ME 5514 VIBRATION MECHANICS SYSTEMS
Homework 2

Luan Cong Doan
luandoan@vt.edu

April 18, 2016

Consider the dx element of the given cantilever beam we have: V and M are the shear force and bending moments respectively.

- Summing the force in Y-direction for the dm elements: $\mathrm{d}m\bar{Y} = \sum F_y \Leftrightarrow \rho A \dfrac{\partial^2 Y}{\partial t^2} = -\dfrac{\partial V}{\partial X}$

with: ρ is the density (mass/volume): $\rho = 2.7\ g/cm^3$

A is the cross-sectional area: $A = 0.32 * 2.45 = 0.784\ cm^2$

The bending moment M is defined by: $M = EI\dfrac{\partial^2 Y}{\partial X^2}$ with I is moment inertial of cross-sectional

$$I = \frac{bh^3}{12} = \frac{2.45 * 10^{-2} * (3.2 * 10^{-3})^3}{12} = 6.69 \times 10^{-11}\ m^4$$

The shear force V is calculated from M: $V = \dfrac{\partial M}{\partial X}$

Because EI is constant, we come up with: $\dfrac{\partial^2 Y}{\partial t^2} + \dfrac{EI}{\rho A}\dfrac{\partial^4 Y}{\partial X^4} = 0$ (1)

Assume the vibration solution is: $Y(X,t) = \bar{Y}(X)\sin\omega t$

(1) become: $\dfrac{\mathrm{d}^4\bar{Y}}{\mathrm{d}X^4} - \beta^4\bar{Y} = 0$ with $\beta^4 = \dfrac{\rho A\omega^2}{EI}$

Solution: $\bar{Y}(X) = a\cosh\beta X + b\sinh\beta X + c\cos\beta X + d\sin\beta X$ (2)

1. Assume the tip mas can be modeled as a point mass $\Rightarrow$ the boundary conditions are:

   - End displacement (on left): $Y(0,t) = 0 \Rightarrow \bar{Y}(0) = 0 \Leftrightarrow \bar{Y}(0) = a + c = 0$ (3)

   - End slope (on left): $Y'(0,t) = 0 \Rightarrow \bar{Y}'(0) = 0$

   with $\bar{Y}'(X,t) = \beta a\sinh\beta X + \beta b\cosh\beta X - \beta c\sin\beta X + \beta d\cos\beta X$

   $\Rightarrow \bar{Y}'(0,t) = \beta b + \beta d = 0 \Leftrightarrow b + d = 0$ (4)

   - End moment (on right): $M(L,t) = EI\dfrac{\partial^2 Y}{\partial X^2}(L,t) = 0 \Leftrightarrow \bar{Y}''(L,t) = 0$

   with $\bar{Y}''(X,t) = \beta^2 a\cosh\beta X + \beta^2 b\sinh\beta X - \beta^2 c\cos\beta X - \beta^2 d\sin\beta X$

   $\Rightarrow \bar{Y}''(L,t) = \beta^2\left(a\cosh\beta L + b\sinh\beta L - c\cos\beta L - d\sin\beta L\right) = 0$

   $\Leftrightarrow a\cosh\beta L + b\sinh\beta L - c\cos\beta L - d\sin\beta L = 0$ (because $\beta \neq 0$)

   $\Leftrightarrow \dfrac{a}{b} = -\dfrac{\sinh\beta L + \sin\beta L}{\cosh\beta L + \cos\beta L}$ (5)

   - Shear force V at right end: $V(L,t) = \bar{M}Y''(L,t) \Leftrightarrow \bar{Y}'''(L,t) = -\dfrac{\bar{M}L}{m}\beta^4\bar{Y}(L)$

   with: $\bar{M}$ is the tip mass: $\bar{M} = (1.13 * \pi * 1.15^2 + 1.17 * \pi * 0.45^2) * 2.7 = 14.69$ g

   m - mass of beam: $m = \mathrm{V}\rho = 0.32 * 2.45 * 46 * 2.7 = 97.4$ g

   $\bar{Y}'''(X,t) = \beta^3 a\sinh\beta X + \beta^3 b\cosh\beta X + \beta^3 c\sin\beta X - \beta^3 d\cos\beta X$

