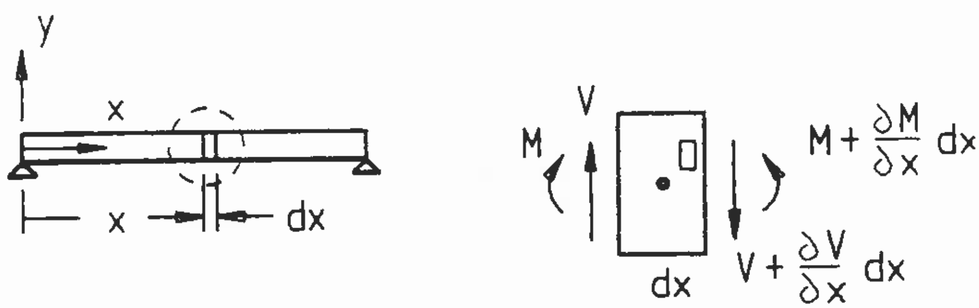
SESSION FIFTEEN

CONTINUOUS SYSTEMS

AS THE NUMBER OF DEGREES OF FREEDOM OF A DISCRETE, N DEGREE-OF-FREEDOM SYSTEM APPROACH INFINITY, WE OBTAIN A (REAL) "CONTINUOUS" SYSTEM. AS AN EXAMPLE OF A CONTINUOUS SYSTEM, LET US CONSIDER THE CASE OF THE LATERAL VIBRATION OF BEAMS (NEGLECTING ROTARY INTERTIA).

CONSIDER THE dX ELEMENT OF THE VIBRATING BEAM SHOWN.



V AND M ARE THE SHEAR FORCE AND BENDING MOMENTS RESPECTIVELY. SUMMING FORCES IN THE Y DIRECTION FOR THE dm ELEMENT GIVES, dm $\ddot{Y} = \Sigma F_Y$, OR,

$$\rho A \, dX \frac{\partial^2 Y}{\partial t^2} = -\left(V + \frac{\partial V}{\partial X} dX \right) + V = \frac{\partial V}{\partial X} dX$$

OR,

$$\rho A \frac{\partial^2 Y}{\partial t^2} = -\frac{\partial V}{\partial X} \qquad (1)$$

WHERE ρ = MASS/VOL., AND A = CROSSECTIONAL AREA

SUMMING MOMENTS ABOUT POINT 0 GIVES,

$$\frac{\partial M}{\partial X} dX + M - M - V\frac{dX}{2} - (V + \frac{\partial V}{\partial X} dX)\frac{dX}{2} = 0$$

(THIS NEGLECTS ROTARY INTERTIA.)

IGNORING $(\frac{\partial V}{\partial X} dX)\frac{dX}{2}$ AS A HIGHER ORDER TERM, WE ARRIVE AT,

$$V = \frac{\partial M}{\partial X} \qquad (2)$$

FROM ELEMENTARY STRENGTH OF MATERIALS,

$$M = EI\frac{\partial^2 Y}{\partial X^2} \qquad (3)$$

SUBSTITUTING EQUATIONS (3) INTO (2) AND THE RESULT INTO EQUATION(1) GIVES,

$$\rho A \frac{\partial^2 Y}{\partial t^2} = -\frac{\partial^2}{\partial X^2}\left(EI\frac{\partial^2 Y}{\partial X^2}\right) \qquad (4)$$

EQUATION (4) IS THE GOVERNING DIFFERENTIAL EQUATION OF A BEAM NEGLECTING ROTARY INERTIA.

NOW, IF EI IS CONSTANT, THEN,

$$\frac{\partial^2 Y}{\partial t^2} + \frac{EI}{\rho A} \frac{\partial^4 Y}{\partial X^4} = 0 \qquad (5)$$

ASSUMING THAT WE HAVE A VIBRATORY SOLUTION,

$$Y(X, t) = \overline{Y}(X) \text{ SIN } \omega t \qquad (6)$$

THEN EQUATION (5) BECOMES,

$$-\omega^2 \overline{Y} + \frac{EI}{\rho A} \frac{\partial^4 \overline{Y}}{\partial X^4} = 0$$

THIS EQUATION NOW ONLY HAS ONE INDEPENDENT VARIABLE SO THAT THE PARTIAL DERIVATIVE BECOMES AN ORDINARY DERIVATIVE. THUS,

$$\frac{d^4 \overline{Y}}{dX^4} - \frac{\rho A \omega^2}{EI} \overline{Y} = 0$$

OR

$$\frac{d^4 \overline{Y}}{dX^4} - \beta^4 \overline{Y} = 0 \qquad (7)$$

WHERE,

$$\beta^4 = \frac{\rho A \omega^2}{EI} \qquad (8)$$

ASSUMING THAT THE SOLUTION IS THE FORM, $\overline{Y} = C_i e^{a_i X}$,

EQUATION (7) BECOMES,

$$C_i \, a_i^4 \, e^{a_i X} - C_i \, \beta^4 \, e^{a_i X} = 0$$

SO THAT,

$$a_i^4 = \beta^4$$

OR,

$$a_i^2 = \pm\sqrt{\beta^4}$$

OR,

$$a_i = \pm\sqrt{\pm\sqrt{\beta^4}}$$

THUS, THE FOUR ROOTS ARE,

$$a_{1,2,3,4} = \beta, \; -\beta, \; i\beta, \; -i\beta$$

THUS, THE SOLUTION TO EQUATION (7) IS,

$$\overline{Y}(X) = C_1 e^{\beta X} + C_2 e^{-\beta X} + C_3 e^{i\beta X} + C_4 e^{-i\beta X} \qquad (9)$$

