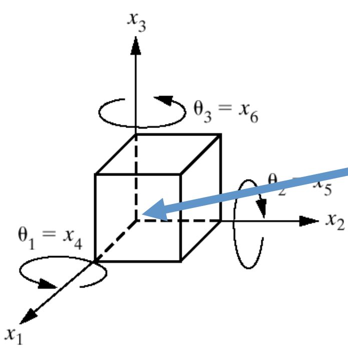
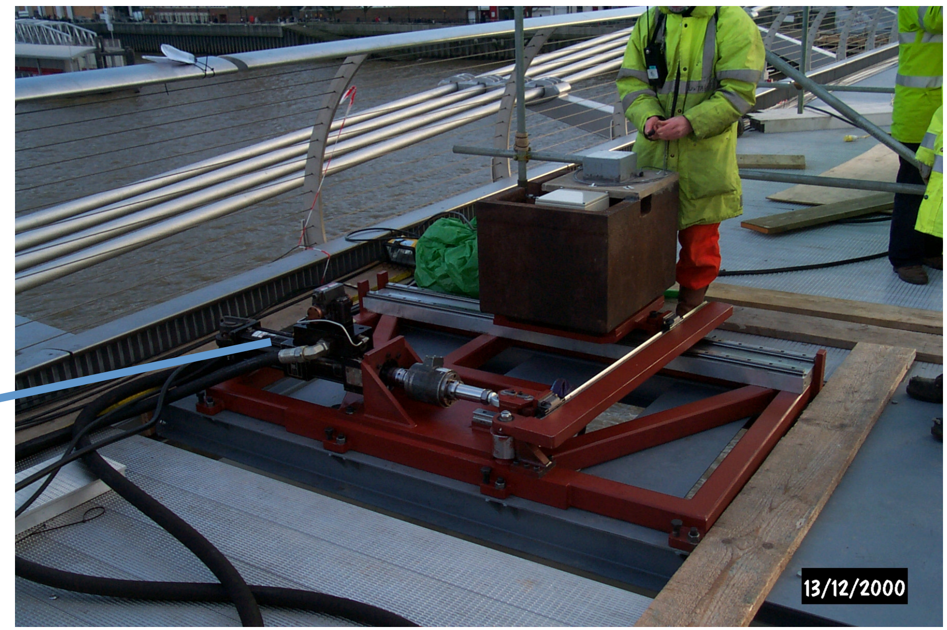
Chapter 4: Multiple Degree of Freedom Systems

The Millennium bridge required many degrees of freedom to model and design with.

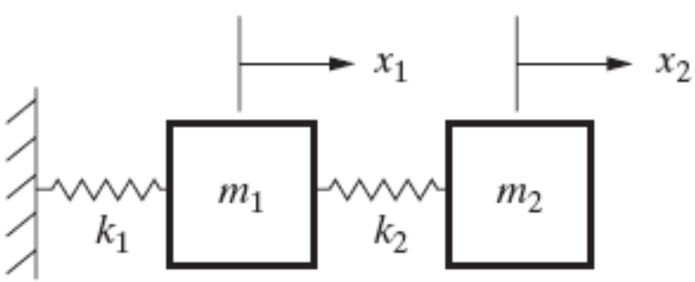


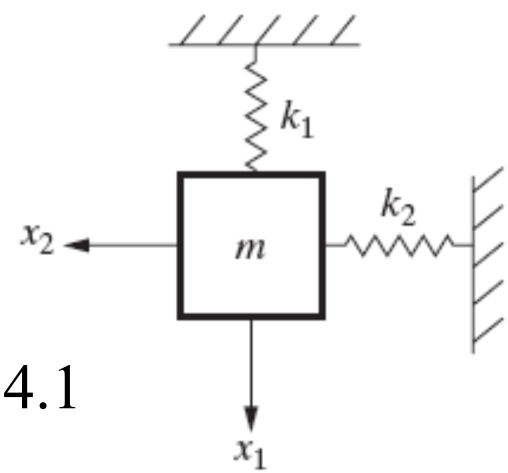


Extending the first 3 chapters to more then one degree of freedom

The first step in analyzing multiple degrees of freedom (DOF) is to look at 2 DOF