   $\Rightarrow \bar{Y}'''(L,t) = \beta^3\left(a\sinh\beta L + b\cosh\beta L + c\sin\beta L - d\cos\beta L\right) = -\dfrac{\bar{M}L}{m}\beta^4\bar{Y}(L)$

   $$\Leftrightarrow a\left[(\sinh\beta L - \sin\beta L) + \frac{\bar{M}}{m}\beta L(\cosh\beta L - \cos\beta L)\right] = -b\left[(\cosh\beta L + \cos\beta L) + \frac{\bar{M}}{m}\beta L(\sinh\beta L - \sin\beta L)\right]$$

   $$\Leftrightarrow \frac{a}{b} = -\frac{(\cosh\beta L + \cos\beta L) + \dfrac{\bar{M}}{m}\beta L(\sinh\beta L - \sin\beta L)}{(\sinh\beta L - \sin\beta L) + \dfrac{\bar{M}}{m}\beta L(\cosh\beta L - \cos\beta L)} \quad (6)$$

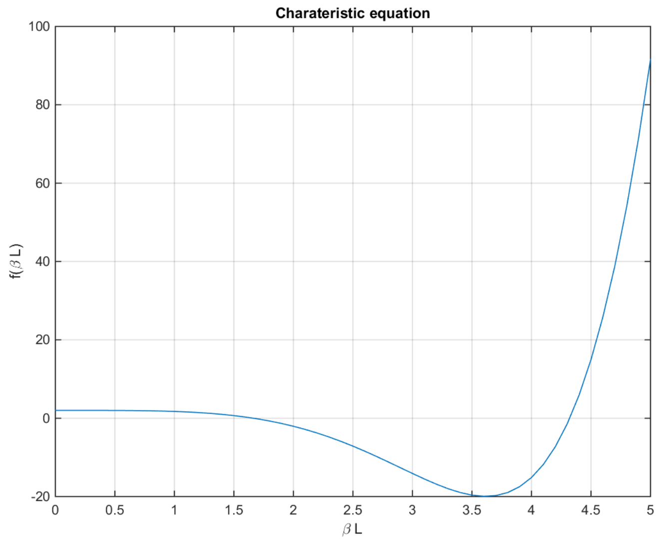
   From (5) and (6) we have the characteristic equation:

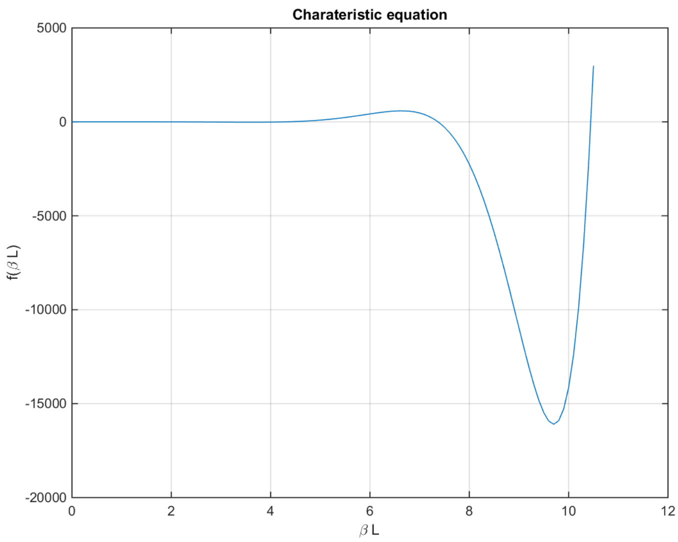
   $$\frac{\sinh\beta L + \sin\beta L}{\cosh\beta L + \cos\beta L} = \frac{(\cosh\beta L + \cos\beta L) + \dfrac{\bar{M}}{m}\beta L(\sinh\beta L - \sin\beta L)}{(\sinh\beta L - \sin\beta L) + \dfrac{\bar{M}}{m}\beta L(\cosh\beta L - \cos\beta L)}$$

$\Leftrightarrow \sinh^2 \beta L - \sin^2 \beta L + \frac{\bar{M}}{m} \beta L \sinh \beta L \cosh \beta L - \frac{\bar{M}}{m} \beta L \sinh \beta L \cos \beta L + \frac{\bar{M}}{m} \beta L \sin \beta L \cosh \beta L - \frac{\bar{M}}{m} \beta L \sin \beta L \cos \beta L = \cosh^2 \beta L + 2 \cosh \beta L \cos \beta L + \cos^2 \beta L + \frac{\bar{M}}{m} \beta L \sinh \beta L \cosh \beta L - \frac{\bar{M}}{m} \beta L \cosh \beta L \sin \beta L + \frac{\bar{M}}{m} \beta L \cos \beta L \sinh \beta L - \frac{\bar{M}}{m} \beta L \cos \beta L \sin \beta L$

