Energy methods

We now look at writing equations of motion based on energy principles

The minimum number of degrees of freedom required to describe a system's behavior are called *generalized coordinates*

$$n = 3N - c$$

n = # of DOF
N = # of particles
c = # of constraints

We will refer to these DOF when writing energy equations

Generalized coordinates may appear as new functions of physical coordinates, such as what we saw with modal coordinates

Principle of virtual work

- Virtual work can be used to obtain equations of motion

$$\delta W = \sum_i \overline{F}_i \bullet \delta_{\overline{r}_i} = 0$$

Where F_i are applied external forces

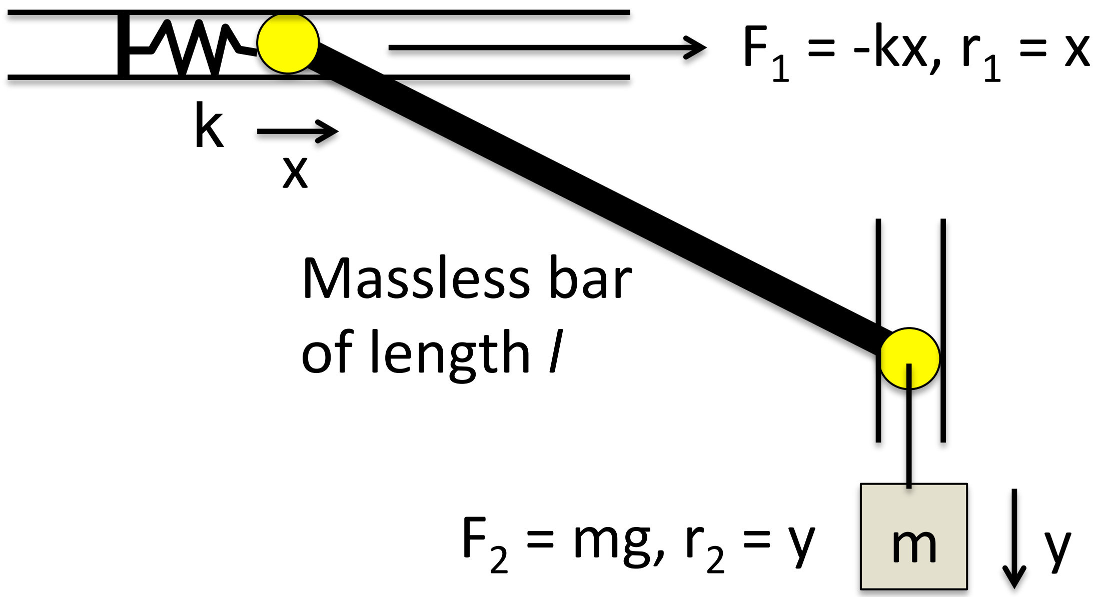
δ_{r_i} are the virtual displacements that result – arbitrary w/ no time base, and infinitesimal

For dynamic problems,

$$\delta W = \sum_i \left(\overline{F}_i - m\ddot{\overline{r}}\right) \bullet \delta_{\overline{r}_i} = 0$$

Also called the generalized principle of D'Alembert

Forces perpendicular to motion do no work (such as guided systems)



- Use the principle of virtual work to determine the equilibrium position for the system shown

Define the virtual displacements (about equilibrium):

$$\delta W = F_1 \bullet \delta r_1 + F_2 \bullet \delta r_2 = -kx\delta x + mg\delta y = 0$$

$$\delta_{r_1} = \delta x$$

$$\delta_{r_2} = \delta y$$

$$x = l(1-\cos\theta), \quad \delta x = l\sin\theta\,\delta\theta$$

$$y = l\sin\theta, \quad \delta y = l\cos\theta\,\delta\theta$$

$$-k\left(l\left(1-\cos\theta\right)\right)l\sin\theta\,\delta\theta + mg\left(l\cos\theta\,\delta\theta\right) = 0$$

$$\left[-kl^2\left(1-\cos\theta\right)\sin\theta + mgl\cos\theta\right]\delta\theta = 0$$

- *Note that the virtual displacement* $\delta\theta$ *is arbitrary, so the rest of the equation must be zero*