- DOF: Minimum number of coordinates to specify the position of a system
- Many systems have more than 1 DOF
- Examples of 2 DOF systems
  - car with sprung and unsprung mass (both heave)
  - elastic pendulum (radial and angular)
  - motions of a ship (roll and pitch)





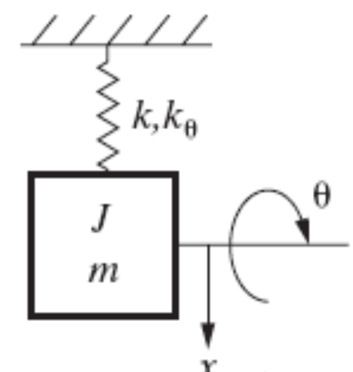
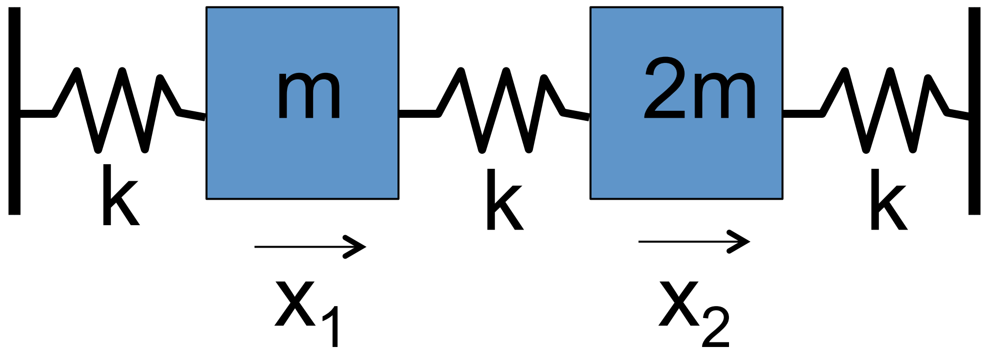


Fig 4.1

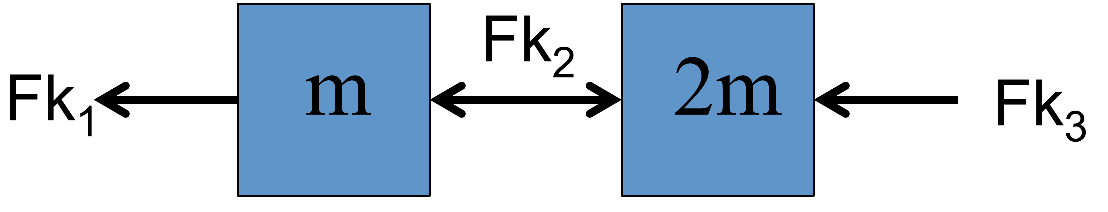
4.1 Two-Degree-of-Freedom Model (Undamped)



A 2 degree of freedom system is used to base much of the analysis and conceptual development of MDOF systems.

FBD:

Assume $x_1 > x_2$



A system of equations can be written from the FBD:

$$\Sigma F_{x_1} = m\ddot{x}_1 = -kx_1 - k\left(x_1 - x_2\right)$$

$$\Sigma F_{x_2} = 2m\ddot{x}_2 = k\left(x_1 - x_2\right) - kx_2$$

In matrix form:

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Assuming harmonic motion:

$$x_1\left(t\right) = X_1 e^{i\omega t},\ x_2\left(t\right) = X_2 e^{i\omega t}$$

$$\ddot{x}_1\left(t\right) = -\omega^2 X_1 e^{i\omega t},\ \ddot{x}_2\left(t\right) = -\omega^2 X_2 e^{i\omega t}$$

Substituting into the matrix equation:

$$\begin{bmatrix} -m\omega^2 + 2k & -k \\ -k & -2m\omega^2 + 2k \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Now, for a non-trivial solution, the determinant of the matrix must be zero (creates dependency in the solution vector)
  - This will lead to an eigenvector / eigenvector problem

$$\begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -2m\omega^2 + 2k \end{vmatrix} = 0$$

