Assumed mode summation and Finite Element Method

- The mode shapes of a system are assumed to be the linear sum of shape functions

$$r(x,t)=\sum_{i=1}^{k}\phi_i(x)q_i(t)$$

These are the generalized coordinates representing a weighting of the mode shapes

These are shape functions for continuous systems (beams, rods, etc).

$$\dot{r}(x,t)=\sum_{i=1}^{k}\phi_i(x)\dot{q}_i(t)$$

Kinetic energy can be expressed as:

$$T=\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\dot{q}_i\dot{q}_j\int\phi_i(x)\phi_j(x)dm$$

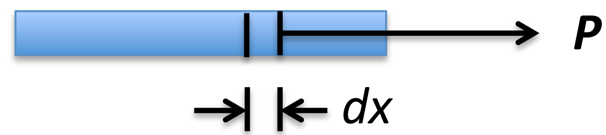
Now we can define generalized mass from the kinetic energy equation:

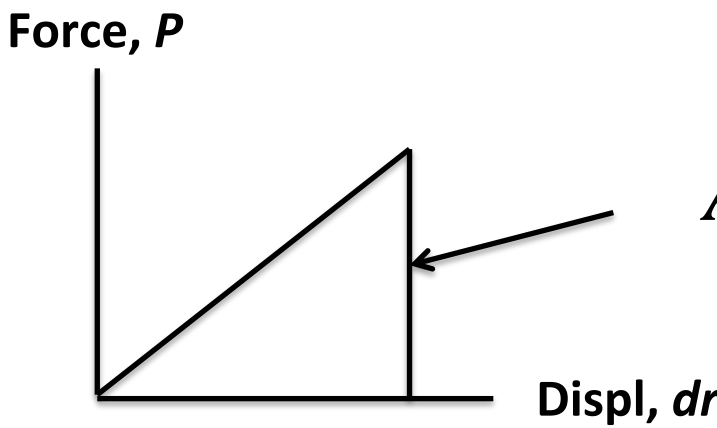
$$T = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\dot{q}_i\dot{q}_j \underbrace{\int \phi_i(x)\phi_j(x)\,dm}$$

$$m_{ij} = \int \phi_i(x)\phi_j(x)\,dm$$

We can now write the kinetic energy as: $T = \frac{1}{2}(\dot{q})^T[m](\dot{q})$

If we consider the axial vibration of a rod, we can define generalized stiffness from the potential energy equation





$$A = \text{area} = \frac{1}{2} P \frac{dr}{dx} dx = dU$$

$$P = EA\frac{dr}{dx} \quad \text{(Hooke's law)}$$

$$dU = \frac{1}{2} EA \left(\frac{dr}{dx} \right)^2 dx$$

Now integrating over the element domain results in the total potential energy:

$$U = \frac{1}{2}\int EA\left(\frac{dr}{dx}\right)^2 dx$$

$$= \frac{1}{2}\int EA\frac{d}{dx}\left(\sum_{i=1}^{k}\phi_i(x)q_i(t)\right)\frac{d}{dx}\left(\sum_{j=1}^{k}\phi_j(x)q_j(t)\right)dx$$

$$= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}q_i q_j \underbrace{\int EA\phi_i'\phi_j' dx}_{K_{ij} = \int EA\phi_i'\phi_j' dx}$$

Note that for beams, *K* would be written: $K_{ij} = \int EA\phi_i''\phi_j'' dx$

Similar to kinetic energy, we can express the potential energy in terms of matrix *k:*

$$T = \frac{1}{2}\left(\dot{q}\right)^T \left[m\right]\left(\dot{q}\right) \qquad U = \frac{1}{2}\left(q\right)^T \left[k\right]\left(q\right)$$

