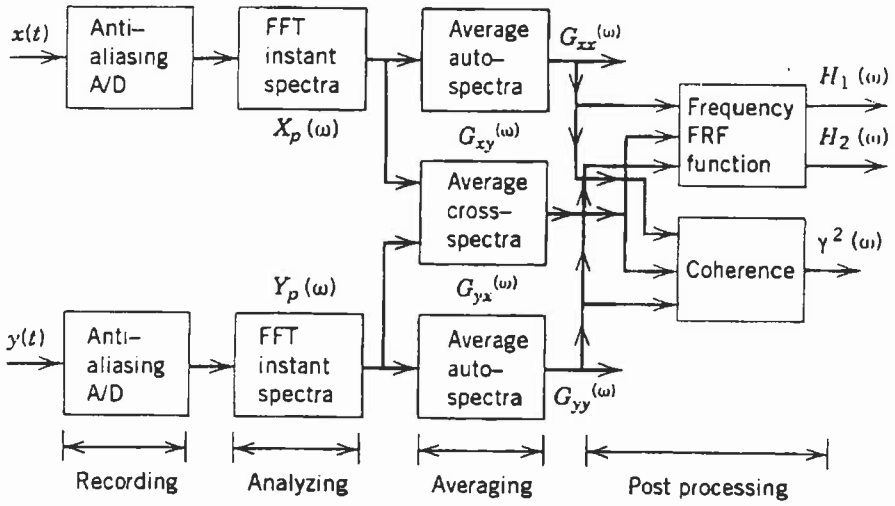
7. Digital Signal Analyzers

Obtaining an FRF experimentally requires the use of a sampling device, a transform from the time domain to the frequency domain, and a calculation of the auto- and cross-spectra. There are many intermediate operations that must occur before the FRF is displayed on the computer screen, and these will be addressed in this section.

A schematic of the experimental measurement system is shown below:



7.1 Anti-Aliasing Filters, A/D Converter

Early analyzers utilized dedicated anti-aliasing filters which were not integrated with the sampling A/D. These filters were expensive because they had to be phase and magnitude matched for each channel.

Today, the anti-aliasing function is integrated with the sampling utilizing a combination of oversampling and high order digital filters to achieve the needed results.

The purpose of the anti-aliasing filter is to eliminate frequency content which is above the highest detectable frequency of the analyzer. From the DFT theory presented in 6.3, this frequency corresponds to the Nyquist Frequency.

- Previously, the Nyquist frequency was defined as the frequency corresponding to the $N/2$ harmonic. Recall that Eqn. 6.50 stated the kth harmonic frequency to be $f_k = k/T$. This results in:

$$\text{Nyquist frequency} = f_N = \frac{N}{2T} \tag{7.1}$$

The Nyquist frequency can alternatively be stated in terms of the sampling frequency, f_s. Sampling frequency is related to the sampling interval, Δt:

$$f_s = \frac{1}{\Delta t} \tag{7.2}$$

And since the sampling interval can be expressed as $\Delta t = T/N$, this results in:

$$f_s = \frac{N}{T} \tag{7.3}$$

Which leads to another form of the Nyquist frequency:

$$\text{Nyquist frequency} = f_N = \frac{f_s}{2} \tag{7.4}$$

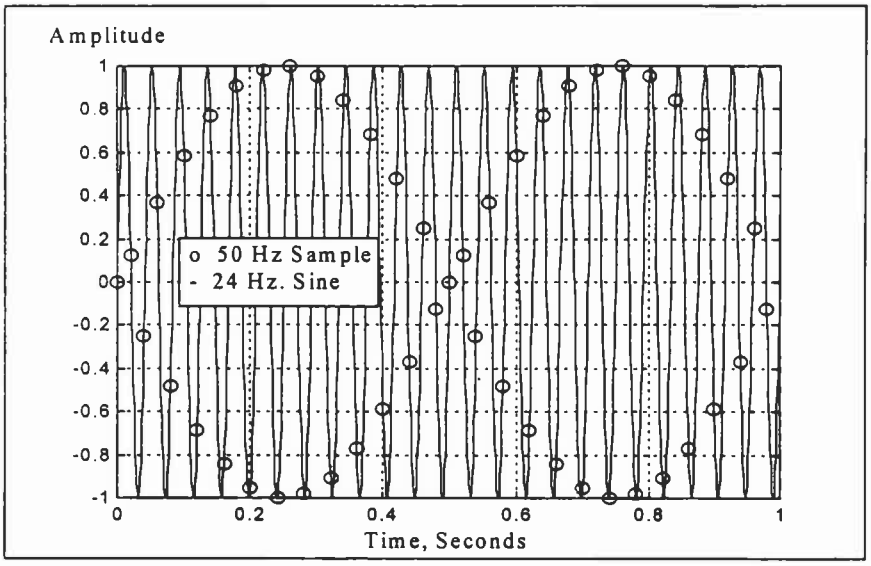
To illustrate what happens if there are harmonics above the sampling frequency of the analyzer, a simple demonstration is given:

Consider the previous example of a 10 Hz sine function sampled at:

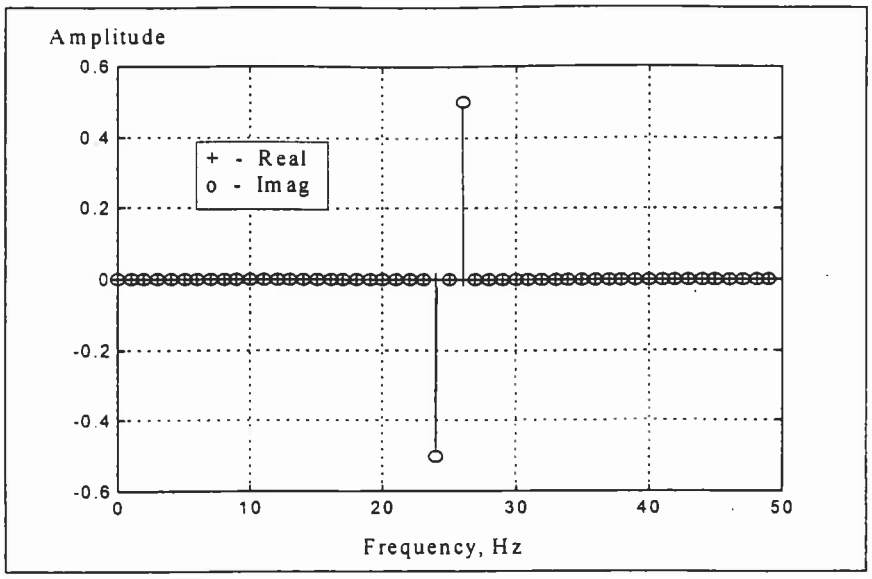
$$\Delta t = T/N = 1/50 = 0.02 \text{ sec}$$

This corresponds to a sampling frequency of $f_s = 1/\Delta t = 50$ Hz.

The Nyquist frequency will be 25 Hz, therefore a 24 Hz signal should be adequately measured using the current sampling rate:

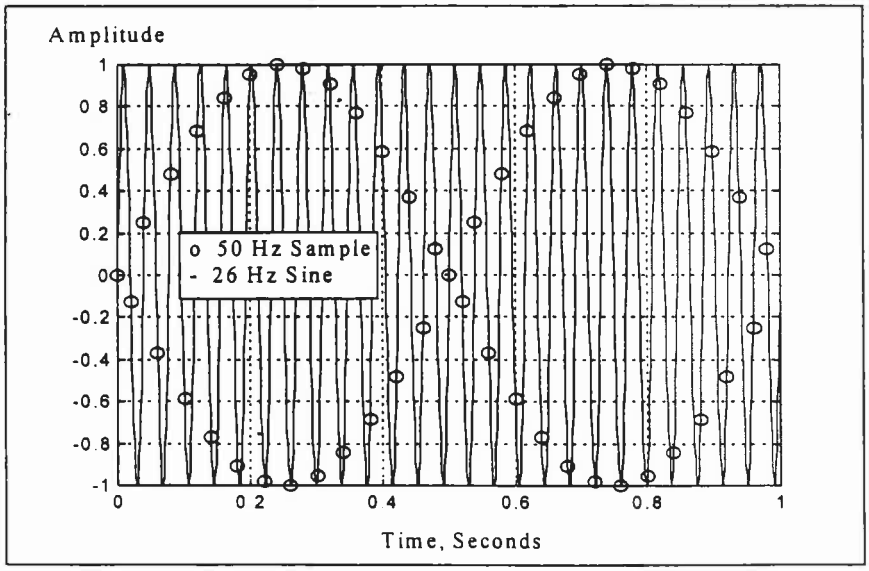


The DFT of the above sampled data points gives:

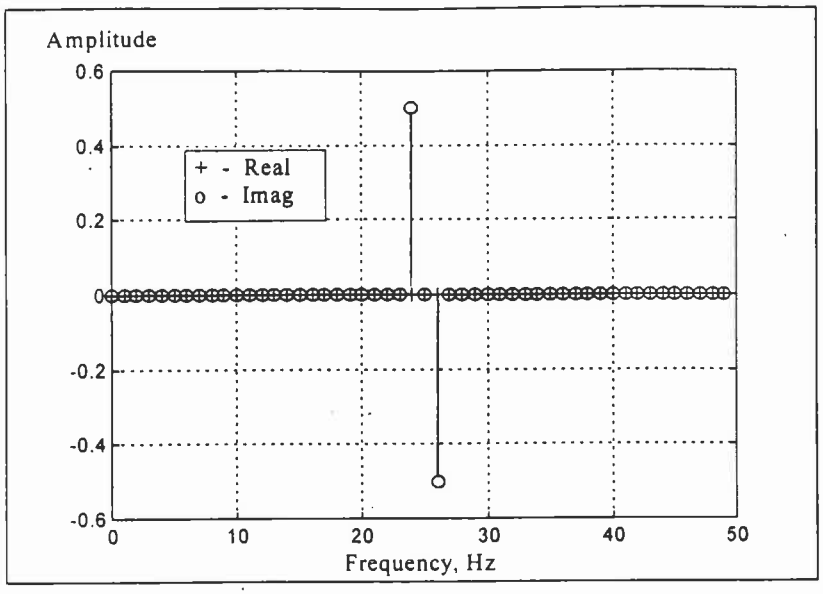


Using Eqn. 6.65 for the IDFT reproduces the waveform *exactly at the sampled points.* We still cannot obtain the original waveform between the samples.

If a *26 Hz sine function* is sampled at 50 Hz, problems with the IDFT will occur. The sampled data appears as below:



And the frequency domain plot looks like:



Note that the sign of the imaginary components has changed. This results in a *computed* signal at 24 Hz using Eqn. 6.65:

$$x_n = 2\left[0 - 0.5\sin\left(\frac{2\pi(24)n}{50}\right)\right] \quad n = 0, N-1 \qquad (7.5)$$

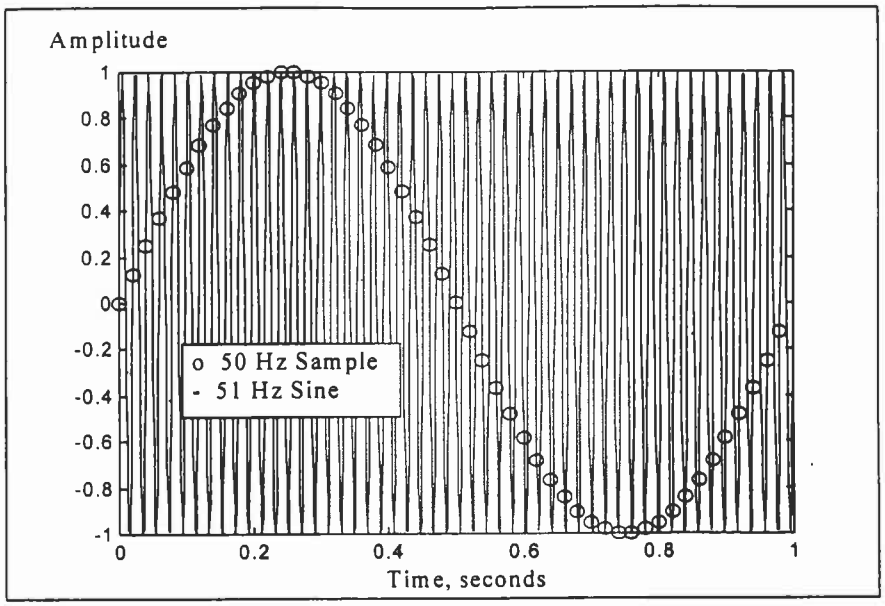
Which is 180 degrees out of phase and *does not actually exist!*

This signal is called an *aliased signal* and is caused by undersampling the data.

