ME 5514 Vibration of Mechanical Systems

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Vibrations are everywhere in our daily life.

Modeling and Degrees of Freedom

The examples on the previous slide many degrees of freedom and many parts, we will start with one degree of freedom and work towards many.

- Recall from your study of statics and physics that a degree of freedom is the independent parameter needed to describe the configuration of a physical system

- So single degree of freedom system, which is where we start, is a system whose position in time and space can be defined by one coordinate, here a displacement or position.

Degrees of Freedom: The Minimum Number of coordinates to specify a configuration

- For a single particle confined to a line, one coordinate suffices so it has one degree of freedom
- For a single particle in a plane two coordinates define its location so it has two degrees of freedom
- A single particle in space requires three coordinates so it has three degrees of freedom

What is this course about?

- We will examine vibration response in several different levels of modelling.
  - Review single degree of freedom systems (SDOF)
  - Lumped multiple degree of freedom systems (MDOF)
  - Distributed mass systems

Real life examples:

- Tacoma Narrows Bridge (Gallopin Gertie)

  What is exciting the structure?

  Winds at 42mph

  Would a 20 or 60 mph wind done the same?

  Could this have been prevented? How?

http://www.youtube.com/watch?v=3mclp9QmCGs

Helicopter flight is a miracle

[Helicopter Loses tail Rotor]

Modelling and degrees of freedom.

- London Millennium Bridge: The penguin walk
  - http://www.youtube.com/watch?v=gQK21572oSU
  - Opened and closed within a few hours!
    - Random vibration and direction in walking assumed wrong
    - Penguin walking
    - The concept of “Locking” took place.
    - Accounted for one direction vertical direction and not lateral.

Google Link

What we hope to achieve in this class is:

- Understand how to analyze vibration using dynamic principles.
  - Understand vibrations
  - Know how to analyze vibrations
  - Predict vibrations
- Using our models we hope to understand:
  - Concepts of natural frequencies (eignvalues)?
  - What are Mode shapes (eigenvectors)?
  - What happens when we have a Forced Response?
  - The concept of ***<u>Resonance</u>***
  - What happens when we have multiple degrees of freedom?

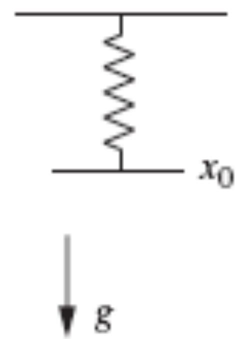
When do we see vibrations in a positive way?

Tools you will need for this class.

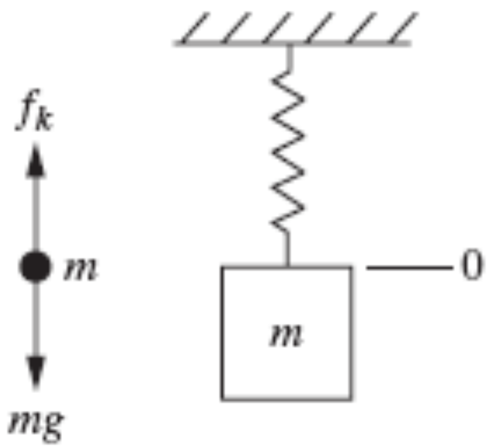
- Mathematics:
  - Differential equation
  - Linear Algebra
- System Dynamics
- Software
  - MatLab (Mathematica, etc)
    - You will need to use these program in order to solve homework problems and the take home final
    - You will also need to attach the code to your HM and Final.
  - Word / LaTex
    - You will have to turn in your assignments in a printed format using word/latex.

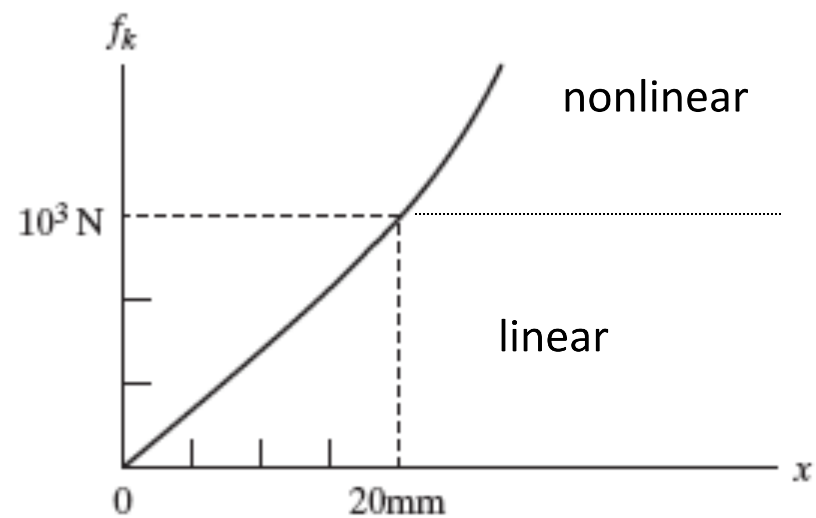
Let us consider a spring mass system

- From strength of materials recall:



FBD:





A plot of force versus displacement:

$$\text{experiment} \Rightarrow f_k = kx$$

Stiffness and Mass

Vibration is caused by the interaction of two different forces, one related to position (stiffness) and one related to acceleration (mass).

Proportional to displacement

Stiffness (k)

$$f_k = -kx(t)$$

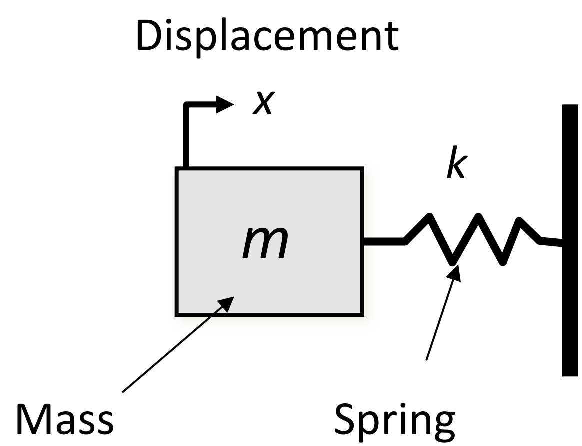
statics

Mass (m)

