HOMEWORK 3

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Solution A

Solution (a)

1. take a derivative for $u(x(t), t)$ using the chain rule:

$$\frac{\partial u(x(t),t)}{\partial t} = \partial_t u(x(t),t) + \partial_t u(x(t),t)x'(t)$$

Since $u(x(t), t) = c$, $\frac{\partial u(x(t),t)}{\partial t} = 0$. Then,

$$\partial_t u(x(t),t) + \partial_t u(x(t),t)x'(t) = 0$$

2. by advection function $\partial_t u(x,t) + a\partial_x u(x,t) = 0$, apply previous answer:

$$\partial_t u(x,t) + a\partial_x u(x,t) = \partial_t u(x(t),t) + \partial_x u(x(t),t)x'(t) = 0$$

Then, we have $x'(t) = a$

3. since $x'(t) = a$, then $x(t) = at + x(0) = at + x_0$. Then, at $t = 0$

$$u(x(t),t) = u(x(0),0) = u_0(x(0)) = u_0(x_0)$$

4. from previous two questions, we have $x(0) = x(t) - at$, hence

$$u(x(t),t) = u_0(x_0) = u_0(x(t) - at) = u_0(x - at)$$

Solution (b)

1. the u(x) should be absolute integrity, which is

$$||u_0||_1 = \int_R |u_0(x)|dx < \infty$$

when u_0 is absolute integrable, then $\hat{u_0}(\xi)$ will be continous and close to zero at infinity. Hence, if $u_0(x)$ has an absolute integral derivative, which $\hat{u}(\xi)$ is differentiable, the $\hat{u_0}(\xi) = 0$ as ξ goes infinity.

2. By dominant convergence theorem (Leibniz), we can put the derivative into the integral:

$$v'(x) = \partial_x(\frac{1}{2\pi}\int_R e^{i\xi x}\hat{v}(\xi)d\xi) = \frac{1}{2\pi}\int_R \partial_x e^{i\xi x}\hat{v}(\xi)d\xi = \frac{1}{2\pi}\int_R e^{i\xi x}i\xi\hat{v}(\xi)d\xi$$

3. replacing the variable x by $x - h$:

$$F[v(\cdot - h)](\xi) = \int_R e^{-i\xi x}v(x-h)dx = e^{-i\xi h}\int_R e^{-i(\xi-h)x}v(x-h)dx = e^{-i\xi h}\hat{v}(\xi)$$

4. the same as question 2 that move derivative into integral:

$$\begin{aligned}\partial_t \hat{u}(\xi,t) &= \int_R e^{-i\xi x}\partial_t u(x,t)dx \\ &= -\int_R e^{-i\xi x} a\partial_x u(x,t)dx \\ &= -\int_R e^{-i\xi x} ad[u(x,t)] \\ &= -(e^{-i\xi x}u(x,t)|_R - \int_R u(x,t)d[e^{-i\xi x}a])\end{aligned}$$

Here, in this equation:

$$e^{-i\xi x}u(x,t)|_R = e^{-i\xi x}u(x,t)|_{-\infty}^{\infty} = e^{-i\xi\infty}u(\infty,t) - e^{-i\xi-\infty}u(-\infty,t) = 0$$

Since $u(x,t) \to 0$ when $x \to \infty$. $u(x,t) \to 0$ when $x \to -\infty$ because u(x, t) has a Fourier transformation. Hence, equation becomes:

$$\partial_t \hat{u}(\xi,t) = \int_R u(x,t)d[e^{-i\xi x}a] = -i\xi a \int e^{-i\xi x}u(x,t)dx = -ai\xi\hat{u}(\xi,t)$$

5. We can solve the integral by Newton-Leibniz:

$$\frac{\partial_t \hat{u}(\xi,t)}{\hat{u}(\xi,t)} = -ai\xi$$

Take integral at both sides of equation:

$$\int \frac{\partial_t \hat{u}(\xi,t)}{\hat{u}(\xi,t)} = \int -ai\xi$$

Then we have:

$$\begin{aligned}log\hat{u}(\xi,t) &= -ai\xi t + c \\ \hat{u}(\xi,t) &= ce^{-ai\xi t}\end{aligned}$$

