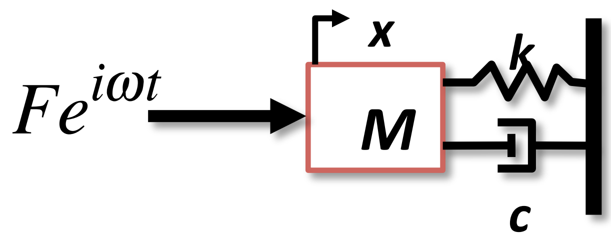
2.3.2 Complex response method

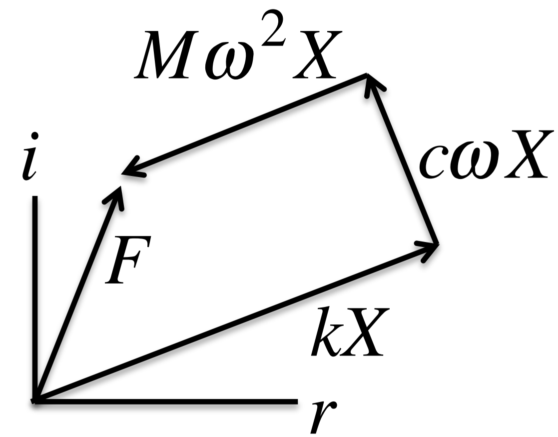
- So far, we have used a sin or cos function to represent a harmonic forcing function
  - ***But actually, there is a better way!***

$$m\ddot{x} + c\dot{x} + kx = F\cos\omega t$$



Can be written:

$$m\ddot{x} + c\dot{x} + kx = Fe^{i\omega t}$$



*Where the **real** part of the harmonic function is the cos function and is a projection on the real axis*

Complex response method

- Now assume the forced response x_p takes the same form as the forcing function:

$$x_p = Xe^{i\omega t}$$

*But in this case, the magnitude of the harmonic response **X** is complex. The derivatives are:*

$$\dot{x}_p = i\omega X e^{i\omega t}$$

$$\ddot{x}_p = -\omega^2 X e^{i\omega t}$$

Complex response method

- Plugging into the original DE:

$$\left(-m\omega^2 + ic\omega + k\right)Xe^{i\omega t} = Fe^{i\omega t}$$

$$\frac{X}{F} = \frac{1}{-m\omega^2 + ic\omega + k}$$

or:

$$X = \frac{F/k}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + \dfrac{ic\omega}{k}}$$

$$\frac{X}{F/k} = \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + i2\zeta\dfrac{\omega}{\omega_n}}$$

Complex response method

Now we can find the real and imaginary parts of ***X*** by multiplying and dividing by the complex conjugate:

$$\frac{X}{F/k}=\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+i2\zeta\frac{\omega}{\omega_n}}\left(\frac{1-\left(\frac{\omega}{\omega_n}\right)^2-i2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2-i2\zeta\frac{\omega}{\omega_n}}\right)=\frac{1-\left(\frac{\omega}{\omega_n}\right)^2-i2\zeta\frac{\omega}{\omega_n}}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

Expressing the real and imaginary parts as magnitude and phase results in:

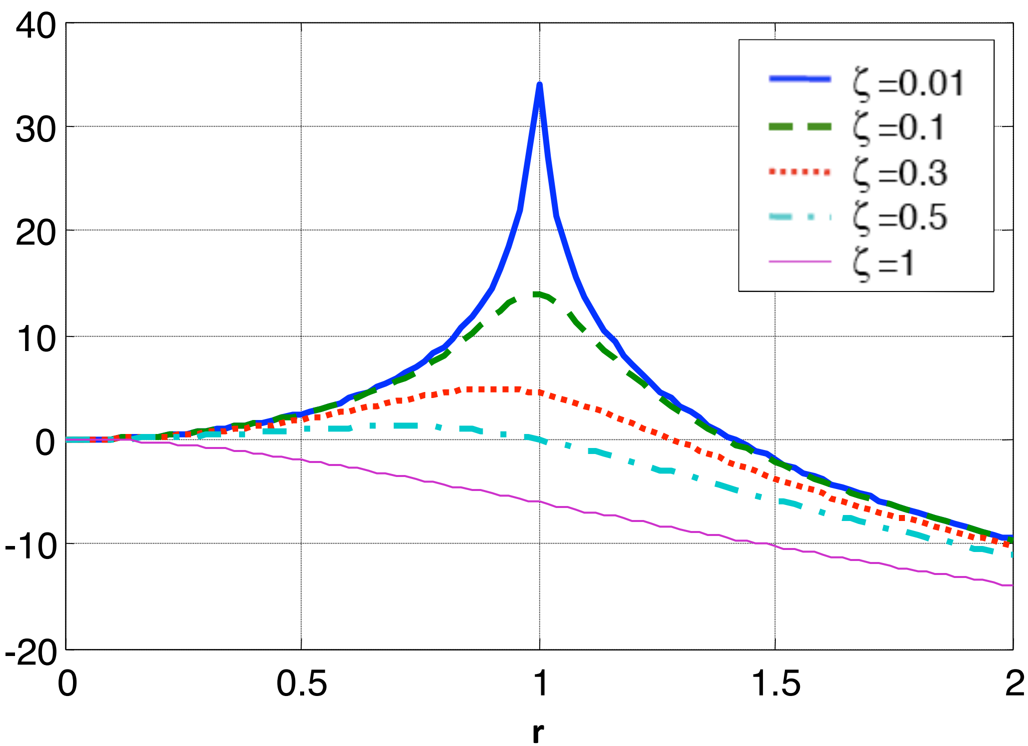
$$\left|\frac{X}{F/k}\right|=\frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}}\qquad \phi=\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)\quad \text{(lag)}$$