THE EXPONENTIALS CAN BE WRITTEN IN A DIFFERENT FORM BY NOTING THAT,

$$e^{\pm \beta X} = \text{COSH } \beta X \pm \text{SINH } \beta X$$

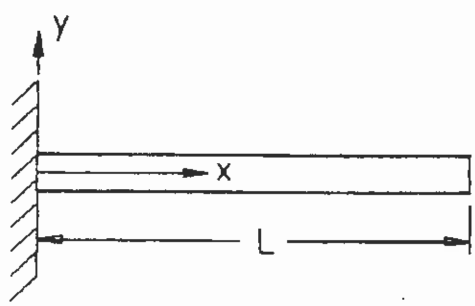
$$e^{\pm i\beta X} = \text{COS } \beta X \pm \text{SIN } \beta X$$

THUS, THE SOLUTION TO EQUATION (7) IS REWRITTEN AS,

$$\overline{Y}(X) = A \text{ COSH } \beta X + B \text{ SINH } \beta X + C \text{ COS } \beta X + D \text{ SIN } \beta X \qquad (10)$$

WHERE A, B, C, AND D ARE CONSTANTS OF INTEGRATION.

AS AN EXAMPLE, LET US CONSIDER A CANTILEVER BEAM.



THE BOUNDARY CONDITIONS AT THE LEFT OF THE BEAM ARE,

$$Y(0, t) = 0 \quad \Rightarrow \quad \overline{Y}(0) = 0 \qquad (a)$$

$$\frac{\partial Y}{\partial X}(0, t) = 0 \quad \Rightarrow \quad Y'(0) = 0 \qquad (b)$$

AT THE RIGHT OF THE BEAM,

$$M(L,t) = EI \frac{\partial^2 Y(L,t)}{\partial X^2} = 0 \quad \Rightarrow \quad \bar{Y}''(L) = 0 \qquad \text{(c)}$$

$$V(L,t) = EI \frac{\partial^3 Y(L,t)}{\partial X^3} = 0 \quad \Rightarrow \quad \bar{Y}'''(L) = 0 \qquad \text{(d)}$$

SUBSTITUTING CONDITIONS (a) – (d) INTO EQUATION (10) GIVES,

(a) $\bar{Y}(0) = 0 = A + C \quad \Rightarrow \quad A = -C \qquad (11)$

(b) $\bar{Y}'(0) = 0 = \beta[A \text{ SINH } \beta X + B \text{ COSH } \beta X - C \text{ SIN } \beta X + D \text{ COS } \beta X]\,\big|_{X=0}$

$$0 = \beta\,[B + D] \quad \Rightarrow \quad B = -D \qquad (12)$$

(c) $\bar{Y}''(L) = 0 = \beta^2[A \text{ COSH } \beta L + B \text{ SINH } \beta L - C \text{ COS } \beta L - D \text{ SIN } \beta L]$

SUBSTITUTION OF $C = -A$ AND $D = -B$ FROM EQUATIONS (9) AND (10) GIVES,

$$A(\text{COSH } \beta L + \text{COS } \beta L) + B(\text{SINH } \beta L + \text{SIN } \beta L) = 0 \qquad (13)$$

(d) $\bar{Y}'''(L) = 0 = \beta^3[A \text{ SINH } \beta L + B \text{ COSH } \beta L + C \text{ SIN } \beta L - D \text{ COS } \beta L]$

AGAIN, $C = -A$ AND $D = -B$, SO THEN,

$$A(\text{SINH } \beta L - \text{SIN } \beta L) + B(\text{COSH } \beta L + \text{COS } \beta L) = 0 \qquad (14)$$

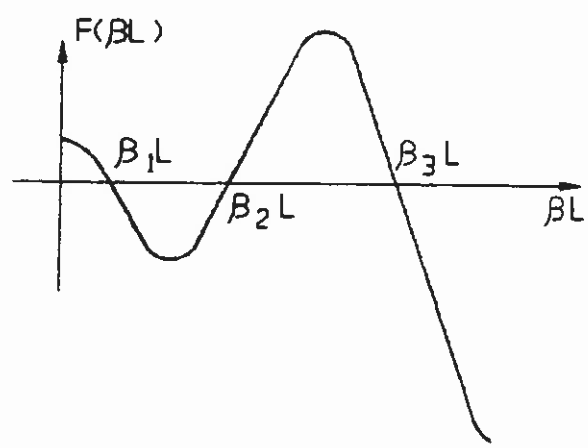
SOLVE FOR $\frac{A}{B}$ FROM EQUATIONS (13) AND (14), AND EQUATING THE RESULTING EQUATION GIVES,

$$\frac{\text{COSH}\,\beta L + \text{COS}\,\beta L}{\text{SINH}\,\beta L - \text{SIN}\,\beta L} = \frac{\text{SINH}\,\beta L + \text{SIN}\,\beta L}{\text{COSH}\,\beta L + \text{COS}\,\beta L}$$

CROSS-MULTIPLY AND SIMPLIFY. THE RESULT IS A TRANSCENDENTAL EQUATION,

$$F(\beta L) = \text{COSH}\,\beta L\ \text{COS}\,\beta L + 1 = 0 \qquad (15)$$

EQUATION (15) IS THE "CHARACTERISTIC EQUATION" OF THE CANTILEVER BEAM. A PLOT OF EQUATION (15) LOOKS LIKE,



THE FIRST TWO βL ROOTS ARE, $\beta_1 L = 1.875$, AND $\beta_2 L = 4.695$.

