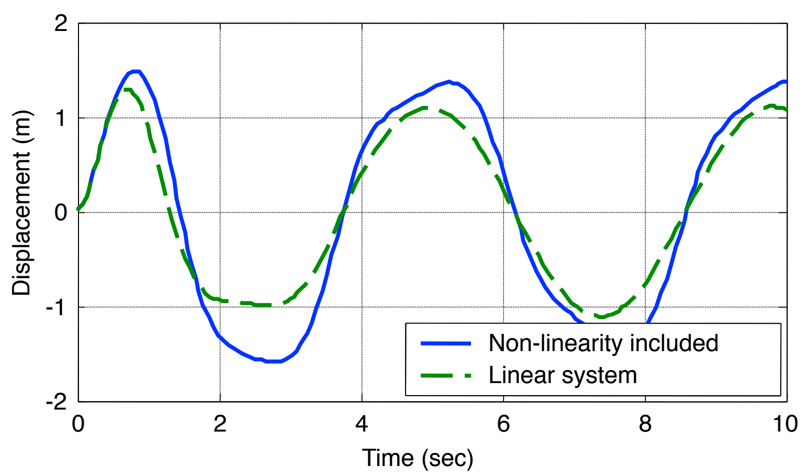
# 2.8 Numerical Simulation and Design

Four things we can do computationally to help solve, understand, and design vibration problems subject to harmonic excitation

- Symbolic manipulation
- Solution of the time response
- Plotting the time response
- Plotting magnitude and phase



# Symbolic manipulation can be done in numerical software packages

Let

$$A = \begin{bmatrix} \omega_n^2 - \omega^2 & 2\zeta\omega_n\omega \\ -2\zeta\omega_n\omega & \omega_n^2 - \omega^2 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} f_0 \\ 0 \end{bmatrix}$$

What is

$$A_n = A^{-1}x$$

This can be solved using Matlab, Mathcad, or Mathematica

Symbolic manipulation can be done in numerical software packages (Mathcad)

Solve equations (2.34) using Mathcad symbolics :

Enter this →

$$\begin{bmatrix} \omega n^2 - \omega^2 & 2\cdot\zeta\cdot\omega n\cdot\omega \\ -(2\cdot\zeta\cdot\omega n\cdot\omega) & \omega n^2 - \omega^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} f0 \\ 0 \end{bmatrix}$$

Choose evaluate under symbolics to get this

$$\begin{bmatrix} \dfrac{(\omega n^2 - \omega^2)}{(\omega n^4 - 2\cdot\omega n^2\cdot\omega^2 + \omega^4 + 4\cdot\zeta^2\cdot\omega n^2\cdot\omega^2)}\cdot f0 \\ 2\cdot\zeta\cdot\omega n\cdot\dfrac{\omega}{(\omega n^4 - 2\cdot\omega n^2\cdot\omega^2 + \omega^4 + 4\cdot\zeta^2\cdot\omega n^2\cdot\omega^2)}\cdot f0 \end{bmatrix}$$

# Symbolic manipulation can be done in numerical software packages (MATLAB)

```
>> syms z wn w f0
>> A=[wn^2-w^2 2*z*wn*w;-2*z*wn*w wn^2-w^2];
>> x=[f0 ;0];
>> An=inv(A)*x
      An =
      [ (wn^2-w^2)/(wn^4-2*wn^2*w^2+w^4+4*z^2*wn^2*w^2)*f0]
      [   2*z*wn*w/(wn^4-2*wn^2*w^2+w^4+4*z^2*wn^2*w^2)*f0]
>> pretty(An)

            [                 2    2                  ]
            [             (wn   - w ) f0              ]
            [ --------------------------------------- ]
            [   4        2  2     4       2    2   2  ]
            [ wn   - 2 wn  w   + w   + 4 z   wn   w   ]
            [                                         ]
            [                z wn w f0                ]
            [2 ---------------------------------------]
            [    4        2  2     4       2    2   2 ]
            [  wn   - 2 wn  w   + w   + 4 z   wn   w  ]
```

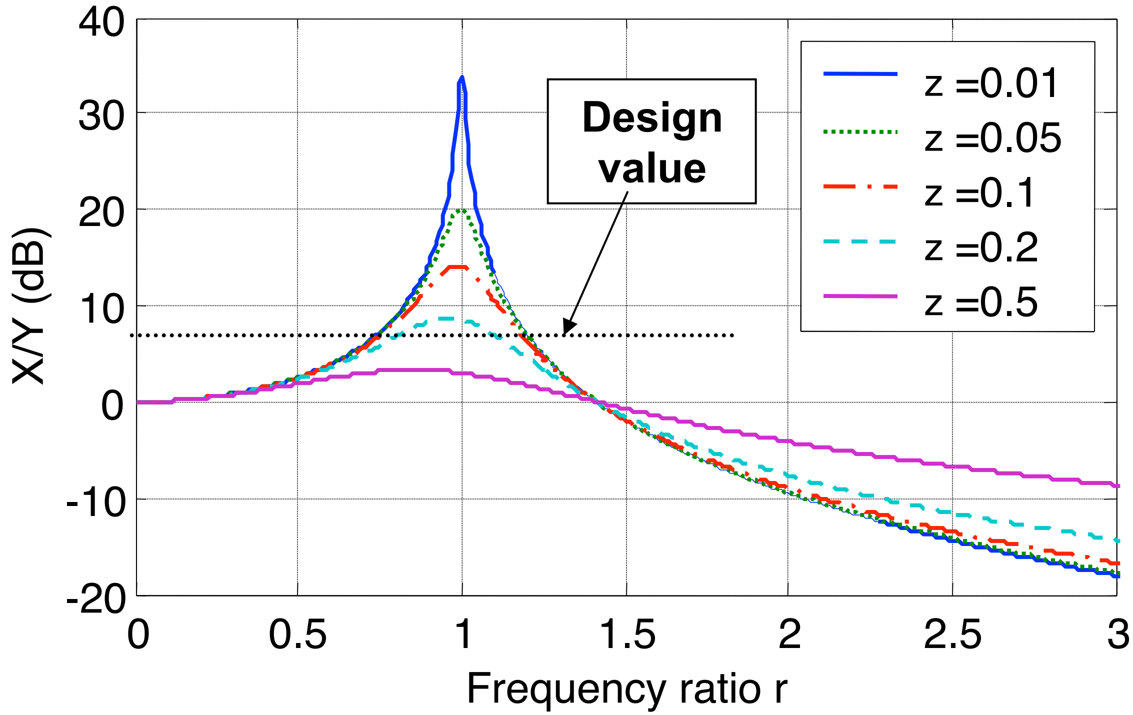
# Matlab plotting example:
# Magnitude plots for base excitation

```
%m-file to plot base excitation to mass vibration
r=linspace(0,3,500);
ze=[0.01;0.05;0.1;0.20;0.50];
X=sqrt( ((2*ze*r).^2+1) ./ ( (ones(size(ze))*(1-r.*r).^2) + (2*ze*r).^2) );
figure(1)
plot(r,20*log10(X))
```

**The values of z can then be chosen directly off of the plot.**

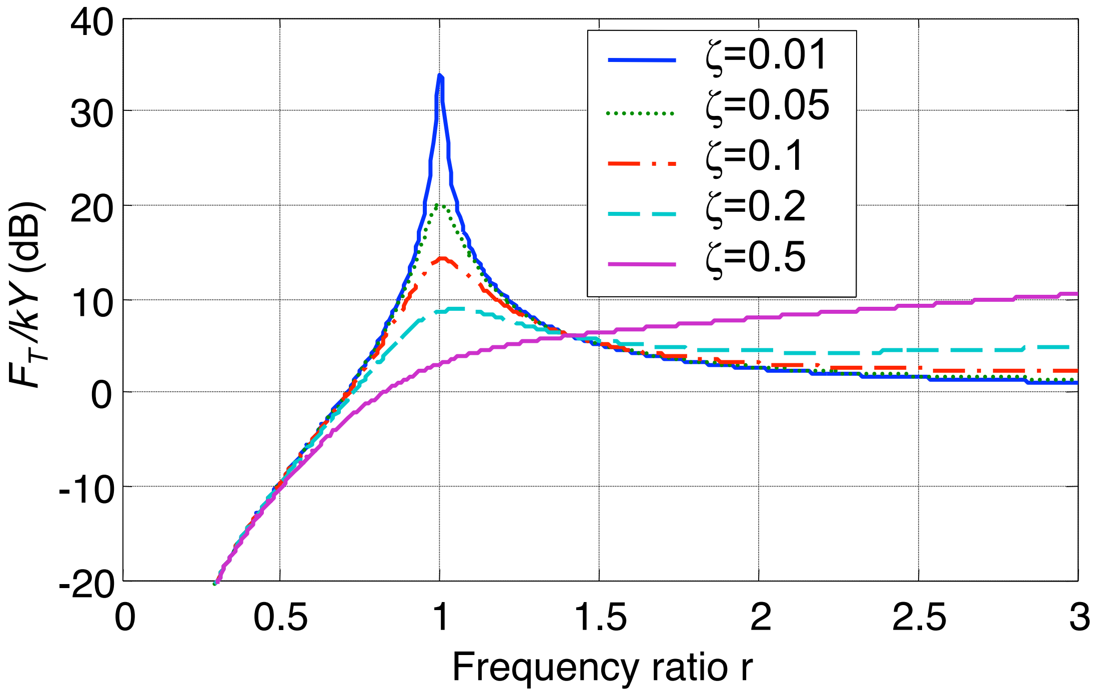
**For Example:**

**If the T.R. needs to be less than 2 (or 6dB) and *r* is close to 1 then z must be more than 0.2 (probably about 0.3).**



