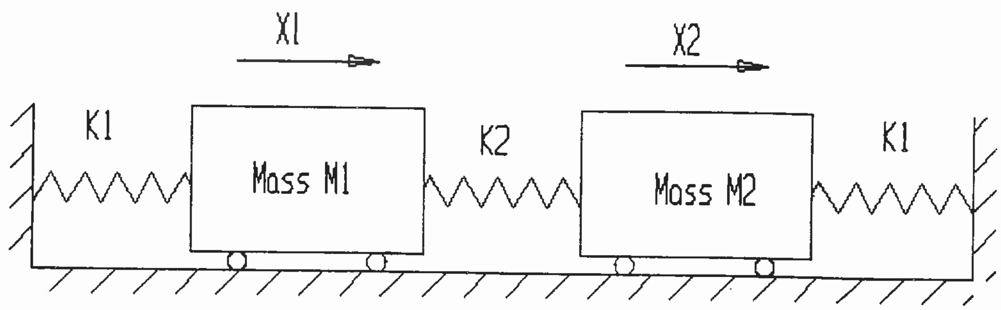
3. Multi-Degree-of-Freedom (MDOF) Systems

Consider the two-degree-of-freedom system below:



If we use Newton's 2nd Law to write the differential equations of motion, we have:

$$\overset{+x1\rightarrow}{\sum F} = M_1\ddot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 = 0$$

$$\overset{+x2\rightarrow}{\sum F} = M_2\ddot{x}_2 + (K_1 + K_2)x_2 - K_2x_1 = 0 \tag{3.1}$$

These can be written in matrix form:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_1 + K_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.2}$$

The system at this point is unforced, so the solution will consist only of the homogeneous part, $x_h(t)$. Assuming harmonic motion of the two masses occurring at the same frequency ω gives:

$$
\begin{aligned}
x_{h1} &= X_1 e^{i\omega t} \\
x_{h2} &= X_2 e^{i\omega t}
\end{aligned}
\tag{3.3}
$$

This results in the matrix equation:

$$
\begin{bmatrix} -\omega^2 M_1 + (K_1 + K_2) & -K_2 \\ -K_2 & -\omega^2 M_2 + (K_1 + K_2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\omega t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.4}
$$

Which has a nontrivial solution when the determinant of the matrix is equal to zero:

$$
\begin{vmatrix} -\omega^2 M_1 + (K_1 + K_2) & -K_2 \\ -K_2 & -\omega^2 M_2 + (K_1 + K_2) \end{vmatrix} = 0 \tag{3.5}
$$

At this point, sample numbers will be substituted to illustrate the free vibration solution:

$$
\begin{aligned}
M_1 &= M_2 = 1 \\
K_1 &= 25 \text{ (consistent units)} \\
K_2 &= 5
\end{aligned}
\tag{3.6}
$$

Taking the determinant of the above matrix results in the following quadratic equation:

$$
\lambda^2 - 60\lambda + 875 = 0 \tag{3.7}
$$

Where $\lambda = \omega^2$. At this point it is interesting to note that when λ is substituted into the original Eqn. (3.4) and terms are rearranged, the following equation results:

$$\begin{bmatrix} (K_1+K_2)/M_1 & -K_2/M_1 \\ -K_2/M_2 & (K_1+K_2)/M_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.8)$$

This is the standard form of an eigenvalue problem, hence the choice of $\lambda = \omega^2$.

