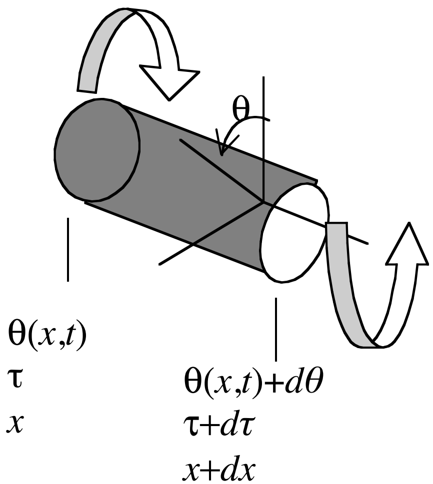
6.4 Torsional Vibrations



$$d\tau = \frac{\partial \tau}{\partial x} dx, \text{ from calculus}$$

$$\tau = GJ \frac{\partial \theta(x,t)}{\partial x}, \text{ from solid mechanics}$$

G=shear modulus

J=polar moment of area cross section

Summing moments on the element dx

$$\tau + \frac{\partial \tau}{\partial x} dx - \tau = \rho J \frac{\partial^2 \theta(x,t)}{\partial t^2} dx$$

Where ρ is the shaft's mass density

Combining these expressions yields;

$$\frac{\partial}{\partial x}\left(GJ \frac{\partial \theta(x,t)}{\partial x} \right) = \rho J \frac{\partial^2 \theta(x,t)}{\partial t^2}, GJ \text{ constant} \Rightarrow$$

$$\boxed{\frac{\partial^2 \theta(x,t)}{\partial t^2} = \frac{G}{\rho} \frac{\partial^2 \theta(x,t)}{\partial x^2}} \qquad (6.66)$$

The initial and boundary conditions for torsional vibration problems are:

- Two spatial conditions (boundary conditions)
- Two time conditions (initial conditions)
- See Table 6.4 for a list of conditions and Equation (6.67) and Table 6.3 for odd cross section
- Clamped-free rod:

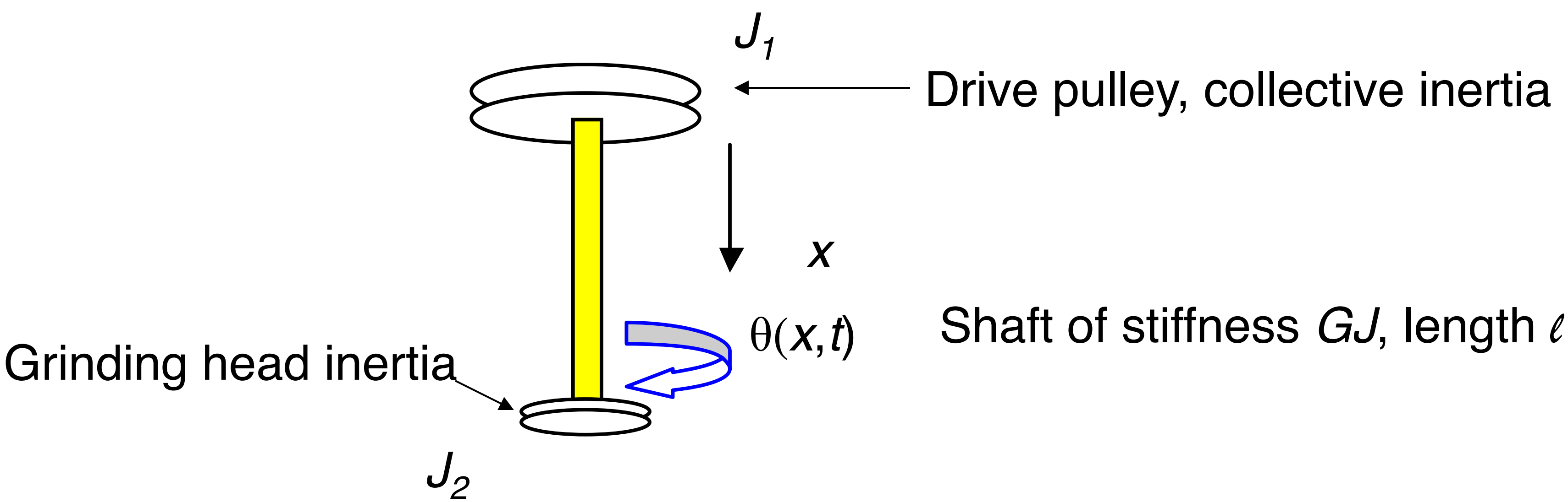
$$\theta(0,t) = 0 \quad \text{Clamped boundary (0 deflection)}$$

$$G\theta_x(\ell,t) = 0 \text{ Free boundary (0 torque)}$$

$$\theta(x,0)=\theta_0(x) \quad \text{and} \quad \theta_t(x,0)=\dot{\theta}_0(x)$$