So that…

$$\left(-m\omega^2 + 2k\right)\left(-2m\omega^2 + 2k\right) - k^2 = 0$$

The determinant equation is simplified with a lambda (eigenvalue) substitution:

$$\text{Let } \omega^2 = \lambda: \qquad 2m\lambda^2 - 6km\lambda + 3k = 0$$

$$\lambda_1 = .634\frac{k}{m}, \quad \lambda_2 = 2.366\frac{k}{m}$$

- Substituting λ_1 into the first equation (or second – it doesn't matter):

$$\left(-m\left(.634\frac{k}{m}\right) + 2k\right)X_1 - kX_2 = 0$$

$$X_1 = .731X_2$$

Two eigenvalue/eigenvector solutions are obtained using either equation of motion

- Substituting λ_2 into the first equation (or second – it doesn't matter):

$$\left(-m\left(2.366\frac{k}{m}\right)+2k\right)X_1 - kX_2 = 0 \qquad X_1 = -2.73X_2$$

Since X_1 is dependent on X_2, we can arbitrarily select a value for X_2 and then X_1 will be scaled from X_2:

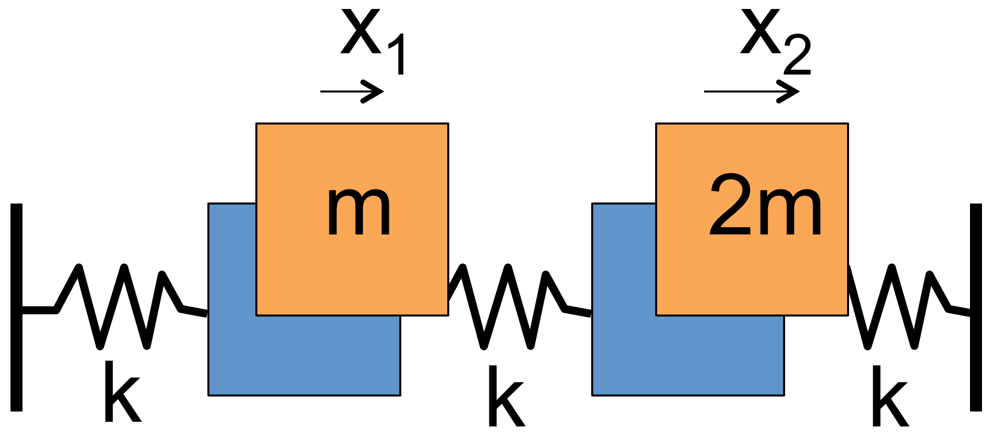
$$\lambda_1 = .634\frac{k}{m} \qquad \lambda_1 = 2.73\frac{k}{m}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} .731 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -2.73 \\ 1 \end{pmatrix}$$

The vectors are called eigenvectors and represent mode shapes of vibration

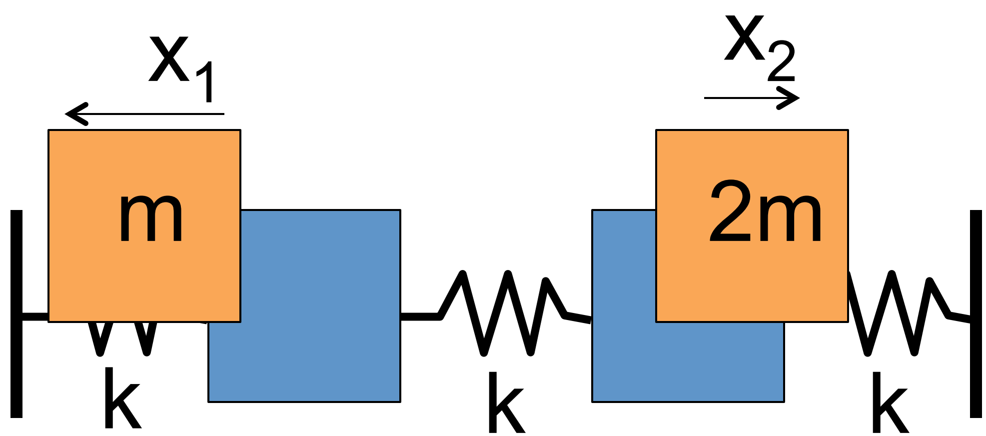
$$\lambda_1 = .634\frac{k}{m}, \omega_1 = .8\sqrt{\frac{k}{m}}$$