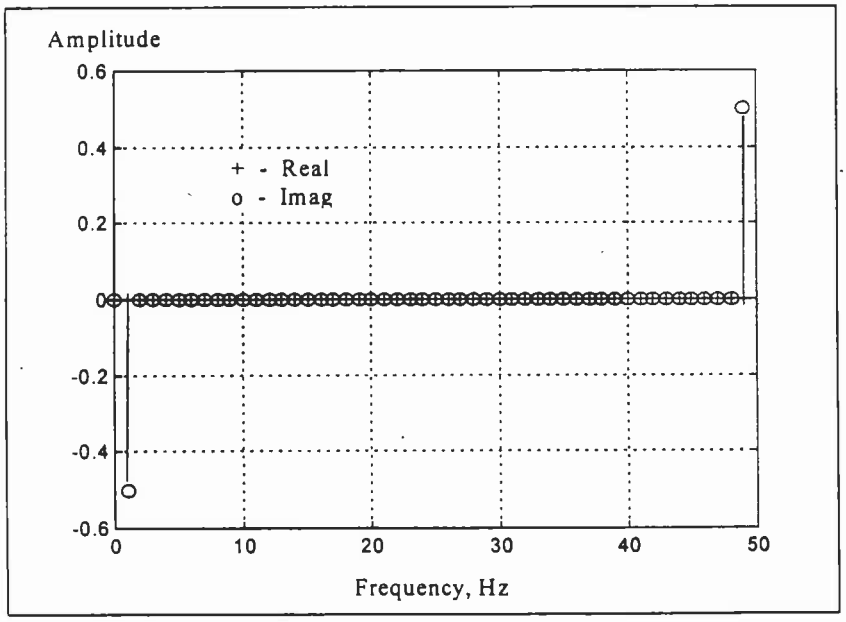
- Any harmonic component above the Nyquist frequency will be aliased to the frequencies below Nyquist in the DFT.

- This applies to harmonics *above the sampling frequency:*

Consider a signal at 51 Hz, which is 1 Hz above the sampling frequency. The sampled data appears as:



The transform of the data results in the frequency domain plot:



Again, using Eqn. 6.65 to reconstruct the data gives:

$$x_n = 2\left[0 + 0.5\sin\left(\frac{2\pi(1)n}{50}\right)\right] \quad n = 0, N-1 \qquad (7.6)$$

Which is a 1 Hz sine function that can easily be seen from the sampled data plot.

To summarize the effects of measuring data above the Nyquist frequency:

- Frequencies above the Nyquist and below the sampling frequency appear as aliased signals at a frequency which is a mirror reflection about Nyquist.

- Frequencies above the sampling frequency can be thought of as being reflected *twice:* once about Nyquist and once about the zero frequency.

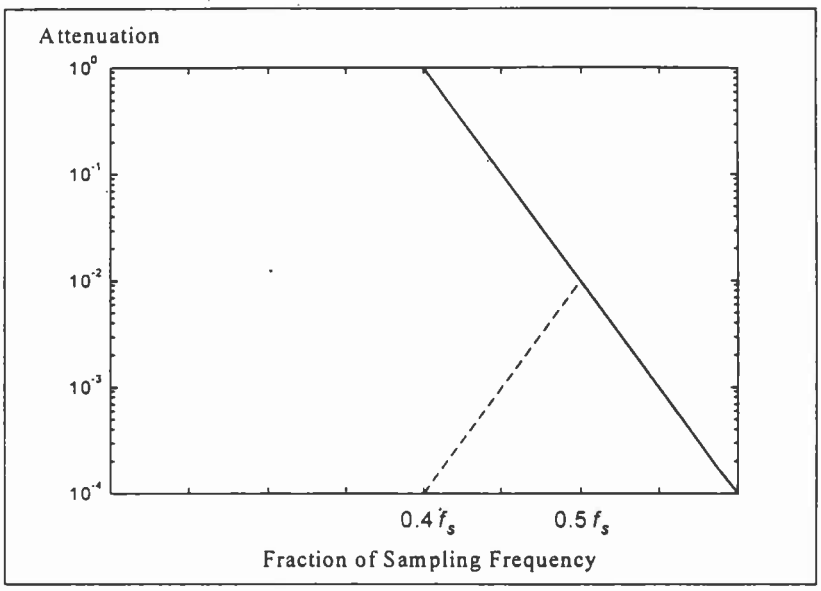
- As frequencies increase, their aliased components continue to be reflected between the Nyquist frequency and zero frequency.

To prevent the possibility of transforming data which has harmonics above the Nyquist frequency, anti-aliasing (low pass) filters are used to eliminate frequency content above Nyquist.

These filters typically have a very high roll-off rate, sometimes as high as 200 dB/octave.

A common anti-aliasing roll-off rate is 120 dB/octave. For this case, the break frequency must be selected so that harmonic components above Nyquist will be in the noise of the analyzer.

Typical analyzer noise floors are -72 dB (not as common anymore). If the break frequency is selected as $0.4 f_s$, then components at and above Nyquist will be attenuated *back* to $0.4 f_s$ <u>below</u> the noise floor:



This can be verified by using a log equation to express the attenuation:

$$dB_{atten} = -120\,dB\left[\log_2 f - \log_2 .4 f_s\right] \qquad (7.7)$$

Obtaining digital data from an analog signal involves the use of an A/D converter. Typical converters are classified as 12-bit or 16-bit. The *resolution* of a converter is based on the number of divisions of full scale that is obtained using the relation:

$$2^{12} = 4096 \text{ divisions for a 12-bit A/D}$$

$$2^{16} = 65536 \text{ divisions for a 16-bit A/D}$$

If full-scale voltage is set at ± 1.0 V, then the resolution possible with a 12-bit A/D is $2/4096$, or $\approx .50$ mV. A 16-bit converter allows $\approx .030$ mV resolution. This produces a *quantization error* of:

$$V_e \approx \pm .25\,\text{mV} \quad \text{for the 12 - bit converter}$$
$$V_e \approx \pm .015\,\text{mV} \quad \text{for the 16 - bit converter}$$

The full scale range of the A/D can be expressed as *dynamic range* using the resolution of the converter:

$$\text{Dynamic range in dB} = \text{Full scale voltage range} / \text{Resolution}$$

For the converter examples presented, dynamic range is:

$$\text{12 - bit:} \quad \frac{2}{2/4096} = 72\,\text{dB}$$

$$\text{16 - bit:} \quad \frac{2}{2/65536} = 96\,\text{dB}$$

7.2 Windowing

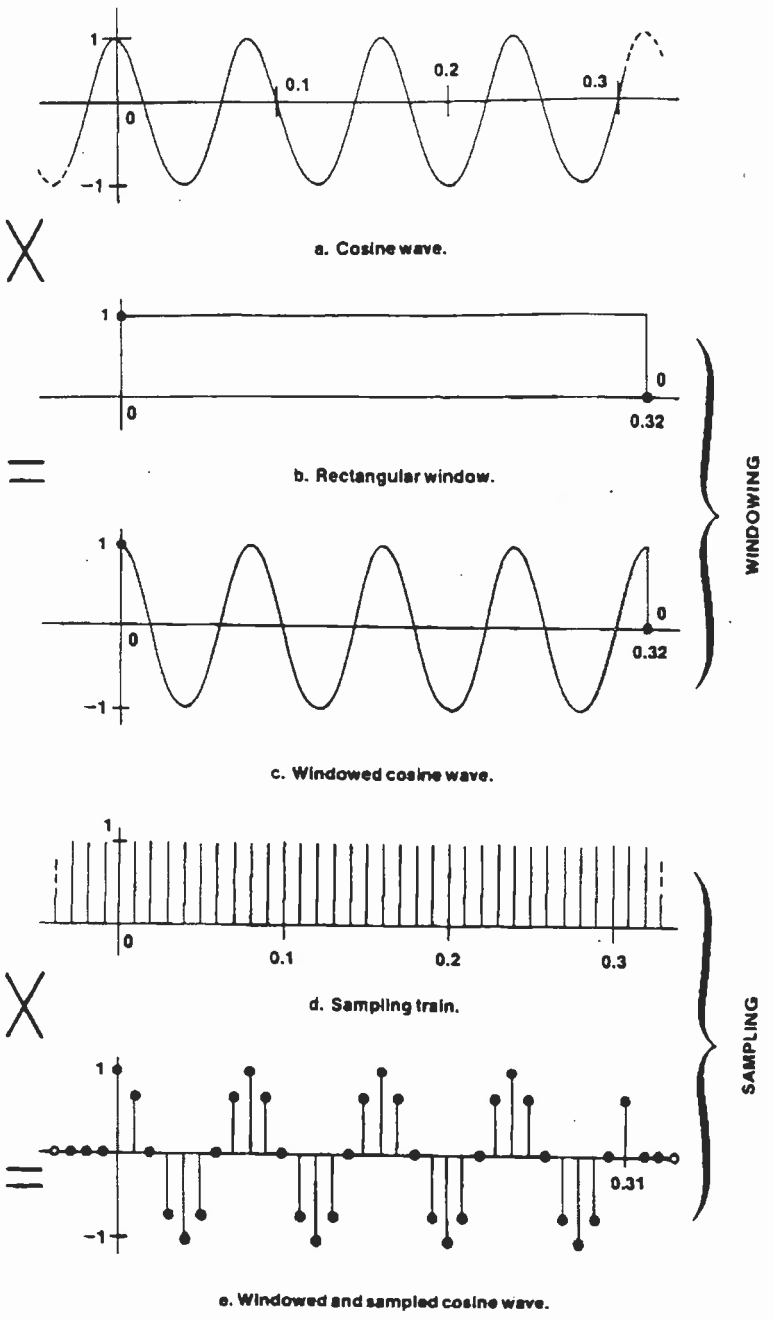
An analyzer works on sampled data which exists only during the time it is sampled. Therefore, a more correct definition of the data being analyzed is:

$$\hat{x}(t) = w(t)\,x(t) \qquad (7.8)$$

The window function *w(t)* in the above example simply represents the rectangular window, so its value will be:

$$\begin{aligned} w(t) &= 0 \quad \text{for} \quad -\infty \le t \le 0 \\ w(t) &= 1 \quad \text{for} \quad 0 \le t \le T \\ w(t) &= 0 \quad \text{for} \quad T \le t \le -\infty \end{aligned} \qquad (7.9)$$

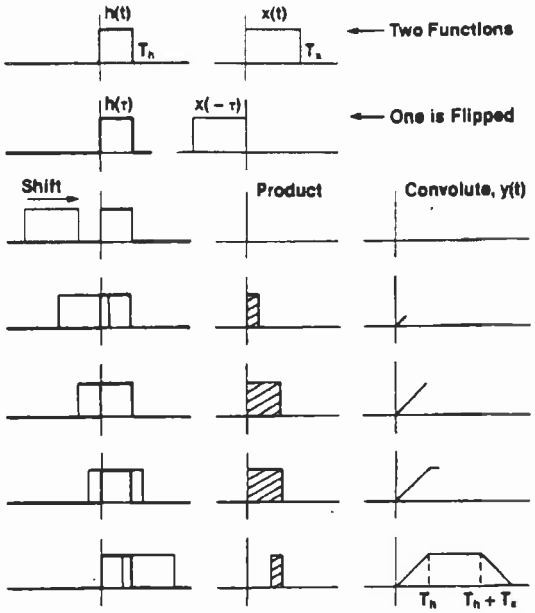
As a result of the window, the analyzer will no longer be transforming raw data $x(t)$, but the windowed function $\hat{x}(t)$.



Recall in section 6.2.1 that it was shown that convolution in the time domain is multiplication in the frequency domain. Conversely, *multiplication in the time domain is convolution in the frequency domain.*

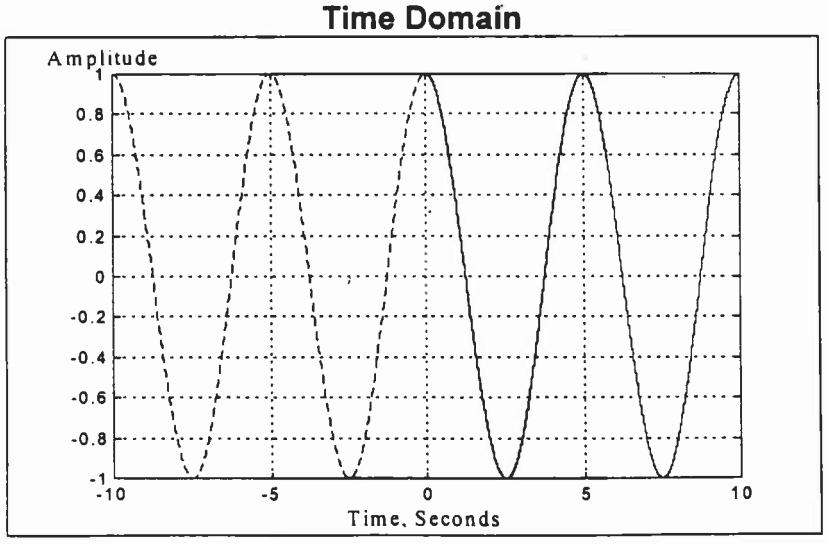
In this case, the window function affects the frequency domain representation of the signal *x(t)* thru a convolution.

