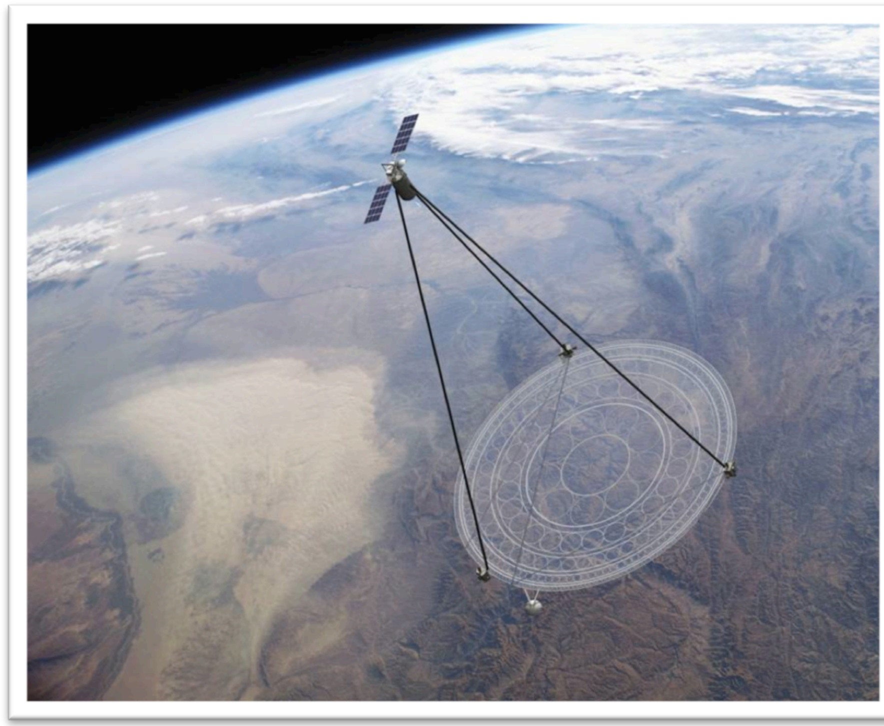
Chapter 6 Distributed Parameter Systems

Extending the first 5 chapters (particularly Chapter 4) to systems with distributed mass and stiffness properties:

Strings, rods and beams



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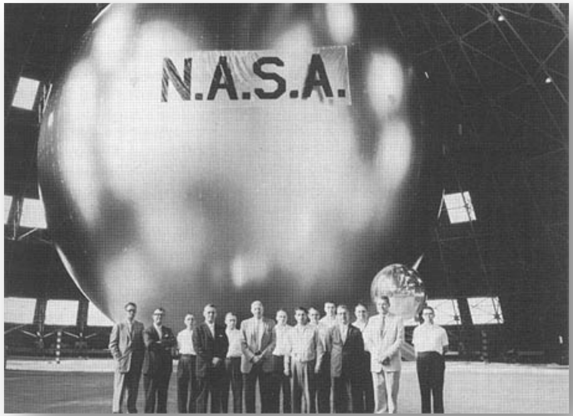
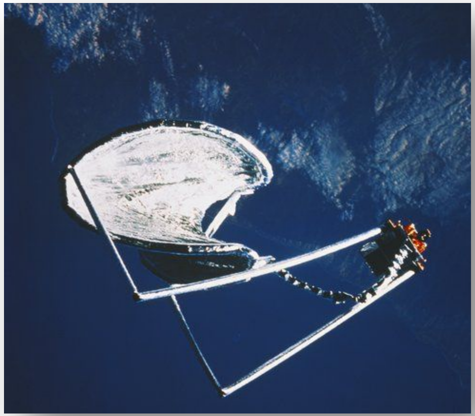
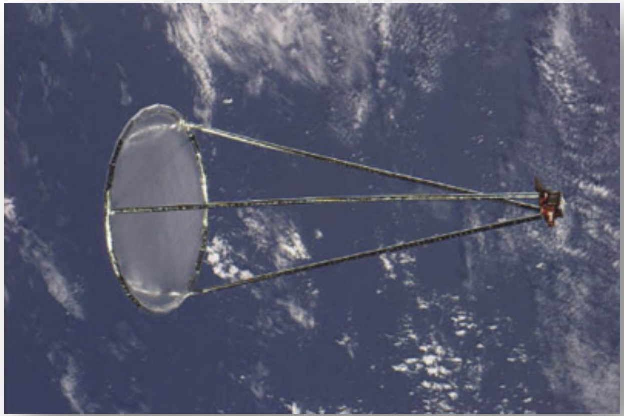
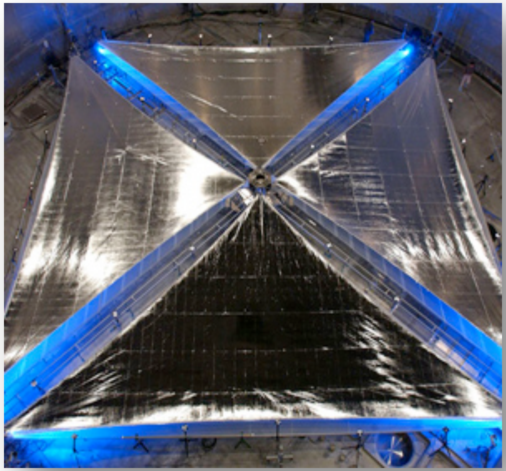
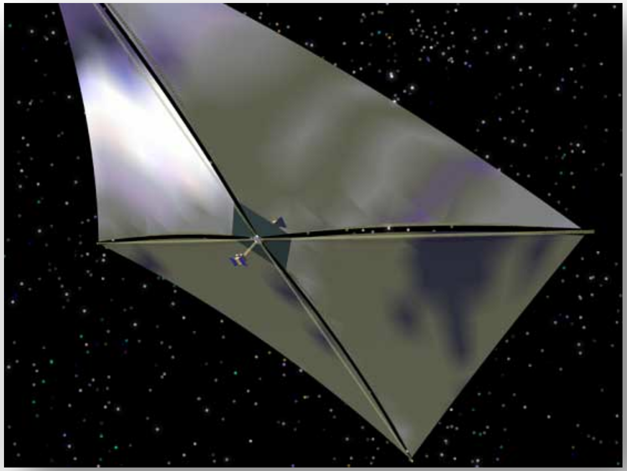
Distributed parameter systems

- The time response of a distributed-parameter system is described spatially by a continuous function of the relative position along the system.
- Specific cases considered here will be: Strings, rods, beams, membranes and plates.

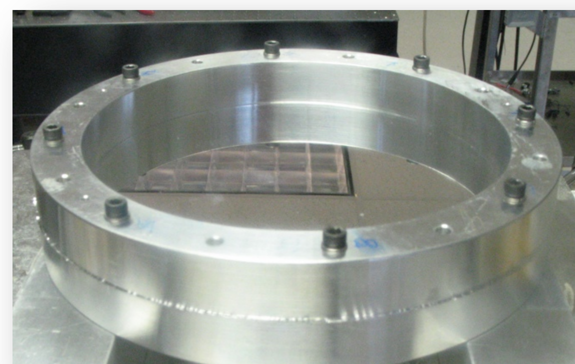
Gossamer Space Structures



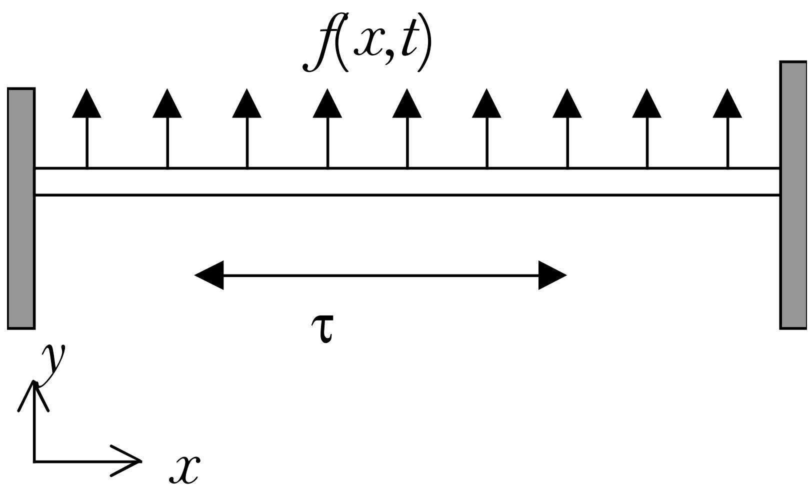






Distributed parameter systems

- Single degree of freedom had one natural frequency
- In multiple degree of freedom systems we denoted ω_i to represent one natural frequency for each degree of freedom.
- Now we move into distributed parameter systems that have infinite natural frequencies.
- Designing against resonance becomes much harder when we have infinite number of natural frequencies that can be excited.
- A note on ω_n