$$\Phi_1 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} .731 \\ 1 \end{pmatrix}$$



$$\lambda_1 = 2.366\frac{k}{m}, \omega_2 = 1.54\sqrt{\frac{k}{m}}$$

$$\Phi_2 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -2.73 \\ 1 \end{pmatrix}$$



We can now write the response equations for both independent degrees of freedom as a superposition of the independent modes

*Note that we discovered in the normal mode solution that there are actually **two** modes of vibration that affect both physical degrees of freedom:*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1 e^{i\omega t} \\ X_2 e^{i\omega t} \end{pmatrix} \Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_1 \Phi_1 e^{i\omega_1 t} + A_2 \Phi_2 e^{i\omega_2 t}$$

*…And the coefficients **A** are complex to account for phase shift*

Looking at just the sine component of the complex response and accounting for phase with ψ results in:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \Phi_1 \sin(\omega_1 t + \psi_1) + C_2 \Phi_2 \sin(\omega_2 t + \psi_2)$$

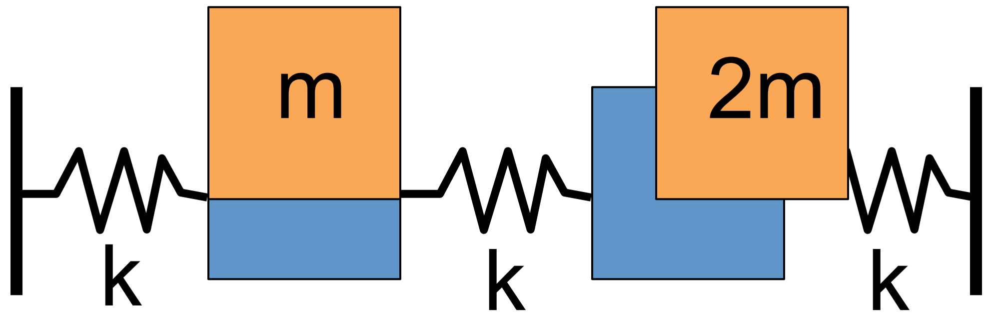
$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \omega_1 C_1 \Phi_1 \cos(\omega_1 t + \psi_1) + \omega_2 C_2 \Phi_2 \cos(\omega_2 t + \psi_2)$$

Now apply initial conditions (t = 0) to solve for the C's and psi's

Using initial conditions:

- Assume the following initial conditions:

$$x_1(0)=\dot{x}_1(0)=0 \qquad x_2(0)=1 \quad \dot{x}_2(0)=0$$



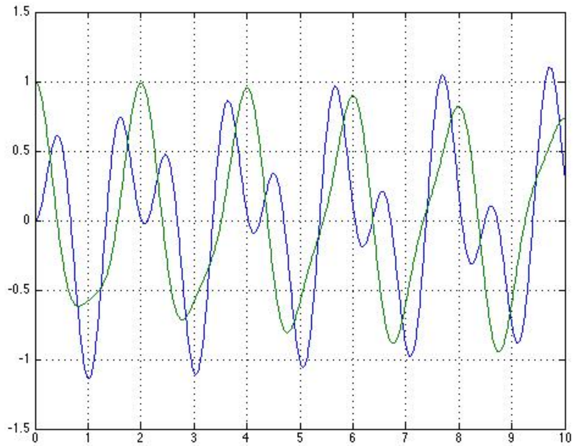
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1\Phi_1 \sin(\psi_1) + C_2\Phi_2 \sin(\psi_2)$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \omega_1 C_1\Phi_1 \cos(\psi_1) + \omega_2 C_2\Phi_2 \cos(\psi_2)$$

