9. Parameter Estimation

Parameter estimation is the process of extracting the modal parameters of natural frequency, damping, and residue from the FRF of a structure.

There are many different types of estimation, some which are based on the time domain response and others on the frequency domain.

9.1 SDOF Curve Fitting

9.1.1 Peak Picking

The simplest method of curve fitting goes back to the SDOF model we have seen many times before:

$$\left|H(\omega)_{ij}\right|_r = \left|\frac{X(\omega)}{F(\omega)}\right|_r = \frac{{}_rA_{ij}}{\sqrt{\left(-\omega^2+\omega_r^2\right)^2+\left(2\zeta\omega\omega_r\right)^2}} \tag{9.1}$$

In this case, only the r^{th} mode is considered since we have assumed that *the influence of neighboring modes is negligible.*

At the resonance of this function, we obtain:

$$\left|H(\omega)_{ij\,p}\right|_r = \frac{{}_rA_{ij}/\omega_r^2}{2\zeta} \tag{9.2}$$

Provided the modes of the structure are well separated and lightly damped (> 0.01 damping ratio), this is a valid estimate.

The half-power frequencies of the FRF correspond to an amplitude of $\left(\text{peak amplitude} \Big/ \sqrt{2} \right)$. The frequencies where these occur can be solved for from Eqn. 9.1:

$$|H(\omega)| = \frac{{}_rA_{ij} \big/ \omega_r^2}{2\zeta\sqrt{2}} = \frac{{}_rA_{ij}}{\sqrt{\left(-\omega^2 + \omega_r^2\right)^2 + (2\zeta\omega\omega_r)^2}} \qquad (9.3)$$

This is rewritten:

$$\frac{1}{2\zeta\sqrt{2}} = \frac{\omega_r^2}{\sqrt{\left(-\omega^2 + \omega_r^2\right)^2 + (2\zeta\omega\omega_r)^2}} \qquad (9.4)$$

Squaring both sides allows the solution for ω:

$$\omega^4 + \left(4\zeta^2\omega_r^2 - 2\omega_r^2\right)\omega^2 + \left(\omega_r^4 - \omega_r^4 8\zeta^2\right) = 0 \qquad (9.5)$$

This results in 2 roots:

$$\omega^2{}_{1,2} = \omega_r^2\left(1 - 2\zeta^2 \pm 2\zeta\sqrt{1+\zeta^2}\right) \qquad (9.6)$$

If ζ is small, then an approximation is obtained with:

$$\omega_{1,2}^2 = \omega_r^2(1 \pm 2\zeta) \tag{9.7}$$

The frequencies $\omega_{1,2}$ are the frequencies at which the amplitude will be $1/\sqrt{2}$ of the peak amplitude, which we assume occurs at ω_r.

The difference in the above frequencies is $\omega_2^2 - \omega_1^2 = \omega_r^2 4\zeta$. This can be expressed as:

$$(\omega_2 - \omega_1)(\omega_2 + \omega_1) = \omega_r^2 4\zeta \tag{9.8}$$

The average of the two frequencies will be approximatley ω_r, so

$$\frac{\omega_2 + \omega_1}{2} = \omega_r \tag{9.9}$$

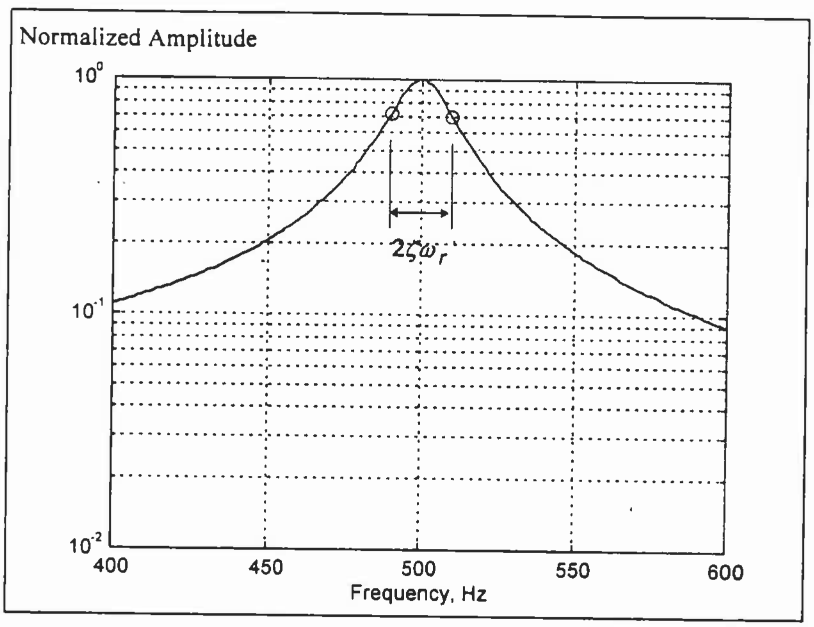
Substituting into Eqn. 9.8 results in:

$$\begin{aligned} (\omega_2 - \omega_1)2\omega_r &= \omega_r^2 4\zeta \\ (\omega_2 - \omega_1) &= \omega_r 2\zeta \end{aligned} \tag{9.10}$$

Finally, Eqn. 9.10 is rewritten as:

$$\frac{(\omega_2 - \omega_1)}{\omega_r} = 2\zeta \qquad (9.11)$$

Graphically, this appears as shown below:



This provides an estimate of ζ given a measurement of the half-power frequencies and the resonance frequency.

The mode vector is obtained from an estimate of the residue ${}_rA_{ij}$ using a form of Eqn. 9.2:

$${}_rA_{ij} = \omega_r^2\, 2\zeta \left| H(\omega)_{ij_p} \right|_r \qquad (9.12)$$

This method is referred to as *peak picking or peak amplitude*, since it relies on selection of the peak amplitude for a particular mode.

9.1.2 SDOF Circle Fit

For an SDOF system, the FRF of mobility $Y = V/F$ plots a circle in the complex (Nyquist) Plane. Consider the FRF expression:

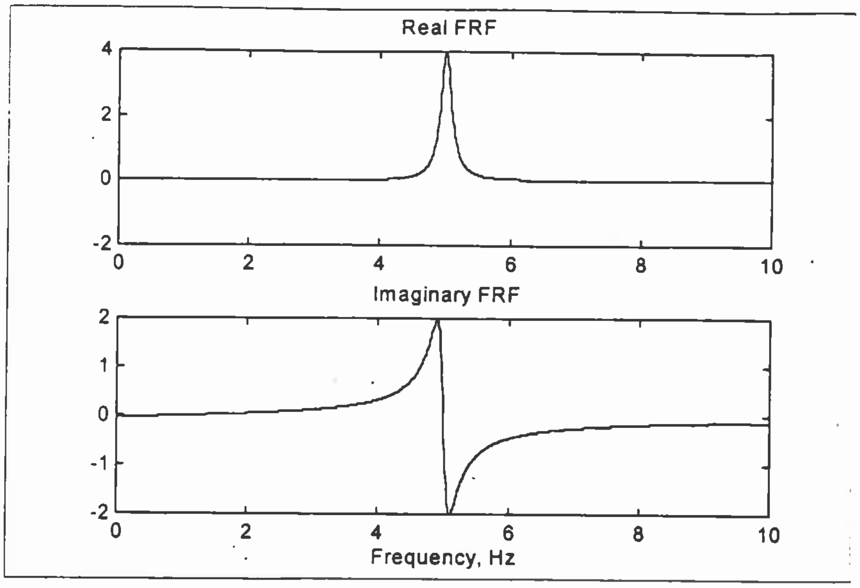
$$\frac{V_i(\omega)}{F_j(\omega)} = \frac{i\omega A_{ij}}{\left(-\omega^2 + \omega_r^2\right) + i\left(2\zeta\omega\omega_r\right)} \tag{9.13}$$

Assuming the residue is real, the real and imaginary parts are:

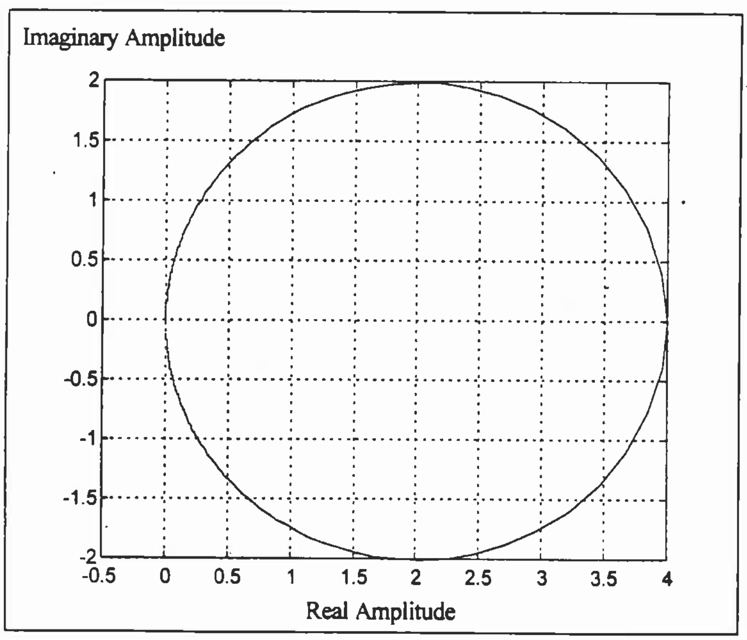
$$\left.\frac{V_i}{F_j}\right|_{real} = \frac{{}_rA_{ij}\, 2\zeta\,\omega^2\omega_r}{\left(-\omega^2 + \omega_r^2\right)^2 + \left(2\zeta\omega\omega_r\right)^2}$$

$$\left.\frac{V_i}{F_j}\right|_{imag} = \frac{{}_rA_{ij}\,\omega\left(-\omega^2 + \omega_r^2\right)}{\left(-\omega^2 + \omega_r^2\right)^2 + \left(2\zeta\omega\omega_r\right)^2} \tag{9.14}$$

These functions are plotted on the real and imaginary axes for $\omega_r = 2\pi 5$ rad/sec and $\zeta = .02$:



And plotting in the Nyquist Plane gives:

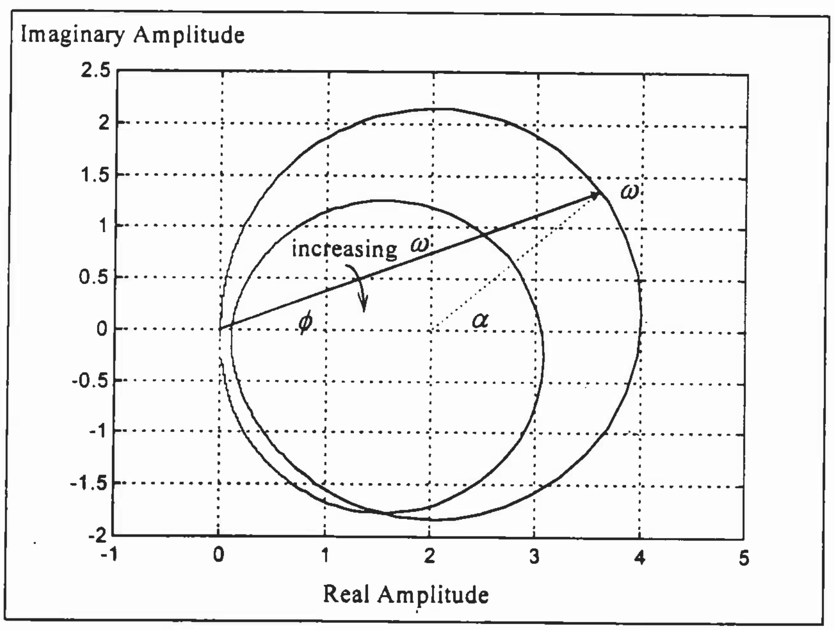


If there are other modes present (and there will be), it will be necesary to account for them.