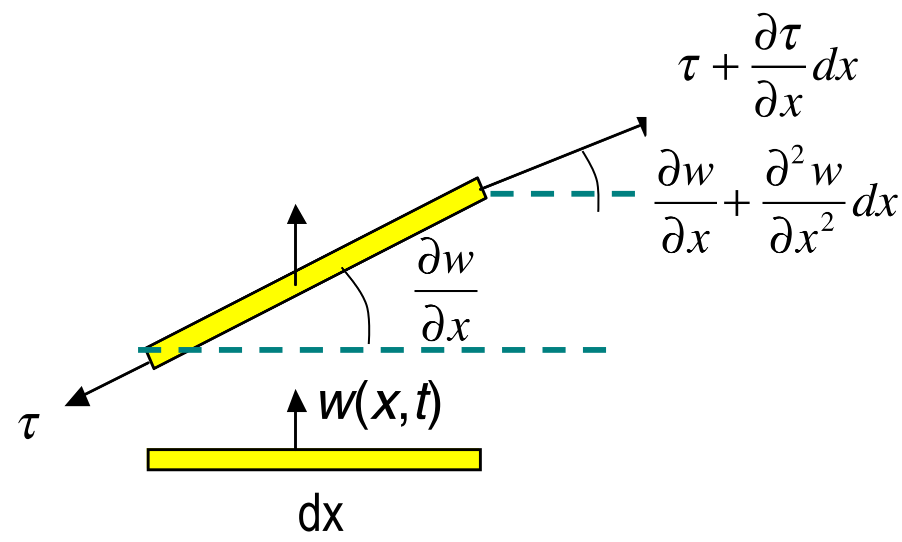


The string/cable equation



- Start by considering a uniform string stretched between two fixed boundaries
- Assume constant, axial tension τ in string
- Let a distributed force $f(x,t)$ act along the string

Examine a small element of the string



- From the above element, a force balance can be written considering the density of the string to be $\rho = \text{mass/length}$

$$\sum F_y = \left[\tau + \frac{\partial \tau}{\partial x} dx\right]\left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx\right] - \tau \frac{\partial w}{\partial x} = \rho dx \frac{\partial^2 w(x,t)}{\partial t^2}$$

Re-writing the force balance equation and eliminating higher-order terms results in:

$$\frac{\partial \tau}{\partial x}\frac{\partial w(x,t)}{\partial x}dx = \rho dx \frac{\partial^2 w(x,t)}{\partial t^2}$$

Dividing by dx results in:

$$\frac{\partial \tau}{\partial x}\frac{\partial w(x,t)}{\partial x} = \rho \frac{\partial^2 w(x,t)}{\partial t^2}$$ Which is true on the domain $0 \le x \le L$

Assuming that the tension τ is constant, $$\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\rho}{\tau}\frac{\partial^2 w(x,t)}{\partial t^2}$$

And let $c = \sqrt{\frac{\tau}{\rho}}$ which gives us: $$\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w(x,t)}{\partial t^2}$$

$$\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w(x,t)}{\partial t^2}$$

This equation is second order in time and second order in space, therefore 4 constants of integration. Two from initial conditions:

$$w(x,0) = w_0(x), \quad w_t(x,0) = \dot{w}_0(x) \;\text{ at } t = 0$$

And two from boundary conditions (e.g. a fixed-fixed string):

$$w(0,t) = w(\ell,t) = 0, \qquad t > 0$$

Each of these terms has a physical interpretation:

- Deflection is $w(x,t)$ in the y-direction
- The slop of the string is $w_x(x,t)$
- The restoring force is $\tau w_{xx}(x,t)$
- The velocity is $w_t(x,t)$
- The acceleration is $w_{tt}(x,t)$

  at any point x along the string at time t

Note that the above applies to cables as well as strings

There are Two Solution Types for Two Situations

- This is called the wave equation and if there are no boundaries, or they are sufficiently far away it is solved as a wave phenomena
  – Disturbance results in propagating waves*
- If the boundaries are finite, relatively close together, then we solve it as a vibration phenomena (focus of *Vibrations*)
  – Disturbance results in vibration

*focus of courses in *Acoustics* and in *Wave Propagation*

Solution of the Wave Equation:

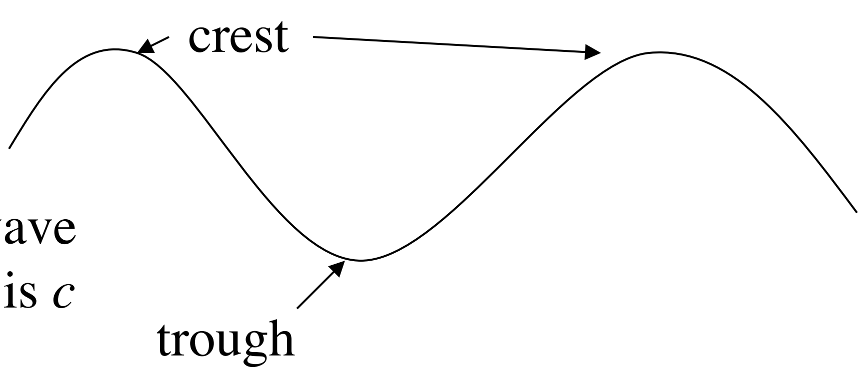
- Interpret $w(x,t)$ as a stress, particle velocity, or displacement to examine the propagation of waves in elastic media
  - Called *wave propagation*
- Interpret $w(x,t)$ as a pressure to examine the propagation of sound in a fluid
  - Called *acoustics*

Solution of the "string equation" as a Wave

A solution is of the form:

$$w(x,t) = w_1(x - ct) + w_2(x + ct)$$

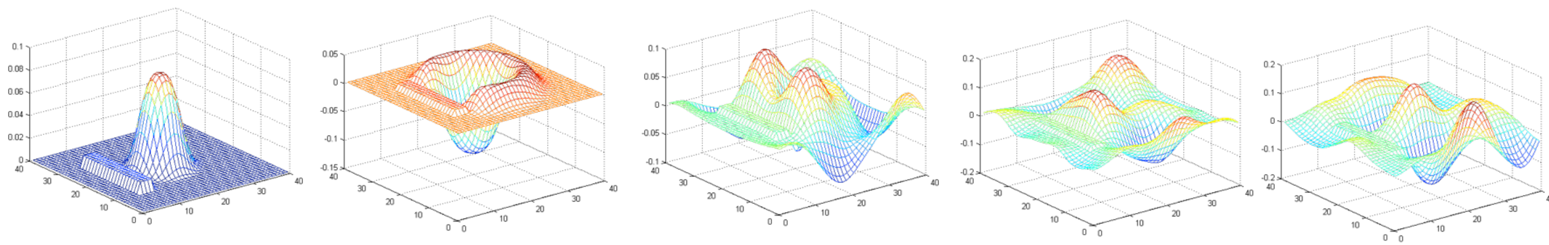
This describes one wave traveling forward and one wave traveling backwards (called traveling waves as the form of the wave moves along the media)