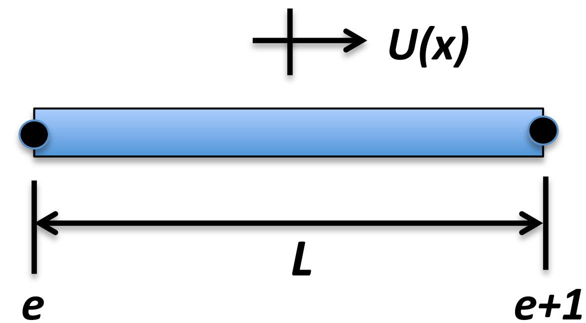
[*m*] and [*k*] represent the generalized mass and stiffnesses for a given deflection shape

Finite element method for axial rod vibration

- FEA is useful for non-regular geometry of distributed systems
  - Complex structures that are not defined by a lumped mass system
  - Systems that are affected by higher frequency mode shapes
    - Electronics, precision instruments
- FEA is also useful as a single tool for solving vibration problems
  - It is available for CAD-generated solid models, making it the tool of choice for dynamic analysis

FE derivation for a rod in axial vibration

- Consider the dynamic equations we derived for a uniform rod in axial vibration:



$$\frac{d^2u(x)}{dx^2}+\beta^2u(x)=0$$

Note – this was our spatial D.E. after separation of variables

$$\frac{d^2u}{dx^2}+\left(\frac{\omega}{c}\right)^2u=0$$

$$c^2\frac{d^2u}{dx^2}+\omega^2u=0 \quad\Rightarrow\quad \frac{E}{\rho}\frac{d^2u}{dx^2}+\omega^2u=0$$

$$\left(\frac{E}{\rho}\frac{d^2u}{dx^2}+\omega^2u=0\right)\rho A \quad\Rightarrow\quad EA\frac{d^2u}{dx^2}+\rho A\omega^2u=0$$

With this DE, we desire to write the weak form of the equation:

$$\int w\left(EA\frac{d^2u}{dx^2} + \rho A\omega^2 u \right)dx = 0$$

Assume that the cross section is constant

$$= EA\int_e^{e+1} w\frac{d^2u}{dx^2}dx + \rho A\omega^2\int_e^{e+1} uwdx = 0$$

$$= EA\left[w\frac{du}{dx}\bigg|_e^{e+1} - \int_e^{e+1}\frac{du}{dx}\frac{dw}{dx}dx \right] + \rho A\omega^2\int_e^{e+1} uwdx = 0$$

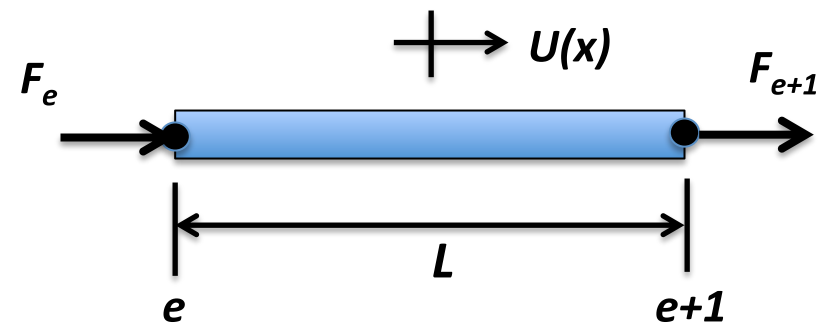
$$= EA\left[w(e+1)\frac{du}{dx}\bigg|_{e+1} - w(e)\frac{du}{dx}\bigg|_e - \int_e^{e+1}\frac{du}{dx}\frac{dw}{dx}dx \right] + \rho A\omega^2\int_e^{e+1} uw\,dx = 0$$