After solving for the unknowns:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = .79\begin{pmatrix} .731 \\ 1 \end{pmatrix}\sin\left(.8\sqrt{\frac{k}{m}}t+\frac{\pi}{2}\right)+.21\begin{pmatrix} -2.73 \\ 1 \end{pmatrix}\sin\left(1.54\sqrt{\frac{k}{m}}t+\frac{\pi}{2}\right)$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} .57 \\ .79 \end{pmatrix}\sin\left(.8\omega_1 t+\frac{\pi}{2}\right)+\begin{pmatrix} -.57 \\ .21 \end{pmatrix}\sin\left(1.54\omega_2 t+\frac{\pi}{2}\right)$$



Example 4.1.7 given the initial conditions compute the time response

$$\text{consider } \mathbf{x}(0)=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{mm}, \ \dot{\mathbf{x}}(0)=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{A_1}{3}\sin\left(\sqrt{2}t+\varphi_1\right) - \frac{A_2}{3}\sin\left(2t+\varphi_2\right) \\ A_1\sin\left(\sqrt{2}t+\varphi_1\right) + A_2\sin\left(2t+\varphi_2\right) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = -\begin{bmatrix} \frac{A_1}{3}\sqrt{2}\cos\left(\sqrt{2}t+\varphi_1\right) - \frac{A_2}{3}2\cos\left(2t+\varphi_2\right) \\ A_1\sqrt{2}\cos\left(\sqrt{2}t+\varphi_1\right) + A_2 2\cos\left(2t+\varphi_2\right) \end{bmatrix}$$

At t = 0 we have

$$\begin{bmatrix} 1\text{ mm} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{A_1}{3}\sin(\phi_1) - \frac{A_2}{3}\sin(\phi_2) \\ A_1\sin(\phi_1) + A_2\sin(\phi_2) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{A_1}{3}\sqrt{2}\cos(\phi_1) - 2\frac{A_2}{3}\cos(\phi_2) \\ A_1\sqrt{2}\cos(\phi_1) + 2A_2\cos(\phi_2) \end{bmatrix}$$

4 equations and 4 unknowns:

$$3 = A_1 \sin(\phi_1) - A_2 \sin(\phi_2)$$

$$0 = A_1 \sin(\phi_1) + A_2 \sin(\phi_2)$$

$$0 = A_1 \sqrt{2} \cos(\phi_1) - A_2 2 \cos(\phi_2)$$

$$0 = A_1 \sqrt{2} \cos(\phi_1) + A_2 2 \cos(\phi_2)$$

Yields:

$$A_1 = 1.5 \text{ mm}, A_2 = -1.5 \text{ mm}, \phi_1 = \phi_2 = \frac{\pi}{2} \text{ rad}$$

The final solution is:

$$\begin{aligned} x_1(t) &= 0.5\cos\sqrt{2}t + 0.5\cos 2t \\ x_2(t) &= 1.5\cos\sqrt{2}t - 1.5\cos 2t \end{aligned} \qquad (4.34)$$

These initial conditions give a response that is a combination of modes. The response of each mass contains both frequencies.