Solving for theta results in:

$$kl^2\left(1-\cos\theta\right)\sin\theta = mgl\cos\theta$$

$$\left(1-\cos\theta\right)\tan\theta = \frac{mg}{kl}$$

- We can also solve this problem with a more generic form of the virtual work equation:

$$\sum_{k=1}^{n} Q_k \delta q_k = 0$$

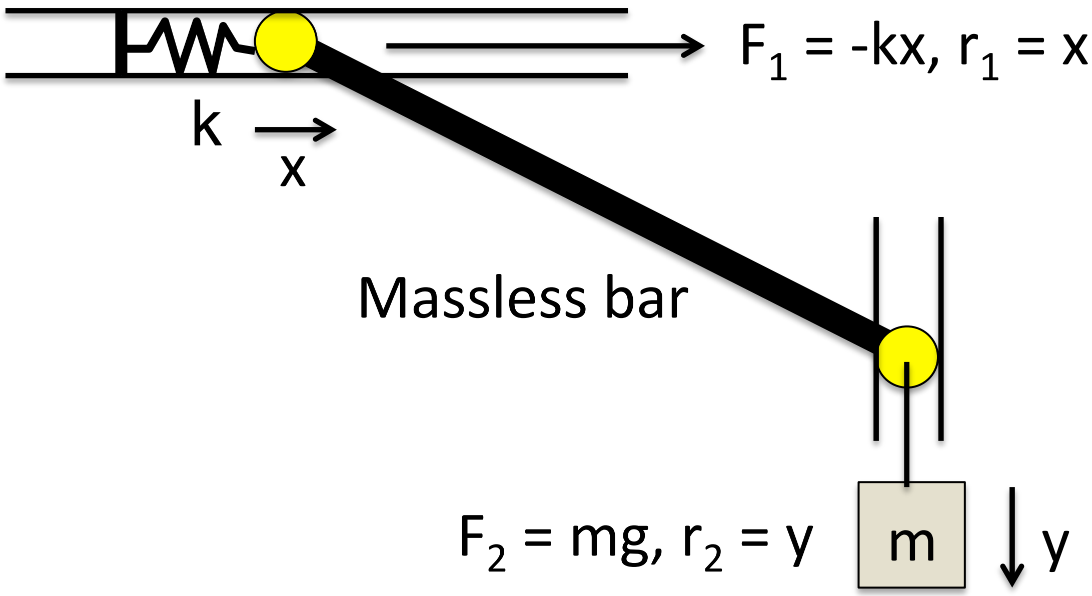
The virtual displacements are now described in terms of generalized coordinates

$$Q_k = \sum_{i=1}^{N} F_i \bullet \frac{\partial r_i}{\partial q_k} \quad k = 1, 2, \cdots, n$$

n degrees of freedom

This implies that $Q_k = 0$ for $k = 1, 2, \cdots, n$ since δq_k is arbitrary

Returning to the previous problem:



We previously used *x* (*r_1*) and *y* (*r_2*) for coordinates, but now we will use θ = *q_1*

$$Q_1 \delta q_1 = 0$$

$$Q_1 = F_1 \bullet \frac{\partial r_1}{\partial q_1} + F_2 \bullet \frac{\partial r_2}{\partial q_1} = 0$$

$$F_1 = -kx_i, \quad F_2 = -mg_j$$

$$r_1 = x_i, \quad r_2 = -y_j$$

$$x = l\left(1-\cos\theta\right), \quad y = l\sin\theta$$

$$\frac{\partial r_1}{\partial\theta} = \frac{\partial\left(l\left(1-\cos\theta\right)\right)}{\partial\theta} i = l\sin\theta i$$

$$\frac{\partial r_2}{\partial\theta} = \frac{\partial\left(l\sin\theta\right)}{\partial\theta} j = \left(-l\cos\theta\right) j$$

$$Q_1 = F_1 \bullet \frac{\partial r_1}{\partial q_1} + F_2 \bullet \frac{\partial r_2}{\partial q_1} = 0$$

$$Q_1 = -kl\left(1-\cos\theta\right)_i \bullet l\sin\theta_i + -mg_j \bullet \left(-l\cos\theta\right)_j$$

$$= kl^2\left(1-\cos\theta\right)\sin\theta + mgl\cos\theta$$

Which leads to the same previous result: $kl^2\left(1-\cos\theta\right)\sin\theta = mgl\cos\theta$

The principle of virtual work requires vector math - it would be nice to have a scalar-only process

- LaGrange's Equation is based on scalar values only, but it is derived from vector mechanics
- Consider *N* particles and *n* DOF:

$$r_p = r_p\left(q_1, q_2, \cdots, q_n, t\right),\ p = 1, 2, \cdots, N$$

The kinetic energy of the system can be written:

$$T = \frac{1}{2}\sum_{p=1}^{N} m_p \dot{r}_p \bullet \dot{r}_p$$

Expanding on the *r* velocities:

$$\dot{r}_p = \frac{dr_p}{dt} = \sum_{k=1}^{n} \frac{\partial r_p}{\partial q_k}\dot{q}_k + \frac{\partial r_p}{\partial t}$$

