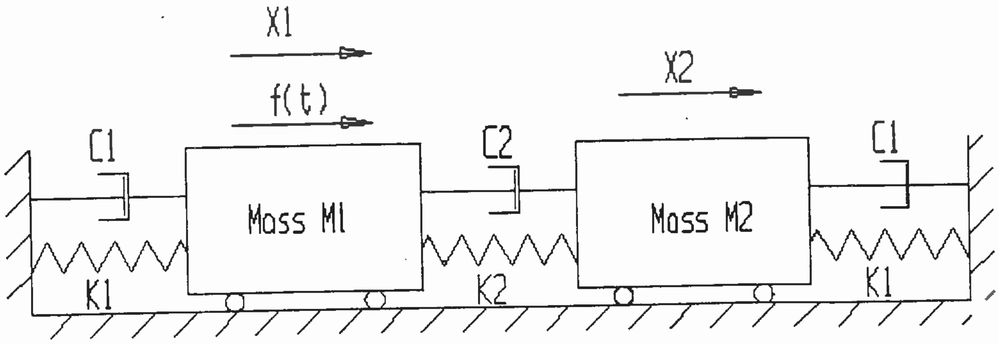
3.4 *The Effect of Proportional Damping on MDOF Systems*

Damping is usually modeled as a viscous element next to a stiffness element:



The differential equations differ only in that a damping matrix has been added to the system:

$$\begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (C_1 + C_2) & -C_2 \\ -C_2 & (C_1 + C_2) \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_1 + K_2) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}. \tag{3.45}$$

Considering a harmonic forcing function $f(t) = F_1 e^{i\omega t}$ and writing the assumed form of the response:

$$\begin{aligned} x_1(t) &= X_1 e^{i\omega t} \\ x_2(t) &= X_2 e^{i\omega t} \end{aligned} \tag{3.46}$$

The equation can be rewritten as:

$$\left(-\omega^2\begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix} + i\omega\begin{bmatrix} C_1 + C_2 & -C_2 \\ -C_2 & C_1 + C_2 \end{bmatrix} + \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_1 + K_2) \end{bmatrix}\right)\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\omega t} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} e^{i\omega t} \qquad (3.47)$$

We have already shown that simplification (diagonalization) can occur with the mass and stiffness matrices if they are pre- and post-multiplied by the modal matrix. If this is true, then the same should hold for the damping matrix *provided* that:

$$[C] = \alpha[M] + \beta[K] \qquad (3.48)$$

Or, *the damping matrix must be a linear combination of the mass and stiffness matrices.*

This is called <u>proportional damping.</u>

For many real systems, the assumption of proportional damping is valid since damping is frequently related to the stiffness of the structure. Examples include:

1. Homogeneous materials such as plates and beams

2. Isolators used on vibrating equipment which provide stiffness as well as damping

3. Viscoelastic damping treatment on a uniform surface

Cases where proportional damping is not a valid assumption, or the system has nonproportional damping:

1. Viscous damping elements that are not matched to a stiffness
2. Fastened joints (usually nonlinear)

The form of the response will be similar to the undamped case:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = [\phi]\left(-\omega^2\begin{bmatrix} \ddots & & \\ & m & \\ & & \ddots \end{bmatrix} + i\omega\begin{bmatrix} \ddots & & \\ & c & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} \ddots & & \\ & k & \\ & & \ddots \end{bmatrix}\right)^{-1}[\phi]^T\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \tag{3.49}$$

And the only change from what was presented previously is the addition of the diagonal damping matrix.

In general, the FRF will have the following form:

$$\frac{X_i}{F_j} = \sum_{r=1}^{n}\left(\frac{\phi_{ir}\phi_{jr}}{-\omega^2 m_r + ic_r\omega + k_r}\right) \tag{3.50}$$

Using the system's rth natural frequency ω_r and rth damping ratio ζ_r to simplify the expression similar to the undamped case:

$$H_{ij}(\omega) = \frac{X_i}{F_j} = \sum_{r=1}^{n}\left(\frac{\hat{\phi}_{ir}\hat{\phi}_{jr}}{-\omega^2 + i2\zeta_r\omega\omega_r + \omega_r^2}\right) \tag{3.51}$$

This is the most common form of the FRF for viscously damped systems. Expressed in matrix-vector form yields:

$$\{X\} = [\phi]\left(\left[\ddots -\omega^2 \ddots\right] + \left[\ddots i2\zeta\omega\omega_r \ddots\right] + \left[\ddots \omega_r^2 \ddots\right]\right)^{-1} [\phi]^T \{F\} \quad (3.52)$$

Note that the mode vectors are mass-normalized; they can be replaced by the residue ${}_rA_{ij}$:

$$H_{ij}(\omega) = \frac{X_i}{F_j} = \sum_{r=1}^{n} \left(\frac{{}_rA_{ij}}{-\omega^2 + i2\zeta_r\omega\omega_r + \omega_r^2} \right) \quad (3.53)$$

In experimental measurements, a curve-fitting program is used to determine the residue and the terms in the denominator.
Is there a way of determining the individual mode vector components?