Now we can define functions for *u* and *w:*

$$u = \sum_i u_i \phi_i \quad \text{and} \quad w_i = \phi_i$$

And define nodal forces on the ends of the rod:

$$EA\frac{du}{dx}\bigg|_e = F_e \qquad EA\frac{du}{dx}\bigg|_{e+1} = F_{e+1}$$

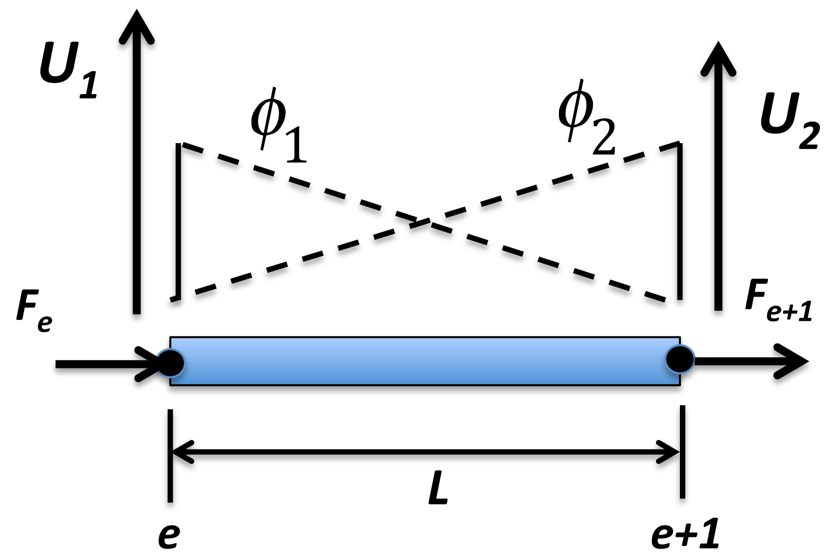


The shape functions for the rod will be defined by the number of degrees of freedom used in the model. For a 2-node rod, the function will be linear:

$$\phi_1 = \left(\frac{L-x}{L}\right) \quad \frac{d\phi_1}{dx} = -\frac{1}{L}$$

$$\phi_2 = \frac{x}{L} \quad \frac{d\phi_2}{dx} = \frac{1}{L}$$

Linear 2-node rod shape functions and BC:



$$w_1 = \phi_1 \quad w_1(e) = 1 \quad w_1(e+1) = 0$$

$$w_2 = \phi_2 \quad w_2(e) = 0 \quad w_2(e+1) = 1$$

Now we can return to the integral equation to find the element mass and stiffnesses:

$$EA\left[w(e+1)\frac{du}{dx}\bigg|_{e+1} - w(e)\frac{du}{dx}\bigg|_{e} - \int_e^{e+1}\frac{du}{dx}\frac{dw}{dx}dx \right] + \rho A\omega^2 \int_e^{e+1} uw\,dx = 0$$

$$EA\left[w(e+1)\frac{du}{dx}\bigg|_{e+1} - w(e)\frac{du}{dx}\bigg|_{e} - \sum_{i=1}^{2}\underbrace{\left(\int_e^{e+1}\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx\right)}u_i \right] + \rho A\omega^2 \sum_{i=1}^{2}\underbrace{\left(\int_e^{e+1}\phi_i\phi_j\,dx\right)}_{m_{ij}}u_i$$

$$K_{11} = EA\int_0^L\left(-\frac{1}{L}\right)\left(-\frac{1}{L}\right)dx = \frac{EA}{L}$$

$$K_{12} = EA\int_0^L\left(-\frac{1}{L}\right)\left(\frac{1}{L}\right)dx = -\frac{EA}{L} = K_{21}$$

$$K_{22} = \frac{EA}{L}$$

Now we can return to the integral equation to find the element masses and stiffnesses:

$$EA\left[w(e+1)\frac{du}{dx}\bigg|_{e+1} - w(e)\frac{du}{dx}\bigg|_{e} - \int_e^{e+1}\frac{du}{dx}\frac{dw}{dx}dx \right] + \rho A\omega^2\int_e^{e+1} uw\,dx = 0$$

$$EA\left[w(e+1)\frac{du}{dx}\bigg|_{e+1} - w(e)\frac{du}{dx}\bigg|_{e} - \underbrace{\sum_{i=1}^{2}\left(\int_e^{e+1}\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx\right)}u_i \right] + \rho A\omega^2 \underbrace{\sum_{i=1}^{2}\left(\int_e^{e+1}\phi_i\phi_j\,dx\right)}_{m_{ij}}u_i$$