$\Leftrightarrow 2 + 2 \cosh \beta L \cos \beta L + 2 \frac{\bar{M}}{m} \beta L \cos \beta L \sinh \beta L - 2 \frac{\bar{M}}{m} \beta L \cosh \beta L \sin \beta L = 0$

$\Leftrightarrow 1 + \cosh \beta L \cos \beta L + \frac{\bar{M}}{m} \beta L \cos \beta L \sinh \beta L - \frac{\bar{M}}{m} \beta L \cosh \beta L \sin \beta L = 0$





The first four nonzero roots of characteristic equation are: $\beta_1 L = 1.665, \beta_2 L = 4.322, \beta_3 L = 7.37$ and $\beta_4 L = 10.45$

Natural frequencies is defined from computed values of β, we have:

since: $\beta^4 = \frac{\rho A \omega^2}{EI} \Rightarrow \omega = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta L)^2 \sqrt{\frac{EI}{\rho A L^4}}$

we first have: $\sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{68.9 \bullet 10^9 \times 6.69 \bullet 10^{-11}}{2.7 \bullet 10^6 \times 78.4 \bullet 10^{-6} \times 0.46^4}} \approx 0.7$

$\beta_1 L = 1.665 \Rightarrow \omega_1 = (\beta_1 L)^2 \bullet 0.7 = 1.665^2 \bullet 0.7 = 1.94$ rad/s

$\beta_2 L = 4.322 \Rightarrow \omega_1 = (\beta_2 L)^2 \bullet 0.7 = 4.322^2 \bullet 0.7 = 13.07$ rad/s

$\beta_3 L = 7.37 \Rightarrow \omega_1 = (\beta_3 L)^2 \bullet 0.7 = 7.37^2 \bullet 0.7 = 38.02$ rad/s

$\beta_4 L = 10.45 \Rightarrow \omega_1 = (\beta_4 L)^2 \bullet 0.7 = 10.45^2 \bullet 0.7 = 76.4$ rad/s

The mode shape can be written as:

$$\bar{Y}_i(X) = \left(\frac{A}{B}\right)_i \left(\cosh(\beta_i L\frac{X}{L}) - \cos(\beta_i L\frac{X}{L})\right) + \sinh(\beta_i L\frac{X}{L}) - \sin(\beta_i L\frac{X}{L})$$

with: $\left(\frac{A}{B}\right)_i = -\frac{\sinh\beta_i L + \sin\beta_i L}{\cosh\beta_i L + \cos\beta_i L}$

we have the table result:

βL	1.575	4.226	7.282	10.371
a/b	-1.341	-0.985	-1.0005	1

$$\begin{aligned}
\bar{Y}_1(X) &= -1.341\left(\cosh(1.665\frac{X}{0.46}) - \cos(1.665\frac{X}{0.46})\right) + \sinh(1.665\frac{X}{0.46}) - \sin(1.6655\frac{X}{0.46}) \\
&= -1.341\left(\cosh(3.62X) - \cos(3.62X)\right) + \sinh(3.62X) - \sin(3.62X) \\
\bar{Y}_2(X) &= -0.985\left(\cosh(4.322\frac{X}{0.46}) - \cos(4.322\frac{X}{0.46})\right) + \sinh(4.322\frac{X}{0.46}) - \sin(4.322\frac{X}{0.46}) \\
&= -0.985\left(\cosh(9.4X) - \cos(9.4X)\right) + \sinh(9.4X) - \sin(9.4X) \\
\bar{Y}_3(X) &= -1.0005\left(\cosh(7.37\frac{X}{0.46}) - \cos(7.37\frac{X}{0.46})\right) + \sinh(7.37\frac{X}{0.46}) - \sin(7.37\frac{X}{0.46}) \\
&= -1.0005\left(\cosh(16.02X) - \cos(16.02X)\right) + \sinh(16.02X) - \sin(16.02X) \\
\bar{Y}_4(X) &= -1\left(\cosh(10.45\frac{X}{0.46}) - \cos(10.45\frac{X}{0.46})\right) + \sinh(10.45\frac{X}{0.46}) - \sin(10.45\frac{X}{0.46}) \\
&= -\cosh(22.72X) + \cos(22.71X) + \sinh(22.72X) - \sin(22.72X)
\end{aligned}$$