The assumption is made that in Eqn. 9.13, the mobility contribution from neighboring modes is *a constant complex value that results in a shift of the center of the circle:*

$$\frac{V_i(\omega)}{F_j(\omega)} = \frac{i\omega A_{ij}}{\left(-\omega^2 + \omega_r^2\right) + i(2\zeta\omega\omega_r)} + R_{ij} \qquad (9.15)$$

This is demonstrated by adding another mode at 7 Hz and again plotting the behavior in the complex plane:



The large circle corresponds to the first mode at 5 Hz. Note that it has shifted up and slightly to the right.

If the shift is accounted for, then the circle can be fit to obtain the modal parameters.

The natural frequency ω_r is determined from the rate of sweep of the circle. That is, the natural frequency is located at the *part of the circle that changes the fastest with increasing* ω. If equal-spaced frequency measurements have been acquired, then the natural frequency will be located between the two points which are farthest apart.

Once the natural frequency is found, two other points (also obtained from the frequency spacing of the analyzer) are selected to determine the damping.

First, however, a relation between the phase angle ϕ and the angle α (angle on the circle) is established:

$$\phi = \frac{\alpha}{2} \tag{9.16}$$

For two frequency points ω_a and ω_b which are below and above the natural frequency, the angle α is related by:

$$\begin{aligned} \tan\left({}^{\alpha_a}\!/_2\right) &= \frac{-\omega_a^2 + \omega_r^2}{2\zeta\omega_a\omega_r} \\ \tan\left({}^{\alpha_b}\!/_2\right) &= \frac{-\omega_b^2 + \omega_r^2}{2\zeta\omega_b\omega_r} \end{aligned} \tag{9.17}$$

The damping ratio can then be solved for:

$$\zeta = \frac{\omega_b^2 - \omega_a^2}{2\omega_r\left(\omega_b \tan\left(\alpha_b/2\right) + \omega_a \tan\left(\alpha_a/2\right)\right)} \qquad (9.18)$$

Finally, the mode vector is determined from the diameter of the circle because at resonance, the amplitude will be:

$$\left.\frac{V_i}{F_j}\right|_{\omega=\omega_r} = \frac{{}_rA_{ij}}{2\zeta\omega_r} = (\text{diameter}) \qquad (9.19)$$

9.2 MDOF Curve-Fitting

SDOF techniques are adequate for well separated modes. For modes which are closely coupled, or weak modes the MDOF techniques are preferred.

9.2.1 MDOF Circle Fitting

As an extension of the SDOF circle fitting method, the MDOF method uses a better model for the modes outside of the fitting range.

Recall that the SDOF circle fit assumed that there was a *constant* displacement of the circle center due to the surrounding modes:

$$\frac{V_i(\omega)}{F_j(\omega)} = \frac{i\omega A_{ij}}{\left(-\omega^2 + \omega_r^2\right) + i\left(2\zeta_r \omega \omega_r\right)} + R_{ij} \qquad (9.20)$$

In the MDOF technique, R_{ij} is modeled as the surrounding modes *and* the residual contribution. The residuals represent modal content outside of the measurement range, and together with the in-range modes results in:

$$R_{ij} = \sum_{r=1}^{n-1} \frac{i\omega\, {}_rA_{ij}}{\left(-\omega^2 + \omega_r^2\right) + i\left(2\zeta_r \omega \omega_r\right)} + {}_lR(\omega)_{ij} + {}_hR(\omega)_{ij} \qquad (9.21)$$

The first term represents the modes within the frequency range of the FRF.

- The last two terms are the residuals and represent the out-of-range contribution. This includes rigid body modes (near zero frequency) and higher frequency modes (above the range of the analysis).