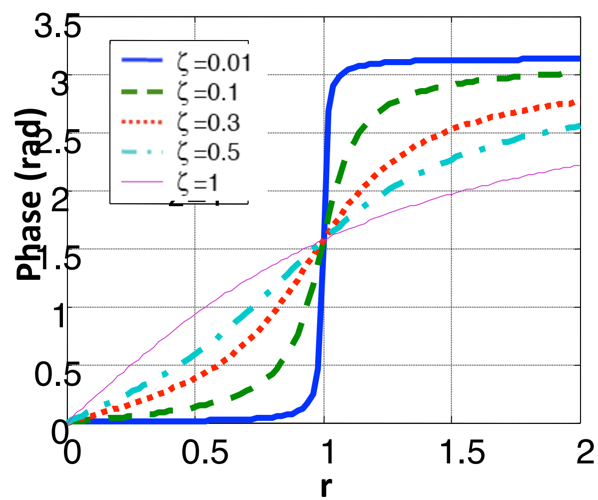
Complex response method

$$\left|\frac{X}{F/k}\right| = \left|\frac{X(\omega)}{F/k}\right| = H(\omega) = \text{Frequency response function, or FRF}$$

$$H(\omega) = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$H(\omega)$ in dB





Section 2.3.3 Transfer Function Methods

The Laplace Transform

- Changes ODE into algebraic equation
- Solve algebraic equation then compute the inverse transform
- Rule and table based in many cases
- Is used extensively in control analysis to examine the response
- Related to the frequency response function

The Laplace Transform approach:

- See appendix B and section 3.4 for details
- Transforms the time variable into an algebraic, complex variable
- Transforms differential equations into an algebraic equation
- Related to the frequency response method

$$X(s) = \mathcal{L}(x(t)) = \int_0^{\infty} x(t) e^{-st} dt$$

The Laplace Transform approach:

Take the transform of the equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \Rightarrow$$

$$(ms^2 + cs + k)X(s) = \frac{F_0 s}{s^2 + \omega^2}$$

Now solve algebraic equation in *s* for *X*(*s*)

$$X(s) = \frac{F_0 s}{(ms^2 + cs + k)(s^2 + \omega^2)}$$

To get the time response this must be "inverse transformed"

Transfer Function Method

With zero initial conditions:

$$(ms^2 + cs + k)X(s) = F(s) \Rightarrow$$

$$\frac{X(s)}{F(s)} = H(s) = \frac{1}{ms^2 + cs + k} \qquad (2.59)$$

The transfer function

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \qquad (2.60)$$

$$= \text{frequency response function}$$

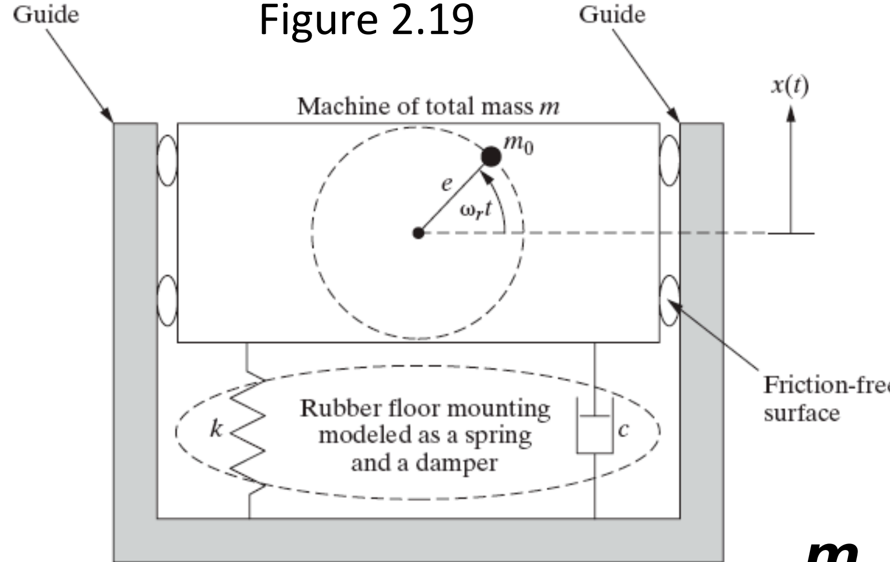
TABLE B.1 PARTIAL LIST OF FUNCTIONS AND THEIR LAPLACE TRANSFORMS WITH ZERO INITIAL CONDITIONS AND $t > 0$