Example 6.4.1: grinding shaft vibrations

- Top end of shaft is connected to pulley ($x = 0$)
- J_1 includes collective inertia of drive belt, pulley and motor



Use torque balance at top and bottom to get the Boundary Conditions:

$$GJ\left.\frac{\partial\theta(x,t)}{\partial x}\right|_{x=0} = J_1 \left.\frac{\partial^2\theta(x,t)}{\partial t^2}\right|_{x=0} \quad \text{at top}$$

$$GJ\left.\frac{\partial\theta(x,t)}{\partial x}\right|_{x=\ell} = -J_2 \left.\frac{\partial^2\theta(x,t)}{\partial t^2}\right|_{x=\ell} \quad \text{at bottom}$$

The minus sign follows from right hand rule.

Again use separation of variables to attempt a solution

$$\theta(x,t) = \Theta(x)T(t) \Rightarrow$$

$$\frac{\Theta''(x)}{\Theta(x)} = \left(\frac{\rho}{G}\right)\frac{\ddot{T}(t)}{T(t)} = -\sigma^2$$

$$c^2 = \left(\frac{G}{\rho}\right)$$

$$\boxed{\Theta''(x) + \sigma^2\Theta(x) = 0,} \quad \boxed{\ddot{T}(t) + \omega^2 T(t) = 0}$$

$$\omega = \sigma c = \sigma\sqrt{\frac{G}{\rho}}$$

The next step is to use the boundary conditions:

Boundary Condition at $x = 0 \Rightarrow$

$$GJ\Theta'(0)T(t) = J_1\Theta(0)\ddot{T}(t) \Rightarrow$$

$$\frac{GJ\Theta'(0)}{J_1\Theta(0)} = \frac{\ddot{T}(t)}{T(t)} = -c^2\sigma^2 \Rightarrow$$

$$\Theta'(0) = -\frac{\sigma^2 J_1}{\rho J}\Theta(0)$$

Similarly the boundary condition at ℓ yields:

$$\Theta'(\ell) = \frac{\sigma^2 J_2}{\rho J}\Theta(\ell)$$

The Boundary Conditions reveal the *Characteristic Equation*

$$\Theta(x) = a_1 \sin\sigma x + a_2 \cos\sigma x \Rightarrow \Theta(0) = a_2$$

$$\Theta'(x) = a_1\sigma \cos\sigma x - a_2\sigma \sin\sigma x \Rightarrow \Theta'(0) = a_1\sigma$$

$$x = 0 \Rightarrow$$

$$\Theta'(0) = -\frac{\sigma^2 J_1}{\rho J}\Theta(0) \Rightarrow \underline{a_1 = -\frac{\sigma J_1}{\rho J}} a_2$$

$$x = \ell \Rightarrow$$

$$\Theta'(\ell) = \frac{\sigma^2 J_1}{\rho J}\Theta(\ell) \Rightarrow a_1\sigma \cos\sigma\ell - a_2\sigma \sin\sigma\ell = \frac{\sigma^2 J_1}{\rho J} a_1 \sin\sigma\ell + a_2 \cos\sigma\ell$$

$$\Rightarrow \underline{\tan(\sigma\ell) = \frac{\rho J\ell(J_1 + J_2)(\sigma\ell)}{J_1 J_2(\sigma\ell)^2 - (\rho J\ell)^2}} \qquad (6.82)$$

THE CHARACTERISTIC EQUATION

Solving the for the first mode shape

$$\tan(\sigma\ell) = \frac{\rho J\ell(J_1 + J_2)(\sigma\ell)}{J_1 J_2(\sigma\ell)^2 - (\rho J\ell)^2} \text{ has 0 as its first solution:}$$

$$\text{Numerically solve for } \sigma_n\ell,\ \ n = 1,2,3,\ldots,\text{and } \ \omega_n = \sigma_n\sqrt{\frac{G}{\rho}}$$

$$\text{Note for } n = 1, \sigma_1 = 0 \Rightarrow \omega_1 = 0 \Rightarrow \ddot{T}(t) = 0 \Rightarrow$$

$$T(t) = a + bt \ \text{ the rigid body mode of the shaft turning}$$

$$\Rightarrow \Theta_1''(x) = 0, \Rightarrow \Theta_1(x) = a_1 + b_1 x \Rightarrow$$

$$x = 0 \Rightarrow b_1 = 0 \Rightarrow \Theta_1(x) = a_1 \text{ the first mode shape}$$