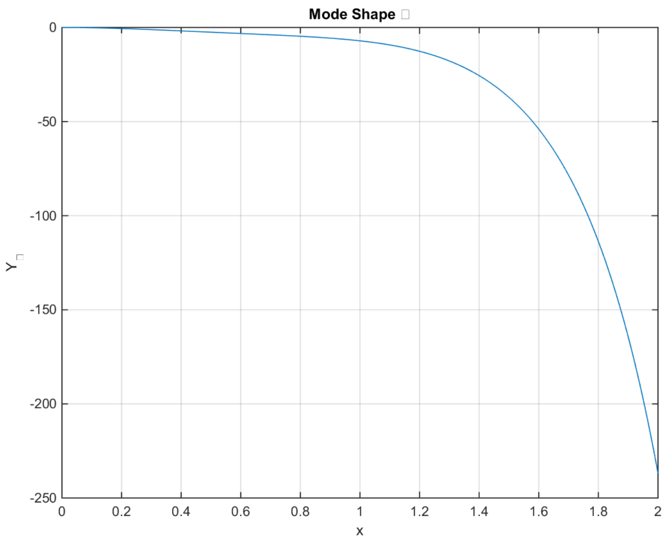


Figure 1: Mode Shape 1

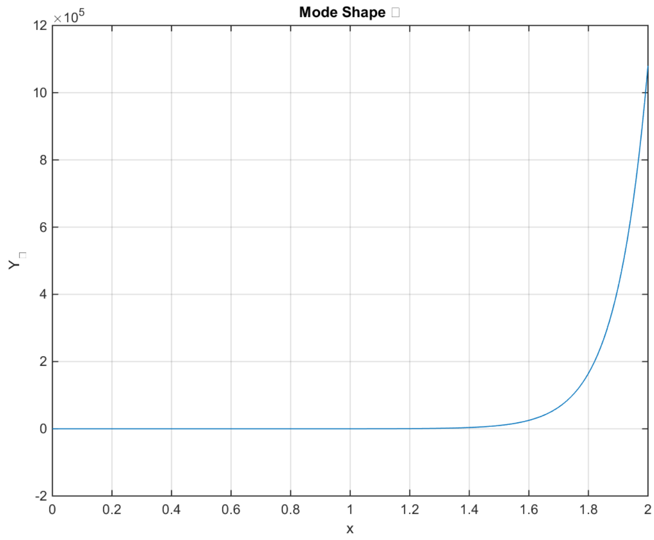


Figure 2: Mode Shape 2

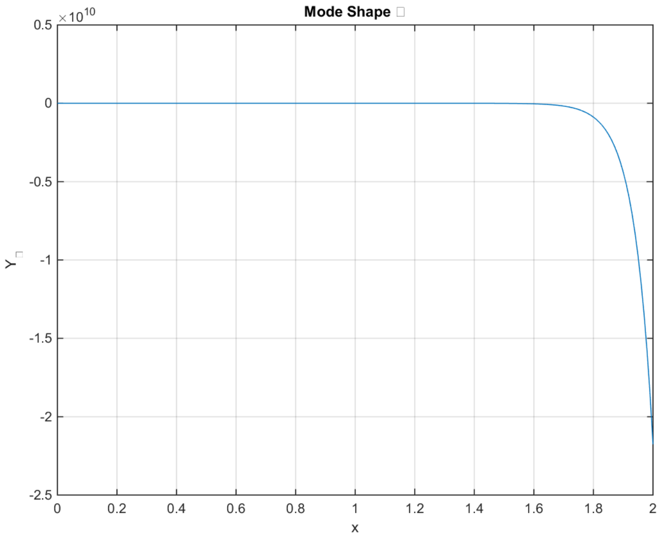


Figure 3: Mode Shape 3

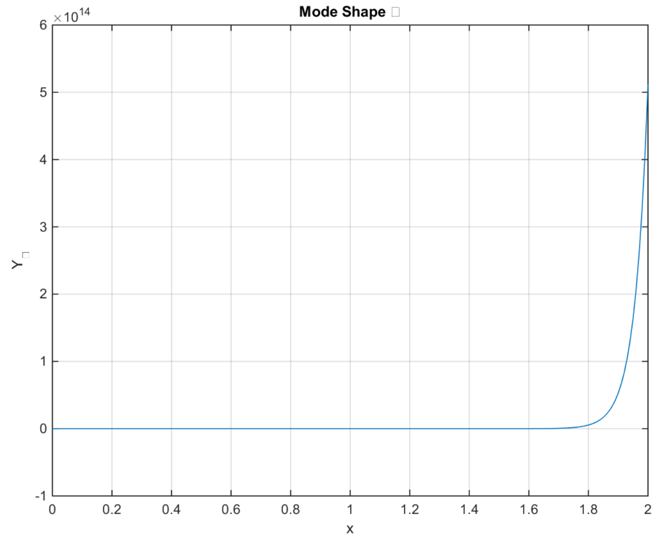


Figure 4: Mode Shape 4

2. Modeled the tip mass as a solid having mass and inertia:

   - The first two bound conditions are same with above: we have equation (3) and (4)

   - From the rotational inertial we form the third and forth condition:

$$\frac{a}{b} = -\frac{k_1(\sinh\beta L - \sin\beta L) + (\cosh\beta L + \cos\beta L)}{k_1(\cosh\beta L + \cos\beta L) + (\sinh\beta L - \sin\beta L)} \tag{7}$$

$$\frac{a}{b} = -\frac{k_2(\cosh\beta L - \cos\beta L) - (\sinh\beta L + \sin\beta L)}{k_2(\sinh\beta L + \sin\beta L) - (\cosh\beta L + \cos\beta L)} \tag{8}$$