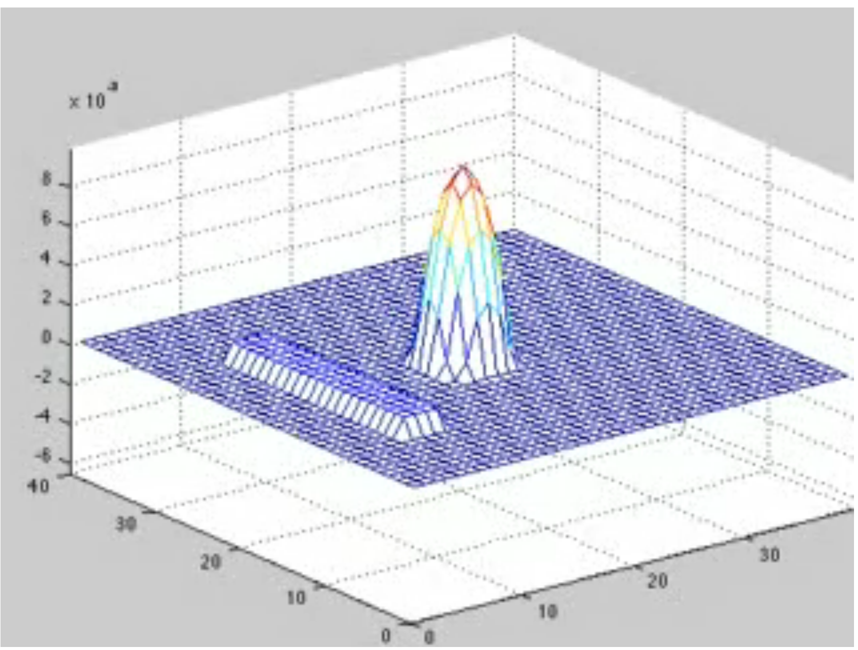


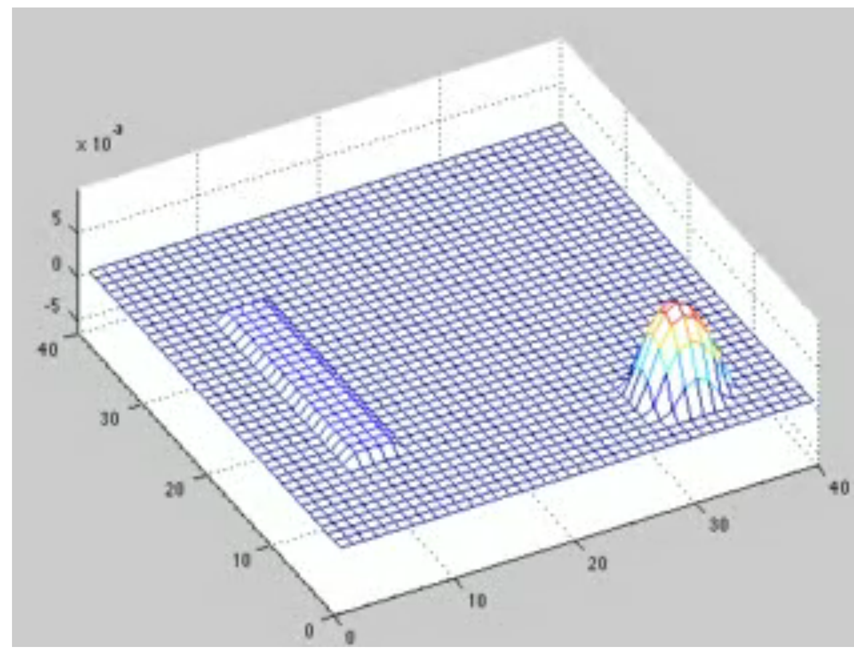
The wave speed is c

Think of waves in a pool of water

Example of wave propagation







Example 6.1.1 what are the boundary conditions for this system?

For the given system,

$$\sum_y F_{x=l} = \tau \frac{\partial w}{\partial x} + kw(l,t)$$

$$\tau \left.\frac{\partial w}{\partial x}\right|_{x=l} = -kw(l,t)\Big|_{x=l}$$

And at $x = 0$, the BC is

$$w(x,t)\Big|_{x=0} = 0$$

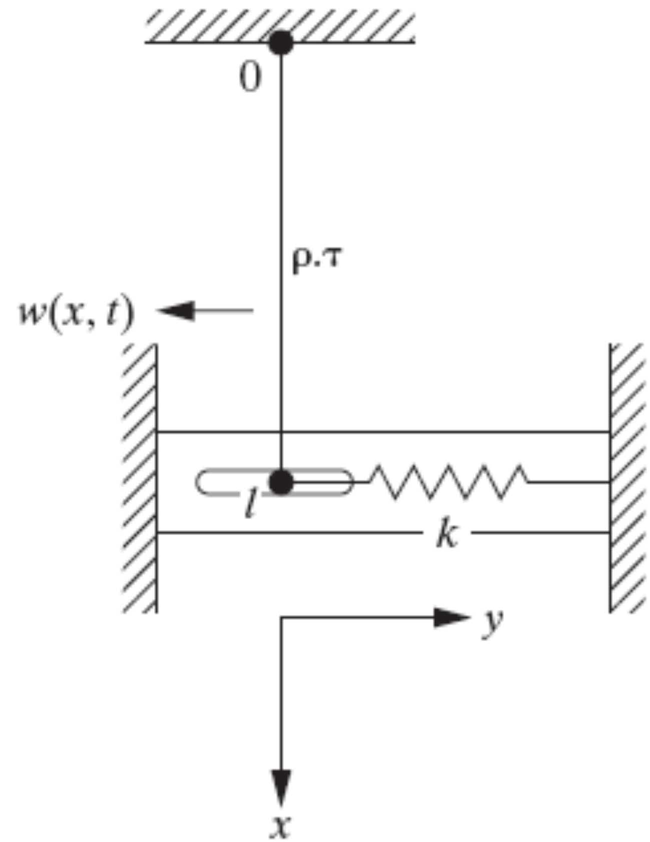


Fig 6.2

6.2 Modes and Natural Frequencies for a pinned-pinned string

We apply separation of variables

$$w(x,t) = X(x)T(t) \Rightarrow \quad c^2 \frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2}$$

$$c^2 X''(x)T(t) = X(x)\ddot{T}(t) \quad \text{where } '' = \frac{d^2}{dx^2} \text{ and } \ddot{} = \frac{d^2}{dt^2}$$

$$\Rightarrow \frac{c^2 X''(x)}{X(x)} = \frac{\ddot{T}(t)}{T(t)} = -\omega^2 \qquad \text{why a constant?}$$

$$\frac{c^2 X''(x)}{X(x)} = -\omega^2$$

$$\Rightarrow \frac{\ddot{T}(t)}{T(t)} = -\omega^2 \quad \Rightarrow \ddot{T}(t) + \omega^2 T(t) = 0$$

$$\Rightarrow X''(x) + \frac{\omega^2}{c^2} X(x) = 0 \Rightarrow X''(x) + \beta^2 X(x) = 0 \qquad \beta = \frac{\omega}{c} \Rightarrow \omega = c\beta$$

Solving the Time Equation

$$\ddot{T}(t) + \omega^2 T(t) = 0 \Rightarrow T(t) = A\sin\omega t + B\cos\omega t$$