$$f_m = ma(t) = m\ddot{x}(t)$$

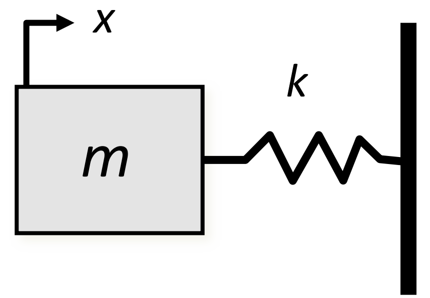
dynamics

Proportional to acceleration

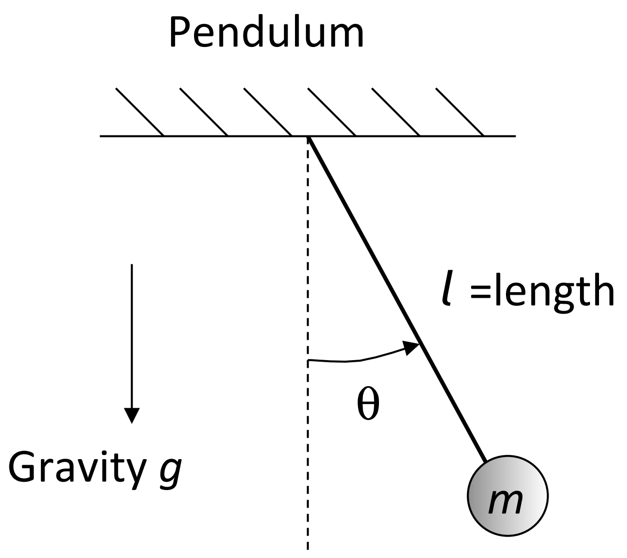


Newton's 2^{nd} law and Euler's 2^{nd} law

$$\sum F_x = M\ddot{x} \quad \text{(1-D motion)}$$
$$\sum F_y = M\ddot{y}$$
$$\sum F_z = M\ddot{z}$$

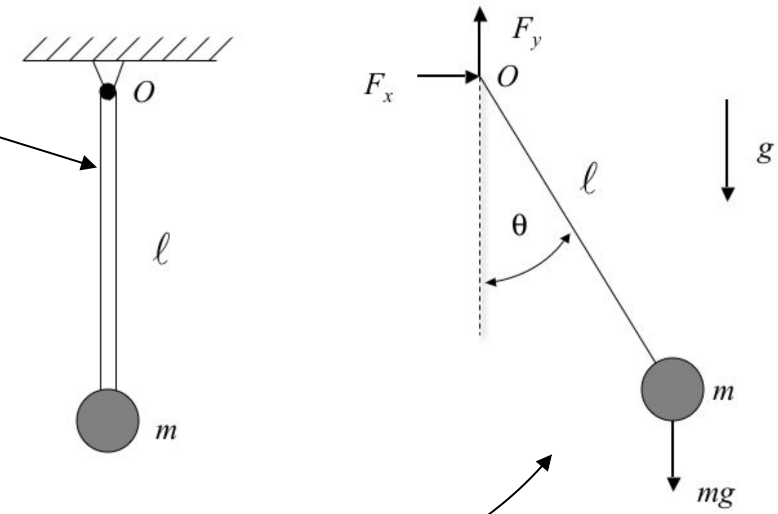


$$\sum T_{\theta x} = J_{xx}\ddot{\theta}_x \quad \text{(1-D motion)}$$
$$\sum T_{\theta y} = J_{yy}\ddot{\theta}_y$$
$$\sum T_{\theta z} = J_{zz}\ddot{\theta}_z$$



Book example 1.1.1: The Pendulum

- Sketch the structure or part of interest
- Write down all the forces and make a "free body diagram"
- Use Newton's Law and/or Euler's Law to find the equations of motion



$$\sum T_0 = J_0 \alpha, \quad J_0 = m\ell^2$$

FBD on board

The problem is one dimensional, hence a scalar equation results

$$J_0\alpha\left(t\right)=-mgl\sin\theta\left(t\right)\Rightarrow m\ell^2\ddot{\theta}(t)+\underbrace{mg\ell\sin\theta(t)}_{\substack{restoring\\ force}}=0$$

Here the over dots denote differentiation with respect to time t

This is a second order, nonlinear ordinary differential equation

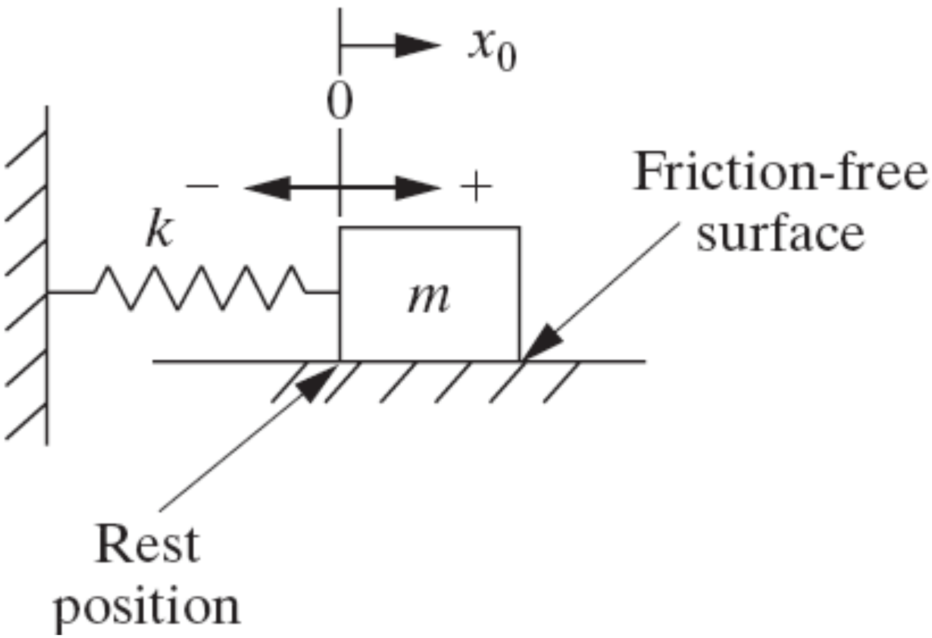
We can *linearize* the equation by using the approximation $\sin\theta\approx\theta$

$$\Rightarrow m\ell^2\ddot{\theta}(t)+mg\ell\theta(t)=0\Rightarrow\ddot{\theta}(t)+\frac{g}{\ell}\theta(t)=0$$

Requires knowledge of $\theta(0)$ and $\dot{\theta}(0)$

the initial position and velocity.