with $k_1 = \frac{\bar{M}}{m}\beta L = 0.1508\beta L$

The rotary inertia is: $I_0 = \frac{1}{4}M_1 r_1^2 + \frac{1}{3}M_1 h_1^2 + M_2 h_2^2$

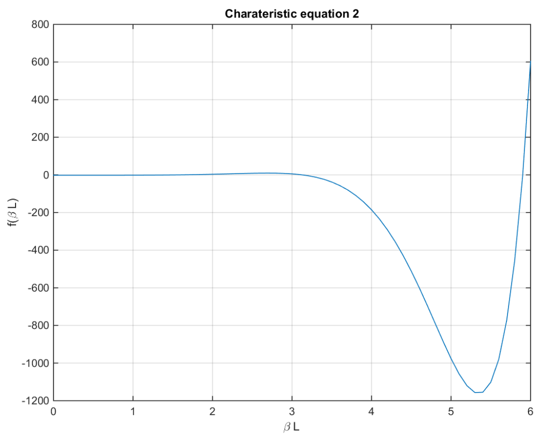
$$= \pi * 1.15^2 \times 10^{-4} * 1.13 \times 10^{-2} * 2.7 \times 10^6 \left(\frac{1}{4}1.15^2 \times 10^{-4} + \frac{1}{3}1.13^2 \times 10^{-4}\right)$$
$$+ \pi * 0.45^2 \times 10^{-4} * 1.17 \times 10^{-2} * 2.7 \times 10^6 * 1.17^2 \times 10^{-4}$$
$$= 1.234 \times 10^{-3}$$

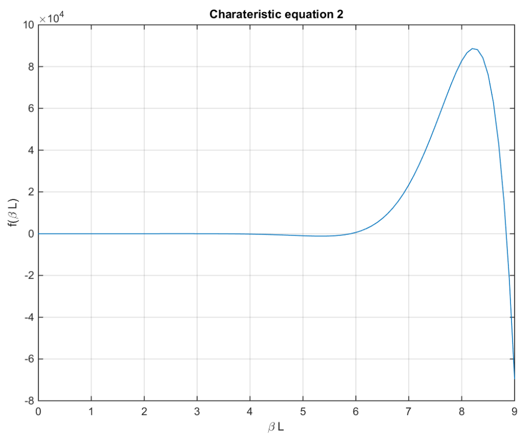
So the rotary term for k_2 is: $k_2 = \frac{I_0}{mL^2}(\beta L)^3 = \frac{1.234 \times 10^{-3}}{97.4 \times 10^{-3} * 0.46^2}(\beta L)^3 = 0.06(\beta L)^3$

Characteristic equation:

$(k_1 k_2 - 1)(\cosh\beta L \cos\beta L) - (k_1 k_2 + 1) + (k_1 + k_2)(\cosh\beta L \sin\beta L) - (k_1 - k_2)(\sinh\beta L \cos\beta L) = 0$

