Bab 11
POLINOMIAL

Tujuan

- Membahas tentang polinomial pangkat satu dua, tiga sampai pangkat ke –n

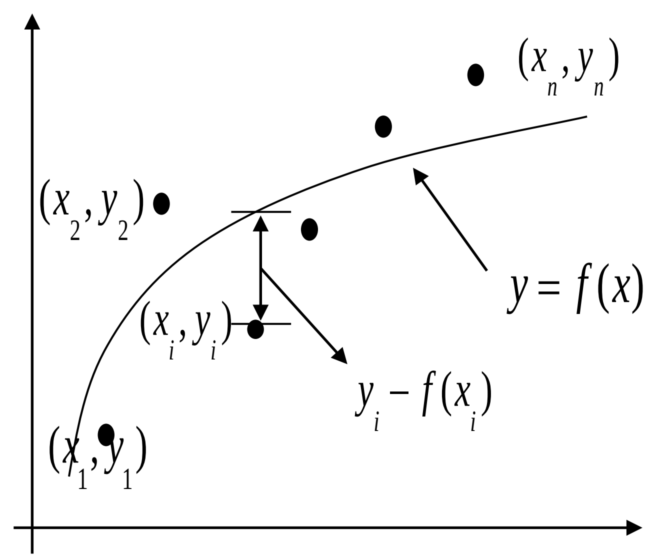
REGRESI NON LINIER

Metode yang popular digunakan :

1. Eksponensial $y = ae^{bx}$
2. Power $y = ax^b$
3. Pertumbuhan Saturasi $y = \frac{ax}{b+x}$
4. Polinomial $y = a_0 + a_1x + \cdots + a_nx^n$

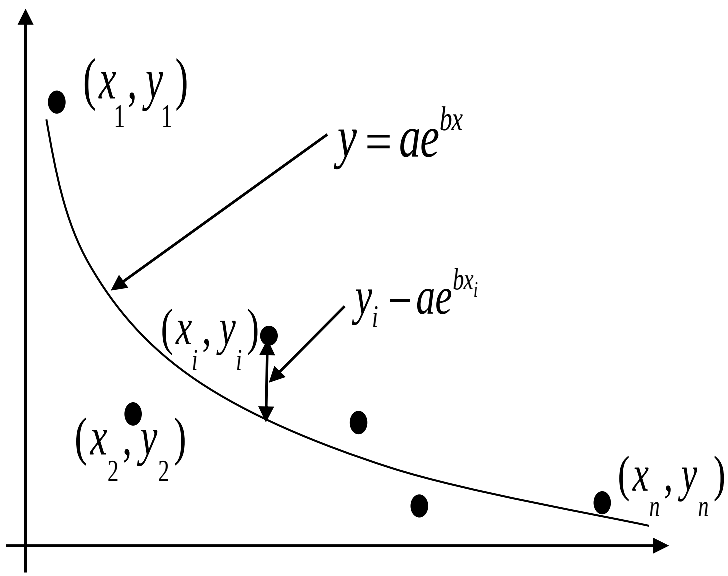
Regresi Non Linier

Contoh ada data sejumlah n titik $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Persamaan data adalah $y = f(x)$. Dimana $f(x)$ adalah fungsi non linier dari x



Eksponensial

Data berupa titik-titik $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, dengan $y = ae^x$



Mencari Konstanta dari Model Eksponensial

Penjumlahan dari selisih yang dikuadratkan :

$$S_r = \sum_{i=1}^{n} (y - ae^{bx_i})^2$$

Persamaan atas didiferensialkan dengan *a* & *b*

$$\frac{\partial S_r}{\partial a} = \sum_{i=1}^{n} 2\left(y_i - ae^{bx_i}\right)\left(-e^{bx_i}\right) = 0$$

$$\frac{\partial S_r}{\partial b} = \sum_{i=1}^{n} 2\left(y_i - ae^{bx_i}\right)\left(-ax_ie^{bx_i}\right) = 0$$