Free-body diagram and equation of motion



- Newton's Law:

$$m\ddot{x}(t) = -kx(t) \Rightarrow m\ddot{x}(t) + kx(t) = 0$$

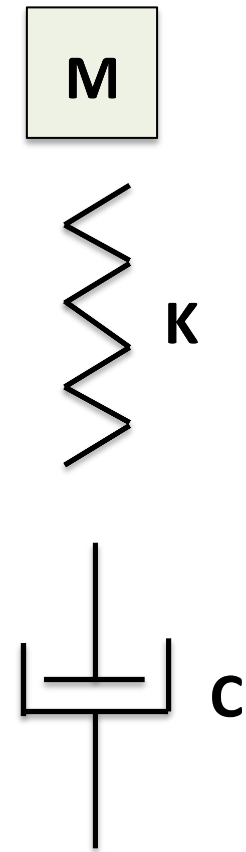
What do we need to solve this problem?

$$I.C.\ :\ \ x(0) = x_0, \dot{x}(0) = v_0 \qquad (1.2)$$

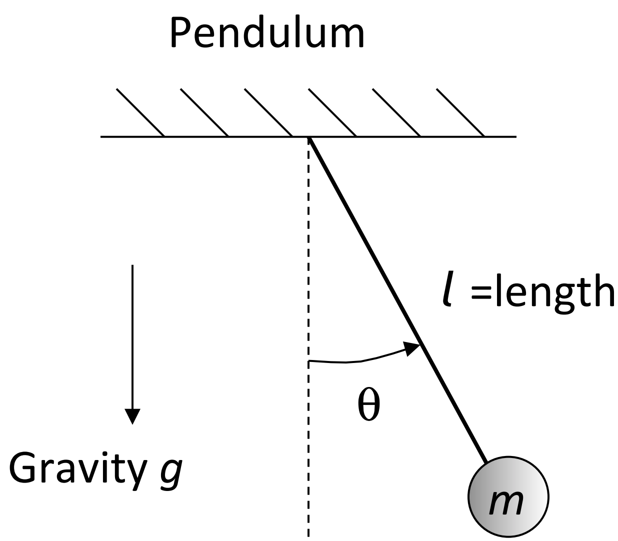
2nd order ordinary differential equation

Review of linear modeling elements

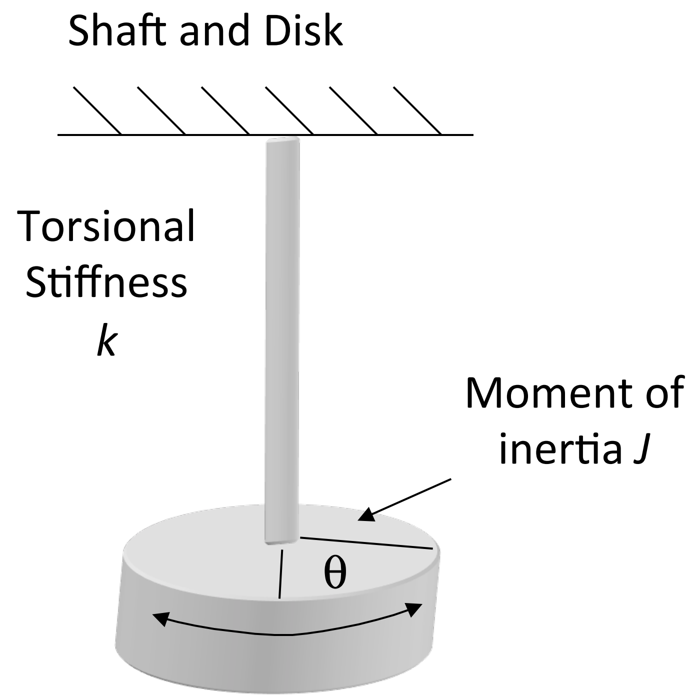
- Mass units are? $\dfrac{lb\sec^2}{in}$ or kg
- Stiffness units are? $\dfrac{lb}{in}$ or $\dfrac{N}{m}$
- Damping units are? $\dfrac{lb\sec}{in}$ or $\dfrac{N\sec}{m}$



Examples of rotary single DOF systems



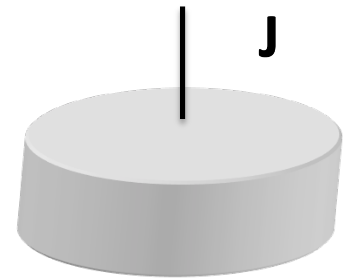
$$\ddot{\theta}(t) + \frac{g}{l}\theta(t) = 0$$



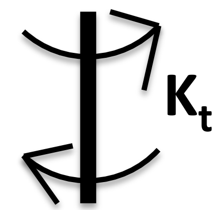
$$J\ddot{\theta}(t) + K_t\theta(t) = 0$$

Review of rotary modeling elements

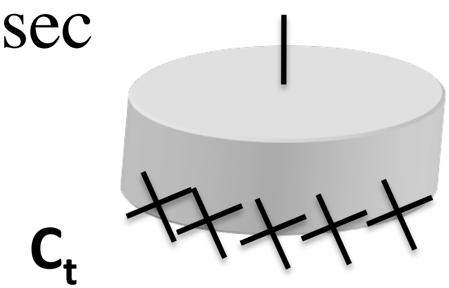
- Inertia units are? $lb\,\sec^2 in$ or $kg\ m^2$



- Stiffness units are? $lb\ in$ or $N\ m$



- Damping units are? $lb\ in\,\sec$ or $N\ m\,\sec$



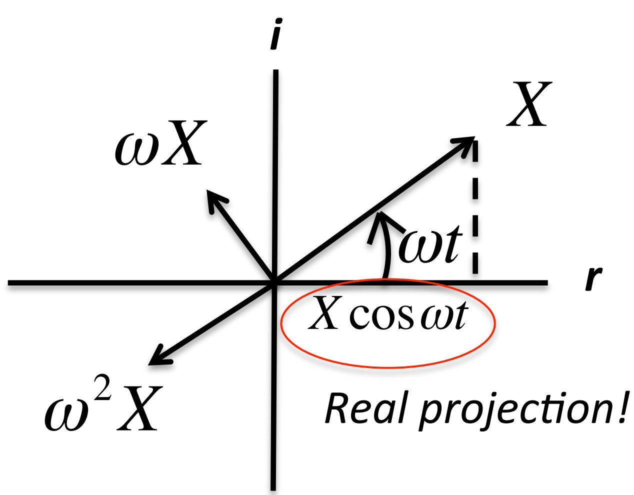
Whiteboard problems

- Springs in parallel and series
- Rotary inertial elements

Harmonic motion

$$
\begin{aligned}
x(t) &= X\cos\omega t \\
\dot{x}(t) &= -\omega X\sin\omega t \qquad \text{Leads by } 90^o \\
\ddot{x}(t) &= -\omega^2 X\cos\omega t \qquad \text{Leads by } 180^o
\end{aligned}
$$