$\Leftrightarrow (9.048 \times 10^{-3}(\beta)^4 - 1)(\cosh\beta L \cos\beta L) - (9.048 \times 10^{-3}(\beta)^4 + 1) + (0.1508\beta L + 0.06(\beta L)^3)(\cosh\beta L \sin\beta L) - (0.1508\beta L - 0.06(\beta L)^3)(\sinh\beta L \cos\beta L) = 0$





The first four nonzero roots of characteristic equation are: $\beta_{21}L = 1.55, \beta_{22}L = 3.13, \beta_{23}L = 5.9$ and $\beta_{24}L = 8.84$

Natural frequencies is defined from computed values of β, we have:

$$\omega_n^2 = \beta^4 \frac{EIL}{m} \Leftrightarrow \omega_n = (\beta L)^2 \sqrt{\frac{EI}{mL^3}}$$

with: $$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{68.9 \bullet 10^9 \times 6.69 \bullet 10^{-11}}{97.4 \bullet 10^{-3} \times 0.46^3}} \approx 22.05$$

The first four natural frequencies of vibration are defined:

βL	1.55	3.13	5.9	8.84
ω_n	53	216.3	767.9	1725

The mode shape can be written as:

$$\bar{Y}_i(X) = \left(\frac{A}{B}\right)_i \left(\cosh(\beta_i L \frac{X}{L}) - \cos(\beta_i L \frac{X}{L})\right) + \sinh(\beta_i L \frac{X}{L}) - \sin(\beta_i L \frac{X}{L})$$

with: $$\left(\frac{a}{b}\right)_i = -\frac{k_{1i}(\sinh \beta_i L - \sin \beta_i L) + (\cosh \beta_i L + \cos \beta_i L)}{k_{1i}(\cosh \beta_i L + \cos \beta_i L) + (\sinh \beta_i L - \sin \beta_i L)}$$

we have the table result:

βL	1.55	3.13	5.9	8.84
a/b	-1.52	-0.97	-1.0002	1

$$\begin{aligned}
\bar{Y}_1(X) &= -1.52 \left(\cosh(1.55\frac{X}{0.46}) - \cos(1.55\frac{X}{0.46})\right) + \sinh(1.55\frac{X}{0.46}) - \sin(1.55\frac{X}{0.46}) \\
&= -1.52 \left(\cosh(3.37X) - \cos(3.37X)\right) + \sinh(3.37X) - \sin(3.37X) \\
\bar{Y}_2(X) &= -0.97 \left(\cosh(3.13\frac{X}{0.46}) - \cos(3.13\frac{X}{0.46})\right) + \sinh(3.13\frac{X}{0.46}) - \sin(3.13\frac{X}{0.46}) \\
&= -0.97 \left(\cosh(6.8X) - \cos(6.8X)\right) + \sinh(6.8X) - \sin(6.8X) \\
\bar{Y}_3(X) &= -1.0002 \left(\cosh(5.9\frac{X}{0.46}) - \cos(5.9\frac{X}{0.46})\right) + \sinh(5.9\frac{X}{0.46}) - \sin(5.9\frac{X}{0.46}) \\
&= -1.0002 \left(\cosh(12.83X) - \cos(12.83X)\right) + \sinh(12.83X) - \sin(12.83X) \\
\bar{Y}_4(X) &= -1 \left(\cosh(8.84\frac{X}{0.46}) - \cos(8.84\frac{X}{0.46})\right) + \sinh(8.84\frac{X}{0.46}) - \sin(8.84\frac{X}{0.46}) \\
&= -\cosh(19.22X) + \cos(19.22X) + \sinh(19.22X) - \sin(19.22X)
\end{aligned}$$