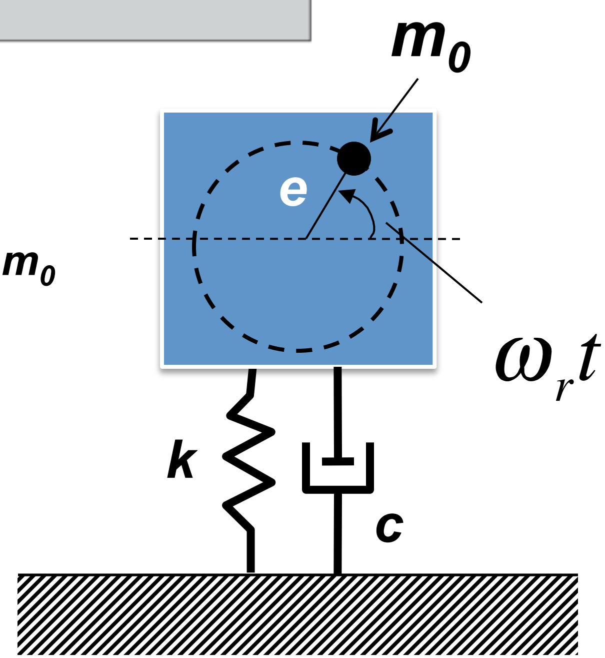
	$F(s)$	$f(t)$
(1)	1	$\delta(t_0)$ unit impulse at t_0
(2)	$\frac{1}{s}$	1, unit step
(3)	$\frac{1}{s+a}\left(\frac{1}{s-a}\right)$	$e^{-at}(e^{at})$
(4)	$\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at}-e^{-bt})$
(5)	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$
(6)	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$
(7)	$\frac{1}{s(s^2+\omega^2)}$	$\frac{1}{\omega^2}(1-\cos \omega t)$
(8)	$\frac{1}{s^2+2\zeta\omega s+\omega^2}$	$\frac{1}{\omega_d} e^{-\zeta\omega t} \sin \omega_d t,\ \zeta < 1,\ \omega_d = \omega\sqrt{1-\zeta^2}$
(9)	$\frac{\omega^2}{s(s^2+2\zeta\omega s+\omega^2)}$	$1 - \frac{\omega}{\omega_d} e^{-\zeta\omega t} \sin(\omega_d t + \phi),\ \phi = \cos^{-1}\zeta,\quad \zeta < 1$
(10)	$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!},\ n = 1, 2 \ldots$
(11)	$\frac{n!}{(s-\omega)^{n+1}}$	$t^n e^{\omega t},\ n = 1, 2 \ldots$
(12)	$\frac{1}{s(s+\omega)}$	$\frac{1}{\omega}(1-e^{-\omega t})$
(13)	$\frac{1}{s^2(s+\omega)}$	$\frac{1}{\omega^2}(e^{-\omega t}+\omega t-1)$
(14)	$\frac{\omega}{s^2-\omega^2}$	$\sinh \omega t$
(15)	$\frac{s}{s^2-\omega^2}$	$\cosh \omega t$
(16)	$\frac{1}{s^2(s^2+\omega^2)}$	$\frac{1}{\omega^3}(\omega t - \sin \omega t)$
(17)	$\frac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$
(18)	$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t}{2\omega} \sin \omega t$

2.5 Rotating Unbalance

- Gyros
- Cryo-coolers
- Tires
- Washing machines

Figure 2.19





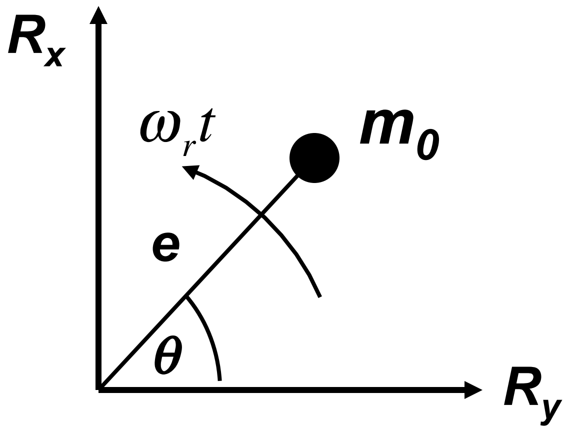
Machine of total mass *m* i.e. *m_0* included in *m*

e = eccentricity

m_o = mass unbalance

$\omega_r t$ =rotation frequency

Rotating Unbalance (cont)