NOW SINCE $\beta^4 = \dfrac{\rho A \omega^2}{EI}$, WE GET,

$$\omega = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta L)^2 \sqrt{\frac{EI}{\rho A L^4}}$$

SUBSTITUTING THE VALUES OF βL FOR THE FIRST TWO EIGENVALUES GIVES,

$$\omega_1 = 3.52 \sqrt{\frac{EI}{\rho A L^4}} \qquad \omega_2 = 22.0 \sqrt{\frac{EI}{\rho A L^4}}$$

TO DETERMINE THE DEFLECTION SHAPE (EIGENFUNCTION) CORRESPONDING TO A PARTICULAR EIGENFREQUENCY, ω_i, CONSIDER EQUATION (10) FOR A SPECIFIC VALUE OF β_i,

$$\overline{Y}_i(X) = A_i \text{ COSH } \beta_i X + B_i \text{ SINH } \beta_i X + C_i \text{ COS } \beta_i X + D_i \text{ SIN } \beta_i X \qquad (10)$$

FROM EQUATIONS (11) AND (12), $C_i = -A_i$ AND $D_i = -B_i$. THUS, EQUATION (10) CAN BE REWRITTEN AS,

$$\overline{Y}_i(X) = A_i \text{ COSH } \beta_i X + B_i \text{ SINH } \beta_i X - A_i \text{ COS } \beta_i X - B_i \text{ SIN } \beta_i X$$

$$= B_i \left(\frac{A_i}{B_i} (\text{COSH } \beta_i X - \text{COS } \beta_i X) + \text{SINH } \beta_i X - \text{SIN } \beta_i X \right)$$

$$(15.8)$$

AS WITH DISCRETE SYSTEMS, WE CAN ONLY OBTAIN THE MODE SHAPE AND NOT THE EXACT FUNCTION $\overline{Y}_i(X)$. DROPPING THE FIRST TERM, B_i, THE MODE SHAPE IS GIVEN BY,

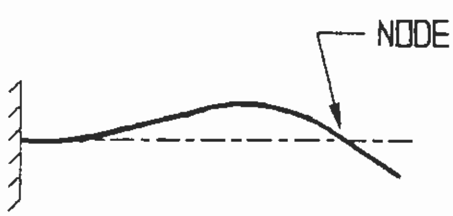
$$\overline{Y}_i(X) = \left(\frac{A}{B}\right)_i \left(\text{COSH}\,\beta_i X - \text{COS}\,\beta_i X\right) + \text{SINH}\,\beta_i X - \text{SIN}\,\beta_i X \qquad (16)$$

EQUATION (16) CAN BE REWRITTEN AS,

$$\overline{Y}_i(X) = \left(\frac{A}{B}\right)_i \left(\text{COSH}\,(\beta_i L\,\frac{X}{L}) - \text{COS}\,(\beta_i L\,\frac{X}{L})\right) + \text{SINH}\,(\beta_i L\,\frac{X}{L}) - \text{SIN}\,(\beta_i L\,\frac{X}{L}) \qquad (17)$$

FOR A GIVEN MODE SHAPE, $\beta_i L$ IS DETERMINED. THE RATIO $(A/B)_i$ CAN BE FOUND FROM EITHER EQUATIONS (13) OR (14) AND ARE FOUND TO BE $(A/B)_1 = -1.376$, AND $(A/B)_2 = -1.020$. EACH MODE SHAPE, $\overline{Y}_i(X)$, CAN THEN BE PLOTTED AS A FUNCTION OF X/L . THE FIRST TWO MODE SHAPES ARE SHOWN BELOW.

MODE 1, $\beta_1 L = 1.875$, $(A/B)_1 = -1.376$



MODE 2, $\beta_2 L = 4.695$, $(A/B)_2 = -1.020$

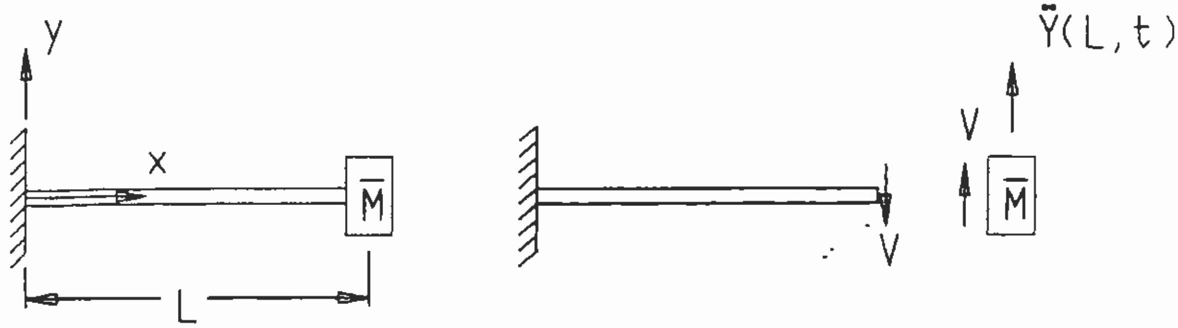
FOR OTHER BOUNDARY CONDITIONS, SEE APPENDIX D OF THE TEXTBOOK. THE COMMON BOUNDARY CONDITIONS ARE,

END CONDITION	DEFLECTION	SLOPE	MOMENT	SHEAR
FIXED	$Y = 0$	$Y' = 0$		
FREE			$Y'' = 0$	$Y''' = 0$
HINGED (S.S)	$Y = 0$		$Y'' = 0$	

THE FOLLOWING TABLE LISTS THE $(\beta_i L)^2$ ROOTS FOR TYPICAL CONFIGURATIONS OF THE COMMON BOUNDARY CONDITIONS.