Solving for λ: $\lambda_{1,2} = 25, 35$

And the natural frequencies of oscillation are:

$$\omega_1 = \sqrt{\lambda_1} = 5 \text{ rad / sec}$$
$$\omega_2 = \sqrt{\lambda_2} = 5.9 \text{ rad / sec}$$

The eigenvectors are obtained by substituting the eigenvalues into one of the system equations. Substituting the first eigenvalue:

$$\left((K_1+K_2)/M_1 \right) X_1 - \left(K_2/M_1 \right) X_2 - \lambda X_1 = 0$$
$$(30-25) X_1 - 5 X_2 = 0 \quad (3.9)$$
$$X_1 = X_2$$

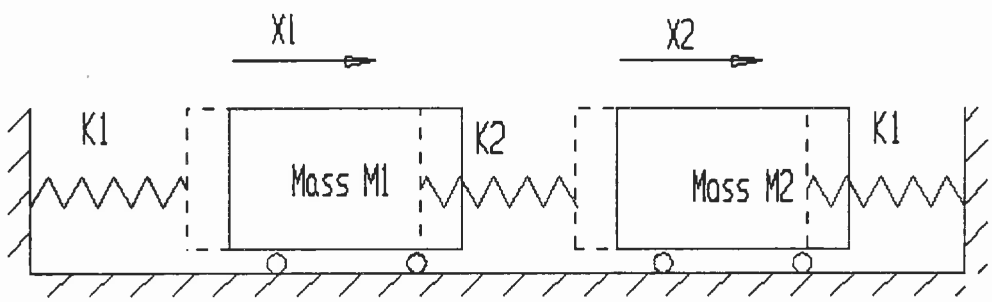
The first eigenvector is:

$$\phi_1 = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{3.10}$$

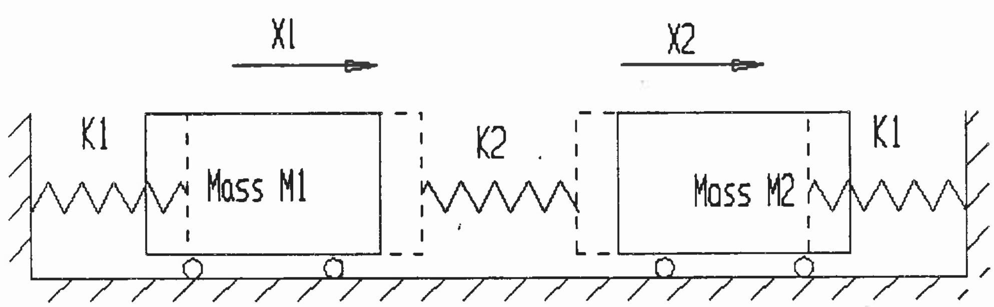
And the second eigenvector is:

$$\phi_2 = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{3.11}$$

The eigenvectors represent the *modes* of vibration at the natural frequencies. The first mode shape can be shown as:



And the second mode shape is:



Note that the mode shapes are *dependent* on each other, and the amplitudes can be arbitrarily scaled.

3.1 Orthogonality of Eigenvectors

The system of equations of (3.4) can be equivalently written as:

$$[K]\{X\} = \lambda[M]\{X\} \tag{3.12}$$

Replacing $\{X\}$ with the *i*th eigenvector ϕ_i and substituting the corresponding eigenvalue results in:

$$[K]\phi_i = \lambda_i[M]\phi_i \tag{3.13}$$

Premultiplying by ϕ_j^T results in:

$$\phi_j^T[K]\phi_i = \lambda_i\,\phi_j^T[M]\phi_i \tag{3.14}$$

Likewise, the eigenvalues / eigenvectors can be switched to obtain:

$$\phi_i^T[K]\phi_j = \lambda_j\,\phi_i^T[M]\phi_j \tag{3.15}$$

For most dynamic systems, $[M]$ and $[K]$ are symmetric, so the following relations can be written:

$$\phi_i^T[K]\phi_j = \phi_j^T[K]\phi_i$$
$$\phi_i^T[M]\phi_j = \phi_j^T[M]\phi_i \qquad (3.16)$$

Equations (3.14) and (3.15) can be rewritten using (3.16):

$$\phi_j^T[K]\phi_i = \lambda_j \phi_i^T[M]\phi_j$$
$$\phi_j^T[K]\phi_i = \lambda_i \phi_i^T[M]\phi_j \qquad (3.17)$$

And taking the difference of these equations results in:

$$0 = (\lambda_i - \lambda_j)\phi_i^T[M]\phi_j \qquad (3.18)$$

For distinct eigenvalues, this implies that $\phi_i^T[M]\phi_j = 0$. Similarly, it can be shown that $\phi_i^T[K]\phi_j = 0$.