Here, the residual c can be determined by plug $t = 0$ into $\hat{u}(\xi,t)$, hence, $c = \hat{u}(\xi,0) = u_0(\xi)$. Finally, we have:

$$\hat{u}(\xi,t) = u_0(\xi)e^{-ai\xi t}$$

6. If $u_0(\cdot) \in L^2(R)$, also notice that $|e^{-ai\xi t}| \leq 1$ for any t, we will have:

$$\int |\hat{u}(\xi,t)|^2 d\xi = \int |u_0(\xi)e^{-ai\xi t}|^2 d\xi \leq \int |u_0(\xi)|^2 d\xi \leq \infty$$

for any fixed t by Plancherel's theorem, we will have:

$$\int_R |u(x,t)|^2 dx = \int |\hat{u}(\xi,t)|^2 d\mathbf{x} \leq \infty$$

7. Combining the Inversion of FT with question 5, we have:

$$u(x,t) = \frac{1}{2\pi}\int_R e^{i\xi x}\hat{u}(\xi,t)d\xi = \frac{1}{2\pi}\int_R e^{i\xi(x-at)}\hat{u_0}(\xi)d\xi$$

8. Similar with what wave did in part (a), we can replace $x - at$ with a new x

$$u(x,t) = \frac{1}{2\pi}\int_R e^{i\xi(x-at)}\hat{u_0}(\xi)d\xi = u_0(x-at)$$

Solution B

Solution (a)

I would choose the negative sign. When choosing the negative sign, the $u(x,t)$ has the following form:

$$u(x,t) = \int_R exp(i\xi(x - \xi^3 t))u_0(\xi)d\xi$$

which can be bounded by some constant times the norm of $u(x,0)$. Also, if we take $u_0 = \sin x$, then a solution for $u_t = u_{xxxx}$, would be $sin(x)exp(t)$, which is diverge obviously to compute out. Hence we must choose the negative sign.

Solution (b)

For the explicit Euler method, we have

$$\frac{y^{n+1} - y^n}{\delta t} = -\lambda y^n$$

Then we can solve this:

$$|\rho| = |1 - \lambda \Delta t| \leq 1$$

which is the stability region. For the λ, since we know that the eigenvalue of the second order differential operator is bounded by $-4/\Delta^2 x$, we can derive the eigenvalue of the forth order differential operator is bounded by $16/\Delta^2 x$. Therefore, we need to meet $\Delta t \leq 1/8\Delta^4 x$ for stability. The stability zone show below:

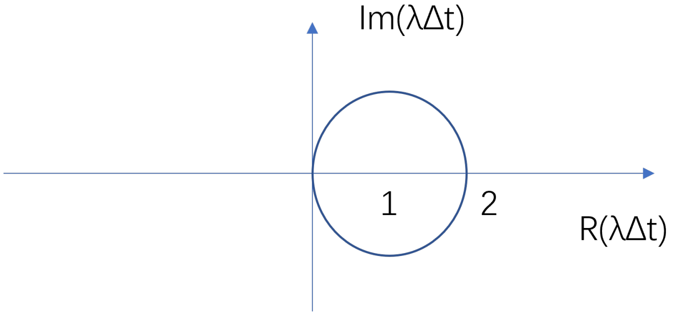

Figure 1. the complex plane and stability zone.

Solution (c)

We can have a better alternative for the Explict Euler scheme. For example, if we use the trapezoidal method instead, we can have more stability zone which is the whole left half pane.

Solution C

Solution (a)

First, use the Taylor Expansion to expand the 2d scheme:

$$u(x,y,t+\Delta t) = u(x,y,t) + \Delta t u_t + \frac{\Delta^2 t}{2}u_{tt} + \frac{\Delta^3 t}{6}u_{ttt} + \frac{\Delta^4 t}{24}u_{tttt} + \cdots$$

$$u(x,y,t-\Delta t) = u(x,y,t) - \Delta t u_t + \frac{\Delta^2 t}{2}u_{tt} - \frac{\Delta^3 t}{6}u_{ttt} + \frac{\Delta^4 t}{24}u_{tttt} + \cdots$$

Since we are going to compute 4th order, then the truncated error of the expansion would be 4th as well. Then we can aggregate the above two equations, we have:

$$u(x,y,t+\Delta t) + u(x,y,t-\Delta t) = 2u(x,y,t) + \Delta^2 t u_{tt} + \frac{\Delta^4 t}{12}u_{tttt}$$

In homework 2 setting, we have:

$$u_{tt} = \Delta u$$
$$u_{tttt} = \Delta(u_{tt}) = \Delta\Delta u = \Delta^2 u$$