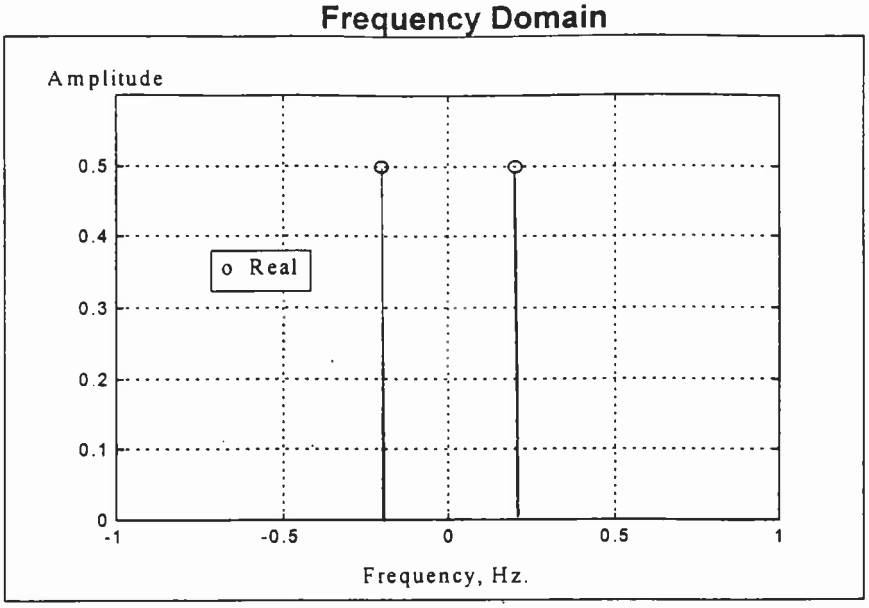
Convolution can be visualized in the time domain below:



Example:

A 0.2 Hz Cosine function is transformed to the frequency domain. The function is represented by two real frequency components, one at -0.2 Hz and one at 0.2 Hz:





These components can be thought of as delta functions.

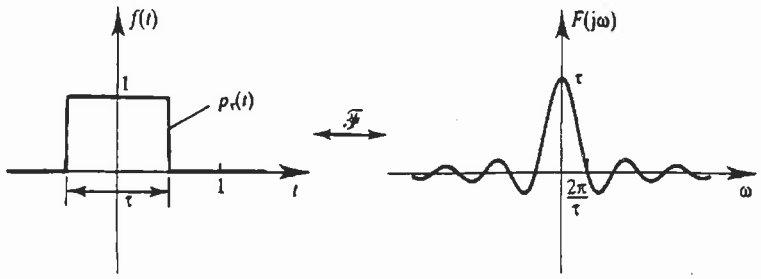
If the transform of the signal is obtained with a *finite length* record, then this is equivalent to windowing the data with Eqn. 7.8.

Again, a multiplication in time domain is equivalent to convolution in the frequency domain. To observe the convolution, we need to convert the window to the frequency domain:

$$W(\omega) = \int_{-\infty}^{\infty} w(t)e^{-i\omega t}\,dt = \int_{-T/2}^{T/2} e^{-i\omega t}\,dt \quad \text{(0 everywhere else)}$$

$$= T\frac{\sin\left(\omega T/2\right)}{\omega T/2} = T\,\mathrm{sinc}\left(\omega T/2\right) \tag{7.10}$$

The limits of integration are symmetrical since this is how the frequency domain data is presented (and the time domain signal is allowed to extend to negative time).

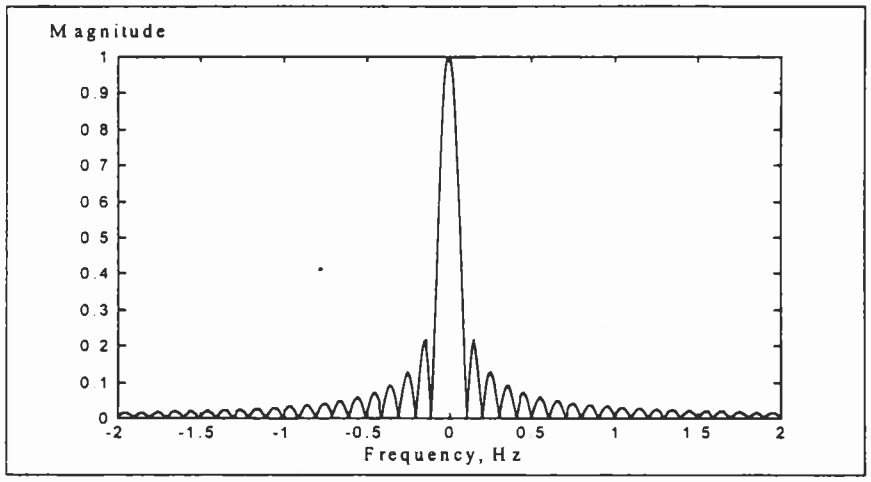


This function has zero crossings at multiples of $2\pi/T$.

In the case of the DFT, the window function is defined slightly different due to the periodic nature of the DFT. This form is seen in Eqn. 6.53:

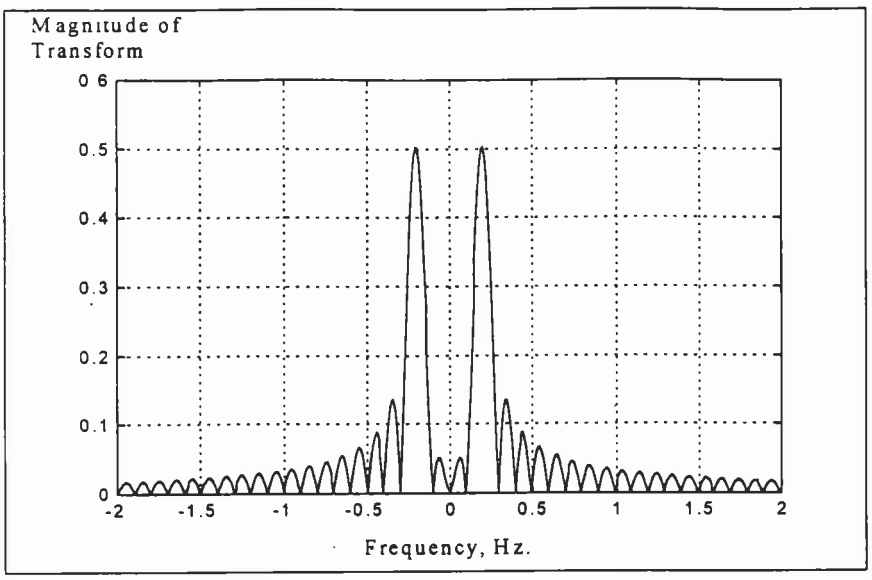
$$W_k = \frac{W(\omega_k, T)}{T} = \text{sinc}\left(\omega_k T/2\right) \qquad (7.11)$$

For the example presented here:



- In this plot, *k* is allowed to take on non-integer values.

Convolving this function with the delta functions of the cosine function with $T = 10$ sec results in the frequency domain plot:

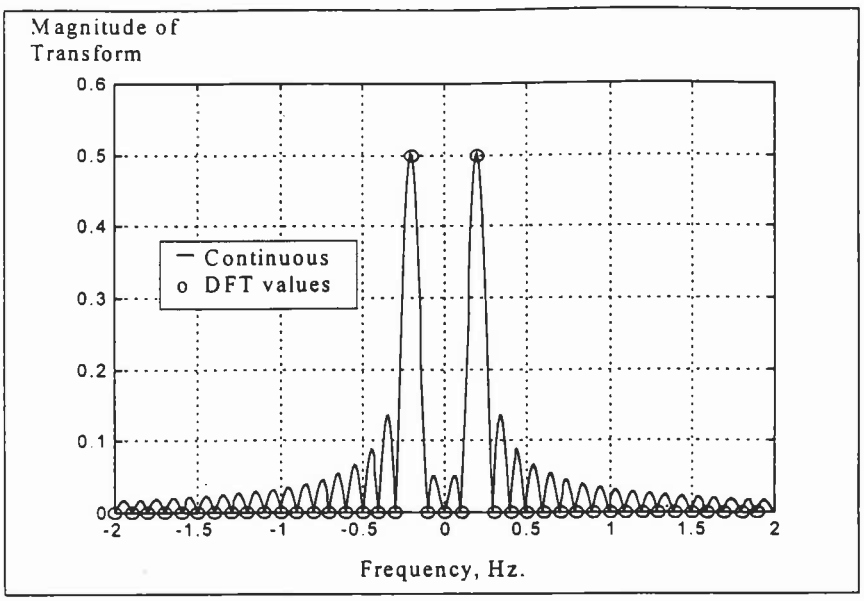


Visually, the above curve is obtained by sliding the window function over the delta functions (in frequency domain) and plotting the products.

- In reality we cannot go thru the calculations presented above because all that we have is a set of sampled points.

- We are dealing with sampled data and there is only a discrete set of frequency domain values that can be determined. These discrete frequencies are given in Eqn. 6.50.

A plot of the actual discrete frequencies obtained in the transform of the windowed data is shown below. For this case, $T = 10\,\text{sec}$ and $\Delta f = 0.1\text{Hz}$:



In this particular case, the *only* frequency components displayed are at ± 0.2 Hz. This happens to be the only ones that really exist, so we are happy : -).

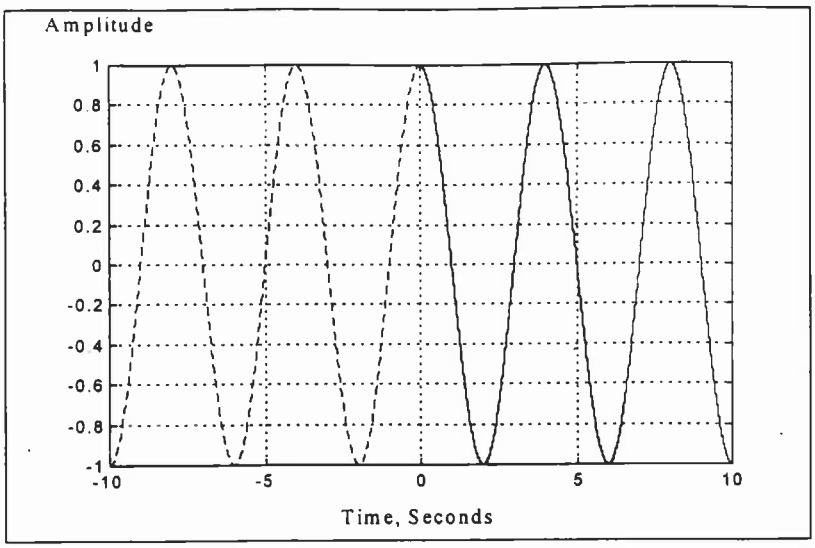
In reality, we most likely won't be measuring signals that land on the discrete frequencies we are calculating. This results in a phenomenon called *leakage.*

7.2.1 Leakage

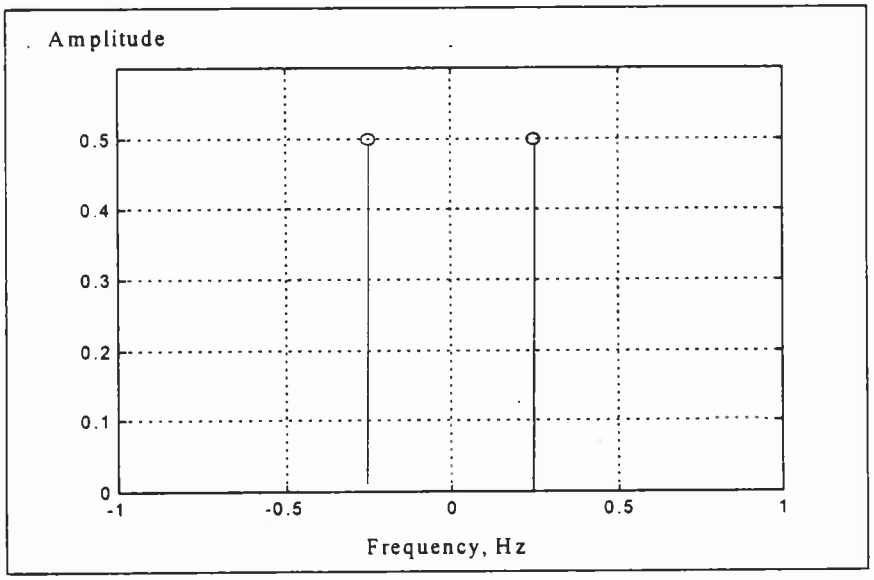
If a signal is not periodic in the sampling window, it will be impossible to represent that signal exactly in the frequency domain. Instead, the signal will "leak" into neighboring frequency bins, giving a false indication of the true frequency spectrum.