This implies that *T* is oscillating with frequency ω

Solving the Spatial Equation:

$$X''(x) + \beta^2 X(x) = 0 \Rightarrow X(x) = a_1 \sin\beta x + a_2 \cos\beta x$$

At this point we have 4 unknowns: *A, B,* a_1, and a_2

Use the Boundary Conditions to determine a_1 and a_2

$$w(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$\Rightarrow a_1 \sin(0) + a_2 \cos(0) = 0 \Rightarrow a_2 = 0$$

$$w(\ell,t) = 0 \Rightarrow X(\ell)T(t) = 0 \Rightarrow X(\ell) = 0$$

$$\Rightarrow a_1 \sin(\beta\ell) = 0$$

$$\Rightarrow \beta\ell = n\pi, \quad \Rightarrow \beta_n = \frac{n\pi}{\ell}, \quad n = 1,2,3,\cdots$$

(But n cannot be zero, why?)

$$X(x) = a_1 \sin(\beta_n x) \Rightarrow X_n(x) = a_n \sin(\beta_n x), \quad n = 1,2,3,\cdots$$

The $X_n(x)$ are called *eigenfunctions*

So one constant determined, but we also found the form of the spatial solution

Constructing the total solution by recombining *X*(*x*) and *T*(*t*)

$$\omega = c\beta \Rightarrow \omega_n = \frac{n\pi c}{\ell} \Rightarrow T_n(t) = A_n \sin(\omega_n t) + B_n \cos(\omega_n t)$$

$$\Rightarrow w_n(x,t) = X_n(x)T_n(t) = c_n \sin\omega_n t \sin\beta_n x + d_n \cos\omega_n t \sin\beta_n x$$

$$= c_n \sin(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x) + d_n \cos(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x)$$

So there are n solutions, since the system is linear we add them up:

$$\boxed{w(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x) + d_n \cos(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x)}$$

We still do not know the constants c_n and d_n but we have yet to use the initial conditions

Orthogonality is used to evaluate the remaining constants from the initial conditions

$$\int_0^{\ell} \sin(\frac{n\pi}{\ell}x)\sin(\frac{m\pi}{\ell}x)dx = \begin{cases} \ell/2, & n=m \\ 0, & n \neq m \end{cases} = \frac{\ell}{2}\delta_{nm} \quad (6.28)$$

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x) + d_n \cos(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x)$$

From the initial position:

$$w(x,0) = w_0(x) = \sum_{n=1}^{\infty} d_n \sin(\frac{n\pi}{\ell}x)\cos(0) \Rightarrow$$

Does this look like something we have seen before?

$$\int_0^{\ell} w_0(x)\underline{\sin(\frac{m\pi}{\ell}x)}dx = \sum_{n=1}^{\infty} d_n \int_0^{\ell} \sin(\frac{n\pi}{\ell}x)\underline{\sin(\frac{m\pi}{\ell}x)}dx \Rightarrow$$

$$d_m = \frac{2}{\ell}\int_0^{\ell} w_0(x)\sin(\frac{m\pi}{\ell}x)dx, \quad m = 1,2,3\ldots \tag{6.31}$$

$$m \to n \Rightarrow$$

$$d_n = \frac{2}{\ell}\int_0^{\ell} w_0(x)\sin(\frac{n\pi}{\ell}x)dx, \quad n = 1,2,3\ldots$$

$$\dot{w}_0(x) = \sum_{n=1}^{\infty} c_n \sigma_n \mathrm{c}\sin(\frac{n\pi}{\ell}x)\cos(0) \tag{6.32}$$

$$c_n = \frac{2}{n\pi c}\int_0^{\ell} \dot{w}_0(x)\sin(\frac{n\pi}{\ell}x)dx, \quad n = 1,2,3\ldots \tag{6.33}$$

The Eigenfunctions become the vibration mode shapes

$$w_0(x) = \sin\frac{\pi}{\ell}x, \quad \text{which is the first eigenfunction } (n{=}1)$$

$$\dot{w}_0(x) = 0, \Rightarrow c_n = 0, \quad \forall n$$

$$d_n = \frac{2}{\ell}\int_0^\ell \sin(\frac{\pi}{\ell}x)\sin(\frac{n\pi}{\ell}x)dx = 0, \quad n = 2,3\ldots$$

$$d_1 = 1 \Rightarrow$$

$$w(x,t) = \sin(\frac{\pi}{\ell}x)\cos\left(\frac{\pi c}{\ell}t\right)$$

Causes vibration in the first mode shape

A more systematic way to generate the *characteristic* equation is write the boundary conditions (6.20) in matrix form

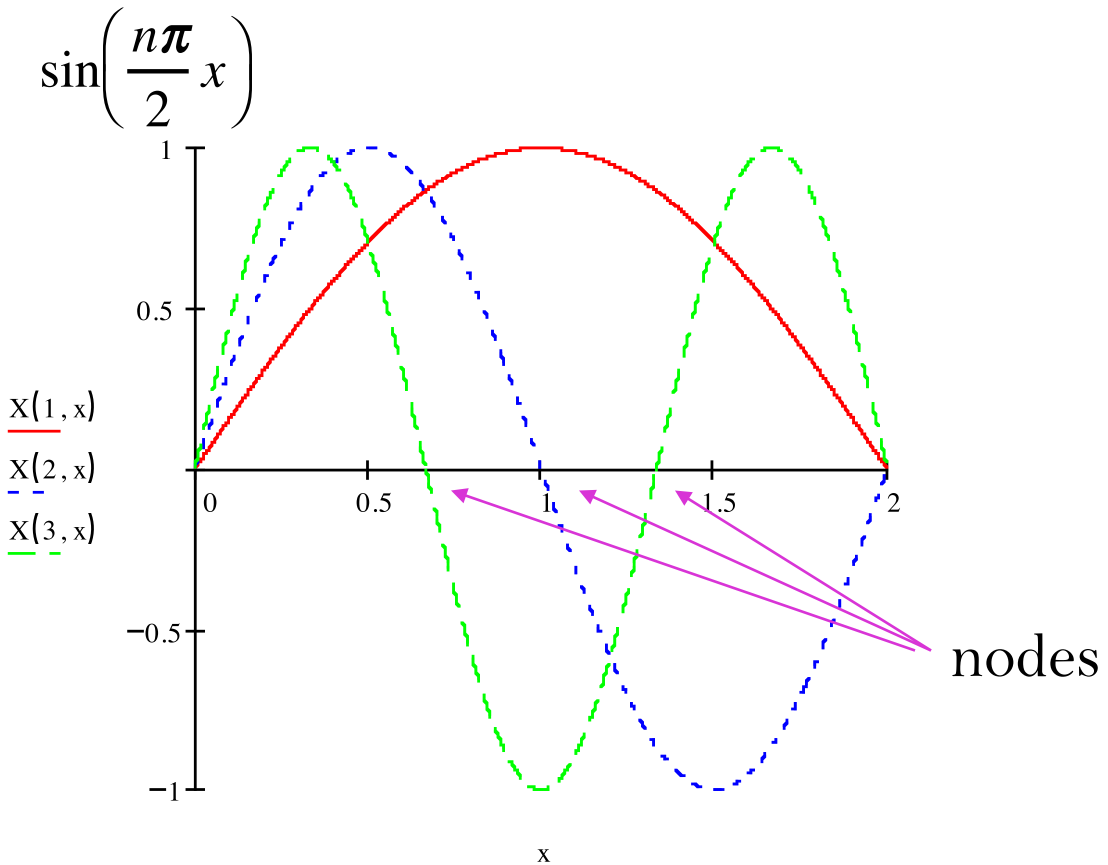
$$a_1 \sin \beta\ell = 0 \quad \text{and} \quad a_2 = 0$$

$$\Rightarrow \begin{bmatrix} \sin \beta\ell & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