$$k_{11} = EA\int_0^L\left(-\frac{1}{L}\right)\left(-\frac{1}{L}\right)dx = \frac{EA}{L}$$

$$k_{12} = EA\int_0^L\left(-\frac{1}{L}\right)\left(\frac{1}{L}\right)dx = -\frac{EA}{L} = k_{21}$$

$$k_{22} = \frac{EA}{L}$$

$$[k] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For the mass matrix, a similar process is used:

$$EA\left[w(e+1)\left.\frac{du}{dx}\right|_{e+1} - w(e)\left.\frac{du}{dx}\right|_{e} - \sum_{i=1}^{2}\left(\int_{e}^{e+1} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx \right) u_i \right] + \rho A \omega^2 \sum_{i=1}^{2}\left(\int_{e}^{e+1} \phi_i \phi_j \, dx \right) u_i = 0$$

$$m_{11} = \rho A \int_0^L \left(\frac{L-x}{L}\right)\left(\frac{L-x}{L}\right) dx = \frac{\rho A L}{3}$$

$$m_{12} = m_{21} = \rho A \int_0^L \left(\frac{L-x}{L}\right)\left(\frac{x}{L}\right) dx = \frac{\rho A L}{6}$$

$$m_{22} = \frac{\rho A L}{3}$$

$$\left[m\right] = \frac{\rho A L}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The boundary conditions fall out of the equation as well:

$$EA\left[w(e+1)\frac{du}{dx}\bigg|_{e+1} - w(e)\frac{du}{dx}\bigg|_{e} - \int_{e}^{e+1}\frac{du}{dx}\frac{dw}{dx}dx \right] + \rho A\omega^2\int_{e}^{e+1} uw\,dx = 0$$

$$w_1(e)=1 \quad w_1(e+1)=0$$
$$w_2(e)=0 \quad w_2(e+1)=1$$

$$EA\frac{du}{dx}\bigg|_{e} = F_e \quad EA\frac{du}{dx}\bigg|_{e+1} = F_{e+1}$$

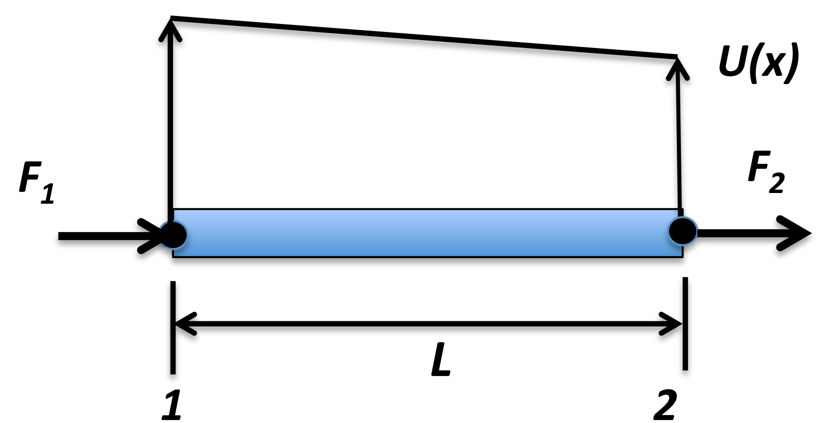
The end forces on the element, F_e and F_{e+1} either are balanced by neighboring elements or they are used in the boundary conditions to solve for the overall response.

Putting these new terms together results in the matrix formulation for the element:

$$[m](\ddot{u})+[k](u)=(F)$$

$$\left(-\omega^2[m]+[k]\right)(u)=(F)$$

$$\left(-\omega^2\frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}+\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$



Once the matrix of equations is formed, any of the previous techniques discussed for solving EVP can be used

- Matrix iteration
- Given’s method
- Jacobi
- Householder reflections
- Guyan reduction
- Etc.