Visually, this can be simulated by the following:

A 0.25 Hz cosine function will appear as below, similar to the 0.2 Hz case:

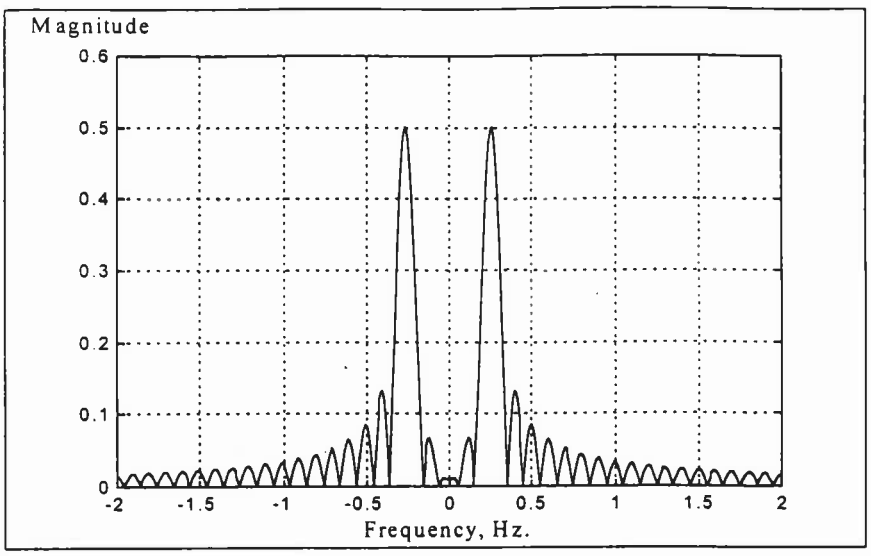


The function is plotted on the frequency axis as two real components of 0.5 magnitude at -0.25 Hz and +0.25 Hz:



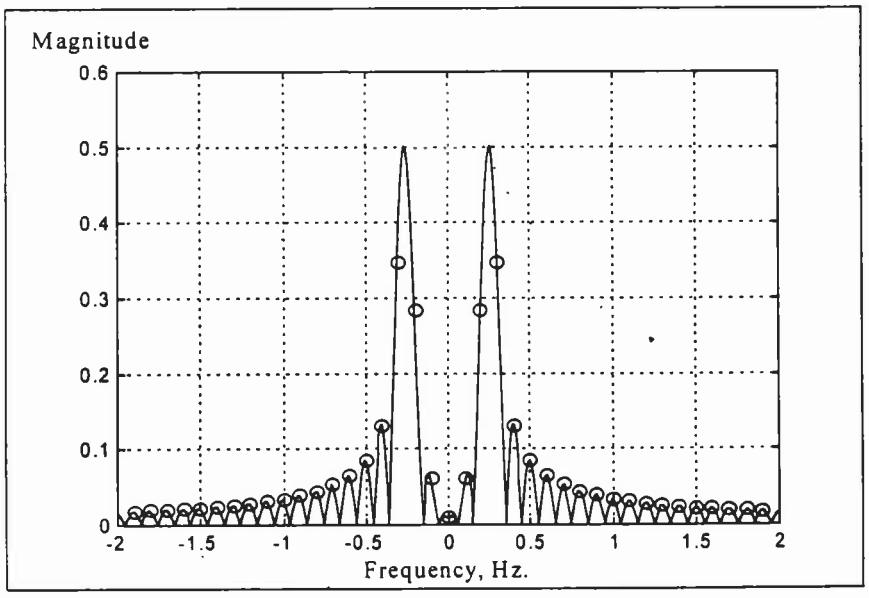
The window function still has a period of 10 seconds; its shape will be the same as in the previous case.

As before, when the window function is slid over the delta functions, a shape similar to the 0.2 Hz case is obtained:

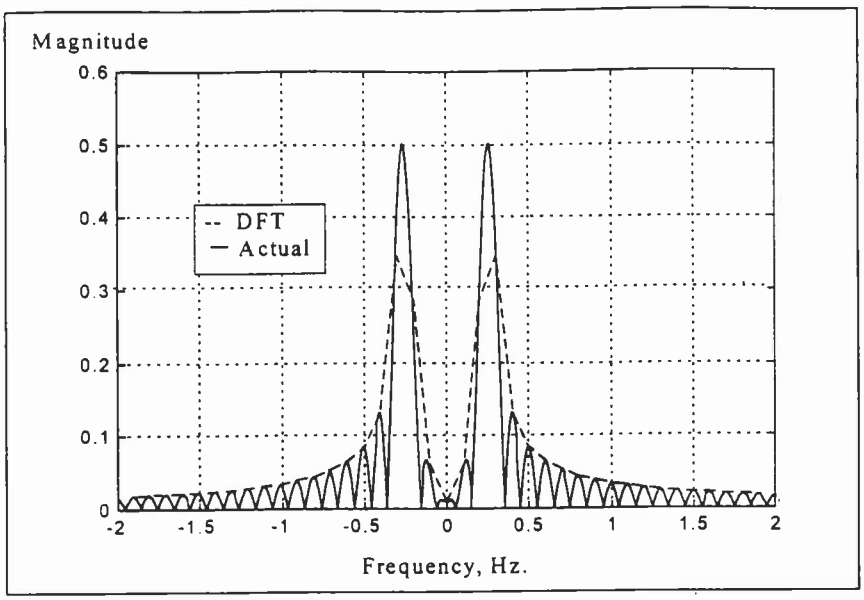


Only now the peaks are at 0.25 Hz.

Again, frequency resolution of the DFT restricts the frequencies obtained from the transform. Overlaying the discrete frequencies calculated gives:

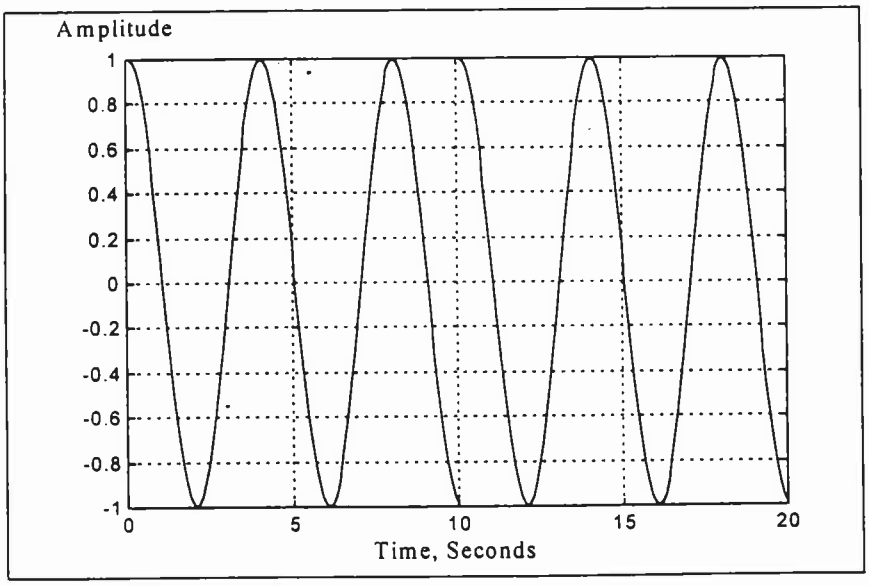


If the discrete points are enveloped, it is observed that the calculated DFT is quite different than the actual:



The error that occurs is due to the time domain signal not being periodic in the sampling window. This causes some of the frequency content of the signal to "leak" into neighboring frequency domain bins, hence the name leakage.

The transform being calculated is on a time sample that is *assumed to repeat itself periodically* on a period equal to the window length. The example presented (0.25 Hz.) would look like:



Which certainly is not the actual signal. The leaked harmonic components can be interpreted to be components necessary to describe the sudden jump from segment to segment.

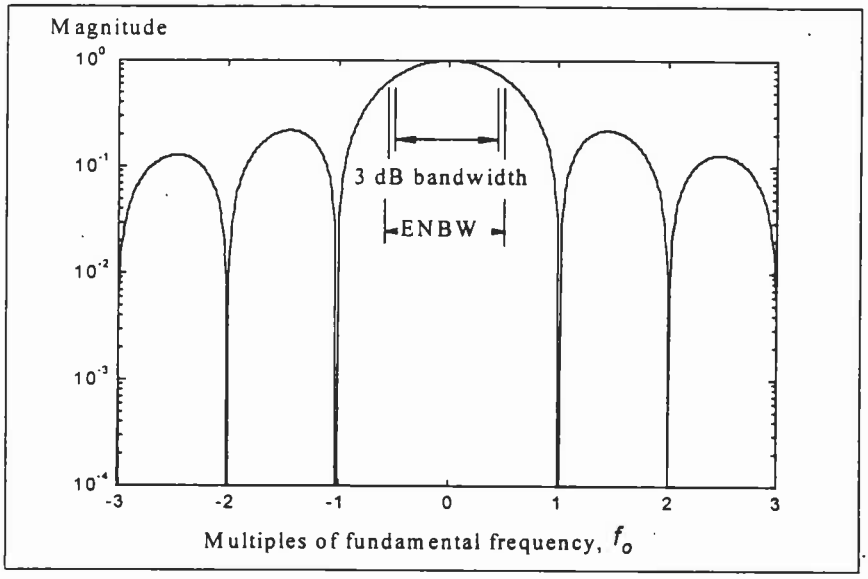
Since we may not have control over the data, it is necessary to do something to minimize the effect of nonperiodicity.

7.2.2 Commonly Used Windows

To reduce the effects of leakage, a "window" can be applied to the sampled data. We have already seen the characteristics of the default window, the rectangular or "boxcar" window.

Windows can be characterized more usefully in the frequency domain. They are actually filters in the frequency domain.

The boxcar window plotted on a log scale in the frequency domain appears as:



Definitions:

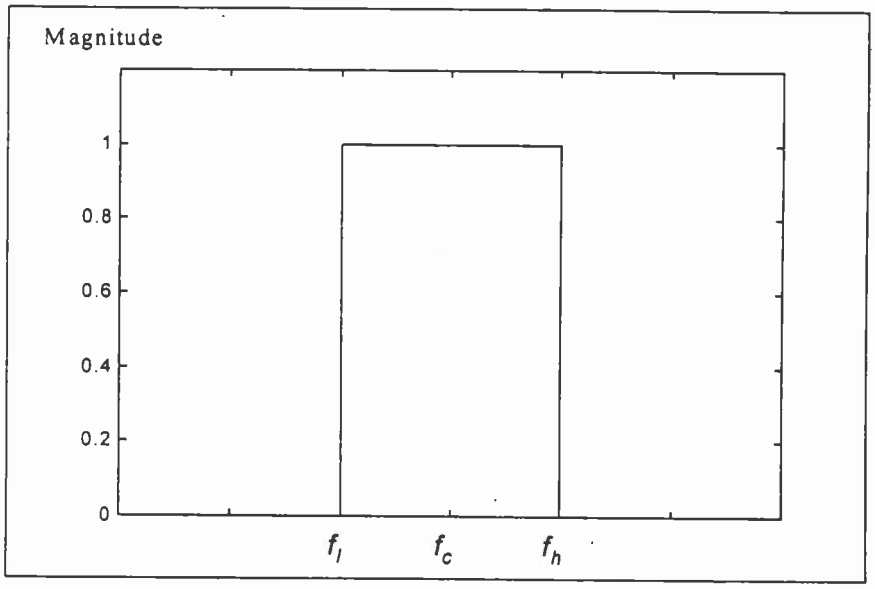
3 dB Bandwidth: The frequency span over which the filter (window) has a magnitude greater than 3 dB.

- The rectangular window has a 3 dB bandwidth of about $0.9\,\omega_o$.

Equivalent noise bandwidth (ENBW): The rectangular window is capturing energy over a broad spectrum because of its frequency-domain side lobes. The ENBW is the bandwidth of a *rectangular filter in the frequency domain* that captures the same broadband energy.

The ENBW is expresses in terms of the fundamental frequency ω_o, or f_o if we are considering Hz. instead of rad/sec.

- A rectangular filter is a theoretical ideal filter that appears as:



It is desirable to minimize the frequency span of the filter $f_h - f_l$ so that a good estimate of the true frequency content is obtained.

- *Recall that the side lobes of a rectangular window caused leakage.*

- Note that the a rectangular window plot only shows *magnitude*; the bandwidth determination accounts for the sign of the function.

If an ideal rectangular filter were feasible, it would be the filter of choice because it concentrates all the measured energy around the spectral line. Note that if the width of the rectangular filter approaches zero: $f_h - f_l \to 0$, the filter becomes a delta function in the frequency domain. What is this in the time domain?