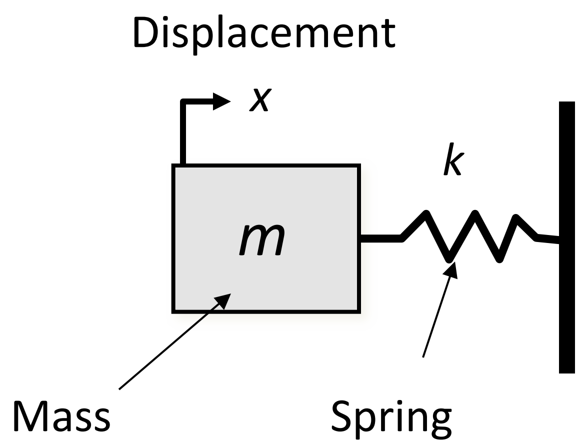
Phasor notation:



Alternative expression in complex notation:

$$
\begin{aligned}
x(t) &= Xe^{i\omega t} \\
\dot{x}(t) &= -i\omega Xe^{i\omega t} \\
\ddot{x}(t) &= -\omega^2 Xe^{i\omega t}
\end{aligned}
$$

Returning to the spring/mass system:

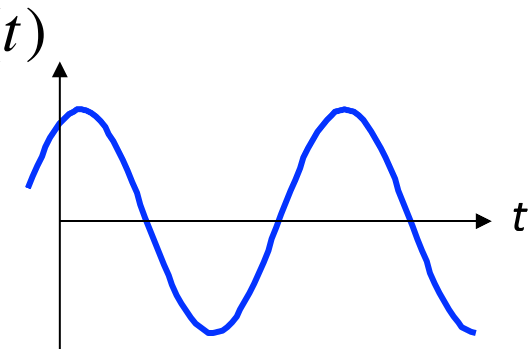


$$m\ddot{x} + kx \quad = \quad 0 \quad \text{(free vibration)}$$

Solution to 2nd order differential equations

Lets assume a solution:

$$x(t) = A\sin(\omega_n t + \phi) \quad (1.3)$$



Differentiating twice gives:

$$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi) \quad (1.4)$$

$$\ddot{x}(t) = -\omega_n^2 A\sin(\omega_n t + \phi) = -\omega_n^2 x(t) \quad (1.5)$$

Substituting back into the equations of motion gives: $m\ddot{x}(t) + kx(t) = 0$

$$-m\omega_n^2 A\sin(\omega_n t + \phi) + kA\sin(\omega_n t + \phi) = 0$$

$$-m\omega_n^2 + k = 0 \quad \text{or} \quad \omega_n = \sqrt{\frac{k}{m}}$$

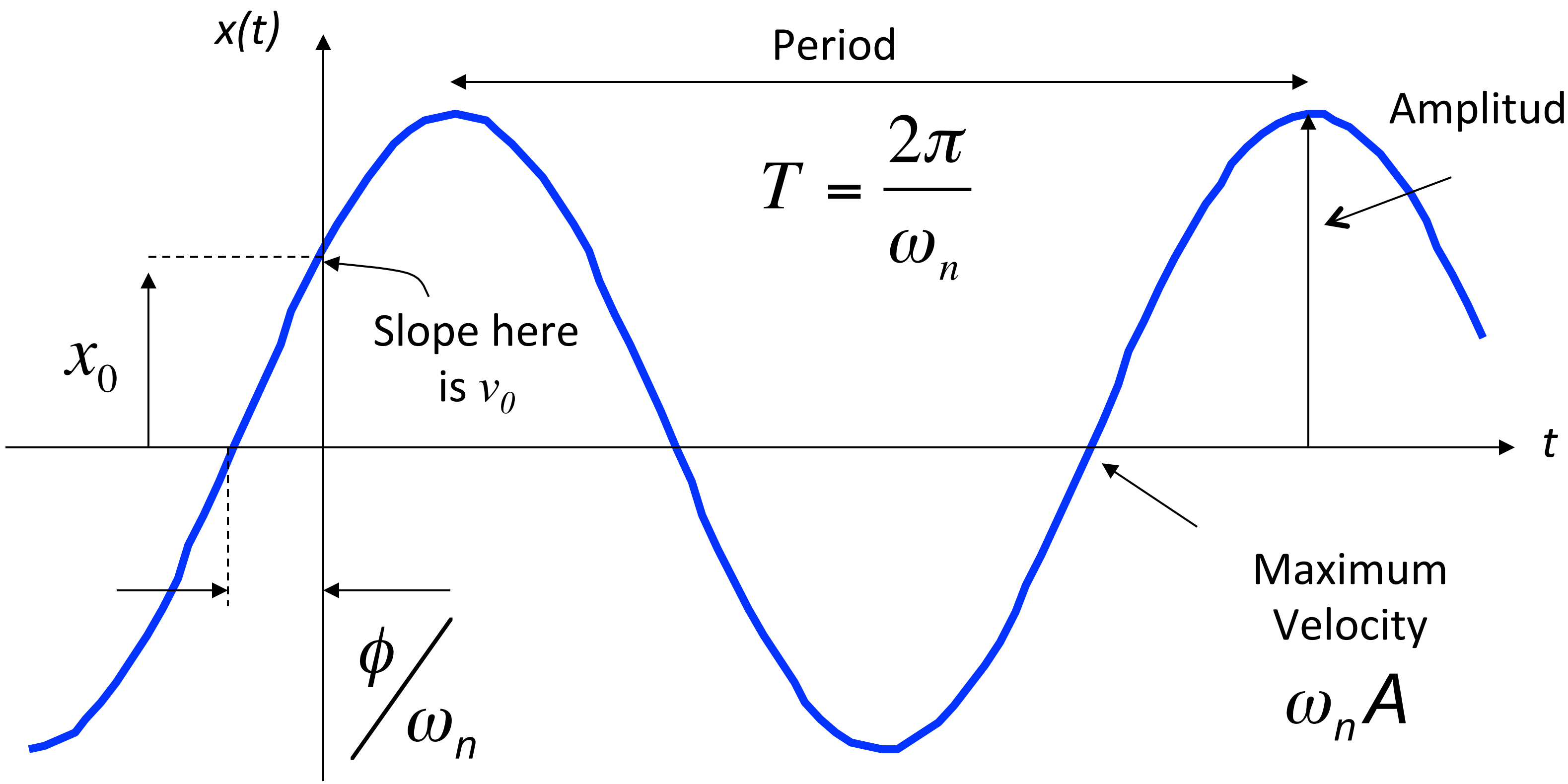
Natural frequency

Units?