k refers to the degrees of freedom, not particles

$$T = \frac{1}{2}\sum_{p=1}^{N} m_p \left(\sum_{r=1}^{n} \frac{\partial r_p}{\partial q_r}\dot{q}_r + \frac{\partial r_p}{\partial t} \right) \bullet \left(\sum_{s=1}^{n} \frac{\partial r_p}{\partial q_s}\dot{q}_s + \frac{\partial r_p}{\partial t} \right)$$

or

$$T\left(q_1, q_2, \cdots, q_{n,} \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{n,} t\right)$$

n = Number of DOF

Back to the generalized principle of D'Alembert:

$$\sum_{p=1}^{N}\left(F_p - m_p\ddot{r}_p\right)\bullet\delta r_p = 0$$

re-arranging:

$$\sum_{p=1}^{N} F_p\bullet\delta r_p = \delta W = \sum_{p=1}^{N} m_p\ddot{r}_p\bullet\delta r_p$$

- And considering the following:

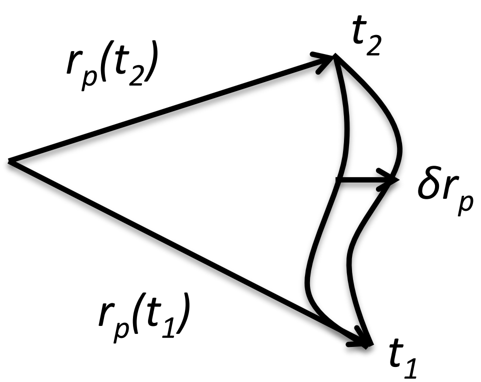
$$\frac{d}{dt}\left(\dot{r}_p\bullet\delta r_p\right) = \ddot{r}\bullet\delta r_p + \delta\left(\frac{1}{2}\dot{r}_p\bullet\dot{r}_p\right)$$

This leads to:

$$\sum_{p=1}^{N} m_p \frac{d}{dt}\left(\dot{r}_p \bullet \delta r_p\right) = \sum_{p=1}^{N} m_p \ddot{r}_p \bullet \delta r_p + \delta\left(\frac{1}{2} m_p \dot{r}_p \bullet \dot{r}_p\right)$$

$$= \delta W + \delta T$$

- Now consider two temporal states of the p^{th} particle:



0 at t_1, t_2

$$\int_{t_1}^{t_2}\left(\delta T + \delta W\right)dt = \int_{t_1}^{t_2}\sum_{p=1}^{N} m_p \frac{d}{dt}\left(\dot{r}_p \bullet \delta r_p\right)dt = \sum_{p=1}^{N} m_p \dot{r}_p \bullet \cancel{\delta r_i}\Big|_{t_1}^{t_2}$$

The integral value is 0 which is the basis for Extended Hamilton's Principle:

$$\int_{t_1}^{t_2} \left(\delta T + \delta W\right) dt = 0$$

Now: $$\delta T = \sum_{k=1}^{n} \left(\frac{\partial T}{\partial q_k} \delta q_k + \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k \right)$$

$$\delta W = \sum_{p=1}^{N} F_p \bullet \delta r_p$$ And: $$\delta r_p = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k} \delta q_k$$

$$\delta W = \sum_{p=1}^{N} F_p \bullet \left(\sum_{k=1}^{n} \frac{\partial r_p}{\partial q_k} \delta q_k \right) = \sum_{k=1}^{n} \left(\sum_{p=1}^{N} F_p \bullet \frac{\partial r_p}{\partial q_k} \right) q_k$$

Generalized forces Q_k

This simplifies to: $$\delta W = \sum_{k=1}^{n} Q_k \, \delta q_k$$

Returning to Extended Hamilton's Principle:

$$\int_{t_1}^{t_2} \sum_{k=1}^{n} \left(\frac{\partial T}{\partial q_k} \delta q_k + \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k + Q_k \delta q_k \right) dt = 0$$

=0 at t_1, t_2

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k \, dt = \int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_k} \frac{d}{dt} \delta q_k \, dt = \cancel{\frac{\partial T}{\partial \dot{q}_k} \delta q_k \Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) \delta q_k dt$$

LaGrange's Equation is derived from the Ext. Hamilton's Principle:

$$\int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) + Q_k \right] \delta q_k \, dt = 0$$

Each k DOF must be satisfied, so

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = Q_k \quad k = 1, 2, \cdots, n$$

Conservative and non-conservative forces

LaGrange's Equation

- Conservative forces can be expressed as:

$$Q_{k_c} = -\frac{\partial V}{\partial q_k} \quad k = 1, 2, \cdots, n$$

- So the final form of LaGrange's Eqn is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_{k_{NC}} \quad k = 1, 2, \cdots, n$$