# Matlab plotting example:
# Force magnitude plots for base excitation

```
%m-file to plot base excitation to mass vibration
r=linspace(0,3,500);
ze=[0.01;0.05;0.1;0.20;0.50];
X=sqrt( ((2*ze*r).^2+1) ./ ( (ones(size(ze))*(1-r.*r).^2) + (2*ze*r).^2) );
F=X.*(ones(length(ze),1)*r).^2;
figure(1)
plot(r,20*log10(F))
```



# A Review Example of State Space

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F_0\cos\omega t$$

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=f_0\cos\omega t$$

Define: $$x_1=x(t)$$

$$x_2=\dot{x}(t)=\dot{x}_1$$

$$\Rightarrow\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-2\zeta\omega_n x_2-\omega_n^2x_1+f_o\cos(\omega t)\end{cases}$$

# Numerical Simulation

We can put the forced case

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos\omega t$$

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = f_0 \cos\omega t$$

into a state space form

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2\zeta\omega_n x_2 - \omega_n^2 x_1 + f_0 \cos\omega t$$

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ f_0 \cos\omega t \end{bmatrix}$$

# Numerical Integration

$$\text{Euler:}\quad \mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + A\mathbf{x}(t_i)\Delta t + \mathbf{f}(t_i)\Delta t$$

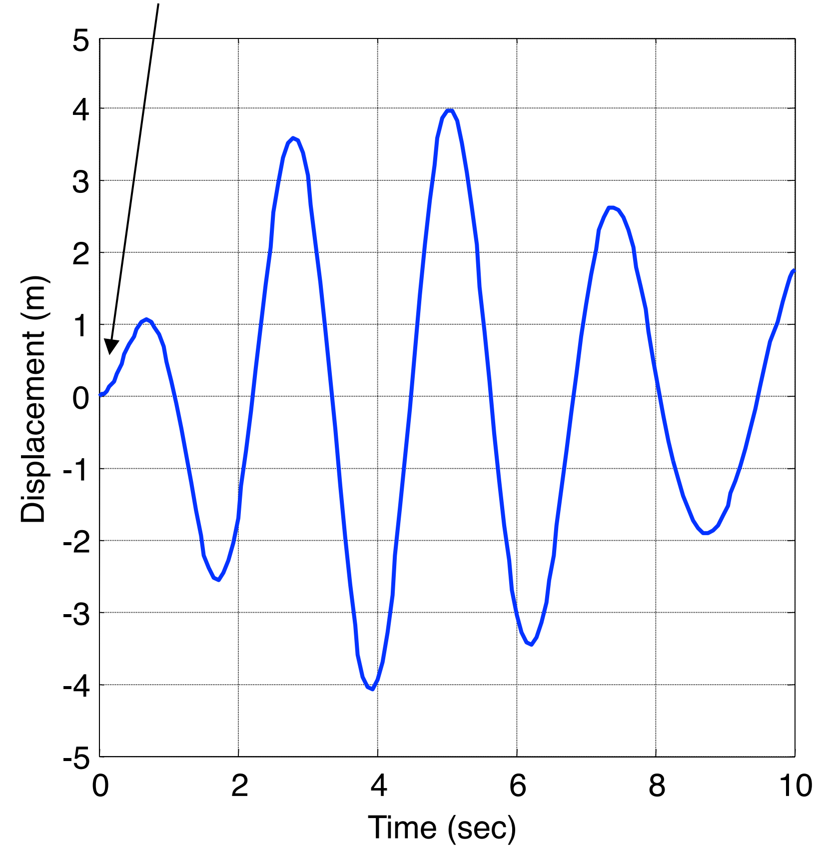
**Using the ODE45 function**

```
>>TSPAN=[0 10];
>>Y0=[0;0];
>>[t,y] =ode45('num_for',TSPAN,Y0);
>>plot(t,y(:,1))
```

**Zero initial conditions**

**Including forcing**

```
function Xdot=num_for(t,X)
m=100;k=1000;c=25;
ze=c/(2*sqrt(k*m));
wn=sqrt(k/m);
w=2.5;F=1000;f=F/m;
f=[0 ;f*cos(w*t)];
A=[0 1;-wn*wn -2*ze*wn];
Xdot=A*X+f;
```



## **Example 2.8.2:** Design damping for an electronics model

- 100 kg mass, subject to 150cos(5t) N
- Stiffness $k$=500 N/m, $c$ = 10kg/s
- Usually $x_0$=0.01 m, $v_0$ = 0.5 m/s
- Find a new $c$ such that the max transient value is 0.2 m

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos\omega t$$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2\zeta\omega_n x_2 - \omega_n^2 x_1 + f_0 \cos\omega t \\
\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ f_0 \cos\omega t \end{bmatrix}
\end{aligned}$$

# To run this use the following file:

**Create function to model forcing**

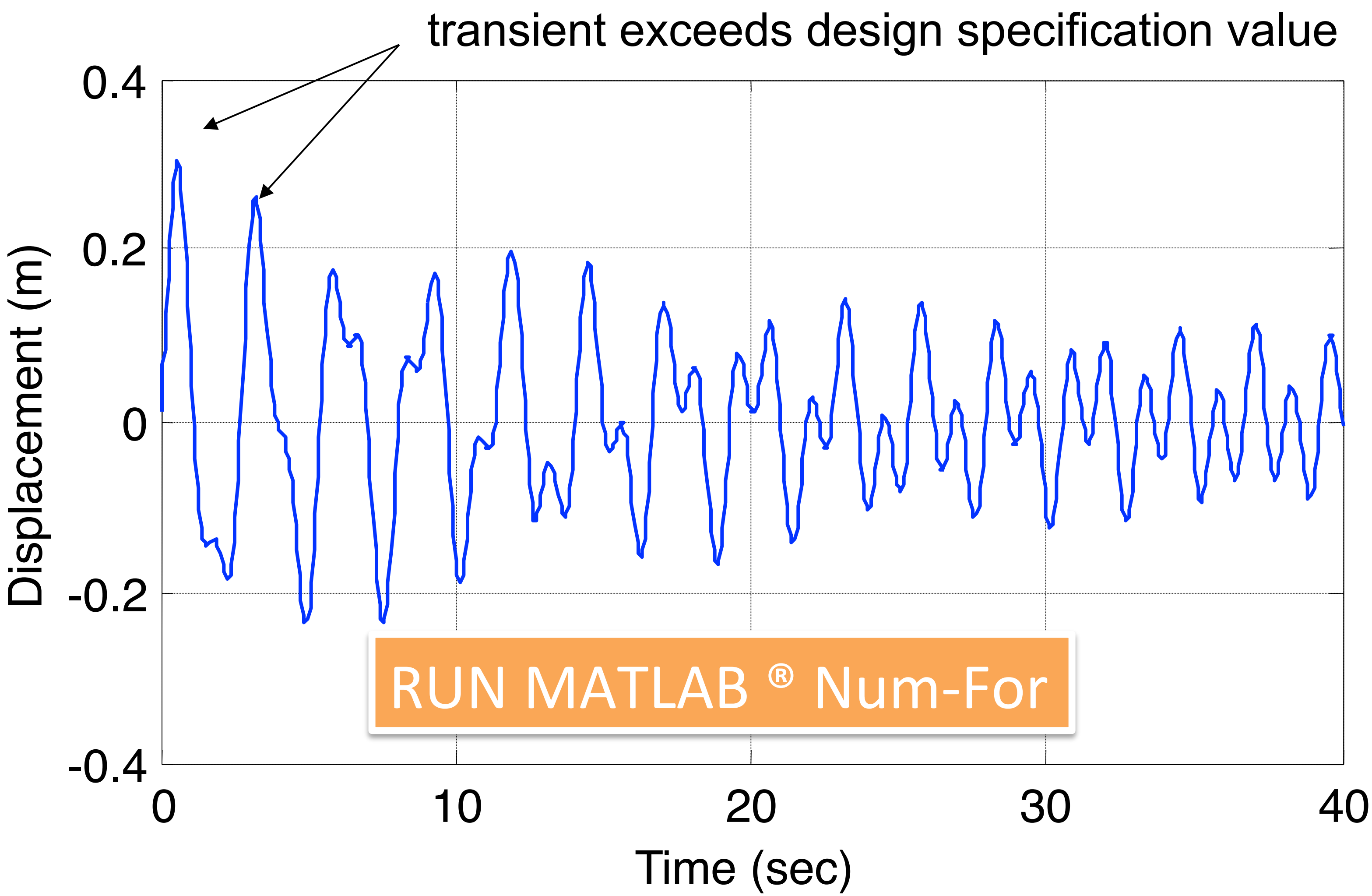
```
function Xdot=num_for(t,X)
m=100;k=500;c=10;
ze=c/(2*sqrt(k*m));
wn=sqrt(k/m);
w=5;F=150;f=F/m;
f=[0 ;f*cos(w*t)];
A=[0 1;-wn*wn -2*ze*wn];
Xdot=A*X+f;
```

