HW 1 - Boyuan Lu

A (a) First, calculate δ^2

$$\delta u(x) = u(x+\frac{h}{2}) - u(x-\frac{h}{2})$$

$$\delta(\delta u(x)) = [u(x+h) - u(x)] - [u(x) - u(x-h)]$$

$$= u(x+h) - 2u(x) + u(x-h)$$

Second, calculate μ^2

$$\mu u(x) = \frac{u(x+\frac{h}{2}) + u(x-\frac{h}{2})}{2}$$

$$\mu(\mu u(x)) = \frac{[\frac{u(x+h)+u(x)}{2}] + [\frac{u(x)+u(x-h)}{2}]}{2} =$$

$$= \frac{1}{4}(u(x+h) + 2u(x) + u(x-h))$$

$$= \frac{1}{4}(u(x+h) - 2u(x) + u(x-h) + 4u(x))$$

$$= \frac{1}{4}(\delta^2 u(x) + 4u(x))$$

$$= \frac{1}{4}(\delta^2 + 4)\, u(x)$$

Therefore, $u^2 = \frac{1}{4}(\delta^2 + 4)$

$= \frac{1}{4}\delta^2 + 1$ ↙ since it won't be zero

$$\frac{\mu^2}{\frac{1}{4}\delta^2 + 1} = 1 \Rightarrow \mu(\frac{1}{4}\delta^2 + 1)^{-\frac{1}{2}} = 1$$

prove end

(b) The $hD=\delta-\frac{\delta^3}{24}+\frac{3\delta^5}{640}-\frac{5\delta^7}{7168}\cdots$ is not very useful because

• $\delta, \delta^3, \delta^5 \cdots$ is mounted at $\frac{h}{2}, \frac{3h}{2}, \frac{5h}{2}$, therefore, it's not on the grid.

• $hD=\delta-\frac{\delta^3}{24}+\frac{3\delta^5}{640}-\frac{5\delta^7}{7168}\cdots$ is on the 1st, ~~2nd~~, 3rd, 5th order derivative, it cannot tell the even order of derivatives (2nd. 4th, ...)

(c) • To make the grid on h, we can convert order δ from odd to even (from 1, 3, 5, ... to 2, 4, 6 ....), using the result $\mu(1+\frac{1}{4}\delta^2)^{-\frac{1}{2}}=1$,

① $hD=\delta-\frac{\delta^3}{24}+\frac{3\delta^5}{640}-\frac{5\delta^7}{7168}\cdots$ combine ① & ②

② $\delta^2+4=4\mu^2$, $\frac{\delta}{\sqrt{4\mu^2-4}}=1$

$$hD=\frac{\delta^2}{\sqrt{4\mu^2-4}}-\frac{\delta^4}{\sqrt{4\mu^2-4}}+\frac{3\delta^6}{\sqrt{4\mu^2-4}}-\cdots$$ all derivatives are on grid now.

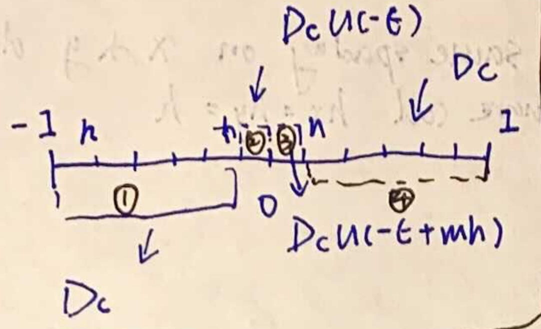
• To make the derivatives cover all orders, we can convert δ from odd series to interger series (from 1, 3, 5 ... to 1, 2, 3, 4, 5 ...)

$$hD=\delta-\delta^2\underbrace{\frac{\delta}{24}}_{\frac{2\sqrt{\mu^2-1}}{24}}+\delta^3\underbrace{\frac{3\delta^2}{640}}_{\frac{3\times4(\mu^2-1)}{640}}-\cdots$$

1st 2nd 3rd

$$=\delta-\frac{\sqrt{\mu^2-1}}{12}\delta^2+\frac{3}{160}(\mu^2-1)\delta^3\cdots$$ covers all orders from 1, 2, 3, ...

