullet 10^9 \times 6.69 \bullet 10^{-11}}{2.7 \bullet 10^6 \times 78.4 \bullet 10^{-6} \times 0.46^4}} \approx 0.7$$

$$\beta_1 L = 1.665 \Rightarrow \omega_1 = (\beta_1 L)^2 \bullet 0.7 = 1.665^2 \bullet 0.7 = 1.94 \text{ rad/s}$$

$$\beta_2 L = 4.322 \Rightarrow \omega_1 = (\beta_2 L)^2 \bullet 0.7 = 4.322^2 \bullet 0.7 = 13.07 \text{ rad/s}$$

$$\beta_3 L = 7.37 \Rightarrow \omega_1 = (\beta_3 L)^2 \bullet 0.7 = 7.37^2 \bullet 0.7 = 38.02 \text{ rad/s}$$

$$\beta_4 L = 10.45 \Rightarrow \omega_1 = (\beta_4 L)^2 \bullet 0.7 = 10.45^2 \bullet 0.7 = 76.4 \text{ rad/s}$$

The mode shape can be written as:

$$\bar{Y}_i(X) = \left(\frac{A}{B}\right)_i \left(\cosh(\beta_i L \frac{X}{L}) - \cos(\beta_i L \frac{X}{L}) \right) + \sinh(\beta_i L \frac{X}{L}) - \sin(\beta_i L \frac{X}{L})$$

$$\text{with: } \left(\frac{A}{B}\right)_i = -\frac{\sinh \beta_i L + \sin \beta_i L}{\cosh \beta_i L + \cos \beta_i L}$$

we have the table result:

βL	1.575	4.226	7.282	10.371
a/b	-1.341	-0.985	-1.0005	1

$$\begin{aligned} \bar{Y}_1(X) &= -1.341 \left(\cosh(1.665 \frac{X}{0.46}) - \cos(1.665 \frac{X}{0.46}) \right) + \sinh(1.665 \frac{X}{0.46}) - \sin(1.665 \frac{X}{0.46}) \\ &= -1.341 (\cosh(3.62X) - \cos(3.62X)) + \sinh(3.62X) - \sin(3.62X) \\ \bar{Y}_2(X) &= -0.985 \left(\cosh(4.322 \frac{X}{0.46}) - \cos(4.322 \frac{X}{0.46}) \right) + \sinh(4.322 \frac{X}{0.46}) - \sin(4.322 \frac{X}{0.46}) \\ &= -0.985 (\cosh(9.4X) - \cos(9.4X)) + \sinh(9.4X) - \sin(9.4X) \\ \bar{Y}_3(X) &= -1.0005 \left(\cosh(7.37 \frac{X}{0.46}) - \cos(7.37 \frac{X}{0.46}) \right) + \sinh(7.37 \frac{X}{0.46}) - \sin(7.37 \frac{X}{0.46}) \\ &= -1.0005 (\cosh(16.02X) - \cos(16.02X)) + \sinh(16.02X) - \sin(16.02X) \\ \bar{Y}_4(X) &= -1 \left(\cosh(10.45 \frac{X}{0.46}) - \cos(10.45 \frac{X}{0.46}) \right) + \sinh(10.45 \frac{X}{0.46}) - \sin(10.45 \frac{X}{0.46}) \\ &= -\cosh(22.72X) + \cos(22.71X) + \sinh(22.72X) - \sin(22.72X) \end{aligned}$$

