ME 5514 VIBRATION MECHANICS SYSTEMS
Final Project

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Problem 1. Consider the dx element of the given cantilever beam we have: V and M are the shear force and bending moments respectively.

- Summing the force in Y-direction for the dm elements:

$$\mathrm{d}m\bar{Y} = \sum F_y \Leftrightarrow \rho A \frac{\partial^2 Y}{\partial t^2} = -\frac{\partial V}{\partial X}$$

with: ρ is the density (mass/volume): $\rho = 2.7\ g/cm^3$

A is the cross-sectional area: $A = 0.32 * 2.45 = 0.784\ cm^2$

The bending moment M is defined by: $M = EI\dfrac{\partial^2 Y}{\partial X^2}$ with I is moment inertial of cross-sectional

$$I = \frac{bh^3}{12} = \frac{2.45 * 10^{-2} * (3.2 * 10^{-3})^3}{12} = 6.69 \times 10^{-11}\ m^4$$

The shear force V is calculated from M: $V = \dfrac{\partial M}{\partial X}$

Because EI is constant, we come up with: $\dfrac{\partial^2 Y}{\partial t^2} + \dfrac{EI}{\rho A}\dfrac{\partial^4 Y}{\partial X^4} = 0$ (1)

Assume the vibration solution is: $Y(X,t) = \bar{Y}(X)\sin\omega t$

(1) become: $\dfrac{\mathrm{d}^4\bar{Y}}{\mathrm{d}X^4} - \beta^4\bar{Y} = 0$ with $\beta^4 = \dfrac{\rho A\omega^2}{EI}$

Solution: $\bar{Y}(X) = a\cosh\beta X + b\sinh\beta X + c\cos\beta X + d\sin\beta X$ (2)

1. Assume the tip mas can be modeled as a point mass $\Rightarrow$ the boundary conditions are:

   - End displacement (on left): $Y(0,t) = 0 \Rightarrow \bar{Y}(0) = 0 \Leftrightarrow \bar{Y}(0) = a + c = 0$ (3)

   - End slope (on left): $Y'(0,t) = 0 \Rightarrow \bar{Y}'(0) = 0$

   with $\bar{Y}'(X,t) = \beta a\sinh\beta X + \beta b\cosh\beta X - \beta c\sin\beta X + \beta d\cos\beta X$

   $\Rightarrow \bar{Y}'(0,t) = \beta b + \beta d = 0 \Leftrightarrow b + d = 0$ (4)

   - End moment (on right): $M(L,t) = EI\dfrac{\partial^2 Y}{\partial X^2}(L,t) = 0 \Leftrightarrow \bar{Y}''(L,t) = 0$

   with $\bar{Y}''(X,t) = \beta^2 a\cosh\beta X + \beta^2 b\sinh\beta X - \beta^2 c\cos\beta X - \beta^2 d\sin\beta X$

   $\Rightarrow \bar{Y}''(L,t) = \beta^2\left(a\cosh\beta L + b\sinh\beta L - c\cos\beta L - d\sin\beta L\right) = 0$

   $\Leftrightarrow a\cosh\beta L + b\sinh\beta L - c\cos\beta L - d\sin\beta L = 0$ (because $\beta \neq 0$)

   $$\Leftrightarrow \frac{a}{b} = -\frac{\sinh\beta L + \sin\beta L}{\cosh\beta L + \cos\beta L} \quad (5)$$

   - Shear force V at right end: $V(L,t) = \bar{M}Y''(L,t) \Leftrightarrow \bar{Y}'''(L,t) = -\dfrac{\bar{M}L}{m}\beta^4\bar{Y}(L)$

   with: $\bar{M}$ is the tip mass: $\bar{M} = (1.13 * \pi * 1.15^2 + 1.17 * \pi * 0.45^2) * 2.7 = 14.69$ g

   m - mass of beam: $m = \mathrm{V}\rho = 0.32 * 2.45 * 46 * 2.7 = 97.4$ g

   $\bar{Y}'''(X,t) = \beta^3 a\sinh\beta X + \beta^3 b\cosh\beta X + \beta^3 c\sin\beta X - \beta^3 d\cos\beta X$

   $\Rightarrow \bar{Y}'''(L,t) = \beta^3\left(a\sinh\beta L + b\cosh\beta L + c\sin\beta L - d\cos\beta L\right) = -\dfrac{\bar{M}L}{m}\beta^4\bar{Y}(L)$

   $$\Leftrightarrow a\left[(\sinh\beta L - \sin\beta L) + \frac{\bar{M}}{m}\beta L(\cosh\beta L - \cos\beta L)\right] = -b\left[(\cosh\beta L + \cos\beta L) + \frac{\bar{M}}{m}\beta L(\sinh\beta L - \sin\beta L)\right]$$

   $$\Leftrightarrow \frac{a}{b} = -\frac{(\cosh\beta L + \cos\beta L) + \frac{\bar{M}}{m}\beta L(\sinh\beta L - \sin\beta L)}{(\sinh\beta L - \sin\beta L) + \frac{\bar{M}}{m}\beta L(\cosh\beta L - \cos\beta L)} \quad (6)$$

   From (5) and (6) we have the characteristic equation:

   $$\frac{\sinh\beta L + \sin\beta L}{\cosh\beta L + \cos\beta L} = \frac{(\cosh\beta L + \cos\beta L) + \frac{\bar{M}}{m}\beta L(\sinh\beta L - \sin\beta L)}{(\sinh\beta L - \sin\beta L) + \frac{\bar{M}}{m}\beta L(\cosh\beta L - \cos\beta L)}$$

$$\Leftrightarrow 1+\cosh\beta L\cos\beta L+\frac{\bar{M}}{m}\beta L\cos\beta L\sinh\beta L-\frac{\bar{M}}{m}\beta L\cosh\beta L\sin\beta L=0$$

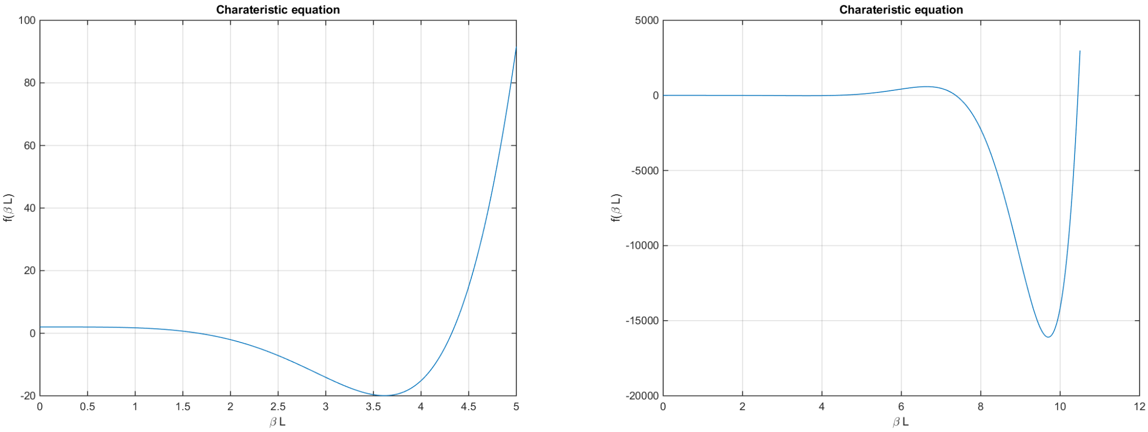

Figure 1: Characteristic equation 1

The first four nonzero roots of characteristic equation are: $\beta_1 L = 1.665, \beta_2 L = 4.322, \beta_3 L = 7.37$ and $\beta_4 L = 10.45$

