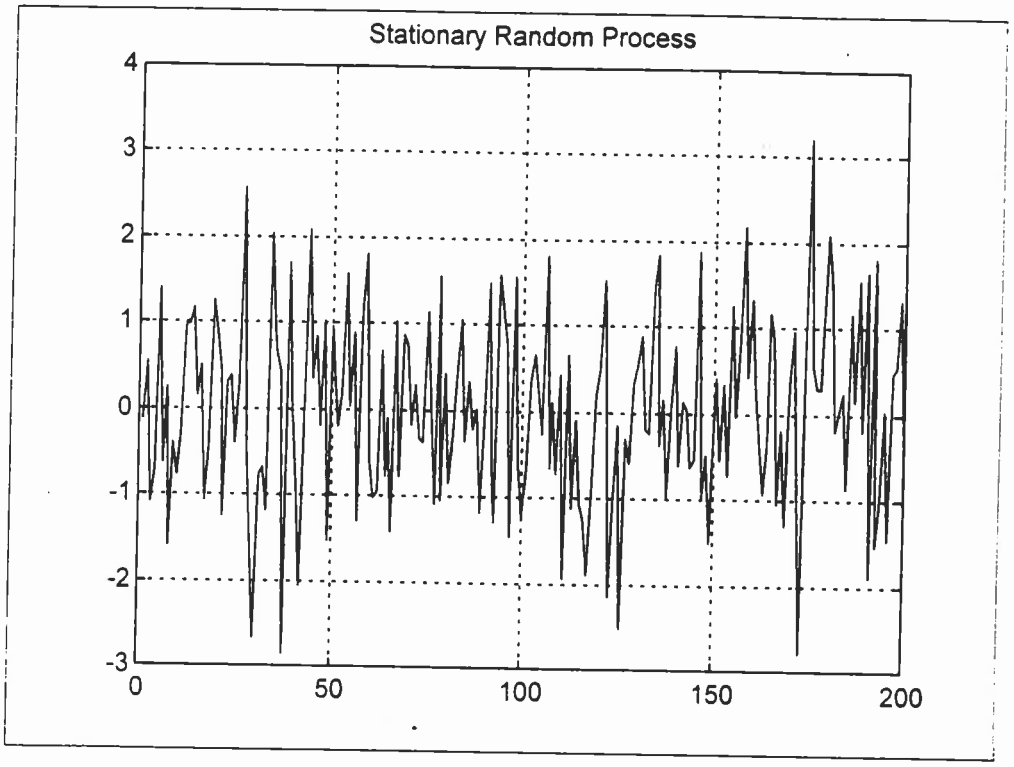
6. Frequency Domain Analysis

6.1 Fourier Transform of the Autocorrelation Function

Consider the naturally occurring random process shown below:



If the function $x(t)$ is represented by the random process, the Fourier transform cannot be obtained because Eqn. 5.18 is not satisfied.

However, If we use the *autocorrelation function* instead, the following are satisfied:

- $R_{xx}(\tau)$ goes to zero as τ becomes large because random data is uncorrelated to events which occur "far away" from the reference time. This leads to:

$$\int_{-\infty}^{\infty} |R_{xx}(\tau)| d\tau < \infty \tag{6.1}$$

- $R_{xx}(\tau)$ is a real, even function so that $R_{xx}(\tau) = R_{xx}(-\tau)$

Provided these are true, a quantity called the autospectral density function, S_{xx} can be defined. Using Eqns. 5.16 and 5.17 to express the transform pair of the *autocorrelation* function:

$$\begin{aligned} S_{xx}(\omega) &= \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau \\ R_{xx}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega\tau} d\omega \end{aligned} \tag{6.2}$$

The *spectral density of the x(t) process,* S_{xx}, can be generalized to include cross-spectral density functions for *x(t)* and *y(t)* as well:

$$\begin{aligned} S_{xy}(\omega) &= \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} d\tau \\ R_{xy}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{i\omega\tau} d\omega \end{aligned} \tag{6.3}$$

These relationships are referred to as the *Wiener-Khinchine relations.* In general, S_{xy} is complex.