BEAM CONFIGURATION	$(\beta_1 L)$ FUNDAMENTAL	$(\beta_2 L)$ SECOND MODE	$(\beta_3 L)$ THIRD MODE
SIMPLY SUPPORTED	9.87	39.5	88.9
CANTILEVER	3.52	22.0	61.7
FREE-FREE	22.4	61.7	121.0
CLAMPED-CLAMPED	22.4	61.7	121.0
CLAMPED-HINGED	15.4	50.0	104.0
HINGED-FREE	0	15.4	50.0

IN LAB 3, WE CONSIDERED A CANTILEVER WITH AN END MASS $\bar{M}$.



THE BOUNDARY CONDITIONS ARE:

END DISPLACEMENT, $\quad Y(0,t) = 0 \quad \Rightarrow \quad \bar{Y}(0) = 0 \qquad$ (a)

END SLOPE, $\quad Y'(0,t) = 0 \quad \Rightarrow \quad \bar{Y}'(0) = 0 \qquad$ (b)

END MOMENT, $\quad M(L,t) = EI\dfrac{\partial^2 Y}{\partial X^2}(L,t) = 0 \quad \Rightarrow \quad \bar{Y}''(L) = 0 \qquad$ (c)

SHEAR, $\quad V(L,t) = EI\dfrac{\partial^3 Y}{\partial X^3}(L,t) = \bar{M}\ddot{Y}(L,t) \quad \Rightarrow \quad \bar{Y}'''(L) = -\dfrac{\omega^2 \bar{M}}{EI}\bar{Y}(L)$

$$= -\frac{\bar{M}}{\rho A}\beta^4 \bar{Y}(L)$$

$$= -\frac{\bar{M}L}{m}\beta^4 Y(L) \qquad \text{(d)}$$

WHERE $m = \rho AL$, THE MASS OF THE BEAM, AND $\beta^4 = \dfrac{\rho A \omega^2}{EI}$.

APPLYING THE FOUR BOUNDARY CONDITIONS TO THE SOLUTION OF EQUATION (10) IN A MANNER SIMILAR TO THAT SHOWN FOR THE SIMPLE CANTILEVER BEAM, GIVES,

$$C = -A \quad \text{AND} \quad D = -B \qquad \text{(e)}$$

$$\bar{Y}''(L) = 0 = \beta^2[A \text{ COSH } \beta L + B \text{ SINH } \beta L - C \text{ COS } \beta L - D \text{ SIN } \beta L] \qquad \text{(f)}$$

AND,

$$\bar{Y}'''(L) = -\frac{\bar{M}L}{m}\beta^4 Y(L) = \beta^3[A \text{ SIN } \beta L + B \text{ COS } \beta L + C \text{ SIN } \beta L - D \text{ COS } \beta L] \qquad \text{(g)}$$

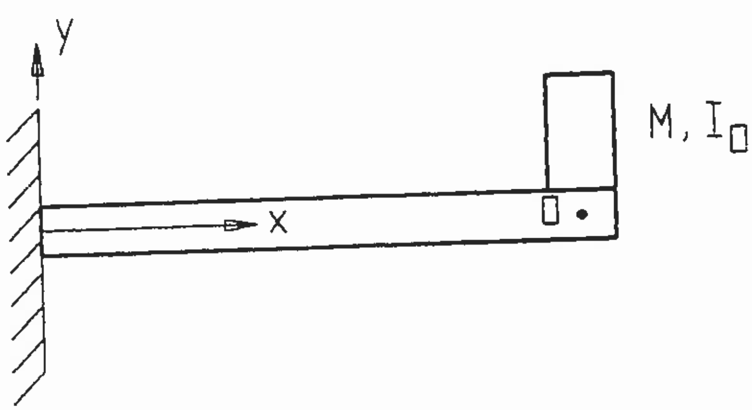
THE REMAINING PROCEDURE IS IDENTICAL TO THE SIMPLE CANTILEVER BEAM. USING EQUATIONS (e) AND (f), ELIMINATE C AND D FROM EQUATIONS (f) AND (g). THEN SOLVE FOR A/B IN EQUATIONS (f) AND (g) AND THEN EQUATE THEM. THIS YIELDS THE CHARACTERISTIC EQUATION,

$$1 + \text{COSH } \beta L \text{ COS } \beta L + \frac{\bar{M}}{m}(\beta L)\left(\text{COS } \beta L \text{ SIN } \beta L - \text{SIN } \beta L \text{ COS } \beta L\right) = 0 \qquad (18)$$

NOTE THAT IF $\bar{M} = 0$, THE ABOVE EQUATION REDUCES TO EQUATION (15).

$\beta_i L$ IN EQUATION (18) IS SOLVED FOR IN THE SAME MANNER THAT EQUATION (15) WAS SOLVED. THE VALUES OF $\beta_i L$ ARE THEN SUBSTITUTED INTO ONE OF THE EQUATIONS FOR A/B. A/B IS THEN SUBSTITUTED INTO EQUATION (17) TO GIVE THE CORRESPONDING MODE SHAPE.

WE WOULD CONTINUE THIS USING THE ACTUAL NUMBERS FOR THE LAB, HOWEVER, WE WOULD NOT OBTAIN VERY SATISFACTORY RESULTS. THE REASON IS THAT THE END MASS IS NOT REALLY A POINT MASS AS WAS ASSUMED. WHAT WE HAVE DONE TO THIS POINT WOULD BE ACCEPTABLE IF THAT WAS THE CASE. HOWEVER, FOR THE LAB, WE WILL GO BACK AND INCLUDE THE ROTARY INERTIA OF THE END MASS.