Natural frequencies is defined from computed values of β, we have:

since: $\beta^4 = \dfrac{\rho A\omega^2}{EI} \Rightarrow \omega = \beta^2\sqrt{\dfrac{EI}{\rho A}} = (\beta L)^2\sqrt{\dfrac{EI}{\rho A L^4}}$

we first have: $\sqrt{\dfrac{EI}{\rho A L^4}} = \sqrt{\dfrac{68.9\bullet 10^9\times 6.69\bullet 10^{-11}}{2.7\bullet 10^6\times 78.4\bullet 10^{-6}\times 0.46^4}} \approx 0.7$

$$\beta_1 L = 1.665 \Rightarrow \omega_1 = (\beta_1 L)^2\bullet 0.7 = 1.665^2\bullet 0.7 = 1.94 \text{ rad/s}$$

$$\beta_2 L = 4.322 \Rightarrow \omega_1 = (\beta_2 L)^2\bullet 0.7 = 4.322^2\bullet 0.7 = 13.07 \text{ rad/s}$$

$$\beta_3 L = 7.37 \Rightarrow \omega_1 = (\beta_3 L)^2\bullet 0.7 = 7.37^2\bullet 0.7 = 38.02 \text{ rad/s}$$

$$\beta_4 L = 10.45 \Rightarrow \omega_1 = (\beta_4 L)^2\bullet 0.7 = 10.45^2\bullet 0.7 = 76.4 \text{ rad/s}$$

The mode shape can be written as:

$$\bar{Y}_i(X) = \left(\frac{A}{B}\right)_i\left(\cosh(\beta_i L\frac{X}{L}) - \cos(\beta_i L\frac{X}{L})\right) + \sinh(\beta_i L\frac{X}{L}) - \sin(\beta_i L\frac{X}{L})$$

with: $\left(\dfrac{A}{B}\right)_i = -\dfrac{\sinh\beta_i L+\sin\beta_i L}{\cosh\beta_i L+\cos\beta_i L}$

we have the table result:

βL	1.575	4.226	7.282	10.371
a/b	-1.341	-0.985	-1.0005	1

$$\bar{Y}_1(X) = -1.341\left(\cosh(1.665\frac{X}{0.46}) - \cos(1.665\frac{X}{0.46})\right) + \sinh(1.665\frac{X}{0.46}) - \sin(1.6655\frac{X}{0.46})$$
$$= -1.341\left(\cosh(3.62X) - \cos(3.62X)\right) + \sinh(3.62X) - \sin(3.62X)$$

$$\bar{Y}_2(X) = -0.985\left(\cosh(4.322\frac{X}{0.46}) - \cos(4.322\frac{X}{0.46})\right) + \sinh(4.322\frac{X}{0.46}) - \sin(4.322\frac{X}{0.46})$$
$$= -0.985\left(\cosh(9.4X) - \cos(9.4X)\right) + \sinh(9.4X) - \sin(9.4X)$$

$$\bar{Y}_3(X) = -1.0005\left(\cosh(7.37\frac{X}{0.46}) - \cos(7.37\frac{X}{0.46})\right) + \sinh(7.37\frac{X}{0.46}) - \sin(7.37\frac{X}{0.46})$$
$$= -1.0005\left(\cosh(16.02X) - \cos(16.02X)\right) + \sinh(16.02X) - \sin(16.02X)$$

$$\bar{Y}_4(X) = -1\left(\cosh(10.45\frac{X}{0.46}) - \cos(10.45\frac{X}{0.46})\right) + \sinh(10.45\frac{X}{0.46}) - \sin(10.45\frac{X}{0.46})$$
$$= -\cosh(22.72X) + \cos(22.71X) + \sinh(22.72X) - \sin(22.72X)$$

Mode shape for the point mass:

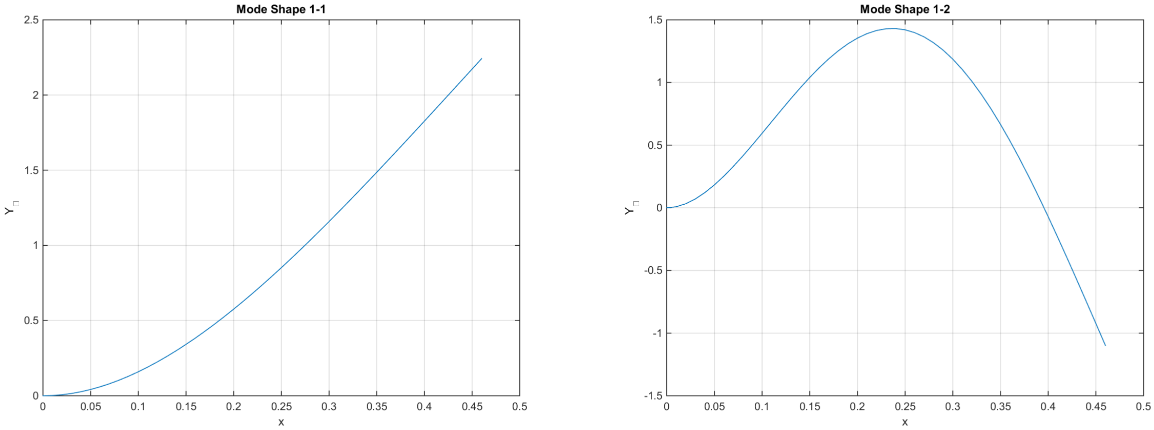

Figure 2: Mode Shape 1 and 2

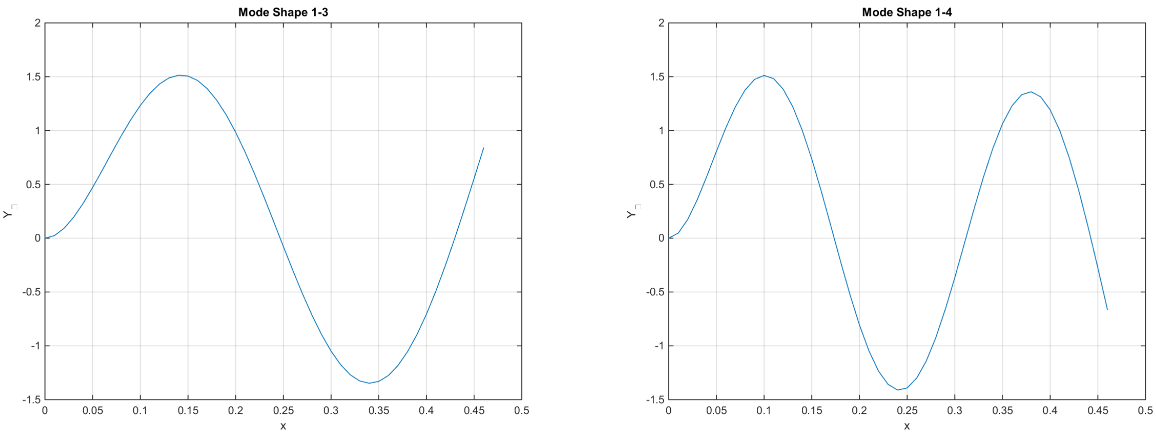

Figure 3: Mode Shape 3 and 4

2. Modeled the tip mass as a solid having mass and inertia:

- The first two bound conditions are same with above: we have equation (3) and (4)

- From the rotational inertial we form the third and forth condition:

$$\frac{a}{b} = -\frac{k_1(\sinh\beta L - \sin\beta L) + (\cosh\beta L + \cos\beta L)}{k_1(\cosh\beta L + \cos\beta L) + (\sinh\beta L - \sin\beta L)} \tag{7}$$

$$\frac{a}{b} = -\frac{k_2(\cosh\beta L - \cos\beta L) - (\sinh\beta L + \sin\beta L)}{k_2(\sinh\beta L + \sin\beta L) - (\cosh\beta L + \cos\beta L)} \tag{8}$$

with $k_1 = \frac{\bar{M}}{m}\beta L = 0.1508\beta L$

The rotary inertia is: $I_0 = \frac{1}{4}M_1 r_1^2 + \frac{1}{3}M_1 h_1^2 + M_2 h_2^2$

$$= \pi * 1.15^2 \times 10^{-4} * 1.13 \times 10^{-2} * 2.7 \times 10^6 \left(\frac{1}{4}1.15^2 \times 10^{-4} + \frac{1}{3}1.13^2 \times 10^{-4}\right)$$
$$+ \pi * 0.45^2 \times 10^{-4} * 1.17 \times 10^{-2} * 2.7 \times 10^6 * 1.17^2 \times 10^{-4}$$
$$= 1.234 \times 10^{-3}$$

