Rayleigh's method

Based on Rayleigh's Principle:

The frequency of vibration of a conservative system vibrating about an equilibrium position has a stationary value in the neighborhood of a natural mode. This stationary value, in fact, is a minimum value in the neighborhood of the fundamental natural mode

The kinetic and potential energies of an *n* degree of freedom discrete system can be expressed as:

$$T = \frac{1}{2}\dot{x}^T \left[m \right] \dot{x} \qquad V = \frac{1}{2} x^T \left[k \right] x$$

Rayleigh's method

To find the natural frequencies, we assume harmonic motion:

$$x = X\cos\omega t$$

…where *X* denote the vector of amplitudes (mode shape) and ω represents the natural frequency of vibration. If the system is conservative, the maximum kinetic energy is equal to the maximum potential energy:

$$T_{\max} = V_{\max}$$

Substituting the harmonic motion assumption into the above equation results in:

$$T_{\max} = \frac{1}{2}X^T\left[m\right]X\omega^2 = V_{\max} = \frac{1}{2}X^T\left[k\right]X$$

A quotient can be formed to solve for ω:

$$\omega^2 = \frac{X^T \left[k \right] X}{X^T \left[m \right] X}$$

This is denoted Raleigh's quotient and can be written as ***R(X)***

- As previously mentioned, ***R(X)*** has a stationary value when the arbitrary vector ***X*** is in the neighborhood of any eigenvector ***X***
- To prove this, we express the the arbitrary *X* vector in terms of the normal modes of the system:

$$X = c_1 X_1 + c_2 X_2 + \cdots$$

The expansion for the quotient terms considering that the cross terms of the form $c_i c_j X_i^T \left[k\right] X_j$ are zero becomes:

$$X^T \left[k\right] X = c_1^2 X_1^T \left[k\right] X_1 + c_2^2 X_2^T \left[k\right] X_2 + c_3^2 X_3^T \left[k\right] X_3 + \cdots$$

$$X^T \left[m\right] X = c_1^2 X_1^T \left[m\right] X_1 + c_2^2 X_2^T \left[m\right] X_2 + c_3^2 X_3^T \left[m\right] X_3 + \cdots$$

Recognizing that $X_i^T \left[k\right] X_i = \omega^2 X_i^T \left[m\right] X_i$

Raleigh's quotient can be written:

$$\omega^2 = \frac{c_1^2 \omega_1^2 X_1^T \left[m\right] X_1 + c_2^2 \omega_2^2 X_2^T \left[m\right] X_2 + \cdots}{c_1^2 X_1^T \left[m\right] X_1 + c_2^2 X_2^T \left[m\right] X_2 + \cdots}$$

For normalized modes, the previous equation becomes:

$$\omega^2 = \frac{c_1^2\omega_1^2 + c_2^2\omega_2^2 + \cdots}{c_1^2 + c_2^2 + \cdots}$$

If *X* differs little from the eigenvector X_r, then the coefficient c_r will be much larger than the remaining coefficients c_i (*i not equal to r)* and we can write the above equation as:

$$\omega^2 = R(X) = \frac{c_r^2\omega_r^2 + c_r^2 \sum_{i=1,2,\cdots(i\neq r)} \left(\frac{c_i}{c_r}\right)^2 \omega_i^2}{c_r^2 + c_r^2 \sum_{i=1,2,\cdots(i\neq r)} \left(\frac{c_i}{c_r}\right)^2 \omega_i^2}$$

Now looking at the coefficient ratios:

$$\left|\frac{c_i}{c_r}\right| = \varepsilon_i << 1$$ where ε_i is a small number for all *i* not equal to *r*

Results in: $$R(X) = \omega_r^2\left[1 + 0(\varepsilon)^2\right]$$

- This result confirms that small changes of the eigenvector are reflected changes to the 2nd order in the eigenvalue.
- The stationary value is a minimum value in the neighborhood of the fundamental mode, X_i.

This can be shown by letting r = 1 (first mode) and this results in:

$$R(X) = \frac{\omega_1^2 + \sum_{i=2,3,\cdots} \left(\frac{c_i}{c_1}\right)^2 \omega_i^2}{1 + \sum_{i=2,3,\cdots} \left(\frac{c_i}{c_1}\right)^2}$$

$$= \omega_1^2 + \sum_{i=2,3,\cdots} \varepsilon_i^2 \omega_i^2 - \omega_1^2 \sum_{i=2,3,\cdots} \varepsilon_i^2$$

$$= \omega_1^2 + \sum_{i=2,3,\cdots} \left(\omega_i^2 - \omega_1^2\right) \varepsilon_1^2$$

In general, $\omega_i^2 > \omega_1^2$ which leads to

$$R(X) \geq \omega_1^2$$

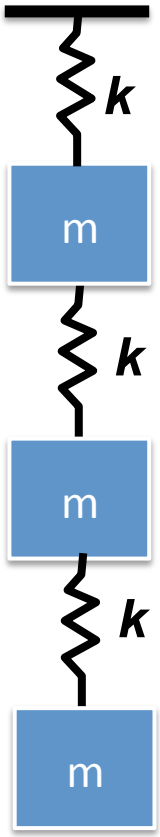
This means that Rayleigh's quotient is never lower than the first eigenvalue.