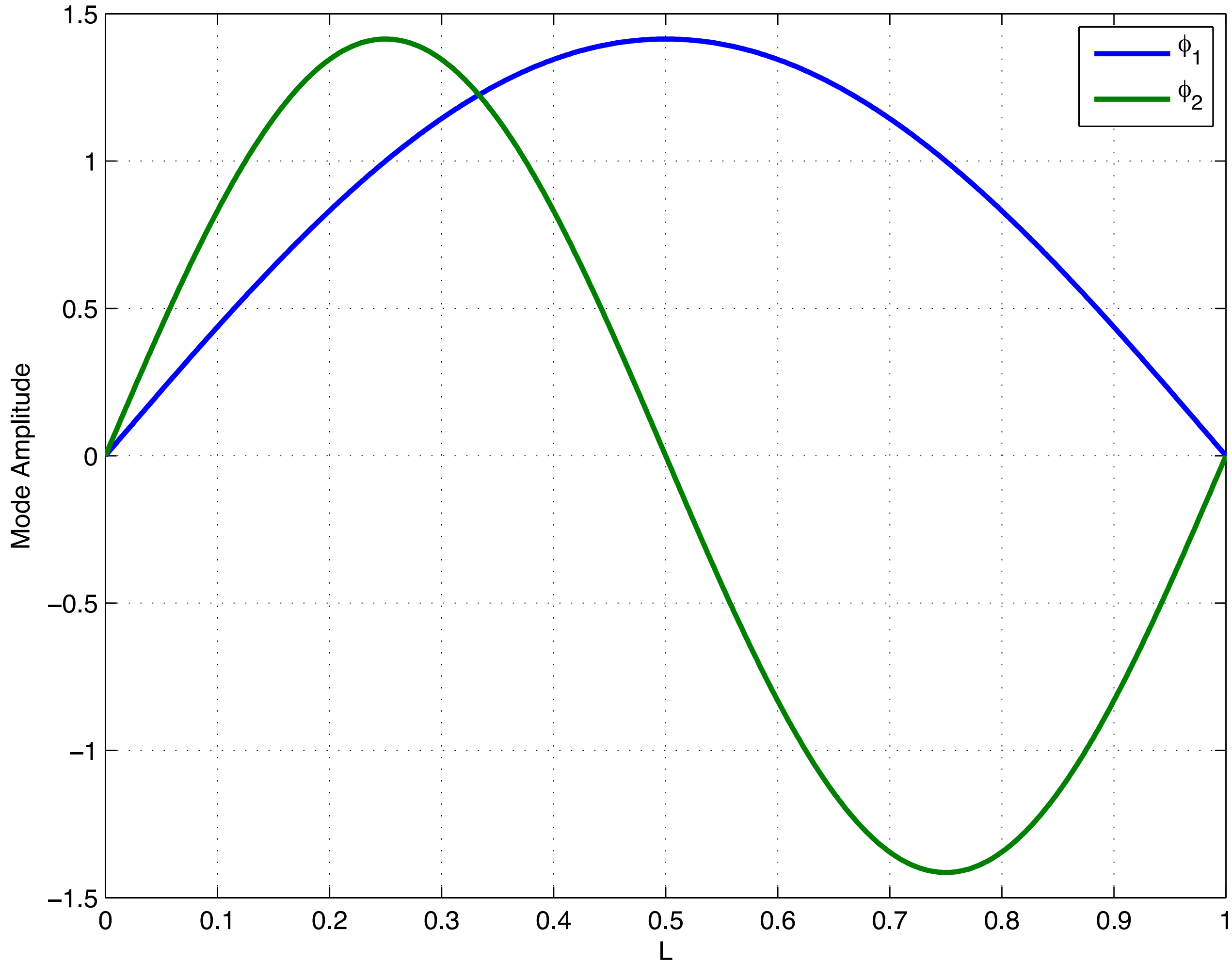
$$\Rightarrow \det\left(\begin{bmatrix} \sin \beta\ell & 0 \\ 0 & 1 \end{bmatrix} \right) = 0 \Rightarrow \underline{\sin \beta\ell = 0}$$

This seemingly longer approach works in general and will be used to compute the characteristic equation in more complicated situations

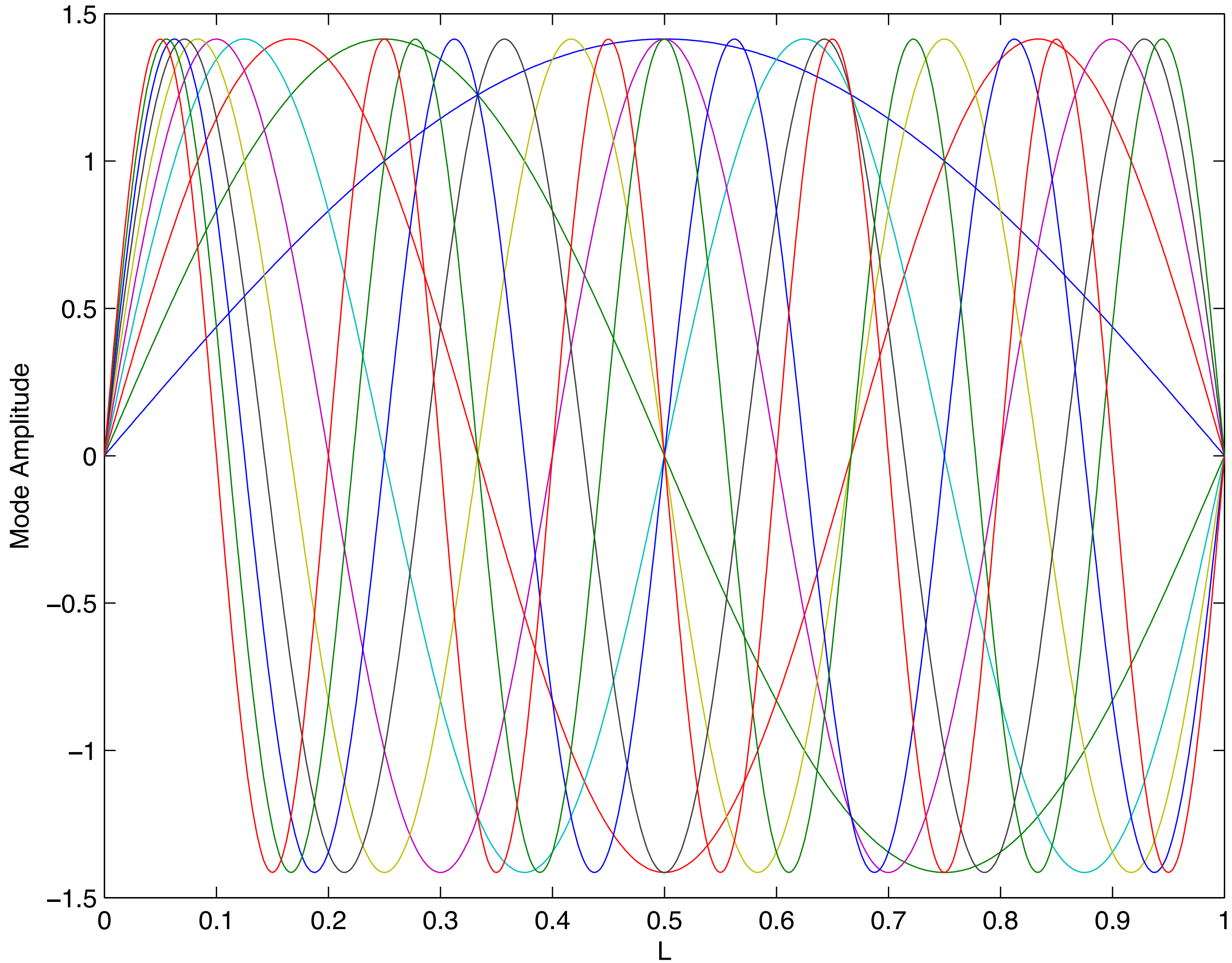
Plots of mode shapes (fig 6.3)



Mode 1 and Mode 2



Modes of a String



Example 6.2.2: Piano wire:

l =1.4 m, T =11.1x10^4 N, m=110 g.

