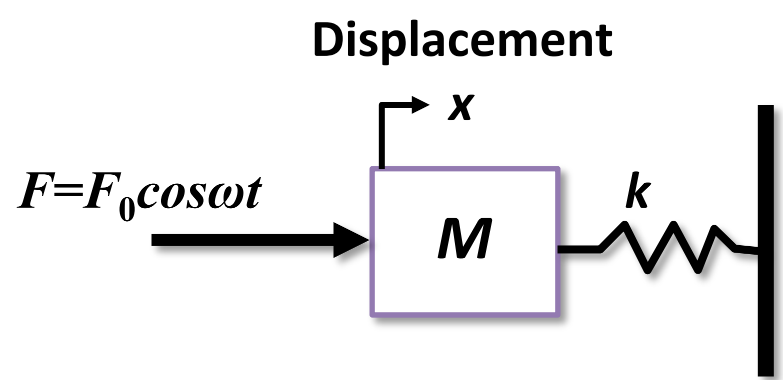
Harmonic excitation of undamped systems

- Consider the usual spring mass damper system with applied force $F(t)=F_0\cos\omega t$
- ω is the driving frequency
- F_0 is the magnitude of the applied force
- We take $c = 0$ to start with



Equation of motion

- **Solution is the sum of *homogenous* and *particular* solution**

$$x_T = x_h(t) + x_p(t)$$

- **The *particular* solution assumes the form of forcing function (physically the input wins):**

$$x_p(t) = X\cos(\omega t)$$

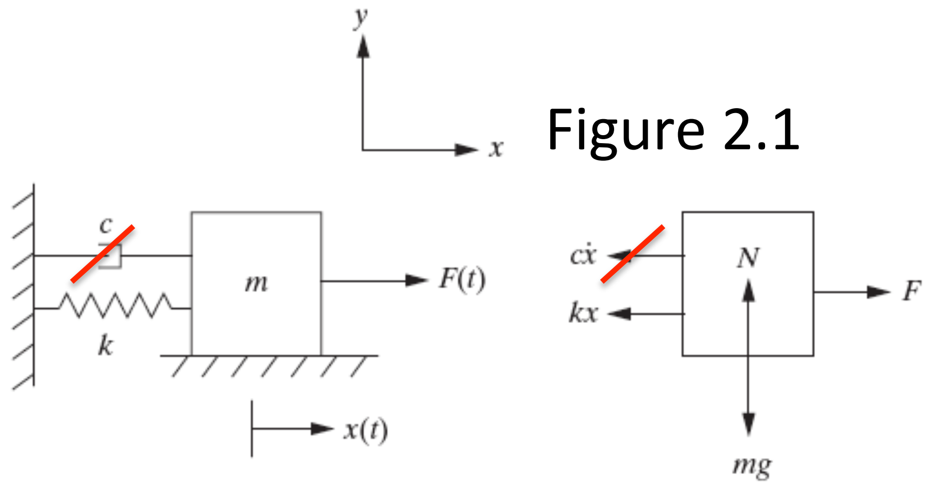


Figure 2.1

$$\frac{m\ddot{x}(t)}{m} = \frac{-kx(t)}{m} + \frac{F_0}{m}\cos(\omega t)$$

$$\ddot{x}(t) + \omega_n^2 x(t) = f_0\cos(\omega t)$$

$$\text{where} \quad f_0 = F_0/m, \quad \omega_n = \sqrt{k/m}$$

Substitute the *particular* solution into the equation of motion:

$$x_p(t) = X\cos(\omega t) \quad \Rightarrow \quad \ddot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t)$$

Lets try this in class:

$$\overbrace{-\omega^2 X\cos\omega t}^{\ddot{x}_p} + \overbrace{\omega_n^2 X\cos\omega t}^{\omega_n^2 x_p} = f_0 \cos\omega t$$

$$\text{solving yields: } X = \frac{f_0}{\omega_n^2 - \omega^2}$$

Thus, the <u>particular</u> solution has the form:

$$x_p(t) = \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t)$$

Full solution: Homogeneous and Particular to the general solution

$$x_T = x_h(t) + x_p(t)$$

$$x(t) = \underbrace{A_1 \sin\omega_n t + A_2 \cos\omega_n t}_{\text{homogeneous}} + \overbrace{\frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t}^{\textit{particular}} \qquad (2.8)$$

A_1 and A_2 are constants of integration.

How do we obtain the constants of integration?

Apply the initial conditions to evaluate the constants

$$x(0) = A_1 \sin 0 + A_2 \cos 0 + \frac{f_0}{\omega_n^2 - \omega^2} \cos 0 = A_2 + \frac{f_0}{\omega_n^2 - \omega^2} = x_0$$

$$\boxed{\Rightarrow \underline{A_2 = x_0 - \frac{f_0}{\omega_n^2 - \omega^2}}}$$

$$\dot{x}(0) = \omega_n (A_1 \cos 0 - A_2 \sin 0) - \frac{f_0}{\omega_n^2 - \omega^2} \sin 0 = \omega_n A_1 = v_0$$

$$\boxed{\Rightarrow \underline{A_1 = \frac{v_0}{\omega_n}}}$$

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right) \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

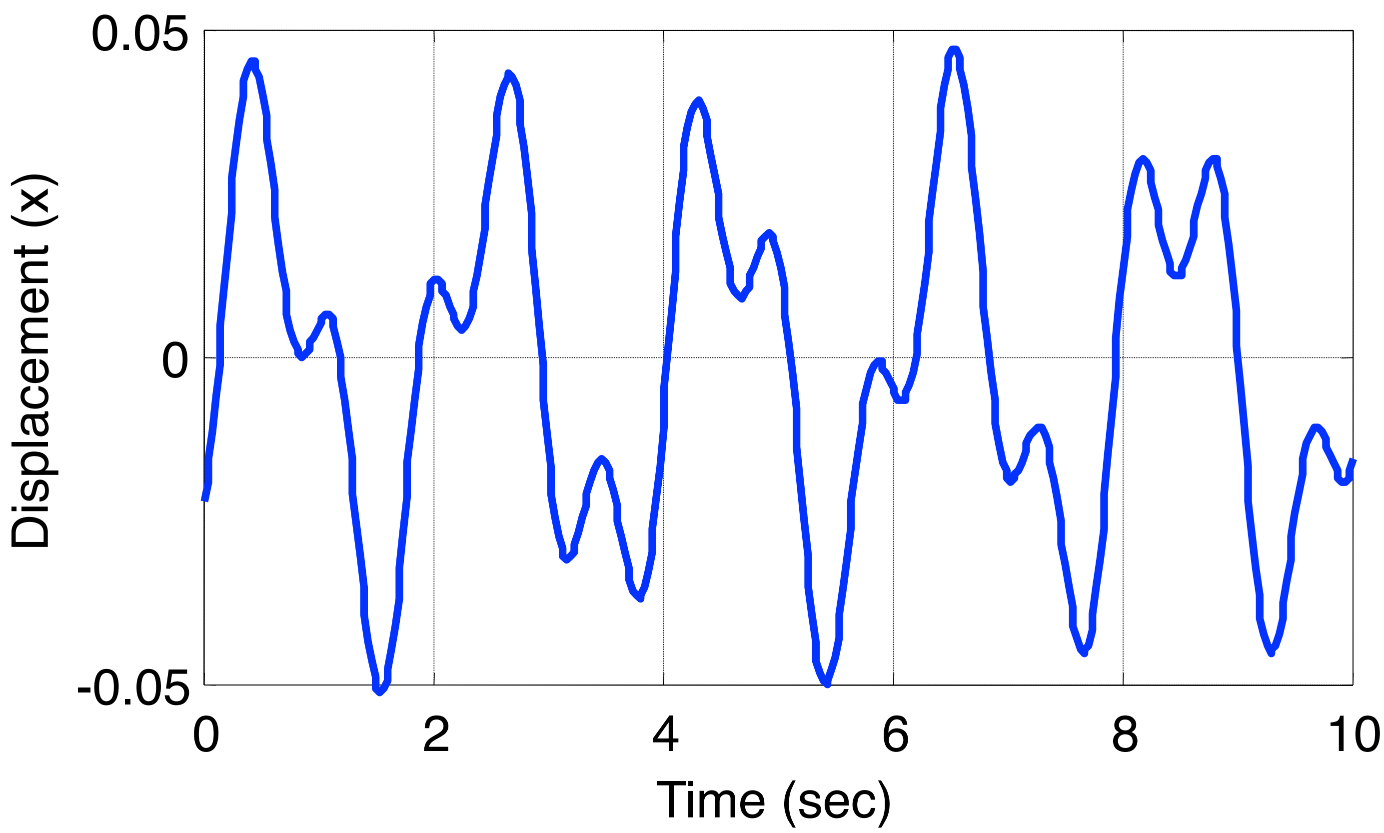
What is interesting about this solution?