Using the inverse Fourier transform:

$$w(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)e^{i\omega t}d\omega \qquad (7.12)$$

Recall the definition of the convolution integral (Eqn. 6.9):

$$x(t) = \int_{0}^{t} f(\xi)\,h(t-\xi)d\xi \qquad (7.13)$$

This equation had its origin in a more general equation defined for a delta function and an arbitrary function *f:*

$$\int_{-\infty}^{\infty}\delta(\omega-\nu)f(\omega)d\omega = f(\nu) \qquad (7.14)$$

Substituting $f(\omega) = e^{i\omega t}$ and $\nu = 0$ simplifies Eqn. 7.12:

$$w(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-0)e^{i\omega t}d\omega = \frac{1}{2\pi}e^{i0t} = \frac{1}{2\pi} \qquad (7.15)$$

If the delta function is scaled by 2π this results in $x(t) = 1$, which states that the *delta function in the frequency domain represents an infinitely long, rectangular time-domain window.*

If data is sampled for an infinite amount of time at infinitesimally small intervals, then we would end up with a very accurate frequency domain representation of the data.

- Since this is not possible, some means of filtering (windowing) the data must be used to account for the finite sampling period.

The most common window used is the Hanning Window, which is represented in the time domain by:

$$w(t) = \frac{1}{2}\left(1 - \cos\frac{2\pi}{T}t\right) \quad \text{for } 0 \le t \le T \qquad (7.16)$$

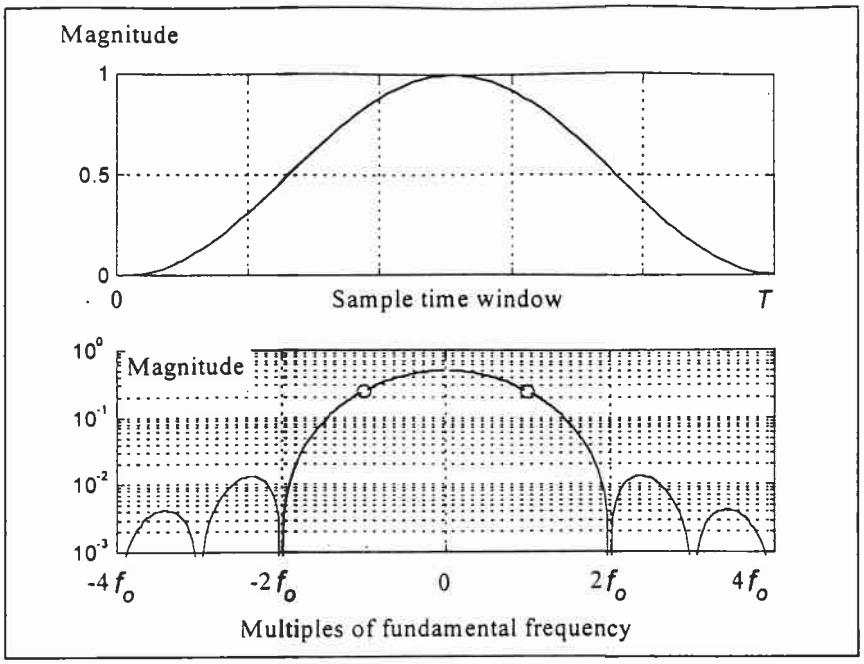
and in the frequency domain by:

$$W(\omega) = \frac{T}{2}\left(\frac{\omega_o^2}{\omega_o^2 - \omega^2}\right)\frac{\sin\left(\omega T/2\right)}{\left(\omega T/2\right)} \qquad (7.17)$$

Or in the form of the DFT:

$$W_k = \frac{1}{2}\left(\frac{\omega_o^2}{\omega_o^2 - \omega_k^2}\right)\frac{\sin\left(\omega_k T/2\right)}{\left(\omega_k T/2\right)} \qquad (7.18)$$

These are graphically shown below:



When the transform of a signal is determined over the sampled period *T,* discrete frequency components will be calculated at multiples of the fundamental frequency of $\omega_o = 2\pi/T$, or $f_o = 1/T$.

In the case of a harmonic centered on a calculated frequency, the window will produce components in the frequency domain *at the neighboring bins.* This is indicated by the circled points on the frequency plot which are *non zero* at the neighboring frequencies, f_{n-1} and f_{n+1}.

This works in the following manner:

- The center lobe of the Hanning filter weighs a bin-centered harmonic f_n by 0.5 at the f_n bin.
- When the filter is moved to the next lower bin (convolution) f_{n-1}, the calculated component will be *0.25 x adjacent f_n bin-centered harmonic.*
- Likewise, moving up to the next highest frequency f_{n+1} results in a component of *0.25 x f_n bin-centered harmonic.*

So for a harmonic centered on a bin, the calculated frequency value that gets recorded is:

$$X(k)_{\text{Hanning}} = \frac{1}{2}\left[X(k) + \frac{1}{2}\left[X(k-1) + X(k+1)\right]\right]_{\text{Rectangular}} \quad (7.19)$$

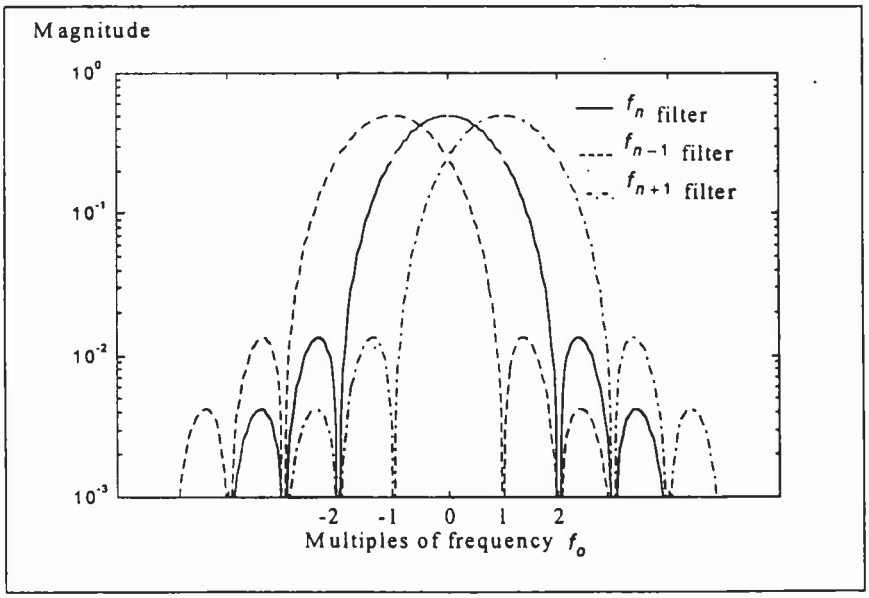
The bin-centered harmonic has been reduced in amplitude and the recorded harmonic is determined from three neighboring bins.

- The reduction in amplitude of the central harmonic is due to the attenuation in the time domain that the window causes. This is called *coherent gain.*

- For the Hanning window, coherent gain = 0.5 and is usually compensated for in the analyzer.

Another way to visualize windowing is to consider the window function as a series of *frequency domain filters centered at the discrete calculated frequencies.*

Three of these are shown below:



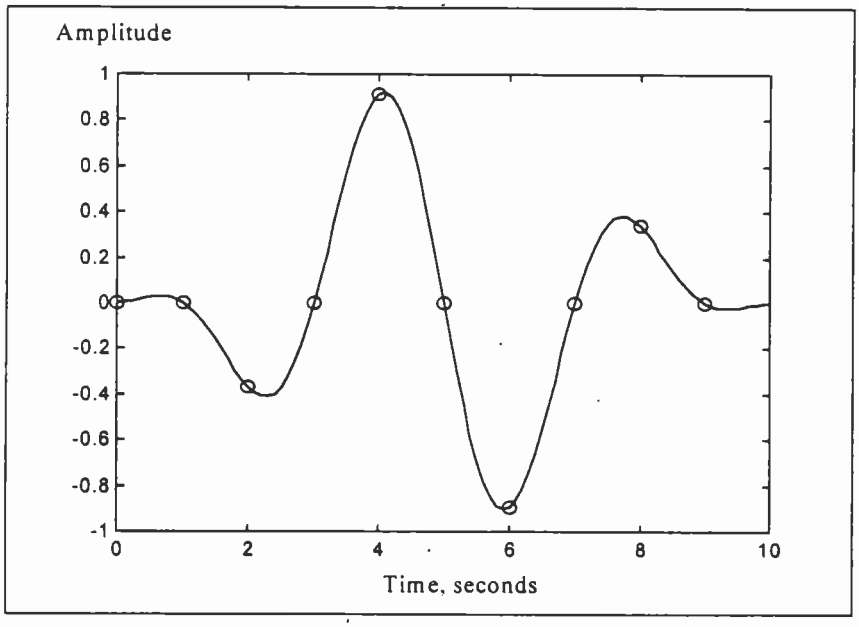
The result of applying a Hanning window to a bin-centered harmonic is to spread the signal out over three bins. This is not good, since it is the leakage phenomenon we saw using the rectangular window.

However, if the Hanning window is applied to a harmonic that is *not* bin centered, the result is *less leakage than if the rectangular window was used. This is very good.*

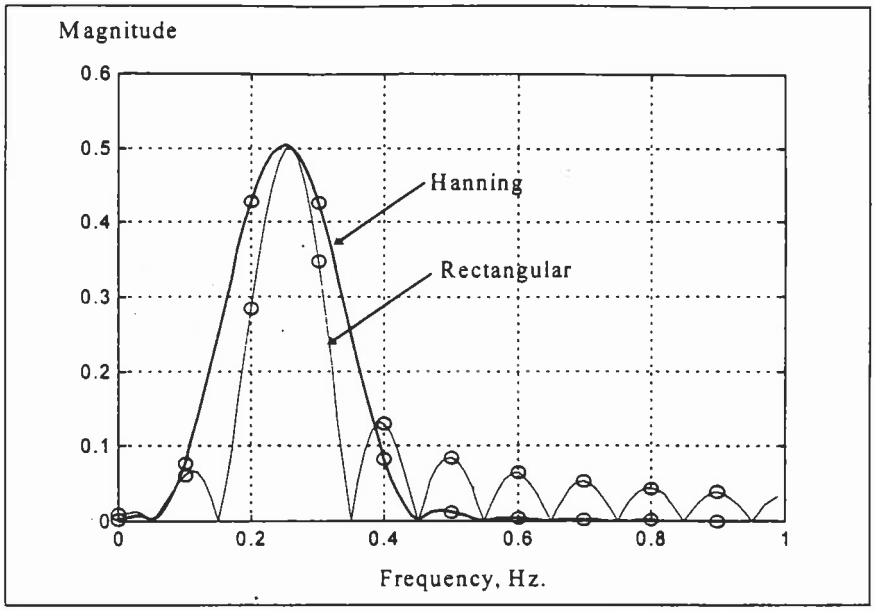
Consider the following two cases:

The same 0.25 Hz cosine signal that was examined before using the rectangular and the Hanning windows.

In the Hanning case, the time domain data appears as:



The following plot is obtained if we look at positive frequencies only:



It can be seen that the Hanning windowed cosine function appoximates the 0.25 Hz signal much better than the rectangularly windowed data. Note that the Hanning data has been coherent gain compensated.

The 0.2 Hz and 0.3 Hz components that approximate the 0.25 Hz function are down -1.4 dB from the true amplitude (.851) for the Hanning window. This is better than the rectangularly-windowed components which are attenuated approximately -3.9 dB.

Definitions cont'd:

Ripple: The uncertainty within an f_o bandwidth. The attenuations above are also known as the filter ripple.

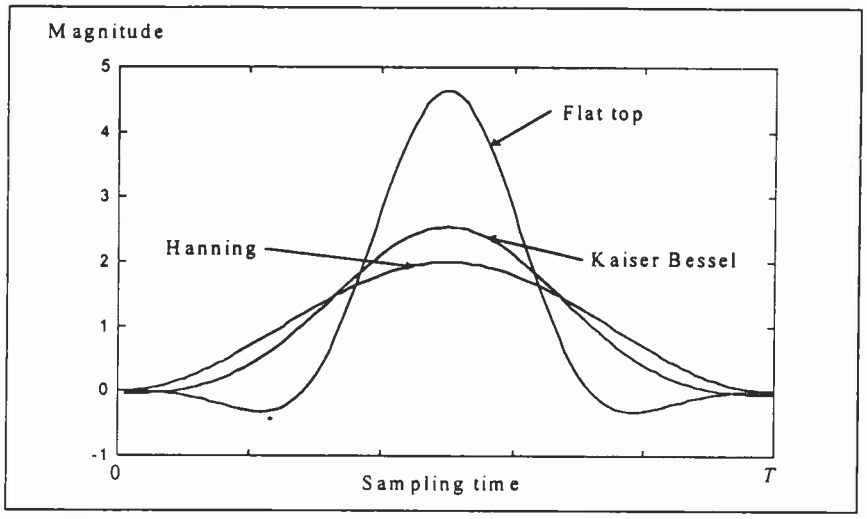
Another interpretation of the window is obtained in the time-domain:

The sampled data for the Hanning example shows that the beginning and end of the record is *driven to zero by the window.* This makes the record periodic, which we have seen is necessary for an accurate frequency domain representation.