So the rotary term for k_2 is: $k_2 = \frac{I_0}{mL^2}(\beta L)^3 = \frac{1.234 \times 10^{-3}}{97.4 \times 10^{-3} * 0.46^2}(\beta L)^3 = 0.06(\beta L)^3$

Characteristic equation:

$$(k_1k_2 - 1)(\cosh\beta L\cos\beta L) - (k_1k_2 + 1) + (k_1 + k_2)(\cosh\beta L\sin\beta L) - (k_1 - k_2)(\sinh\beta L\cos\beta L) = 0$$

$$\Leftrightarrow (9.048{\times}10^{-3}(\beta)^4{-}1)(\cosh\beta L\cos\beta L){-}(9.048{\times}10^{-3}(\beta)^4{+}1){+}(0.1508\beta L{+}0.06(\beta L)^3)(\cosh\beta L\sin\beta L){-}$$
$$(0.1508\beta L - 0.06(\beta L)^3)(\sinh\beta L\cos\beta L) = 0$$

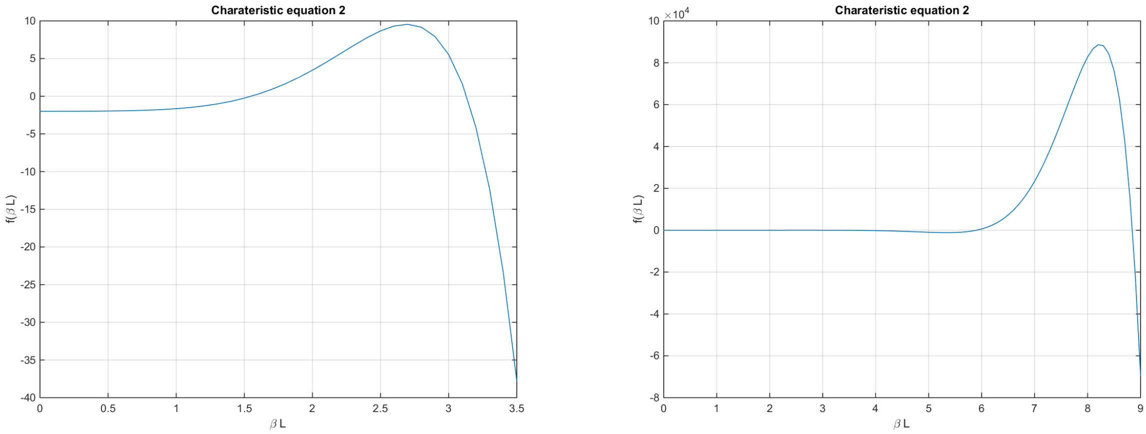

Figure 4: Characteristic equation 2

The first four nonzero roots of characteristic equation are: $\beta_{21}L = 1.55, \beta_{22}L = 3.13, \beta_{23}L = 5.9$ and $\beta_{24}L = 8.84$

Natural frequencies is defined from computed values of β, we have:

$$\omega_n^2 = \beta^4\frac{EIL}{m} \Leftrightarrow \omega_n = (\beta L)^2\sqrt{\frac{EI}{mL^3}}$$

with: $$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{68.9 \bullet 10^9 \times 6.69 \bullet 10^{-11}}{97.4 \bullet 10^{-3} \times 0.46^3}} \approx 22.05$$

The first four natural frequencies of vibration are defined:

βL	1.55	3.13	5.9	8.84
ω_n	53	216.3	767.9	1725

The mode shape can be written as:

$$\bar{Y}_i(X) = \left(\frac{A}{B}\right)_i\left(\cosh(\beta_i L\frac{X}{L}) - \cos(\beta_i L\frac{X}{L})\right) + \sinh(\beta_i L\frac{X}{L}) - \sin(\beta_i L\frac{X}{L})$$

with: $$\left(\frac{a}{b}\right)_i = -\frac{k_{1i}(\sinh\beta_i L - \sin\beta_i L) + (\cosh\beta_i L + \cos\beta_i L)}{k_{1i}(\cosh\beta_i L + \cos\beta_i L) + (\sinh\beta_i L - \sin\beta_i L)}$$

we have the table result:

βL	1.55	3.13	5.9	8.84
a/b	-1.52	-0.97	-1.0002	1

$$\begin{aligned}
\bar{Y}_1(X) &= -1.52\left(\cosh(1.55\frac{X}{0.46}) - \cos(1.55\frac{X}{0.46})\right) + \sinh(1.55\frac{X}{0.46}) - \sin(1.55\frac{X}{0.46})\\
&= -1.52\left(\cosh(3.37X) - \cos(3.37X)\right) + \sinh(3.37X) - \sin(3.37X)\\
\bar{Y}_2(X) &= -0.97\left(\cosh(3.13\frac{X}{0.46}) - \cos(3.13\frac{X}{0.46})\right) + \sinh(3.13\frac{X}{0.46}) - \sin(3.13\frac{X}{0.46})\\
&= -0.97\left(\cosh(6.8X) - \cos(6.8X)\right) + \sinh(6.8X) - \sin(6.8X)\\
\bar{Y}_3(X) &= -1.0002\left(\cosh(5.9\frac{X}{0.46}) - \cos(5.9\frac{X}{0.46})\right) + \sinh(5.9\frac{X}{0.46}) - \sin(5.9\frac{X}{0.46})\\
&= -1.0002\left(\cosh(12.83X) - \cos(12.83X)\right) + \sinh(12.83X) - \sin(12.83X)\\
\bar{Y}_4(X) &= -1\left(\cosh(8.84\frac{X}{0.46}) - \cos(8.84\frac{X}{0.46})\right) + \sinh(8.84\frac{X}{0.46}) - \sin(8.84\frac{X}{0.46})\\
&= -\cosh(19.22X) + \cos(19.22X) + \sinh(19.22X) - \sin(19.22X)
\end{aligned}$$

Mode shape for the mass with int...:

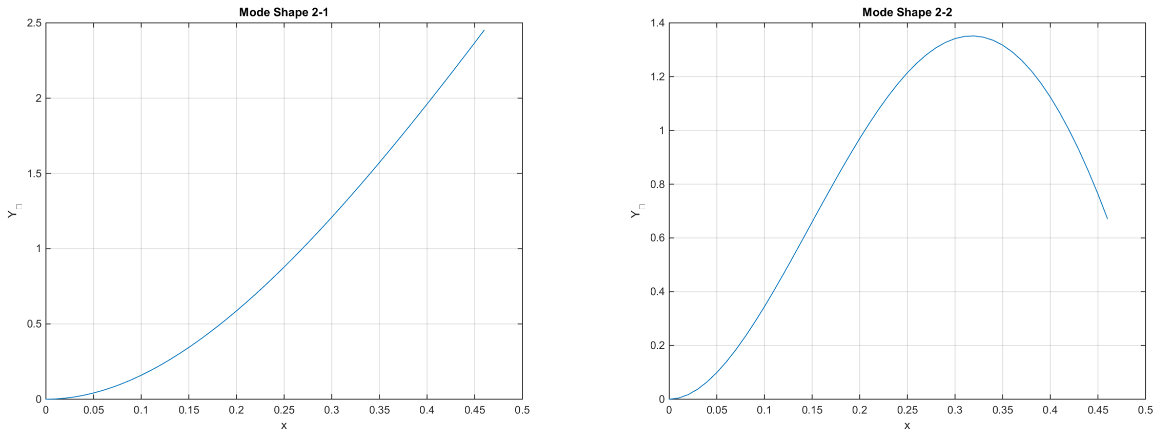

Figure 5: Mode Shape 1 and 2

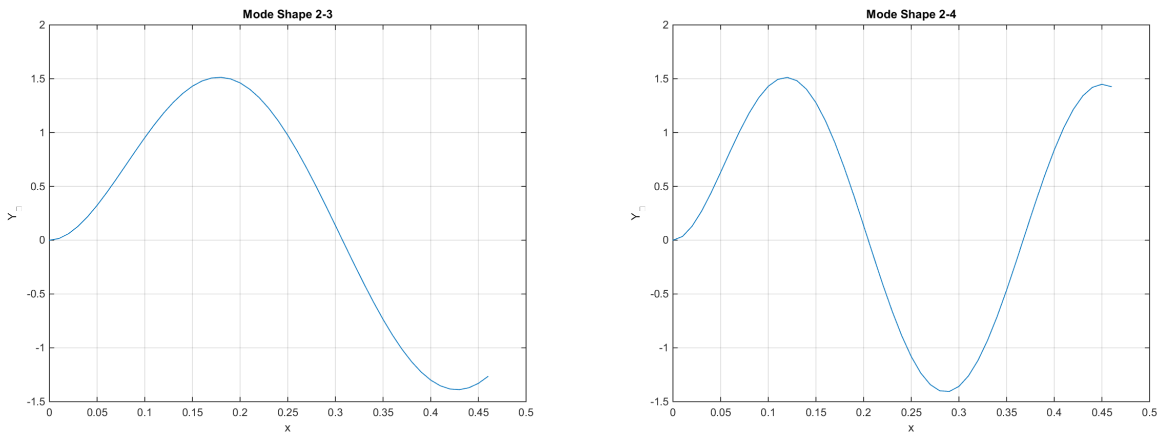

Figure 6: Mode Shape 3 and 4

Problem 2. Consider the given cantilever beam:

1. Determine the first and second mode shapes and natural frequencies using the closed form solution:

   Applied result from **Problem 1** we have the general solution is equation (2):

$$\bar{Y}(X) = a\cosh\beta X + b\sinh\beta X + c\cos\beta X + d\sin\beta X \tag{2}$$

   The boundary conditions are defined as:

   - End displacement (on left): $Y(0,t) = 0 \Rightarrow \bar{Y}(0) = 0 \Leftrightarrow \bar{Y}(0) = a + c = 0$ (9)

   - End slope (on left): $Y'(0,t) = 0 \Rightarrow \bar{Y}'(0) = 0$

$$\text{with } \bar{Y}'(X,t) = \beta a\sinh\beta X + \beta b\cosh\beta X - \beta c\sin\beta X + \beta d\cos\beta X$$

$$\Rightarrow \bar{Y}'(0,t) = \beta b + \beta d = 0 \Leftrightarrow b + d = 0 \tag{10}$$

   - End moment (on right): $M(L,t) = EI\dfrac{\partial^2 Y}{\partial X^2}(L,t) = 0 \Leftrightarrow \bar{Y}''(L,t) = 0$

$$\text{with } \bar{Y}''(X,t) = \beta^2 a\cosh\beta X + \beta^2 b\sinh\beta X - \beta^2 c\cos\beta X - \beta^2 d\sin\beta X$$

$$\Rightarrow \bar{Y}''(L,t) = \beta^2\left(a\cosh\beta L + b\sinh\beta L - c\cos\beta L - d\sin\beta L\right) = 0$$

$$\Leftrightarrow a\cosh\beta L + b\sinh\beta L - c\cos\beta L - d\sin\beta L = 0 \qquad (\text{because } \beta \neq 0)$$

$$\Leftrightarrow \frac{a}{b} = -\frac{\sinh\beta L + \sin\beta L}{\cosh\beta L + \cos\beta L} \tag{11}$$

- Shear force V at right end: $V(L,t) = \bar{M}Y''(L,t) \Leftrightarrow \bar{Y}'''(L,t) = 0$

$$\text{with: } \bar{Y}'''(X,t) = \beta^3 a\sinh\beta X + \beta^3 b\cosh\beta X + \beta^3 c\sin\beta X - \beta^3 d\cos\beta X$$

$$\Rightarrow \bar{Y}'''(L,t) = \beta^3\left(a\sinh\beta L + b\cosh\beta L + c\sin\beta L - d\cos\beta L\right) = 0$$

$$\Leftrightarrow a(\sinh\beta L - \sin\beta L) = -b(\cosh\beta L + \cos\beta L)$$

$$\Leftrightarrow \frac{a}{b} = -\frac{(\cosh\beta L + \cos\beta L)}{(\sinh\beta L - \sin\beta L)} \tag{12}$$

From (11) and (12) we have characteristic equation:

$$\frac{\sinh\beta L + \sin\beta L}{\cosh\beta L + \cos\beta L} = \frac{(\cosh\beta L + \cos\beta L)}{(\sinh\beta L - \sin\beta L)} \Leftrightarrow \cosh\beta L\sin\beta L + 1 = 0$$

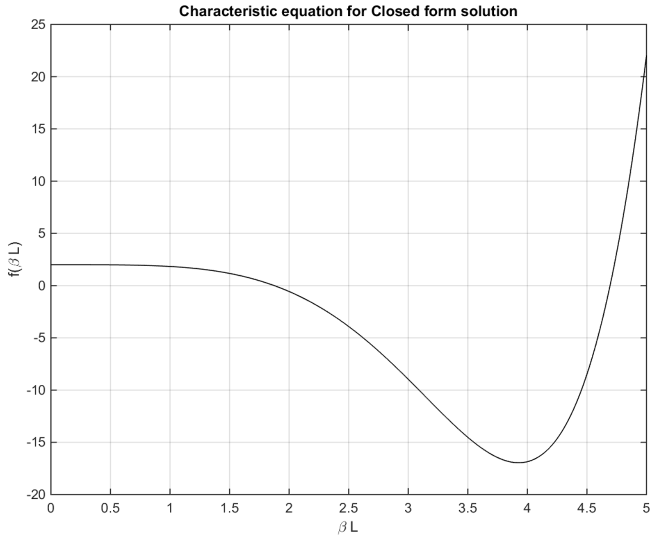

Figure 7: Characteristic equation

The first 2 nonzero roots of characteristic equation are: $\beta_1 L = 1.8751, \beta_2 L = 4.4941$

Natural frequencies is defined from computed values of β, we have:

since: $\beta^4 = \dfrac{\rho A\omega^2}{EI} \Rightarrow \omega = \beta^2\sqrt{\dfrac{EI}{\rho A}} = (\beta L)^2\sqrt{\dfrac{EI}{\rho AL^4}}$

we first have: $\sqrt{\dfrac{EI}{\rho AL^4}} = \sqrt{\dfrac{68.9 \bullet 10^9 \times 6.69 \bullet 10^{-11}}{2.7 \bullet 10^6 \times 78.4 \bullet 10^{-6} \times 0.46^4}} \approx 22.0499$

$$\beta_1 L = 1.665 \Rightarrow \omega_1 = (\beta_1 L)^2 \bullet 22.0499 = 1.8751^2 \bullet 22.0499 = 77.53 \text{ rad/s}$$

$$\beta_2 L = 4.322 \Rightarrow \omega_1 = (\beta_2 L)^2 \bullet 22.0499 = 4.6941^2 \bullet 22.0499 = 485.86 \text{ rad/s}$$

The mode shape can be written as:

$\bar{Y}_i(X) = \left(\dfrac{A}{B}\right)_i \left(\cosh(\beta_i L\dfrac{X}{L}) - \cos(\beta_i L\dfrac{X}{L})\right) + \sinh(\beta_i L\dfrac{X}{L}) - \sin(\beta_i L\dfrac{X}{L})$

with: $\left(\dfrac{A}{B}\right)_i = -\dfrac{\sinh\beta_i L + \sin\beta_i L}{\cosh\beta_i L + \cos\beta_i L}$

we have the table result:

βL	1.8751	4.6941
a/b	-1.3622	-0.9819

$$\bar{Y}_1(X) = -1.3622\left(\cosh(1.8751\frac{X}{0.46}) - \cos(1.8751\frac{X}{0.46})\right) + \sinh(1.8751\frac{X}{0.46}) - \sin(1.8751\frac{X}{0.46})$$

$$= -1.3622\left(\cosh(4.0763X) - \cos(4.0763X)\right) + \sinh(4.0763X) - \sin(4.0763X)$$

$$
\begin{aligned}
\bar{Y}_2(X) &= -0.9819\left(\cosh(4.6941\frac{X}{0.46}) - \cos(4.6941\frac{X}{0.46})\right) + \sinh(4.6941\frac{X}{0.46}) - \sin(4.6941\frac{X}{0.46}) \\
&= -0.985\left(\cosh(10.2046X) - \cos(10.2046X)\right) + \sinh(10.2046X) - \sin(10.2046X)
\end{aligned}
$$