What force is imparted on the structure? Note it rotates with x component:

$$x_r = e\sin\omega_r t$$

$$\Rightarrow a_x = \ddot{x}_r = -e\omega_r^2 \sin\omega_r t$$

From sophomore dynamics,

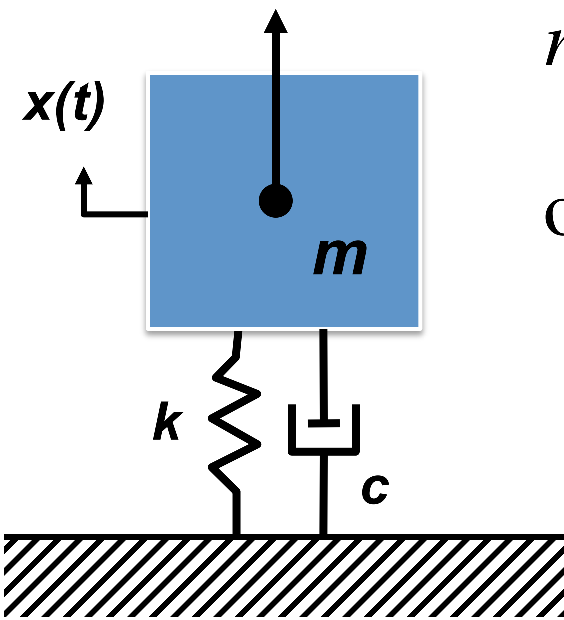
$$R_x = m_0 a_x = -m_o e\omega_r^2 \sin\theta = -m_o e\omega_r^2 \sin\omega_r t$$

$$R_y = m_0 a_y = -m_o e\omega_r^2 \cos\theta = -m_o e\omega_r^2 \cos\omega_r t$$

Rotating Unbalance (cont)

The problem is now just like any other SDOF system with a harmonic excitation

$$m_0 e\omega_r^2 \sin(\omega_r t)$$



$$m\ddot{x} + c\dot{x} + kx = m_o e\omega_r^2 \sin\omega_r t \qquad (2.82)$$

or $$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{m_o}{m} e\omega_r^2 \sin\omega_r t$$

Note the influences on the forcing function (we are assuming that the mass *m* is held in place in the *y* direction as indicated in Figure 2.19)

Rotating Unbalance (cont)

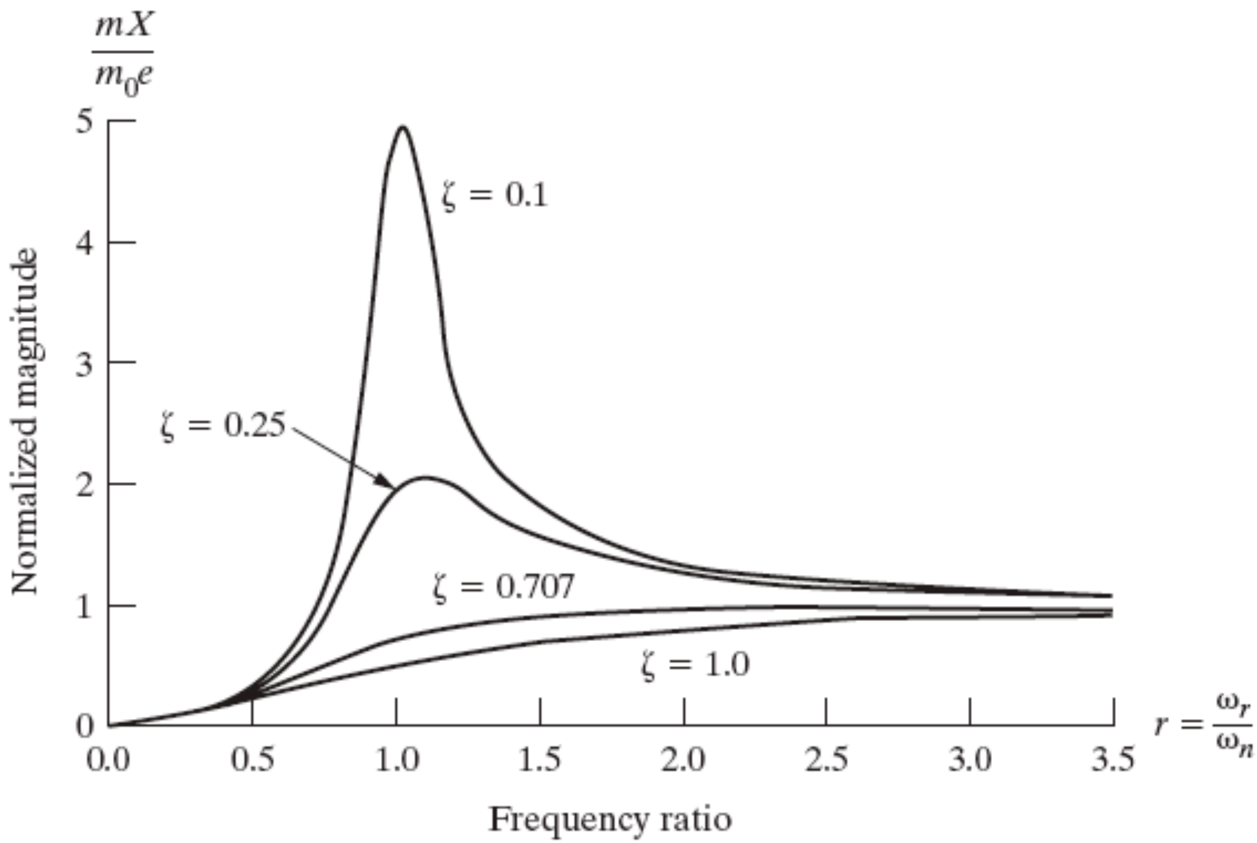
- Just another SDOF oscillator with a harmonic forcing function
- Expressed in terms of frequency ratio *r*