Recall the properties of the auto- and cross-correlation functions:

$$\begin{aligned} R_{xx}(-\tau) &= R_{xx}(\tau) \\ R_{xy}(-\tau) &= R_{yx}(\tau) \end{aligned} \tag{6.4}$$

Using these, additional relationships for the auto- and cross-spectral density functions can be written:

$$\begin{aligned} S_{xx}(-\omega) &= S_{xx}^{*}(\omega) = S_{xx}(\omega) \\ S_{xy}(-\omega) &= S_{xy}^{*}(\omega) = S_{yx}(\omega) \end{aligned} \tag{6.5}$$

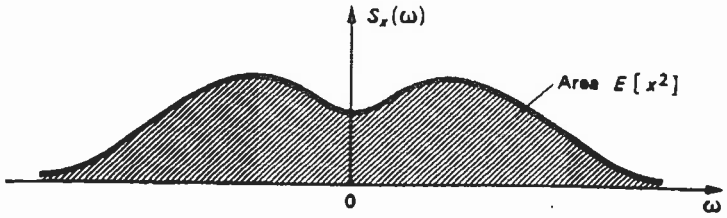
An important outcome of the auto- equations (6.2) is realized when $\tau = 0$. This leads to:

$$R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega \tag{6.6}$$

From the definition of $R_{xx}(\tau)$, we can relate the mean square value of a random process to the autospectral density by:

$$\Psi_x^2 = E\left[x^2\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega \tag{6.7}$$

Graphically, this is shown below:



The mean square value of a stationary random process *x(t)* is given by the area under the S_{xx} curve. The units of S_{xx} are $(\text{units})^2/\text{Hz}$ and its *more accurate name* will be *mean square spectral density.*

- By its definition, S_{xx} will be real, even and non-negative.
- Because of this, we can define another spectrum by:

$$\begin{aligned} G_{xx}(\omega) &= 2S_{xx}(\omega) \quad \text{for} \quad 0 < \omega < \infty \\ G_{xx}(0) &= S_{xx}(0) \quad \text{for} \quad \omega = 0 \end{aligned} \tag{6.8}$$

Some additional comments:

- Section 4 showed that the auto- and cross-correlation functions can be used to characterize a signal over a period *T*. Random processes that are stationary are treated the same as periodic processes except that $T \to \infty$.

- Expressing a signal's characteristics in the frequency domain requires that $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$, which does not hold for stationary random processes.

- To avoid this problem, the spectral density function S is introduced which is based on the correlation function of the signal.

In the next section, we will see that the spectral density function is *directly related* to the frequency content of the signal.

6.2 *Relation between* S_{xx} *and* $X(\omega)$

Recall Eqn. 4.11 that is used to compute the autocorrelation function:

$$R_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x(t+\tau)\,dt \qquad (6.9)$$

This was defined for periodic data. If we consider $T \to \infty$, then a substitution can be made for $x(t+\tau)$ using the Fourier transform:

$$\underset{T\to\infty}{R_{xx}(\tau)} = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}e^{i\omega\tau}\,d\omega\right)dt \qquad (6.10)$$

The integrals can be rearranged to give:

$$\underset{T\to\infty}{R_{xx}(\tau)} = \frac{1}{2\pi T}\int_{-\infty}^{\infty} X(\omega)e^{i\omega\tau}\int_{-\infty}^{\infty} x(t)e^{i\omega t}\,dt\,d\omega \qquad (6.11)$$

The last integral in (6.11) is noted to be $X(\omega)^*$, resulting in:

$$\underset{T\to\infty}{R_{xx}(\tau)} = \frac{1}{2\pi T}\int_{-\infty}^{\infty} X(\omega)^* X(\omega)e^{i\omega\tau}\,d\omega \qquad (6.12)$$

By convention the conjugate term appears first.

Finally, referring to Eqn. 6.2:

$$S_{xx}(\omega) = \frac{X(\omega)^* X(\omega)}{T} \qquad (6.13)$$

Which is another way of stating the autospectral density function.

A few notes:

- $X(\omega)$ is expressed in $^{\text{units}}/_{\text{Hz}}$, which can be seen from Eqns. 5.16 and 5.17. This results in the correct units for S_{xx}.

- We just said that the Fourier transform of a random signal cannot be determined because $\int_{-\infty}^{\infty} |x(t)|\,dt \geq \infty$, preventing the calculation of S_{xx} using Eqn. 6.13. However, by forcing a continuous signal to have finite duration using a window, we will be able to obtain S_{xx} in this manner.