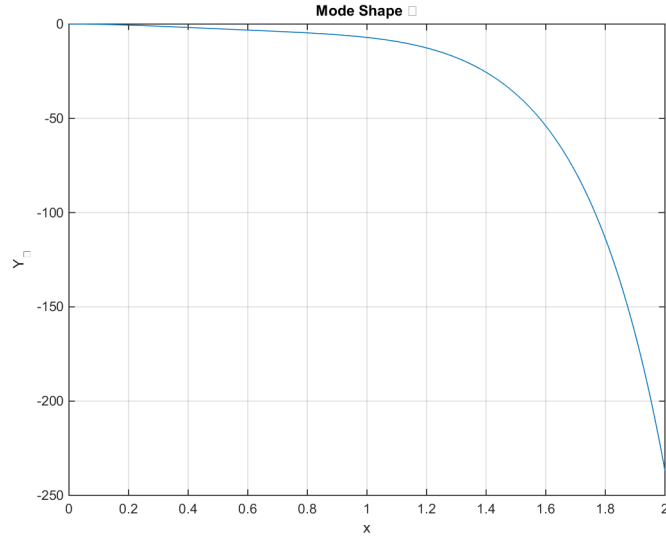


Figure 1: Mode Shape 1

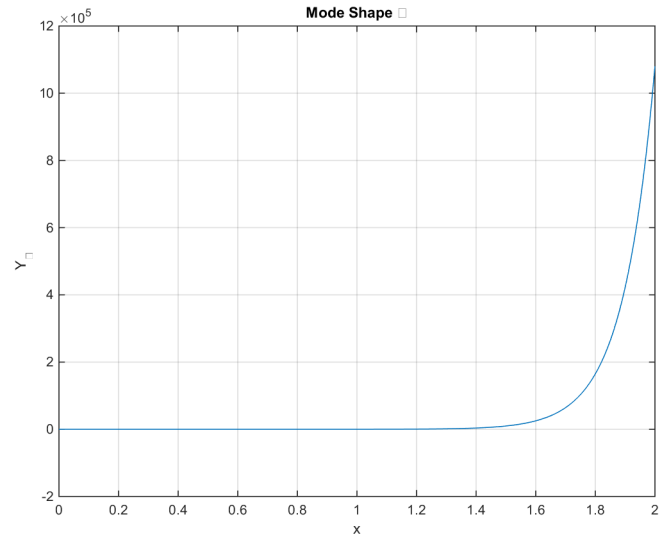


Figure 2: Mode Shape 2

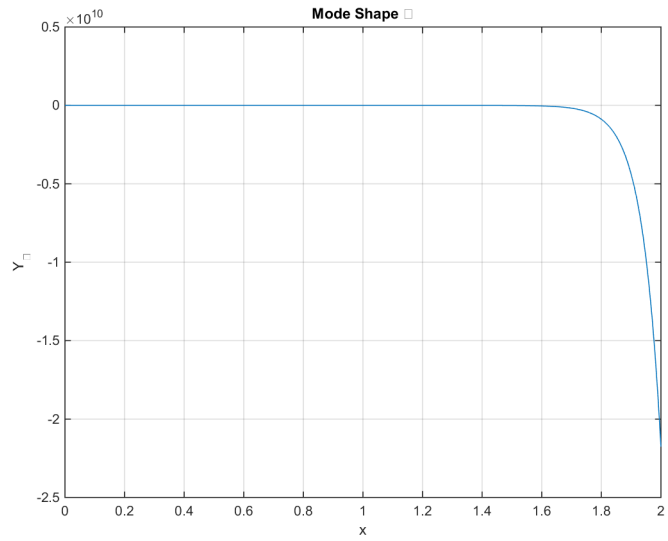


Figure 3: Mode Shape 3

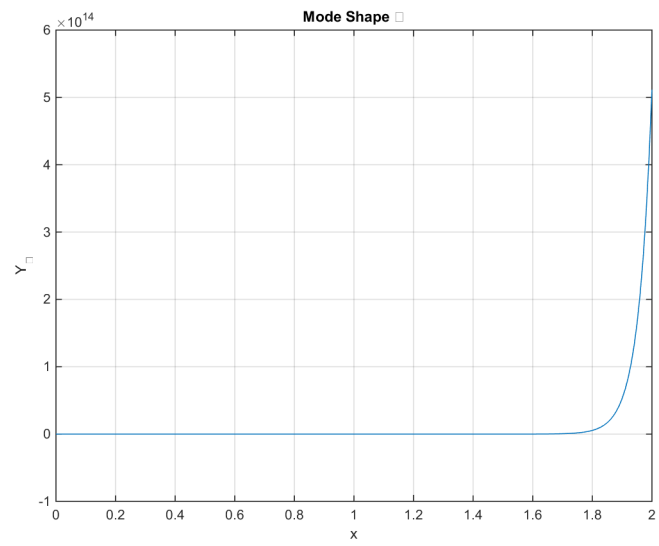


Figure 4: Mode Shape 4

2. Modeled the tip mass as a solid having mass and inertia:

- The first two bound conditions are same with above: we have equation (3) and (4)