Solutions of the Characteristic Equation involve solving a transcendental equation

$$(bx^2 - a)\tan x = x$$

$$x = \sigma\ell, \quad a = \frac{\rho J \ell}{J_1 + J_2}, \quad b = \frac{J_1 J_2}{(J_1 + J_2)\rho J \ell}$$

$$J_1 = J_2 = 10 \text{ kg}\cdot\text{m}^2/\text{rad}, \rho{=}2700 \text{ kg/m}^3,$$

$$J{=}5 \text{ kg}\cdot\text{m}^2/\text{rad}, \ \ell{=}0.25 \text{ m}$$

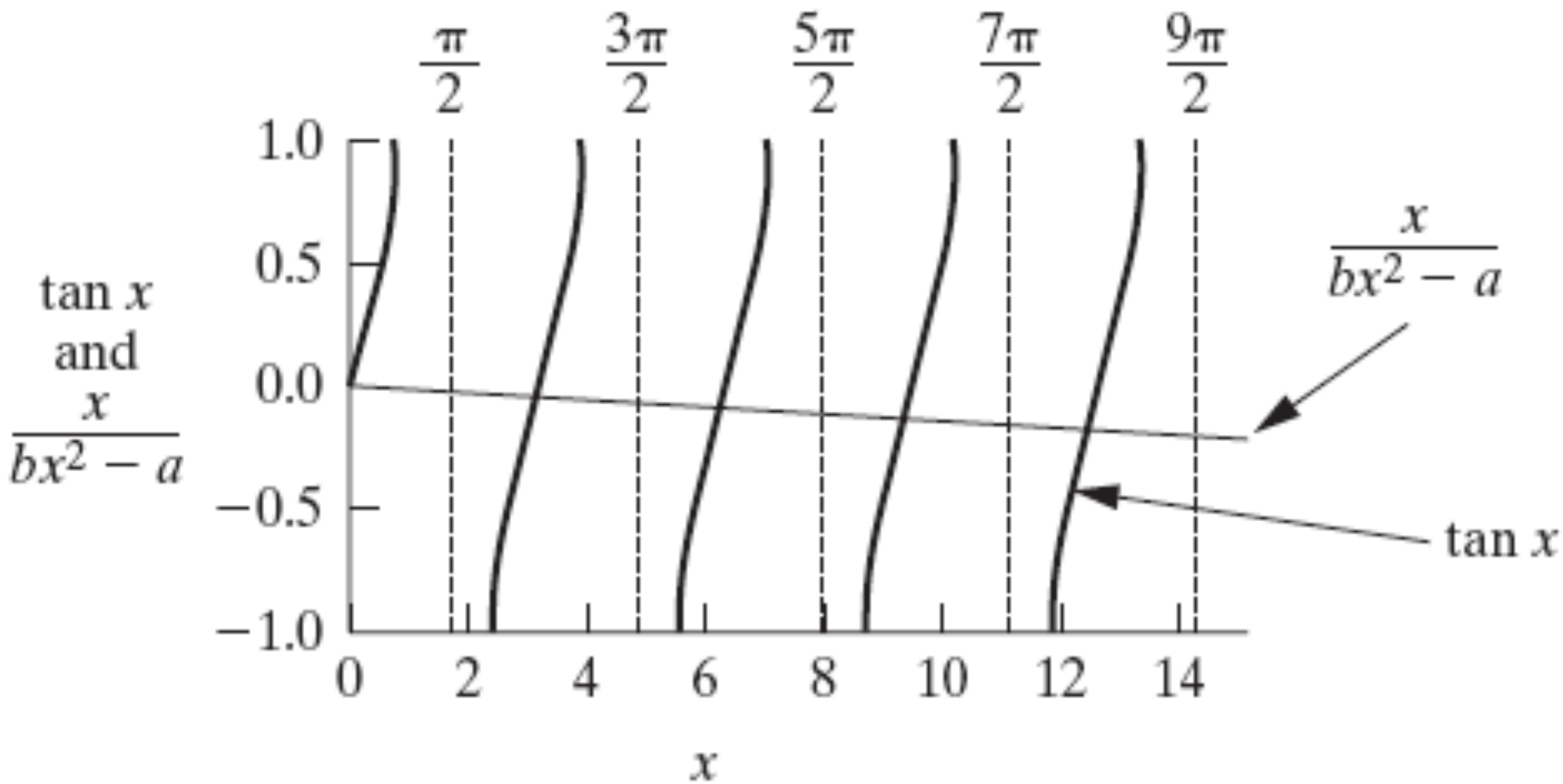
$$G = 25\times 10^9\,\text{Pa}$$

$$\Rightarrow$$

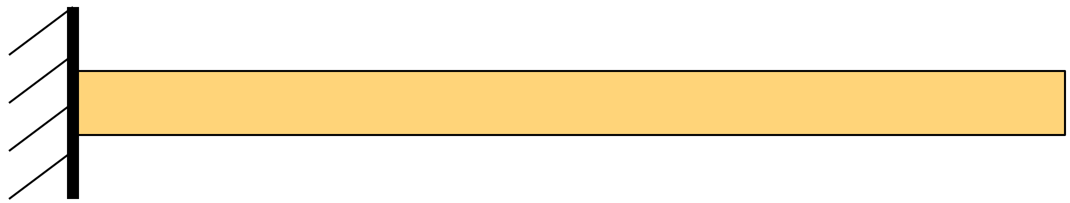
$$f_1 = 0 \text{ Hz}, \ f_2 = 38{,}013 \text{ Hz},$$

$$f_3 = 76{,}026 \text{ Hz}, \quad f_4 = 114{,}039 \text{ Hz},$$

Fig 6.9 Plots of each side of eq (6.82) to assist in find initial guess for numerical routines used to compute the roots.