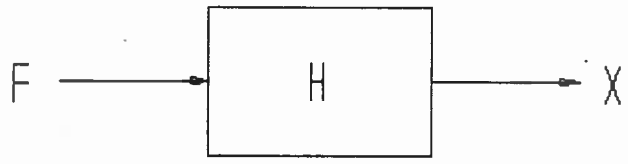
Likewise, the *cross-spectral density function* S_{xy} can be obtained from:

$$S_{xy}(\omega) = \frac{X(\omega)^* Y(\omega)}{T} \qquad (6.14)$$

6.3 Single Input - Single Output Relationships

6.3.1 Frequency Response Function $H(\omega)$ from Experimental Data

Consider the simple input-output model shown below which represents a dynamic system. The input force $f(t)$ is transformed by the system dynamics resulting in an output response, $x(t)$.



As was previously derived, the frequency response function consists *analytically* of the superposition of single-degree-of-freedom responses:

$$H(\omega) = \frac{X_i}{F_j} = \sum_{r=1}^{n} \left(\frac{\hat{\phi}_{ir}\hat{\phi}_{jr}}{-\omega^2 + i2\zeta_r\omega\omega_r + \omega_r^2} \right) \qquad (6.15)$$

Experimental estimates of a system's behavior in the frequency domain are typically accomplished by the following:

1. Excitation of the structure

2. Measurement of the input force and output response; the response being either acceleration, displacement or velocity

3. Transformation of the time domain data to the frequency domain, filtering as needed

4. Determination of the frequency response function (FRF) using an estimator which will minimize bias error

If we look at the frequency domain response, then the input-output relationship is:

$$X(\omega) = H(\omega)F(\omega) \qquad (6.16)$$

The FRF, $H(\omega)$, can exist as one of three specific forms depending on the units of $X(\omega)$:

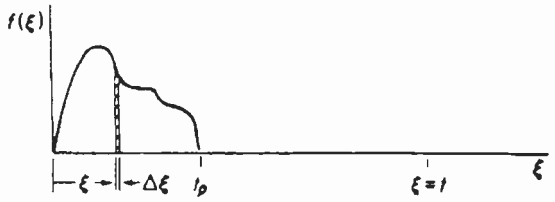
- If the output is displacement, $X(\omega)$ is referred to as *receptance*
- If the output is velocity, $X(\omega) = Y(\omega)$ and is referred to as *mobility*
- If the output is acceleration, $X(\omega) = A(\omega)$ and is referred to as *accelerance*

6.3.2 Relationship Between Time and Frequency Domains

Since data is acquired in the time domain and analyzed in the frequency domain, the relationships between the two domains should be understood.

- Equation 6.15 can be derived from the time domain response of a system using the convolution integral.

Recall from vibrations that the response of a system to an arbitrary input could be described by the summation of the responses to a series of short pulses. A transient input is shown below:



The response to this input can be related to the impulse response function, $h(t)$. The impulse response function is the response of a system to an impulse, or Dirac Delta Function:

$$\delta(t-\xi) = 0 \quad \text{For all } t \neq \xi$$

$$\delta(t-\xi) = \infty \quad \text{For } t = \xi$$

$$\int_0^\infty \delta(t-\xi)\,dt = 1$$

For a pulse of short duration, the response can be described by $f(\xi)\Delta\xi\,h(t-\xi)$, where the impulse response function is *scaled* by the area of the pulse. For a series of pulses, the response becomes:

$$x(t) = \sum_{r=1}^{n} f(\xi)\,h(t-\xi)\,\Delta\xi \qquad (6.17)$$

Letting the summation become an integral results in:

$$x(t) = \int_o^t f(\xi)\,h(t-\xi)\,d\xi \qquad (6.18)$$

This is called the *convolution integral.*

It can be shown by a variable substitution that an equivalent form is:

$$x(t) = \int_{o}^{t} f(t-\xi)\,h(\xi)\,d\xi \qquad (6.19)$$

Consider a case where the input signal is a harmonic of frequency ν with unity amplitude: $f(t) = e^{i\nu t}$. The response can be written as:

$$x(t) = \int_{-\infty}^{\infty} e^{i\nu(t-\xi)}\,h(\xi)\,d\xi \qquad (6.20)$$

The limits of integration indicate that the input is actually a continuous signal, not a transient. In this special case, $h(\xi)$ is non-causal.

Rewriting the equation by isolating the exponential of *t*:

$$x(t) = e^{i\nu t} \int_{-\infty}^{\infty} h(\zeta)e^{-i\nu\zeta}\,d\zeta \qquad (6.21)$$

The integral expression is the formal definition of the Fourier transform, or:

$$H(\nu) = \int_{-\infty}^{\infty} h(\zeta)e^{-i\nu\zeta}\,d\zeta \qquad (6.22)$$

So the response *to a single frequency* can be written as:

$$x(t) = e^{i\nu t} H(\nu) \tag{6.23}$$

This says that the output to a harmonic is simply the input scaled in magnitude and phase by the frequency response function.