A MORE ACCURATE MODEL OF THE LABORATORY PROBLEM

LET US NOW INCLUDE THE MASS ROTARY INERTIA TOGETHER WITH THE TRANSLATIONAL INTERTIA.

RECALL THE GENERAL SOLUTION WHICH IS,

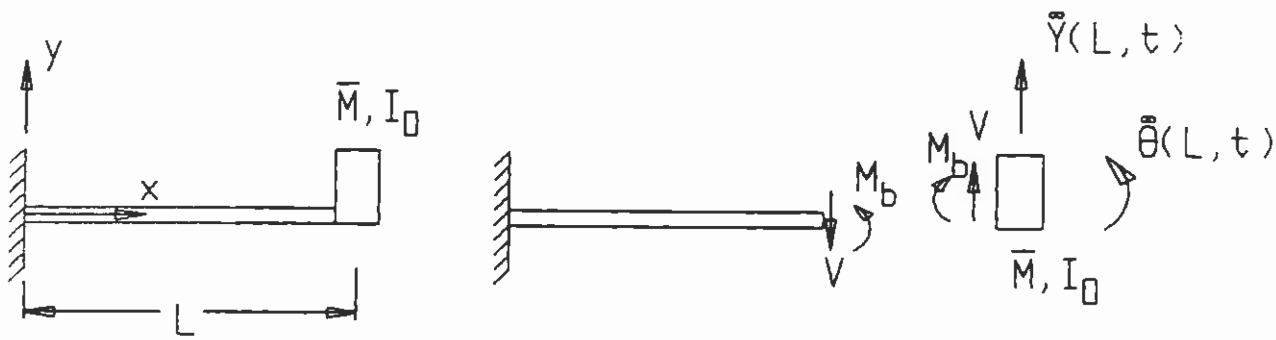
$$\overline{Y}(X) = A \text{ COSH } \beta X + B \text{ SINH } \beta X + C \text{ COS } \beta X + D \text{ SIN } \beta X$$

FOR A CANTILEVER BEAM, $\overline{Y}(0) = \overline{Y}'(0) = 0$ LEADS TO $C = -A$, $D = -B$

THUS,

$$\overline{Y}(X) = A(\text{COSH } \beta X - \text{COS } \beta X) + B(\text{SINH } \beta X - \text{SIN } \beta X) \qquad (19)$$

THE ROTATIONAL INERTIA B.C. IS,



$$\Sigma M_0 = -M_b \Big|_{X=L} = I_0 \ddot{\theta} \Big|_{X=L}$$

BUT

$$\theta = \frac{dY(X,t)}{dX} = \frac{d\overline{Y}(X)}{dX} \text{ SIN } \omega t$$

THUS,

$$-M_b \Big|_{X=L} = -I_0 \omega_n{}^2 \frac{d\bar{Y}(X)}{dX} \Big|_{X=L} \text{ SIN } \omega t \qquad \text{(a)}$$

AGAIN, FROM STRENGTH OF MATERIALS, $M_b = EI \dfrac{d^2Y}{dX^2}$, SO,

$$M_b = EI \frac{d^2\bar{Y}}{dX^2} \text{ SIN } \omega t$$

EQUATING THIS TO EQUATION (a) GIVES,

$$EI \frac{d^2\bar{Y}}{dX^2} \Big|_{X=L} = I_0\, \omega_n{}^2 \frac{d\bar{Y}}{dX} \Big|_{X=L} \qquad \text{(b)}$$

FROM THE PREVIOUS PROBLEM WITH THE CONCENTRATED END MASS, EQUATIONS (e) AND (g) (PAGE (15.12) ARE STILL VALID. USING EQUATIONS (e) EQUATION (g) CAN BE WRITTEN AS,

$$\beta^3 EI\, [A(\text{SINH } \beta L - \text{SIN } \beta L) + B(\text{COSH } \beta L + \text{COS } \beta L)] =$$

$$-\, \omega_n{}^2\, M\, [A(\text{COSH } \beta L - \text{COS } \beta L) + B(\text{SINH } \beta L - \text{SIN } \beta L)]$$

SOLVING FOR A/B YIELDS,

$$\frac{A}{B} = -\frac{K_1(\text{SINH } \beta L - \text{SIN } \beta L) + (\text{COSH } \beta L + \text{COS } \beta L)}{K_1(\text{COSH } \beta L - \text{COS } \beta L) + (\text{SINH } \beta L - \text{SIN } \beta L)} \qquad (20)$$

WHERE $K_1 = \dfrac{\omega_n{}^2 \overline{M}}{\beta^3 EI}$. BUT $\omega_n{}^2 = \dfrac{\beta^4 EIL}{m}$ THUS,

$$K_1 = \frac{\overline{M}}{m} \beta L \qquad (21)$$

SUBSTITUTING EQUATION (b) FROM THE PREVIOUS PAGE INTO (20) GIVES,

$$\beta^2 EI\,[A(\cosh \beta L + \cos \beta L) + B(\sinh \beta L + \sin \beta L)] =$$

$$I_0\, \omega_n{}^2 \beta\, [A(\sinh \beta L + \sin \beta L) + B(\cosh \beta L - \cos \beta L)]$$

AGAIN, SOLVING FOR A/B GIVES,

$$\frac{A}{B} = -\frac{K_2(\cosh \beta L - \cos \beta L) - (\sinh \beta L + \sin \beta L)}{K_2(\sinh \beta L + \sin \beta L) - (\cosh \beta L + \cos \beta L)} \qquad (22)$$