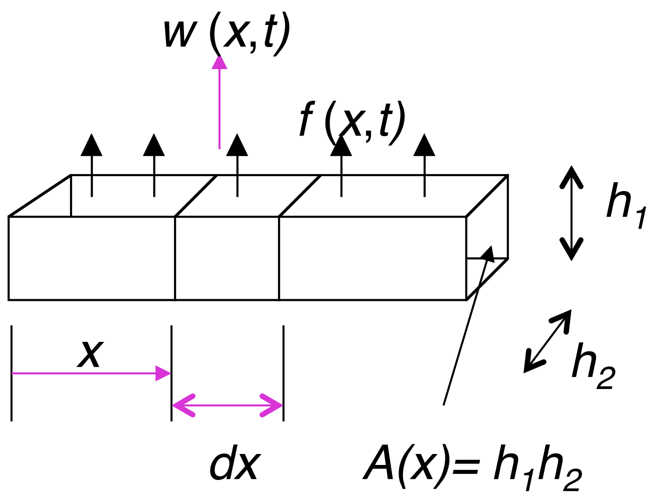


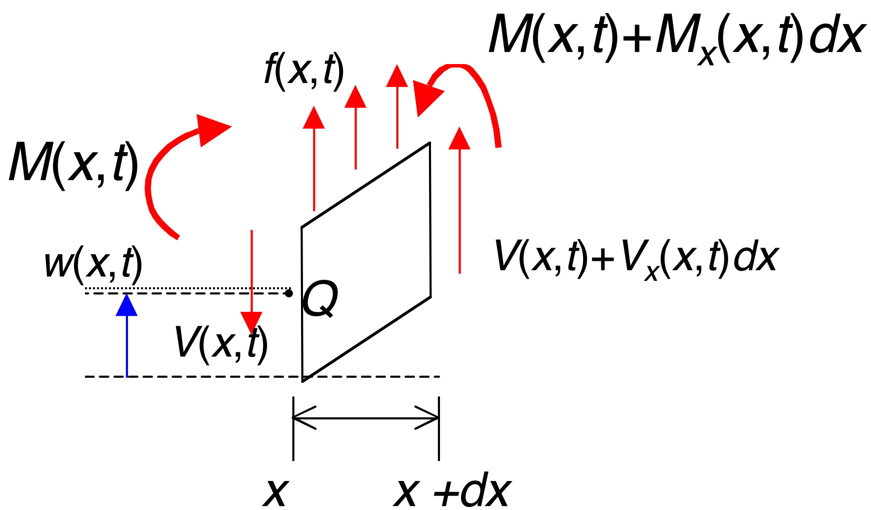
Euler-Bernoulli Beam Analysis



- Uniform along its span and slender
- Linear, homogenous, isotropic elastic material without axial loads
- Plane sections remain plane
- Plane of symmetry is plane of vibration so that rotation & translation decoupled
- Rotary inertia and shear deformation neglected

6.5 Bending vibrations of a beam





bending stiffness $= EI(x)$

$E =$ Youngs modulus

$I(x) =$ cross-sect. area moment of inertia about z

$$M(x,t) = EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}$$

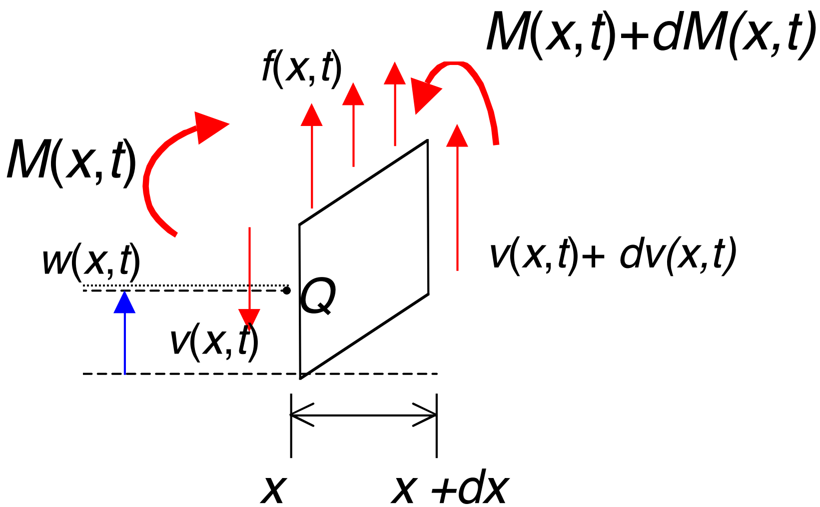
Next sum forces in the y-direction (up, down)