Next, consider the inverse Fourier transform of $f(t)$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \tag{6.24}$$

This integral can be interpreted as an *infinite sum of harmonic inputs,* each one scaled by $F(\omega)d\omega / 2\pi$. By Eqn. 6.23 we can write an expression for the response to one of these harmonic components of frequency $\omega = \nu$:

$$x(t)_\nu = \left(\frac{1}{2\pi} F(\nu) d\omega \right) e^{i\nu t} H(\nu) \tag{6.25}$$

Adding up all contributions over the frequency range results in:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) H(\omega) e^{i\omega t} d\omega \tag{6.26}$$

This leads to the inverse Fourier transform,

$$X(\omega) = H(\omega)F(\omega) \qquad (6.27)$$

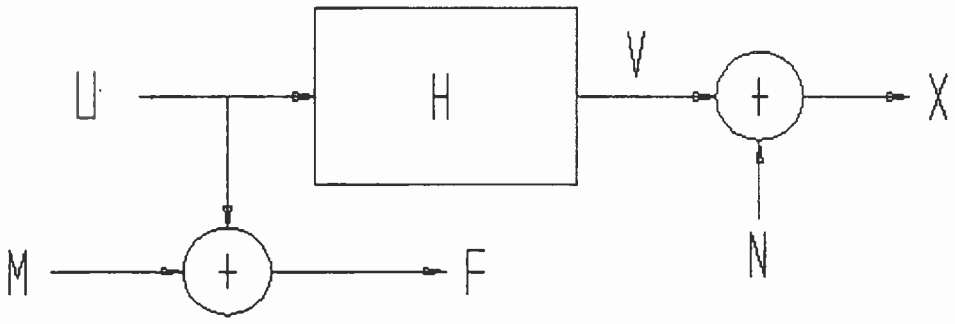
Thus, an important relationship has been demonstrated: *Convolution in the time domain is equivalent to multiplication in the frequency domain.* It can also be shown that convolution in the frequency domain is equivalent to multiplication in the time domain. This principle will be used when filtering is discussed.

6.3.3 Estimation of $H(\omega)$ in the presence of noise.

The previous input / output model shown assumed that $f(t)$ was a noiseless input signal passing thru *H* to give a noiseless output signal $x(t)$. In general, the signals measured will have noise on both the input and output.

Not only is noise a factor, but if the input signal is random its amplitude may at times become very low. Some means of averaging the data will be necessary to get good estimates of the FRF.

A more correct model for FRF estimation is shown below:



In this model, the measured input *F* is comprised of a true signal *U* and a noise signal *M*. The output is comprised of a true output *V* and a noise signal *N*.

It is desired to obtain good estimates of $H(\omega)$ in the presence of the noise. These estimates can be obtained from Eqn. 6.27 by premultiplying the equation by $F(\omega)^* / T$:

$$\frac{F(\omega)^* X(\omega)}{T} = H(\omega)\frac{F(\omega)^* F(\omega)}{T} \qquad (6.28)$$

From Eqn. 6.13, this can be expressed as:

$$S_{FX}(\omega) = H(\omega) S_{FF}(\omega) \qquad (6.29)$$

Recall that the Fourier transform is defined over frequencies from $-\infty$ to $+\infty$. Likewise, the spectral density functions are also defined over this domain. More common in experimental work is to use the *single-sided spectrum, G,* which is defined over positive frequencies only:

$$\begin{aligned} G_{XX}(\omega) &= 2S_{XX}(\omega) \quad \text{for } 0 < \omega < \infty \\ G_{XX}(\omega) &= S_{XX}(0) \quad \text{for } \omega = 0 \end{aligned} \qquad (6.30)$$

This relationship also holds for the cross-spectra:

$$\begin{aligned} G_{XY}(\omega) &= 2S_{XY}(\omega) \quad \text{for } 0 < \omega < \infty \\ G_{XY}(\omega) &= S_{XY}(0) \quad \text{for } \omega = 0 \end{aligned} \qquad (6.31)$$