The first residual term is obtained by observing that as the frequency gets much larger than a resonance frequency, the mobility becomes:

$$\left(\frac{V_i(\omega)}{F_j(\omega)}\right)_{\frac{\omega}{\omega_{nat}} \to \infty} = \frac{i\omega A_{ij}}{-\omega^2} = \frac{{}_lK_{ij}}{\omega} = {}_lR(\omega)_{ij} \qquad (9.22)$$

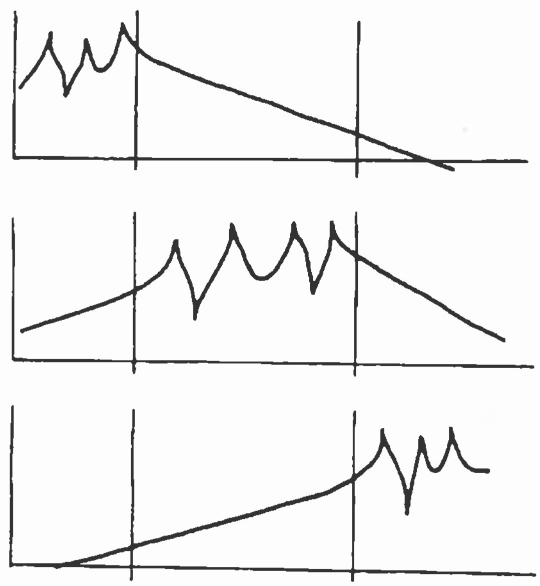
This would be the case if we wanted to model modes which were *below* the frequency range of interest.

And as the frequency becomes much less than a resonance frequency, the mobility becomes:

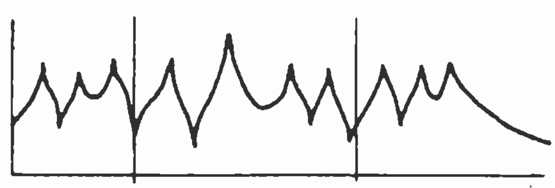
$$\left(\frac{V_i(\omega)}{F_j(\omega)}\right)_{\frac{\omega}{\omega_{nat}} \to 0} = \frac{i\omega A_{ij}}{\omega_{nat}^2} = {}_hK_{ij}\omega = {}_hR(\omega)_{ij} \qquad (9.23)$$

Which is the model for modes *above* the frequency range of interest.

These residuals can be visualized as representing the out-of-range modes shown below:



All modes taken together results in the whole FRF:



- The residuals are functions in ω.

With the improved model of Eqn. 9.20, the fitting involves an iterative process:

1. The modes identified over the measured frequency range are fit using the SDOF circle fitting technique. This is a first estimate of the modes.

2. Each mode is again fit after the neighboring modes are subtracted. With this estimate, the low frequency residual term is estimated by:

$$\left({}_{l}R_{ij}\right)_{0<\omega<\omega_h} = Y_{ij} - \sum_{r=lo}^{hi} \frac{i\omega\, {}_{r}A_{ij}}{\left(-\omega^2 + \omega_r^2\right) + i\left(2\zeta_r \omega \omega_r\right)} \tag{9.24}$$

The fitting is performed on a frequency range from D.C. to a frequency ω_h that adequately represents the residual contribution.

The high frequency residual is estimated by:

$$\left({}_{h}R_{ij}\right)_{\omega_l<\omega<\omega_{max}} = Y_{ij} - \sum_{r=lo}^{hi} \frac{i\omega\, {}_{r}A_{ij}}{\left(-\omega^2 + \omega_r^2\right) + i\left(2\zeta_r \omega \omega_r\right)} \tag{9.25}$$

Again, the limiting frequencies ω_l and ω_{max} are chosen to adequately represent the high frequency contributions.

Finally, Eqn. 9.20 is used a third time to obtain a refined estimate of the modes.

9.2.2 Frequency Domain Curve Fitting - Orthogonal Polynomials

More common than the Circle Fitting is Frequency Domain Curve Fitting. This method is easier to implement although it requires more computational time.

- Sub-regions of the FRF are identified and fit progressively until the entire FRF is fit.