Comparison of the free and forced response

- **Sum of two harmonic terms of different frequency**
- **Free response has amplitude and phase affected by forcing function**
- **Our solution is <u>not</u> defined for $\omega_n = \omega$ because it produces division by 0.**
- **If forcing frequency is close to natural frequency the amplitude of *particular* solution is very large**

$$x(t) = \frac{v_0}{\omega_n}\sin\omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right)\cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

Response for m=100 kg, k=1000 N/m, F=100 N, ω = ωn +5
v_0=0.1m/s and x_0= -0.02 m.



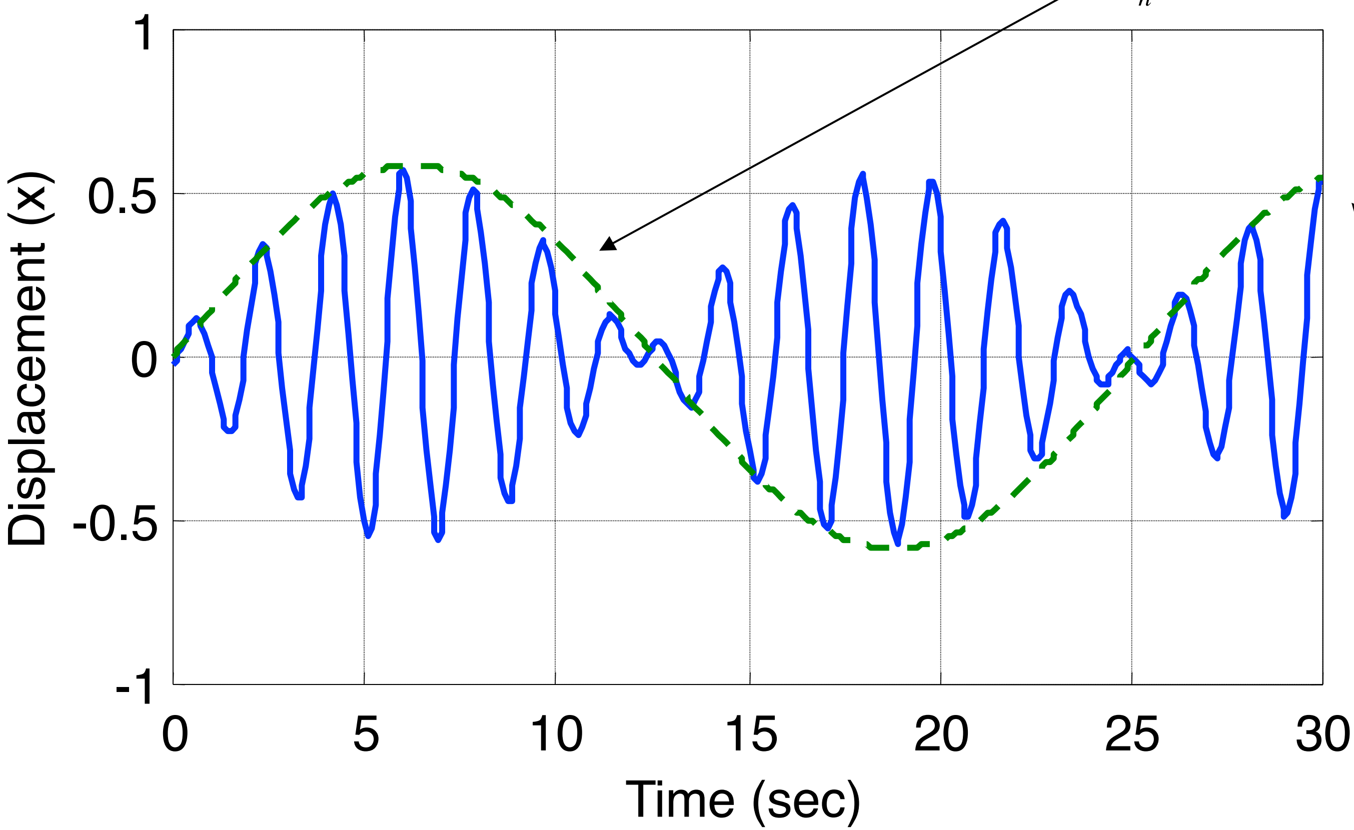
Note the obvious presence of two harmonic signals

What happens when ω is near ω_n?

$$x(t) = \frac{2f_0}{\omega_n^2 - \omega^2}\sin\left(\frac{\omega_n - \omega}{2}t\right)\sin\left(\frac{\omega_n + \omega}{2}t\right) \quad (2.13)$$

When the drive frequency and natural frequency are close a beating phenomena occurs

$$\frac{2f_0}{\omega_n^2 - \omega^2}\sin\left(\frac{\omega_n - \omega}{2}t\right)$$



Larger amplitude

Called beating

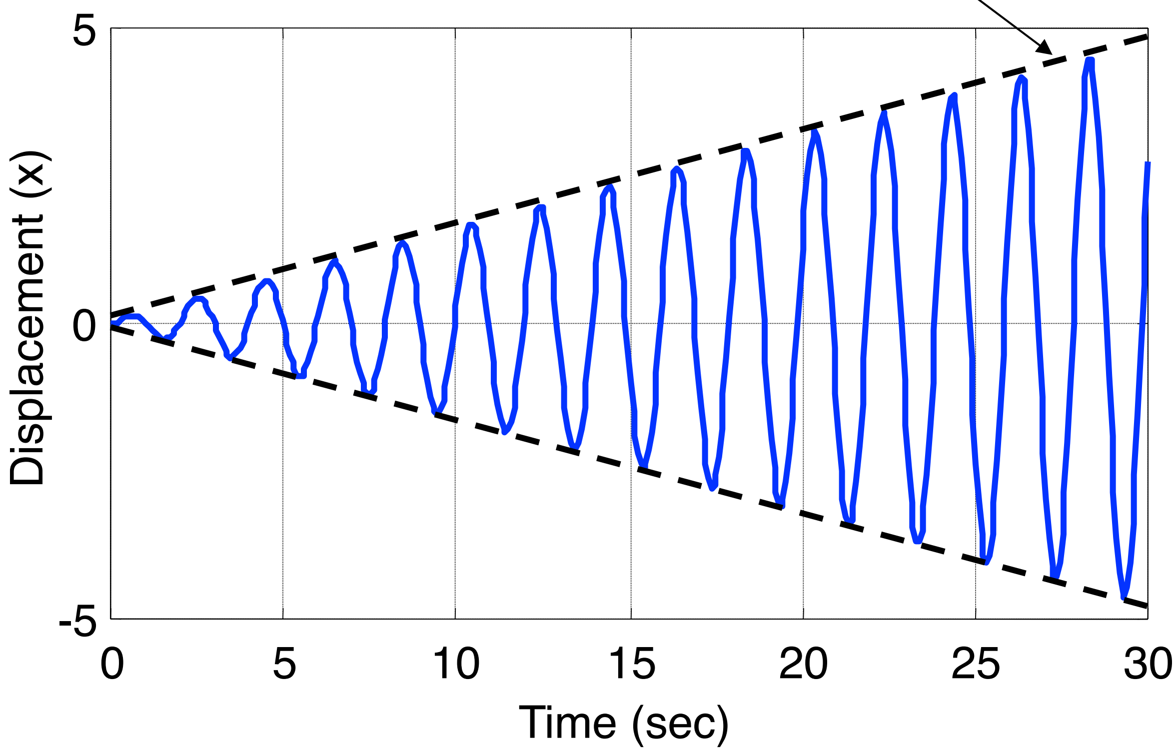
What happens when ω is ω_n?

$$x_p(t) = tX\sin(\omega t)$$

substitute into eq. and solve for X

$$X = \frac{f_0}{2\omega}$$

$$x(t) = A_1 \sin\omega t + A_2 \cos\omega t + \overbrace{\frac{f_0}{2\omega} t \sin(\omega t)}^{\text{grows with out bound}}$$



When the drive frequency and natural frequency are the same the amplitude of the vibration grows without bounds. This is known as a *resonance* condition. The most important concept in Chapter 2!