**MATLAB command window**

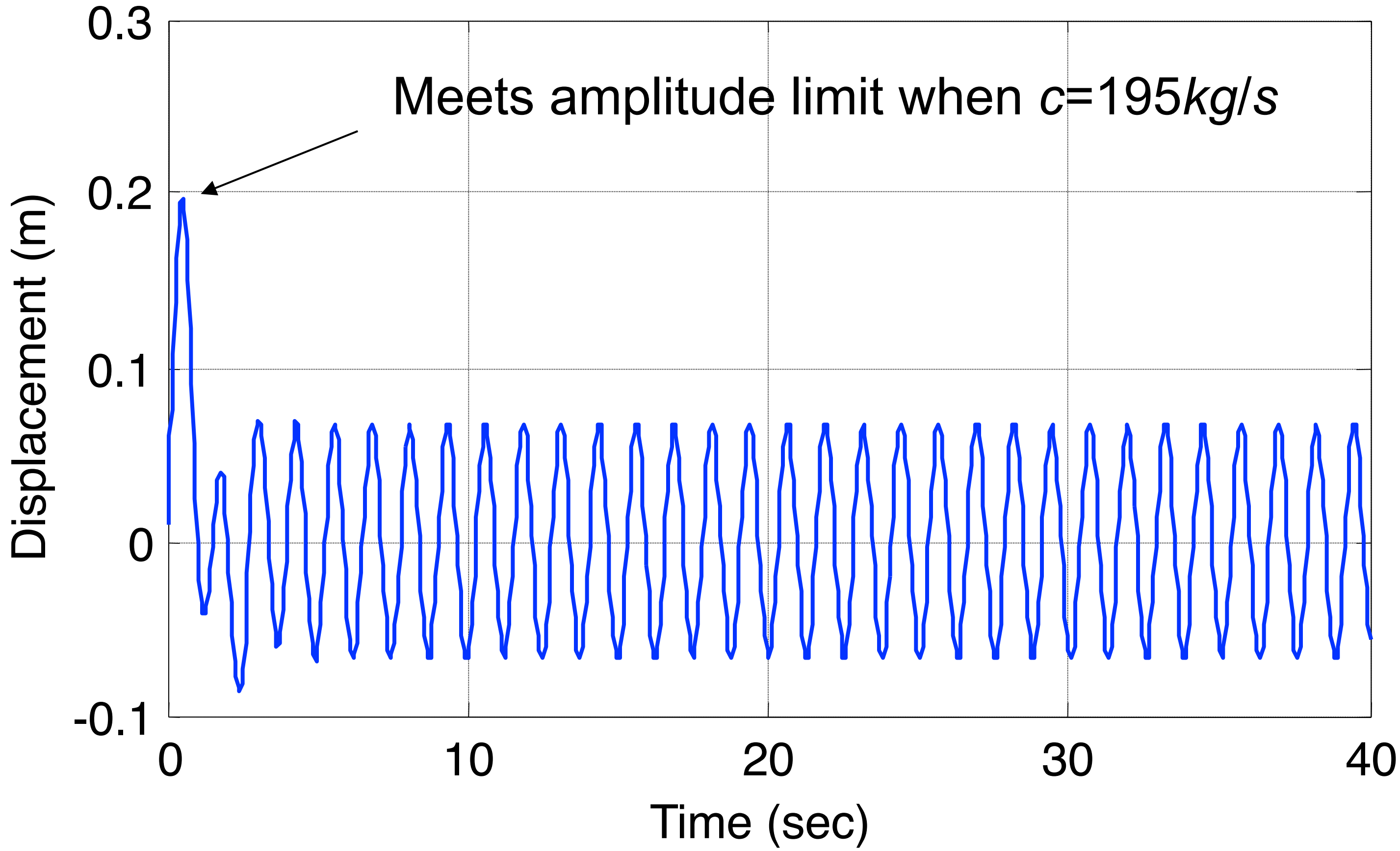
```
>>TSPAN=[0 40];
>> Y0=[0.01;0.5];
>>[t,y] = ode45('num_for',TSPAN,Y0);
>> plot(t,y(:,1))
>> xlabel('Time (sec)')
>> ylabel('Displacement (m)')
>> grid
```

**Rerun this code, increasing *c* each time until a response that satisfies the design limits results.**

# Response of the board is:



Solution: code it, plot it, and change $c$ until the desired response bound is obtained.



# 2.9 Nonlinear Response Properties

- More than one equilibrium
- Steady state depends on initial conditions
- Period depends on I.C. and amplitude
- Sub and super harmonic resonance
- No superposition
- Harmonic input resulting in nonperiodic motion
- Jumps appear in response amplitude

# Computing the forced response of a non-linear system

A non-linear system has a equation of motion given by:

$$\ddot{x}(t) + f(x, \dot{x}) = f_0 \cos \omega t$$

Put this expression into state-space form:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -f(x_1, x_2) + f_0 \cos \omega t \end{cases}$$

In vector form:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}) + \mathbf{f}(t)$$

# Numerical Form

Vector of nonlinear dynamics

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x_2(t) \\ -f(x_1, x_2) \end{bmatrix}$$

Input force vector

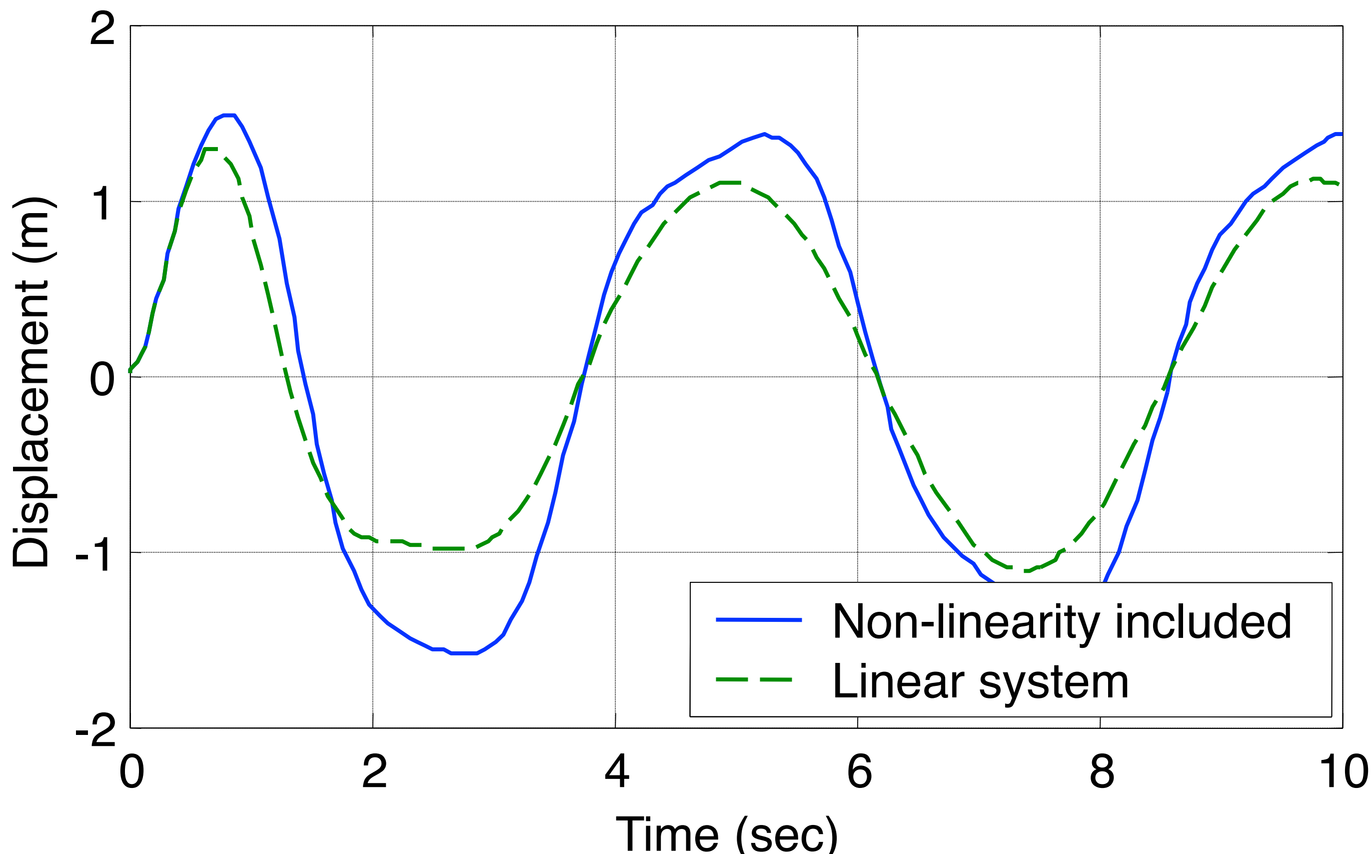
$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ f_0 \cos(\omega t) \end{bmatrix}$$