- If the ith driving point FRF is measured, the experimentally determined residue is $\left(\hat{\phi}_{ir}\right)^2$ for the the rth natural frequency. Therefore, we can determine the driving point mode amplitude by taking the square root of this term. All other measured residues will be in the form of $\hat{\phi}_{ir}\hat{\phi}_{jr}$, so to obtain the jth amplitude, the residue is divided by $\hat{\phi}_{ir}$.

3.4.1 Some Notes About the Eigenvectors of Proportionally Damped Systems

It has been stated that the eigenvectors (mode vectors) of an undamped system will diagonalize the mass, stiffness and damping matrices of a proportionally damped system.

These undamped eigenvectors will be *different* from the eigenvectors obtained from the damped system in that they will be *rotated by some constant amount relative to the forcing function.* Because all of the responses are shifted, they maintain the same *relative* phase relationship. It is the relative relationship that enforces the orthogonality condition of the vectors.

When there is damping, the problem of finding the eigenvectors (mode vectors) becomes a *complex eigenproblem*, where both the eigenvalues and eigenvectors will be complex. The eigenvalues will occur in complex conjugate pairs and hence there will be *2n* eigenvalues.

To solve for the eigenvectors and eigenvalues, the homogeneous form of Eqn. 3.45 must be recast in a state-vector form:

$$\begin{bmatrix} [\overline{m}] & [\overline{c}] \\ [0] & [\overline{m}] \end{bmatrix} \begin{Bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{Bmatrix} + \begin{bmatrix} [0] & [\overline{k}] \\ -[\overline{m}] & [0] \end{bmatrix} \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3.54)$$

Where the state vector is defined as $\{\bar{x}\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix}$. As usual, the solution is assumed to have a complex exponential form:

$$\{\bar{x}\} = \{\bar{X}\} e^{rt} \qquad (3.55)$$

To examine the results using this form of the equation, the previous problem is analyzed with the following damping values:

$$\begin{aligned} C_1 &= 0.5 \\ C_2 &= 0.3 \end{aligned} \qquad \text{(consistent units)}$$

Substituting into Eqn. 3.45:

$$\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0.8 & -0.3 \\ -0.3 & 0.8 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 30 & -5 \\ -5 & 30 \end{bmatrix}\right)\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} e^{i\omega t} \qquad (3.56)$$

Rewriting in the form of Eqn 3.54:

$$\left(\lambda\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0.8 & -0.3 \\ -0.3 & 0.8 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 30 & -5 \\ -5 & 30 \end{bmatrix} \\ -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}\right)\left\{ \bar{X} \right\} = \left\{ 0 \right\} \qquad (3.57)$$

Note that this state vector form can be rewritten in standard eigenproblem form, $[A]\{X\} - \lambda[I]\{X\} = \{0\}$, by premultiplying by the inverse of the first matrix and -1:

$$\begin{aligned} \left(-\lambda[F]^{-1}[F] - [F]^{-1}[G]\right)\{\bar{X}\} &= \{0\} \\ \left([H] - \lambda\,[I]\right)\{\bar{X}\} &= \{0\} \end{aligned} \qquad (3.58)$$

The resulting eigenvalues and eigenvectors are:

$$\phi_1 = \begin{pmatrix} -0.0892 - 0.688i \\ -0.0892 - 0.688i \\ 0.138 - 0.0109i \\ 0.138 - 0.0109i \end{pmatrix} \quad \lambda_1 = 0.25 + 4.99i \qquad \phi_2 = \begin{pmatrix} -0.0892 + 0.688i \\ -0.0892 + 0.688i \\ 0.138 + 0.0109i \\ 0.138 + 0.0109i \end{pmatrix} \quad \lambda_2 = 0.25 - 4.99i$$

$$\phi_3 = \begin{pmatrix} -0.552 - 0.426i \\ 0.552 + 0.426i \\ 0.0804 - 0.0862i \\ -0.0804 + 0.0862i \end{pmatrix} \quad \lambda_3 = 0.55 + 5.89i \qquad \phi_4 = \begin{pmatrix} -0.552 + 0.426i \\ 0.552 - 0.426i \\ 0.0804 + 0.0862i \\ -0.0804 - 0.0862i \end{pmatrix} \quad \lambda_4 = 0.55 - 5.89i$$

(3.59)

Notes:

- The eigenvectors and eigenvalues occur in complex conjugate pairs. Only values for modes one and three are needed to describe the problem.
- The top two entries of each eigenvector are the lower entries scaled by λ.