Mencari Konstanta dari Model Eksponensial

Persamaan atas juga bisa ditulis :

$$-\sum_{i=1}^{n} y_i e^{bx_i} + a\sum_{i=1}^{n} e^{2bx_i} = 0$$

$$\sum_{i=1}^{n} y_i x_i e^{bx_i} - a\sum_{i=1}^{n} x_i e^{2bx_i} = 0$$

Mencari Konstanta dari Model Eksponensial

Penyelesaian persamaan pertama :

$$a = \frac{\sum_{i=1}^{n} y_i e^{bx_i}}{\sum_{i=1}^{n} e^{2bx_i}}$$

Lalu subtitusikan kembali ke persamaan sebelumnya :

$$\sum_{i=1}^{n} y_i x_i e^{bx_i} - \frac{\sum_{i=1}^{n} y_i e^{bx_i}}{\sum_{i=1}^{n} e^{2bx_i}} \sum_{i=1}^{n} x_i e^{2bx_i} = 0$$

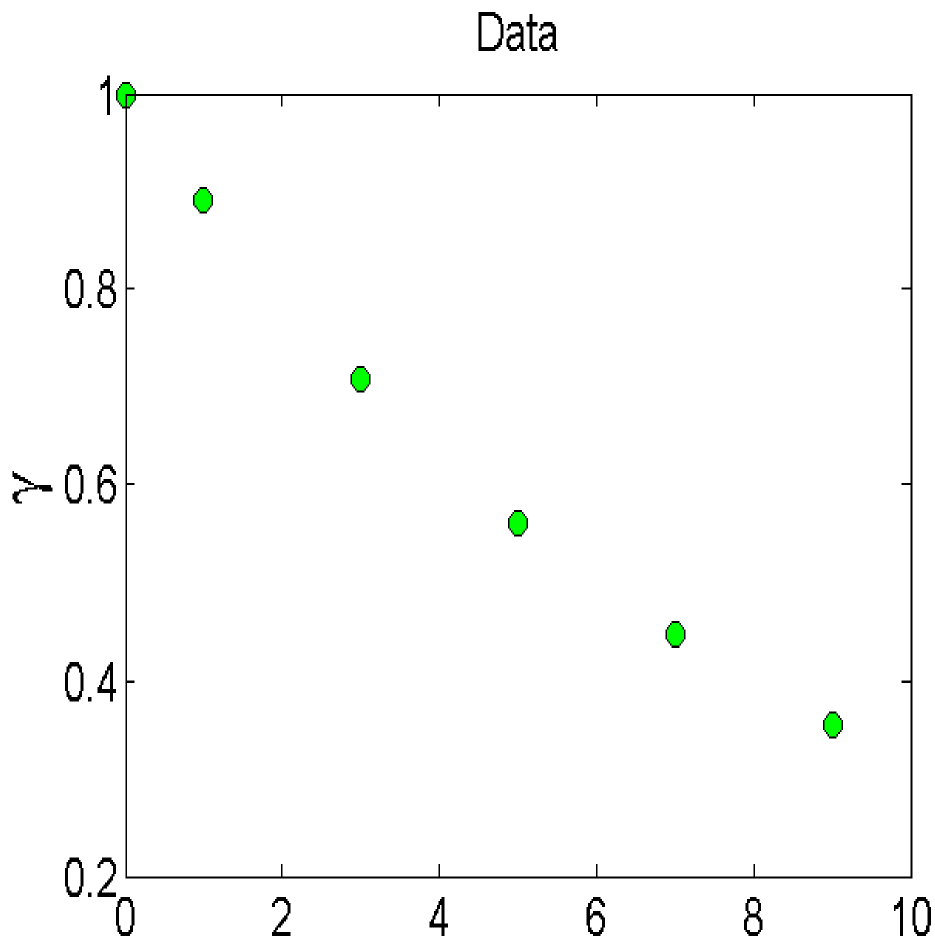
Konstanta *b* bisa dicari dengan metode numerik biseksi