Other common windows include the Kaiser-Bessel, flat top and Exponential:

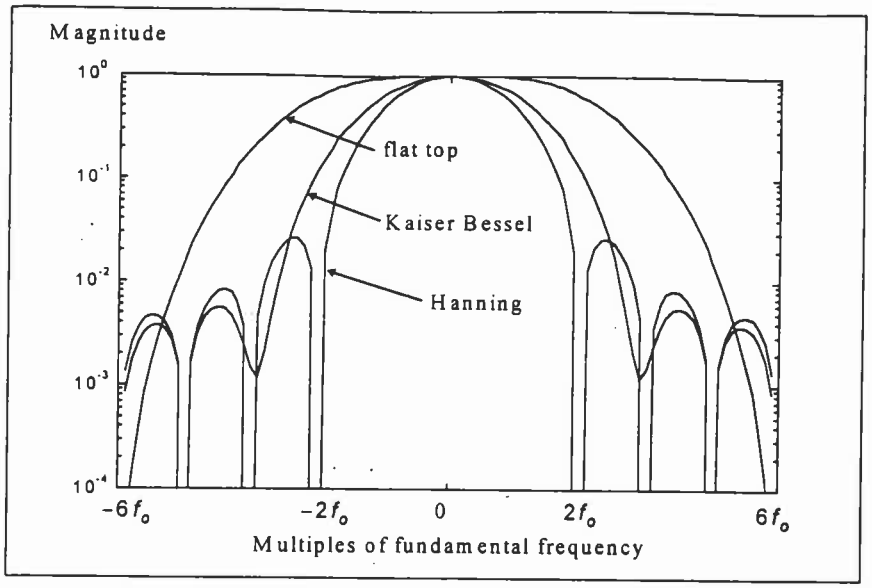
The Kaiser-Bessel and flat top windows have applications similar to the Hanning window: they force continuous data to be periodic in the sampling window.

All three windows in time domain are shown below:



Note that the coherent gain has been accounted for in these windows, so the frequency domain amplitude will be correct.

These windows can also be compared in the frequency domain:



The window selected for the sampling depends on the type of signal being sampled:

- Flat top has the smallest attenuation over a range of frequencies. It is good for single sine signals.
- Kaiser Bessel is effective if neigboring sinusoids are at least 3 bins apart.
- Hanning is a good window if the signals are separated by less than 3 bins. This applies to random signals.

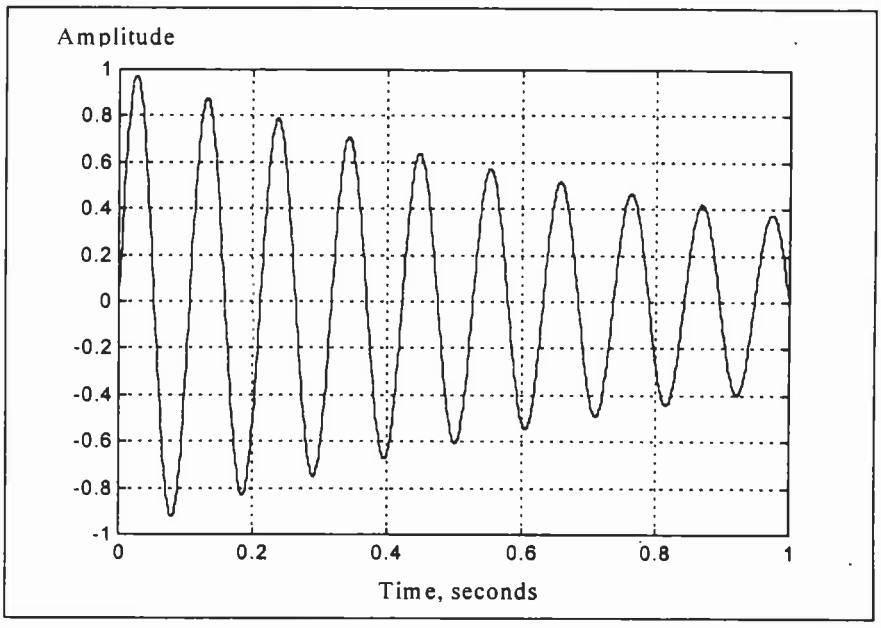
Common Windows

	Bandwidth		
Window Type	ENBW	3 dB	Ripple (dB)
rectangular	$1.0f_o$	$0.89f_o$	3.92
Hanning	$1.5f_o$	$1.44f_o$	1.42
Kaiser-Bessel	$1.8f_o$	$1.71f_o$	1.02
flat top	$3.77f_o$	$3.72f_o$	0.01

Another common window is the exponential window. The exponential window is used when excitation is transient.

Recall that windows drive the signal to be periodic in the time-domain. This applies to transient signals as well, although the transient nature of the signal itself will cause other harmonics to occur.

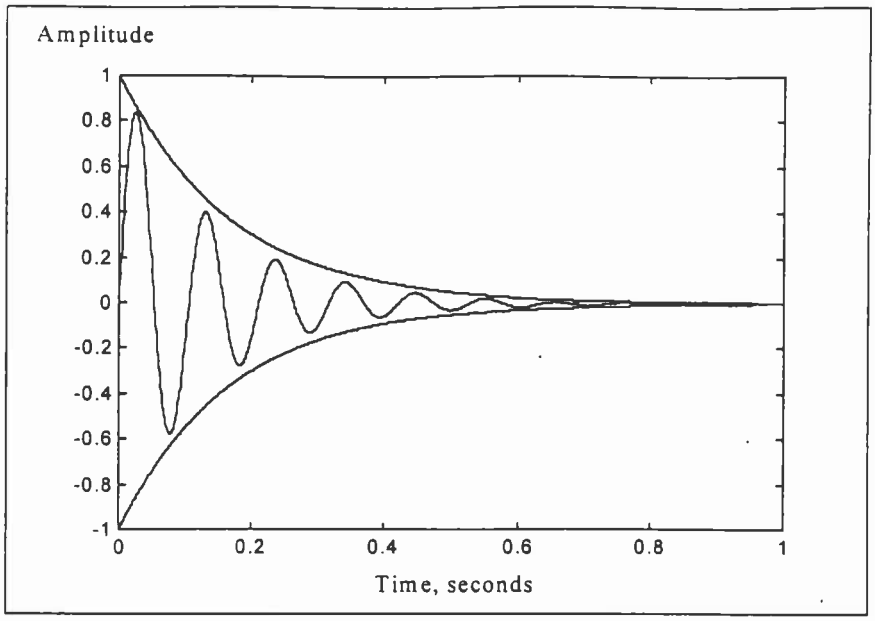
An impact test will create a "ring down" response of the structure, which is characterized by an exponential decay over a harmonic response:



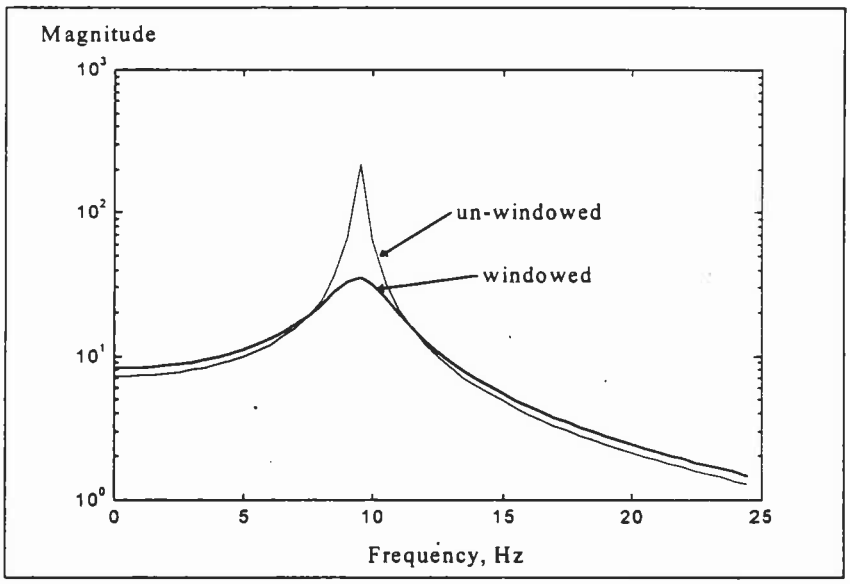
In this case, only one harmonic is shown (at 9.5 Hz). This would be the case for a single degree of freedom system.

The harmonic response is not periodic in the window, which will cause a leakage effect.

To correct this, an exponential window can be applied to this type of data to drive the ring-down to a zero amplitude:



Looking at the frequency domain of these two cases shows that even with the window, there is leakage:



Not only is there leakage, but the peak has decreased.

In order to sort this out, the dynamics of the system must be analyzed.

The ring-down behavior of a system is caused by the damping in the structure. For a single degree of freedom system, the time response can be written as:

$$x(t) = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \qquad (7.20)$$

In the above example, the following parameters were chosen:

$$\begin{aligned} \zeta &= 0.0167 \\ \omega_n &= 59.699 \text{ rad / sec} \\ C &= 1 \end{aligned}$$

Recall that in the frequency domain, an SDOF system will behave as:

$$\left|\frac{X}{F/K}\right| = \frac{1}{\sqrt{\left(1-\left(\omega/\omega_n\right)^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}} \qquad (7.21)$$

So the ring-down in the time domain is observed to be the *apparent leakage in the frequency domain.* The damping term causes both the ring-down and frequency-dependent response.

The decrease in the frequency domain peak is due to the *artificial damping applied to the system.*

If the system parameters are to be extracted from the frequency domain data, some means of correcting for the artificial damping must be used.

It can be shown that the actual system damping can be obtained from:

$$\zeta = \zeta' - a/\omega_n \qquad (7.22)$$

Where ζ' is the damping obtained from the curve-fit of the frequency domain data, and a is related to the exponential window by:

$$w(t) = e^{-at} \qquad (7.23)$$

In the case of processing data in an analyzer, there are considerations to be made regarding the windowing of the input and output. This is covered in the next section.

$2(.5)(.5)$

7.3 Estimating Spectra for Real Structures: part II

7.3.1 Spectral and FRF Estimates

Once the data has been digitized and brought into the analyzer, the spectral quantities discussed previously are determined.

Analyzers typically redefine the spectral quantities presented in Section 6.3 around the definition of the periodic discrete Fourier transform (section 5.3). Furthermore, the DFT is usually calculated for *positive frequencies only*. We have already seen that we only need positive frequencies to regain the original data exactly.

Recall that the *DFT* was defined from the Fourier transform of a variable $x(t)$:

$$X_k = \frac{X(\omega_k, T)}{T} \tag{7.24}$$

Since the analyzer is a discrete sampling device, this will be the same definition implemented by the analyzer.

The single-sided autospectra and cross-spectra for discrete data is likewise defined differently:

$$\begin{aligned} G_{XX_k} &= 2S_{XX_k} = 2E\left[\hat{S}_{XX_k}\right] = 2E\left[\hat{X}_k^* \hat{X}_k\right] \\ G_{XY_k} &= 2S_{XY_k} = 2E\left[\hat{S}_{XY_k}\right] = 2E\left[\hat{X}_k^* \hat{Y}_k\right] \end{aligned} \quad \text{for } k \geq 1$$

(7.25)

$$\begin{aligned} G_{XX_0} &= E\left[\hat{S}_{XX_0}\right] = E\left[\hat{X}_0^* \hat{X}_0\right] \\ G_{XY_0} &= E\left[\hat{S}_{XY_0}\right] = E\left[\hat{X}_0^* \hat{Y}_0\right] \end{aligned} \quad \text{for } k = 0$$

This relates to the spectral density terms defined in Section 5 by:

$$G_{XX_k} = 2\left(X(\omega_k,T)^*/T\right)\left(X(\omega_k,T)/T\right) = \frac{G_{XX}(\omega_k,T)}{T}$$
$$G_{XY_k} = 2\left(X(\omega_k,T)^*/T\right)\left(Y(\omega_k,T)/T\right) = \frac{G_{XY}(\omega_k,T)}{T} \quad (7.26)$$

(for $k \geq 1$)

Note that X_k has the same units as $x(t)$. Furthermore, G_{XX_k} and G_{XY_k} are expressed as $(\text{units})^2$. This differs from the autocorrelation-derived auto and cross-spectra which were expressed as $(\text{units})^2/\text{Hz}$.

Typically, the analyzer has many options for displaying spectral information. A few of these are:

1. Spectrum $\sqrt{G_{XX_k}}$: This is simply the magnitude of the autospectrum as defined above.

2. Power or Energy Spectral Density (PSD or ESD) $\hat{G}_{XX_k}$:
   This is the form of the auto and cross-spectral densities $G(\omega)$ that we originally derived. It comes in $(\text{units})^2/\text{Hz}$. The calculation is simple:
   $$\breve{G}_{XX_k} = G_{XX_k}T$$

Calculating the FRF is the same as presented in Section 6.3:

$$H_{1_k} = \frac{G_{FX_k}}{G_{FF_k}}$$
$$H_{2_k} = \frac{G_{XX_k}}{G_{XF_k}} \qquad (7.27)$$

Where X is the input and F is the output.