- From the rotational inertial we form the third and forth condition:

$$\frac{a}{b} = -\frac{k_1(\sinh \beta L - \sin \beta L) + (\cosh \beta L + \cos \beta L)}{k_1(\cosh \beta L + \cos \beta L) + (\sinh \beta L - \sin \beta L)} \quad (7)$$

$$\frac{a}{b} = -\frac{k_2(\cosh \beta L - \cos \beta L) - (\sinh \beta L + \sin \beta L)}{k_2(\sinh \beta L + \sin \beta L) - (\cosh \beta L + \cos \beta L)} \quad (8)$$

with $k_1 = \frac{\bar{M}}{m}\beta L = 0.1508\beta L$

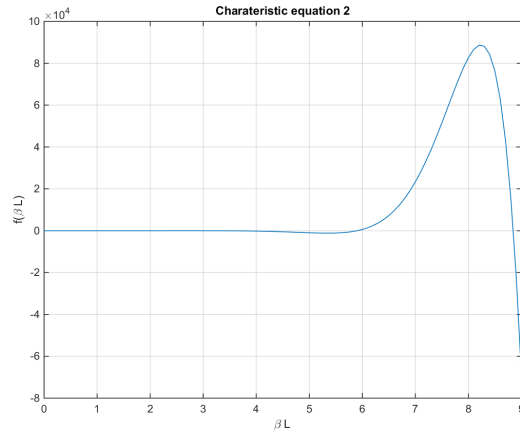
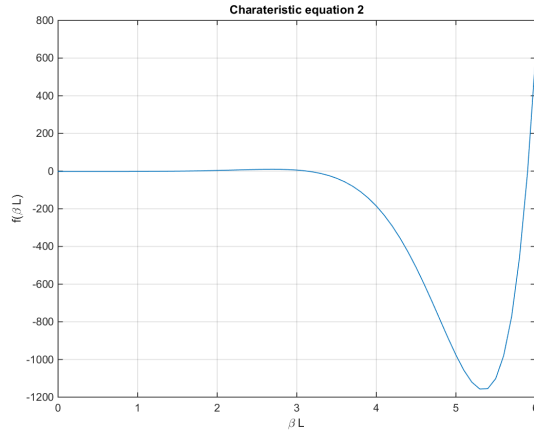
$$\begin{aligned} \text{The rotary inertia is: } I_0 &= \frac{1}{4}M_1r_1^2 + \frac{1}{3}M_1h_1^2 + M_2h_2^2 \\ &= \pi * 1.15^2 \times 10^{-4} * 1.13 \times 10^{-2} * 2.7 \times 10^6 \left(\frac{1}{4} 1.15^2 \times 10^{-4} + \frac{1}{3} 1.13^2 \times 10^{-4} \right) \\ &\quad + \pi * 0.45^2 \times 10^{-4} * 1.17 \times 10^{-2} * 2.7 \times 10^6 * 1.17^2 \times 10^{-4} \\ &= 1.234 \times 10^{-3} \end{aligned}$$

$$\text{So the rotary term for } k_2 \text{ is: } k_2 = \frac{I_0}{mL^2}(\beta L)^3 = \frac{1.234 \times 10^{-3}}{97.4 \times 10^{-3} * 0.46^2}(\beta L)^3 = 0.06(\beta L)^3$$

Characteristic equation:

$$(k_1k_2 - 1)(\cosh \beta L \cos \beta L) - (k_1k_2 + 1) + (k_1 + k_2)(\cosh \beta L \sin \beta L) - (k_1 - k_2)(\sinh \beta L \cos \beta L) = 0$$

$$\Leftrightarrow (9.048 \times 10^{-3}(\beta)^4 - 1)(\cosh \beta L \cos \beta L) - (9.048 \times 10^{-3}(\beta)^4 + 1) + (0.1508\beta L + 0.06(\beta L)^3)(\cosh \beta L \sin \beta L) - (0.1508\beta L - 0.06(\beta L)^3)(\sinh \beta L \cos \beta L) = 0$$