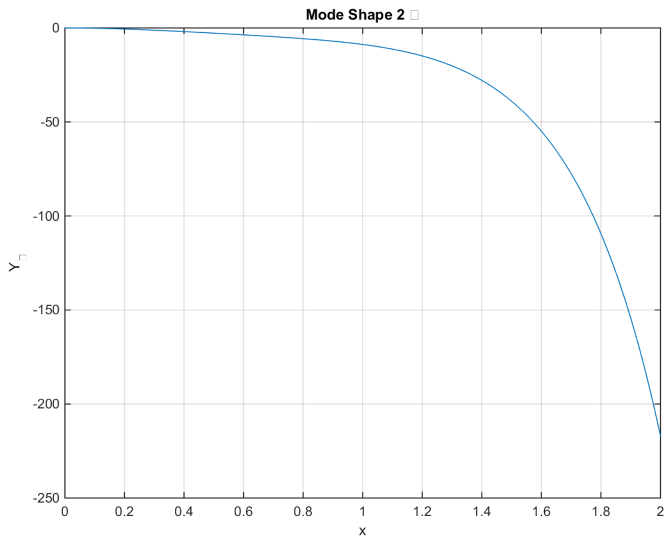


Figure 5: Mode Shape 2-1

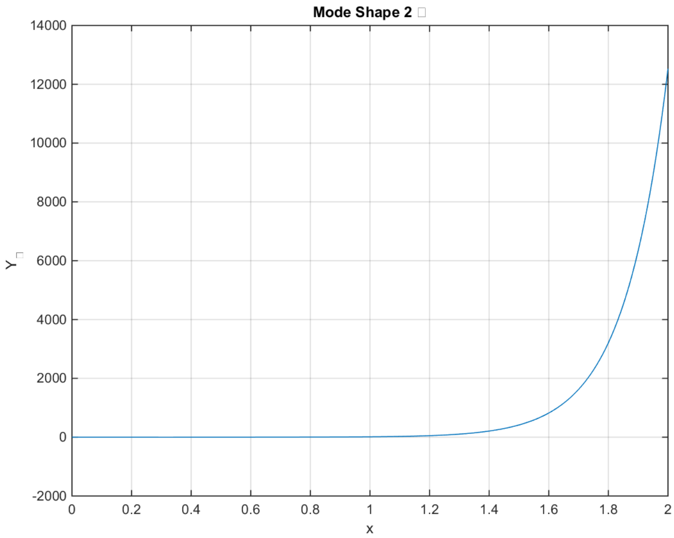


Figure 6: Mode Shape 2-2

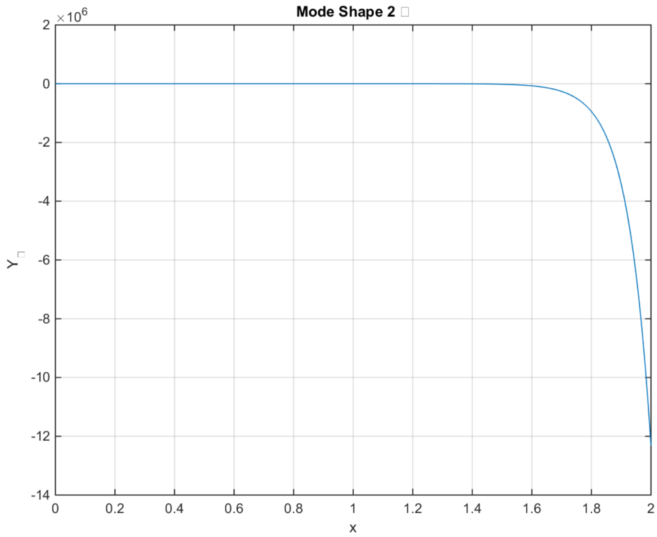


Figure 7: Mode Shape 2-3

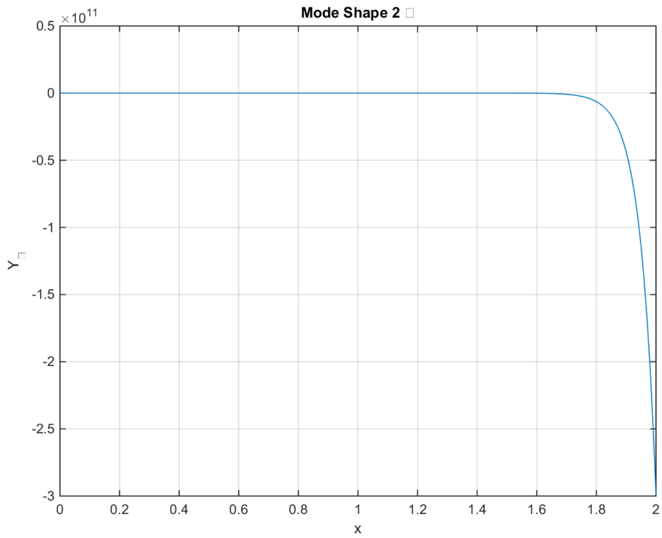


Figure 8: Mode Shape 2-4