B. (a)



define $\tau = Du(x) - u'(x_j)$

① $-1 < x_j < -h$

$$\tau = \frac{u(x_j+h) - u(x_j-h)}{2h} - u'(x_j) = 0$$

② $h < x_j < 1$

$$\tau = \frac{u(x_j+h) - u(x_j-h)}{2h} - u'(x_j)$$

$$u(x+h) = u(x) + u'(x)h + \frac{u''(x)h^2}{2} + O(h^3) \quad u(x-h) = u(x) - u'(x)h + \frac{u''(x)h^2}{2} + O(h^3)$$

$$\frac{u(x_j+h) - u(x_j-h)}{2h} = u'(x_j) + O(h^2) \qquad \tau \in O(h^2)$$

③ $-h < x_j < 0$ $\qquad Du(x) = \dfrac{u(x+h) - u(x-h)}{2h}$

$$\tau = \frac{u(x_j+h) - u(x_j-h)}{2h} - u'(x_j) = \frac{(x_j+h)^n}{2h} \in O(h^{n-1})$$

$$\tau = \begin{cases} 0, & -1 < x_j < -h \\ h^2, & h < x_j < -1 \\ h^{n-1}, & -h < x_j < 0 \\ h^{n-1} + h^2, & 0 < x_j < h \end{cases}$$

④ $0 < x_j < h$ $\qquad D_c u(x) = \dfrac{u(x+mh+h) - u(x+mh-h)}{2h}$

$$u(x+mh+h) = u(x) + u'(x)(m+1)h + \frac{u''(x)}{2}(m+1)^2h^2 + O(h^3)$$

$$u(x+mh-h) = u(x) - u'(x)(m-1)h + \frac{u''(x)}{2}(m-1)^2h^2 + O(h^3)$$

$$\frac{u(x+mh+h) - u(x+mh-h)}{2h} = \frac{2hu'(x) + \frac{h^2u''(x)}{2}4m + O(h^3)}{2h} = u'(x) + hmu''(x) + O(h^2)$$

$$\tau = D_c u(x) - u'(x_j) = hm[(x_j+mh)^n]'' + O(h^2) \in O(h^{n-1}) + O(h^2)$$

(a) $\|E\|_\infty = \sum_{-1}^{1} \|\tau_j\|_{max}$, $n \geq 2$, D_c is consistent

Order of accuracy: $\begin{cases} O(h), & n=2 \\ O(h^2), & n>2 \end{cases}$

(b) $\|E\|_1 = h\sum_j |E_j|$, for any integer $n \geq 1$, D_c is consistent

Order of accuracy: $\begin{cases} O(h), & n=1 \\ O(h^2), & n>1 \end{cases}$

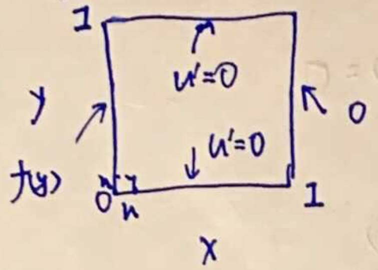
(c) Taylor expansion is right for part (a) but too pessimistic for part (b)

$$\frac{(x_j+h)^n - (x_j+h)^n}{2h} = \frac{(x_j+h)^3 - (x_j-h)^3}{2h} \in 3x_j^2 + O(h^2), \text{ just the part (a)}$$

$$\frac{(x_j+h)^3 - (x_j-h)^3}{2h} \in 3x_j^2 + O(h^5) \quad \text{error: } O(h^5) + O(h^2) = O(h^3) \xleftrightarrow{\text{permissive}} O(h^2) \text{ part (b)}$$

c. $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$

(a)