Sum moments about the point Q

Use the moment given from stenght of materials

Assume sides do not bend (no shear deformation)

Consider equilibrium on the *dx* section of the beam:



$$\overset{\uparrow +}{\sum} F_y = -v + v + dv = \rho dV \frac{\partial^2 w}{\partial t^2}$$

Re-writing the differential volume in terms of area and differential length,

$$\rho dV = \rho A dx$$

$$\rho A dx \frac{\partial^2 w}{\partial t^2} = dv \qquad \frac{dv}{dx} = \rho A \frac{\partial^2 w}{\partial t^2} \quad (1)$$

$$\sum M_Q = M - M - dM - v dx = 0$$

$$v = -\frac{dM}{dx} \quad (2)$$

$$\frac{dv}{dx} = -\frac{d^2 M}{dx^2} = \rho A \frac{\partial^2 w}{\partial t^2} \qquad \frac{d^2 M}{dx^2} = -\rho A \frac{\partial^2 w}{\partial t^2} \quad (3)$$

Euler-Bernoulli beam theory says that $M = EI\dfrac{\partial^2 w}{\partial x^2}$

Also, this is a good time to review all the derivative equations for Euler-Bernoulli Beams:

$$\theta = \frac{dw}{dx}, \quad \frac{M}{EI} = \frac{d^2 w}{dx^2}, \quad \frac{v}{EI} = \frac{d^3 w}{dx^3}, \quad \frac{q}{EI} = \frac{d^4 w}{dx^4}$$

Now when we have to consider boundary conditions on the vibrating beam problem, we can refer to these equations

Euler-Bernoulli beam theory says that $M = EI\dfrac{\partial^2 w}{\partial x^2}$

Using the previous equation for *M* results in:

$$\frac{d^2}{dx^2}\left(M = EI\frac{\partial^2 w}{\partial x^2} \right) = -\rho A\frac{\partial^2 w}{\partial t^2} = EI\frac{\partial^4 w}{\partial x^4} \quad (4)$$

For constant *E*, *I*

Using separation of variables again allows the displacement to be expressed as: $w(x,t) = X(x)T(t)$

Assuming harmonic motion, $\dfrac{\partial^2 w}{\partial t^2} = X(x)\dfrac{\partial^2 T(t)}{\partial t^2} = -\omega^2 X(x)T(t)$

$$T(t) = T\sin\omega t$$

Substituting equation 4 into the last equation results in:

$$-\rho A\left(\frac{\partial^2 w}{\partial t^2}\right)=-\rho A\left(-\omega^2 X(x)T(t)\right)=\rho A\omega^2 X(x)T(t)$$

Therefore, equation 4 becomes:

$$EI\frac{d^4X(x)}{dx^4}T(t)=\rho A\omega^2 X(x)T(t) \qquad 5$$

Now redefine the beam density to be density/length:

$$\frac{d^4X}{dx^4}=\frac{\bar{\rho}\omega^2}{EI}X \qquad \text{or} \qquad \frac{d^4X}{dx^4}-\frac{\bar{\rho}\omega^2}{EI}X=0$$

Now we can define a new term β:

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\bar{\rho}\omega^2}{EI}$$ (Again, the density is per unit length)

$$\frac{d^4 X}{dx^4} - \beta^4 X = 0 \quad (6)$$

Assuming a solution of the form $X = e^{ax}$ results in $\left(a^4 - \beta^4\right)X = 0$

To solve this equation there will be four roots: $a^4 = \beta^4, \quad a = \pm\beta, \pm i\beta$

$$X(x) = a_1 \sin\beta x + a_2 \cos\beta x + a_3 \sinh\beta x + a_4 \cosh\beta x \quad (7)$$

We can write the derivative equations from equation 7 and then solve for the four unknowns

$$X' = \left(a_1 \cos\beta x - a_2 \sin\beta x + a_3 \cosh\beta x + a_4 \sinh\beta x \right)\beta$$

$$X'' = etc.$$

Boundary conditions are as follows:

Free end

bending moment $= EI\dfrac{\partial^2 w}{\partial x^2} = 0$

shear force $= \dfrac{\partial}{\partial x}\left[EI\dfrac{\partial^2 w}{\partial x^2}\right] = 0$

Clamped (or fixed) end

deflection $= w = 0$

slope $= \dfrac{\partial w}{\partial x} = 0$

Pinned (or simply supported) end

deflection $= w = 0$

bending moment $= EI\dfrac{\partial^2 w}{\partial x^2} = 0$

Sliding end

slope $= \dfrac{\partial w}{\partial x} = 0$

shear force $= \dfrac{\partial}{\partial x}\left[EI\dfrac{\partial^2 w}{\partial x^2}\right] = 0$

Solution of the time equation yields the oscillatory nature:

$$c^2 \frac{X''''(x)}{X(x)} = -\frac{\ddot{T}(t)}{T(t)} = \omega^2 \Rightarrow$$

$$\ddot{T}(t) + \omega^2 T(t) = 0 \Rightarrow$$

$$T(t) = A \sin \omega t + B \cos \omega t$$

Two initial conditions:

$$w(x,0) = w_0(x), w_t(x,0) = \dot{w}_0(x)$$

Example 6.5.1: compute the mode shapes and natural frequencies for a clamped-pinned beam.

