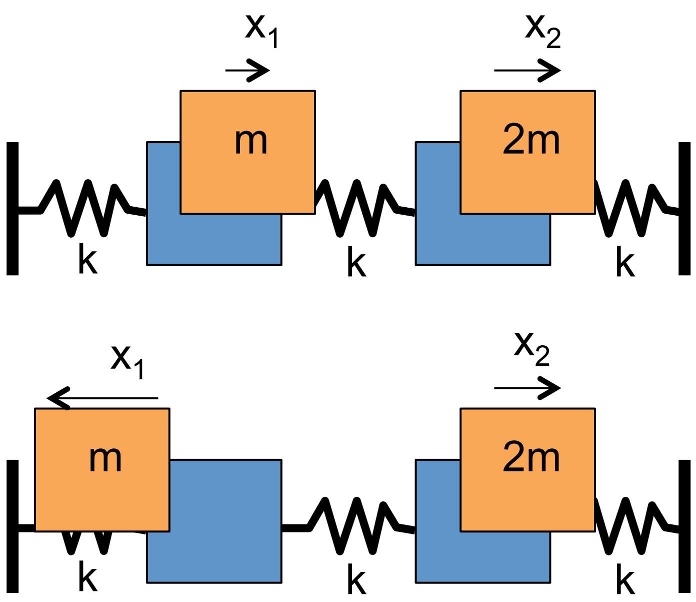
Some notes about eigenvector orthogonality

- **They are orthogonal with when evaluated with the mass and/or stiffness matrices**

$$\Phi_i^T\left[M\right]\Phi_j = \begin{bmatrix} m_{11} & & \\ & m_{22} & \\ & & m_{nn} \end{bmatrix}$$

$$\Phi_i^T\left[K\right]\Phi_j = \begin{bmatrix} k_{11} & & \\ & k_{22} & \\ & & k_{nn} \end{bmatrix}$$



…they are ***not*** orthogonal with respect to each other

- From previous example:

$$\Phi_i^T \Phi_j = \begin{bmatrix} .731 & 1 \\ -2.73 & 1 \end{bmatrix} \begin{bmatrix} .731 & -2.73 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} .533 & -1 \\ -1 & 8.45 \end{bmatrix}$$

- Using orthogonality, we can rewrite the eigenvalue problem (EVP) in a couple of different ways

$$\left(-\lambda\left[M\right]+\left[K\right]\right)\left(X\right)=\left(0\right)$$

Variations on the EVP

$$\Phi^T\left(-\lambda\left[M\right]+\left[K\right]\right)\left(X\right)\Phi=\left(0\right)$$

$$\left(-\lambda\left[M_{diag}\right]+\left[K_{diag}\right]\right)\left(X\right)=\left(0\right)$$

Using mass normalized eigenvectors: $\hat{\phi}_i=\frac{1}{\sqrt{m_{ii}}}\phi_i$

Results in:

$$\hat{\Phi}^T\left[M\right]\hat{\Phi}=\left[I\right]$$

$$\hat{\Phi}^T\left[K\right]\hat{\Phi}=\left[\lambda_{diag}\right]$$

$$\left(-\left[\lambda_{diag}\right]+\left[\omega_{diag}^2\right]\right)\left(X\right)=\left(0\right)$$

Which is a trivial equation that just says each of the eigenvalues is equal to the independent squared natural frequencies of the system

Solving eigenvalue problems with Matlab

eig: Eigenvalues and eigenvectors.
E = eig(A) produces a column vector E containing the eigenvalues of a square matrix A.

[V,D] = eig(A) produces a diagonal matrix D of eigenvalues and a full matrix V whose columns are the corresponding eigenvectors so that A*V = V*D.

Recall:
$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}+\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can work with this original equation to put it in a form recognized by the eig command by dividing eqn 1 by m and eqn 2 by 2m:

$$\begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}\left(X\right)+\begin{bmatrix} {}^{2k}\!/\!_{m} & {}^{-k}\!/\!_{m} \\ {}^{-k}\!/\!_{2m} & {}^{2k}\!/\!_{2m} \end{bmatrix}\left(X\right)=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[A\right]\left(X\right)-\lambda\left(X\right)=\left(0\right)$$

$$\left[A\right]\left(X\right)=\lambda\left(X\right)$$

or: $\left[A\right]\left(V\right)=\left(V\right)d$

Where

$$\left[A\right]=\begin{bmatrix} 2\omega^2 & -\omega^2 \\ -\frac{1}{2}\omega^2 & \omega^2 \end{bmatrix}$$

Substituting values for k, m:

Let m = 1, k = 10 (consistent units)

$$\omega = \frac{k}{m} = 10 \qquad \left[A\right] = \begin{bmatrix} 20 & -10 \\ -5 & 10 \end{bmatrix}$$

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>> [V,d] = eig([20 -10;-5 10])

V =
   0.9391   0.5907
  -0.3437   0.8069

d =
  23.6603        0
        0   6.3397
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Previously, we had:

$$\lambda_1 = .634\frac{k}{m}, \omega_1 = .8\sqrt{\frac{k}{m}}$$

$$\Phi_1 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} .731 \\ 1 \end{pmatrix}$$

Are they consistent?:

$$\lambda_1 = 2.366\frac{k}{m}, \omega_2 = 1.54\sqrt{\frac{k}{m}}$$

$$\Phi_2 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -2.73 \\ 1 \end{pmatrix}$$