To return to the eigenvectors and eigenvalues of the *original* problem, we need to extract the $\{x\}$ vector from $\begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix}$. Keeping the last two entries of ϕ_1 and ϕ_3 results in:

$$\phi = \begin{bmatrix} 0.138 - 0.0109i & 0.0804 - 0.0862i \\ 0.138 - 0.0109i & -0.0804 + 0.0862i \end{bmatrix} \qquad (3.60)$$

And now the diagonal mass, damping and stiffness can be determined by pre-multiplying by transpose of the eigenvectors and post-multiplying by the eigenvectors:

$$\begin{aligned}
\left[\ddots m_{ii} \ddots\right] &= \begin{bmatrix} 0.0384 & \\ & 0.0278 \end{bmatrix} \\
\left[\ddots c_{ii} \ddots\right] &= \begin{bmatrix} 0.0192 & \\ & 0.0306 \end{bmatrix} \\
\left[\ddots k_{ii} \ddots\right] &= \begin{bmatrix} 0.961 & \\ & 0.973 \end{bmatrix}
\end{aligned} \tag{3.61}$$

The mass-normalized eigenvector matrix can be determined, and using it to again diagonalize the matrices results in:

$$\begin{aligned}
\left[\ddots \hat{m}_{ii} \ddots\right] &= \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \\
\left[\ddots \hat{c}_{ii} \ddots\right] &= \begin{bmatrix} 0.50 & \\ & 1.10 \end{bmatrix} \\
\left[\ddots \hat{k}_{ii} \ddots\right] &= \begin{bmatrix} 25.0 & \\ & 35.0 \end{bmatrix}
\end{aligned} \tag{3.62}$$

- It should be recognized that the natural frequencies for this system are identical to those for the undamped system.

The eigenvalues and eigenvectors contain information about system damping; the eigenvalue has information about *both* damping and natural frequency. Recall that SDOF vibration can be described by a complex quantity that has a decay term and a frequency term:

$$y(t) = Ye^{\lambda t} \tag{3.63}$$

where:

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d \tag{3.64}$$

Since we are dealing with the same form of the solution, the damping ratio ζ_r and the natural frequency ω_r can be identified from the terms above:

$$\omega_1 = 5.0 \text{ rad / sec} \qquad \omega_2 = 5.9 \text{ rad / sec}$$

$$\zeta_1 = 0.05 \qquad \zeta_2 = 0.093$$

These values can be derived from the mass, damping and stiffness matrices as well (using the form of Eqn. 3.52).

Previously it was stated that the diagonal matrices of Eqn. 3.62 can also be obtained using the *undamped* eigenvectors. The only difference between the damped and undamped eigenvectors is a phase angle; the relative motion between the masses does not change. Recall that the mass-normalized undamped eigenvectors are:

$$\hat{\phi} = \begin{bmatrix} .707 & .707 \\ .707 & -.707 \end{bmatrix} \tag{3.65}$$

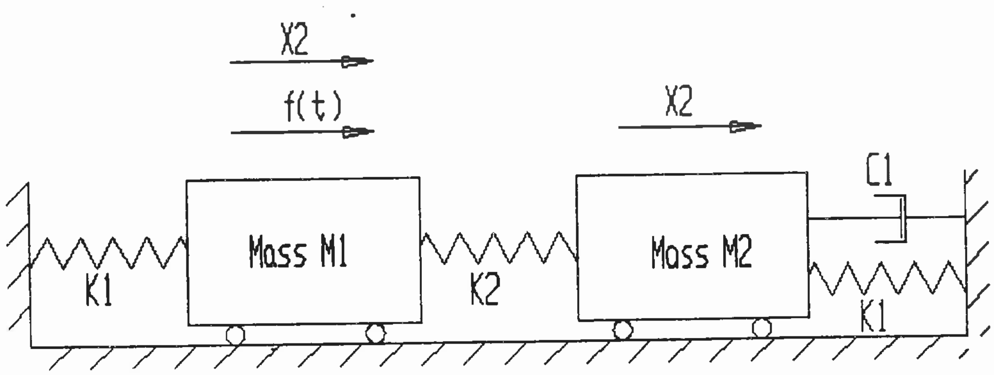
Using this matrix, Eqn. 3.62 is again obtained.

Notes:

- The eigenvectors are the phase-shifted undamped eigenvectors. They are complex but have *relative* phase angles of 0 or 180 degrees.
- The natural frequencies correspond to the undamped natural frequencies.
- The diagonalized mass, damping and stiffness matrices are real.