An estimate of $H(\omega)$ will be obtained from:

$$H_1(\omega) = \frac{G_{FX}(\omega)}{G_{FF}(\omega)} \tag{6.32}$$

This form of the estimator is identified by the subscript "1", and has unique characteristics in the presence of noise. If we include the noise in the model, we will obtain the following:

$$\begin{aligned} G_{FX}(\omega) &= 2\frac{\left(U^*(\omega)+M^*(\omega)\right)\left(V(\omega)+N(\omega)\right)}{T} \\ &= 2\frac{U^*V+M^*V+U^*N+M^*N}{T} \end{aligned}$$

and (6.33)

$$\begin{aligned} G_{FF}(\omega) &= 2\frac{\left(U^*(\omega)+M^*(\omega)\right)\left(U(\omega)+M(\omega)\right)}{T} \\ &= 2\frac{U^*U+M^*U+U^*M+M^*M}{T} \end{aligned}$$

These can be simplified to give:

$$\begin{aligned} G_{FX}(\omega) &= G_{UV}+G_{MV}+G_{UN}+G_{MN} \\ G_{FF}(\omega) &= G_{UU}+G_{MU}+G_{UM}+G_{MM} \end{aligned} \tag{6.34}$$

Since the spectral quantities are obtained experimentally, they must be interpreted accordingly.

We will see shortly that it is possible to obtain *a form* of $X(\omega)$ and $F(\omega)$ using finite-length records, giving us an estimate of the spectra:

$$\begin{aligned} G_{FX}(\omega) &= E\left[\hat{G}_{FX}(\omega)\right] \\ G_{FF}(\omega) &= E\left[\hat{G}_{FF}(\omega)\right] \end{aligned} \tag{6.35}$$

The hat above the frequency domain estimates indicates that they are measured quantities. Again, because we are dealing with experimental data, we are obtaining *expected* values of the spectra.

When the expected value notation is used on Eqn. 6.34, we obtain:

$$\begin{aligned} G_{FX}(\omega) &= E\left[\hat{G}_{UV}\right] + E\left[\hat{G}_{MV}\right] + E\left[\hat{G}_{UN}\right] + E\left[\hat{G}_{MN}\right] \\ G_{FF}(\omega) &= E\left[\hat{G}_{UU}\right] + E\left[\hat{G}_{MU}\right] + E\left[\hat{G}_{UM}\right] + E\left[\hat{G}_{MM}\right] \end{aligned} \tag{6.36}$$

Recalling that the true signals are *U* and *V*, and the noise signals are *M* and *N*, we can simplify Eqn. 6.36:

- For G_{FX}, uncorrelated signal pairs include *M* and *V*, *U* and *N*, *M* and *N*. This results in $E\left[\hat{G}_{MV}\right] = E\left[\hat{G}_{UN}\right] = E\left[\hat{G}_{MN}\right] = 0$.

- For G_{FF}, uncorrelated signal pairs include *M* and *U*, *U* and *M*. This results in $E\left[\hat{G}_{MU}\right] = E\left[\hat{G}_{UM}\right] = 0$.

The resulting calculation of $H(\omega)$ becomes:

$$H_1(\omega) = \frac{G_{FX}(\omega)}{G_{FF}(\omega)} = \frac{E\left[\hat{G}_{UV}\right]}{E\left[\hat{G}_{UU}\right] + E\left[\hat{G}_{MM}\right]} \qquad (6.37)$$

And this can be written as:

$$H_1(\omega) = \frac{G_{UV}(\omega)}{G_{UU}(\omega) + G_{MM}(\omega)} \qquad (6.38)$$

The *true* FRF is $H(\omega) = {G_{UV}(\omega)}/{G_{UU}(\omega)}$, so from Eqn. 6.38 it can be seen that a bias error from noise on the input is introduced.

- The estimator biases the true FRF *low* if there is noise on the input.
- This may not be a problem if noise on the input is low; this estimator *assumes* there is no noise on the input

If noise on the input is a problem, then another form of the FRF can be used. Premultiplying Eqn. 6.16 by ${X(\omega)^*}/{T}$ results in:

$$S_{XX}(\omega) = H(\omega) S_{XF}(\omega) \qquad (6.39)$$

Using the same procedure as before, we obtain a new estimate of $H(\omega)$ indicated by the subscript "2":

$$H_2(\omega) = \frac{G_{XX}}{G_{XF}} = \frac{G_{VV}(\omega) + G_{NN}(\omega)}{G_{VU}(\omega)} \qquad (6.40)$$

- This estimator biases the true FRF *high.*
- This may not be a problem if noise on the output is low; this estimator *assumes* there is no noise on the output.

As a result of these findings, typical analyzers allow the user to select either $H_1(\omega)$ or $H_2(\omega)$, depending on the measurement noise magnitudes.

