MATH 5315 Homework 3 Andy Lu

3.) Proof:

Suppose $f \in \mathbb{P}_K$. Pick $n \in \mathbb{N}$ s.t. $n > K$. Then by theorem 4 in section 6.2 of the book $f[x_0, x_1, \dots x_n] = 0$, since the nth derivative of f equals zero.

4.) Proof: Given $A_i(x) = [1 - 2(x - x_i)\ell_i'(x_i)]\ell_i^2(x) \quad (0 \le i \le n)$

$B_i(x) = (x - x_i)\ell_i^2(x) \quad (0 \le i \le n)$

where $\ell_i(x) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{x - x_j}{x_i - x_j} \quad (0 \le i \le n)$

Note: $\ell_i(x_i) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{(x_i - x_j)}{(x_i - x_j)} = \prod_{\substack{j=0 \\ j \ne i}}^{n} 1 = 1 \quad$ (Eq. 1)

$\ell_i(x_j) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{(x_j - x_j)}{(x_i - x_j)} = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{(0)}{(x_i - x_j)} = 0 \quad$ (Eq. 2)

i. $A_i(x) = \delta_{ij}$

Case I: $x = x_i$

$A_i(x_i) = [1 - 2(x_i - x_i)\ell_i'(x_i)]\ell_i^2(x_i)$

$= 1$ since $(x_i - x_i) = 0$ and $\ell_i(x_i) = 1$ from (Eq. 1)

Case II: $x = x_j$

$A_i(x_j) = [1 - 2(x_j - x_i)\ell_i'(x_i)]\ell_i^2(x_j)$

$= 0$ since the square brackets multiply $\ell_i^2(x_j) = (0)^2 = 0$ by (Eq. 2).

Thus $A_i(x) = \delta_{ij}$.

ii. $A_i'(x_j) = 0$

$$A_i'(x_j) = \frac{d}{dx}\left(\left[1 - 2(x_j - x_i)\,l_i'(x_i)\right] l_i^2(x)\right)$$

$$= \frac{d}{dx}\left(l_i^2(x) + 2x_i\,l_i'(x_i)\,l_i^2(x) - 2\,l_i'(x_i)\,l_i^2(x)\,x\right) \quad \left[\begin{matrix}\text{Distribute}\\ \text{terms}\end{matrix}\right]$$

$$= \frac{d}{dx}\left(l_i^2(x)\left[1 + 2x_i\,l_i'(x_i)\right] - 2\,l_i'(x_i)\,l_i^2(x)\,x\right)$$

The first term is a constant times $l_i^2(x)$, thus only a chain rule is needed. The second term is a constant times $l_i^2(x)$ times x; thus, a product rule is needed.

$$= 2(l_i(x))\,l_i'(x)\left[1 + 2x_i\,l_i'(x_i)\right] - 2\,l_i'(x_i)\left[x\,2\,l_i(x)\,l_i'(x) + l_i^2(x)\right]$$

Evaluating at $x = x_j$ and using Eq. 2:

$$= 2(0)\,l_i'(x_j)\left[1 + 2x_i\,l_i'(x_i)\right] - 2\,l_i'(x_i)\left[x_j\,2\,\cancel{l_i(x_j)}^{0}\,l_i'(x_j) + \cancel{l_i^2(x_j)}^{0}\right]$$

$$= 0 - 0 = 0$$

Thus $A_i'(x_j) = 0$

iii. $B_i(x_j) = 0$

Using Eq. 2, $B_i(x_j) = (x - x_i)\,l_i^2(x_j) = (x_j - x_i)(0)^2 = 0$

Thus $B_i(x_j) = 0$

iv. $B_i'(x_j) = \delta_{ij}$

$$B_i'(x_j) = x(2)\,l_i(x)\,l_i'(x) + l_i^2(x) - x_i(2)\,l_i(x)\,l_i'(x)$$

Case I: $x = x_i$

Evaluating $B_i'(x)$ at $x = x_i$ and using Eq. 1:

$$B_i'(x_i) = 2x_i\,l_i'(x_i) + 1 - 2x_i\,l_i'(x_i) = 1$$

Case II: $x = x_j$

Evaluating $B_i'(x)$ at $x = x_j$, since every term has a $l_i(x_j)$, then by Eq. 2, $B_i'(x_j) = 0$

Thus $B_i'(x_j) = \delta_{ij}$

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