Let's have the same spacing on x & y direction, which we have square cell $h_x = h_y = h$

on x-direction

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{bmatrix}, \quad F = \begin{bmatrix} -f\cos(2\pi y) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

on y-direction

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \\ & & 0 & h^2 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

(b) interior node: (Assume $h_x = h_y = h$, square cell)

$$\cancel{f(x,y)} = -\frac{\partial u^2(x,y)}{\partial x^2} - \frac{\partial u^2(x,y)}{\partial y^2} = 0$$

$$-\frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} - \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2} = 0$$

$$u(x,y) = \frac{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)}{4}$$

Dirichlet BC: $u(x=0,y) = f(y)$, $u(x=1,y) = 0$

Neumann BC: $u(x, y=0) = -h\overset{0}{u'(x)} + u(y+h) = u(y+h)$

$u(x, y=1) = h\overset{0}{u'(y)} + u(y-h) = u(y-h)$ (Appendix A)

see the shape in figure 1, and the error plot in figure 2.

(c) According to figure 2, the error ~~of~~ between analytical solution and numerical solution is $O(h^2)$, and it's close to 0 as $h \to 0$. It's consistent

(d) Let's say $Au = \lambda u$, $Bv = \tau v$,

where u, λ are the eigenvector, eigenvalue for A

v, τ are the eigenvector, eigenvalue for B

$$A \otimes I \neq I \otimes B = A \otimes B$$

Let's time eigenvector:

$$(A \otimes B)(u \otimes v)$$

By mixed product property:

$$(A \otimes B)(u \otimes v) = (Au) \otimes (Bv)$$

$$= (\lambda u) \otimes (\tau v)$$

suppose λ_i, τ_i are the eigenvalues for A & B

$$= (\lambda_i \tau_i)(u \otimes v) \quad i = 1, 2, \sim N$$

Then, for $A \otimes I + I \otimes B$ the eigenvalue is $(\lambda_i \tau_i)$ $i = 1 \cdots n$

the eigenvector is $u \otimes v$

(e) After scaling, the eigenvalue of B are real and positives, $\tau_1, \tau_2, \tau \cdots \tau_n \geq 0$

eigenvalue of matrix A is bounded by $\frac{1}{\pi^2}$,

In lecture, we have been shown that

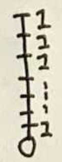
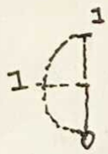
$$\lambda_{min}(-\tilde{\Delta}) = \lambda_{min} \underbrace{\tau_{min}}_{\text{non-negative}} \geq \lambda_{min} \geq \frac{1}{\pi^2} - O(h^2) > 0 \text{ as } \frac{1}{\pi^2} \text{ is c independent of } h$$

(in h is small)

(f) the figure 3 shows the relationship between number of grid N and number of iteration C. The red curve is $O(N^2)$ line, showing that my convergency is slower than $O(m^2)$, which is $O(m^2 \log n)$. Somehow it matches the theory of convergency

(g) the figure 4 shows the convergence speed under new boundary condition $\hat{f} = \begin{cases} 1 \\ 0 \end{cases}$

The order is approximately the same, which is between $O(m^2)$ $O(m^2 \log m)$ but $\hat{f}$ is slightly slower than the convergence speed for $\hat{f}$. ~~The step function in boundary will take more steps to converge than the smooth function~~

It looks strange to me that a discontinuous function kind of converge faster than a smooth function for Boundary Condition. It might be due to consistent value across the boundary  takes less steps to converge than the sin function . Generally, I think they are roughly the same.

D. The figure 5 shows the convergency speed between the Jacobi iteration and multigrid iteration. For Jacobi iteration, we use both high and low frequency components so that it decays slow and converge for long time. If we apply multigrid, we can ultitize the low frequency zone and exclude the high and very high. Therefore, leading to a acceleration of convergence. In figure 5, the multgrid method shows faster in terms of convergency speed than Jacobi iteration.

E. I think I have already done this part in problem C (g).

The figure 4 will be the error analysis result.