A measure of the amount of noise contaminating a signal is obtained with the ordinary coherence function, γ_{XF}^2. This is defined as:

$$\gamma_{XF}^2(\omega) = \frac{|G_{XF}(\omega)|^2}{G_{XX}(\omega)G_{FF}(\omega)} = \frac{|S_{XF}(\omega)|^2}{S_{XX}(\omega)S_{FF}(\omega)} \qquad (6.41)$$

- The coherence function has its origin in the correlation function $\rho_{XF}^2(\tau)$ defined for time domain signals.
- Coherence has limits of $0 \le \gamma_{XF}^2 \le 1$.

The interpretation of coherence is that it represents the fractional part of the mean square value at the output *x(t)* that is contributed by the input, *f(t)*. A coherence of 1 implies that 100% of the output is due to the input.

- Note that coherence is based on *statistical averages,* so if we acquire only one average, the coherence will have a value of unity regardless of the noise present!

Computationally, there are some other forms of the coherence function that are common:

$$\gamma^2_{XF}(\omega) = \frac{1}{1+\alpha(\omega)+\beta(\omega)+\alpha(\omega)\beta(\omega)}$$

$$\alpha(\omega) = G_{MM}(\omega) \Big/ G_{UU}(\omega) \qquad (6.42)$$

$$\beta(\omega) = G_{NN}(\omega) \Big/ G_{VV}(\omega)$$

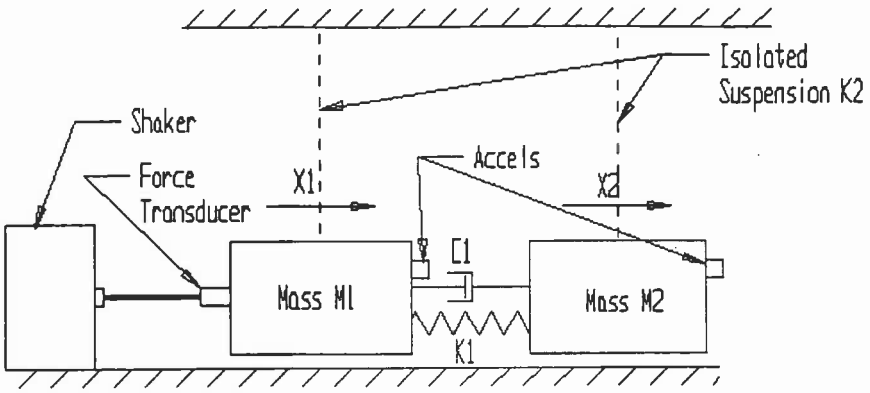
and:

$$\gamma^2_{XF}(\omega) = \frac{H_1(\omega)}{H_2(\omega)} \qquad (6.43)$$

Equation 6.43 indicates that H_1 will always underestimate the true FRF and H_2 will overestimate it.

6.3.4 Examples of the FRF Estimates Using H_1 and H_2

Consider the two-DOF system presented in section 3.3 with added damping of $C_1 = 10.4$ $\text{lb sec}/\text{in}$ between masses 1 and 2:



The parameters chosen for the system are:

$$K_1 = 20{,}000 \text{ lb/in}$$

$$C_1 = 5.2 \text{ lb sec/in}$$

$$M_1 = M_2 = 0.0674 \frac{\text{lb sec}^2}{\text{in}}$$

After diagonalizing the mass and stiffness matrices, the receptance response function is obtained:

$$H(\omega) = \frac{V_i}{U_j} = \frac{1}{-.135\omega^2} + \frac{1}{-.135\omega^2 + i77\omega + 80{,}000} \qquad (6.44)$$

This FRF equation can be converted to accelerance simply by multiplying by $i\omega$ once for each derivative (from the assumption of $x(t) = X e^{i\omega t}$). Because accelerance represents two derivatives of the receptance FRF, the accelerance is expressed as:

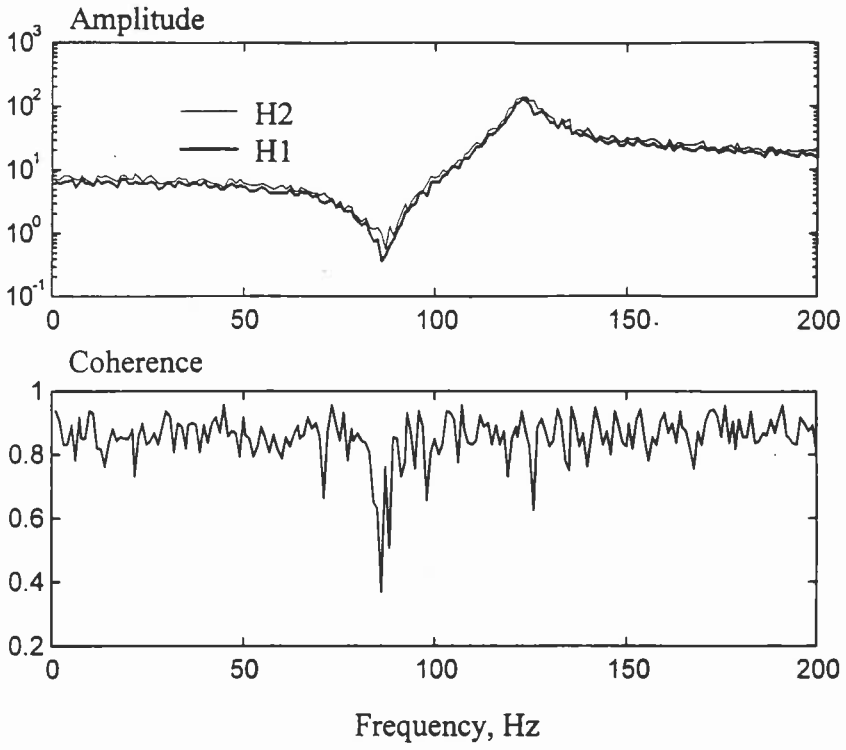
$$H(\omega) = -\omega^2 \left(\frac{1}{-.135\omega^2} + \frac{1}{-.135\omega^2 + i77\omega + 80{,}000} \right) \qquad (6.45)$$

To simulate a random process in the frequency domain, it is necessary to consider the transformation from the time domain to the frequency domain. Since Fourier transformation is a linear operation, the following is true:

- If $x(t)$ is a Gaussian random variable, then $X_r(\omega)$ and $X_i(\omega)$ will be Gaussian random variables as well.

The simulation consists of generating random real and imaginary components for the true input *U* and the noise signals *M* and *N* (with reference to the system model in Section 6.3.3). The true output *V* is generated from Eqn. 6.45.

For 10 averages with noise signals M and N that are 40% of U, the FRF plot below is obtained:



Some points to make:

- As previously stated, H_1 tends to be biased *low* and H_2 tends to be biased *high*.
- The coherence is low near the antiresonance because the true output signal G_{VV} is low. This is apparent in Eqn. 6.42. The estimator H_2 will have more error at this point than H_1 because of the bias error term G_{NN}.
- In reality, force drop-off at resonance can occur because of the shaker dynamics. This will result in low coherence at the resonance and cause the H_1 estimator to be biased low. More on this later.

6.3.5 Errors in Measurements of Random Data

If we are dealing with experimental data which is random, all of the estimates we have been discussing should be related to a statistical probability.

The normalized random error ε_r of a variable φ can be defined as:

$$\varepsilon[\varphi] = \frac{\sigma[\hat{\varphi}]}{\varphi} \tag{6.46}$$

Where $\sigma[\hat{\varphi}]$ is the standard deviation, standard error, or random error:

$$\sigma[\hat{\varphi}] = \sqrt{\left(E[\hat{\varphi}] - \hat{\varphi}\right)^2} \tag{6.47}$$

For estimates of a spectral density function, the error can be expressed as:

$$\varepsilon\left[\left|\hat{G}_{XY}\right|\right] = \frac{1}{|\gamma_{XY}|\sqrt{n_d}} \tag{6.48}$$

Where X and Y represent input or output signals (could be autospectral density), and n_d is the number of averages.

This equation says that *even* if $\gamma_{XY} = 1$, There is still an uncertainty in one average of:

$$\sigma\left[\hat{G}_{XY}\right] = |G_{XY}| \tag{6.49}$$

Which is large! This is due to the nature of random data.

Two other observations are apparent in Eqn. 6.48:

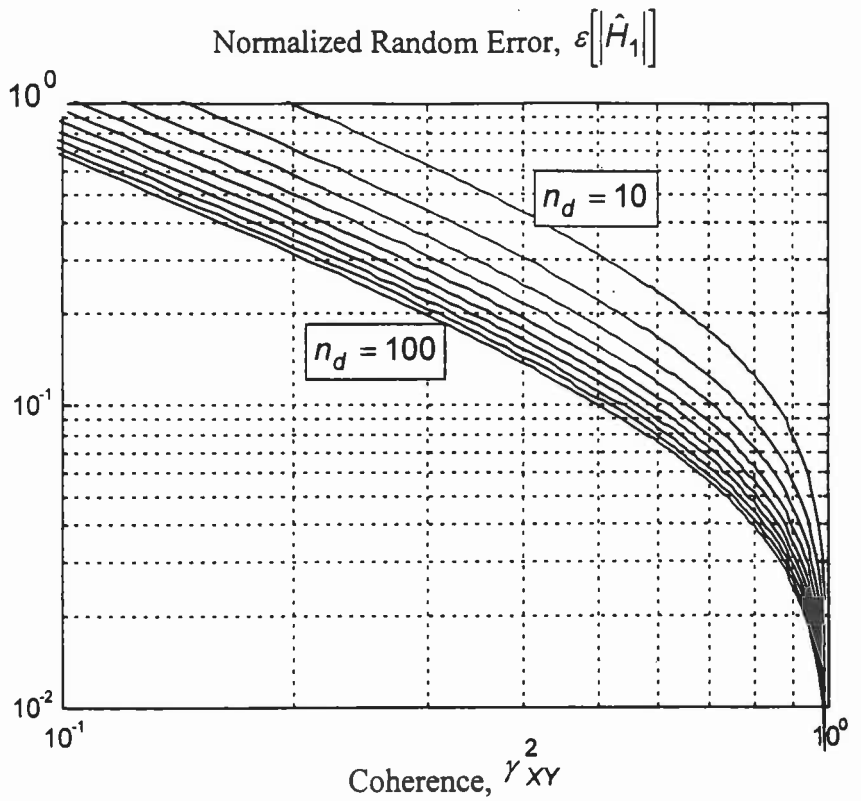
- Even with low coherence, a large number of averages can be used to obtain a good estimate
- Averaging is always necessary with random data

For the FRF magnitude estimate $|H_1|$ with *no noise on the input*, the following relationship for error can be shown:

$$\varepsilon\left[\left|\hat{H}_1\right|\right] = \frac{\left(1-\gamma_{XY}^2\right)^{1/2}}{|\gamma_{XY}|\sqrt{2n_d}} \tag{6.50}$$

- Note that as $\gamma_{XY}^2 \to 1$, the error $\varepsilon\left[\left|\hat{H}_1\right|\right] \to 0$ regardless of the number of averages.
- Regardless of the coherence, $\varepsilon\left[\left|\hat{H}_1\right|\right] \to 0$ as the number of averages becomes large.

Plots of this error function are shown below for different numbers of averages:



As can be seen, a high coherence (> 0.90) is desirable to keep the error at reasonable levels.

Along with a magnitude error, a phase error in radians for the estimator is also given:

$$\sigma\left[\hat{\phi}_{XY}\right] \approx \varepsilon\left[\left|\hat{H}_{XY}\right|\right] \qquad (6.51)$$

So that the above figure also plots the standard deviation of the phase based on averages and coherence.

An uncertainty relationship can also be developed for the coherence:

$$\varepsilon\left[\hat{\gamma}^2_{XY}\right] \approx \frac{\sqrt{2}\left(1-\gamma^2_{XY}\right)}{\left|\gamma^2_{XY}\right|\sqrt{n_d}} \qquad (6.52)$$

With these error estimates, it is possible to determine the number of averages required for an accurate estimate of the magnitude and phase of the FRF.

An interesting fact:

The coherence for one average is always one, misleading some people to believe that they have really good data with one average.

This is due to the fact that coherence is based on *expected values* of measured signals, which is much different than the measured signal. For one average, $E[\varphi] = \varphi$ and as a result:

$$\begin{aligned} G_{FX}(\omega) &= E\left[\hat{G}_{FX}(\omega)\right] = 2\frac{\hat{F}^*(\omega)\hat{X}(\omega)}{T} \\ G_{FF}(\omega) &= E\left[\hat{G}_{FF}(\omega)\right] = 2\frac{\hat{F}^*(\omega)\hat{F}(\omega)}{T} \\ G_{XX}(\omega) &= E\left[\hat{G}_{XX}(\omega)\right] = 2\frac{\hat{X}^*(\omega)\hat{X}(\omega)}{T} \end{aligned} \qquad (6.53)$$

leading to:

$$\gamma^2_{XF}(\omega) = \frac{\left|G_{XF}(\omega)\right|^2}{G_{XX}(\omega)G_{FF}(\omega)} = \frac{G_{XX}(\omega)G_{FF}(\omega)}{G_{XX}(\omega)G_{FF}(\omega)} = 1 \qquad (6.54)$$