- Similarly, we can show that $R\left(X\right) \leq \omega_n^2$
  - Which means that Rayleigh's quotient is never higher than the highest eigenvalue. Thus, the quotient provides an upper bound for ω_1^2 and a lower bound for ω_n^2

- Now, Rayleigh's quotient can be used to find approximate values of the first natural frequency for a system.

Consider a 3-DOF system:

- Estimate the fundamental natural frequency for an estimated first mode shape of $X = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$



$$[k] = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad [m] = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$R(X) = \omega^2 = \frac{X^T[k]X}{X^T[m]X} = \frac{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} m \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = 0.214\frac{k}{m}$$

This approximation is close:

- Computed: $\omega_1 = 0.463\sqrt{\frac{k}{m}}$
- The actual natural frequency is $\omega_1 = 0.445\sqrt{\frac{k}{m}}$
- The actual mode shape is $X = \begin{Bmatrix} 1 \\ 1.8 \\ 2.25 \end{Bmatrix}$

Other methods used to estimate the first natural frequency include: Matrix iteration, Holzer's method

Hotelling's deflation with matrix iteration can be used to estimate the other natural frequencies

- The process requires the EVP to be reformulated
  - It can be shown that if we solve for the highest frequency eigenvalue/eigenvector, a method exists to eliminate that eigenvalue/eigenvector from the EVP and what remains are all the lower eigenvalue/eigenvectors
  - By solving the inverse EVP, we are actually finding the lowest eigenvalue/eigenvector first, and then the next lowest, and so on
- Similar to the previous process, we will use trial vectors. However, this time the trial vectors must be mass normalized:

$$X_{i-1}^{T}\left[m\right]X_{i-1}=1$$

Hotelling's deflation

- Recall how the standard form of the EVP was obtained:

$$\left[m\right]\left(\ddot{x}\right)+\left[k\right]\left(x\right)=\left(0\right)$$

$$\left[-\lambda\left[m\right]+\left[k\right]\right]\left(X\right)=\left(0\right)$$

$$\left[-\lambda+\left[m\right]^{-1}\left[k\right]\right]\left(X\right)=\left(0\right)$$

$$\left[-\lambda+\left[A\right]\right]\left(X\right)=\left(0\right)$$

Hotelling's deflation

- What we are interested in solving is an ***inverse EVP*:**

$$\left[m\right]\left(\ddot{x}\right)+\left[k\right]\left(x\right)=\left(0\right)$$

$$\left[-\lambda\left[k\right]^{-1}\left[m\right]+\left[I\right]\right]\left(X\right)=\left(0\right)$$

$$\left[\left[k\right]^{-1}\left[m\right]-\left[\frac{I}{\lambda}\right]\right]\left(X\right)=\left(0\right)$$

$$\left[\left[D\right]-\lambda^{*}\right]\left(X\right)=\left(0\right)$$

Where ***D*** represents a matrix for the inverse problem and is iteratively "deflated" to solve for the subsequent next lowest modes in the modified equation

Hotelling's deflation

- We will solve this modified equation using a similar process to Raleigh's quotient where a trial vector is defined
  - *As mentioned previously, the trial vector must always be mass normalized*
- Each subsequent deflated matrix is computed from:

$$\left[D_i\right] = \left[D_{i-1}\right] - \lambda_{i-1}^{*} \hat{X}_{i-1} \hat{X}_{i-1}^{T} \left[m\right]$$

Matrix iteration by the power method is used to find system eigenvalues, eigenvectors

- Define a trial vector for the "inverted" system
- Using matrix iteration, find the best fit eigenvector from the following equation:

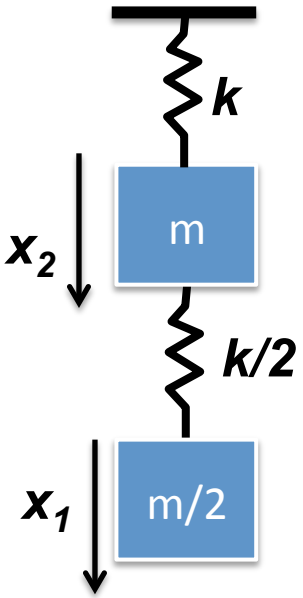
$$V_p = \left[D \right] V_{p-1}$$

Start with a trial vector here

- When convergence occurs, $\lambda_i = \lim_{p \to \infty} \frac{V_{p+1}}{V_p}$

Example

- For the system shown, the EOM is:



$$-\lambda m\begin{bmatrix} 1/2 & \\ & 1 \end{bmatrix}\{\ddot{x}\}+k\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}\{x\}=\{0\}$$

- The standard form of the eigenvalue problem is:

$$\left[-\lambda\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}+\frac{k}{m}\begin{bmatrix} 1 & -1 \\ -1/2 & 3/2 \end{bmatrix}\right]\{X\}=\{0\}$$

Example

- The inverse form of the problem is:

$$\left[-\frac{k}{m\lambda}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}+\begin{bmatrix} 3/2 & 1 \\ 1/2 & 1 \end{bmatrix}\right](X)=(0)$$

- Or

$$\left[-\lambda^*[I]+\begin{bmatrix} 3/2 & 1 \\ 1/2 & 1 \end{bmatrix}\right](X)=(0)$$

  - Where $\lambda^*=\frac{k}{m\lambda}$ and $D=\begin{bmatrix} 1.5 & .5 \\ .5 & 1 \end{bmatrix}$

Matrix iteration example

- Now choose a trial vector: $V_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$V_1 = [D]\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 1.5 \end{pmatrix}$$

$$V_2 = [D]\begin{pmatrix} 2.5 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 5.25 \\ 2.75 \end{pmatrix}$$

$$V_3 = [D]\begin{pmatrix} 5.25 \\ 2.75 \end{pmatrix} = \begin{pmatrix} 10.625 \\ 5.375 \end{pmatrix}$$

Computing the 10th and 11th iterations with normalization of the trial vector results in:

$$V_{10} = \begin{pmatrix} 1 \\ .5 \end{pmatrix} \quad \text{and} \quad V_{11} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The eigenvector is stable after a few iterations

- Computing the eigenvalue results in:

$$\lambda^* = \frac{V_{11_i}}{V_{10_i}} = 2 = \frac{1}{\lambda}$$

The real eigenvalue is: $\lambda = 0.5, \quad \omega = \sqrt{\frac{0.5k}{m}}$

- Some notes:
  - The real system eigenvalue is the inverse of the modified eigenvalue
  - The eigenvector does not change – it is the same for the modified and real systems:

$$X = V_{10} = \begin{pmatrix} 1 \\ .5 \end{pmatrix}$$

Now to find the next eigenvalue/eigenvector, use Hotelling's deflation to obtain a modified *D* matrix

- Determine the new deflated matrix using the new eigenvector:

$$\left[D_i\right] = \left[D_{i-1}\right] - \lambda^*_{i-1}\hat{X}_{i-1}\hat{X}^T_{i-1}\left[m\right]$$

- Repeat the process:

$$V_p = \left[D_i\right]V_{p-1} \qquad \lambda_i = \lim_{p\to\infty}\frac{V_{p+1_j}}{V_{p_j}}$$

In order to compute the 2nd mode, we must first find the modal mass for mode 1, then compute the mass-normalized eigenvector:

$$\hat{m}_{11} = X_1^T [m] X_1 = .75$$

$$\hat{X}_1 = \begin{pmatrix} 1/\sqrt{.75} \\ .5/\sqrt{.75} \end{pmatrix}$$

$$[D_2] = [D_1] - \lambda_1^* \hat{X}_1 \hat{X}_1^T [m] = \begin{bmatrix} .16 & -.34 \\ -.17 & .34 \end{bmatrix}$$

Now proceeding through the same process as before results in:

$$X = V_{10} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \lambda^* = 0.5, \quad \lambda = 2.0$$

- And the system natural frequency is:

$$\omega = \sqrt{\frac{2k}{m}}$$

Other techniques for obtaining eigenvalues, eigenvectors for large systems

- **Jacobi method**
  - Diagonalizes [*A*] using similarity transformations:

$$\left[A_k \right] = R_k^T R_{k-1}^T R_{k-2}^T A R_{k-2} R_{k-1} R_k = \left[\lambda_{diag} \right]$$

$$\left[\Phi \right] = R_{k-2} R_{k-1} R_k$$

Given's method

- Produces a tri-diagonal matrix from [*A*]
- Once in tri-diagonal form, all elements of the eigenvector can be found
- Other techniques:
  - Jacobi method
  - Householder reflections
- All of these numerical techniques are designed to handle a large number of degrees of freedom in a computationally efficient manner