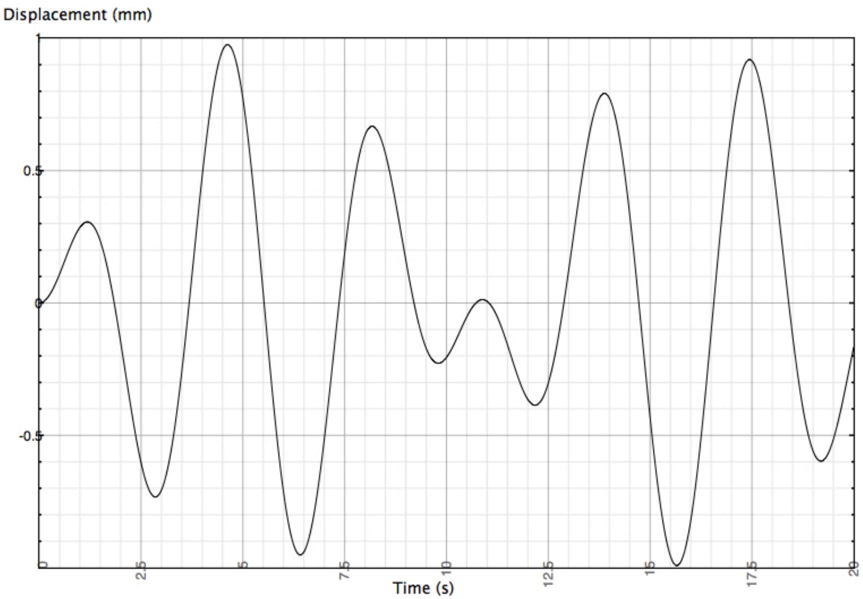


Figure 4.3a $x_1(t)$

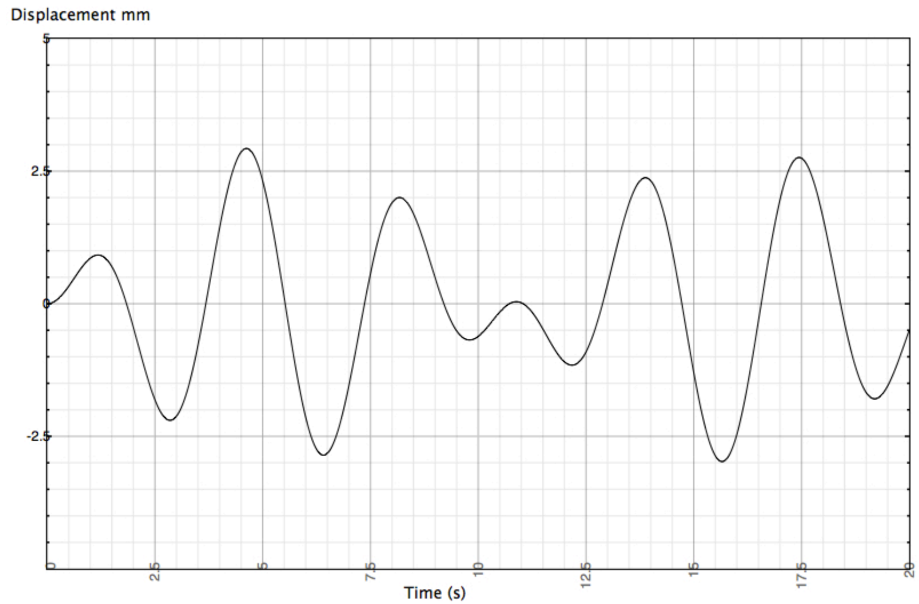


Figure 4.3b $x_2(t)$

Solution as a sum of modes

$$\mathbf{x}(t) = a_1\mathbf{u}_1 \sin\omega_1 t + a_2\mathbf{u}_2 \sin\omega_2 t$$

Determines how the first frequency contributes to the response

Determines how the second frequency contributes to the response

Things to note

- Two degrees of freedom implies two natural frequencies
- Each mass oscillates with these two frequencies present in the response and beats could result
- Frequencies are not those of two component systems

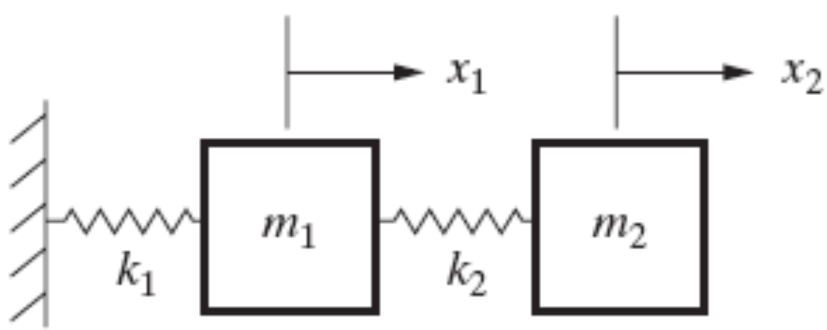
$$\boxed{\omega_1 = \sqrt{2} \neq \sqrt{\frac{k_1}{m_1}} = 1.63, \omega_2 = 2 \neq \sqrt{\frac{k_2}{m_2}} = 1.732}$$

- The above is not the most efficient way to calculate frequencies as the following describes

Summary:

A 2 Degree-of-Freedom system has

- Two equations of motion!
- Two natural frequencies (as we shall see)!