Euler equation is

$$\mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + \mathbf{F}(\mathbf{x}(t_i))\,\Delta t + \mathbf{f}(t_i)\,\Delta t$$

# **Example 2.9.1.** Cubic nonlinear spring

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x + \beta x^3 = f_0 \cos\omega t$$

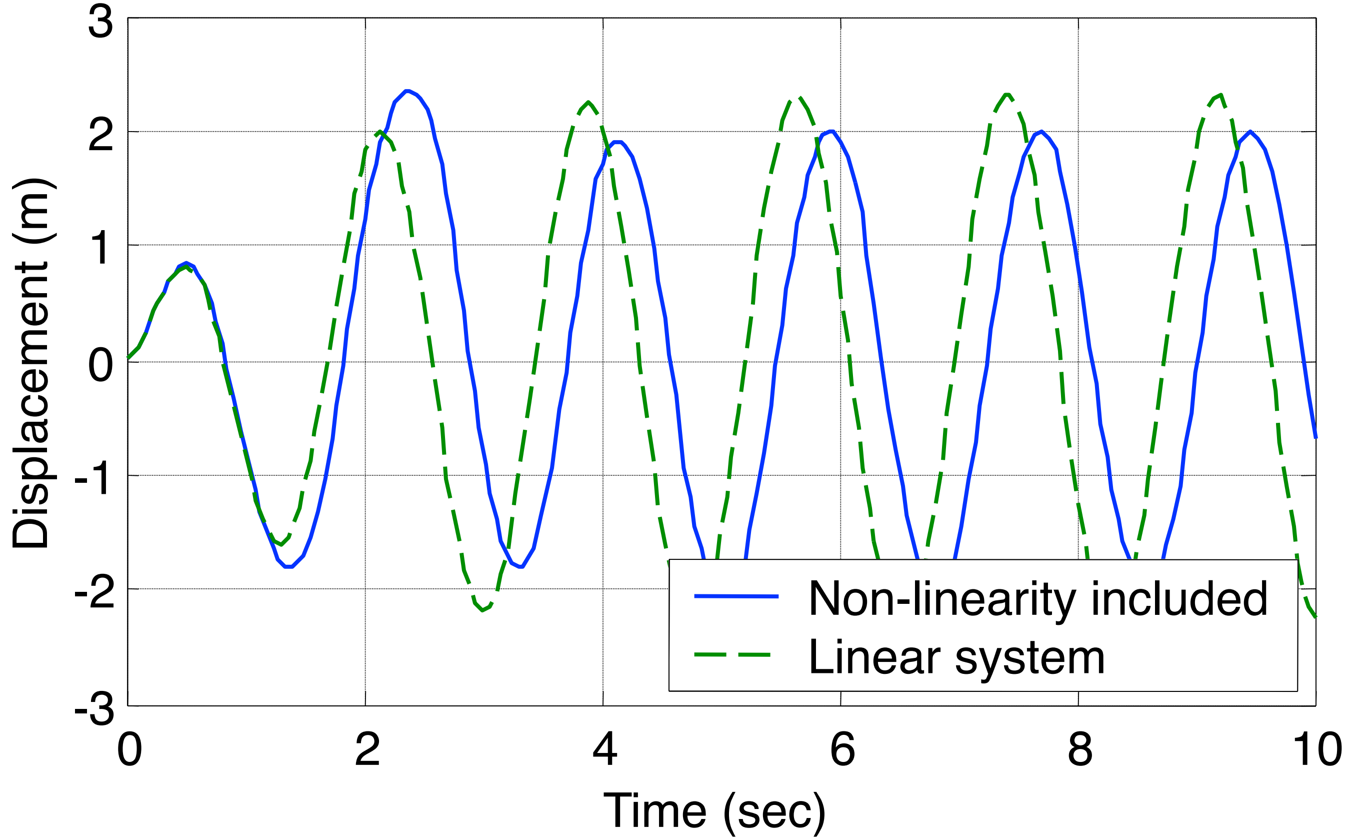



Subharmonic resonance

$$\omega = \frac{\omega_n}{2.964}$$

## **Example.** Cubic nonlinear spring near resonance

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x + \beta x^3 = f_0 \cos\omega t$$




Response near linear resonance $\qquad \omega = \dfrac{\omega_n}{1.09}$

# Chapter 3: General forced response

So far, all of the driving forces have been sine or cosine excitations

In this chapter we examine the response to any form of excitation such as

- Impulse
- Sums of sines and cosines
- Any integrable function

<u>Linear Superposition</u> allows us to break up complicated forces into sums of simpler forces, compute responses, and add to get the total solution

If $x_1$ and $x_2$ are solutions of a linear homogeneous equation, then

$$x = a_1 x_1 + a_2 x_2$$

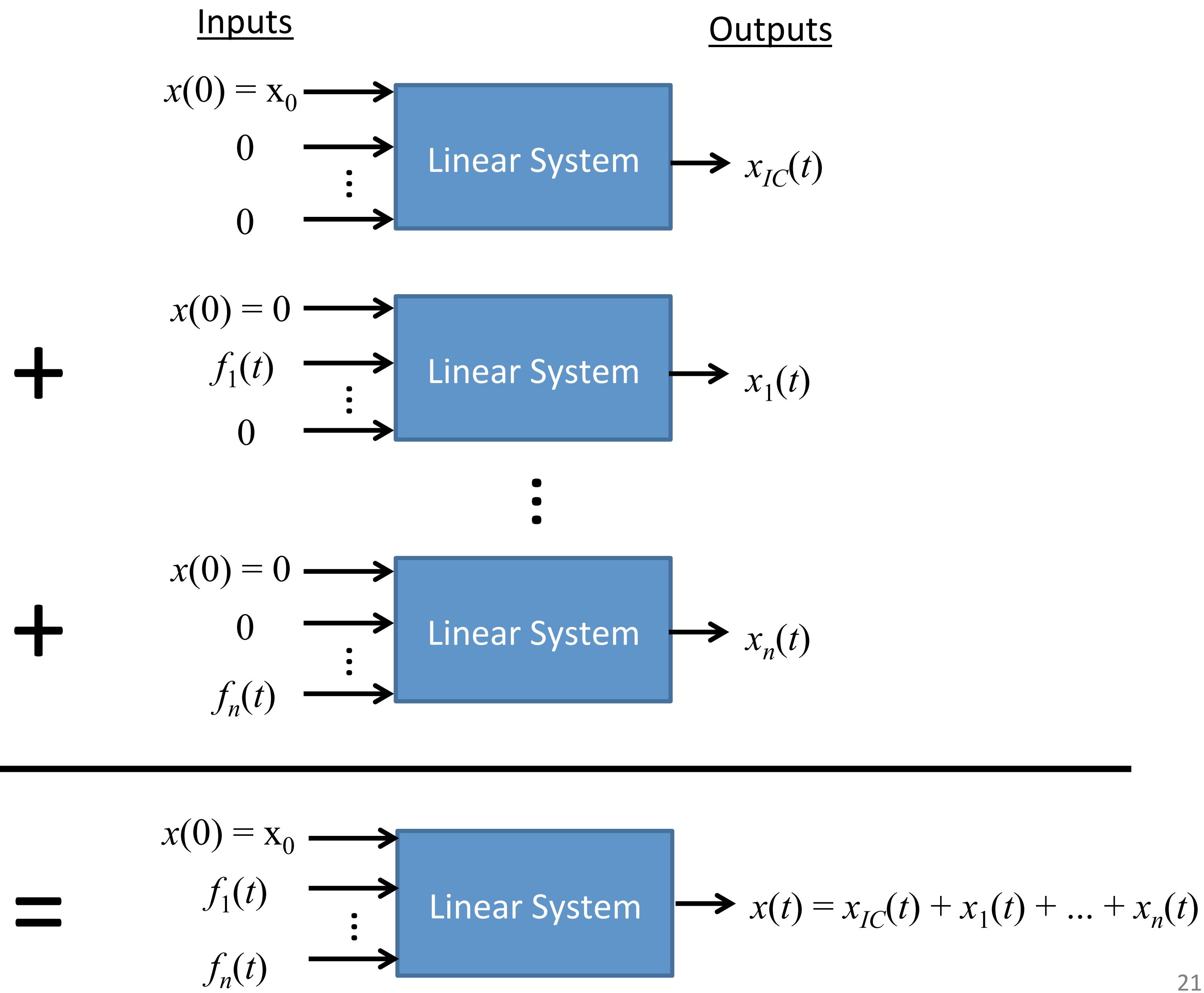
is also a solution.