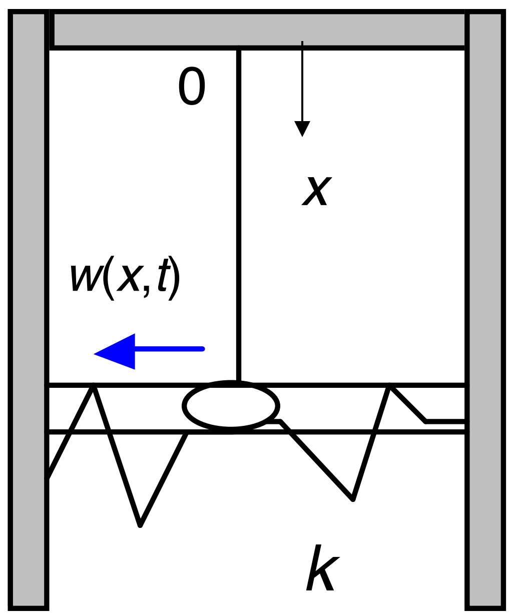
Compute the first natural frequency.

$$\rho A = 110 \text{ g per } 1.4 \text{ m} = 0.0786 \text{ kg/m}$$

$$\omega_1 = \frac{\pi c}{\ell} = \frac{\pi}{1.4}\sqrt{\frac{T}{\rho A}} = \frac{\pi}{1.4}\sqrt{\frac{11.1\times 10^4 \text{ N}}{0.0786 \text{ kg/m}}}$$

$$= 2666.69 \text{ rad/s} \quad \text{or } 424 \text{ Hz}$$

Example 6.2.3: Compute the mode shapes and natural frequencies for the following system:



A cable hanging from the top and attached to a spring of tension τ and density ρ

In this case the characteristic equation must be solved numerically

$$\sum F_y\Big|_{x=\ell} = 0 \Rightarrow T\sin\theta + kw(\ell,t) = 0 \Rightarrow$$

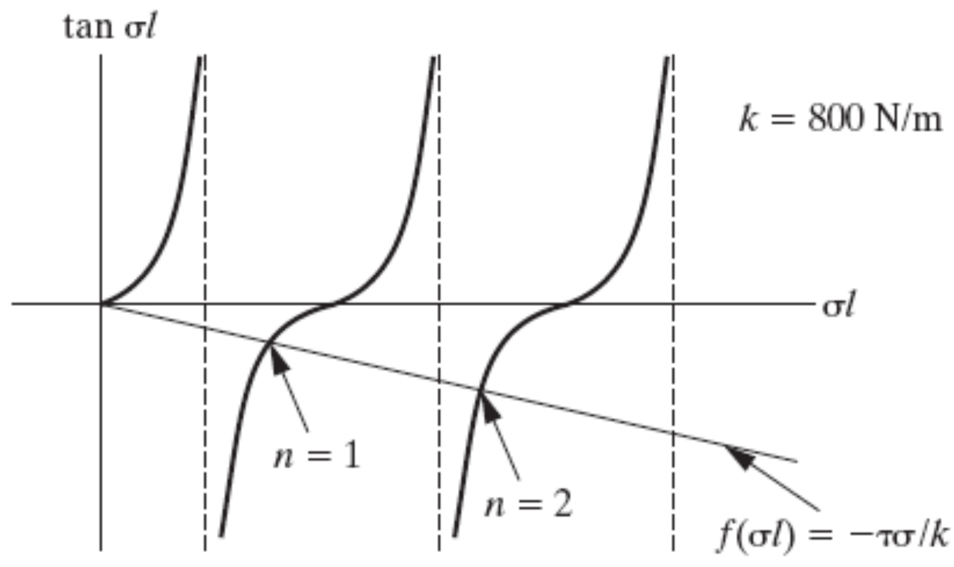
$$T\frac{\partial w(x,t)}{\partial x}\Bigg|_{x=\ell} = -kw(\ell,t)$$

$$X(x) = a_1\sin\beta x + a_2\cos\beta x,$$

$$X(0) = 0 \Rightarrow a_2 = 0 \text{ and } X(x) = a_1\sin\beta x$$

$$\Rightarrow T\beta\cos\beta\ell = -k\sin\beta\ell$$

$$\Rightarrow \tan\beta\ell = -\frac{T\beta}{k}$$



The characteristic equation must be solved numerically for β_n

Fig 6.4

Note text uses σ not β

The Mode Shapes and Natural Frequencies are:

$$k = 1000, \tau = 10, \ell = 2$$

$$\beta_n = 1.497, 2.996, 4.501$$

$$6.013 \ldots \frac{(2n-1)\pi}{2\ell}$$

$$X_n = a_n \sin(\beta_n x)$$

- The values of β_n must be found numerically
- The eigenfunctions are again sinusoids
- The value shown for β_n is for large *n*
- See Window 6.4 for a summary of method

Example: Compute the response of the piano wire to :

$$\text{initial conditions: } w_0(x) = \sin\frac{3\pi x}{\ell}, \dot{w}_0(x) = 0$$

$$\text{Solution: } w(x,t) = \sum_{i=1}^{\infty}\left(c_n \sin\frac{n\pi}{\ell}x \sin\frac{n\pi c}{\ell}t + d_n \sin\frac{n\pi}{\ell}x \cos\frac{n\pi c}{\ell}t\right)$$

$$\dot{w}_0(x) = 0 = \sum_{i=1}^{\infty}\left(c_n \frac{n\pi c}{\ell}\sin\frac{n\pi}{\ell}x \cos(0)\right) \Rightarrow c_n = 0, \forall n$$

$$w(x,t) = \sum_{i=1}^{\infty} d_n \sin\frac{n\pi}{\ell}x \cos\frac{n\pi c}{\ell}t, \text{ at } t = 0, \text{ Eq.(6.31)} \Rightarrow$$

$$d_n = \frac{2}{\ell}\int_0^{\ell} \sin\frac{3\pi}{\ell}x \sin\frac{m\pi}{\ell}x dx, \quad m = 1,2,3\ldots$$

$$= 0, \text{ for all } m \text{ except } m{=}3, d_3 = 1$$

$$w(x,t) = \sin\frac{3\pi}{\ell}x \sin\frac{3\pi c}{\ell}t$$

Some calculation details:

$$\int_0^{\ell} w_0(x) \sin\frac{m\pi}{\ell} x dx$$

$$= \sum_{n=1}^{\infty} d_n \int_0^{\ell} \sin\frac{n\pi}{\ell} x \sin\frac{m\pi}{\ell} x dx$$