At fixed end $x = 0$ and

$$X(0) = 0 \Rightarrow a_2 + a_4 = 0$$

$$X'(0) = 0 \Rightarrow \beta(a_1 + a_3) = 0$$

At the pinned end, $x = \ell$ and

$$X(\ell) = 0 \Rightarrow$$

$$a_1 \sin\beta\ell + a_2 \cos\beta\ell + a_3 \sinh\beta\ell + a_4 \cosh\beta\ell = 0$$

$$EIX''(\ell) = 0 \Rightarrow$$

$$\beta^2(-a_1 \sin\beta\ell - a_2 \cos\beta\ell + a_3 \sinh\beta\ell + a_4 \cosh\beta\ell) = 0$$

The 4 boundary conditions in the 4 constants can be written as the matrix equation:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \\ \sin\beta\ell & \cos\beta\ell & \sinh\beta\ell & \cosh\beta\ell \\ -\beta^2\sin\beta\ell & -\beta^2\cos\beta\ell & \beta^2\sinh\beta\ell & \beta^2\cosh\beta\ell \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{\mathbf{a}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B\mathbf{a} = \mathbf{0}, \mathbf{a} \neq \mathbf{0} \Rightarrow \det(B) = 0 \Rightarrow$$

$$\tan\beta\ell = \tanh\beta\ell$$

The characteristic equation

Solve numerically (fsolve) to obtain solution to transcendental (characteristic) equation

$$\beta_1\ell = 3.926602 \qquad \beta_2\ell = 7.068583 \qquad \beta_3\ell = 10.210176$$

$$\beta_4\ell = 13.351768 \qquad \beta_5\ell = 16.493361 \qquad \ldots$$

$$n > 5 \Rightarrow$$

$$\beta_n\ell = \frac{(4n+1)\pi}{4}$$

Next solve $B\mathbf{a}=0$ for 3 of the constants:

With the eigenvalues known, now solve for the eigenfunctions:

$B\mathbf{a} = \mathbf{0}$ yields 3 constants in terms of the 4th:

$\underline{a_1 = -a_3}$ from the first equation

$\underline{a_2 = -a_4}$ from the second equation

$(\sinh \beta_n \ell - \sin \beta_n \ell) a_3 + (\cosh \beta_n \ell - \cos \beta_n \ell) a_4 = 0$

