6.6 Vibration of membranes and plates

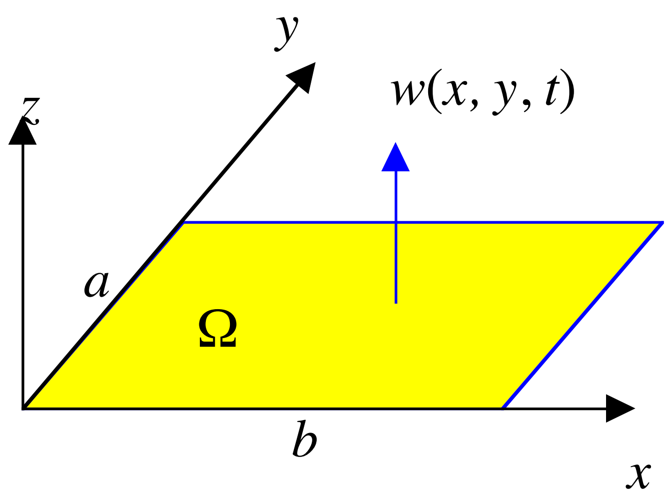


Figure 6.13

- The domain (Ω) is now a plane rather then a line: Two dimensional
- Membrane is a two dimensional string
- Plate is a two dimensional beam

The membrane equation:

$$\tau\nabla^2 w(x,y,t) = \rho w_{tt}(x,y,t), \quad x,y \in \Omega$$

Tension per unit length density(mass/area)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$ The *Laplace* operator

$$\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 w(x,y,t)}{\partial t^2}$$

$$c = \sqrt{\tau/\rho}$$

Boundary and initial conditions

$w(x, y, t) = 0$ for some part of the boundary $\partial\Omega$

$$\frac{\partial w(x, y, t)}{\partial n} = 0 \text{ for some other part of the boundary } \partial\Omega$$

where this derivative donotes the derivative normal to the plane of the membrane

plus the usual initial conditions

Example 6.6.1 Compute the natural frequencies of a square membrane of 1 m side.

$$c^2\left[\frac{\partial^2 w}{\partial^2 x}+\frac{\partial^2 w}{\partial^2 y}\right]=\frac{\partial^2 w}{\partial^2 t},\quad x,y\in\Omega$$

$$w(x,y,t)=X(x)Y(y)T(t)\Rightarrow$$

$$X''YT+XY''T=\frac{1}{c^2}XY\ddot{T}\Rightarrow\frac{X''Y+XY''}{XY}=\overbrace{\frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)}}^{\text{Temporal equation}}=-\omega^2$$

$$\frac{X''Y+XY''}{XY}=\frac{X''}{X}+\frac{Y''}{Y}=-\omega^2\Rightarrow\frac{X''}{X}=-\frac{Y''}{Y}-\omega^2=-\alpha^2$$

$$\Rightarrow\begin{cases}\dfrac{X''}{X}=-\alpha^2\\[2ex]\dfrac{Y''}{Y}=-\gamma^2\end{cases}$$ Two spatial equations

where $\omega^2=\alpha^2+\gamma^2$

$$X''(x) + \alpha^2 X(x) = 0 \Rightarrow A\sin\alpha x + B\cos\alpha x$$

$$Y''(y) + \gamma^2 Y(y) = 0 \Rightarrow C\sin\gamma y + D\cos\gamma y$$

$$X(x)Y(y) = A_1 \sin\alpha x \sin\gamma y + A_2 \sin\alpha x \cos\gamma y$$
$$+ A_3 \cos\alpha x \sin\gamma y + A_4 \cos\alpha x \cos\gamma y$$

Now apply the boundary conditions:

along $x = 0$:

$$T(t)X(0)Y(y) = T(t)(A_3 \sin\gamma y + A_4 \cos\gamma y) = 0 \Rightarrow$$

$$A_3 \sin\gamma y + A_4 \cos\gamma y = 0$$ which must hold for any y

$$\Rightarrow \underline{A_3 = A_4 = 0}$$

Now we have:

$$X(x)Y(y) = A_1 \sin\alpha x \sin\gamma y + A_2 \sin\alpha x \cos\gamma y$$

Along $x = 1$, $w(1, y, t) = 0$:

$$A_1 \sin\alpha \sin\gamma y + A_2 \sin\alpha \cos\gamma y = 0 \Rightarrow$$

$$\sin\alpha (A_1 \sin\gamma y + A_2 \cos\gamma y) = 0 \Rightarrow$$

$$\text{either } \sin\alpha = 0 \quad \text{or } A_1 = A_2 = 0$$

$$\Rightarrow \sin\alpha = 0 \Rightarrow \underline{\alpha_n = n\pi,\ n = 1,2,3,4\ldots}$$