Observation: *The eigenvectors are orthogonal with respect to the mass and stiffness matrices.*

When the same eigenvector is used to pre- and post-multiply the mass and stiffness matrices, the resulting mass and stiffness values are referred to as the *modal* or *generalized* mass and stiffness:

$$\phi_i^T[M]\phi_i = m_{ii}$$
$$\phi_i^T[K]\phi_i = k_{ii} \qquad (3.19)$$

A matrix of mode shapes can be created, $\left[\{\phi_1\} \quad \{\phi_2\}\right] = [\phi]$, and when this matrix is used to pre- and post-multiply the mass and stiffness matrices, diagonal mass and stiffness matrices are obtained:

$$[\phi] = \left[\{\phi_1\} \quad \{\phi_2\}\right] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$[\phi]^T[M][\phi] = \left[\ddots m_{ii} \ddots\right] \qquad (3.20)$$

$$[\phi]^T[K][\phi] = \left[\ddots k_{ii} \ddots\right]$$

Inputting the actual mass and stiffness values results in:

$$\left[\ddots m_{ii} \ddots\right] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\left[\ddots k_{ii} \ddots\right] = \begin{bmatrix} 50 & 0 \\ 0 & 70 \end{bmatrix}$$

Once this is determined, the eigenvectors can be redefined to be *mass normalized eigenvectors*:

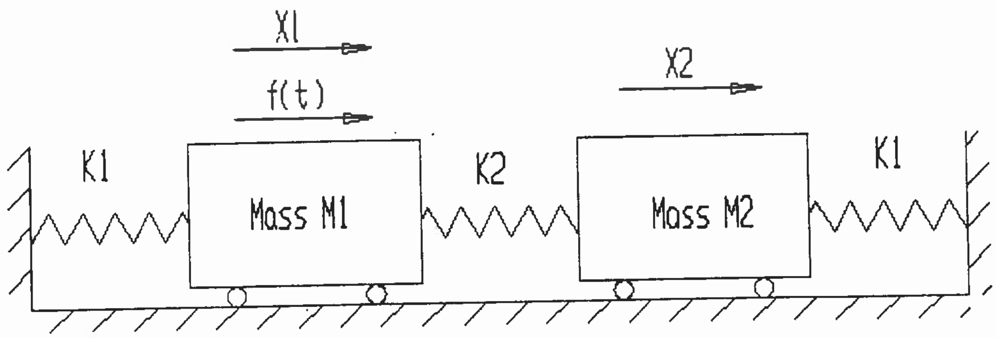
$$\hat{\phi}_i = \frac{1}{\sqrt{m_{ii}}}\phi_i \qquad (3.21)$$

And when the original mass and stiffness matrices are pre- and post-multiplied by the mass normalized eigenvector matrix, the result is:

$$
\begin{aligned}
[\hat{\phi}]^T[M][\hat{\phi}] &= \left[\ddots I \ddots\right] \\
[\hat{\phi}]^T[K][\hat{\phi}] &= \left[\ddots \lambda \ddots\right]
\end{aligned}
\tag{3.22}
$$

3.1 *Harmonically Forced Vibration of a Multi-Degree-Of-Freedom System*

If a harmonic forcing function is added to the first mass in the figure below, the response is found in a manner similar to that for the free vibration case:



$$
\begin{aligned}
f(t) &= F_1 e^{i\omega t} \\
x_1(t) &= X_1 e^{i\omega t} \\
x_2(t) &= X_2 e^{i\omega t}
\end{aligned}
\tag{3.23}
$$

In this case, the frequency ω corresponds to the forcing frequency and *not* the natural frequency of vibration.