3.4.2 The Effect of Nonproportional Damping on MDOF Systems

For the case of nonproportional damping, $[C] \neq \alpha[M]+\beta[K]$, the same solution technique as was used for proportional damping is employed. The *differences* in the results are demonstrated by using the following model variation:



In the above model, only the right-hand damper has been left in. The resulting equations of motion in matrix form are:

$$\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 30 & -5 \\ -5 & 30 \end{bmatrix}\right)\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} e^{i\omega t} \qquad (3.66)$$

The eigenproblem becomes:

$$\left(\lambda\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 30 & -5 \\ -5 & 30 \end{bmatrix} \\ -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}\right)\left\{\bar{X}\right\} = \left\{0\right\} \qquad (3.67)$$

And the solution gives the following eigenvalues and eigenvectors:

$$\begin{aligned} \phi &= \begin{bmatrix} 0.122 - 0.0648i & 0.118 - 0.0134i \\ 0.135 - 0.0324i & -0.108 + 0.0473i \end{bmatrix} \\ \lambda_1 &= 0.126 + 5.01i \\ \lambda_3 &= 0.124 + 5.90i \end{aligned}$$

Notes:

- The eigenvectors are *not* the phase-shifted undamped eigenvectors. They are fully complex, having phase angles that are not 0 or 180 degrees.
- The natural frequencies *do not* correspond to the undamped natural frequencies.

- The mass, damping and stiffness matrices cannot be diagonalized for this form of the problem. The best that can be achieved is to have real diagonal terms and complex off-diagonal terms.

3.4.3 Structural or Hysteretic Damping

Another type of damping commonly assumed in modal analysis is called hysteretic, or structural damping. This differs from viscous damping in that the energy dissipated is not proportional to frequency, but only to the square of the amplitude of oscillation:

$$W_d = \alpha X^2 \tag{3.68}$$

An equivalent damping element is obtained for the dynamic model by setting this quantity equal to the viscous damping energy loss:

$$W_d = \pi c_{eq} \omega X^2 = \alpha X^2 \tag{3.69}$$

Solving for c_{eq}:

$$c_{eq} = \frac{\alpha}{\pi\omega} \tag{3.70}$$

Which results in the SDOF equation:

$$m\ddot{x} + \left(\frac{\alpha}{\pi\omega}\right)\dot{x} + kx = f(t) \tag{3.71}$$

For harmonic excitation, $f(t) = F e^{i\omega t}$ and the response will be $x = X e^{i\omega t}$ which leads to:

$$\begin{aligned} \left(-\omega^2 m + i\omega\left(\frac{\alpha}{\pi\omega}\right) + k\right) X e^{i\omega t} &= F e^{i\omega t} \\ \left(-\omega^2 m + ih + k\right) X &= F \end{aligned} \qquad (3.72)$$

The new term h is the hysteretic damping term. A more common way to express this is by combining the damping with the stiffness to form a *complex stiffness:*

$$\left(-\omega^2 m + k(1 + i\eta)\right) X = F \qquad (3.73)$$

Using this background, it is now possible to derive a relationship for forced MDOF systems.

Recall Eqn. 3.24 that expressed the response of an undamped, MDOF system:

$$\left(\begin{bmatrix} -\omega^2 M_1 & \\ & -\omega^2 M_2 \end{bmatrix} + \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_1 + K_2) \end{bmatrix}\right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{i\omega t} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} e^{i\omega t} \qquad (3.74)$$

This will result in an FRF of the form:

$$H_{ij}(\omega) = \frac{X_i}{F_j} = \sum_{r=1}^{n} \left(\frac{\hat{\phi}_{ir} \hat{\phi}_{jr}}{-\omega^2 + \lambda_r} \right) \tag{3.75}$$

If the stiffness is complex, then the resulting FRF will be:

$$H_{ij}(\omega) = \frac{X_i}{F_j} = \sum_{r=1}^{n} \left(\frac{\hat{\phi}_{ir} \hat{\phi}_{jr}}{-\omega^2 + \omega_r^2 (1 + i\eta)} \right) \tag{3.76}$$

Where the eigenvalues for the problem $\lambda_r = \omega_r^2 (1 + i\eta)$ are now complex. The eigenvectors for this problem are solved for in the usual way and will *be real.*

Complex stiffness not only can be a better representation of the true damping characteristics, but it will also simplify parameter estimation.

3.5 Summary

We have shown that if damping is assumed to be viscous (or vicious) and proportional, then the response of a point to a harmonic force is obtained using Eqn. 3.49. For the non-proportional case, a state vector form of the EVP must be used (Eqn. 3.50).

If the damping is hysteretic (or hysterical), then the response is obtained using Eqn. 3.72. The eigenvectors (mode vectors or scaled mode shapes) for all cases are obtained by solving the determinant of the EVP.