rad/s

demo

Summary of simple harmonics



$$f_n = \frac{\omega_n \text{ rad/s}}{2\pi \text{ rad/cycle}} = \frac{\omega_n \text{ cycles}}{2\pi \text{ s}} = \frac{\omega_n}{2\pi}\text{Hz}$$

Initial conditions

If a system is vibrating then we must assume that something must have (in the past) transferred energy into to the system and caused it to move. For example the mass could have been:

- moved a distance x_0 and then released at $t = 0$ (i.e. given Potential energy) or
- given an initial velocity v_0 (i.e. given some kinetic energy) or
- Some combination of the two above cases

From our earlier solution we know that:

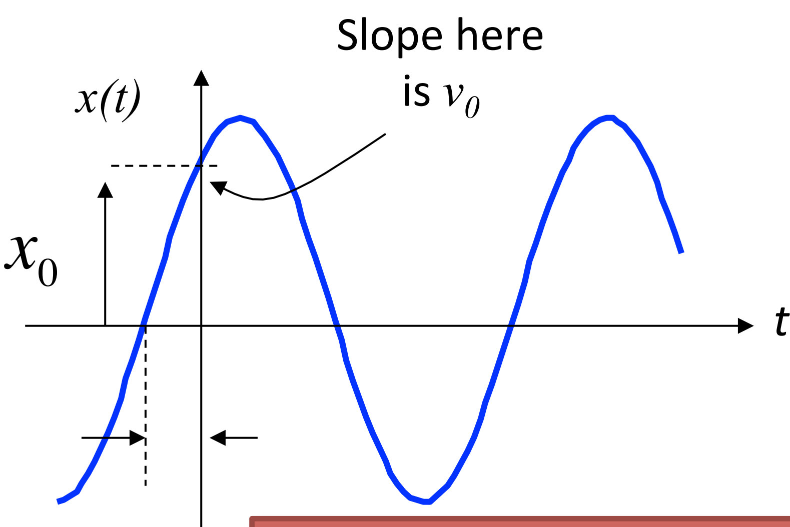
$$\left.\begin{aligned} x_0 &= x(0) = A\sin(\omega_n 0 + \phi) = A\sin(\phi) \\ v_0 &= \dot{x}(0) = \omega_n A\cos(\omega_n 0 + \phi) = \omega_n A\cos(\phi) \end{aligned}\right\}$$

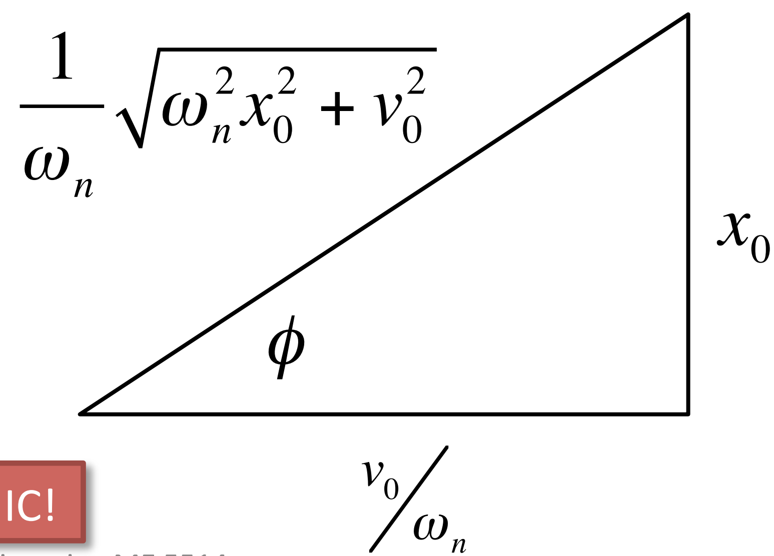
What do we do with these relationships

Initial conditions determine the constants of integration.

Solving these two simultaneous equations for A and ϕ gives:

$$\underbrace{A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}}_{\text{Amplitude}}\,,\quad \underbrace{\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)}_{\text{Phase}}$$





Your solution must match your IC!

Thus the total solution for the spring mass system becomes:

$$x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \sin\left(\omega_n t + \tan^{-1} \frac{\omega_n x_0}{v_0} \right) \qquad (1.10)$$

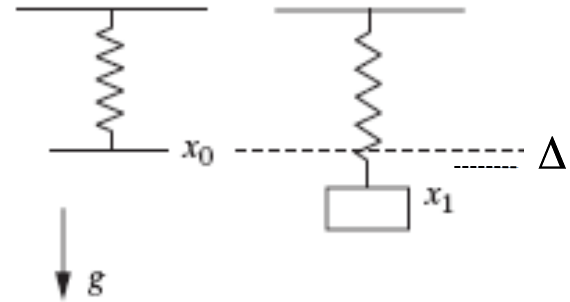
Called the solution to a simple harmonic oscillator
and describes oscillatory motion, or *simple harmonic motion*.

What is the natural frequency of the pendulum?

Wright 1902 glider example

Does gravity matter in spring problems?

Let Δ be the deflection caused by hanging a mass on a spring ($\Delta = x_1 - x_0$ in the figure)



Then from static equilibrium:

$$\boxed{mg = k\Delta}$$

Next sum the forces in the vertical for some point $x > x_1$ measured from Δ

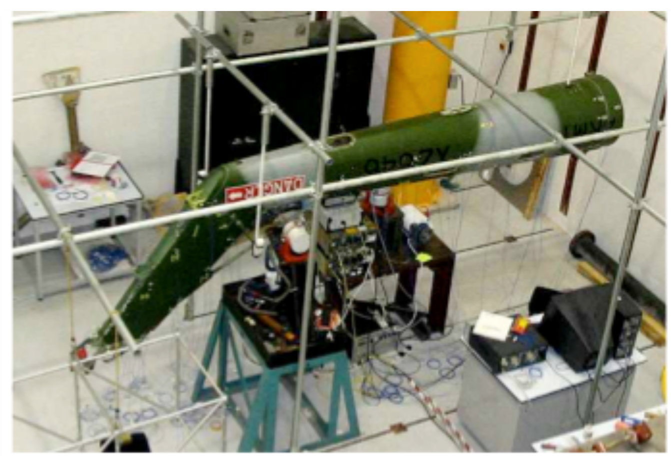
$$m\ddot{x} = -k\left(x + \Delta\right) + mg = -kx + \underbrace{mg - k\Delta}_{=0}$$

$$\Rightarrow m\ddot{x}(t) + kx(t) = 0$$

So no, gravity does not have an effect on the vibration

(note that this is not the case if the spring is nonlinear)

Example of gravity making a difference.