Using the above form of the solution and writing the equations in matrix form:

$$
\left(\begin{bmatrix} -\omega^2 M_1 & \\ & -\omega^2 M_2 \end{bmatrix} + \begin{bmatrix} (K_1+K_2) & -K_2 \\ -K_2 & (K_1+K_2) \end{bmatrix}\right)\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\omega t} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} e^{i\omega t} \tag{3.24}
$$

Recall that we can diagonalize the mass and stiffness matrices by using the modal matrix. This will be accomplished by substituting a new variable into the equation:

$$[\phi]\{Y\} = \{X\} \tag{3.25}$$

Where $\{Y\}$ consists of new coordinates Y_1 and Y_2 which are called *modal* or *generalized* coordinates.

These values represent the amount of mode 1 and mode 2 that are contained in the motion of the system.

Substituting into Eqn. 3.24:

$$\left(-\omega^2[M]+[K]\right)[\phi]\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} e^{i\omega t} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} e^{i\omega t} \tag{3.26}$$

And pre-multiplying by $[\phi]^T$ results in:

$$\begin{aligned} [\phi]^T\left(-\omega^2[M]+[K]\right)[\phi]\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} e^{i\omega t} &= [\phi]^T\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} e^{i\omega t} \\ \left(-\omega^2\left[\ddots m \ddots\right]+\left[\ddots k \ddots\right]\right)\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} e^{i\omega t} &= [\phi]^T\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} e^{i\omega t} \end{aligned} \tag{3.27}$$

With the diagonal left-hand matrix, it becomes easy to solve for the Y vector:

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \left(-\omega^2 \begin{bmatrix} \ddots & & \\ & m & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} \ddots & & \\ & k & \\ & & \ddots \end{bmatrix}\right)^{-1} [\phi]^T \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \qquad (3.28)$$

And obtaining the X vector is a matter of returning to the original transformation of Eqn. 3.25:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = [\phi]\left(-\omega^2 \begin{bmatrix} \ddots & & \\ & m & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} \ddots & & \\ & k & \\ & & \ddots \end{bmatrix}\right)^{-1} [\phi]^T \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \qquad (3.29)$$

This can be simplified:

$$\boldsymbol{X} = \boldsymbol{HF} \qquad (3.30)$$

Where $\boldsymbol{H}$ is the frequency response function (FRF) matrix.

Writing out the mode vectors and solving for X_1 results in:

$$X_1 = \left(\phi_{11}\left(\frac{1}{-\omega^2 m_{11} + k_{11}}\right) \phi_{12}\left(\frac{1}{-\omega^2 m_{22} + k_{22}}\right)\right)\begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix}\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

$$X_1 = \left(\phi_{11}^2\left(\frac{1}{-\omega^2 m_{11} + k_{11}}\right) + \phi_{12}^2\left(\frac{1}{-\omega^2 m_{22} + k_{22}}\right)\right)F_1 \qquad (3.31)$$

Recall that the indices of the mode vector components are identified by ϕ_{kr}, where k is the spatial coordinate and r is the mode number. This allows a general definition of the response at the ith degree of freedom (position) to a force at the jth degree of freedom:

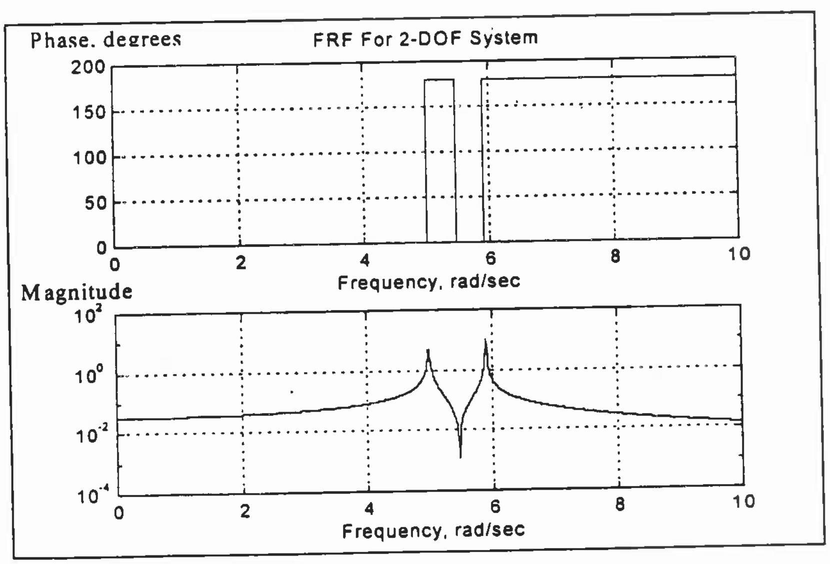
$$\frac{X_i}{F_j} = \sum_{r=1}^{n} \left(\frac{\phi_{ir}\phi_{jr}}{-\omega^2 m_r + k_r} \right) \tag{3.32}$$

Where the number of modes n is *equal* to the number of discrete degrees of freedom.

Returning to the original example and using the mode shapes derived:

$$\frac{X_1}{F_1} = \sum_{r=1}^{2} \left(\frac{\phi_{1r}\phi_{1r}}{-\omega^2 m_r + k_r} \right) = \frac{1}{-\omega^2(2)+50} + \frac{1}{-\omega^2(2)+70} \tag{3.33}$$

A plot of this frequency response function (FRF) as a magnitude and phase angle is shown below:



Some notes about the plot:

1. Typically, FRF's are presented as magnitude and phase to indicate how the degree of freedom behaves relative to the force. A zero-phase angle indicates that the response coincides with the force; a 180-degree phase indicates the force is always directed *against* the direction of the motion.

2. The magnitude at the natural frequencies of the system (5.0 and 5.9 rad/sec) is infinite. The plot resolution has limited the amplitude.

3. This is called a *driving point* FRF, because the point being measured is coincident with the force. A characteristic of a driving point FRF is the antiresonance, seen at approximately 5.5 rad/sec. The FRF for mass 2 does not show a zero amplitude point.

There are two common variations of equation (3.32):

1. The mass-normalized eigenvectors are frequently used in this equation, resulting in a simpler form:

$$\frac{X_i}{F_j} = \sum_{r=1}^{n}\left(\frac{\hat{\phi}_{ir}\hat{\phi}_{jr}}{-\omega^2 + \omega_r^2}\right) = \sum_{r=1}^{n}\left(\frac{\hat{\phi}_{ir}\hat{\phi}_{jr}}{-\omega^2 + \lambda_r}\right) \tag{3.34}$$

The term ω_r is recognized as the r^{th} natural frequency.

Using the specific example above, this becomes:

$$\frac{X_1}{F_1} = \sum_{r=1}^{2}\left(\frac{\hat{\phi}_{1r}\hat{\phi}_{1r}}{-\omega^2 + \omega_r^2}\right) = \frac{1/2}{-\omega^2 + 25} + \frac{1/2}{-\omega^2 + 35} \qquad (3.35)$$

2. If the FRF is obtained from experimental measurements, then it may not be possible to identify the individual mode vector components, so one term is used to describe both:

$$\frac{X_i}{F_j} = \sum_{r=1}^{n}\left(\frac{\hat{\phi}_{ir}\hat{\phi}_{jr}}{-\omega^2 + \lambda_r}\right) = \sum_{r=1}^{n}\left(\frac{{}_rA_{ij}}{-\omega^2 + \lambda_r}\right) \qquad (3.36)$$

The term A is called a *residue*, or *modal constant*. The term residue comes from pole-zero terminology where the *zero*th order numerator term is a residue of the function.

Note that we have now bridged analysis with measurement, through the use of the mode shapes. A discrete system can be experimentally measured and its actual parameters determined from the measurements.