WHERE,

$$K_2 = \frac{I_0 \omega_n{}^2}{\beta EI}$$

BUT AGAIN, $\omega_n{}^2 = \beta^4 \dfrac{EIL}{m}$, SO,

$$K_2 = \frac{I_0 (\beta L)^3}{m L^2} \qquad (23)$$

EQUATING EQUATIONS (20) AND (22) GIVES,

$$\frac{K_1(\text{SINH } \beta L - \text{SIN } \beta L) + (\text{COSH } \beta L + \text{COS } \beta L)}{K_1(\text{COSH } \beta L - \text{COS } \beta L) + (\text{SINH } \beta L - \text{SIN } \beta L)} =$$

$$\frac{K_2(\text{COSH } \beta L - \text{COS } \beta L) - (\text{SINH } \beta L + \text{SIN } \beta L)}{K_2(\text{SINH } \beta L + \text{SIN } \beta L) - (\text{COSH } \beta L + \text{COS } \beta L)}$$

CROSS-MULTIPLYING GIVES,

$$K_1K_2[\text{SINH}^2 \beta L - \text{SIN}^2 \beta L]$$

$$+ K_2(\text{SINH } \beta L \text{ COSH } \beta L + \text{SINH } \beta L \text{ COS } \beta L + \text{COSH } \beta L \text{ SIN } \beta L + \text{SIN } \beta L \text{ COS } \beta L)$$

$$- K_1(\text{SINH } \beta L \text{ COSH } \beta L + \text{SINH } \beta L \text{ COS } \beta L - \text{COSH } \beta L \text{ SIN } \beta L - \text{SIN } \beta L \text{ COS } \beta L)$$

$$- (\text{COSH}^2 \beta L + 2 \text{ COSH } \beta L \text{ COS } \beta L + \text{COS}^2 \beta L)$$

$$= K_1K_2(\text{COSH}^2 \beta L - 2 \text{ COSH } \beta L \text{ COS } \beta L + \text{COS}^2 \beta L)$$

$$+ K_2(\text{COSH } \beta L \text{ SINH } \beta L - \text{SINH } \beta L \text{ COS } \beta L - \text{COSH } \beta L \text{ SIN } \beta L + \text{SIN } \beta L \text{ COS } \beta L)$$

$$- K_1(\text{COSH } \beta L \text{ SINH } \beta L + \text{COSH } \beta L \text{ SIN } \beta L - \text{SINH } \beta L \text{ COS } \beta L - \text{SIN } \beta L \text{ COS } \beta L)$$

$$- (\text{SINH}^2 \beta L - \text{SIN}^2 \beta L)$$

REARRANGING YIELDS,

$$K_1K_2(\text{SINH}^2 \beta L - \text{COSH}^2 \beta L - \text{SIN}^2 \beta L - \text{COS}^2 \beta L + 2 \text{ COSH } \beta L \text{ COS } \beta L)$$

$$- K_2(2 \text{ SINH } \beta L \text{ COS } \beta L + 2 \text{ COSH } \beta L \text{ SIN } \beta L)$$

$$- K_1(2 \text{ SINH } \beta L \text{ COS } \beta L - 2 \text{ COSH } \beta L \text{ SIN } \beta L)$$

$$- (\text{COSH}^2 \beta L - \text{SINH}^2 \beta L + \text{COS}^2 \beta L + \text{SIN}^2 \beta L + 2 \text{ COSH } \beta L \text{ COS } \beta L) = 0$$

BUT, $COSH^2 \beta L - SINH^2 \beta L = 1$ AND $SIN^2 \beta L + COS^2 \beta L = 1$. THUS,

$$K_1K_2(\text{COSH } \beta L \text{ COS } \beta L - 1) + (K_1 + K_2)(\text{COSH } \beta L \text{ SIN } \beta L)$$

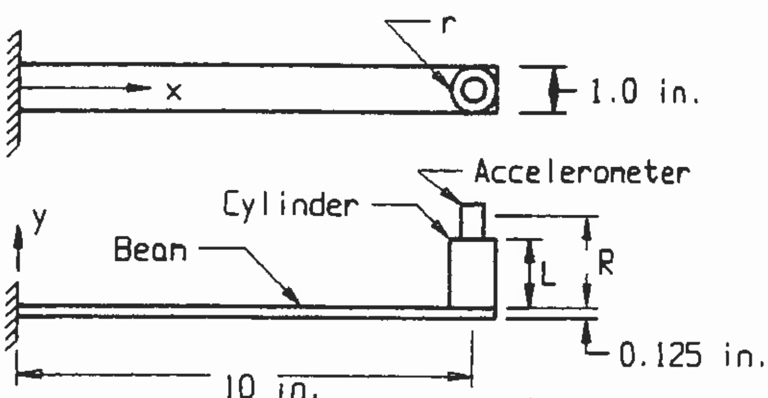
$$- (K_1 - K_2)(\text{SINH } \beta L \text{ COS } \beta L) - 1 - \text{COSH } \beta L \text{ COS } \beta L = 0$$

FURTHER MANIPULATION YIELDS,

$$(K_1K_2 - 1)(\text{COSH } -L \text{ COS } \beta L) - (K_1K_2 + 1)$$

$$+ (K_1 + K_2)(\text{COSH } \beta L \text{ SIN } \beta L) - (K_1 - K_2)(\text{SINH } \beta L \text{ COS } \beta L) = 0 \tag{23}$$

FOR THE LABORATORY PROBLEM,



L = 1.465 in.