Contoh Eksponensial

Hasil pengukuran bahan radioaktif sebagai fungsi waktu dengan menggunakan beberapa tetes Technetium dengan isotop 99m dengan pemindaian gallbladder. Technetium akan menghilang setengah setelah 6 jam. Tapi butuh 24 jam agar radiasi mencapai ambang batas aman. Berikut intensitas relative radiasi sebagai fungsi waktu

t(jam)	0	1	3	5	7	9
γ	1	0.891	0.708	0.562	0.447	0.355

Gambarkan Data jadi Grafik



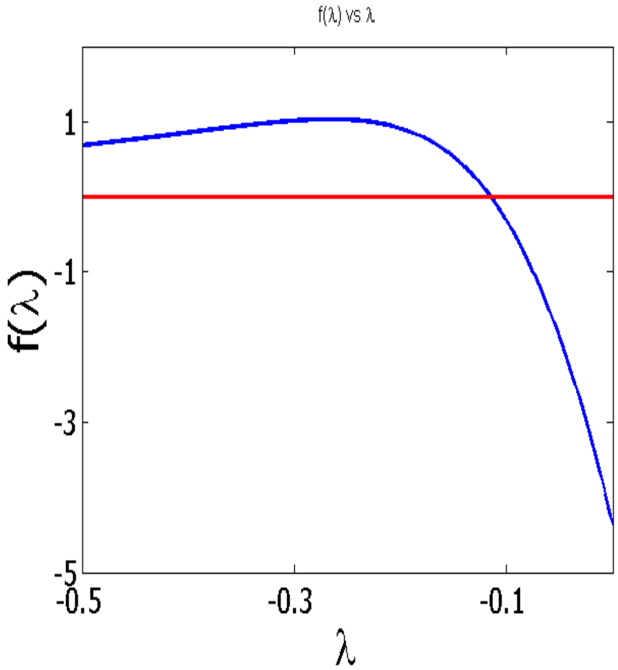
Mencari Konstanta

λ dapat dicari dengan penyelesaian non linier

$$f(\lambda) = \sum_{i=1}^{n} \gamma_i t_i e^{\lambda t_i} - \frac{\sum_{i=1}^{n} \gamma_i e^{\lambda t_i}}{\sum_{i=1}^{n} e^{2\lambda t_i}} \sum_{i=1}^{n} t_i e^{2\lambda t_i} = 0$$

$$A = \frac{\sum_{i=1}^{n} \gamma_i e^{\lambda t_i}}{\sum_{i=1}^{n} e^{2\lambda t_i}}$$

Simulasi persamaan di MATLAB



$$f\left(\lambda\right)=\sum_{i=1}^{n}\gamma_i t_i e^{\lambda t_i}-\frac{\sum_{i=1}^{n}\gamma_i e^{\lambda t_i}}{\sum_{i=1}^{n}e^{2\lambda t_i}}\sum_{i=1}^{n}t_i e^{2\lambda t_i}=0$$

t (hrs)	0	1	3	5	7	9
γ	1.000	0.891	0.708	0.562	0.447	0.355

Simulasi persamaan di MATLAB

$$f(\lambda) = \sum_{i=1}^{n} \gamma_i t_i e^{\lambda t_i} - \frac{\sum_{i=1}^{n} \gamma_i e^{\lambda t_i}}{\sum_{i=1}^{n} e^{2\lambda t_i}} \sum_{i=1}^{n} t_i e^{2\lambda t_i} = 0$$

$$\lambda = -0.1151$$

```
t=[0 1  3  5  7  9]
gamma=[1  0.891  0.708  0.562  0.447  0.355]
syms lamda
sum1=sum(gamma.*t.*exp(lamda*t));
sum2=sum(gamma.*exp(lamda*t));
sum3=sum(exp(2*lamda*t));
sum4=sum(t.*exp(2*lamda*t));
f=sum1-sum2/sum3*sum4;
```