Ref: GARTEUR AG14: Methods for Refinement of Structural Dynamic Finite Element Models

Example 1.2.2: Pendulums and measuring "g"

- A 2 m pendulum swings with a period of 2.893 s
- What is the acceleration due to gravity at that location?

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\ell}{g}}$$

$$g = \frac{4\pi^2}{T^2}\ell = \frac{4\pi^2}{2.893^2\,\mathrm{s}^2}2\ \mathrm{m}$$

$$\Rightarrow g = 9.796\ \mathrm{m/s^2}$$

This is *g* in Denver, CO USA, at 1638m
and a latitude of 40°

Relationship between Displacement, Velocity and Acceleration.

Displacement

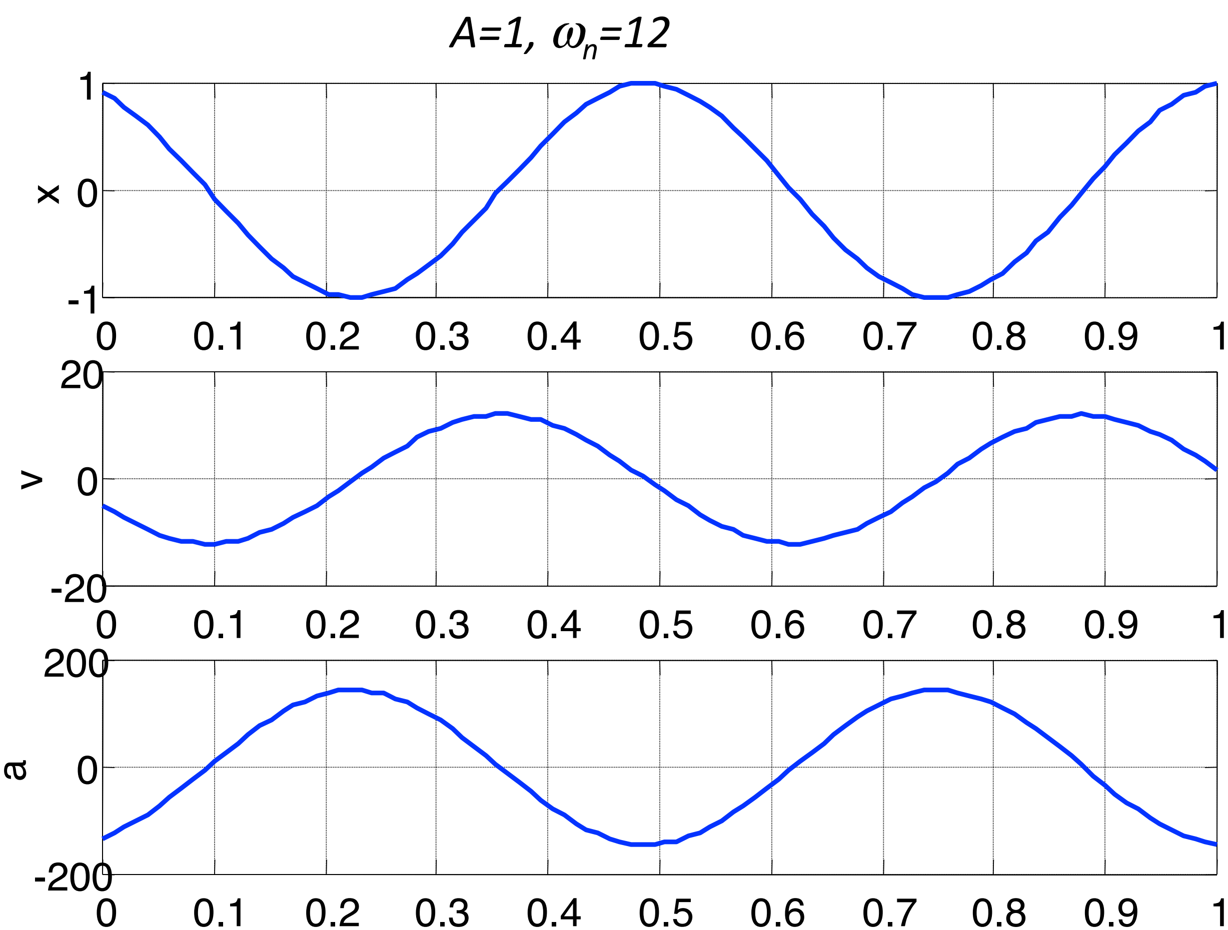
$$x(t) = A\sin(\omega_n t + \phi)$$

Velocity

$$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi)$$

Acceleration

$$\ddot{x}(t) = -\omega_n^2 A\sin(\omega_n t + \phi)$$



Note how the relative magnitude of each increases for $\omega_n > 1$

Equivalent solutions to 2nd order differential equations (see window 1.4)

All of the following solutions are equivalent:

$$x(t) = A\sin(\omega_n t + \phi)$$ Called the magnitude and phase form

$$x(t) = A_1 \sin\omega_n t + A_2 \cos\omega_n t$$ Sometimes called the Cartesian form

$$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t}$$ Called the polar form

The relationships between A and φ, A_1 *and* A_2*, and* a_1 *and* a_2 can be found in Window 1.4 of the course text, page 17.

- Each is useful in different situations
- Each represents the same information
- Each solves the equation of motion

Derivation of the solution

Substitute $\boxed{x(t) = ae^{\lambda t}}$ into $m\ddot{x} + kx = 0 \Rightarrow$

$$m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \Rightarrow$$

$$m\lambda^2 + k = 0 \Rightarrow$$

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}\,j = \pm\omega_n j \Rightarrow$$

$$x(t) = a_1 e^{\omega_n jt} \quad \text{and} \quad x(t) = a_2 e^{-\omega_n jt} \Rightarrow$$

