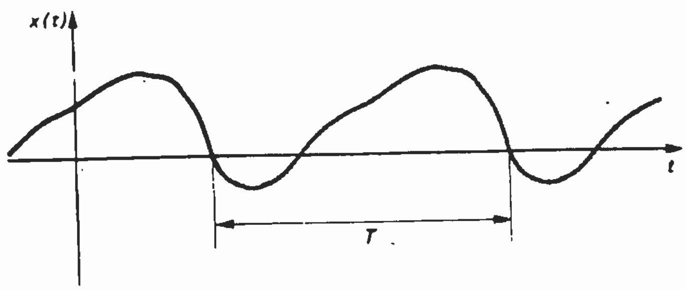
5. Fourier Series and Fourier Transform

5.1 *Fourier Series*

The Fourier Series can be used to describe a periodic function:



$$
\begin{aligned}
x(t) &= a_o + a_1 \cos \omega_o t + a_2 \cos 2\omega_o t + \cdots \\
&\qquad + b_1 \sin \omega_o t + b_2 \sin 2\omega_o t + \cdots \\
&= a_o + \sum_{k=1}^{\infty} (a_k \cos k\omega_o t + b_k \sin k\omega_o t)
\end{aligned} \tag{5.1}
$$

Where $\omega_o = \dfrac{2\pi}{T}$. The coefficients are defined by:

$$
\begin{aligned}
a_o &= \frac{1}{T}\int_0^T x(t)\,dt \\
a_k &= \frac{2}{T}\int_0^T x(t)\cos k\omega_o t\,dt \qquad \text{for } k = 1, \infty \\
b_k &= \frac{2}{T}\int_0^T x(t)\sin k\omega_o t\,dt
\end{aligned} \tag{5.2}
$$

We can rewrite the Fourier series in complex form which will simplify the set of equations.

A harmonic of index k is described by:

$$x_k(t) = a_k \cos k\omega_o t + b_k \sin k\omega_o t \tag{5.3}$$

Using Euler's trig formulas, Eqn. 5.3 can be rewritten:

$$\begin{aligned} x_k(t) &= \frac{a_k}{2}\left(e^{ik\omega_o t} + e^{-ik\omega_o t}\right) + \frac{b_k}{2i}\left(e^{ik\omega_o t} - e^{-ik\omega_o t}\right) \\ &= \frac{1}{2}(a_k - ib_k)e^{ik\omega_o t} + \frac{1}{2}(a_k + ib_k)e^{-ik\omega_o t} \end{aligned} \tag{5.4}$$

Simplifcation occurs by substituting for the coefficients of the exponentials:

$$c_k = \frac{1}{2}(a_k - ib_k) \quad c_{-k} = \frac{1}{2}(a_k + ib_k) \qquad (\text{for } k > 0) \tag{5.5}$$

Then assigning $c_o = a_o$ results in:

$$x(t) = c_o + \sum_{k=1}^{\infty}\left(c_k e^{ik\omega_o t} + c_{-k}e^{i(-k)\omega_o t}\right) \tag{5.6}$$

Further simplification occurs by realizing that if the index k is allowed to extend to $-\infty$ then:

$$\sum_{k=1}^{\infty} c_{-k} e^{i(-k)\omega_o t} = \sum_{k=-1}^{-\infty} c_k e^{ik\omega_o t} \tag{5.7}$$

Which is seen from Eqn. 5.2 where b_k is an odd function and a_k is an even function.

Furthermore, if we let $c_o = c_o e^{i(0)}$, then:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{ik\omega_o t} \tag{5.8}$$

Which is called the *complex form of the Fourier series.*

Likewise, rewriting Eqn. 5.5 using Eqn. 5.2 results in:

$$c_k = \frac{1}{T}\int_0^T x(t) e^{-ik\omega_o t} dt \qquad \text{for} \quad k = -\infty, \infty \tag{5.9}$$

Equations 5.8 and 5.9 represent the *Fourier Series Pair.* The term c_k is frequently referred to as the kth Fourier Coefficient, and represents a weighting of the kth harmonic, $e^{-ik\omega_o t}$, to the data.

- The coefficients are *complex* and can be described by:

$$
\begin{aligned}
|c_k| &= \sqrt{a_k^2 + b_k^2} \\
\phi_k &= \tan^{-1}\left(\frac{b_k}{a_k}\right)
\end{aligned}
\tag{5.10}
$$

The phase angle represents the shift of the *k*th harmonic to match the data.

- It should be noted that the $-k$ terms represent *negative frequency* components, which can be visualized as a clockwise-rotating phasor.

5.2 *Fourier Integral or Fourier Transform*

The above results were obtained for periodic data. Data which is transient requires that a modification due to the infinite length of the period, *T*. Note that Eqn. 5.9 approaches zero as the period approaches ∞.