This method first involves writing the mobility in the form of a rational fraction:

$$Y_{ij} = \frac{i\omega\left(a_0 + a_1(i\omega)^1 + \cdots + a_m(i\omega)^m\right)}{b_0 + b_1(i\omega)^1 + \cdots + b_n(i\omega)^n} = \frac{P_{num}}{P_{den}} \qquad (9.26)$$

This form is obtained directly from the sum of SDOF terms that we have been using.

Next, a weighted least squares problem is solved:

$$\{\varepsilon\}_p = \{W\}_p \dot{*} \left\{Y - \frac{P_{num}}{P_{den}}\right\}_p \qquad (9.27)$$

The p refers to the length of the vector and the operator $\dot{*}$ is a row-by-row multiplication. The vector $\{W\}$ is a weighting to the error formed from the difference of the measurement and the model.

If Eqn. 9.27 is pre-multiplied by the denominator polynomial, a new error definition is created:

$$\{\varepsilon'\}_p = \{W\}\dot{*}\{P_{den}\}\dot{*}\{Y\} - \{W\}\dot{*}\{P_{num}\}_p \qquad (9.28)$$

Note that if $W = \dfrac{1}{P_{den}}$, the new error becomes the original error.

The polynomials are modeled as functions with coefficients:

$$\begin{aligned}[\phi](a) &= \{P_{num}\} \\ [\psi](b) &= \{P_{den}\}\end{aligned} \tag{9.29}$$

And the error becomes:

$$\begin{aligned}\{\varepsilon'\}_p &= \{W\}\dot{*}\{Y\}\dot{*}[\psi](b) - \{W\}\dot{*}[\phi](a)_p \\ &\quad \text{or} \\ \{\varepsilon'\}_p &= \{W_b\}\dot{*}[\psi](b) - \{W_a\}\dot{*}[\phi](a)_p\end{aligned} \tag{9.30}$$

Where:

$$\begin{aligned}\{W_b\} &= \{W\}\dot{*}\{Y\} \\ \{W_a\} &= \{W\}\end{aligned}$$

Expanding this equation in real and imaginary components and writin :e squared error as $E = \varepsilon'\varepsilon'^*$ results in:

$$\begin{aligned}E|_\omega = {} & \left(W_{b\omega r}[\psi_\omega]\{b_r\} - W_{b\omega i}[\psi_\omega]\{b_i\} - W_{a\omega r}[\phi_\omega]\{a_r\} + W_{a\omega i}[\phi_\omega]\{a_i\}\right)^2 \\ & + \left(W_{b\omega i}[\psi_\omega]\{b_r\} + W_{b\omega r}[\psi_\omega]\{b_i\} - W_{a\omega i}[\phi_\omega]\{a_r\} - W_{a\omega r}[\phi_\omega]\{a_i\}\right)^2\end{aligned} \tag{9.31}$$

This is the error determined at each frequency ω. Simplifying this to:

$$E|_{\omega} = g_1^2 + g_2^2 \tag{9.32}$$

Results in a solution for the coefficients by *minimizing the error:*

$$\begin{aligned}
\left.\frac{\partial E}{\partial a_r}\right|_{\omega} &= -2W_{a\omega r}[\phi_{\omega}]^T g_1 - 2W_{a\omega i}[\phi_{\omega}]^T g_2 = 0 \\
\left.\frac{\partial E}{\partial a_i}\right|_{\omega} &= 2W_{a\omega i}[\phi_{\omega}]^T g_1 - 2W_{a\omega r}[\phi_{\omega}]^T g_2 = 0 \\
\left.\frac{\partial E}{\partial b_r}\right|_{\omega} &= 2W_{b\omega r}[\psi_{\omega}]^T g_1 + 2W_{b\omega i}[\psi_{\omega}]^T g_2 = 0 \\
\left.\frac{\partial E}{\partial b_i}\right|_{\omega} &= -2W_{b\omega i}[\psi_{\omega}]^T g_1 + 2W_{b\omega i}[\psi_{\omega}]^T g_2 = 0
\end{aligned} \tag{9.33}$$