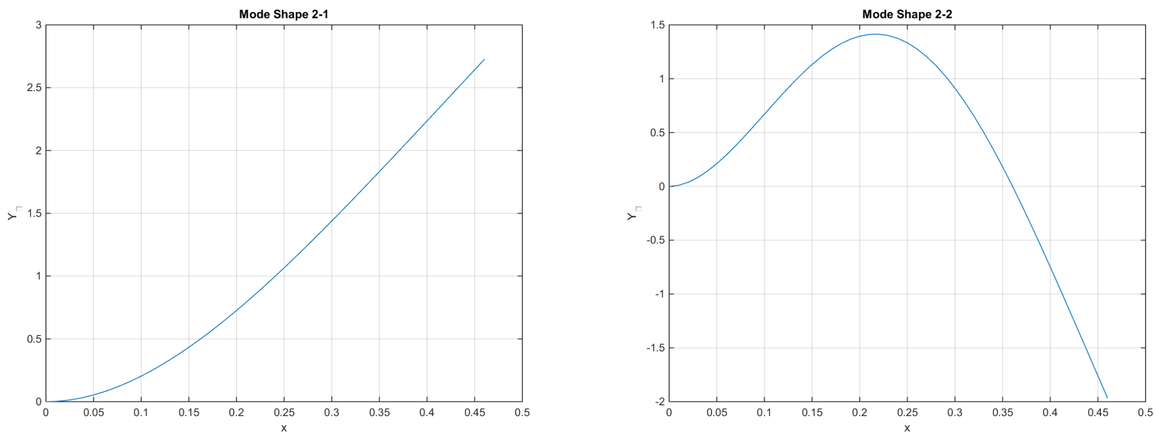

Figure 8: Mode Shape 1 and 2 - no mass applied

2. Mode the beam with three finite elements:

   Beam is divided by three element and four-node mesh (length of each element is $l_e = 46/3$ cm).

   We will have 4 linear coordinates $u_1(t), u_3(t), u_5(t), u_7(t)$ and 4 rotational coordinates $u_2(t), u_4(t), u_6(t), u_8(t)$

   For each element, the transverse static displacement must satisfy: $\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 u(x,t)}{\partial x^2}\right] = 0$

   with: $u(x,t) = c_1(t)x^3 + c_2(t)x^2 + c_3(t)x + c_4(t)$

   $c_i(t)$ are constants of integration with respect to spatial variable x

   The unknown nodal displacement $u_i(t)$ of each element must satisfy the boundary condition:

   $u(0,t) = u_1(t); u_x(0,t) = u_2(t); u(l_e,t) = u_3(t); u(l_e,t) = u_4(t)$

$$
\begin{aligned}
c_4(t) &= u_1(t) \qquad\qquad c_3(t) = u_2(t) \\
c_2(t) &= \frac{1}{l_e^2}[3(u_3 - u_1) - l_e(2u_2 + u_4)] \\
c_1(t) &= \frac{1}{l_e^3}[2(u_1 - u_3) - l_e(u_2 + u_4)]
\end{aligned}
$$

$$
\Rightarrow u(x,t) = \left[1 - 3\frac{x^2}{l_e^2} + 2\frac{x^3}{l_e^3}\right]u_1(t) + l_e\left[\frac{x}{l_e} - 2\frac{x^2}{l_e^2} + \frac{x^3}{l_e^3}\right]u_2(t) + \left[3\frac{x^2}{l_e^2} - 2\frac{x^3}{l_e^3}\right]u_3(t) + l_e\left[-\frac{x^2}{l_e^2} + \frac{x^3}{l_e^3}\right]u_4(t)
$$

   For the first element:

   + The kinetic energy of element: $T(t) = \frac{1}{2}\int_0^{l_e} \rho\mathbf{A}[u_t(x,t)]^2 \mathrm{d}x = \frac{1}{2}\dot{\mathbf{u}}_1^T M \dot{\mathbf{u}}_1$

$$
\text{with: } \mathbf{u}_1(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} \qquad \text{and} \qquad M = \frac{\rho\mathbf{A}l_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -12l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}
$$

   + The strain energy is given by: $V(t) = \int_0^{l_e} EI[u_{xx}(x,t)]^2 \mathrm{d}x = \frac{1}{2}\mathbf{u}_1^T K \mathbf{u}_1$

when $\mathbf{u}_1$ is defined above and $K = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$ The equation for the first element is:

$$M\ddot{\mathbf{u}}_1 + K\mathbf{u}_1 = \frac{\rho \mathbf{A} l_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -12l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \\ \ddot{u}_4(t) \end{bmatrix} + \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} =$$

Because the left end of rod is clamped so both the deflection and slope is 0, we have: $u_1 = u_2 = 0$

Equation (13) become:

$$\frac{\rho \mathbf{A} l_e}{420}\begin{bmatrix} 156 & -22l_e \\ -22l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3(t) \\ \ddot{u}_4(t) \end{bmatrix} + \frac{EI}{l_e^3}\begin{bmatrix} 12 & -6l_e \\ -6l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} u_3(t) \\ u_4(t) \end{bmatrix} = 0 \qquad (14)$$

Similarity, we have for the second element:

$$M\ddot{\mathbf{u}}_2 + K\mathbf{u}_2 = \frac{\rho \mathbf{A} l_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -12l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3(t) \\ \ddot{u}_4(t) \\ \ddot{u}_5(t) \\ \ddot{u}_6(t) \end{bmatrix} + \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} u_3(t) \\ u_4(t) \\ u_5(t) \\ u_6(t) \end{bmatrix} =$$

For the third element:

$$M\ddot{\mathbf{u}}_3 + K\mathbf{u}_3 = \frac{\rho \mathbf{A} l_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -12l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_5(t) \\ \ddot{u}_6(t) \\ \ddot{u}_7(t) \\ \ddot{u}_8(t) \end{bmatrix} + \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} u_5(t) \\ u_6(t) \\ u_7(t) \\ u_8(t) \end{bmatrix} =$$

From (14) (15) and (16) we have the global equation:

$$\frac{\rho \mathbf{A} l_e}{420}\begin{bmatrix} 312 & 0 & 54 & -13l_e & 0 & 0 \\ 0 & 8l_e^2 & 13l_e & -3l_e^2 & 0 & 0 \\ 54 & 13l_e & 312 & 0 & 54 & -13l_e \\ -13l_e & -3l_e^2 & 0 & 8l_e^2 & 13l_e & -3l_e^2 \\ 0 & 0 & 54 & 13l_e & 156 & -22l_e \\ 0 & 0 & -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3(t) \\ \ddot{u}_4(t) \\ \ddot{u}_5(t) \\ \ddot{u}_6(t) \\ \ddot{u}_7(t) \\ \ddot{u}_8(t) \end{bmatrix} + \frac{EI}{l_e^3}\begin{bmatrix} 24 & 0 & -12 & 6l_e & 0 & 0 \\ 0 & 8l_e^2 & -6l_e & 2l_e^2 & 0 & 0 \\ -12 & -6l_e & 24 & 0 & -12 & 6l_e \\ 6l_e & 2l_e^2 & 0 & 8l_e^2 & -6l_e & 2l_e^2 \\ 0 & 0 & -12 & -6l_e & 12 & -6l_e \\ 0 & 0 & 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}\begin{bmatrix} u_3(t) \\ u_4(t) \\ u_5(t) \\ u_6(t) \\ u_7(t) \\ u_8(t) \end{bmatrix}$$

Let $\mathbf{u} = \mathbf{x}e^{-\omega t}$, we have: $\ddot{\mathbf{u}} = -\omega^2 \mathbf{x} e^{\omega t}$

$$\frac{\rho \mathbf{A} l_e}{420} = \frac{2.7 \bullet 10^3 \times 3.2 \bullet 10^{-3} \times 2.45 \bullet 10^{-2} \times 0.46}{3 * 420} = 0.00773$$

and $\frac{EI}{l_e^3} = \frac{68.9 \bullet 10^9 \times 6.69 \bullet 10^{-11} \times 3^3}{0.46^3} = 1278.6$

(17) will become: $\left(-\omega^2[\mathbf{M}] + [\mathbf{K}]\right)\mathbf{x}e^{-\omega t} = 0;$