Eqn. 5.9 can be rewritten using symmetrical integration limits (a periodic function will be periodic in negative time):

$$
c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-ik\omega_o t}\,dt \tag{5.11}
$$

If this expression is substituted into Eqn. 5.8:

$$x(t) \quad = \quad \sum_{k=-\infty}^{\infty} \left(\frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-ik\omega_o t} dt \right) e^{ik\omega_o t} \tag{5.12}$$

Note that the period can be rewritten $T = {2\pi}/{\omega_o}$, where ω_o is the fundamental frequency. The fundamental frequency *is also the frequency resolution* of the series, or $\omega_o = \Delta\omega$, so that:

$$x(t) \quad = \quad \sum_{k=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} x(t) e^{-ik\omega_o t} dt \right) e^{ik\omega_o t} \tag{5.13}$$

If the period is allowed to extend to $\pm\infty$, the fundamental frequency will become $\Delta\omega \rightarrow d\omega$. Two other substitutions are made which represent the transition from discrete to continuous:

$$\sum_{k=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty} \tag{5.14}$$

$$k\,\omega_o \rightarrow \omega$$

This results in:

$$x(t) \quad = \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \right) e^{i\omega t} d\omega \tag{5.15}$$

The expression in the brackets is called the *Fourier Transform of x(t):*

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{5.16}$$

This is expressed in $\text{units}/_{\text{Hz}}$, and is complex in general.

The expression in Eqn 5.15 is the *Inverse Fourier Transform:*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \tag{5.17}$$

These equations are valid when $x(t)$ is bounded, defined over the time region $-\infty \le t \le \infty$, and satisfies:

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty \tag{5.18}$$

5.3 *Discrete Fourier Transform / Inverse Discrete Fourier Transform*

Because experimental data must be sampled before it can be processed, a *discrete* transformation is used to obtain the frequency domain data.

The discrete transformation is derived from the Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{5.19}$$

This was *not* defined for stationary random processes (that continue forever) because Eqn. 5.18 is violated:

$$\int_{-\infty}^{\infty} |x(t)|dt < 0 \tag{5.20}$$

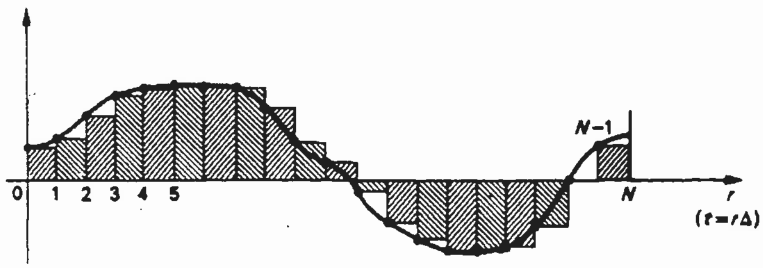
Instead, a *finite time interval* Fourier transform can be defined to handle a random process:

$$X(\omega,T) = \int_{0}^{T} x(t)e^{-i\omega t}dt \tag{5.21}$$

If $x(t)$ is sampled at equally spaced intervals Δt, then time can be represented by $t_n = n\,\Delta t$ and $x(t_n) = x(n\,\Delta t)$. Starting with $n = 0$, the integral can be replaced by a summation:

$$X(\omega,T) = \sum_{n=0}^{N-1} x_n\, e^{-i\omega n\Delta t}\Delta t \tag{5.22}$$

Each x_n represents a discrete sample of the signal where its value is assumed to be constant throughout the sampling interval. This is shown below:



From the figure, it is seen that $T = N\Delta t$.

The transform is calculated at discrete frequencies only, based on the sampling period and the number of time domain points in the record:

$$\omega_k = 2\pi f_k = \frac{2\pi k}{T} = \frac{2\pi k}{N\Delta t} \qquad (5.23)$$

Substituting the above form of the frequency into Eqn. 5.22 results in:

$$X(\omega_k, T) = \sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi kn}{N}} \Delta t \qquad (5.24)$$

The lowest non-zero frequency possible is $\omega_1 = 2\pi/N\Delta t$, or $f_1 = 1/N\Delta t$ if the frequency domain is expressed in Hertz and not rad/sec. This is equivalent to the *frequency resolution* of the transform, ω_o or f_o.

The frequency resolution is simply the fundamental frequency that is fit in the sampling time window of period T. Each higher frequency represents a multiple of the fundamental, $\omega_k = k\,\omega_1$.

- The number of frequency components equals the number of time domain data points, N.