Mencari Konstanta Lainnya

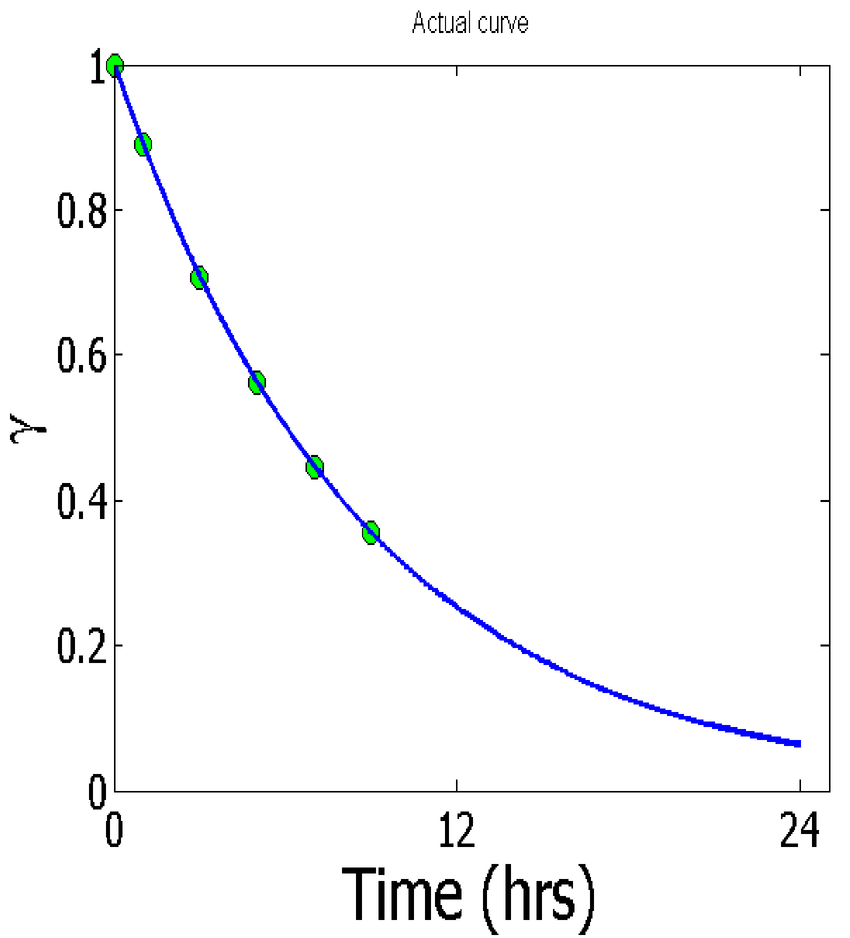
Sekarang *a* dapat dihitung

$$A = \frac{\sum_{i=1}^{6} \gamma_i e^{\lambda t_i}}{\sum_{i=1}^{6} e^{2\lambda t_i}} = 0.9998$$

Nilai dari regresi eksponensial adalah

$$\gamma = 0.9998\, e^{-0.1151 t}$$

Grafik dari eksponensial



Intensitas Relatif Setelah 24 jam

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$$\gamma = 0.9998 \times e^{-0.1151(24)}$$

$$= 6.3160 \times 10^{-2}$$

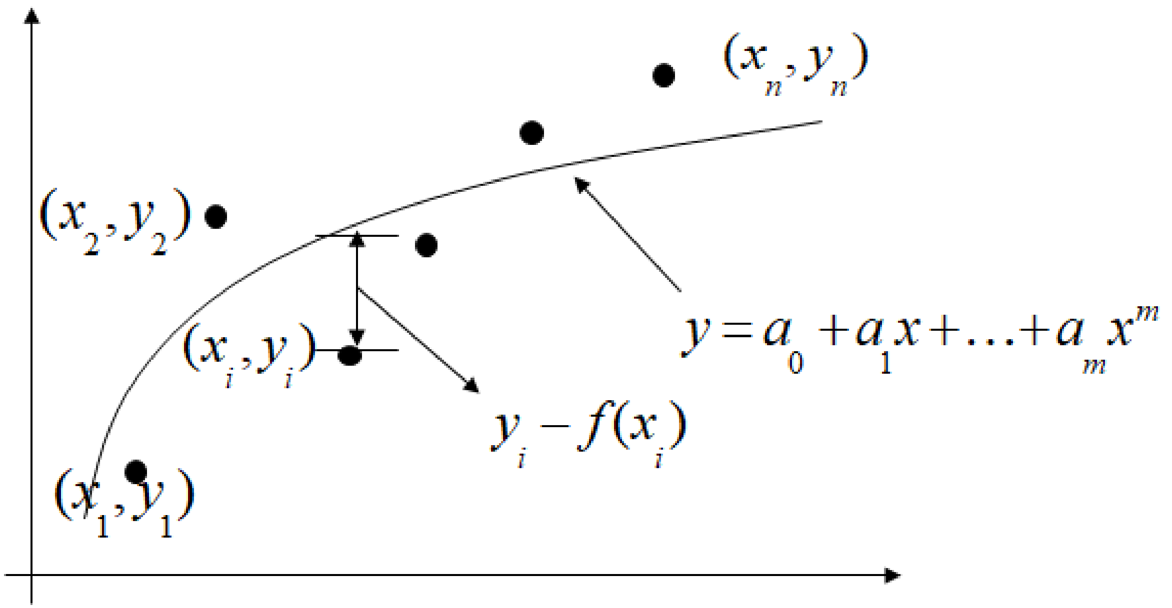
Intensitas relatif radioaktif setelah 24 jam :

$$\frac{6.316 \times 10^{-2}}{0.9998} \times 100 = 6.317\%$$

Polinomial

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \Rightarrow y = a_0 + a_1 x + \ldots + a_m x^m$$