Utilize `MATLAB` to solve this equation, we come up with two first natural frequency (value of ω):

$\omega_1 = 76.16$ and $\omega_2 = 478.81$

Characteristic equation for matrix $\left[-\omega^2[\mathbf{M}] + [\mathbf{K}]\right]$

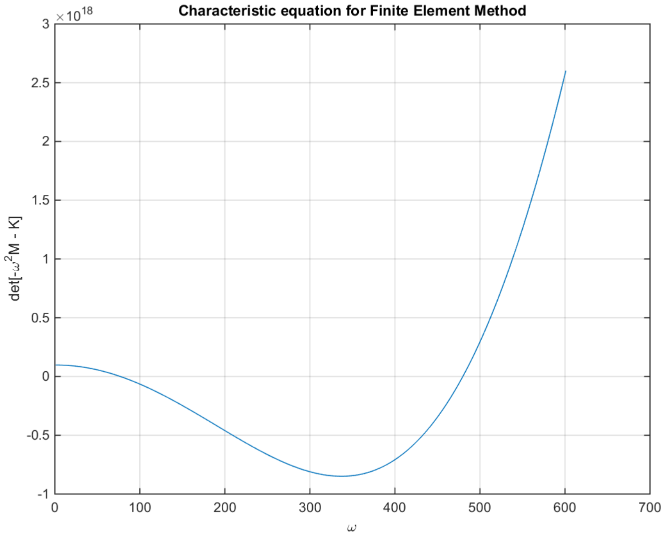

Figure 9: Characteristic equation for finite element method

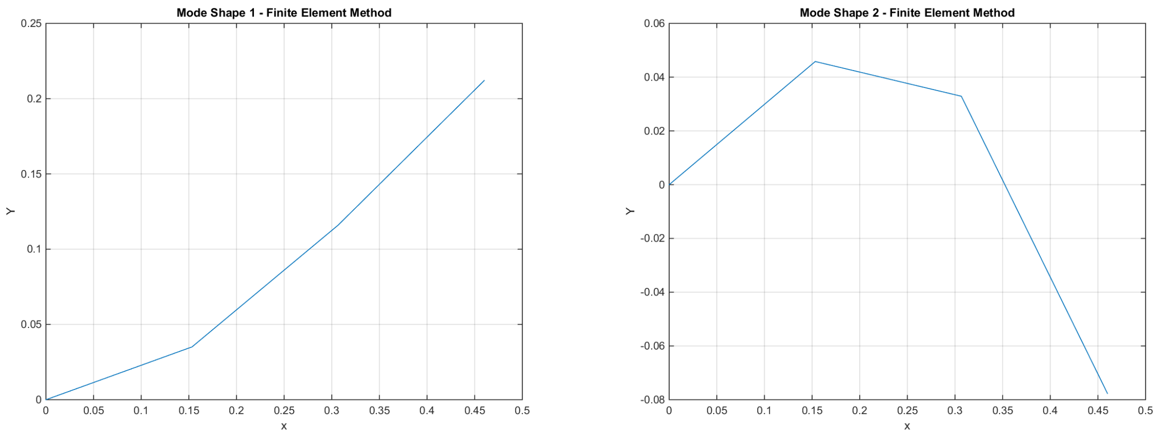

Figure 10: Mode Shape 1 and 2 - no mass applied

Problem 3. Use Discrete Fourier Transform equation to transform the square wave data sample to the frequency domain:

1. Use a sampling rate of 25 Hz ($\Delta t = 0.04$) over a 1-second sampling period and determine:

   The *magnitude frequency* domain component up to the Nyquist frequency: The inverse discrete Fourier

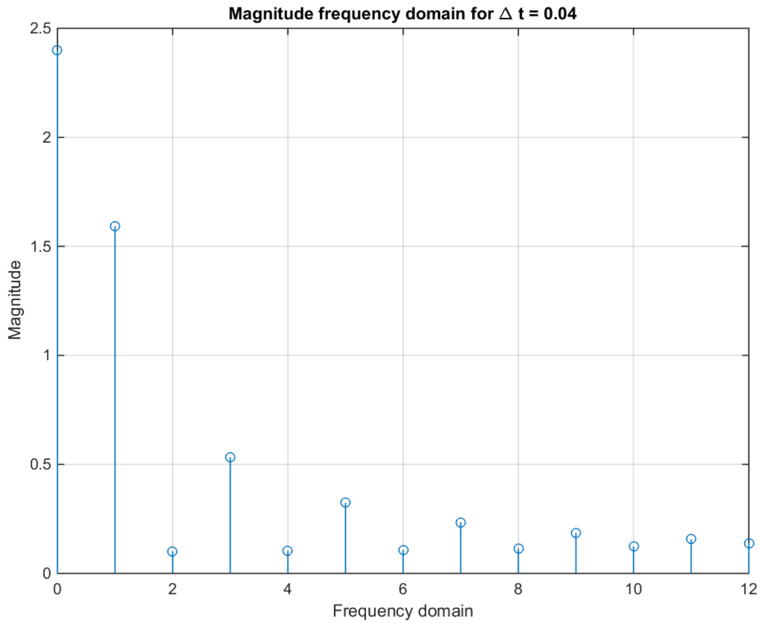

Figure 11: The *magnitude frequency* domain component $\Delta t = 0.04$

   Transform (IDFT) using the harmonic component:

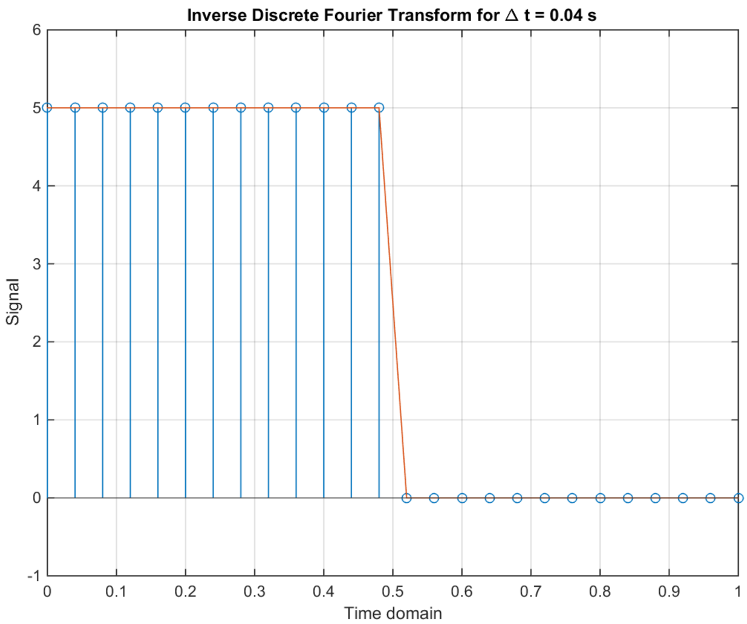

Figure 12: The inverse discrete Fourier Transform for $\Delta t = 0.04$

2. Change sampling rate to 16.67 Hz ($\Delta t = 0.06$):

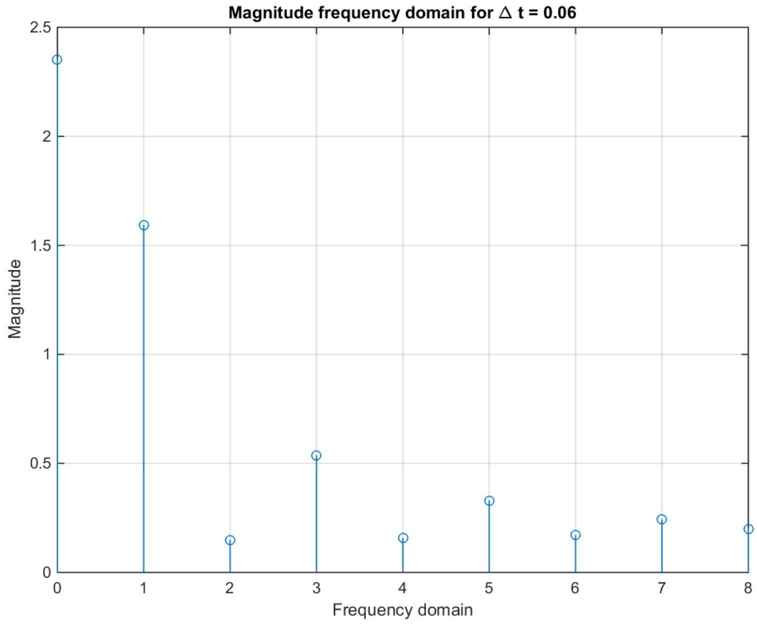

Figure 13: The *magnitude frequency* domain component $\Delta t = 0.06$

The inverse discrete Fourier Transform (IDFT) using the harmonic component:

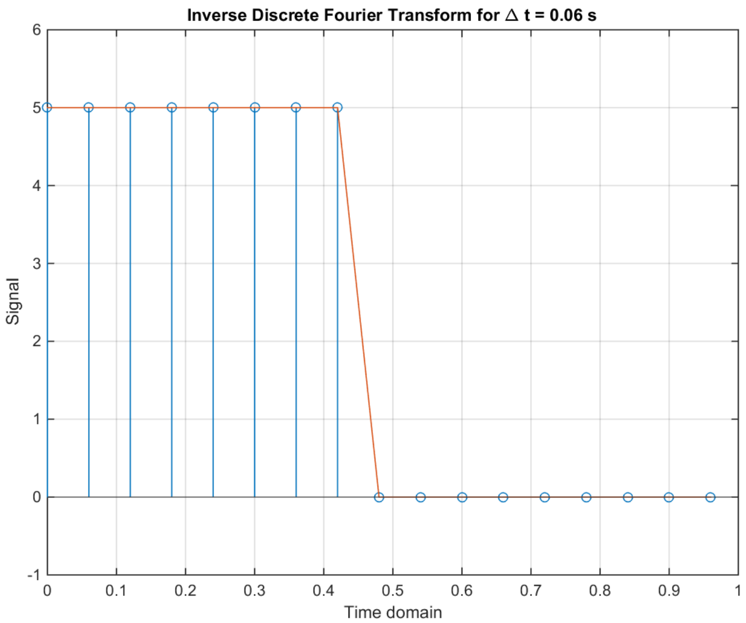

Figure 14: The inverse discrete Fourier Transform for $\Delta t = 0.06$

When overlay frequency in both case:

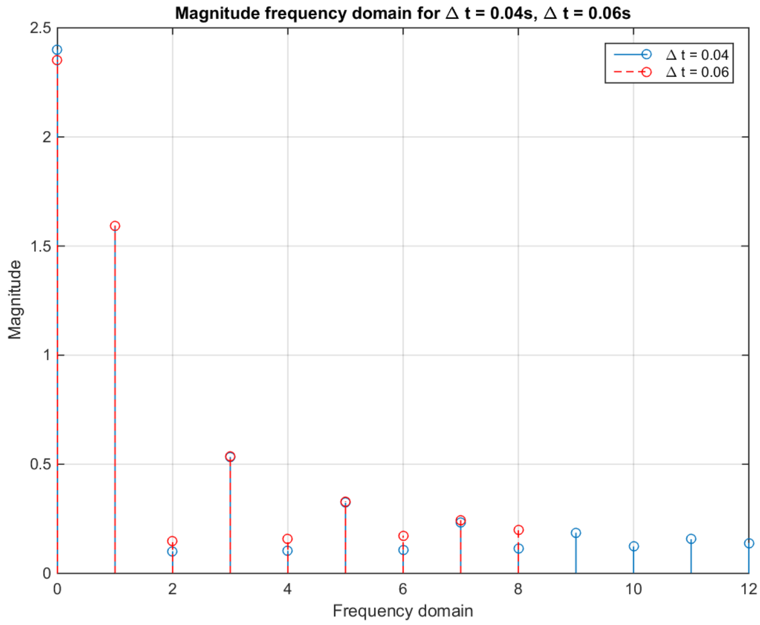

Figure 15: The *magnitude frequency* domain component: $\Delta t = 0.04$ and $\Delta t = 0.06$

From above figure we see that with odd number (on frequency domain) we have no leakage, with even number (on frequency domain), we will have leakage (it seems increasing).

The inverse discrete Fourier Transform (IDFT) using the harmonic component:

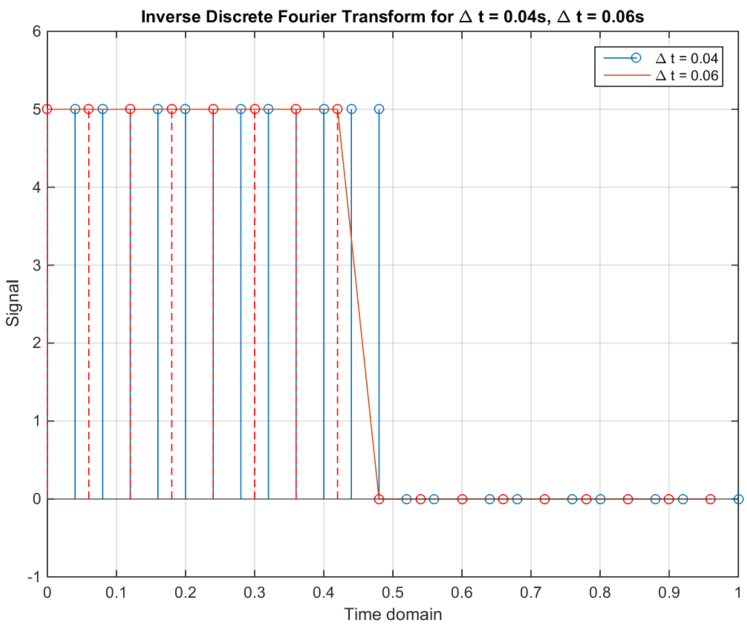

Figure 16: The inverse discrete Fourier Transform for $\Delta t = 0.04$ and $\Delta t = 0.06$

Why aliasing is a problem with this signal?

Because the sampled frequency is higher than the Nyquist frequency, so it will lead to the aliasing.

Problem 4. Consider the provided model to estimate the FRF for a system:

1. The FRF H_c magnitude only with true FRF:

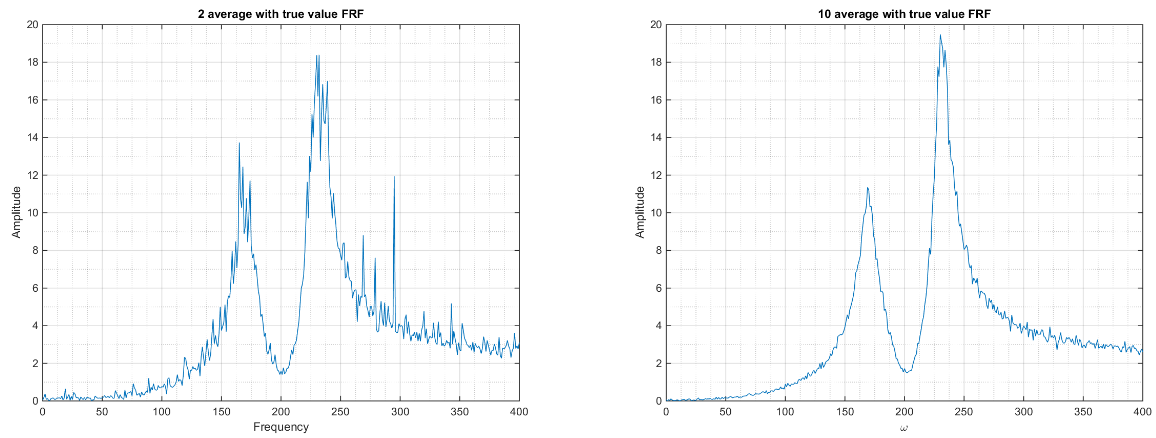

Figure 17: FRF H_c for 2 and 10 average with true FRF

2. The coherence γ^2_{FX}

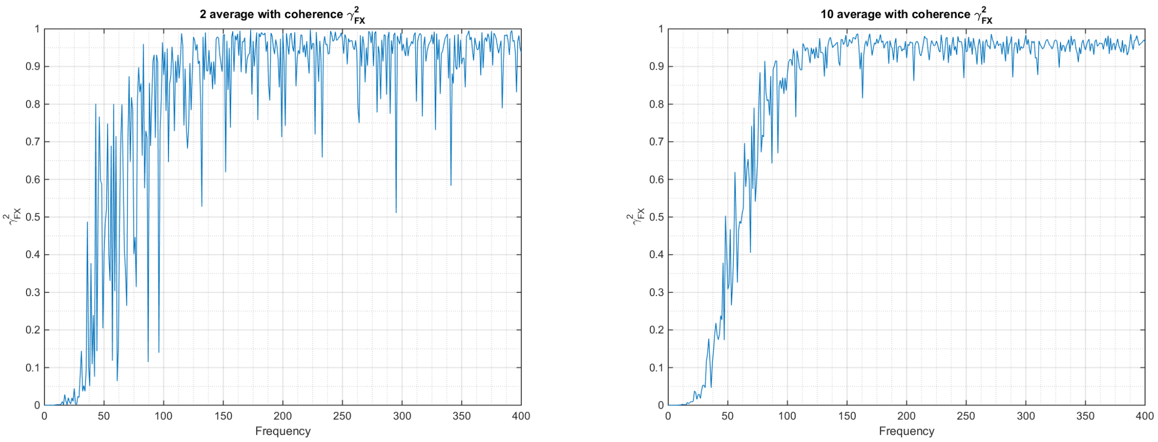

Figure 18: Coherence γ^2_{FX} for 2 and 10 average

3. The single-sided spectral densities

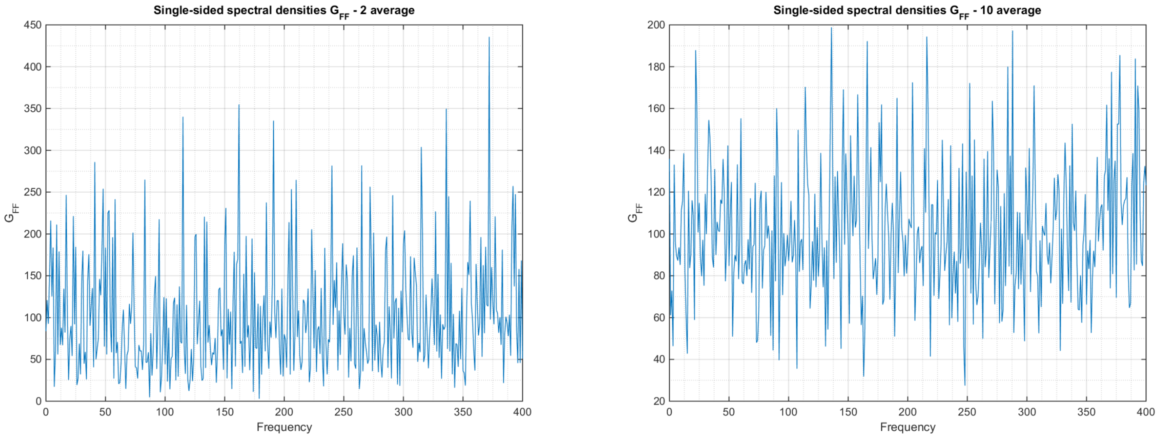

Figure 19: The single-sided spectral G_{FF} for 2 and 10 average

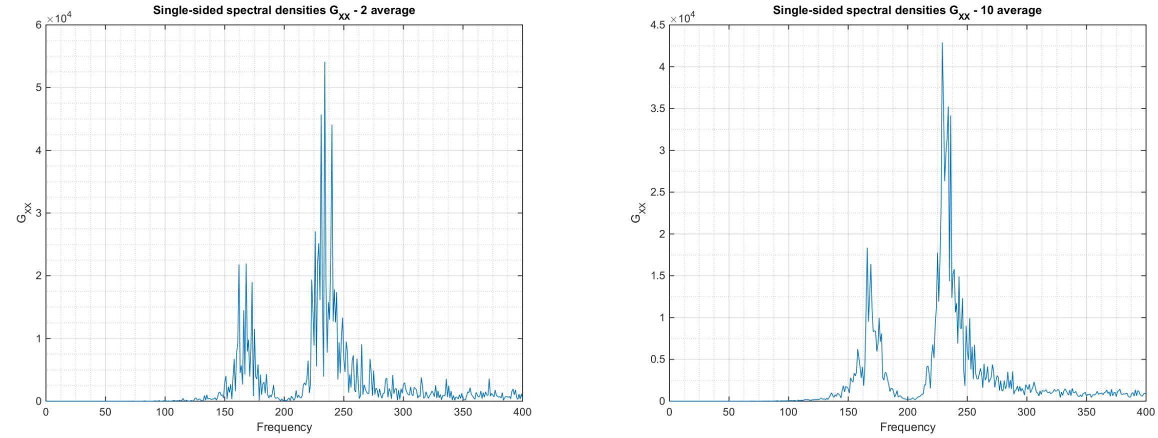

Figure 20: The single-sided spectral G_{XX} for 2 and 10 average

4. Overlay the value of H_c for 2 and 10 average on true value FRF we have:

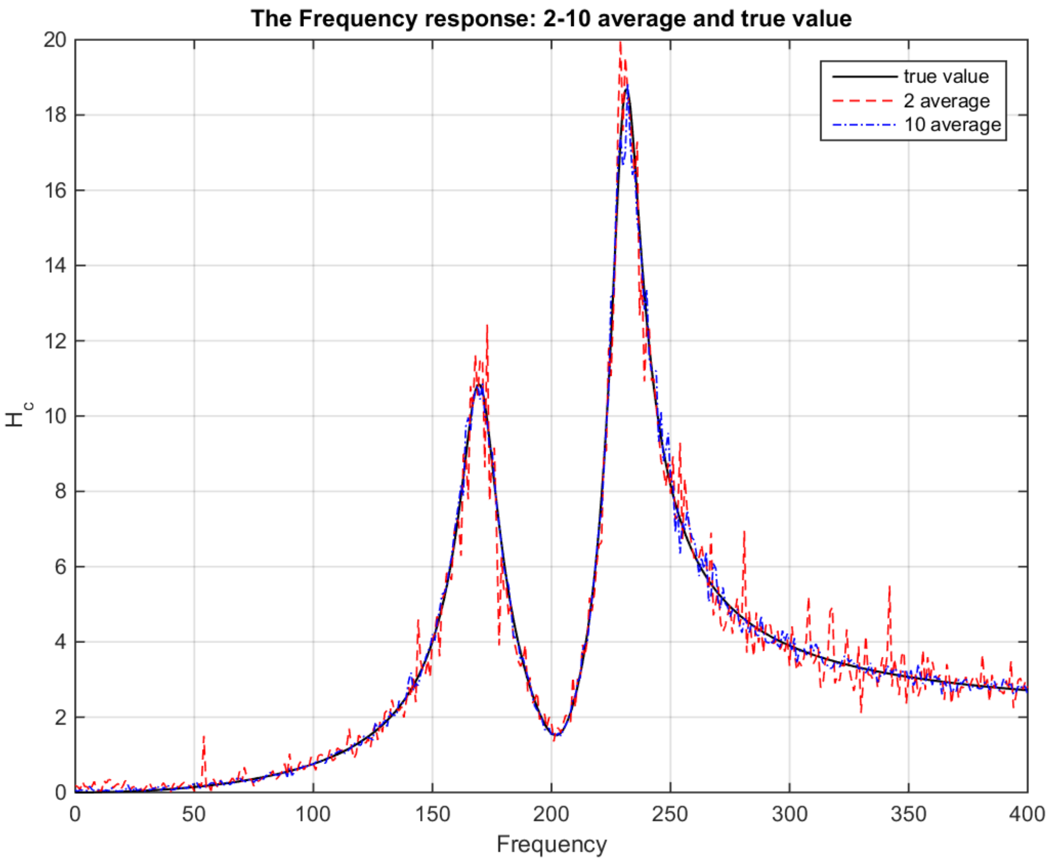

Figure 21: The H_c of true FRF and 2-10 average estimated

We see that in both estimated (with 2-average and 10-average), the mean of H_c still very closed to the true value of FRF $\Rightarrow H_c$ is an unbiased estimator.