Since we defined the limits of time for $x(t)$ to be non-negative, the same holds true for the frequency components:

$$0 \;\le\; \omega_k \;\le\; \frac{2\pi(N-1)}{N\,\Delta t} \tag{5.25}$$

or
$$0 \;\le\; f_k \;\le\; \frac{N-1}{N\,\Delta t}$$

- *Unlike* the frequencies that were obtained with the Fourier series, there are no negative frequency components. We will see shortly, however, that it is equally valid to develop equations that have negative frequency components.

The transform $X(\omega_k, T)$ represents the harmonic coefficients of the data at the kth frequency. Commonly, the form for the Fourier series coefficients c_k (Eqn. 5.9) is used for the transform, resulting in a *periodic discrete Fourier transform:*

$$X_k \;=\; \frac{X(\omega_k, T)}{T} \;=\; \frac{1}{T}\sum_{n=0}^{N-1} x_n\, e^{-\frac{i2\pi k n}{N}}\,\Delta t \tag{5.26}$$

Observing that $\Delta t/T = 1/N$ results in the formal definition of the Discrete Fourier Transform (DFT):

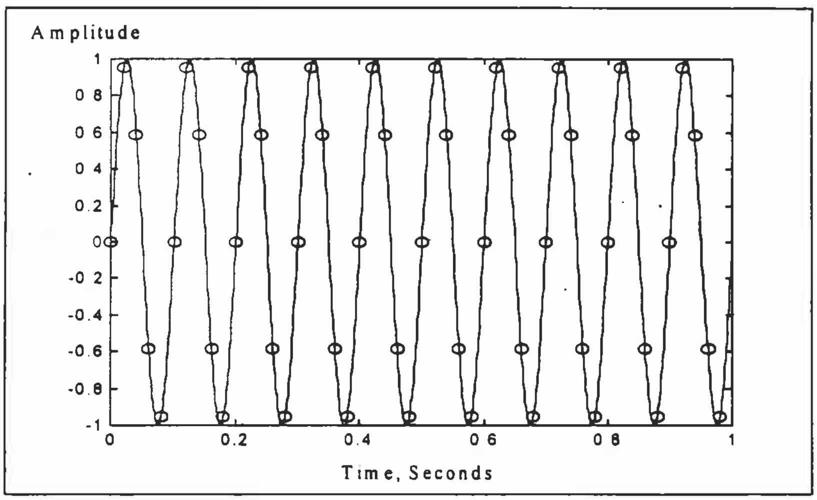
$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi k n}{N}} \qquad \text{for } k = 0, N-1 \qquad (5.27)$$

Likewise, the Inverse Discrete Fourier Transform (IDFT) is defined by the form used for the Fourier series (Eqn. 5.8):

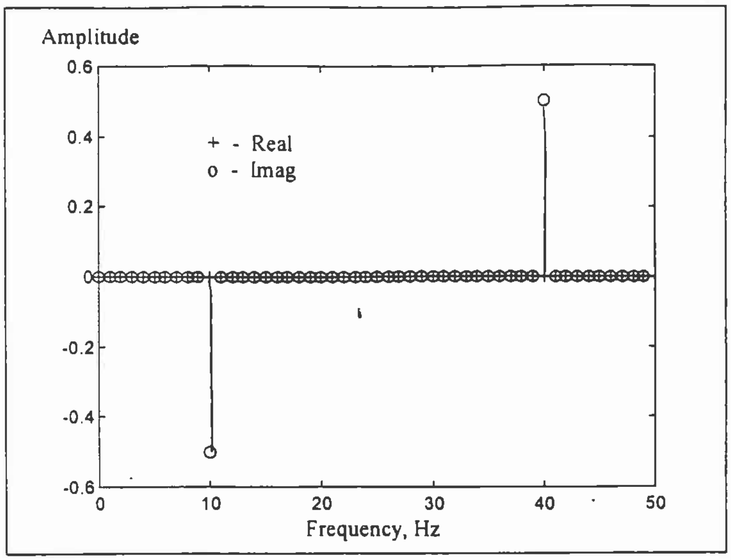
$$x_n = \sum_{k=0}^{N-1} X_k e^{\frac{i2\pi k n}{N}} \qquad n = 0, N-1 \qquad (5.28)$$

To demonstrate the operation of the DFT / IDFT, a sine function can be sampled and transformed. An observation about the frequency range selected will be made.

Consider the 10 Hz sine function shown below:



The circled points represent 50 discrete samples taken on a 1-second period. The transform of the signal produces the frequency domain plot:



Note that the first component is at *zero* Hz, representing the DC offset of the data (which is zero).

There are 50 harmonic components, of which only two are non-zero: one at 10 Hz and one at 40 Hz.