The first four nonzero roots of characteristic equation are: $\beta_{21}L = 1.55, \beta_{22}L = 3.13, \beta_{23}L = 5.9$ and $\beta_{24}L = 8.84$

Natural frequencies is defined from computed values of β , we have:

$$\omega_n^2 = \beta^4 \frac{EIL}{m} \Leftrightarrow \omega_n = (\beta L)^2 \sqrt{\frac{EI}{mL^3}}$$

with: $\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{68.9 \bullet 10^9 \times 6.69 \bullet 10^{-11}}{97.4 \bullet 10^{-3} \times 0.46^3}} \approx 22.05$

The first four natural frequencies of vibration are defined:

βL	1.55	3.13	5.9	8.84
ω_n	53	216.3	767.9	1725

The mode shape can be written as:

$$\bar{Y}_i(X) = \left(\frac{A}{B}\right)_i \left(\cosh(\beta_i L \frac{X}{L}) - \cos(\beta_i L \frac{X}{L}) \right) + \sinh(\beta_i L \frac{X}{L}) - \sin(\beta_i L \frac{X}{L})$$

with: $\left(\frac{a}{b}\right)_i = -\frac{k_{1i}(\sinh \beta_i L - \sin \beta_i L) + (\cosh \beta_i L + \cos \beta_i L)}{k_{1i}(\cosh \beta_i L + \cos \beta_i L) + (\sinh \beta_i L - \sin \beta_i L)}$

βL	1.55	3.13	5.9	8.84
a/b	-1.52	-0.97	-1.0002	1

$$\begin{aligned} \bar{Y}_1(X) &= -1.52 \left(\cosh(1.55 \frac{X}{0.46}) - \cos(1.55 \frac{X}{0.46}) \right) + \sinh(1.55 \frac{X}{0.46}) - \sin(1.55 \frac{X}{0.46}) \\ &= -1.52 (\cosh(3.37X) - \cos(3.37X)) + \sinh(3.37X) - \sin(3.37X) \end{aligned}$$

$$\begin{aligned} \bar{Y}_2(X) &= -0.97 \left(\cosh(3.13 \frac{X}{0.46}) - \cos(3.13 \frac{X}{0.46}) \right) + \sinh(3.13 \frac{X}{0.46}) - \sin(3.13 \frac{X}{0.46}) \\ &= -0.97 (\cosh(6.8X) - \cos(6.8X)) + \sinh(6.8X) - \sin(6.8X) \end{aligned}$$

$$\begin{aligned} \bar{Y}_3(X) &= -1.0002 \left(\cosh(5.9 \frac{X}{0.46}) - \cos(5.9 \frac{X}{0.46}) \right) + \sinh(5.9 \frac{X}{0.46}) - \sin(5.9 \frac{X}{0.46}) \\ &= -1.0002 (\cosh(12.83X) - \cos(12.83X)) + \sinh(12.83X) - \sin(12.83X) \end{aligned}$$

$$\begin{aligned} \bar{Y}_4(X) &= -1 \left(\cosh(8.84 \frac{X}{0.46}) - \cos(8.84 \frac{X}{0.46}) \right) + \sinh(8.84 \frac{X}{0.46}) - \sin(8.84 \frac{X}{0.46}) \\ &= -\cosh(19.22X) + \cos(19.22X) + \sinh(19.22X) - \sin(19.22X) \end{aligned}$$