Averaging is performed according to the coherence γ^2_{XY}:

$$G_{XX_k} = \frac{2}{n_d} \sum_{l=1}^{n_d} \hat{X}_k^* \hat{X}_k$$
$$G_{XF_k} = \frac{2}{n_d} \sum_{l=1}^{n_d} \hat{X}_k^* \hat{F}_k \qquad k \geq 1 \qquad (7.28)$$

7.3.2 Methods of Excitation

Another consideration for estimating spectral quantities is the type of excitation used on the structure.

The following will be addressed:

1. Sine Dwell Excitation
2. Swept Sine
3. Stepped Sine
4. Pure Random
5. Burst Random
6. Multisine or Periodic Random
7. Pseudo or Synchrononous Random
8. Impact

1. Sine:

Sine excitation is simply the input of a single-frequency force and the measurement of a single-frequency response.

If the input frequency is bin-centered, there is no leakage and a rectangular window can be used.

If the input is not bin-centered, then a flat top window can be used to get the best estimate of the center frequency of the signal.

Advantages:
- Very high signal to noise ratio.
- Good for checking linearity of the structure.
- Usually there is no leakage.

Disadvantages:
- Frequency of excitation may not be compatible with analyzer frequency
- Only one frequency being excited

Resonance sine dwell is a technique of exciting a structure at a natural frequency by tuning the force signal to be 90° out of phase with the response.

The theory is based on the general form of the FRF with viscous damping:

$$\frac{X_i}{F_j} = \sum_{r=1}^{n}\left(\frac{\hat{\phi}_{ir}\,\hat{\phi}_{jr}}{-\omega^2 + i2\zeta_r\omega\omega_r + \omega_r^2}\right) \qquad (7.29)$$

If the excitation is at the r^{th} natural frequency, the response will be dominated by the r^{th} mode:

$$H_{ij}(\omega_r) = \frac{X_i(\omega_r)}{F_j(\omega_r)} \cong \frac{\hat{\phi}_{ir}\,\hat{\phi}_{jr}}{i2\zeta_r\omega_r^2} = \frac{-i\,\hat{\phi}_{ir}\,\hat{\phi}_{jr}}{2\zeta_r\omega_r^2} \qquad (7.30)$$

Which says that the response either leads or lags the force by 90°, depending on the sign of $\hat{\phi}_{ir}\hat{\phi}_{jr}$.

The single-mode response can be enhanced by using multiple shakers.
If a finite element model of the stucture is used, exciter locations can be chosen and phased to excite a single mode. The frequency of excitation is chosen to obtain a 90° relative phase between response and force.

In the absense of FEA, experimental techniques can be used to tune modes.

Asher[1] proposed a set of force vectors that would isolate a single mode by driving the real response to zero.

The admittance matrix can be defined as:

$$\boldsymbol{D} = \left(-\omega^2 \boldsymbol{M} + \boldsymbol{K} + i\boldsymbol{C}\right)^{-1} \tag{7.31}$$

Where:

$$\left[\boldsymbol{D}_{real} + i\boldsymbol{D}_{imag}\right]\bar{\boldsymbol{F}} = \bar{\boldsymbol{X}} \tag{7.32}$$

Assuming proportional damping, excitation at the natural frequency will cause the real part of the response to go to zero:

$$\boldsymbol{D}_{real}\,\bar{\boldsymbol{F}} = \boldsymbol{0} \tag{7.33}$$

The admittance function can be experimentally determined with trial force vectors. Natural frequencies are found from $\left|\boldsymbol{D}_{real}\right| = \boldsymbol{0}$ and the forcing vector required to excite the modes is the eigenvector $\boldsymbol{F}$ of Eqn. 7.33.

Another way to derive a forcing vector that excites only one mode is to recall Eqn. 3.47:

$$\bar{\boldsymbol{X}} = \phi\left(-\omega^2 \boldsymbol{M} + i\omega\boldsymbol{C} + \boldsymbol{K}\right)^{-1} \phi^T \bar{\boldsymbol{F}} \tag{7.34}$$

---

[1] Asher, G.W., "A Method of Normal Mode Excitation Utilizing Admittance Measurements," *Proceedings of the National Specialists Meeting on Dynamics and Aeroelasticity, Institute of the Aeronautical Sciences,* Fort Worth, TX, Nov. 1958, pp. 69 - 76.

At resonance, the equation reduces to:

$$\bar{X} = \phi(i\omega C)^{-1} \phi^T \bar{F} \qquad (7.35)$$

But if we are interested in exciting the natural modes of vibration, $X = \phi$, and we end up with a *set of forcing vectors given by:*

$$F = \phi^{-T} \qquad (7.36)$$

Iterative techniques exist[2] to find the mode shapes experimentally using trial forcing vectors.

Sine dwell techniques were popular from the late 50's to the mid 80's because of the high signal-to-noise ratio and the ability to drive large structures.

Large structures are typically subject to nonlinearities, and having a constant amplitude input has an advantage over a random response in that the structure characteristics won't change during the test.

2. Swept Sine:

The swept sine test is a continually increasing- or decreasing-frequency sine function that covers the frequency range of interest in the test.

[2] Morosow, G., and Ayre, R.S., "Force Apportioning for Modal Vibration Testing Using Incomplete Excitation," *Shock and Vibration Bulletin,* No. 48, Pt. 1, 1978, pp. 39 - 48.

The sweep rate is slow enough so that the system is quasi-steady state during data acquistion. A rule for determing sweep rate is:

$$S_{max} < 216 f_r^2 \zeta_r^2 \quad \text{in Hz / min} \qquad (7.37)$$

Too fast a sweep rate will distort the FRF by moving the resonance. One way to check the sweep rate is to sweep up and then down and compare the FRF's; they should be the same.

Advantages:
- Fast
- High signal-to-noise ratio
- Entire FRF is obtained

Disadvantages:
- May not be an accurate FRF

A typical application of swept sine is in acquiring a quick estimate of resonances on a structure. Once the resonances are found, a sine dwell technique or random excitation can be applied around the frequencies of interest.

3. Stepped Sine

A variation of the swept sine test is the stepped sine test. This test dwells at bin-centered frequencies and waits until the structure reaches an equilibrium before data is acquired. The coherence can be determined, and when it exceeds a certain value the spectral quantities are recorded.

Stepped sine testing is becoming more popular as vibration-control software is incorporated in analyzers.

Advantages:
- High signal-to-noise ratio
- Very high coherence

Disadvantages:
- Slow

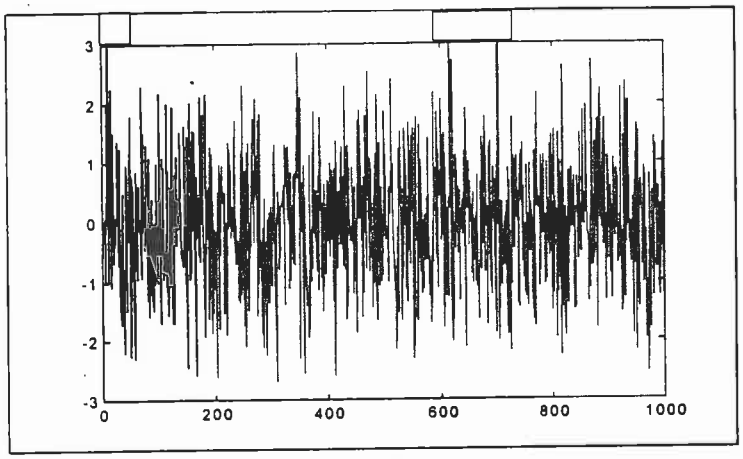
4. Pure Random

A pure random signal consists of a signal formed from harmonics that include all frequencies (∞) in the range of interest with amplitudes and phases that vary randomly. Random signals are frequently referred to as noise.

Different categories of random noise:

White noise: A random signal that has frequency content from D.C. to well above the range being measured.

In the time domain, the signal will appear as:



This signal is distributed $x(t)\left[\mu = 0, \sigma^2 = 1\right]$.

Variance is related to the mean square value $E\left[x^2\right]$ by:

$$\sigma^2 = E\left[x^2\right] - \left(E[x]\right)^2 \qquad (7.38)$$

If the mean is zero, then $\sigma^2 = E\left[x^2\right]$, and the frequency domain content will appear as given in Eqn. 6.7:

$$\sigma^2 = \int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega \qquad (7.39)$$

This relates to the analyzer by Eqn. 7.26:

$$\sigma^2 = \sum_{k=0}^{N-1} G_{XX}(\omega_k, T)\frac{1}{T} = \sum_{k=0}^{N-1} G_{XX_k} \qquad (7.40)$$

This can also be related to a sine function in the following manner:

Let A_k be the amplitude of the sine. The DFT of the signal gives a component $X_k = A_k/2$. From Eqn. 7.25, the autospectra is given by:

$$G_{XX_k} = 2\left(\frac{A_k^*}{2}\right)\left(\frac{A_k}{2}\right) = \frac{A_k^2}{2} \qquad (7.41)$$

Which is the correct value of the mean square.

Referring to the determination of G_{XX_k}:

- In the long run average, the spectral content of the signal will be constant over the range of interest.

- Because of the randomly varying amplitude and phase, *averaging must be used to guarantee that all frequency components are represented.*

The number of averages are in part determined from the coherence plot in Section 6.3.

Again, in the long run average, all values of the autospectra will be described by:

$$G_{XX} = \frac{\sigma^2}{N} \qquad (7.42)$$

Which will be plotted as a constant value over the range of frequencies.

- It should be noted that the input and response values of G_{XX_k} will represent an *average value* over the bin width. This may cause a problem as illustrated below.

Band-limited White Noise

This is simply a random signal that has content within a limited frequency range.

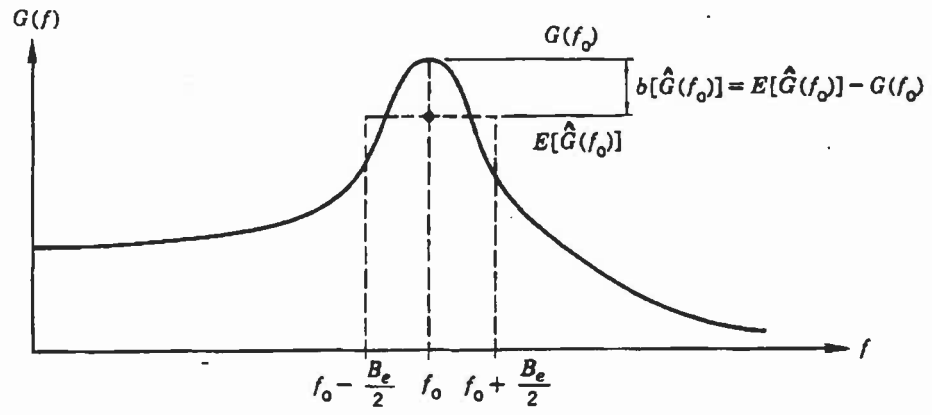
It is used if only a particular frequency range of interest needs to be excited.

The response of the structure will also behave in a random fashion, however the amplitudes will be a function of the system dynamics.

- A common problem with random excitation and response is *resolution bias error*.

This is the error that results in estimating a high-Q system using finite-width frequency bins. Systems with low damping will have a large response at the resonance. The estimated response will be an *average* over the frequency bin.

This is illustrated below:



The quantity $b\left[\hat{G}(f_o)\right]$ is the bias error caused by measuring the spectral content over the bin bandwidth of B_e.

A conservative estimate of the error is given by Bendat and Piersol[3] to be:

$$b\left[\hat{G}_{XX_k}\right] \approx \frac{B_e^2}{24} G'_{XX_k} \qquad (7.43)$$

[3] Bendat, J.S., and Piersol, A.G., *Random Data,* 2nd Ed., John Wiley and Sons, Inc., 1986, p 280.

Some other points concerning random excitation:

- Random signals require the use of windows because they will not be periodic, in general.

- Again, the coherent gain of the window is usually taken into account, so the amplitudes will not have to be corrected.

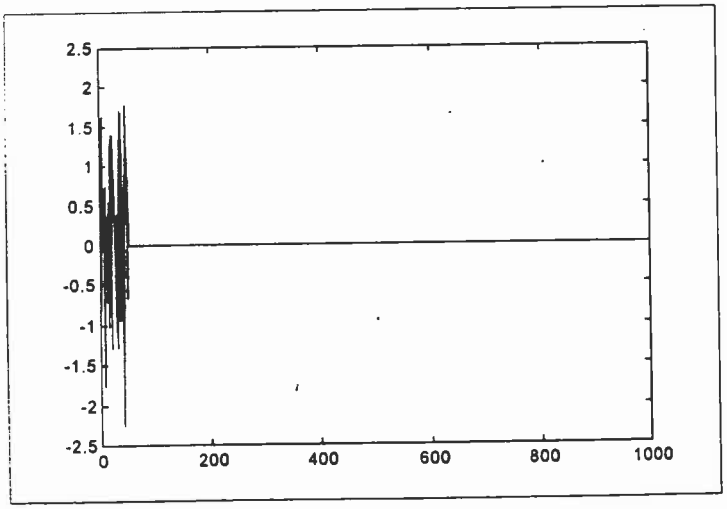
Advantages:
- Excites all frequencies of interest
- Fast
- Linear estimate of non-linear systems

Disadvantages:
- Leakage
- Resolution bias error
- Mandatory use of windows

5. Burst Random

Similar to an impact test, burst random excites a structure with a transient signal consisting of a burst of noise.