from the third (or fourth) equation

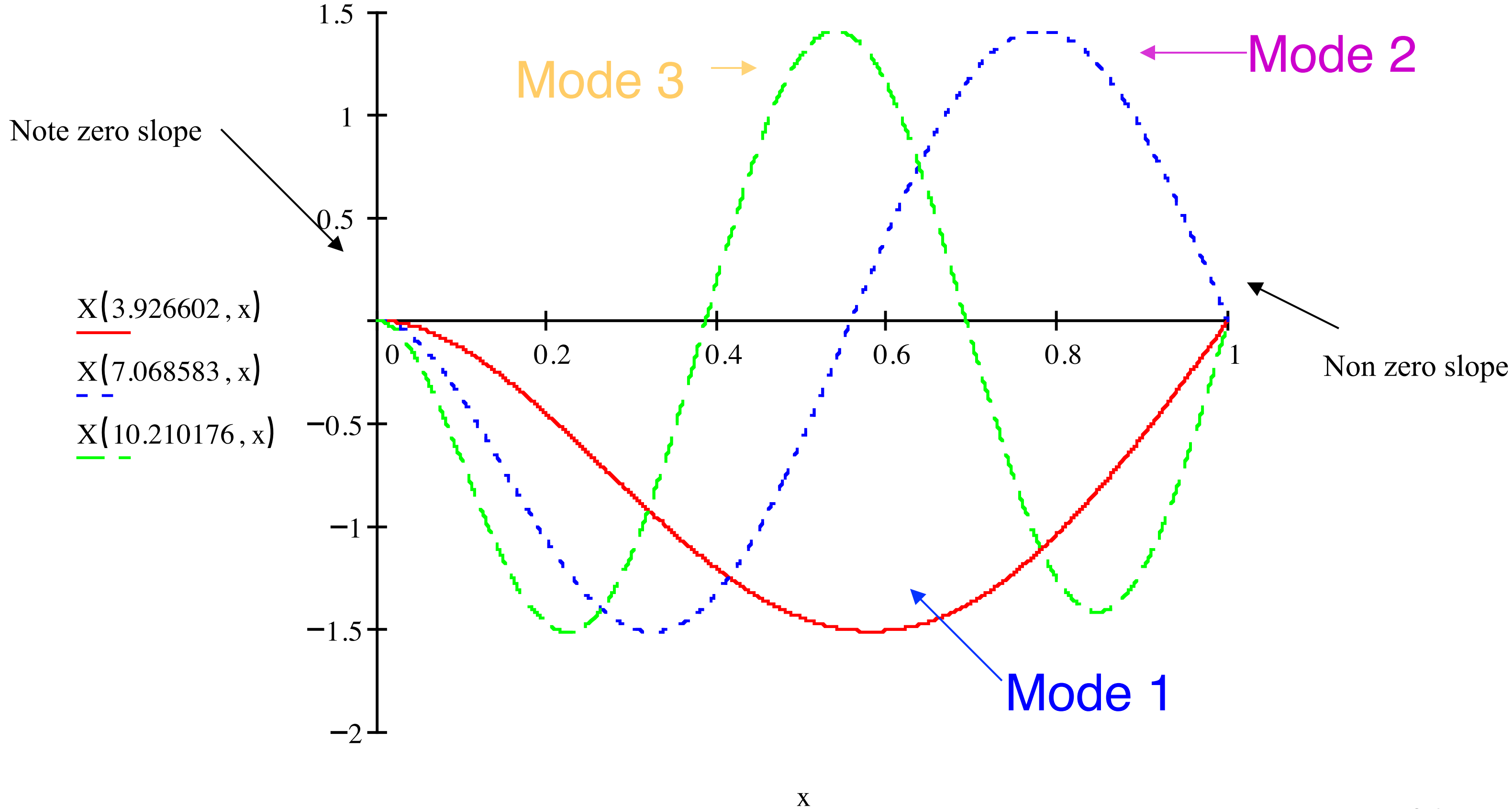
Solving yields:

$$\underline{a_3 = -\frac{\cosh \beta_n \ell - \cos \beta_n \ell}{\sinh \beta_n \ell - \sin \beta_n \ell} a_4}$$

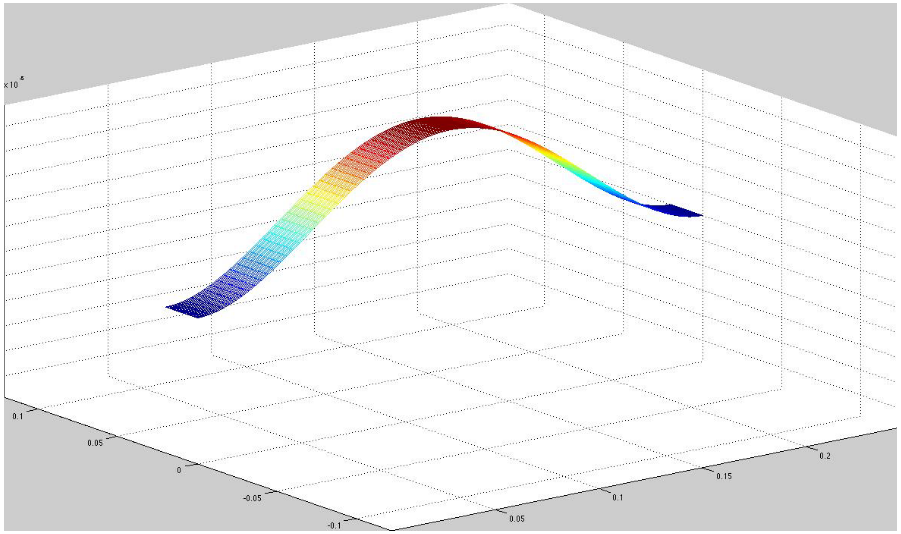
$$\underline{\Rightarrow X_n(x) = (a_4)_n \left[\frac{\cosh \beta_n \ell - \cos \beta_n \ell}{\sinh \beta_n \ell - \sin \beta_n \ell} (\sinh \beta_n \ell x - \sin \beta_n \ell x) - \cosh \beta_n \ell x + \cos \beta_n \ell x \right]}$$