Dengan $(m \leq n-2)$



Polinomial

Selisih data dari tiap titik :

$$E_i = y_i - a_0 - a_1 x_i - \ldots - a_m x_i^m$$

Penjumlahan dari selisih kuadrat :

$$S_r = \sum_{i=1}^{n} E_i^2$$

$$= \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - \ldots - a_m x_i^m \right)^2$$

Polinomial

$$\frac{\partial S_r}{\partial a_0} = \sum_{i=1}^{n} 2.\left(y_i - a_0 - a_1 x_i - . \;.\; . - a_m x_i^m\right)(-1) = 0$$

$$\frac{\partial S_r}{\partial a_1} = \sum_{i=1}^{n} 2.\left(y_i - a_0 - a_1 x_i - . \;.\; . - a_m x_i^m\right)(-x_i) = 0$$

$$\vdots \qquad \vdots \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$\frac{\partial S_r}{\partial a_m} = \sum_{i=1}^{n} 2.\left(y_i - a_0 - a_1 x_i - . \;.\; . - a_m x_i^m\right)(-x_i^m) = 0$$

Polinomial

Persamaan mencari konstanta dalam matrix

Pangkat m

$$\begin{bmatrix} n & \left(\sum_{i=1}^{n} X_i\right) & \cdots & \left(\sum_{i=1}^{n} X_i{}^m\right) \\ \left(\sum_{i=1}^{n} X_i\right) & \left(\sum_{i=1}^{n} X_i{}^2\right) & \cdots & \left(\sum_{i=1}^{n} X_i{}^{m+1}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \left(\sum_{i=1}^{n} X_i{}^m\right) & \left(\sum_{i=1}^{n} X_i{}^{m+1}\right) & \cdots & \left(\sum_{i=1}^{n} 2m\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdots \\ a_m \end{bmatrix} = \begin{bmatrix} \left(\sum_{i=1}^{n} X_i\right) \\ \left(\sum_{i=1}^{n} X_i Y_i\right) \\ \cdots \\ \left(\sum_{i=1}^{n} X_i{}^m Y_i\right) \end{bmatrix}$$

Persamaan diatas untuk mencari konstanta $a_0, a_1, \ldots, a_m$

Polinomial

Polinomial

Pangkat $m=1$ → $y = a_0 + a_1 x$

$m=2$ → $y = a_0 + a_1 x + a_2 x^2$

$m=3$ → $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Polinomial (m=2)

$$\mathbf{y = \alpha_0 + \alpha_1 x + \alpha_2 x^2}$$

$$\alpha_0 n + \alpha_1 \sum_{i=1}^{n} x_i + \alpha_2 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i$$

$$\alpha_0 \sum_{i=1}^{n} x_i + \alpha_1 \sum_{i=1}^{n} x_i^2 + \alpha_2 \sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{n} x_i \,.\, \sum_{i=1}^{n} y_i$$

$$\alpha_0 \sum_{i=1}^{n} x_i^2 + \alpha_1 \sum_{i=1}^{n} x_i^3 + \alpha_2 \sum_{i=1}^{n} x_i^4 = \sum_{i=1}^{n} x_i^2 .\, \sum_{i=1}^{n} y_i$$

Matriksnya :

$$\begin{pmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} y_i & \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} y_i \end{pmatrix}$$

Polinomial (m=3)