If $x_1$ is the particular solution of $\ddot{x} + \omega_n^2 x = f_1$

and $x_2$ is the particular solution of $\ddot{x} + \omega_n^2 x = f_2$

then the solution to $\ddot{x} + \omega_n^2 x = a_1 f_1 + a_2 f_2$ is $a_1 x_1 + a_2 x_2$

# Linear superposition in a nutshell



# 3.1 Impulse Response Function

Consider the function $F(t)$ defined as

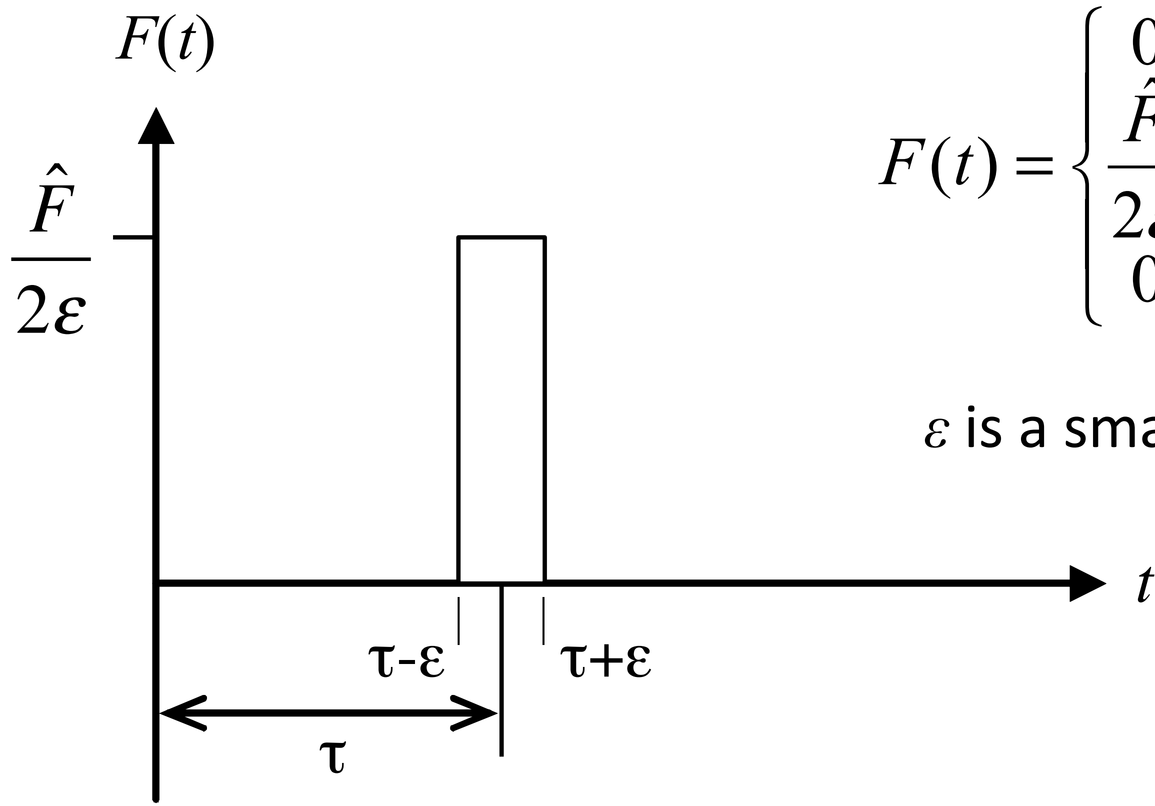
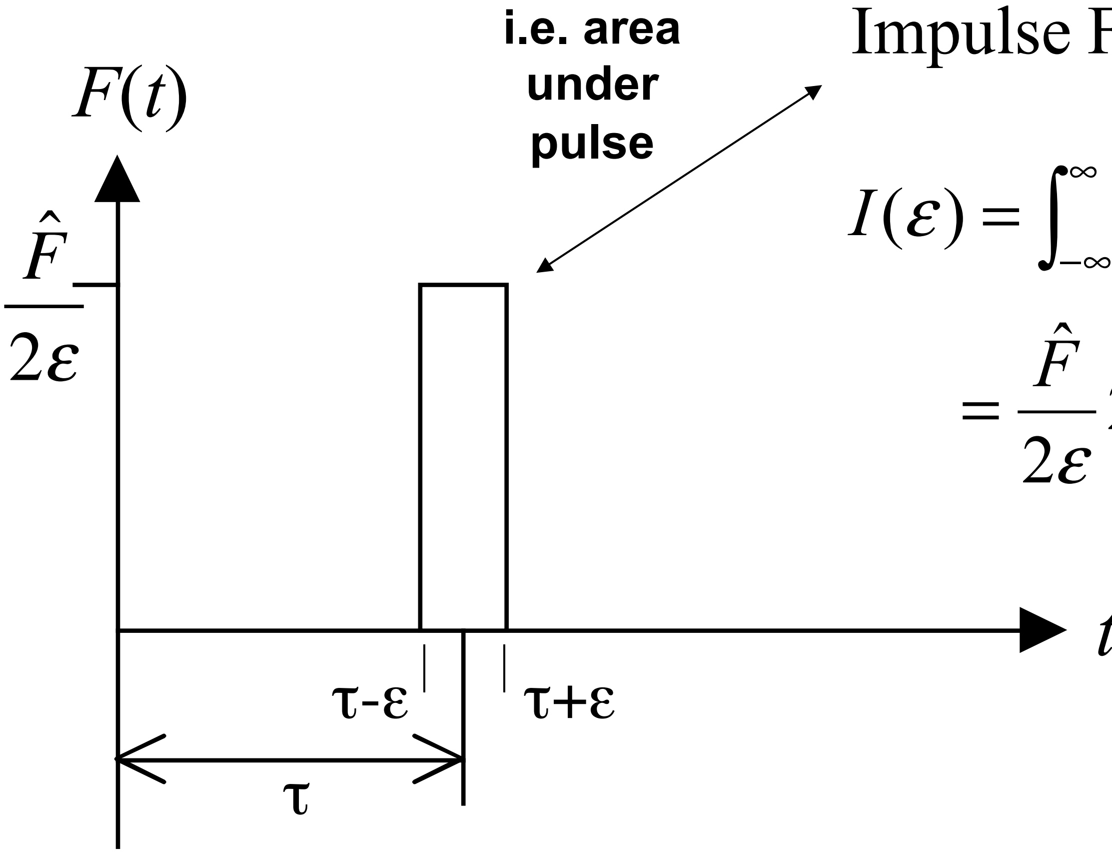



Figure 3.1

$$F(t) = \begin{cases} 0 & t < \tau - \varepsilon \\ \dfrac{\hat{F}}{2\varepsilon} & \tau - \varepsilon < t < \tau + \varepsilon \\ 0 & t > \tau + \varepsilon \end{cases}$$

$\varepsilon$ is a small positive number

From sophomore dynamics, the impulse imparted to an object is equal to the change in the object's momentum

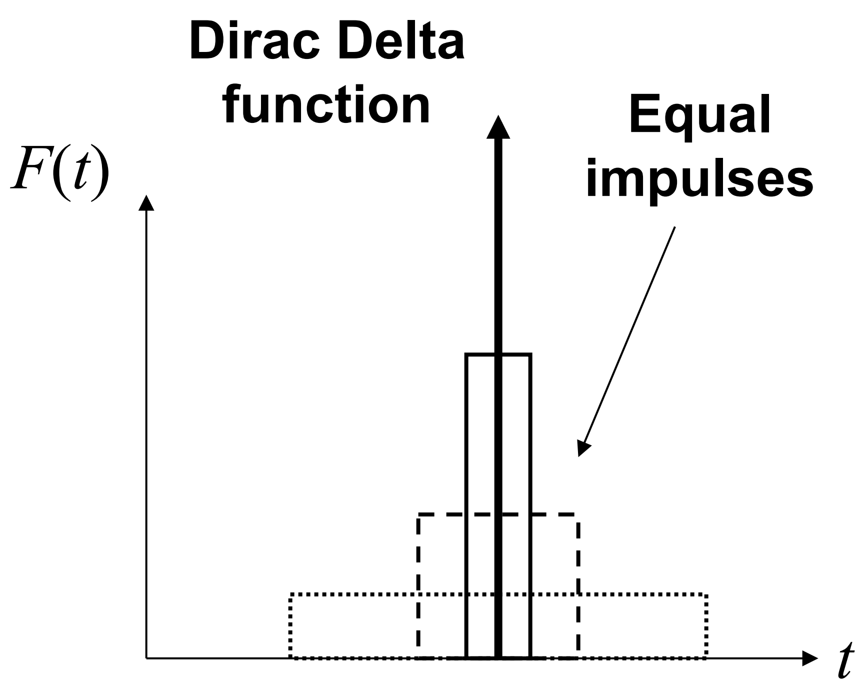



$$\text{Impulse Force} = \int F(t)dt$$

$$I(\varepsilon) = \int_{-\infty}^{\infty} F(t)dt = \int_{\tau-\varepsilon}^{\tau+\varepsilon} F(t)dt$$