$$x_p(t) = X\sin(\omega_r t - \phi) \qquad (2.83)$$

$$X = \boxed{\frac{m_o e}{m}} \frac{r^2}{\sqrt{(1-r^2)^2 + \left(2\zeta r\right)^2}} \qquad (2.84)$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \qquad (2.85)$$

Figure 2.21: Displacement magnitude vs frequency caused by rotating unbalance



Example 2.5.1: Given the deflection at resonance (0.1m), ζ = 0.05 and a 10% out of balance, compute *e* and the amount of added mass needed to reduce the maximum amplitude to 0.01 m.

At resonance *r* = 1 and

$$\frac{mX}{m_0 e} = \frac{1}{2\zeta} = \frac{1}{2(0.05)} \Rightarrow 10\frac{0.1\text{ m}}{e} = \frac{1}{2\zeta} = 10 \Rightarrow \underline{e = 0.1\text{ m}}$$

Now to compute the added mass, again at resonance;

$$\frac{m}{m_0}\left(\frac{X}{0.1\text{ m}}\right) = 10$$

Use this to find Δm so that *X* is 0.01:

$$\frac{m + \Delta m}{m_0}\left(\frac{0.01\text{ m}}{0.1\text{ m}}\right) = 10 \Rightarrow \frac{m + \Delta m}{(0.1)m} = 100 \Rightarrow \underline{\Delta m = 9m}$$

Here m_0 is $10\%m$ or $0.1m$

Example 2.5.2 Helicopter rotor unbalance

Given

$$k = 1\times10^5 \text{ N/m}$$

$$m_{tail} = 60 \text{ kg}$$

$$m_{rot} = 20 \text{ kg}$$

$$m_0 = 0.5 \text{ kg}$$

$$\zeta = 0.01$$

Fig 2.22

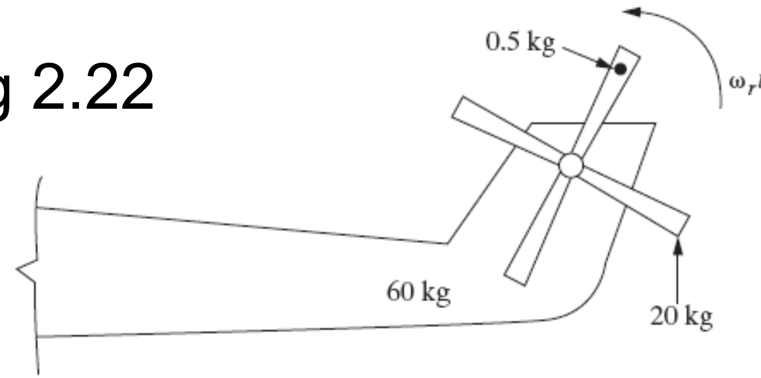
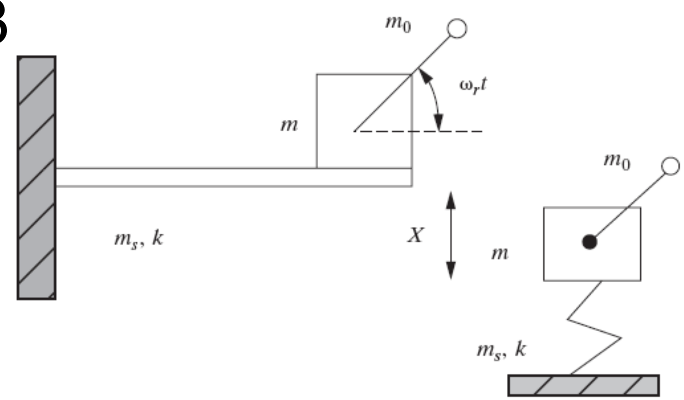