At this point:

$$X(x)Y(y) = A_1 \sin\alpha x \sin\gamma y + A_2 \sin\alpha x \cos\gamma y$$

At y=0, $w(x,0,t) = 0 \Rightarrow$

$$A_1 \sin\alpha x \sin 0 + A_2 \sin\alpha x \cos 0 = 0 \Rightarrow A_2 = 0$$

So $X(x)Y(y) = A_1 \sin\alpha x \sin\gamma y$

At y=1,$w(x,1,t) = 0 \Rightarrow$

$A_1 \sin\alpha x \sin\gamma 1 = 0 \Rightarrow \sin\gamma = 0$

which gives $\gamma_m = m\pi,\ m = 1,2,3,4...$

Frequencies and Mode Shapes

$$\sigma_n = n\pi, \gamma_m = m\pi \Rightarrow \omega_{nm} = \sqrt{\gamma_m^2 + \sigma_n^2} \Rightarrow$$

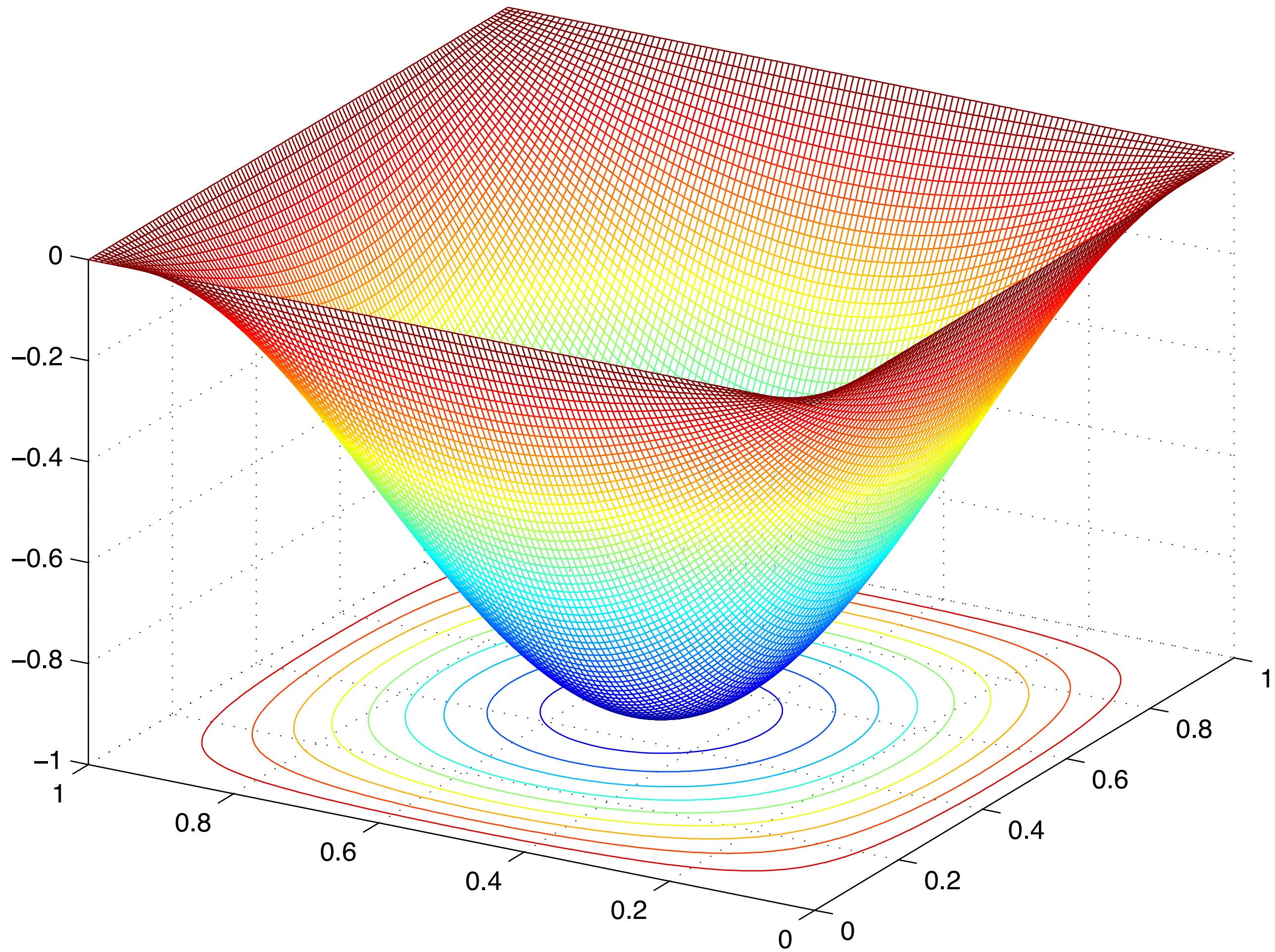
frequencies are

$$\omega_{nm} = \pi\sqrt{n^2 + m^2}$$

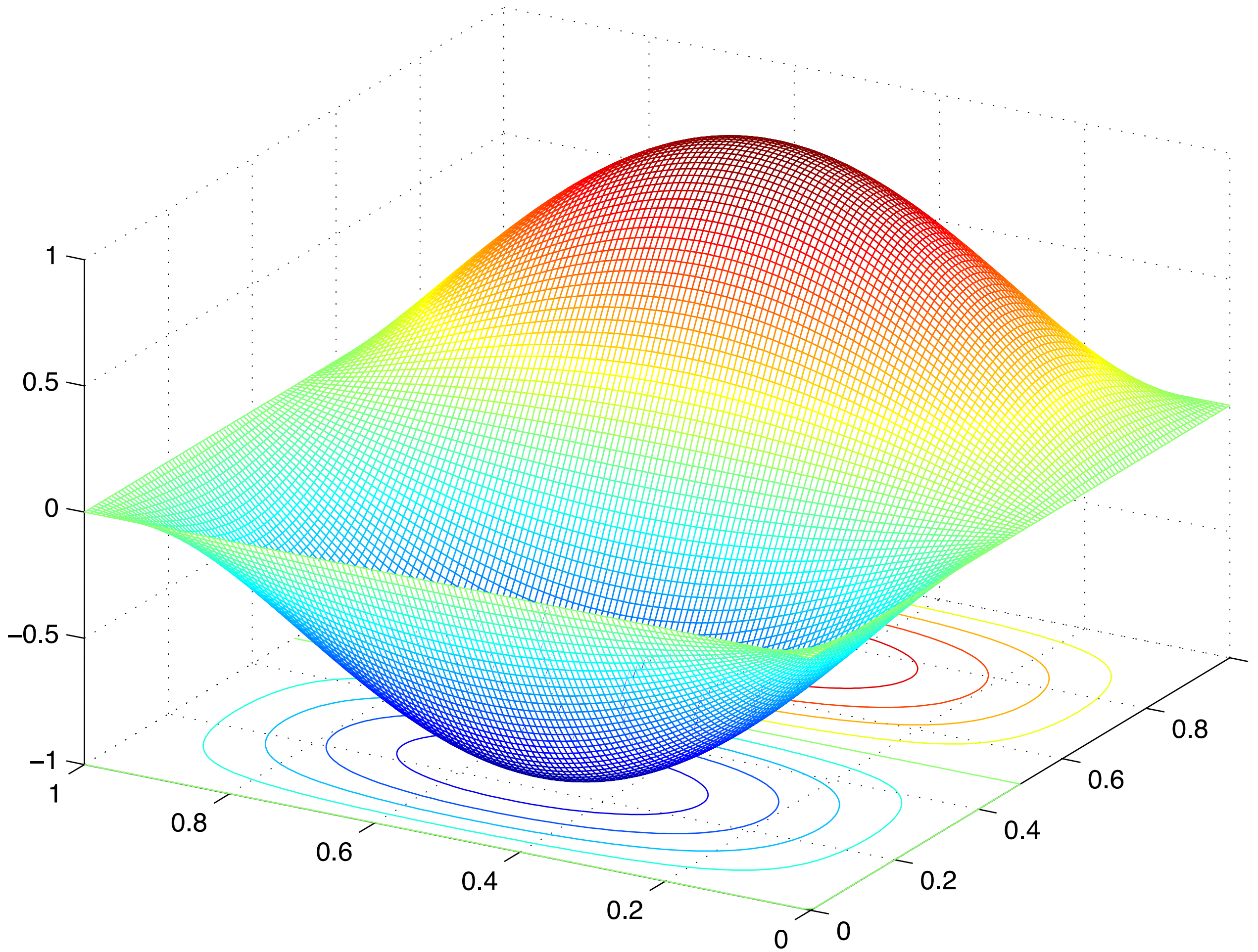
mode shapes are

$$\left\{\sin n\pi x \sin m\pi y\right\}_{n,m=1}^{\infty}$$

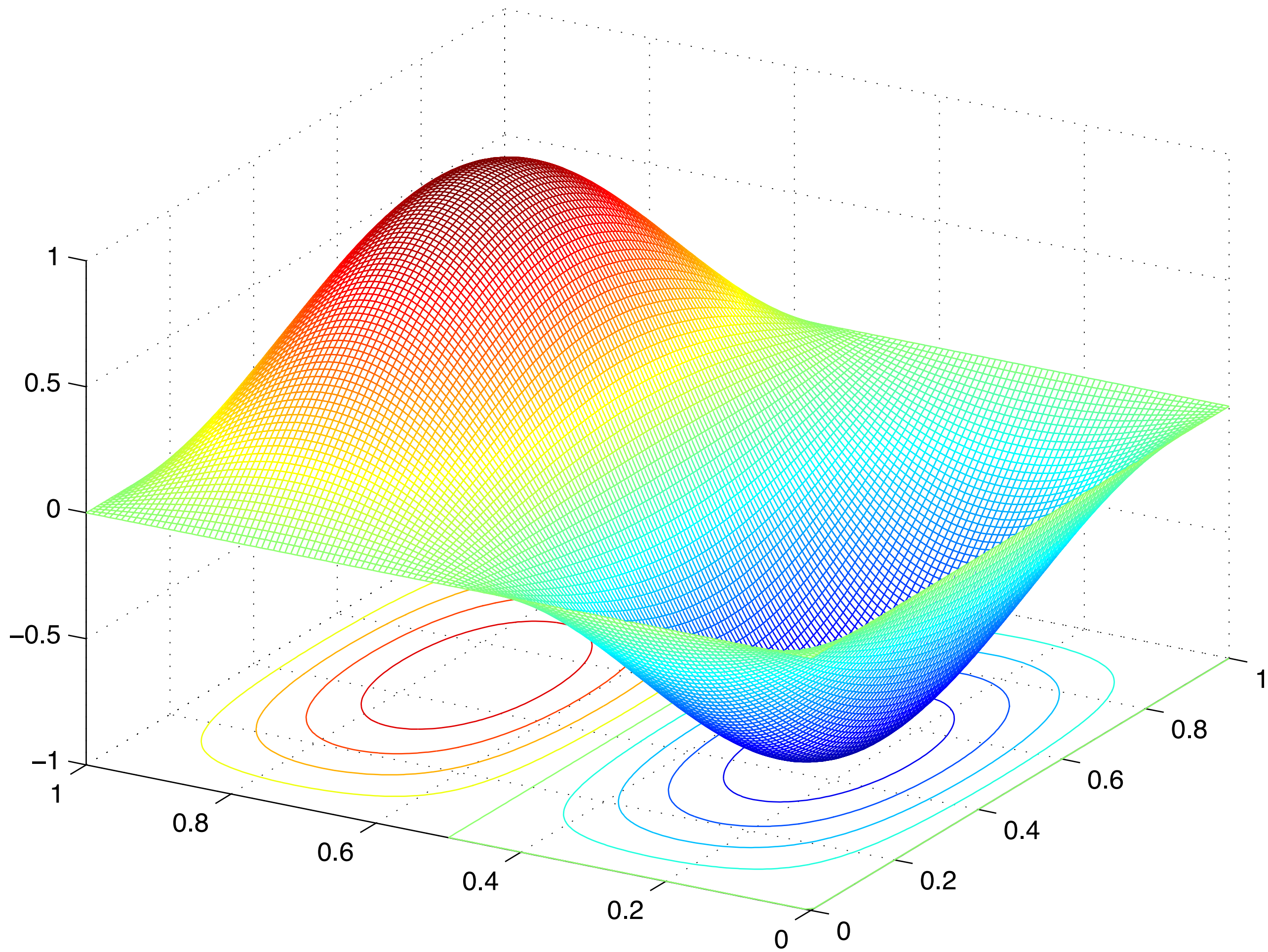
- Mode 11



- Mode 12



- Mode 21



$\omega_{12} = \omega_{21}$

- Mode 22

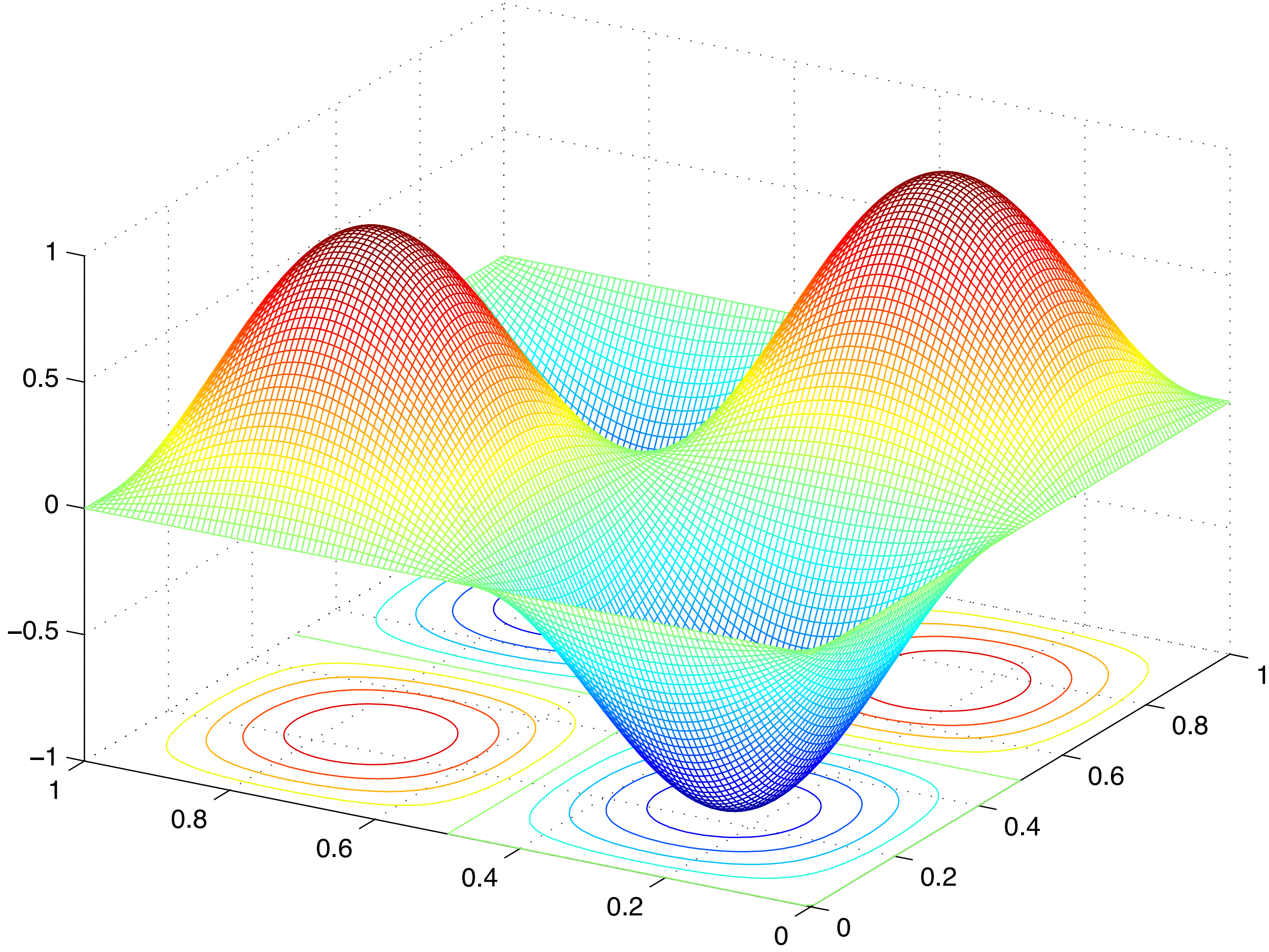


Plate vibration:

$$D_E \nabla^4 w(x,y,t)$$
$$= \rho w_{tt}(x,y,t), \;\; x,y \in \Omega$$

$$D_E = \frac{Eh^3}{12(1-v^2)}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

- Think of it as adding bending stiffness to the membrane
- Assume small deflections with respect to the thickness *h*
- Plane through middle is called neutral plane and does not deform in bending
- No thickness stretch

Boundary Conditions:

- This is a 2D analog of the EB beam theory and is called *thin plate* theory
- Must be enough boundary conditions for the 4th order derivatives:

$$w(x,y,t) = 0 \quad \text{and} \quad \frac{\partial w(x,y,t)}{\partial n} = 0 \ \text{ for } x,y \in \partial\Omega$$

For a simply supported (pinned) rectangular plate

$$w(x,y,t) = 0, \ \text{ along } x = 0, y = 0, x = \ell_1, y = \ell_2$$

$$\frac{\partial^2 w(x,y,t)}{\partial x^2} = 0, \ \text{ along } x = 0, x = \ell_1$$

$$\frac{\partial^2 w(x,y,t)}{\partial y^2} = 0, \ \text{ along } y = 0, y = \ell_2$$