The dynamics of a 2 DOF system consists of 2 homogeneous and coupled equations

Free vibrations, thus homogeneous eqs.

Equations are coupled:

- Both have x_1 and x_2.
- If only one mass moves, the other follows
- Example: pitch and heave of a car model

In this case the coupling is due to k_2.

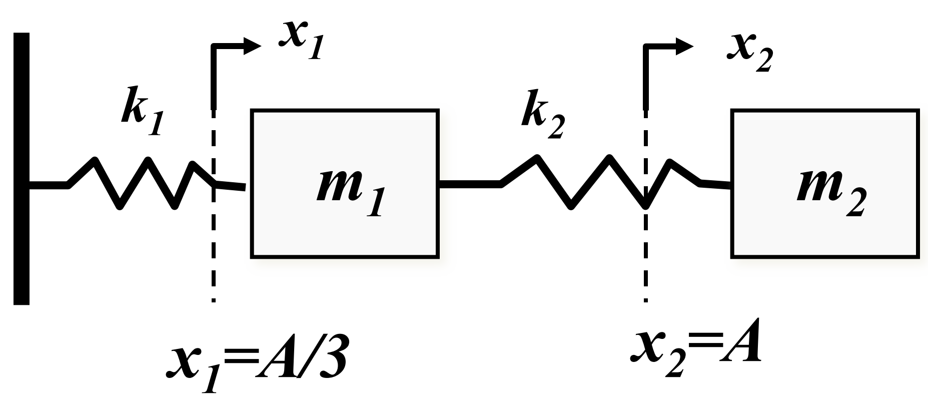
- Mathematically and Physically
- If the coupling stiffness is removed then no coupling occurs and can be solved as two independent SDOF systems

If the initial conditions of displacement match a mode shape…

- **The response will consist of 100% of that mode and 0% of the other mode**
- So if $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = C_1\Phi_1$
- Then: $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1\Phi_1 \sin\left(\omega_1 t + \cancelto{\frac{\pi}{2}}{\psi_1}\right) + \cancelto{0}{C_2}\Phi_2 \sin\left(\omega_2 t + \psi_2\right)$
- Thus each mass oscillates at (one) frequency ω_1 with magnitudes proportional to $C_1\Phi_1$, the 1st *mode shape*

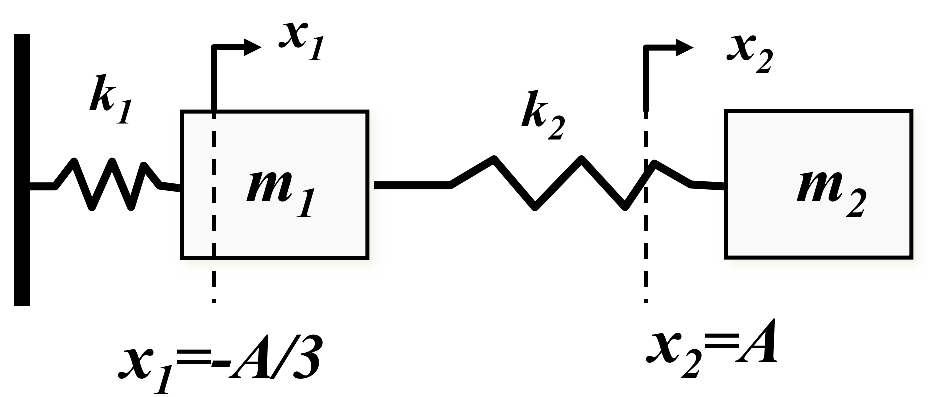
If IC's correspond to mode 1 or 2, then the response is purely in mode 1 or mode 2.