Fig 2.23



Compute the deflection at 1500 rpm and find the rotor speed at which the deflection is maximum

Example 2.5.2 Solution

The rotating mass is 20 + 0.5 or 20.5. The stiffness is provided by the Tail section and the corresponding mass is that determined in the example of a heavy beam. So the system natural frequency is

$$\omega_n = \sqrt{\frac{k}{m + \frac{33}{140} m_{tail}}} = \sqrt{\frac{10^5 \text{ N/m}}{20.5 + \frac{33}{140} 60 \text{ kg}}} = 53.72 \text{ rad/s}$$

The frequency of rotation is

$$\omega_r = 1500 \text{ rpm} = 1500 \frac{\text{rev}}{\text{min}} \frac{\text{min}}{60 \text{ s}} \frac{2\pi \text{ rad}}{\text{rev}} = 157 \text{ rad/s}$$

$$\Rightarrow \; r = \frac{157 \text{ rad/s}}{53.96 \text{ rad/s}} = 2.92$$

Now compute the deflection at r = 2.91 and ζ =0.01 using eq (2.84)

$$X = \frac{m_0 e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$= \frac{(0.5\text{ kg})(0.15\text{ m})}{34.64\text{ kg}}\frac{(2.92)^2}{\sqrt{\left(1-(2.92)^2\right)^2-\left(2(0.01)(2.92)\right)^2}} = \underline{0.002\text{ m}}$$

At around r = 1, the max deflection occurs:

$$r = 1 \Rightarrow \omega_r = 53.72\text{ rad/s} = 53.72\frac{\text{rad}}{\text{s}}\frac{\text{rev}}{2\pi\text{ rad}}\frac{60\text{ s}}{\text{min}} = \underline{515.1\text{ rpm}}$$

At r = 1:

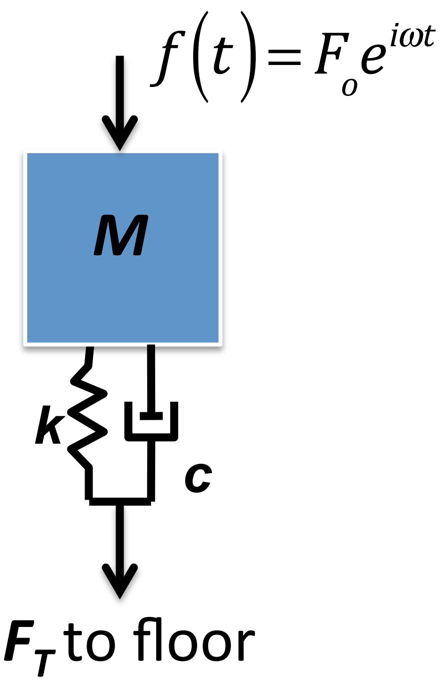
$$X = \frac{m_0 e}{m_{\text{eq}}}\frac{1}{2\zeta} = \frac{(0.5\text{ kg})(0.15\text{ m})}{34.34\text{ kg}}\frac{1}{2(0.01)} = 0.108\text{ m or }\underline{10.8\text{ cm}}$$

Force transmitted for the self-excited case

- A piece of machinery may be vibrating and transmitting force into the floor on which it is mounted
  - This is a problem if the facility cannot tolerate vibration
    - *Example: I consulted at the University of Rochester Laser Lab for Energetics (LLE) that had a problem with ventilation fan vibration causing high precision lasers to become misaligned.*

Laser drivers at LLE

f(t) could be caused by a rotating imbalance



$$F_T = \sqrt{(kX)^2 + (c\omega X)^2}$$

Force transmitted for the self-excited case

$$F_T = \sqrt{\left(kX\right)^2 + \left(c\omega X\right)^2} = kX\sqrt{1+\left(2\zeta r\right)^2}$$

and:

$$\frac{F_o}{kX} = \sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2} \text{ so that } F_o = kX\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}$$

$$\frac{F_T}{F_o} = \frac{kX\sqrt{1+\left(2\zeta r\right)^2}}{kX\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} = \frac{\sqrt{1+\left(2\zeta r\right)^2}}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}}$$

So the transmitted force ratio is the same relationship as for the displacement ratio with base motion