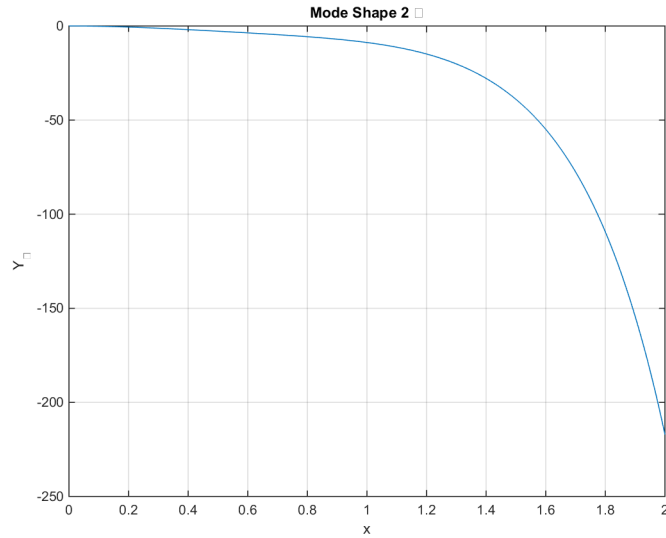


Figure 5: Mode Shape 2-1

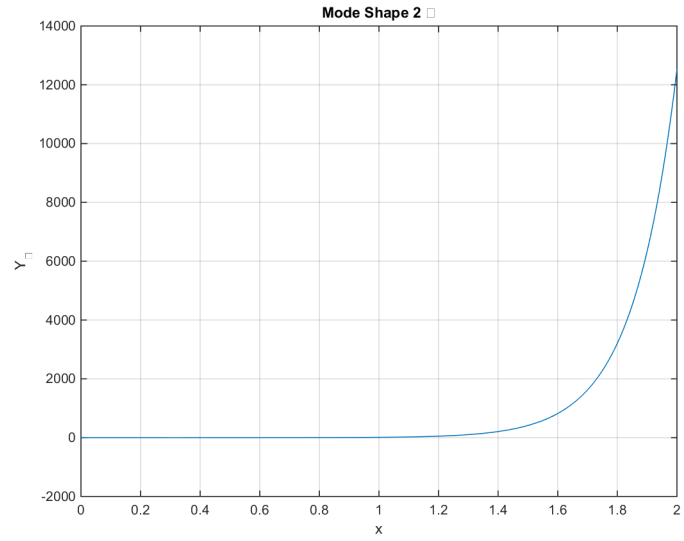


Figure 6: Mode Shape 2-2

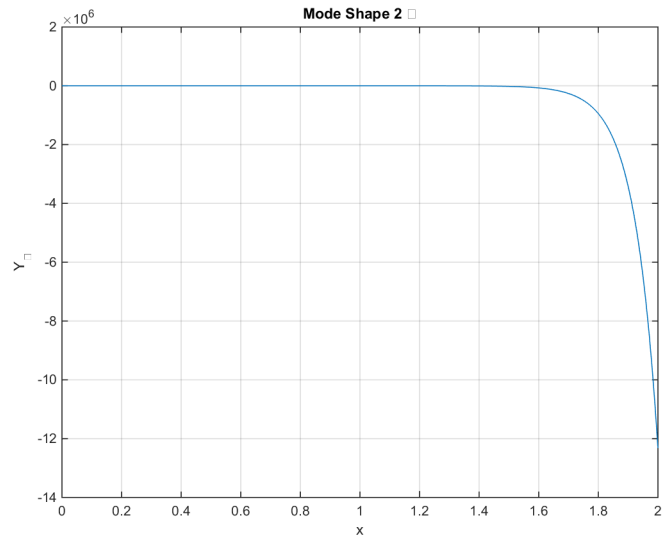


Figure 7: Mode Shape 2-3

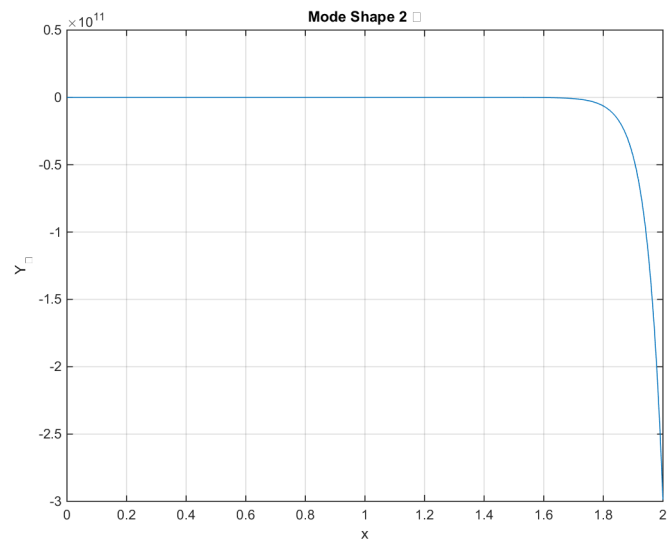


Figure 8: Mode Shape 2-4