$$= d_m \frac{\ell}{2}$$

$$w(\frac{\ell}{2}, t) = \sin 6\pi \sin\frac{3\pi c}{\ell} t$$

$$= 0$$

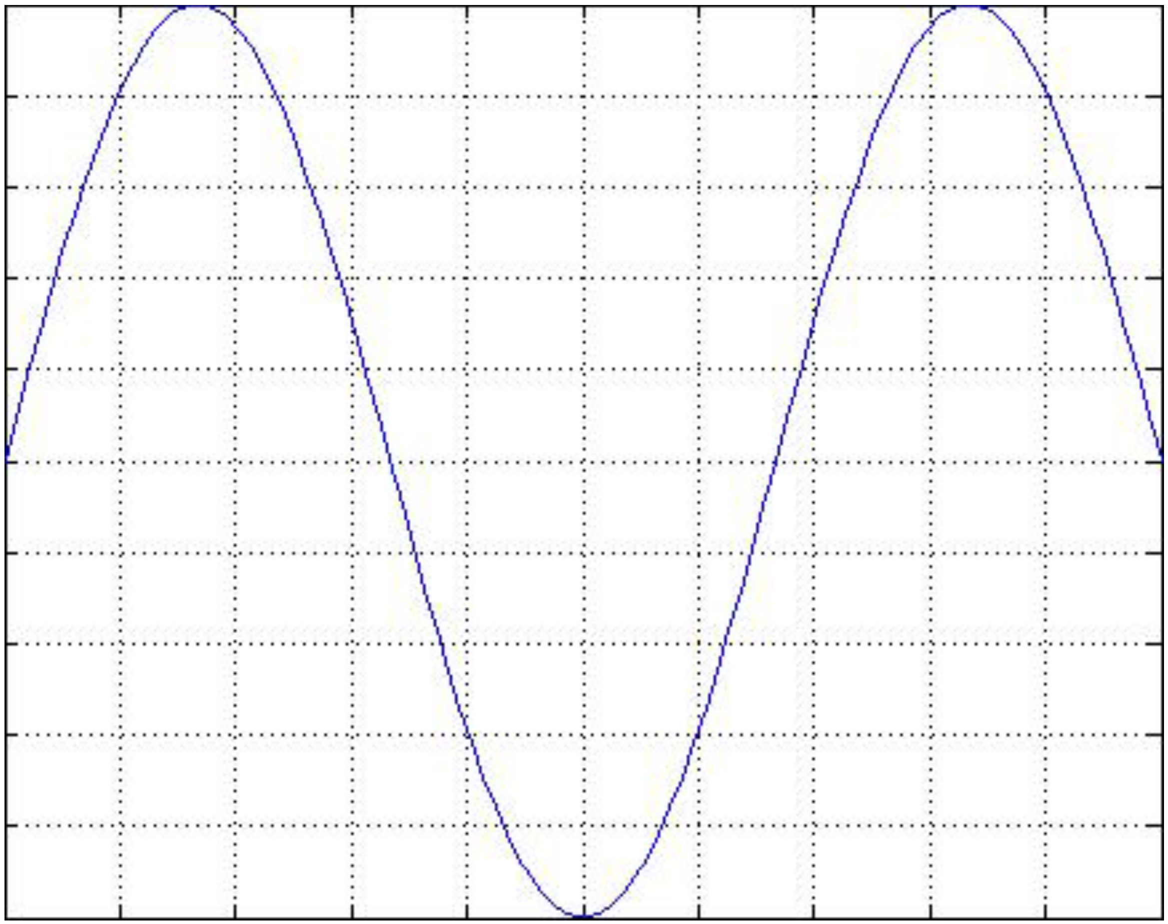
$$w(\frac{\ell}{4}, t) = \sin\frac{3\pi}{4} \sin\frac{3\pi c}{\ell} t$$

$$= 0.707 \sin\frac{3\pi c}{\ell} t$$

$$\ell = 1.4 \text{ m}, c = 11.89 \Rightarrow$$

$$w(\frac{1.4}{4}, t) = 0.707 \sin 80.4t$$

Solution



Summary of Separation of Variables

- Substitute $w(x,t)=X(x)T(t)$ into equation of motion and boundary conditions

- Manipulate all x dependence onto one side and set equal to a constant ($-\beta^2$)

- Solve this spatial equation which results in eigenvalues β_n and eigenfunctions X_n

- Next solve the temporal equation to get $T_n(t)$ in terms of β_n and two constants of integration

- Re form the product $w_n(x,t)=X_n(x)T_n(t)$ in terms of the $2n$ constants of integration

- Form the sum

$$w(x,t)=\sum_{n=1}^{\infty} X_n(x)(A_n \sin\omega_n t + B_n \cos\omega_n t)$$

- Use the initial displacement, initial velocity and the orthogonality of $X_n(x)$ to compute A_n and B_n.

6.3 Vibration of Rods and Bars

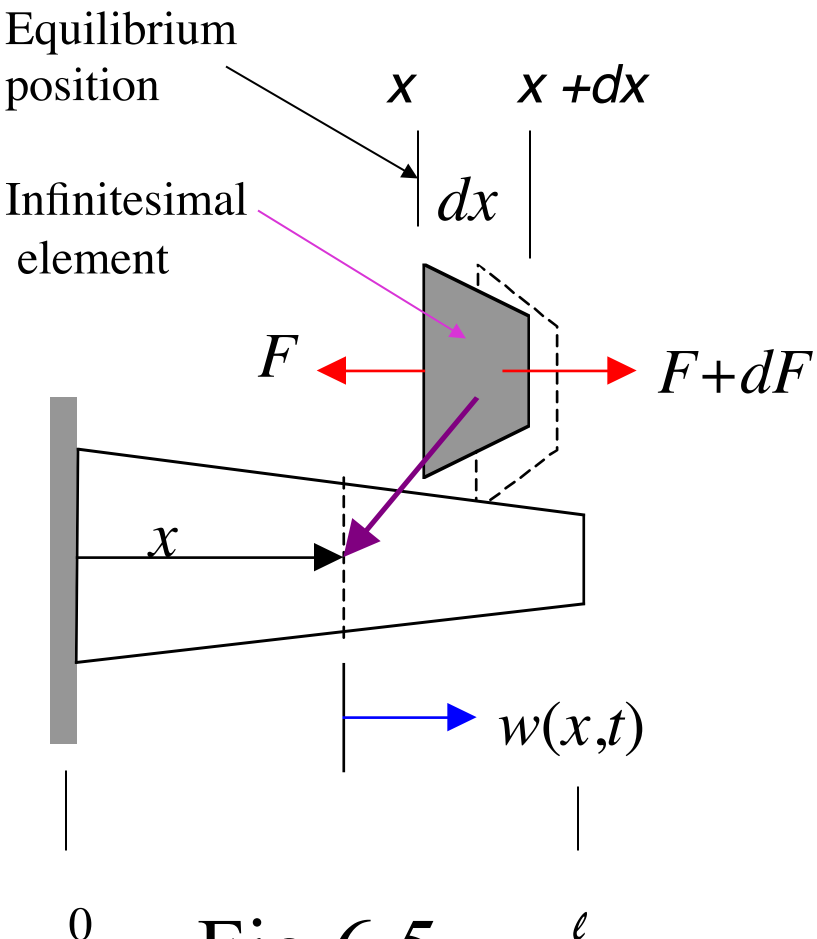


Fig 6.5

- Consider a small element of the bar
- Deflection is now along x (called longitudinal vibration)
- $F = ma$ on small element yields the following:

Force balance:

$$F + dF - F = \rho A(x)dx \frac{\partial^2 w(x,t)}{\partial t^2} \qquad (6.53)$$

Constitutive relation:

$$F = EA(x)\frac{\partial w(x,t)}{\partial x} \Rightarrow dF = \frac{\partial}{\partial x}\left(EA(x)\frac{\partial w(x,t)}{\partial x}\right)dx$$

$$\boxed{\frac{\partial}{\partial x}\left(EA(x)\frac{\partial w(x,t)}{\partial x}\right) = \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}} \qquad (6.55)$$

$$A(x) = \text{constant} \Rightarrow \frac{E}{\rho}\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} \qquad (6.56)$$

B.C.