Mode 1:



$$\mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

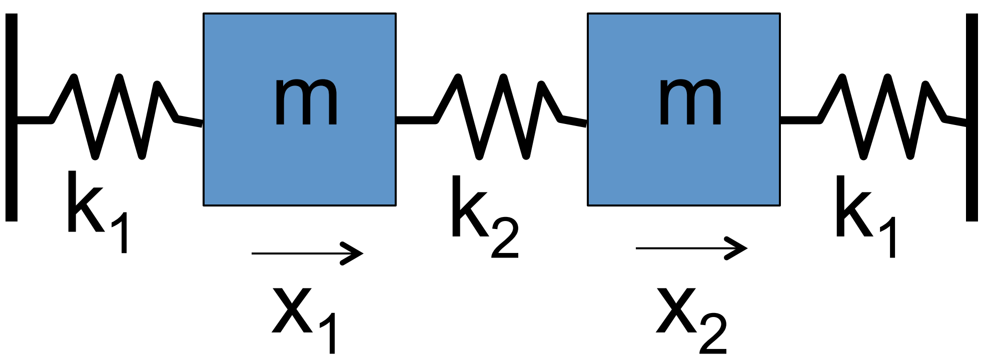
Mode 2:



$$\mathbf{u}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

Beating phenomenon

- Beating phenomenon is observed when the two natural frequencies of a 2-DOF system are close
- It is indicated by an amplitude modulated response behavior on the higher frequency oscillations
- Consider the system below:



$x_1 = 1$
$x_2 = 0$
$xdot_1 = 0$
$xdot_2 = 0$

$m = 1,\ k_1 = 25,\ k_2 = 5$

Solving for the mode shapes and natural frequencies:

$$\omega_1 = 5.0, \quad \omega_2 = 5.9$$

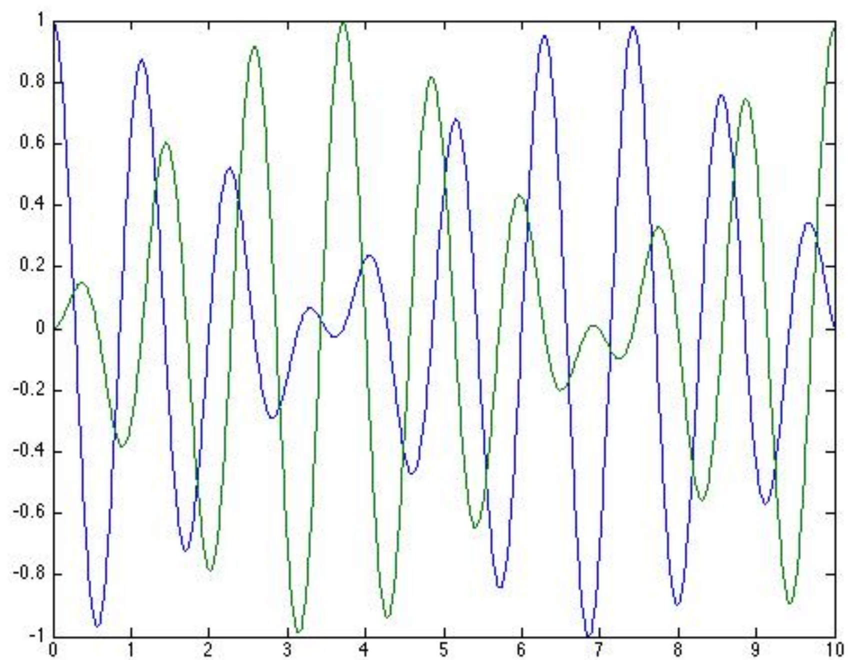
$$\Phi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{2}\Phi_1 \cos 5t - \frac{1}{2}\Phi_2 \cos 5.9t$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \cos 5t + \cos 5.9t \\ \cos 5t - \cos 5.9t \end{pmatrix}$$

Using trig identities, this can be rewritten as:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \cos 5.5t \cos 0.45t \\ \sin 5.5t \sin .45t \end{pmatrix}$$



Don't sit near the engines!