Example 2.1.1: Compute and plot the response for m=10 kg, k=1000 N/m, x_0=0,v_0=0.2 m/s, F=23 N, ω =2ω_n.

$$x(t) = \frac{v_0}{\omega_n}\sin\omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right)\cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000 \text{ N/m}}{10 \text{ kg}}} = 10 \text{ rad/s}, \; \omega = 2\omega_n = 20 \text{ rad/s}$$

$$f_0 = \frac{F}{m} = \frac{23 \text{ N}}{10 \text{ kg}} = 2.3 \text{ N/kg}, \quad \frac{v_0}{\omega_n} = \frac{0.2 \text{ m/s}}{10 \text{ rad/s}} = 0.02 \text{ m}$$

$$\frac{f_0}{\omega_n^2 - \omega^2} = \frac{2.3 \text{ N/kg}}{(10^2 - 20^2) \text{ rad}^2/\text{s}^2} = -7.9667 \times 10^{-3} \text{ m}$$

Equation (2.11) then yields:

$$\underline{x(t) = 0.02\sin 10t + 7.9667 \times 10^{-3}(\cos 10t - \cos 20t)}$$

Example 2.1.2: Given zero initial conditions a harmonic input of 10 Hz with 20 N magnitude and k= 2000 N/m, and measured response amplitude of 0.1m, compute the mass of the system.

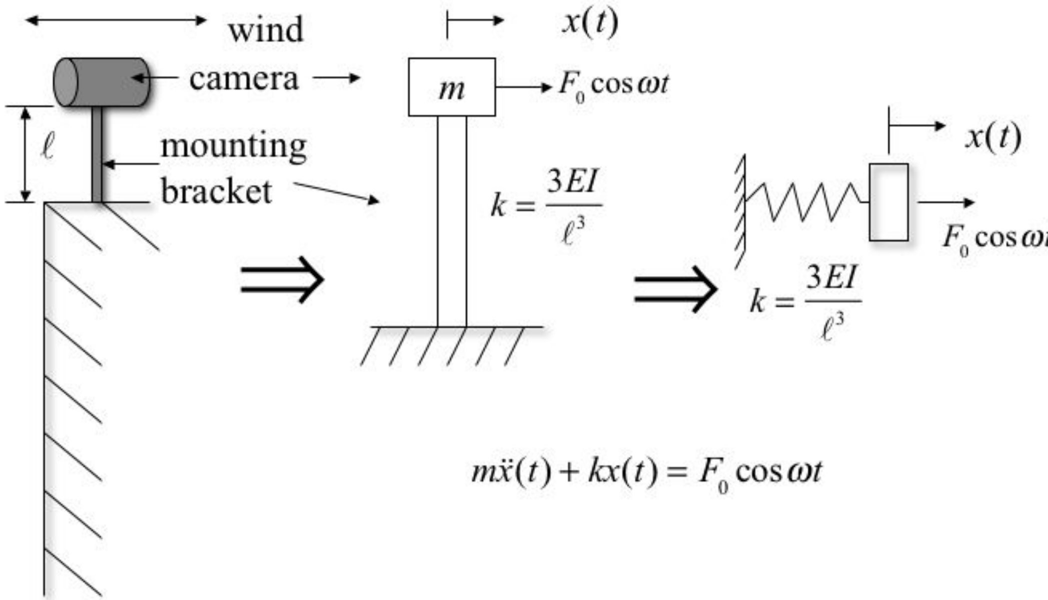
$$x(t) = \frac{f_0}{\omega_n^2 - \omega^2}\left(\cos 20\pi t - \cos\omega_n t\right) \text{ for zero initial conditions}$$

$$\text{trig identity} \Rightarrow x(t) = \underbrace{\frac{2f_0}{\omega_n^2 - \omega^2}}_{0.1\text{ m}} \sin\left(\frac{\omega_n - \omega}{2}t\right)\sin\left(\frac{\omega_n + \omega}{2}t\right)$$

$$\Rightarrow \frac{2f_0}{\omega_n^2 - \omega^2} = 0.1 \Rightarrow \frac{2(20/m)}{\left(2000/m\right) - (20\pi)^2} = 0.1$$

$$\underline{m = 0.45\text{ kg}}$$

Example 2.1.3: Design a rectangular mount for a security camera.



0.02 x 0.02 m
in cross section

Compute $\ell > 0.5$ m so that the mount keeps the camera from vibrating more then 0.01 m of maximum amplitude under a wind load of 15 N at 10 Hz. The mass of the camera is 3 kg.

Solution: Modeling the mount and camera as a beam with a tip mass, and the wind as harmonic, the equation of motion becomes:

$$m\ddot{x} + \frac{3EI}{\ell^3}x(t) = F_0 \cos\omega t$$

From strength of materials: $I = \dfrac{bh^3}{12}$

Thus the frequency expression is: $\omega_n^2 = \dfrac{3Ebh^3}{12m\ell^3} = \dfrac{Ebh^3}{14m\ell^3}$

Here we are interested computing ℓ that will make the amplitude less then 0.01m:

$$\left|\frac{2f_0}{\omega_n^2 - \omega^2}\right| < 0.01 \Rightarrow \begin{cases} (a) \quad -0.01 < \dfrac{2f_0}{\omega_n^2 - \omega^2}, \text{ for } \omega_n^2 - \omega^2 < 0 \\ (b) \quad \dfrac{2f_0}{\omega_n^2 - \omega^2} < 0.01, \text{ for } \omega_n^2 - \omega^2 > 0 \end{cases}$$

Case (a) (assume aluminium for the material)

$$-0.01 < \frac{2f_0}{\omega_n^2 - \omega^2} \Rightarrow 2f_0 < 0.01\omega^2 - 0.01\omega_n^2 \Rightarrow 0.01\omega^2 - 2f_0 > 0.01\frac{Ebh^3}{4m\ell^3}$$

$$\Rightarrow \ell^3 > 0.01\frac{Ebh^3}{4m(0.01\omega^2 - 2f_0)} = 0.321 \Rightarrow \ell > 0.6848 \text{ m}$$

Case (b) (assume aluminium for the material)

$$\frac{2f_0}{\omega_n^2 - \omega^2} < 0.01 \Rightarrow 2f_0 < 0.01\omega_n^2 - 0.01\omega^2 \Rightarrow 2f_0 + 0.01\omega^2 < 0.01\frac{Ebh^3}{4m\ell^3}$$

$$\Rightarrow \ell^3 < 0.01\frac{Ebh^3}{4m(2f_0 + 0.01\omega^2)} = 0.191 \Rightarrow \ell < 0.576 \text{ m}$$

Remembering the constraint that the length must be at least 0.5 m, (a) and (b) yield

$$0.5 < \ell < 0.576, \quad \text{or} \quad \ell > 0.6848 \text{ m}$$

Less material is usually desired, so chose case b, say ℓ = 0.55 m.

To check, note that

$$\omega_n^2 - \omega^2 = \frac{3Ebh^3}{12m\ell^3} - \left(20\pi\right)^2 = 1742 > 0$$

Thus the case a condition is met.