$Y = a_0 + a_1 X + a_2 X^2 + a_3 X^3$

$$a_0 \; n + a_1 \sum_{i=1}^{n} x_i + a_2 \sum_{i=1}^{n} x_i^2 + a_3 \sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{n} y_i$$

$$a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 + a_2 \sum_{i=1}^{n} x_i^3 + a_3 \sum_{i=1}^{n} x_i^4 = \sum_{i=1}^{n} y_i \, x_i$$

$$a_0 \sum_{i=1}^{n} x_i^2 + a_1 \sum_{i=1}^{n} x_i^3 + a_2 \sum_{i=1}^{n} x_i^4 + a_3 \sum_{i=1}^{n} x_i^5 = \sum_{i=1}^{n} y_i \; x_i^2$$

$$a_0 \sum_{i=1}^{n} x_i^3 + a_1 \sum_{i=1}^{n} x_i^4 + a_2 \sum_{i=1}^{n} x_i^5 + a_3 \sum_{i=1}^{n} x_i^6 = \sum_{i=1}^{n} y_i \; x_i^3$$

Polinomial (m=3)

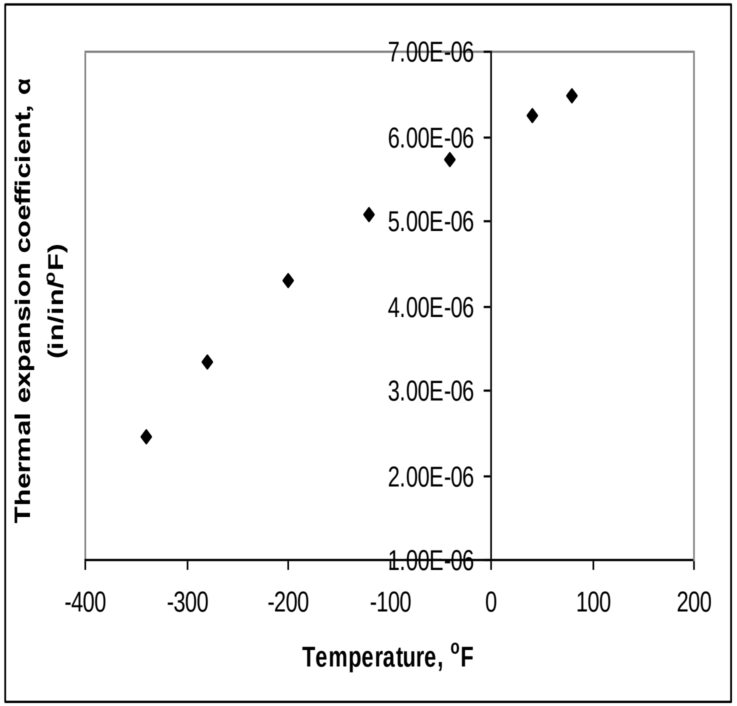
$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 & \sum_{i=1}^{n} x_i^5 \\ \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 & \sum_{i=1}^{n} x_i^5 & \sum_{i=1}^{n} x_i^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i \, x_i \\ \sum_{i=1}^{n} y_i \, x_i^2 \\ \sum_{i=1}^{n} y_i \, x_i^3 \end{bmatrix}$$

Contoh Polinomial

Regresi perkembangan koefisien panas vs data temperature dalam model polinomial

Data temperatur vs α

Temperature, T (ºF)	Coefficient of thermal expansion, α (in/in/ºF)
80	6.47×10^{-6}
40	6.24×10^{-6}
−40	5.72×10^{-6}
−120	5.09×10^{-6}
−200	4.30×10^{-6}
−280	3.33×10^{-6}
−340	2.45×10^{-6}



Polinomial

Persamaan data dengan regresi polinomial

$$\alpha = a_0 + a_1 T + a_2 T^2$$

Koefisien a_0, a_1, a_2

$$\begin{bmatrix} n & \left(\sum_{i=1}^{n} T_i\right) & \left(\sum_{i=1}^{n} T_i^2\right) \\ \left(\sum_{i=1}^{n} T_i\right) & \left(\sum_{i=1}^{n} T_i^2\right) & \left(\sum_{i=1}^{n} T_i^3\right) \\ \left(\sum_{i=1}^{n} T_i^2\right) & \left(\sum_{i=1}^{n} T_i^3\right) & \left(\sum_{i=1}^{n} T_i^4\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \alpha_i \\ \sum_{i=1}^{n} T_i \, \alpha_i \\ \sum_{i=1}^{n} T_i^2 \alpha_i \end{bmatrix}$$

TERIMA KASIH