R = 1.465 + 0.375

r = 0.5 in.

$$m = \frac{1}{386.4}[(0.284)(0.999)(0.1235)(10)] = 9.068 \ (10^{-4}) \ \frac{\text{LBF-SEC}^2}{\text{IN}}$$

$$\overline{M} = \frac{161.5}{386.4(454)} = 9.206 \ (10^{-4}) \ \frac{\text{LBF-SEC}^2}{\text{IN}}$$

THUS,

$$\frac{\overline{M}}{m} = \frac{9.206\ (10^{-4})}{9.068\ (10^{-4})} = 1.0152$$

THE ROTARY INERTIA IS,

$$I_0 = \frac{1}{4}\ Mr^2 + \frac{1}{3}\ ML^2 + M_{ACCEL}\ R^2$$

$$= \frac{140.8}{386.4(454)}\left[\frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{3}(1.465)^2\right] + \frac{14.3(1.465 + 0.375)^2}{386.4(454)}$$

$$= 9.0035\ (10^{-4}) \qquad \text{IBF-IN-SEC}^2$$

SO THE ROTARY TERM FOR K_2 IS,

$$\frac{I_0}{mL^2} = \frac{9.0035\ (10^{-4})}{9.068\ (10^{-4})\ (10)^2} = 9.929\ (10^{-3})$$

THUS, $K_1 = 1.0152\ \beta L$ AND $K_2 = 9.929\ (10^{-3})(\beta L)^3$. THEIR PRODUCT IS, $K_1K_2 = (1.0152\ \beta L)\ 9.929\ (10^{-3})(\beta L)^3 = 1.008\ (10^{-2})\ (\beta L)^4$. SO EQUATION (23) THEN BECOMES,

$$[1.008\ (10^{-2})(\beta L)^4 - 1]\ \text{COSH}\ \beta L\ \text{COS}\ \beta L - [1.008\ (10^{-2})(\beta L)^4 + 1]$$

$$+ [1.0152\ \beta L + 9.929(10^{-3})(\beta L)^3]\ \text{COSH}\ \beta L\ \text{SIN}\ \beta L$$

$$- [1.0152\ \beta L - 9.929(10^{-3})(\beta L)^3]\ \text{SINH}\ \beta L\ \text{COS}\ \beta L = 0 \qquad (24)$$

EQUATION (24) IS THE CHARACTERISTIC EQUATION FOR THE PROBLEM.

THERE ARE A NUMBER OF NUMERICAL METHODS FOR SOLVING FOR THE ROOTS $\beta_i L$ OF EQUATION (24). PROBABLY THE SIMPLIST METHOD IS TO DETERMINE THE ROOT CROSSINGS OF THE EQUATION. THAT IS, IF THE LEFT SIDE OF THE EQUATION IS LABELLED F(βL), WE SIMPLY DETERMINE WHERE F(βL) = 0. THE SECOND FROM THE LAST PAGE OF THIS SESSION SHOWS A PLOT OF F(βL) VS. βL. F(βL) IS NORMALIZED AS NON-ZERO VALUES OF F(βL) GET QUITE LARGE AS βL INCREASES. THE FIRST FIVE ROOTS OF EQUATION (24) ARE,

$(\beta L)_1 = 1.2388 \qquad (\beta L)_2 = 3.6407 \qquad (\beta L)_3 = 5.6670$

$(\beta L)_4 = 8.1753 \qquad (\beta L)_5 = 11.1537$

THE FREQUENCIES ARE FOUND FROM,

$$f = \frac{(\beta L)^2}{2\pi}\sqrt{\frac{EI}{mL^3}} = \frac{(\beta L)^2}{2\pi}\sqrt{\frac{(29)(10^6)\frac{1}{12}(0.999)(0.1235)^3}{\frac{(0.284)(0.1235)(10)}{386.4}(10)^3}}$$

$$= 11.265\ (\beta L)^2$$

THE TABLE BELOW GIVES THE FREQUENCIES FOR THE FIRST FIVE MODES TOGETHER WITH THE TEST RESULTS.

MODE	DIFFERENTIAL EQUATION SOLUTIONS (HZ)	EXPERIMENTAL (HZ)
1	17.3	16.3
2	149.3	141.3
3	361.8	343.2
4	752.9	728.5
5	1401.4	1300.0 1352.7 1481.2

<u>MODE SHAPES (EIGENVECTORS)</u>

EQUATION (17) IS STILL VALID. RECALL,

$$\hat{Y}_i(X) = \frac{A}{B}\left(\text{COSH}\,(\beta_i L\,\frac{X}{L}) - \text{COS}\,(\beta_i L\,\frac{X}{L})\right) + \text{SINH}\,(\beta_i L\,\frac{X}{L}) - \text{SIN}\,(\beta_i L\,\frac{X}{L}) \qquad (17)$$

TO DETERMINE THE FIRST MODE SHAPE, WE FIRST DETERMINE A/B FROM EQUATION (20),

$$\frac{A}{B} = \frac{1.0152\ \beta L(\text{SINH}\ \beta L - \text{SIN}\ \beta L) + (\text{COSH}\ \beta L + \text{COS}\ \beta L)}{1.0152\ \beta L(\text{COSH}\ \text{BL} - \text{COS}\ \beta L) + (\text{SINH}\ \beta L - \text{SIN}\ \beta L)} \qquad (20)$$

FOR MODE 1, $\beta_1 L = 1.2381$. SUBSTITUTING THIS INTO EQUATION (20) GIVES $\left(\frac{A}{B}\right)_1 = -1.16199$. THUS EQUATION (17) GIVE MODE 1 AS,

$$\hat{Y}_1(X) = -1.16199\left(\cosh\left(1.2381\frac{X}{L}\right) - \cos\left(1.2381\frac{X}{L}\right)\right)$$

$$+ \sinh\left(1.2381\frac{X}{L}\right) - \sin\left(1.2381\frac{X}{L}\right) \qquad (23)$$

THE SHAPE IS NOW DETERMINED. SOME VALUES ARE GIVEN BELOW.