$$x(t) = a_1 e^{\omega_n jt} + a_2 e^{-\omega_n jt} \qquad (1.18)$$

This approach will be used again for more complicated problems

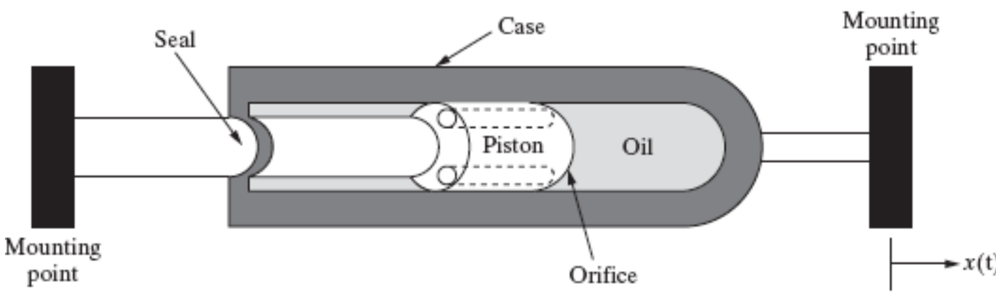
Viscous Damping

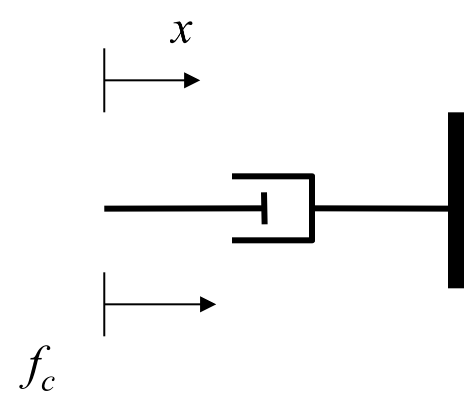
All real systems dissipate energy when they vibrate. To account for this we must consider damping. The most simple form of damping (from a mathematical point of view) is called viscous damping. A viscous damper (or dashpot) produces a force that is proportional to velocity.

Mostly a mathematically motivated form, allowing
a solution to the resulting equations of motion that predicts
reasonable (observed) amounts of energy dissipation.

Damper (c)

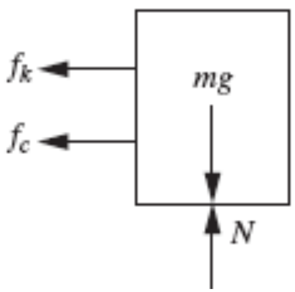
$$\boxed{f_c = -cv(t) = -c\dot{x}(t)}$$





Single degree of Freedom: free response and linear

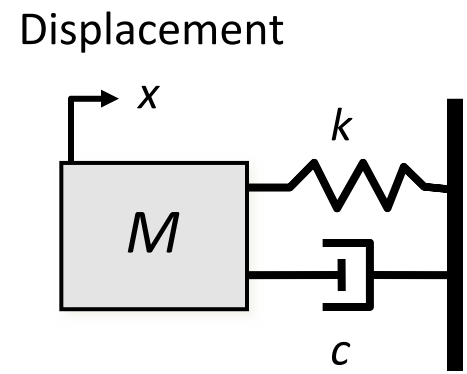
For this damped single degree of freedom system the force acting on the mass is due to the spring and the dashpot i.e. $f_m = -f_k - f_c$.



$$m\ddot{x}(t) = -kx(t) - c\dot{x}(t)$$

or

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \quad (1.25)$$



To solve this for of the equation it is useful to assume a solution of the form (again):

$$x(t) = ae^{\lambda t}$$

What do we do once we assume a solution?

Solution to DE with damping included (dates to 1743 by Euler)

The velocity and acceleration can then be calculated as:

$$\dot{x}(t) = \lambda a e^{\lambda t}$$

$$\ddot{x}(t) = \lambda^2 a e^{\lambda t}$$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0$$

If this is substituted into the equation of motion we get:

$$\underbrace{ae^{\lambda t}}_{=0}\underbrace{(m\lambda^2 + c\lambda + k)}_{=0} = 0 \qquad (1.26)$$

Divide equation by *m*, substitute for natural frequency and assume a non-trivial solution

$$ae^{\lambda t} \neq 0 \quad \Rightarrow \quad (\lambda^2 + \frac{c}{m}\lambda + \omega_n^2) = 0$$

Solution to our differential equation with damping included:

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \qquad (1.28)$$

For convenience we will define a term known as the *damping ratio* as:

$$\zeta = \frac{c}{2\sqrt{km}} \qquad (1.30)$$

Lower case Greek zeta
NOT ξ as some like to use

The equation of motion then becomes:

$$(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2) = 0$$

Solving for λ then gives,

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad (1.31)$$

Damping

There are several cases depending on the value of the damping ratio:

$$-\lambda_{1,2} = -\omega_n \zeta \pm \omega_n \sqrt{\zeta^2 - 1}$$

Case	Name	Damping Ratio	Poles	Response
1	Un-damped	$\zeta = 0$	$\pm i\omega_n$ Complex pair (purely imaginary)	Harmonic constant amplitude
2	Under-damped	$0 < \zeta < 1$	$-\omega_n \zeta \pm i\omega_n \sqrt{1-\zeta^2}$ Complex pair	Harmonic Amplitude decays exponentially
3	Critically-damped	$\zeta = 1$	$-\omega_n$ Real, negative and repeated	Non-oscillatory response
4	Over-damped	$\zeta > 1$	$-\omega_n \zeta \pm \omega_n \sqrt{\zeta^2 - 1}$ Real, negative and distinct	Non-oscillatory response

Damping: Critically damped, under-damped, over-damped and no damping.

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$\zeta = 0 \quad \zeta = 1 \quad 0 < \zeta < 1 \quad \zeta > 1$$

- For the case of no damping (ζ =0)

$$x(t) = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}\ \sin(\omega_n + \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right))$$

$$\omega_n = \sqrt{\frac{k}{m}},\ \text{rad/s} \quad \text{natural frequency}$$

$$x_0 = \text{ the initial position}$$
$$v_0 = \text{ the initial velocity}$$

Possibility 1: Critically damped motion

Critical damping occurs when ζ =1. The damping coefficient *c* in this case is given by:

$$\zeta = 1 \Rightarrow \qquad c = \underbrace{c_{cr} = 2\sqrt{km}}_{\text{definition of critical damping coefficient}} = 2m\omega_n$$