At the clamped end: $w(0,t) = 0,$ (6.57)

At the free end: $EA\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=\ell} = 0$ (6.58)

Example 6.3.1: compute the mode shapes and natural frequencies of a cantilevered bar with uniform cross section.

$$w(x,t) = X(x)T(t) \rightarrow c^2 w_{xx}(x,t) = w_{tt}(x,t)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{c^2 T(t)} = -\sigma^2 \qquad (6.59)$$

$$\Rightarrow \begin{cases} X''(x) + \sigma^2 X(x) = 0, \quad X(0) = 0, \quad AEX'(\ell) = 0 & \text{(i)} \\ \ddot{T}(t) + c^2\sigma^2 T(t) = 0, \quad \text{initial conditions} & \text{(ii)} \end{cases}$$

From equation (i) the form of the spatial solution is

$$X(x) = a\sin\sigma x + b\cos\sigma x$$

Next apply the boundary conditions in (i) to get the characteristic equation and the form of the eigenfunction:

Apply the boundary conditions to the spatial solution to get

$$a\sin(0) + b\cos(0) = 0$$
$$a\cos(\sigma\ell) - b\sin(\sigma\ell) = 0$$

Matrix Form

$$\Rightarrow b = 0 \text{ and } \det\begin{bmatrix} 0 & 1 \\ \cos(\sigma\ell) & -\sin(\sigma\ell) \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} \cos\sigma\ell = 0 \Rightarrow \sigma_n = \dfrac{2n-1}{2\ell}\pi, & n = 1,2,3,\cdots \\ X_n(x) = a_n \sin(\dfrac{(2n-1)\pi x}{2\ell}), & n = 1,2,3,\cdots \end{cases}$$

Next consider the time response equation (ii):

$$\ddot{T}_n(t) + c^2\left(\frac{2n-1}{2\ell}\pi\right)^2 T(t) = 0$$

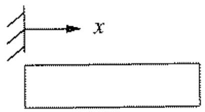
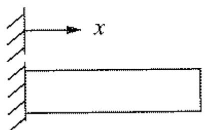
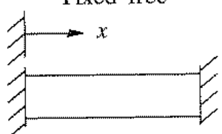
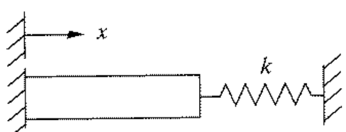
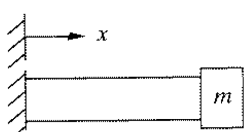
$$\Rightarrow T_n(t) = A_n \sin\frac{(2n-1)c\pi}{2\ell}t + B_n \cos\frac{(2n-1)c\pi}{2\ell}t$$

Thus the solution implies oscillation with frequencies:

$$\omega_n = \frac{(2n-1)c\pi}{2\ell} = \frac{(2n-1)\pi}{2\ell}\sqrt{\frac{E}{\rho}}, \qquad n = 1,2,3\cdots \qquad (6.63)$$

Examples with different B.C.

TABLE 6.2 VARIOUS CONFIGURATIONS OF A UNIFORM BAR OF LENGTH *l* IN LONGITUDINAL VIBRATION ILLUSTRATING THE NATURAL FREQUENCIES AND MODE SHAPES[a]

Configuration	Frequency (rad/s) or characteristic equation	Mode shape
 Free–free	$\omega_n = \dfrac{n\pi c}{l}, n = 0, 1, 2, \ldots$	$\cos \dfrac{n\pi x}{l}$[b]
 Fixed–free	$\omega_n = \dfrac{(2n-1)\pi c}{2l}, n = 1, 2, \ldots$	$\sin \dfrac{(2n-1)\pi x}{2l}$
 Fixed–fixed	$\omega_n = \dfrac{n\pi c}{l}, n = 1, 2, \ldots$	$\sin \dfrac{n\pi x}{l}$
 Fixed–spring	$\lambda_n \cot \lambda_n = -\left(\dfrac{kl}{EA}\right)$ $\omega_n = \dfrac{\lambda_n c}{l}$	$\sin \dfrac{\lambda_n x}{l}$
 Fixed–mass	$\cot \lambda_n = \left(\dfrac{m}{\rho A l}\right) \lambda_n$ $\omega_n = \dfrac{\lambda_n c}{l}$	$\sin \dfrac{\lambda_n x}{l}$

Example 6.3.2 Given $v_0(L)$=3 cm/s, ρ =8x10^3 kg/m^3 and E=20x10^{10} N/m^2, L=5m, compute the response.

$$w(x,t) = \sum_{n=1}^{\infty} (c_n \sin \sigma_n ct + d_n \cos \sigma_n ct) \sin \frac{(2n-1)}{2\ell} \pi x$$

$$d_n = \frac{2}{\ell} \int_0^{\ell} \cancelto{0}{w_0(x)} \sin \frac{(2n-1)}{2\ell} \pi x dx = 0 \Rightarrow$$

$$w_t(x,0) = 0.03\delta(x-\ell) = \sum_{n=1}^{\infty} c_n \sigma_n c \cos(0) \sin \frac{(2n-1)}{2\ell} \pi x$$

Multiply by the mode shape indexed m and integrate:

$$\Rightarrow 0.03\int_0^{\ell}(\sin\frac{(2m-1)}{2\ell}\pi x)\delta(x-\ell)dx$$

$$\boxed{\int_{\Omega} f(x)\delta(x-a)dx = f(a)}$$

$$=\sum_{n=1}^{\infty}\int_0^{\ell} c_n\sigma_n c\sin\frac{(2m-1)}{2\ell}\pi x\sin\frac{(2n-1)}{2\ell}\pi x dx$$

$$\Rightarrow 0.03\sin\frac{(2m-1)}{2}\pi = \frac{c\sigma_m\ell}{2}c_m \Rightarrow c_m = \frac{1}{\pi}\sqrt{\frac{\rho}{E}}\frac{0.06(-1)^{m+1}}{(2m-1)}$$

$$c_n = \sqrt{\frac{8\times 10^3}{210\times 10^9}}\frac{0.12(-1)^{n+1}}{\pi(2n-1)} = 7.455\times 10^{-6}\frac{(-1)^{n+1}}{(2n-1)}\text{ m}$$

$$w(x,t) = 7.455\times 10^{-6}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\sin\left(\frac{2n-1}{10}\pi x\right)\sin\left(512.348(2n-1)t\right)\text{ m}$$

Various Examples Follow From Other Boundary Conditions

- Table 6.1 page 485 gives a variety of different boundary conditions. The resulting frequencies and mode shapes are given in Table 6.2 page 486.
- Various problems consist of computing these values
- Once modes/frequencies are determined, use *mode summation* to compute the response from IC
- See the book by Blevins:*Mode Shapes and Natural Frequencies* for more boundary conditions

TABLE 6.1 A SUMMARY OF VARIOUS BOUNDARY CONDITIONS FOR THE LONGITUDINAL VIBRATION OF THE BAR OF FIGURE 6.5

Fixed at left end:	$w(x,t)\big\rvert_{x=0} = 0$
Fixed at right end:	$w(x,t)\big\rvert_{x=l} = 0$
Free at left end:	$EA\dfrac{\partial w(x,t)}{\partial x}\bigg\rvert_{x=0} = 0$
Fixed at right end:	$EA\dfrac{\partial w(x,t)}{\partial x}\bigg\rvert_{x=l} = 0$
Attached to a mass of mass m at left end:	$AE\dfrac{\partial w(x,t)}{\partial x}\bigg\rvert_{x=0} = m\dfrac{\partial w(x,t)}{\partial t}\bigg\rvert_{x=0}$
Attached to a mass of mass m at right end:	$AE\dfrac{\partial w(x,t)}{\partial x}\bigg\rvert_{x=l} = -m\dfrac{\partial w(x,t)}{\partial t}\bigg\rvert_{x=l}$
Attached to a spring of stiffness k at left end:	$AE\dfrac{\partial w(x,t)}{\partial x}\bigg\rvert_{x=0} = kw(x,t)\big\rvert_{x=0}$
Attached to a spring of stiffness k at right end:	$AE\dfrac{\partial w(x,t)}{\partial x}\bigg\rvert_{x=l} = -kw(x,t)\big\rvert_{x=l}$