$$= \frac{\hat{F}}{2\varepsilon} 2\varepsilon = \hat{F}$$

# We use the properties of impulse to define the impulse function




$$F(t-\tau)=0, \quad t\neq\tau$$

$$\int_{-\infty}^{\infty} F(t-\tau)dt=\hat{F}$$

If $\hat{F}$ = 1, this is the Dirac Delta $\delta(t)$

$$\delta(t-\tau)=0, \quad t\neq\tau$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)dt=1$$

$$\int_{-\infty}^{\infty} \hat{F}\ \delta(t-\tau)dt=\hat{F}$$

The effect of an impulse on a system is related to its change in momentum

$$\overbrace{F\Delta t = \Delta(mv)}^{\text{Impulse = change in momentum}} = m[v(\overset{\text{Just after impulse}}{t_0^+}) - \cancel{v(\overset{\text{Just before impulse}}{t_0^-})}]$$

$$\hat{F} = mv_0 \Rightarrow v_0 = \frac{\hat{F}}{m} = \frac{F\Delta t}{m}$$

**Thus the *response to impulse* with zero IC is equal to the *free response* with IC: $x_0$=0 and $v_0$ =$F\Delta t/m$**

Recall that the free response of a mass-spring-damper system to just non-zero initial conditions

The solution of:

$$m\ddot{x} + c\dot{x} + kx = 0 \quad x(0) = x_0 \quad \dot{x}(0) = v_0$$

in underdamped case:

$$x(t) = \frac{\sqrt{\left(v_0 + \zeta\omega_n x_0\right)^2 + \left(x_0\omega_d\right)^2}}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t + \tan^{-1} \frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0})$$

For $x_0 = 0$ this becomes:

$$x(t) = \frac{v_0 e^{-\zeta\omega_n t}}{\omega_d} \sin\omega_d t$$

# Next compute the response to $x(0)=0$ and $v(0) = F\Delta t/m$

The solution of

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x(0) = 0$$

$$\dot{x}(0) = F\Delta t / m = \hat{F} / m$$

in the underdamped case from the previous slide is

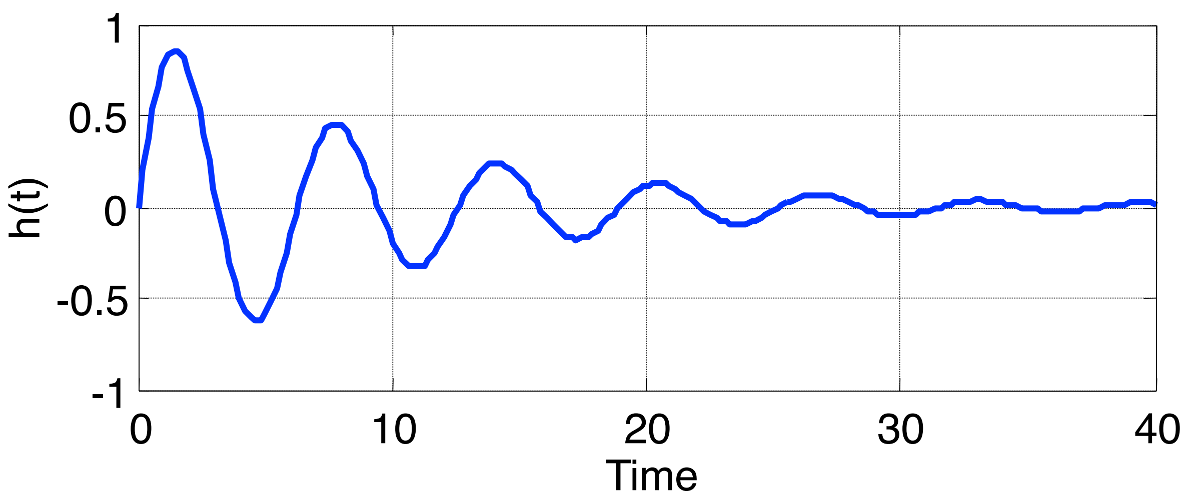
$$x(t) = \hat{F} \underbrace{\frac{e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t}_{\text{Impulse response: } h(t)}$$

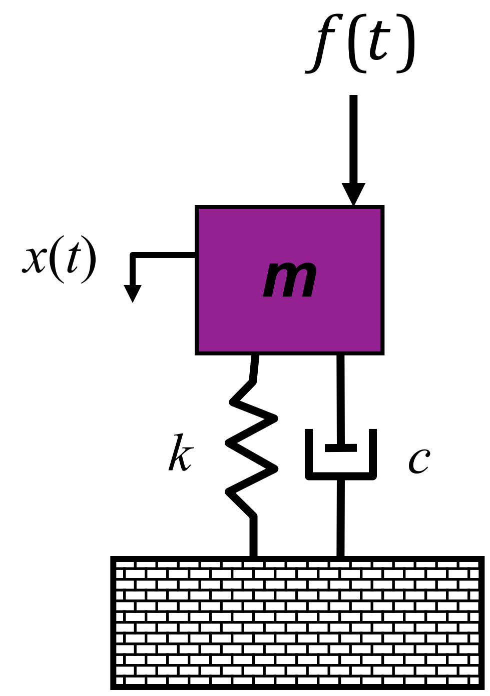
$$\Rightarrow x(t) = \hat{F}\, h(t)$$

So for an underdamped system the impulse response is $(x_0 = 0)$

$$x(t) = \frac{\hat{F} e^{-\zeta\omega_n t}}{m\omega_d} \sin\omega_d t \quad \text{(response to } \hat{F}\text{)} \qquad (3.6)$$

$$x(t) = \hat{F} h(t), \text{ where } \underbrace{h(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} \sin\omega_d t}_{\text{unit impulse response function}} \qquad (3.8)$$




**Response to an impulse at $t = 0$ and zero initial conditions**

The response to an impulse is thus defined in terms of the impulse response function, $h(t)$.

So, the response to $\delta(t)$ is given by $h(t)$

$$h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

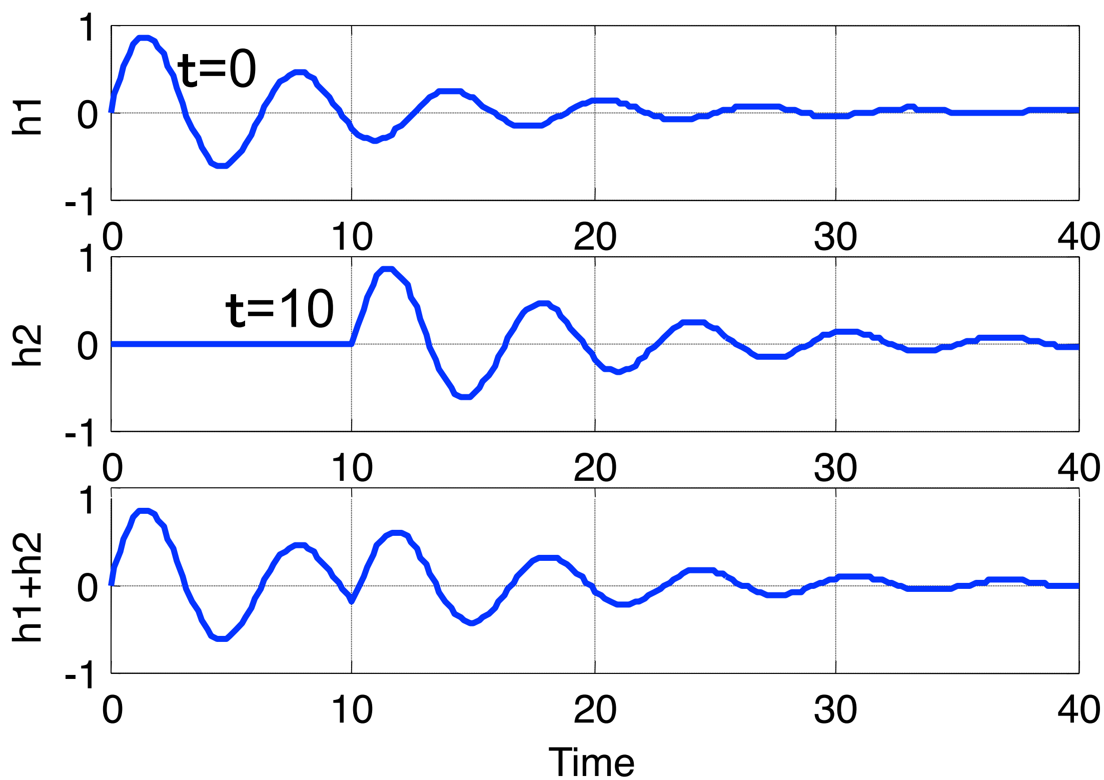
What is the response to a unit impulse applied at a time different than zero?