Next check the mass of the designed beam to insure it does not change the frequency. Note it is less then *m.*

$$\begin{aligned} m &= \rho \ell b h \\ &= (2.7 \times 10^3)(0.55)(0.02)(0.02) \\ &= 0.594 \text{ kg} \end{aligned}$$

A harmonic force may also be represented by sine or a complex exponential. How does this change the solution?

$$m\ddot{x}(t) + kx(t) = F_0 \sin \omega t \quad \text{or} \quad \ddot{x}(t) + \omega_n^2 x(t) = f_0 \sin \omega t \qquad (2.18)$$

The particular solution then becomes a sine:

$$x_p(t) = X \sin \omega t \qquad (2.19)$$

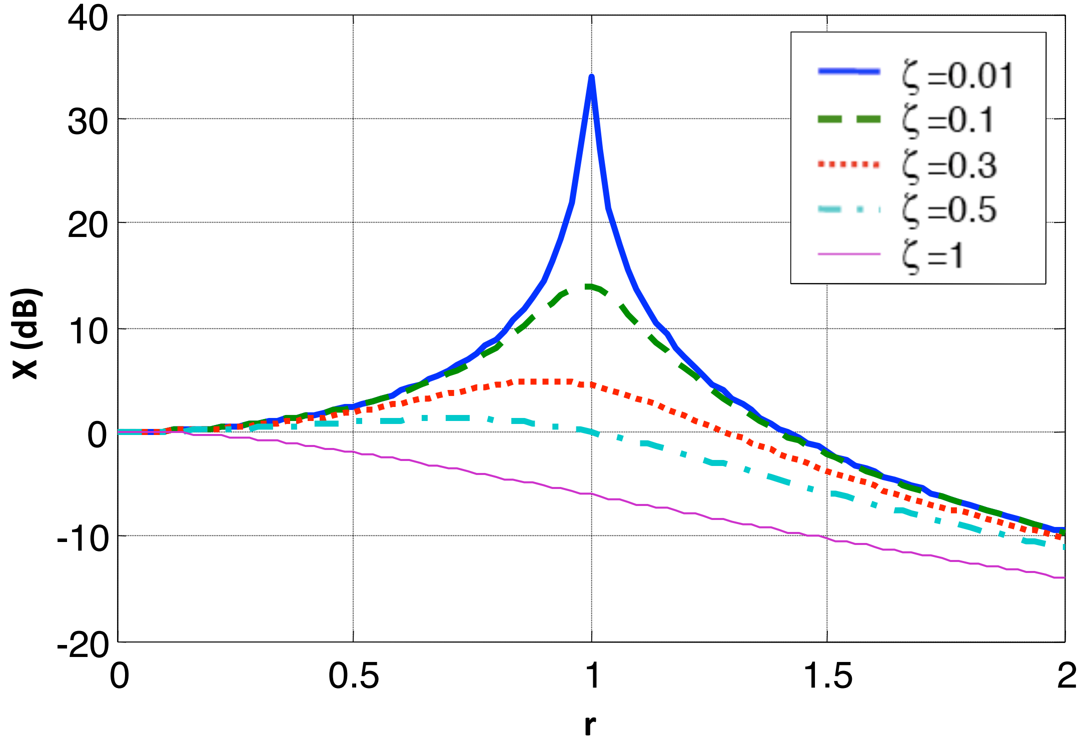
Substitution of (2.19) into (2.18) yields:

$$x_p(t) = \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t$$

Solving for the homogenous solution and evaluating the constants yields

$$x(t) = x_0 \cos \omega_n t + \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \frac{f_0}{\omega_n^2 - \omega^2} \right) \sin \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t \qquad (2.25)$$

Section 2.2 Harmonic Excitation of Damped Systems



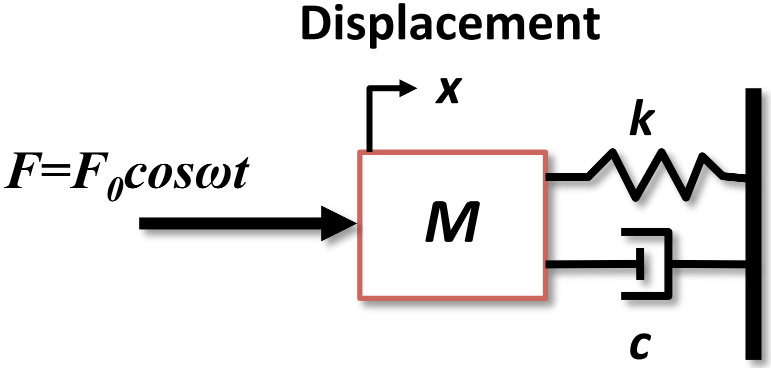
Extending resonance and response calculation to damped systems

Harmonic excitation of damped systems

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$$

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=f(t)$$

$$f(t)=F(t)/m$$



Assumed solution: $x_p(t)=\boxed{\underbrace{X\cos(\omega t-\theta)}_{\text{now includes a phase shift}}}$

Let x_p have the form

$$x_p = X\cos(\omega t - \theta) \Rightarrow \boxed{x_p(t) = A_s\cos\omega t + B_s\sin\omega t}$$

$$X = \sqrt{{A_s}^2 + {B_s}^2}, \theta = \tan^{-1}\left(\frac{B_s}{A_s}\right)$$

Take the derivatives of the assumed solution with respect to *t*

$$\dot{x}_p = -\omega A_s\sin\omega t + \omega B_s\cos\omega t$$

$$\ddot{x}_p = -\omega^2 A_s\cos\omega t - \omega^2 B_s\sin\omega t$$

Note that we are using the rectangular form, but we could use one of the other forms of the solution.

Substitute into the equations of motion

$$\dot{x}_p = -\omega A_s \sin\omega t + \omega B_s \cos\omega t$$
$$\ddot{x}_p = -\omega^2 A_s \cos\omega t - \omega^2 B_s \sin\omega t$$

$$\Rightarrow \quad \ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = f(t)$$

$$(-\omega^2 A_s + 2\zeta\omega_n \omega B_s + \omega_n^2 A_s - f_0)\cos\omega t$$
$$+\left(-\omega^2 B_s + 2\zeta\omega_n \omega A_s + \omega_n^2 B_s\right)\sin\omega t = 0$$

for all time. Specifically for $t = 0, 2\pi/\omega \Rightarrow$

$$(\omega_n^2 - \omega^2)A_s + (2\zeta\omega_n\omega)B_s = f_0$$
$$(-2\zeta\omega_n\omega)A_s + (\omega_n^2 - \omega^2)B_s = 0$$

Write as a matrix equation:

$$\begin{bmatrix} (\omega_n^2 - \omega^2) & 2\zeta\omega_n\omega \\ -2\zeta\omega_n\omega & (\omega_n^2 - \omega^2) \end{bmatrix} \begin{bmatrix} A_s \\ B_s \end{bmatrix} = \begin{bmatrix} f_0 \\ 0 \end{bmatrix}$$

Solving for A_s and B_s:

$$A_s = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$B_s = \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

Substitute the values of A_s and B_s into x_p:

$$x_p(t) = \underbrace{\frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}}_{X} \cos(\omega t - \underbrace{\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)}_{\theta})$$

Add homogeneous and particular to get total solution: $x_T = x_h(t) + x_p(t)$

$$x(t) = \underbrace{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}_{\text{homogeneous or transient solution}} + \underbrace{X\cos(\omega t - \theta)}_{\text{particular or steady state solution}}$$

See Eqn. 2.38

Note: that A and ϕ will not have the same values as in Ch 1, for the free response. Also as t gets large, transient dies out.

Response of a linear system to harmonic inputs.

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos(\omega t) \quad and \quad IC : x_o \, , \dot{x}_o$$

$X(t)$ = transient response (t) + steady-state response (t)

Transient part:

a) will die out with damping

b) For under-damped case, response is an exponentially decaying harmonic function with frequency ω_d (damped natural frequency)

This part of the response is harmonic with the same frequency as the input, *ω*

<u>Frequency Domain Analysis is the computation of the steady-state response under harmonic inputs.</u>

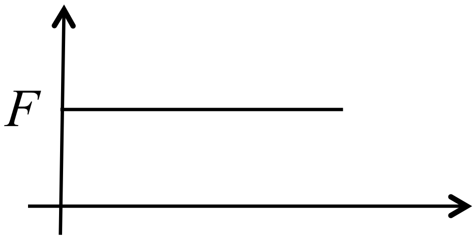
Things to notice about damped forced response:

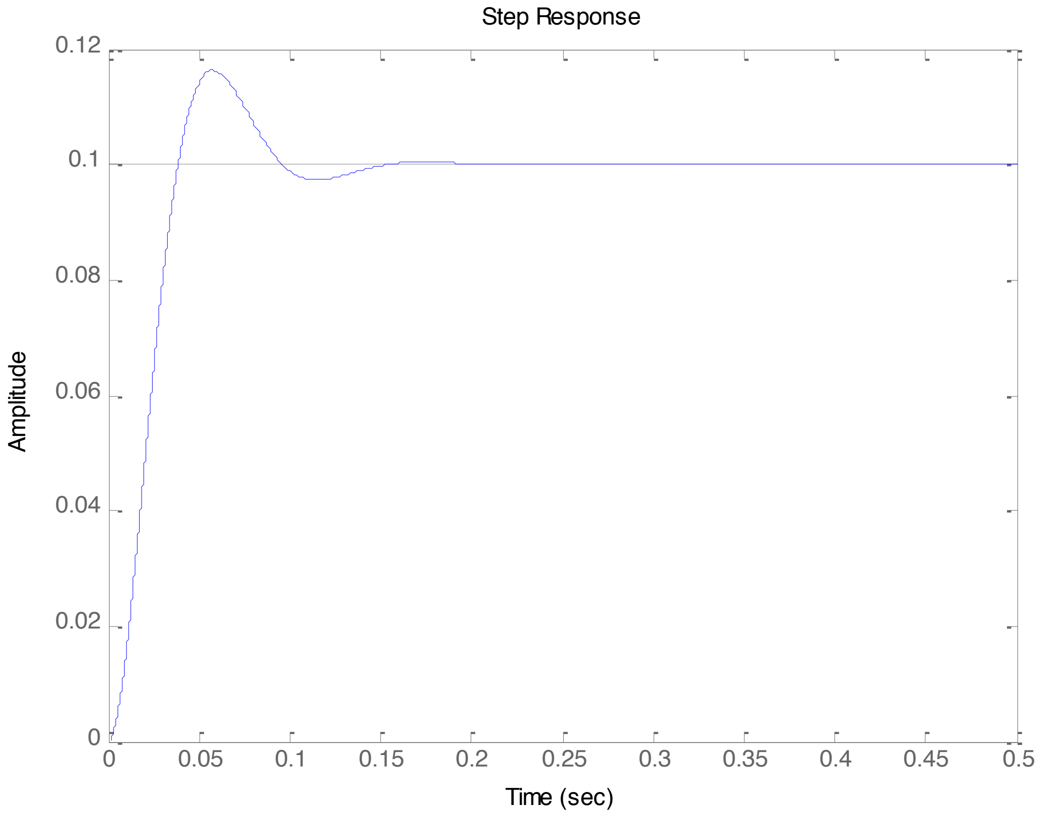
- If $\zeta = 0$, undamped equations result
- Steady state solution prevails for large t
- Often we ignore the transient term (how large is ζ, how long is t?)
- Coefficients of transient terms (constants of integration) are affected by the initial conditions AND the forcing function
- For underdamped systems at resonance the, amplitude is finite.

Before we move to FDA:

Example:

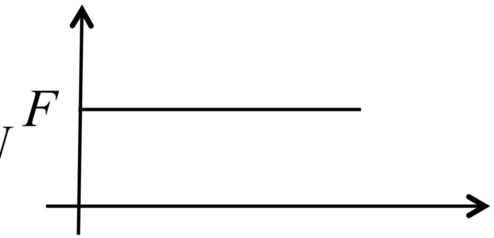
$$\zeta = 0.5\,(50\%)\ ,\ \omega_n = 10\times 2\pi \frac{rad}{\sec}\ ,\ k = 1000\frac{N}{m}\ ,\ F = 100N$$

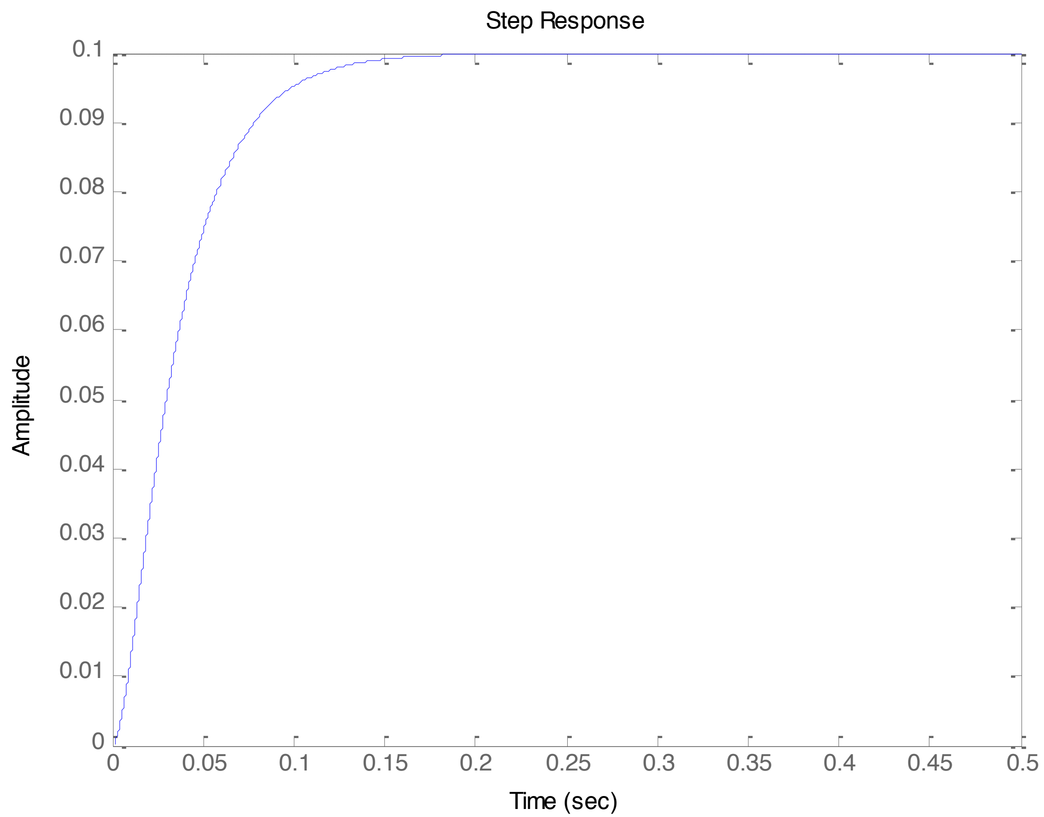




Example:

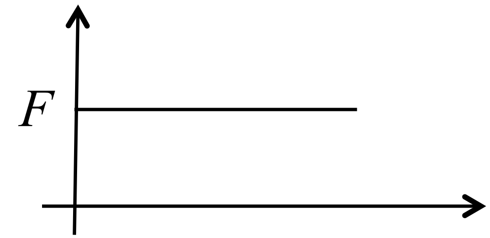
$$\zeta = 1.2\ (120\%)\ ,\ \omega_n = 10 \times 2\pi \frac{rad}{\sec}\ ,\ k = 1000 \frac{N}{m}\ ,\ F = 100N$$



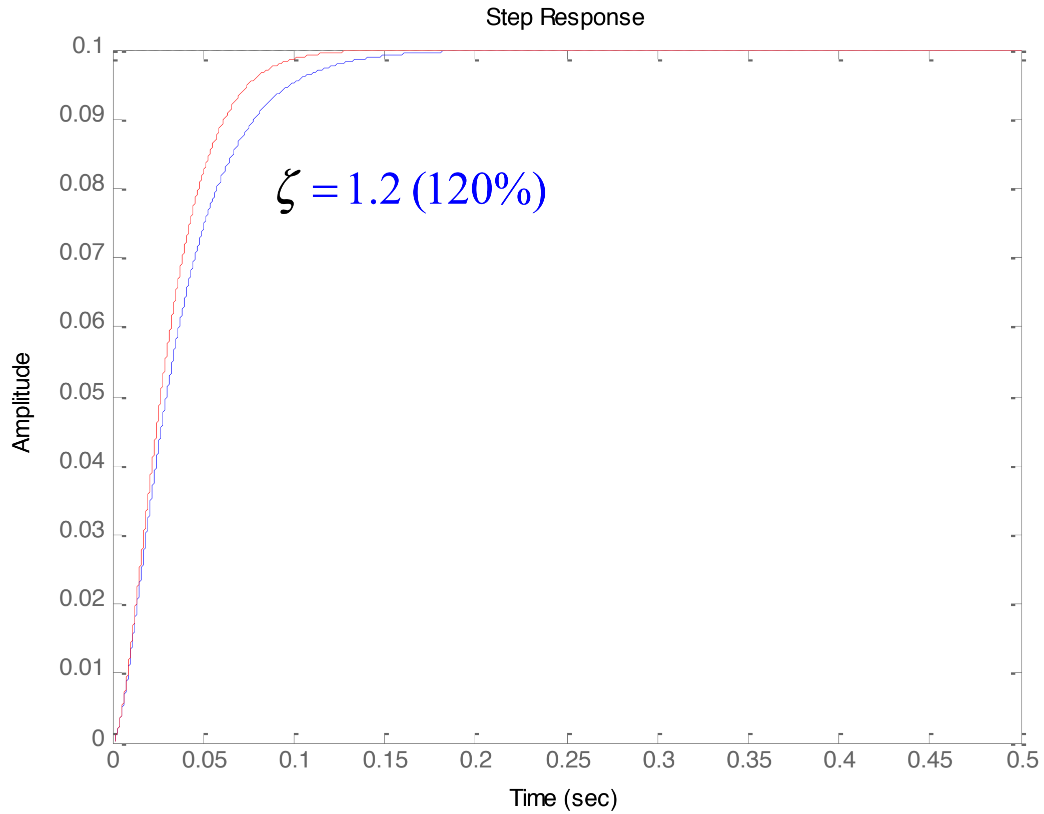


Example:

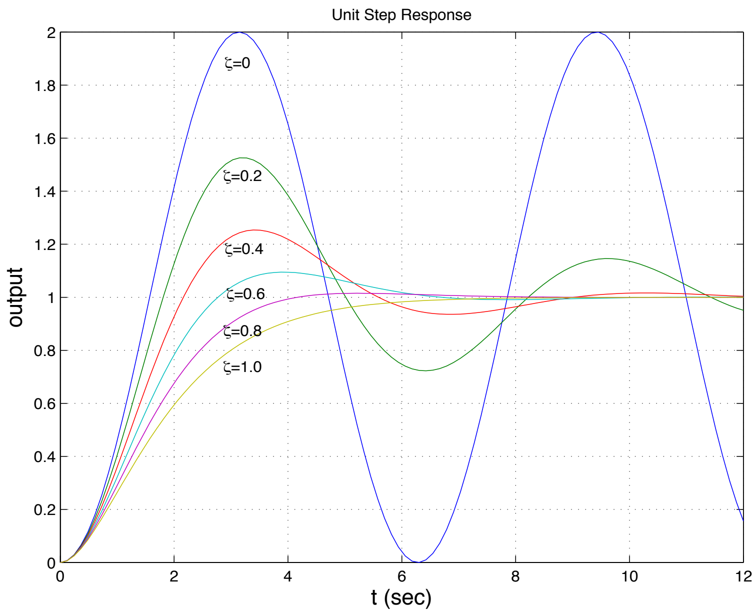
$$\omega_n = 10 \times 2\pi \frac{rad}{\sec}\ ,\ k = 1000\frac{N}{m}\ ,\ F = 100N$$

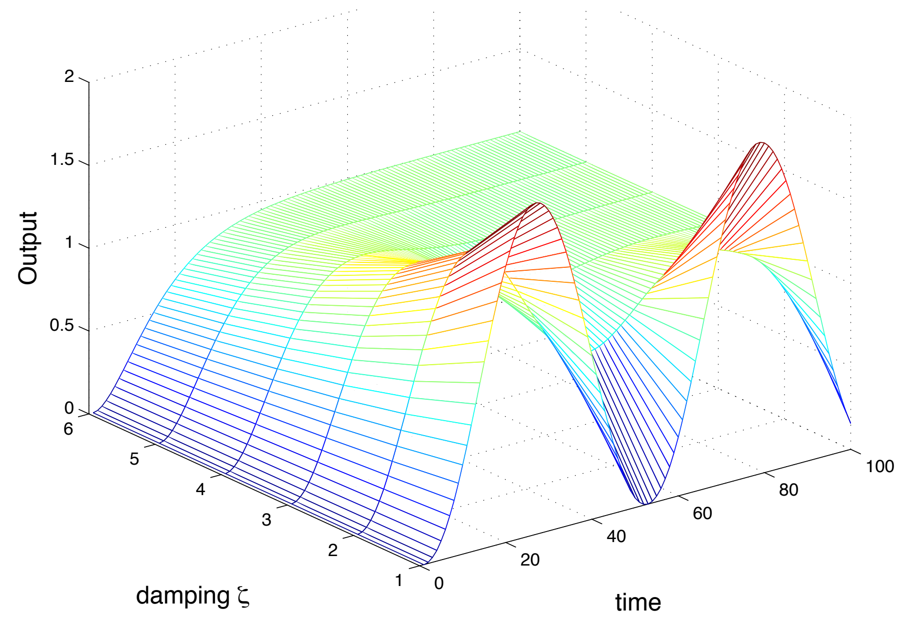


$\zeta = 1.0\ (100\%)$



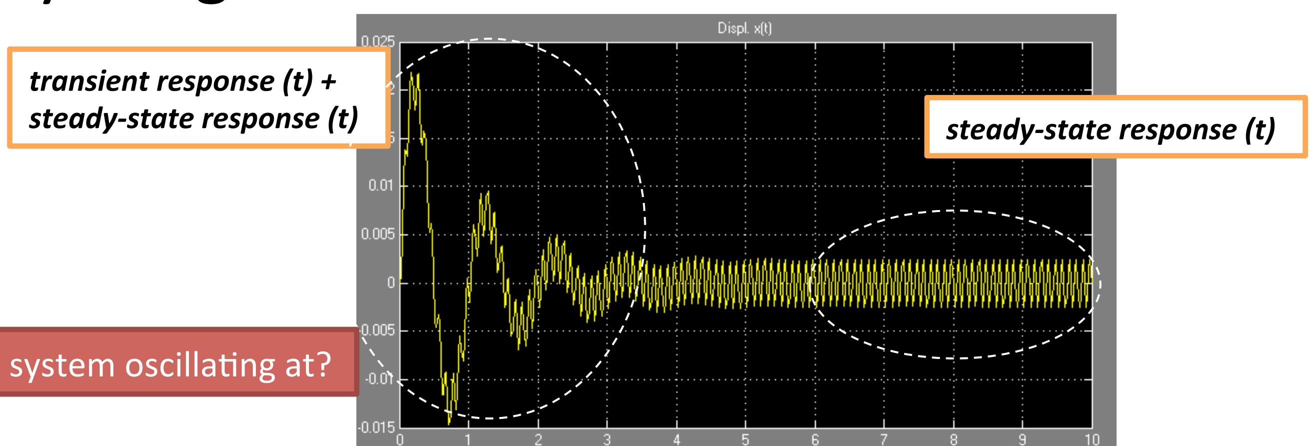
Multiple cases:





Proceeding with ignoring the transient

- Always check to make sure the transient is not significant
- For example, transients are very important in earthquakes
- However, in many machine applications transients may be ignored



What is this system oscillating at?

Proceeding with ignoring the transient

Magnitude:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \qquad (2.39)$$

Frequency ratio:

$$r = \frac{\omega}{\omega_n}$$

Non dimensional Form:

$$\frac{Xk}{F_0} = \frac{X\omega_n^2}{f_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

(2.40)

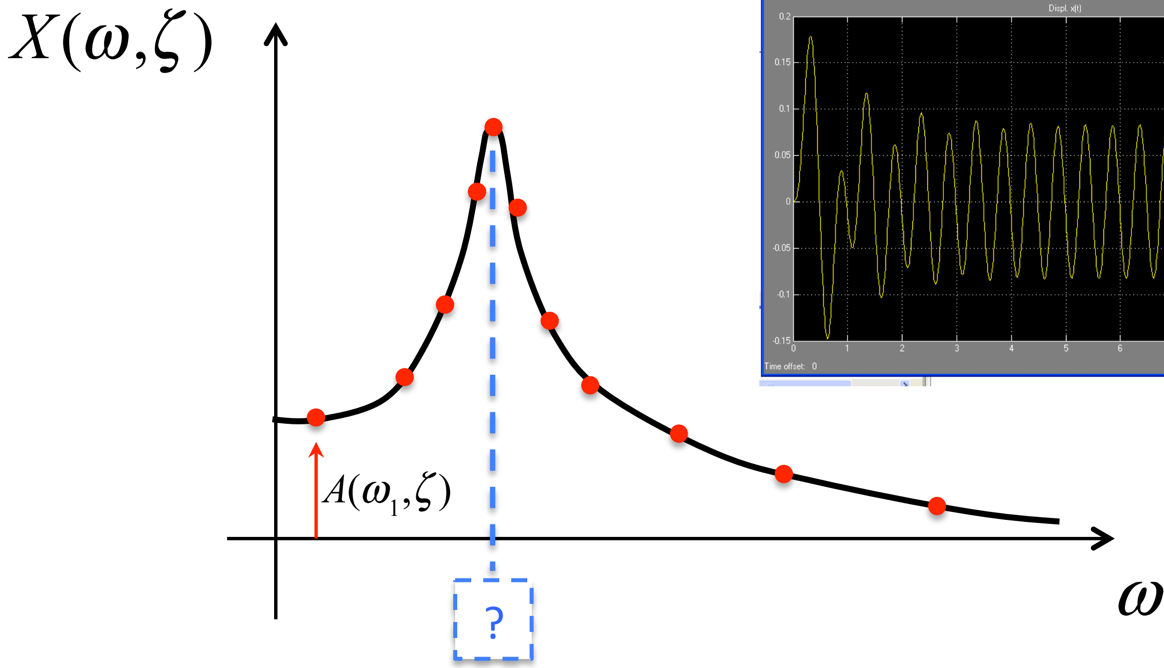
Phase:

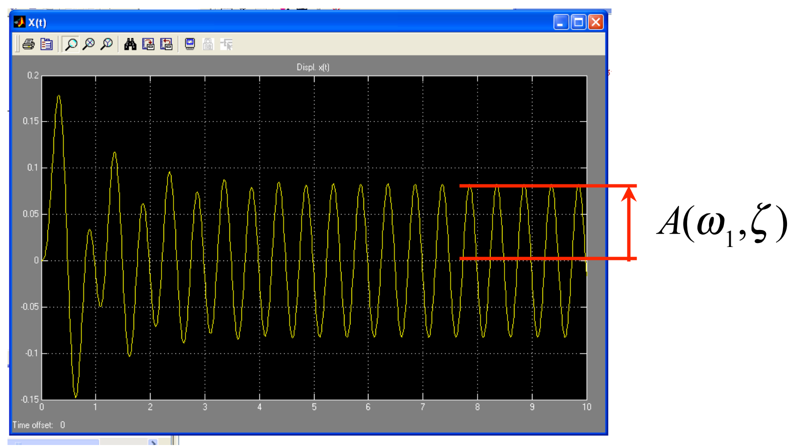
$$\theta = \tan^{-1}\left(2\zeta r \big/ 1 - r^2 \right)$$

$$x_p(t) = \underbrace{\frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}}_{X} \cos(\omega t - \underbrace{\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)}_{\theta})$$

Frequency domain analysis is to compute amplitude (and phase) as a function of input frequency:

$$X = \frac{F_0}{\sqrt{\left(k-\omega^2 m\right)^2 + \left(c\omega\right)^2}} = \frac{F_0}{k}\frac{1}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\zeta\,\omega/\omega_n\right)^2}}$$





Further examination of the magnitude plot reveals some interesting facts

- Resonance is close to $r = 1$
- For $\zeta = 0$, $r = 1$ defines resonance
- As ζ grows resonance moves to $r < 1$, and X decreases
- The exact value of r *for resonance*, can be found by differentiating the magnitude with respect to r

$$\frac{Xk}{F_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

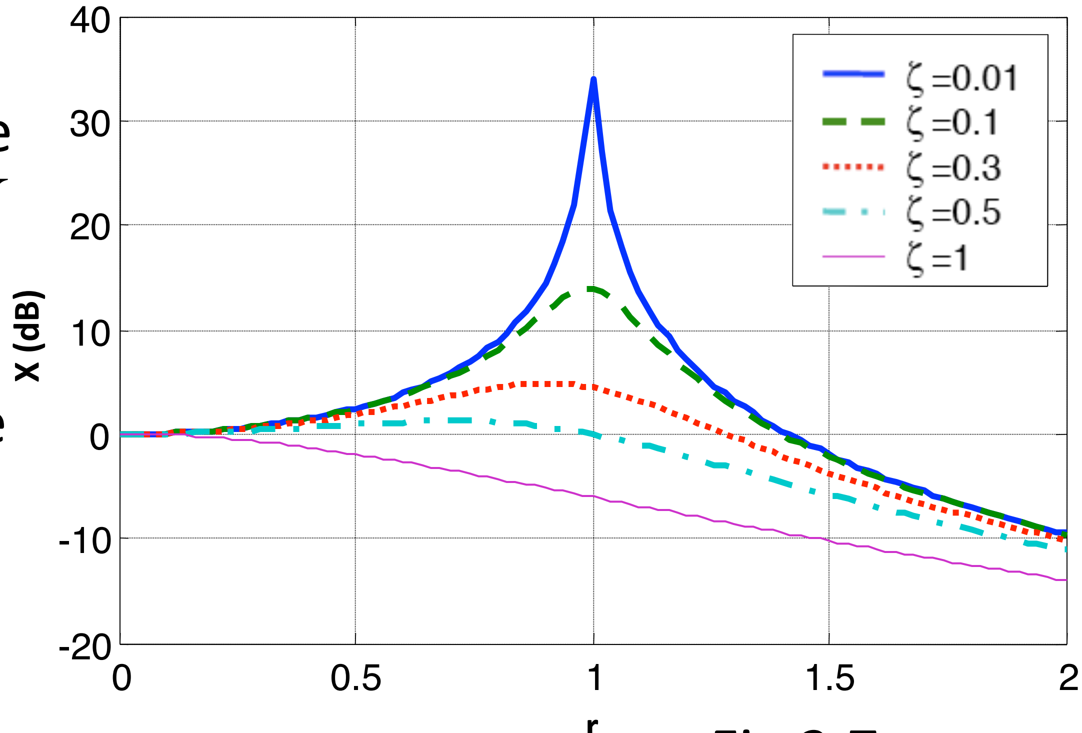


Fig 2.7

Phase plot also reveals characteristics of the response

- Resonance occurs at $\theta = \pi/2$
- The phase changes more rapidly when the damping is small
- From low to high values of r the phase always changes by 180^0 or π radians

$$\theta = \tan^{-1}\left({}^{2\zeta r}\!\big/_{1-r^2} \right)$$

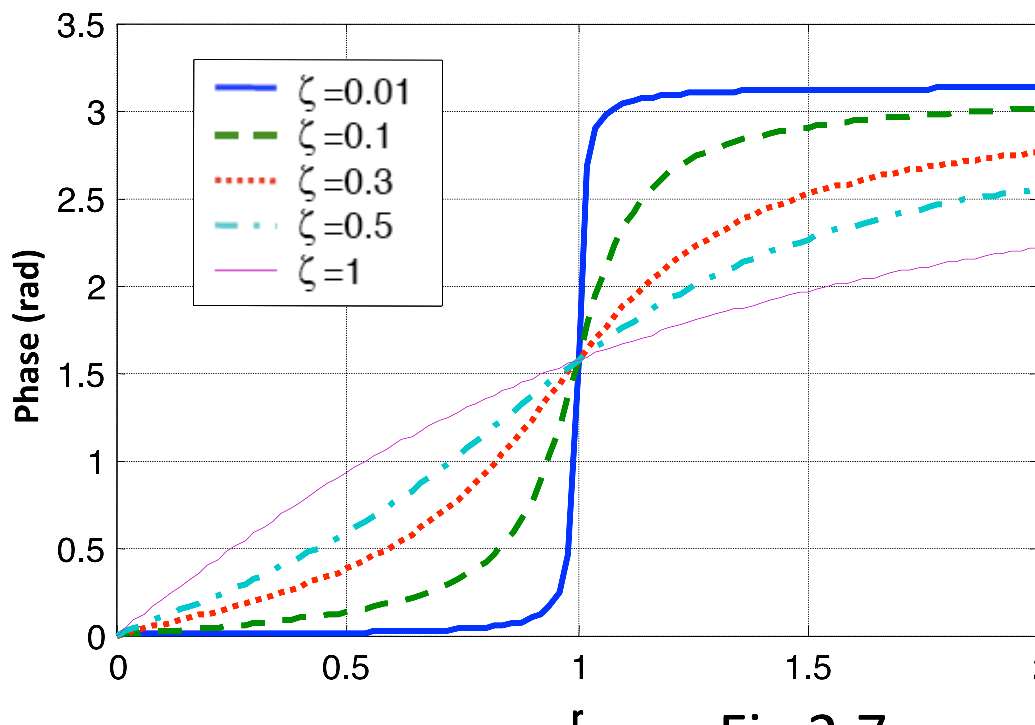
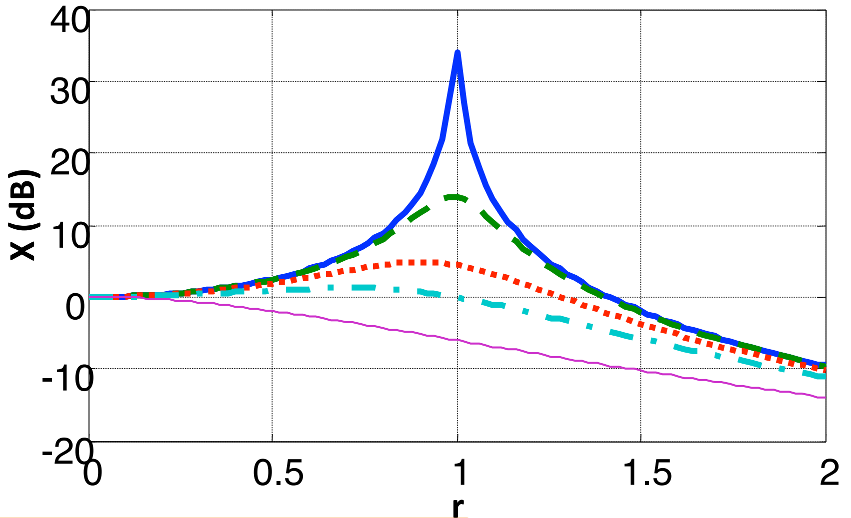