The amplitude of both non-zero components is one-half of the original sine function and will reproduce the original *sampled data points exactly.*

There are some other observations about this result:

- The 40 Hz component does not exist in the time domain.
- The contribution of the 40 Hz component in the IDFT can be substituted by either:

  1. Using a -10 Hz component, obtained by changing the limits of k in Eqn. 6.54 to $-N/2 \rightarrow (N-1)/2$.

2. Doubling the non-zero harmonic components and considering only one-half the frequency range, from $0 \rightarrow {(N-1)}/{2}$.

To demonstrate how these work we must rewrite Eqn. 5.27 considering only the fundamental harmonic and the $N-1$ harmonic:

$$X_1 = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi n}{N}}$$

$$X_{N-1} = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi(N-1)n}{N}}$$

The $N-1$ component can be rewritten as:

$$X_{N-1} = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-i2\pi n} e^{\left(i2\pi n/N\right)}$$

Which is simplified to:

$$X_{N-1} = \frac{1}{N}\sum_{n=0.}^{N-1} x_n e^{\left(i2\pi n/N\right)}$$

And this is observed to be the complex conjugate of X_1.

Likewise, if X_2 and X_{N-2} are considered,

$$X_2 = \frac{1}{N}\sum_{n=0}^{N-1} x_n\, e^{-\frac{i2\pi 2n}{N}}$$

$$X_{N-2} = \frac{1}{N}\sum_{n=0}^{N-1} x_n\, e^{-i2\pi n} e^{\left(i2\pi 2n/N\right)}$$

And again, $X_{N-2} = X_2^*$. If we continue to look at these harmonics, we would see that *all of the harmonics above the $N/2$ harmonic are simply complex conjugates of the the set from $1 \rightarrow (N-1)/2$*. In other words, there is a "folding" of the harmonic components about the $N/2$ frequency. This frequency is called the *Nyquist Frequency* and represents a limit to the unique frequencies obtained in the transform.

This fact will allow us to express the harmonic components in two alternative forms:

1. Use of negative frequency components

Recall the IDFT:

$$x_n = \sum_{k=0}^{N-1} X_k\, e^{\frac{i2\pi kn}{N}} \qquad n = 0, N-1 \qquad (5.29)$$

If we consider a typical harmonic component *above* the Nyquist frequency, its contribution to the time domain response will be:

$$\hat{x}_{nk} = X_k^* e^{\frac{i2\pi(N-k)n}{N}} \qquad n = 0, N-1 \qquad (5.30)$$

This simplifies to:

$$\hat{x}_{nk} = X_k^* e^{-\frac{i2\pi kn}{N}} \qquad n = 0, N-1 \qquad (5.31)$$

Now if we consider a component in the *negative* frequency range, we get:

$$X_{-k} = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{\frac{i2\pi kn}{N}} = X_k^* \qquad (5.32)$$

So Eqn. 5.31 can be written:

$$\hat{x}_{nk} = X_{-k} e^{-\frac{i2\pi kn}{N}} \qquad n = 0, N-1 \qquad (5.33)$$

This says that the components *above* the Nyquist Frequency can be handled equivalently by changing the range of harmonic components to $-N/2 \leq k \leq (N-1)/2$:

$$x_n = \sum_{k=-N/2}^{(N-1)/2} X_k e^{\frac{i2\pi k n}{N}} \qquad n = 0, N-1 \qquad (5.34)$$

Where k is rounded towards zero if the limits in the summation are not integer.

Electrical engineers tend to use negative frequency components as opposed to the components above the Nyquist Frequency.

2. Doubling the components from $1 \rightarrow (N-1)/2$:

The IDFT can be rewritten using the complex conjugate form as:

$$x_n = X_0 + \sum_{k=1}^{(N-1)/2} \left(X_k e^{\frac{i2\pi k n}{N}} + X_k^* e^{\frac{i2\pi (N-k) n}{N}} \right) \qquad n = 0, N-1 \qquad (5.35)$$

Which simplifies to:

$$x_n = X_0 + \sum_{k=1}^{(N-1)/2} \left(X_k e^{\frac{i2\pi k n}{N}} + X_k^* e^{-\frac{i2\pi k n}{N}} \right) \qquad n = 0, N-1 \qquad (5.36)$$

If the real and imaginary parts of X_k and the exponent are written out, the result becomes:

$$x_n = X_0 + 2 \sum_{k=1}^{(N-1)/2} \left(X_{k\,\mathrm{real}} \cos\frac{2\pi k n}{N} - X_{k\,\mathrm{imag}} \sin\frac{2\pi k n}{N} \right) \quad n = 0, N-1 \quad (5.37)$$

Which is a convenient computational form.

For the example presented above, the IDFT becomes:

$$x_n = 2\left[0 + 0.5 \sin\left(\frac{2\pi(10)n}{50}\right)\right] \quad n = 0, N-1 \quad (5.38)$$

Which reproduces the original sampled data points exactly.