This problem can be expressed in matrix form:

$$\begin{bmatrix} [f_1] & [f_2] \\ [f_2]^T & [f_3] \end{bmatrix} \begin{Bmatrix} \{b_r\} \\ \{b_i\} \\ \{a_r\} \\ \{a_i\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix} \tag{9.34}$$

where:

$$[f_1] = \begin{bmatrix} 2\left(W_{b_r}^2 + W_{b_i}^2\right)[\psi]^T[\psi] & 0 \\ 0 & 2\left(W_{b_r}^2 + W_{b_i}^2\right)[\psi]^T[\psi] \end{bmatrix} \tag{9.35}$$

$$[f_2] = \begin{bmatrix} 2\left(W_{a_r}W_{b_r} + W_{a_i}W_{b_i}\right)[\psi]^T[\phi] & 2\left(W_{a_i}W_{b_r} - W_{a_r}W_{b_i}\right)[\psi]^T[\phi] \\ 2\left(W_{a_r}W_{b_i} - W_{a_i}W_{b_r}\right)[\psi]^T[\phi] & -2\left(W_{a_i}W_{b_i} + W_{a_r}W_{b_r}\right)[\psi]^T[\phi] \end{bmatrix} \tag{9.36}$$

$$[f_3] = \begin{bmatrix} 2\left(W_{a_r}^2 + W_{a_i}^2\right)[\phi]^T[\phi] & 0 \\ 0 & 2\left(W_{a_i}^2 + W_{a_r}^2\right)[\phi]^T[\phi] \end{bmatrix} \tag{9.37}$$

Note that ideally $W_a = \frac{1}{P_d}$ and $W_b = \frac{Y}{P_d}$ so that the problem we are solving *is* the minimization of the original error equation, Eqn. 9.27.

Since the poles are not known apriori, an iterative process can be used so that the minimization *approaches* the correct problem.

Typically, a *QR* method such as Gram Schmidt will be used to determine the coefficients *a* and *b*.

Since power polynomials become less orthogonal with order, other types of polynomials are commonly used in frequency domain curve fitting.

Chebycheff and Forsythe are two commonly used polynomials for fitting.

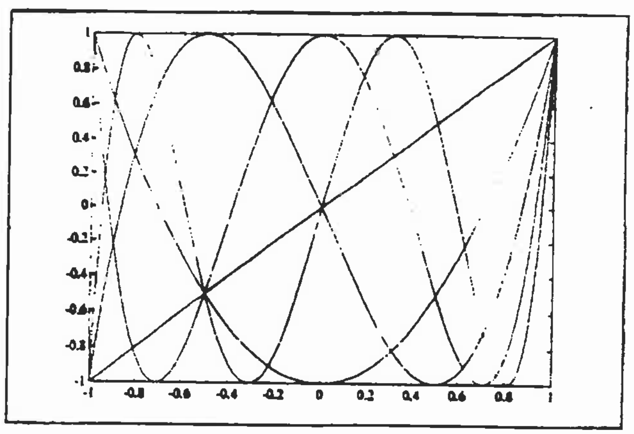
The mobility equation is modified to look like:

$$Y_{ij} = \frac{a_0 + a_1 T_1 + \cdots + a_m T_m}{b_0 + b_1 T_1 + \cdots + b_n T_n} \tag{9.38}$$

The ith order Chebycheff Polynomial is T_i. The polynomials are generated by the following recursive algorithm:

$$
\begin{aligned}
&T_o = 1 \\
&T_1 = \omega \\
&T_2 = 2\omega T_{n-1} - T_{n-2} \\
&\vdots
\end{aligned}
\tag{9.39}
$$

These polynomials exist on a domain from $-1 < \omega < 1$, and appear as shown below:



Plot of the first six Chebycheff Polynomials between −1 and 1.

Sub-band curve fitting is accomplished by selecting a region of the FRF for fitting, and subtracting out the neighboring modes and residuals similar to the MDOF circle fitting technique.

By altering the weighting equations, W_a and W_b, so that the out-of-band weight is zero, subrange fitting is easily accomplished.