It appears as:



Burst random allows sufficient energy to be imparted to the structure because of the duration and amplitude of the burst.

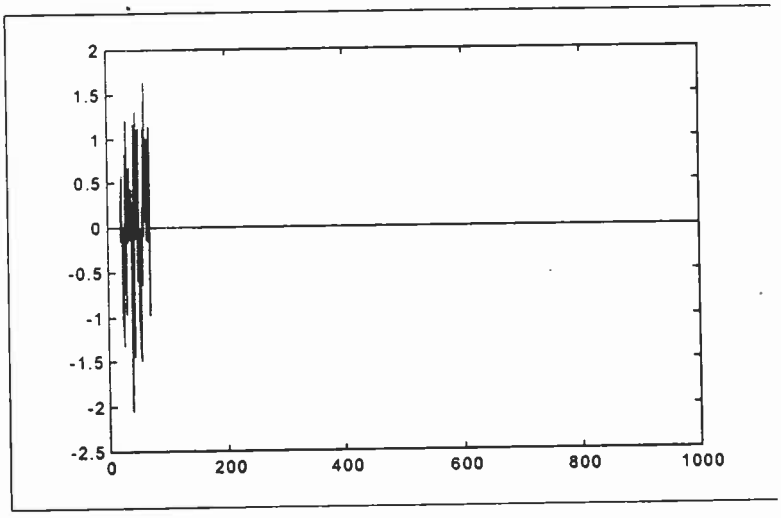
- It has the advantage of being a periodic excitation, beginnning at zero amplitude and ending at zero amplitude.

- The structural response may exist beyond the sampling window, requiring an exponential window to be applied.

Normally, we would expect the analyzer to trigger the start of its data acquisition from the burst signal.

If the analyzer is set to trigger on 10% of full scale, positve slope, then *part of the signal will be lost.*
To avoid this, a *pretrigger delay* is set in the analyzer to begin acquiring data before it detects the trigger condition.

The burst random input will appear as:



The input and output will have a zero response at the beginning.

- Like the pure random case, averaging must occur.

- The time delay between bursts must be long enough so that the structure is not ringing at the start of a new burst.

Advantages:
- If the structure damping is high enough, or the sample is long enough, then no window is required.
- All frequencies of interest are excited

Disadvantages:
- May not have capability with equipment

6. Multisine or Periodic Random

This is a synthesized signal that has bin-centered frequency components and randomly varying magnitude and phase.

The advantage of this signal is that it has no leakage since all components are periodic. Because each sample is uncorrelated to the previous one, the excitation cannot be continuous.

Advantages:
- No leakage
- Best linear approximation for nonlinearities

Disadvantages:
- Longer test time
- Sophisticated equipment

7. Pseudo or Synchronous Random

This is similar to Periodic Random except that each subsequent record is a repeat of the previous. This results in an excitation "loop" which is truly periodic. The amplitudes are constant and only the phase is allowed to vary randomly. The excitation can be continuous.

Advantages:
- Fast
- No leakage

Disadvantages:
- Poor nonlinear estimator

8. Impact excitation

An impact on a structure is a special case of random excitation where all of the harmonic components are input with the same phase.

In the time domain, the impact is represented by a delta function, $f(t) = \delta(t)$. The transform of the delta function is given by:

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt \qquad (7.44)$$

Recall Eqn. 7.13 expressed as the convolution integral in t:

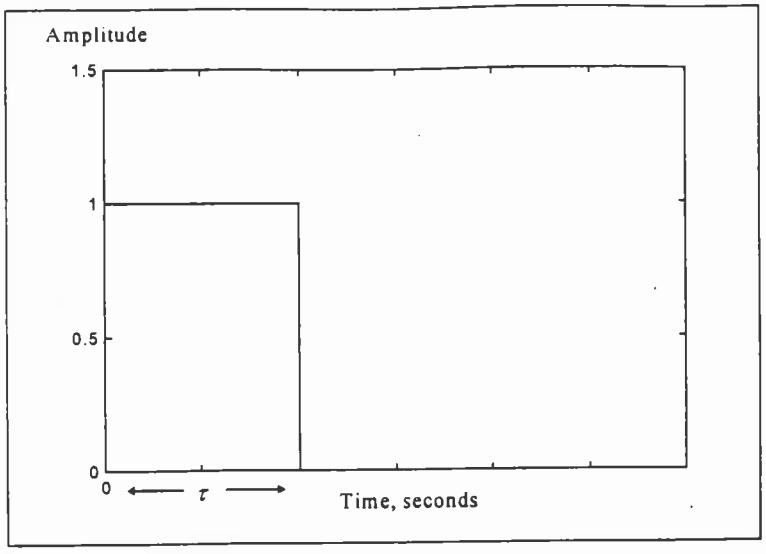
$$\int_{-\infty}^{\infty} \delta(t - \nu) f(t) dt = f(\nu) \qquad (7.45)$$

If $\nu = 0$, then $f(t) = e^{-i\omega t}$ and we obtain $F(\omega) = 1$. This says that an *ideal impulse represents a uniform distribution of energy over the entire frequency spectrum.*

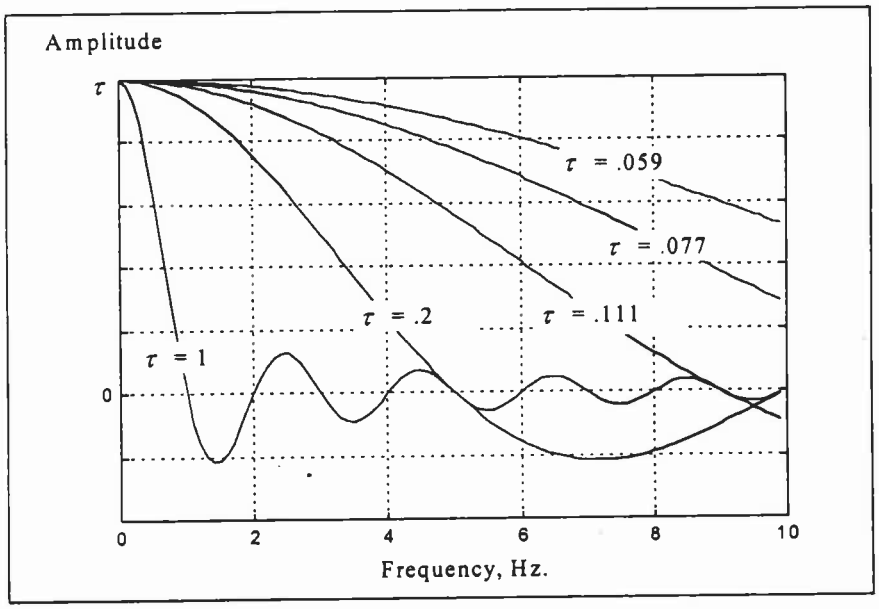
This is good if we wish to excite all frequencies in a structure.

In reality, an impulse is impossible to achieve because it implies infinite height and zero width.

The best we can do is to create a *pulse* of small width and large height:



This has the corresponding frequency domain representation of:



It can be seen that as the pulse width decreases, the frequencies excited increases.

- An impact test is an effective way to excite frequencies in the range of interest

For structures with low natural frequencies, the pulse can be wider which will concentrate the energy around lower frequencies.

- This is achieved with a soft tip on the hammer.

For structures with high natural frequencies, the pulse width should be narrower.

- This is achieved with a hard tip hammer.

The most common combination of windows for an impact test is the force/rectangular window on the force input, and the exponential window in the response.

- The force/rectangular window on the impulse input prevents erroneous signals such as double hits and cable noise from corrupting the data. The window is open only for a short period, long enough to record the impact but nothing else.

- The exponential window on the output is necessary if the structure is lightly damped and continues to ring after the sampling period.

It is desired to have *no window* on the output if the structure response decays to zero at the end of the sampling period.

Triggering is usually done on a positive slope of the input, with some delay.

Advantages:
- Quick
- No shaker setup needed
- Cheap

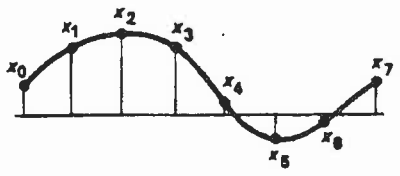
Disadvantages:
- Damage to structure possible
- Not repeatable
- Sometimes very difficult to obtain good results

7.3.3 Crunching the Numbers: the Fast Fourier Transform (FFT)

We have been describing the spectral estimation process in terms of DFT's, but in reality, the analyzer is using something different.

Instead of the tedious calculation of the DFT given in Section 6, a much quicker operation is accomplished using data sets that are 2^p long.

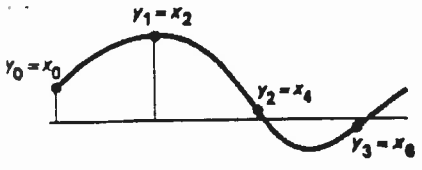
Consider a record which is first sampled as a sequence in *x:*

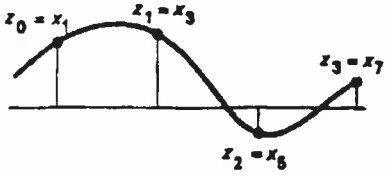


The DFT for this record is:

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{\frac{-i2\pi kn}{N}} \qquad \text{for } k = 0, 1, \ldots N-1 \qquad (7.46)$$

The sample set *x* can be broken down into two other sample sets, *y* and *z:*





Where:

$$y_n = x_{2n}$$
$$z_n = x_{2n+1} \qquad \text{for } n = 0, 1, \ldots N/2 - 1 \qquad (7.47)$$

This assumes that N is even, and leads to a new definition of the DFT:

$$X_k = \frac{1}{N}\left(\sum_{n=0}^{N/2-1} y_n e^{\frac{-i2\pi k(2n)}{N}} + \sum_{n=0}^{N/2-1} z_n e^{\frac{-i2\pi k(2n+1)}{N}} \right) \qquad (7.48)$$

If DFT's are defined for y and z:

$$Y_k = \frac{1}{N/2} \sum_{n=0}^{N/2-1} y_n e^{\frac{-i2\pi kn}{N/2}}$$
$$\text{for } k = 0, 1, \ldots N/2 - 1 \qquad (7.49)$$
$$Z_k = \frac{1}{N/2} \sum_{n=0}^{N/2-1} z_n e^{\frac{-i2\pi kn}{N/2}}$$

These values can be substituted into Eqn. 7.48 to give:

$$X_k = \frac{1}{2} Y_k + \frac{e^{-i2\pi k/N}}{2} Z_k \qquad (7.50)$$

Which can be simplified to:

$$X_k = \frac{1}{2}\left(Y_k + e^{-i2\pi k/N} Z_k\right) \quad \text{for } k = 0, 1, \ldots N/2 - 1 \qquad (7.51)$$

This says that the DFT for x can be obtained from the DFT's of two other variables, y and z which are alternating values of x.

For a record of length 2^p, the partitioning can continue until the sequence only has one value.

Note that we only have X_k for $k = 0, 1, \cdots, N/2 - 1$.

The remaining X_k terms for $k = N/2, \cdots, N-1$ are obtained by realizing that *the frequency domain components Y and Z will repeat themselves because they represent an <u>entire set</u> of harmonic components from* $0 \rightarrow N/2 - 1$*:*

$$\begin{aligned} Y_{k-N/2} &= Y_K \\ Z_{k-N/2} &= Z_k \end{aligned} \qquad (7.52)$$

This results in a complete set of components for X:

$$\begin{aligned} X_k &= \frac{1}{2}\left(Y_k + e^{-i2\pi k/N} Z_k\right) \\ & \qquad \text{for } k = 0, 1, \ldots N/2 - 1 \qquad (7.53) \\ X_{k+N/2} &= \frac{1}{2}\left(Y_k - e^{-i2\pi k/N} Z_k\right) \end{aligned}$$

Which can be rewritten as:

$$X_k = \frac{1}{2}\left(Y_k + W^k Z_k\right)$$

$$X_{k+N/2} = \frac{1}{2}\left(Y_k - W^k Z_k\right) \qquad \text{for } k = 0, 1, \ldots N/2 - 1 \qquad (7.54)$$

Where $W^k = e^{-i2\pi k/N}$. By partitioning Y and Z and further partitioning those partitioned components, a "butterfly" diagram is created outlining the calculations that are made to obtain X. Ultimately, the number of calulations is $N \log_2 N$ vs. N^2 for the DFT.