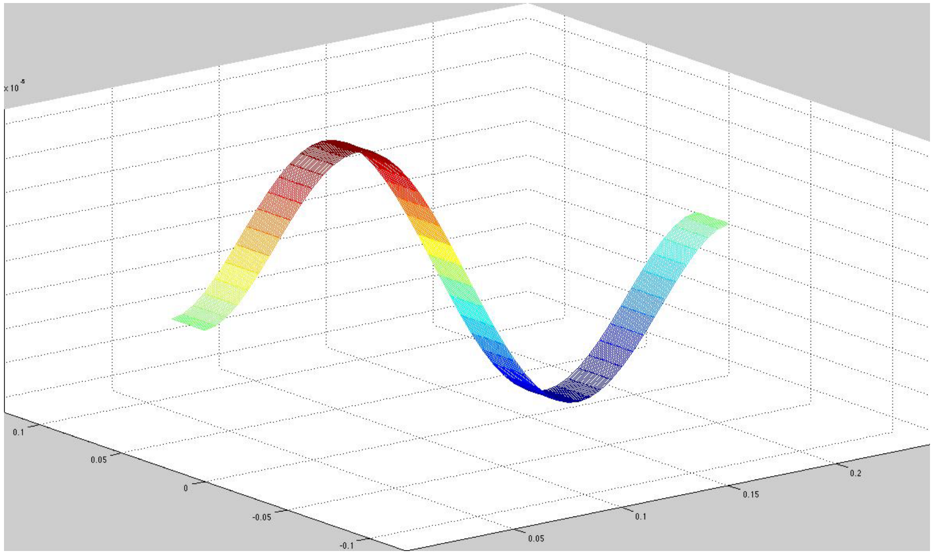
Plot the mode shapes to help understand the system response

$$X(n,x) := \frac{\cosh(n) - \cos(n)}{\sinh(n) - \sin(n)} \cdot (\sinh(n \cdot x) - \sin(n \cdot x)) - \cosh(n \cdot x) + \cos(n \cdot x)$$



Beam modes





Again, the mode shape orthogonality becomes important and is computed as follows:

Write the eigenvalue problem twice, once for *n* and once for *m*.

$$X_n''''(x) = \beta_n^4 X_n(x) \quad \text{and} \quad X_m''''(x) = \beta_m^4 X_m(x)$$

Multiply by $X_m(x)$ and $X_n(x)$ respectively, integrate and subtract to get:

$$\int_0^{\ell} X_n''''(x) X_m(x)\,dx - \int_0^{\ell} X_m''''(x) X_n(x)\,dx = (\beta_n^4 - \beta_m^4)\int_0^{\ell} X_n(x) X_m(x)\,dx$$

Then integrate the left hand side twice by parts to get:

Use integration by parts to evaluate the integrals in the orthogonality condition.

apply $\int udv=uv-\int vdu$ twice:

$$\int_0^{\ell} \underbrace{X_m(x)}_{u}\underbrace{X_n''''(x)dx}_{dv} = \underbrace{X_m}_{u}\underbrace{X_n'''\Big|_0^{\ell}}_{v} - \int_0^{\ell} \underbrace{X_n'''}_{v}\underbrace{X_m'dx}_{du}$$

$$= \cancelto{0}{X_m(\ell)}X_n'''(\ell) - \cancelto{0}{X_m(0)}X_n'''(0) - \int_0^{\ell} \underbrace{X_m'}_{u}\underbrace{X_n'''dx}_{dv}$$

$$-\int_0^\ell \underbrace{X_m'}_{u}\,\underbrace{X_n'''dx}_{dv} = -X_m'(x)\,X_n''(x)\Big|_0^\ell + \int_0^\ell X_n''(x)X_m''(x)dx$$

$$= -X_m'(\ell)\underbrace{X_n''(\ell)}_{0} + X_m'(0)\underbrace{X_n''(0)}_{0} + \int_0^\ell X_n''(x)X_m''(x)dx$$

Thus

$$\int_0^\ell X_n''''(x)X_m(x)dx - \int_0^\ell X_m''''(x)X_n(x)dx = (\beta_n^4 - \beta_m^4)\int_0^\ell X_n(x)X_m(x)dx$$

$$\Rightarrow \underbrace{\int_0^\ell X_n''(x)X_m''(x)dx - \int_0^\ell X_n''(x)X_m''(x)dx}_{0} = \underbrace{(\beta_n^4 - \beta_m^4)}_{\neq 0}\int_0^\ell X_n(x)X_m(x)dx$$

$$\Rightarrow \int_0^\ell X_n(x)X_m(x)dx = 0, \forall n,m, n\neq m$$

The solution can be computed via modal expansion based on orthogonality of the modes.

$$\int_0^{\ell} X_n(x) X_m(x)\,dx = \delta_{nm}$$

$$w(x,t) = \sum_{n=1}^{\infty} (A_n \sin\omega_n t + B_n \cos\omega_n t) X_n(x)$$

$$w(x,0) = w_0(x) = \sum_{n=1}^{\infty} B_n X_n(x) \Rightarrow \underline{B_n = \int_0^{\ell} w_0(x) X_n(x)\,dx}$$

$$w_t(x,0) = \dot{w}_0(x) = \sum_{n=1}^{\infty} \omega_n A_n X_n(x) \Rightarrow \underline{A_n = \frac{1}{\omega_n} \int_0^{\ell} \dot{w}_0(x) X_n(x)\,dx}$$

Car antenna vibration

Beam Second Mode (ODS)