Force transmitted for the case of a rotating unbalance

- We can find the force transmitted due to a rotating unbalance using a procedure similar to the previous case:

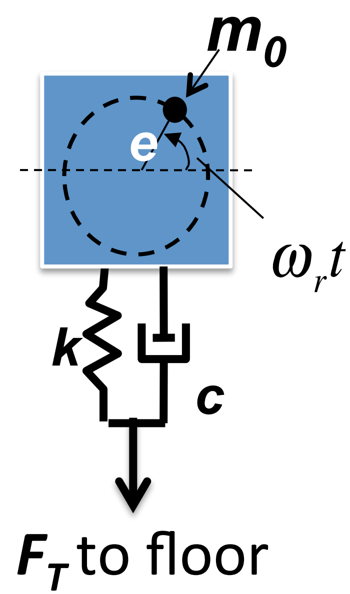
$$F_T = \sqrt{(kX)^2 + (c\omega X)^2} = kX\sqrt{1 + (2\zeta r)^2}$$

and:

$$X = \frac{e m_o r^2}{m\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

which leads to:

$$F_T = \frac{e m_o r^2 k\sqrt{1+(2\zeta r)^2}}{m\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



Accelerometers: a great application of relative motion

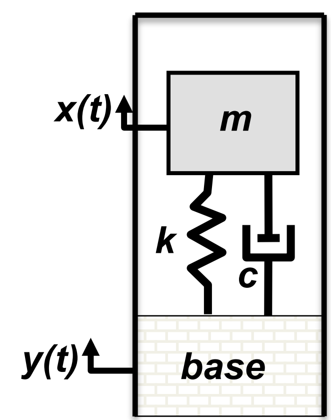
- Accelerometers work on the basis of relative motion:
  - Relative motion is the ***difference*** between the input motion (y) and the output motion (x):

$$\sum F_x = m\ddot{x} = c\left(\dot{y} - \dot{x}\right) + k\left(y - x\right)$$

$$\text{let } z = x - y$$

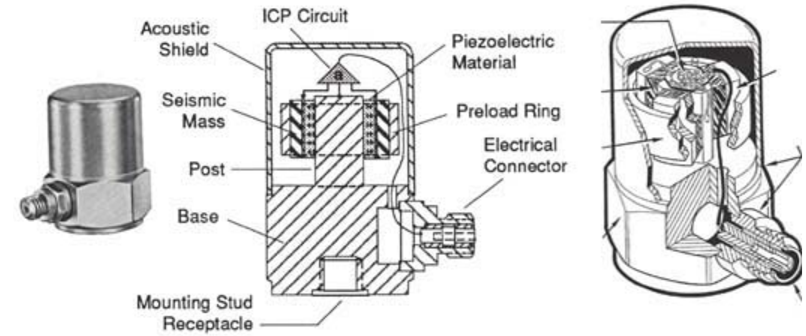
$$m\left(\ddot{z} + \ddot{y}\right) + c\dot{z} + kz = 0$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$



An accelerometer mounts to a structure and measures the acceleration of the base motion

Accelerometer function



http://www.industrial-electronics.com/DAQ/images/10_117.jpg

Now let $y = Ye^{i\omega t}$ and $z = Ze^{i\omega t}$ resulting in:

$$\left(-m\omega^2 + ic\omega + k\right)Z = -m\omega^2 Y$$

We can write the ratio of relative motion to input motion as:

$$\left|\frac{Z}{Y}\right| = \frac{r^2}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}}$$

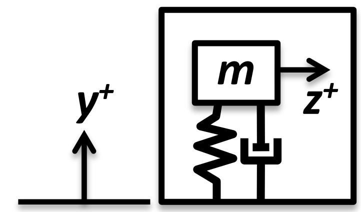
Which we note is the same equation as for a rotating unbalance

Seismic accelerometer function

- The seismic accel contains internal electronics that measure the relative motion between the accel case and an inertial mass inside the case
  – *You can think of this system as consisting of a displacement transducer connected to the inertial mass*
  – If the system natural frequency is **low** compared to the frequencies it will measure, then $r >> 1$, and if the damping is low, we have:

$$\left|\frac{Z}{Y}\right| \cong 1$$

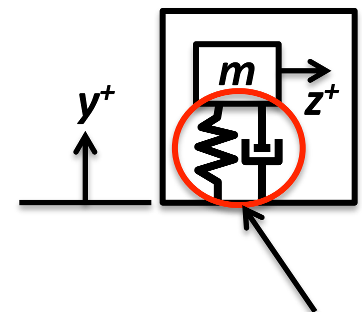
Thus, a seismic accel can estimate the input y directly by measuring z



http://www.wilcoxon.com/vi_index.cfm?PD_ID=34

Piezo-electric accelerometers

- Similar to the seismic accelerometer, the piezo-electric accelerometer infers acceleration by measuring relative motion
  - However, in this case we can determine acceleration directly