Fig 2.7

Example 2.2.3: Compute max peak by differentiating:

$$\frac{d}{dr}\left(\frac{Xk}{F_0}\right) = \frac{d}{dr}\left(\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\right) =$$

$$r_{\text{peak}} = \sqrt{1-2\zeta^2} < 1 \;\Rightarrow\; \zeta < 1/\sqrt{2}$$

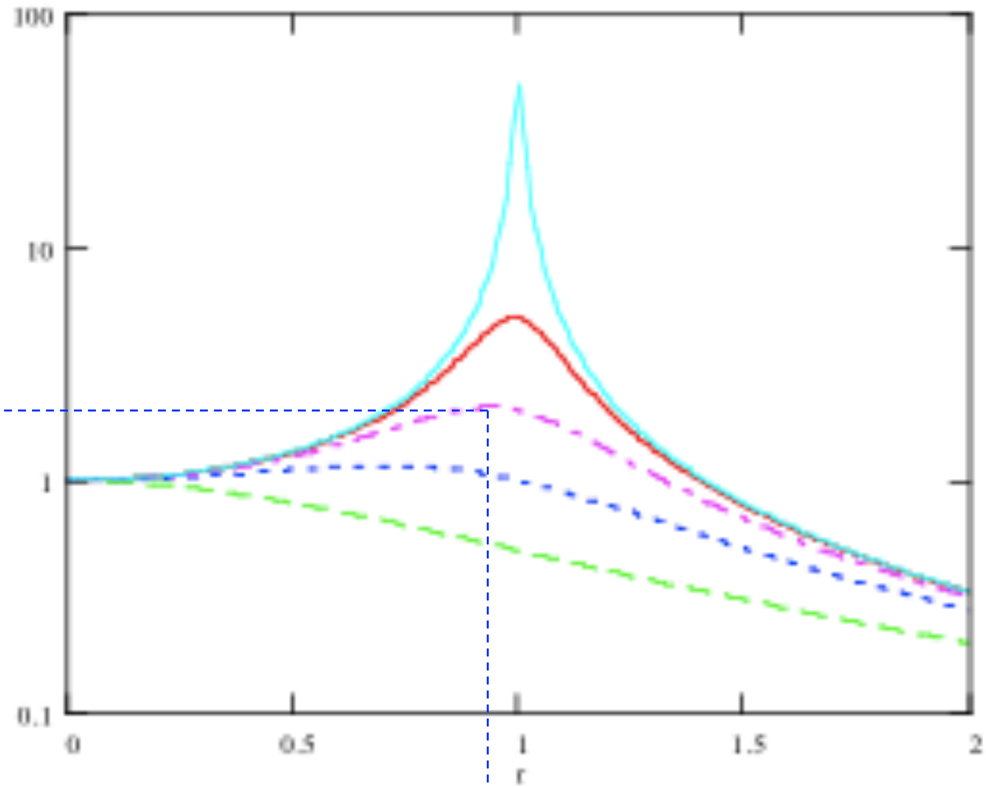


$$\left(\frac{Xk}{F_0}\right)_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\boxed{r_{\text{peak}} = \sqrt{1-2\zeta^2} \;\Rightarrow\; \omega_p = \omega_n\sqrt{1-2\zeta^2}}$$

Bandwidth and Peak resonance

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$



$$\frac{\omega_p}{\omega_n} = \sqrt{1-2\zeta^2}$$

Effect of Damping on peak value

- The top plot shows how the peak value becomes very large when the damping level is small
- The lower plot shows how the frequency at which the peak value occurs reduces with increased damping
- Note that the peak value is only defined for values $\zeta<0.707$

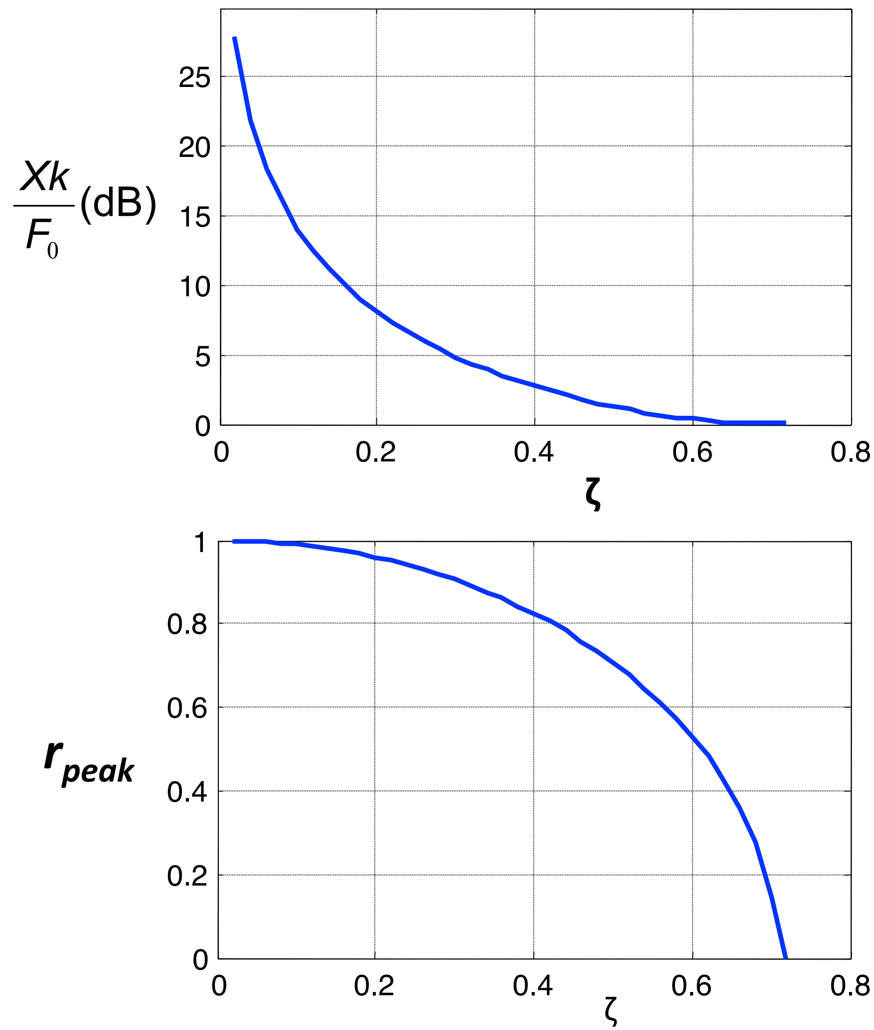


Fig 2.9