The response to $\delta(t\text{-}\tau)$ is $h(t\text{-}\ \tau)$

$$h(t) = \begin{cases} 0 & t < \tau \\ \dfrac{1}{m\omega_d} e^{-\zeta\omega_n (t-\tau)} \sin(\omega_d (t-\tau)) & t > \tau \end{cases}$$

For the case that the impulse occurs at $\tau$ note that the effects of non-zero initial conditions and other forcing terms must be super imposed on this solution (see Equation 3.9)

**For example: If two pulses occur at two different times then their impulse responses will superimpose**



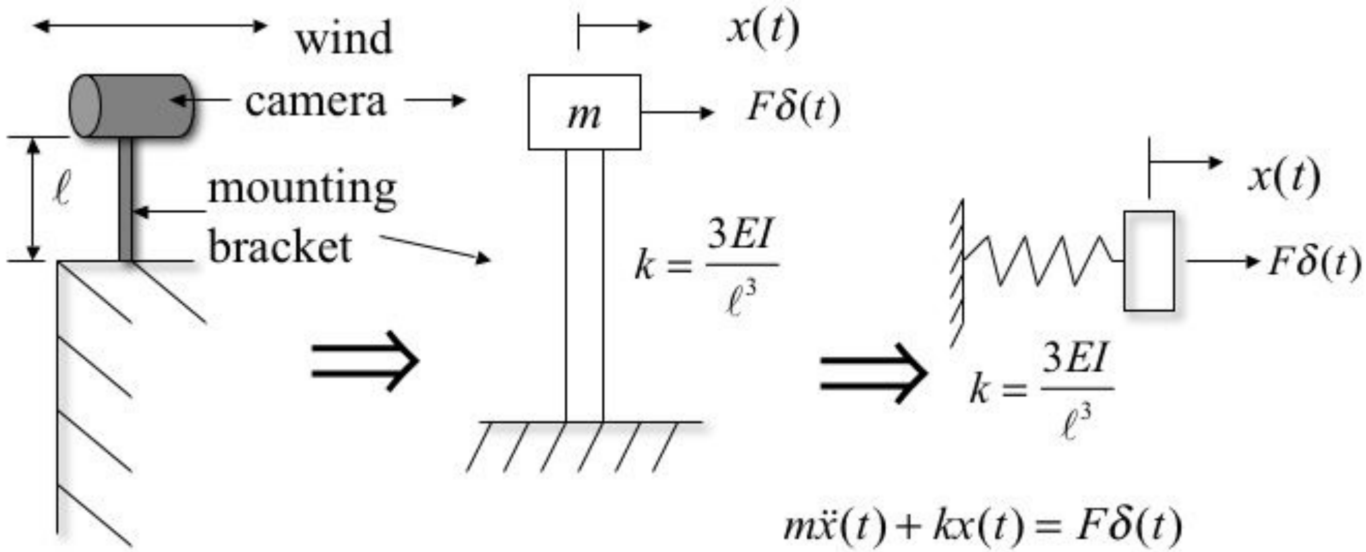
# Consider the undamped impulse response

Setting $\zeta = 0$ in Equation 3.8, the response to a unit impulse applied at $t = \tau$, i.e. $\delta(t\text{-}\tau)$ is

$$h(t-\tau) = \frac{1}{m\omega_n}\sin(\omega_n(t-\tau))$$

## **Example 3.1.2.** Design a camera mount with a vibration constraint

Consider Example 2.1.3 of the security camera again only this time with an impulsive load




<u>Camera</u>
$m_c = 3$ kg

<u>Mount</u>
$l = 0.55$ m
$b = 0.02$ m
$h = 0.02$ m

<u>Bird</u>
$m_b = 1$ kg
$v = 72$ kmh

Camera displacement cannot exceed 0.01 m

Using the stiffness and mass parameters of Example 2.1.3, does the system stay with in vibration limits if hit by a 1 kg bird traveling at 72 kmh?

The natural frequency of the camera system is

$$\omega_n = \sqrt{\frac{k}{m_c}} = \sqrt{\frac{3EI}{m_c \ell^3}} = \sqrt{\frac{3Ebh^3}{12 m_c \ell^3}}$$

$$\omega_n = \sqrt{\frac{3\,(7.1\times 10^{10}\,\mathrm{N/m})(0.02\,\mathrm{m})(0.02\,\mathrm{m})^3}{12\,(3\,\mathrm{kg})(0.55\,\mathrm{m})^3}} \quad = 75.4 \text{ rad/s}$$

Magnitude of the impulse is $F\Delta t = mv_b$

From equations (3.7) and (3.8) with $\zeta = 0$, the impulsive response is:

$$x(t) = \frac{F\Delta t}{m_c \omega_n} \sin \omega_n t = \frac{m_b v}{m_c \omega_n} \sin \omega_n t$$

The magnitude of the response due to the impulse is thus $X = \left| \dfrac{m_b v}{m_c \omega_n} \right|$

Next compute the momentum of the bird to complete the magnitude calculation

$$m_b v = 1\text{ kg} \cdot 72\,\frac{\text{km}}{\text{hr}} \cdot \frac{1000\text{ m}}{\text{km}} \cdot \frac{1\text{ hr}}{3600\text{ s}} = 20\text{ kg m/s}$$

Next use this value in the expression for the maximum value:

$$X = \left|\frac{m_b v}{m_c \omega_n}\right| = \left|\frac{20\text{ kg m/s}}{3\text{ kg} \cdot 75.45\text{ rad/s}}\right| = \underline{0.088\text{ m}}$$

This max value exceeds the camera tolerance

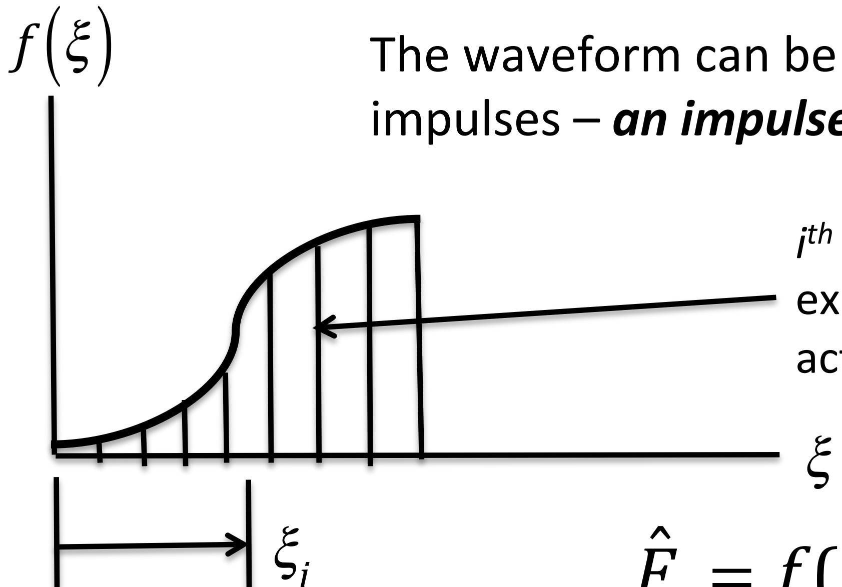
# Using the impulse response function to define response to an arbitrary input

- For any arbitrary input, we can use the impulse response function to define the system response.
- Recall the principle of linear superposition:

$$x(t) = \sum_{i=1}^{N} x_i(t) \qquad \text{where}$$

$$x_i(t) = \text{response to the } i^{th} \text{ input}$$

Consider an ***arbitrary*** waveform that is broken into discrete time-based elements:




The waveform can be described as a series of impulses – ***an impulse train***

$i^{th}$ impulse (width exaggerated – it is actually a delta function)

$$\hat{F}_i = f(\xi_i)(2\varepsilon)$$

**Where $\hat{F}_i$ is the magnitude (area) of the *$i^{th}$* impulse**

As the width of the impulse decreases to a delta function, we can express the forcing function as:

$$f(\xi)(2\varepsilon)_{t=\xi} \Rightarrow \hat{F}(\xi)\delta(t-\xi)$$

The response to the arbitrary waveform can be determined by the superposition of impulse response functions:

For an underdamped system, the response to a ***single*** impulse at $\xi$ is:

$$x(t) = \hat{F}(\xi)h(t-\xi) = \frac{\hat{F}(\xi)}{m\omega_d}e^{-\zeta\omega_n(t-\xi)}\sin\omega_d(t-\xi)$$

Applying calculus and considering in the limit:

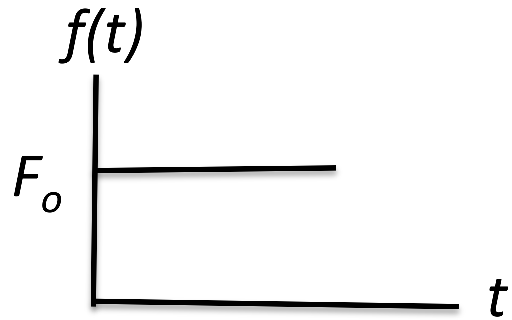
$$\hat{F}(\xi) = f(\xi)d\xi$$

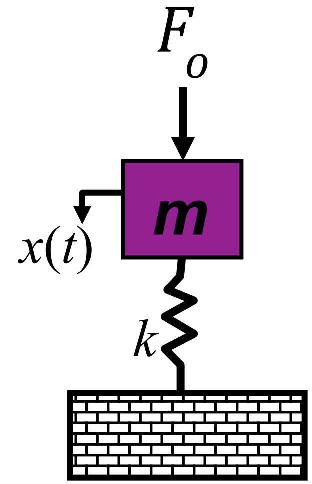
The response to an arbitrary waveform becomes:

$$x(t) = \int_{\xi=0}^{t} f(\xi)h(t-\xi)d\xi$$

Which is called the ***convolution integral***

Consider the step response for an undamped system analyzed using the convolution integral:

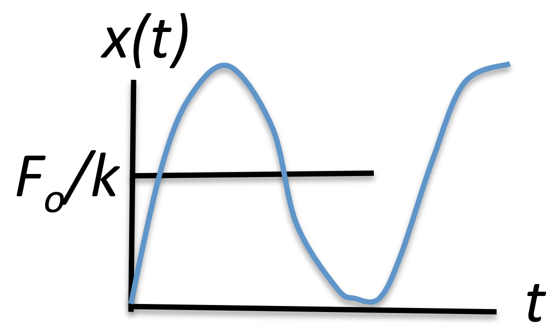


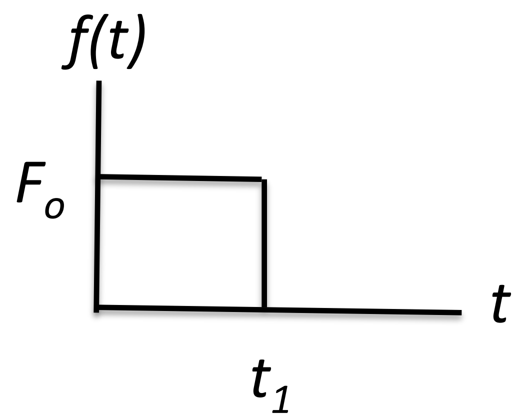

$$x(t) = \int_{\xi=0}^{t} f(\xi)h(t-\xi)d\xi$$

$$= \frac{1}{m\omega_n}\int_0^t F_o \sin\omega_n(t-\xi)d\xi = \frac{F_o}{m\omega_n}\left(\frac{1}{\omega_n}\cos\omega_n(t-\xi)\Big|_0^t\right)$$

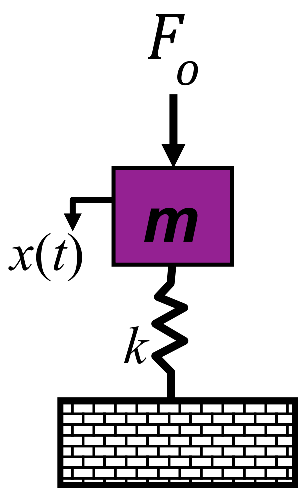
$$= \frac{F_o}{k}\left(1-\cos\omega_n t\right)$$



# Write the convolution integral equation for a pulse (undamped system):

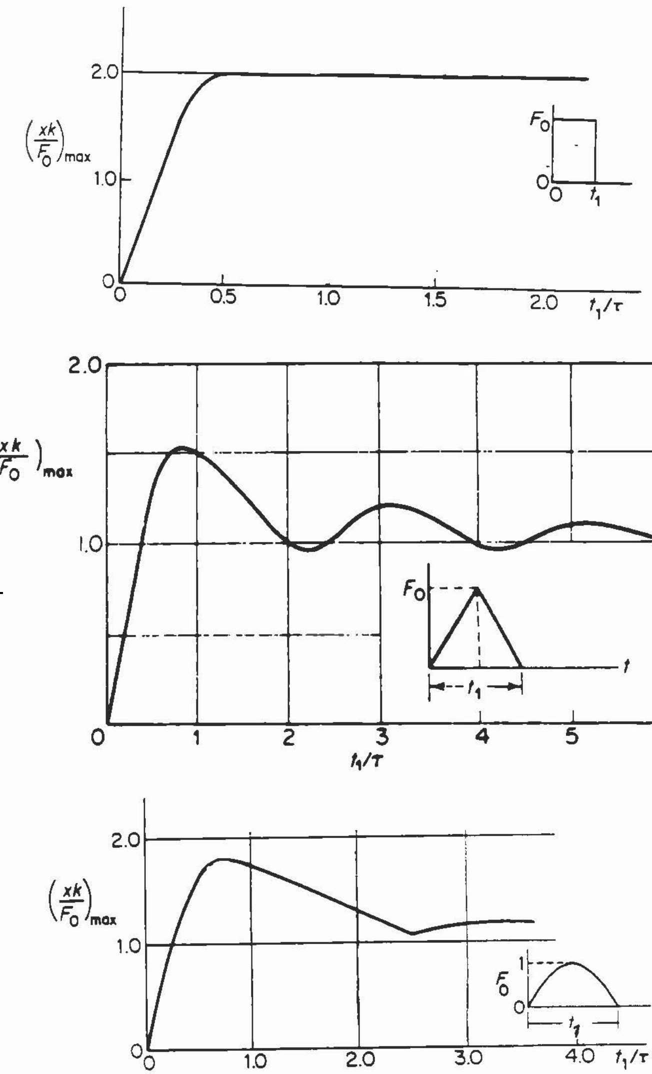


# The shock response spectra define a maximum response envelope for a given input waveform




$$\tau = \frac{2\pi}{\omega_n}$$

SHOCK RESPONSE SPECTRUMS FOR OTHER IMPULSE EXCITATIONS ARE GIVEN BELOW.

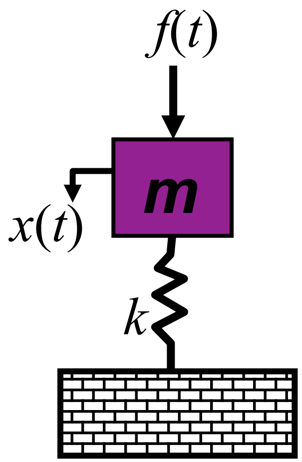


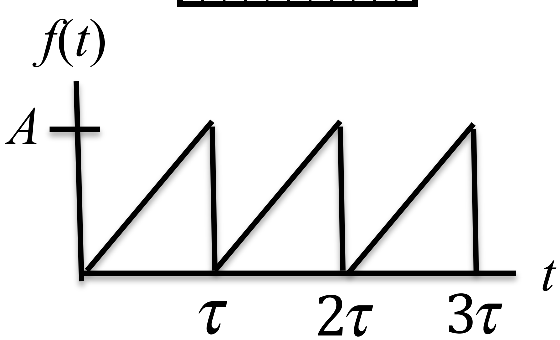

(11.19)

# Use of the Fourier Series for periodic excitation

***Recall that any periodic waveform can be described by the Fourier series as:***

$$f(t) = \frac{A_o}{2} + \sum_{p=1}^{\infty} A_p \cos p\omega t + \sum_{p=1}^{\infty} B_p \sin p\omega t$$




$$A_o = \frac{\omega}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\,dt$$

$$A_p = \frac{\omega}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\cos p\omega t\,dt$$

$$B_p = \frac{\omega}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\sin p\omega t\,dt$$

# Use of the Fourier Series for periodic excitation

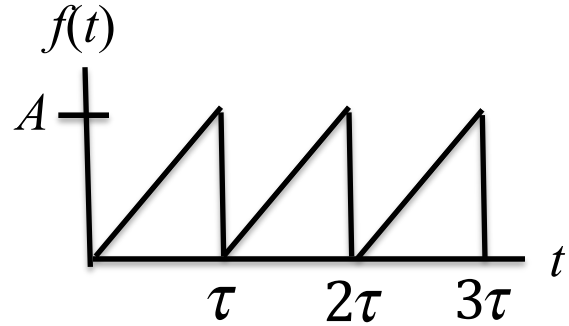
A total solution that includes both homogeneous and particular forms can be generically written:

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \frac{A_o}{2k} + \sum_{p=1}^{\infty} x_{\cos p}(t) + \sum_{p=1}^{\infty} x_{\sin p}(t)$$

*A and phi are determined from the initial conditions and also the forcing function*

*The principle of superposition is used again to define the resulting response*

Example for sawtooth wave:




**Fourier series results in:**

$$f(t) = \frac{A}{2} - \frac{A}{\pi}\sin\omega t - \frac{A}{2\pi}\sin 2\omega t - \ldots$$

and

$$x(t) = x_1(t) + x_2(t) + x_3(t) + \ldots$$

Where the response to the first harmonic is:

$$x_2(t) = \frac{A}{\pi M \omega_n}\left[\frac{\omega_n \sin\omega t - \omega \sin\omega_n t}{\omega_n^2 - \omega^2}\right]$$