$$\boxed{\zeta = \frac{c}{2\sqrt{km}}}$$

Solving for λ then gives,

$$\lambda_{1,2} = -1\omega_n \pm \omega_n\sqrt{1^2 - 1} = -\omega_n$$

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$$

A repeated, real root

The solution then takes the form

$$x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t}$$

Needs two independent solutions, hence the *t* in the second term

Recall: Assumed solution

$$x(t) = ae^{\lambda t}$$

Possibility 1: Critically damped motion

a_1 and a_2 can be calculated from initial conditions *(t=0),*

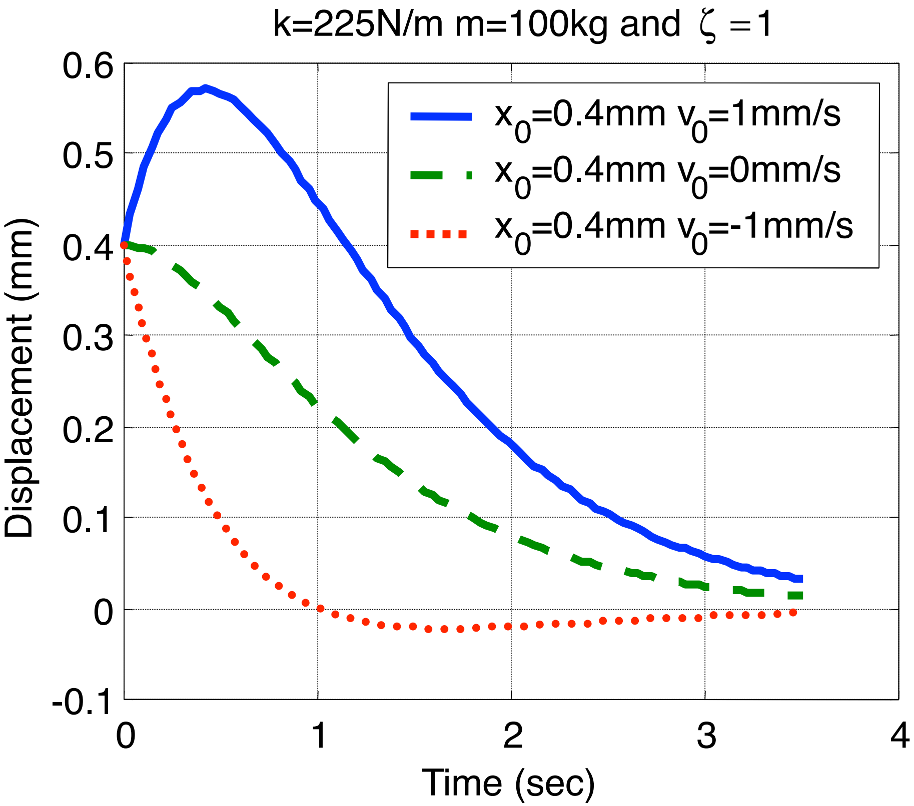
$$x = (a_1 + a_2 t)e^{-\omega_n t}$$

$$\Rightarrow a_1 = x_0$$

$$v = (-\omega_n a_1 - \omega_n a_2 t + a_2)e^{-\omega_n t}$$

$$v_0 = -\omega_n a_1 + a_2$$

$$\Rightarrow a_2 = v_0 + \omega_n x_0$$



- No oscillation occurs
- Useful in door mechanisms, analog gauges

Possibility 2: Over damped motion

An overdamped case occurs when $\zeta > 1$. Both of the roots of the equation are again real.

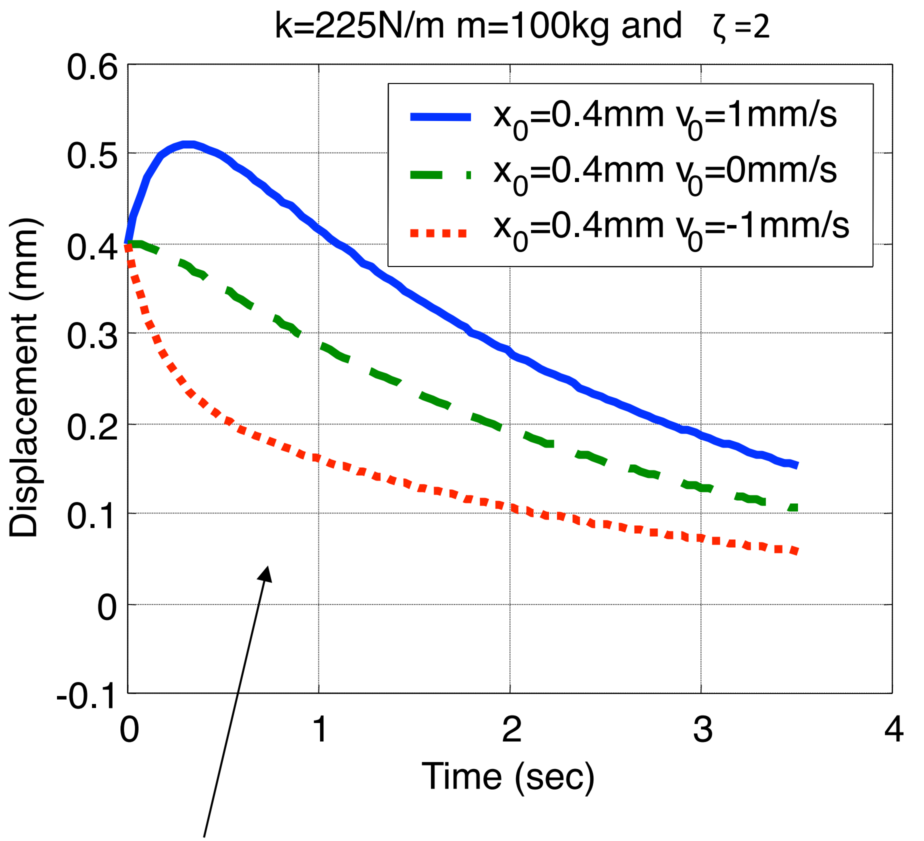
$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_n t\sqrt{\zeta^2-1}} + a_2 e^{\omega_n t\sqrt{\zeta^2-1}}\right)$$

a_1 and a_2 can again be calculated from initial conditions *(t=0)*,

$$a_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$a_2 = \frac{v_0 + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$



Slower to respond than critically damped case

Possibility 3: Under-damped motion

An under-damped case occurs when $0<\zeta<1$. The roots of the equation are complex conjugate pairs. This is the most common case and the only one that yields oscillation.

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n j\sqrt{1-\zeta^2}$$

$$x(t) = e^{-\zeta\omega_n t}(a_1 e^{j\omega_n t\sqrt{1-\zeta^2}} + a_2 e^{-j\omega_n t\sqrt{1-\zeta^2}})$$

$$= Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

The frequency of oscillation ω_d is called the *damped natural frequency* and is given by,

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad (1.37)$$

How do we get
A and ϕ

Possibility 3: Under-damped motion, constants of integration

As before *A* and ϕ can be calculated from initial conditions (*t=0*),

$$A = \frac{1}{\omega_d}\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)$$

- Gives an oscillating response with exponential decay
- Most natural systems vibrate with and underdamped response
- See Window 1.5 for details and other representations