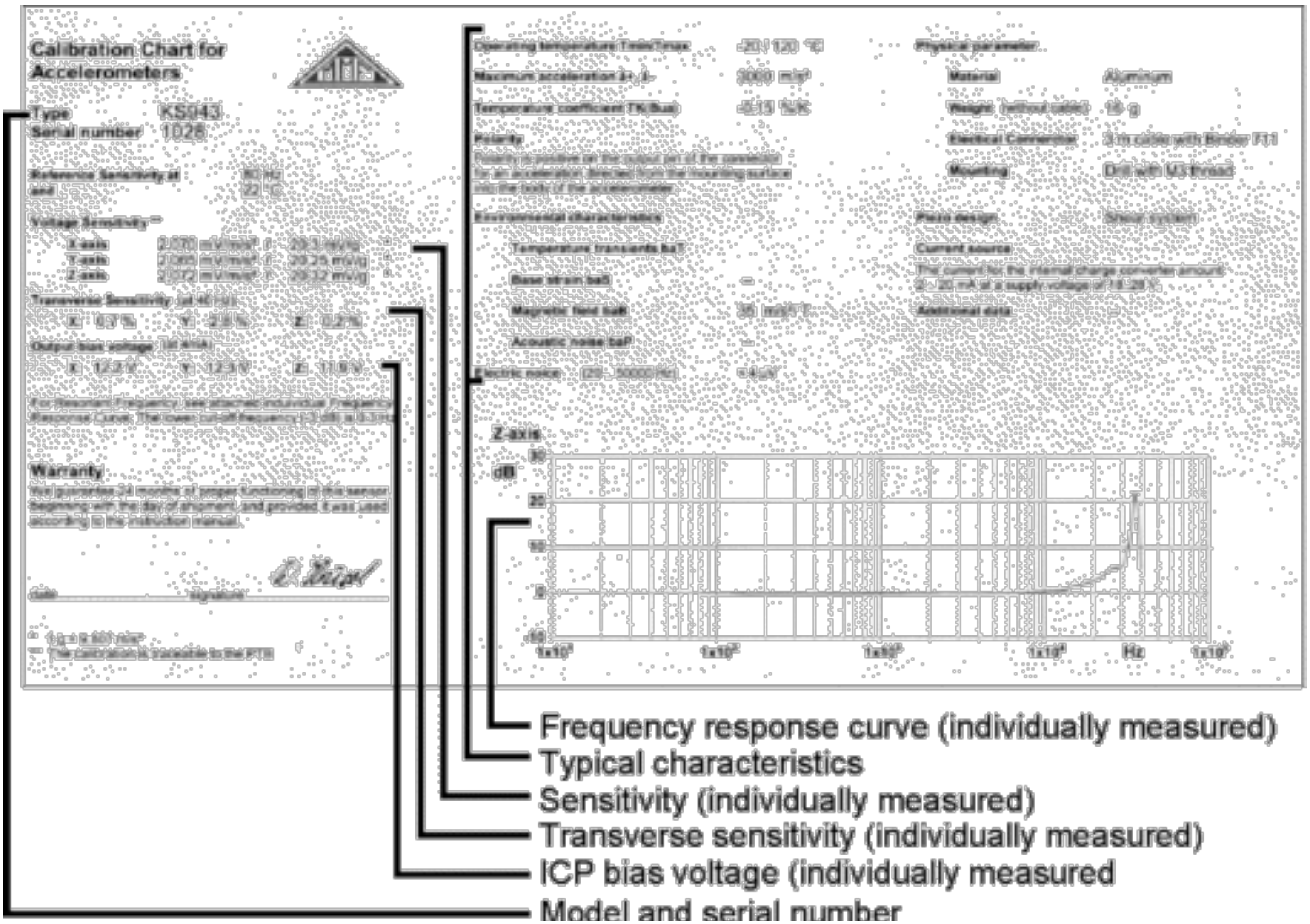
Piezo-electric crystal

$$\frac{\text{charge}}{\text{base accel}} \propto \left|\frac{z}{\ddot{y}}\right| = \left|\frac{Z}{\omega^2 Y}\right| = \frac{1}{\omega^2}\left(\frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}\right)$$

$$\left|\frac{z}{\ddot{y}}\right| = \frac{1/\omega_n^2}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}$$

And for $\frac{\omega}{\omega_n} << 1$, $\left|\frac{z}{\ddot{y}}\right| \cong \frac{1}{\omega_n^2}$

Piezo-electric accel calibration curve is based on linear response << natural frequency



http://www.mmf.de/images/drawings/figure8.gif