MODE 1 (f = 17.3 HZ)

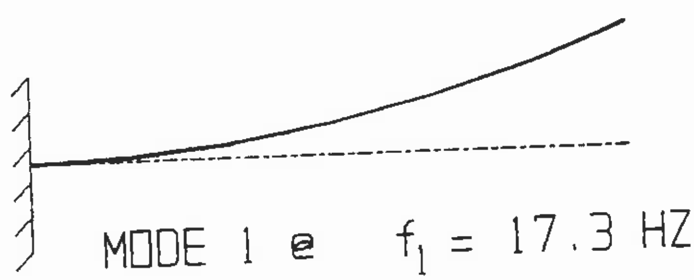
X/L	$\hat{Y}_1(X)$
0	0
0.2	−0.06626
0.4	−0.24480
0.6	−0.50557
0.8	−0.81949
1.0	−1.15942

THIS SHAPE CAN BE UNIT NORMALIZED BY DIVIDING BY THE LARGEST VALUE, −1.15942 . THIS GIVES,

MODE 1 (f = 17.3 HZ)

X/L	Y_1(X)
0	0
0.2	0.0571
0.4	0.2111
0.6	0.4361
0.8	0.7068
1.0	1.0000

A PLOT OF THIS SHAPE IS,

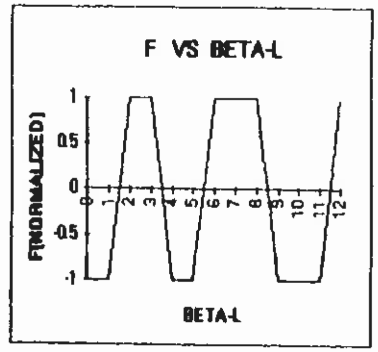


THE NEXT TWO PAGES GIVE THE RESULTS FOR THE FIRST FIVE MODE SHAPES.

EIGEN VALUES TERMS IN EQUATION (24)

BETA-L	TERM 1	TERM 2	TERM 3	TERM 4	BETA-L	F(BETA-L)
0	-1	1	0	0	0	-2
1	-0.825	1.0101	1.33108652	0.6383	1	-1.143
2	1.3131	1.1613	7.21764004	-2.945	2	10.314
3	1.8291	1.8165	4.70791029	-27.55	3	32.267
4	-28.21	3.5805	-97.057235	-61.1	4	-67.75
5	111.57	7.3	-449.53749	88.719	5	-426
6	2336.5	14.064	-464.19368	764.36	6	1093.9
7	9591.2	25.202	3788.82768	1529.8	7	11823
8	-8737	42.288	19472.6944	-658.8	8	11352
9	-2E+05	67.135	27341.6576	-7008	9	-2E+05
10	-9E+05	101.8	-120313.93	-2061	10	-1E+06
11	19421	148.58	-729939.44	-271.4	11	-7E+05
12	1E+07	210.02	-1281115.8	-3E+05	12	1E+07
1.2388	-0.595	1.0237	2.25734993	0.6383		9E-05
3.6407	-12.91	2.7709	-38.114748	-53.8		-1E-04
5.667	1188.7	11.396	-631.72017	465.62		0.0019
8.1753	-24694	46.027	23128.2969	-1612		-0.008
11.154	852000	157	-865331.85	-13489		-9E-04

BETA-L	F(NORM)
0	-1
1	-1
2	1
3	1
4	-1
5	-1
6	1
7	1
8	1
9	-1
10	-1
11	-1
12	1



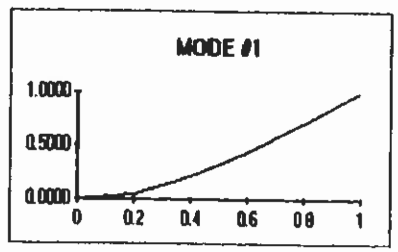
MODE #1

BETA-L			A/B
1.2388	2.9957	2.5781	-1.1619902

EIGENVECTOR #1

x/L	BET-L	A/B	Y
0	1.2388	-1.162	0
0.2	1.2388	-1.162	-0.0662599
0.4	1.2388	-1.162	-0.2448034
0.6	1.2388	-1.162	-0.5055749
0.8	1.2388	-1.162	-0.8194932
1	1.2388	-1.162	-1.1594155

x/L	Y(NORM)
0	0.0000
0.2	0.0571
0.4	0.2111
0.6	0.4361
0.8	0.7068
1	1.0000



MODE #2

BETA-L			A/B
3.6407	90.359	93.262	-0.9688678

EIGENVECTOR #2

x/L	BET-L	A/B	Y
0	3.6407	-0.969	0
0.2	3.6407	-0.969	-0.3853563
0.4	3.6407	-0.969	-1.0454364
0.6	3.6407	-0.969	-1.3479521
0.8	3.6407	-0.969	-0.9376168
1	3.6407	-0.969	0.19549795

x/L	Y(NORM)
0	0.000
0.2	0.286
0.4	0.776
0.6	1.000
0.8	0.696
1	-0.145