Replacing these into above equation, we can have,

$$u(x,y,t+\Delta t) + u(x,y,t-\Delta t) = 2u(x,y,t) + \Delta^2 t \Delta u + \frac{\Delta^4 t}{12}\Delta^2 u$$

Rearrange:

$$u(x,y,t+\Delta t) = -u(x,y,t-\Delta t) + 2u(x,y,t) + \Delta^2 t \Delta u + \frac{\Delta^4 t}{12}\Delta^2 u$$

Solution (b)

Initial at $t = 0$, we can write the Taylor expansion as:

$$u(x,y,\Delta t) = u(x,y,0) + \Delta t u_t + \frac{\Delta^2 t}{2}u_{tt} + \frac{\Delta^3 t}{6}u_{ttt} + \frac{\Delta^4 t}{24}u_{tttt}$$
$$u(x,y,-\Delta t) = u(x,y,0) - \Delta t u_t + \frac{\Delta^2 t}{2}u_{tt} - \frac{\Delta^3 t}{6}u_{ttt} + \frac{\Delta^4 t}{24}u_{tttt}$$

Subtract above two equations:

$$u(x,y,\Delta t) - u(x,y,-\Delta t) = 2\Delta t u_t + \frac{\Delta^3 t}{3}u_{ttt}$$

In the equation, u_t is known, $u_t tt$ can be computed by u_t,and the $u(x,y,-\Delta)$ can be computed by following trick:

$$u(x,y,-\Delta t) = -u(x,y,\Delta t) + 2u(x,y,0) + \Delta^2 t \Delta u + \frac{\Delta^4 t}{12}\Delta^2 u$$

we have $u(x,y,0) = 0$,Δu and $\Delta^2 u$ is also 0. Then, we have

$$u(x,y,-\Delta t) = -u(x,y,\Delta t)$$

Then $u(x,y,\Delta t$ can be initialized. Notice that all the expansions above have the same truncate error which is 4th order.

Solution (c)

Below the figure 2 shows the error plot for the different grid size. The errors were computed by a fine grid of 2^7 over 2^7. Notice that the slope ratio of log error vs log grid size is equal to 4 (parallel to slope 4 line), which means that the method is in 4th order accuracy.

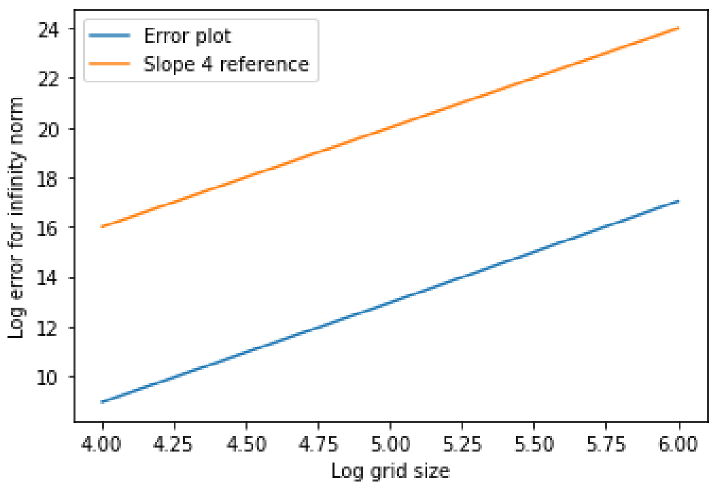

Figure 2. the log error plot for the varied grid size (blue line) compared with slope ratio of 4 reference line (orange line) using Chebyshev grid interpolation.

Solution (d)

According to the scheme I have in homework 2:

$$\frac{y_{i,j}^{t+1} - 2y_{i,j}^{t} + y_{i,j}^{t-1}}{\Delta t^2} = \lambda y^t$$

I have derived the CFL condition:

$$-4 \leq Re(\lambda \Delta t^2) \leq 0$$

Let's denote the Chebyshev different matrix as D_N as the Trefethen's book. Then the matrix forms by:

$$L_N = D_N^2 \otimes I + I \otimes D_N^2$$

Find the ranges of eigenvalue of matrix from Trefethen's book. The largest eigenvalue is approximately $-0.048N^4$. Therefore, the eigenvalue of L_N will be $2*-0.048N^4 = -0.096N^4$, and eigenvalue of L_N^2 will be $(-0.096N^4)^2$. Then, we can write the CFL condition as following:

$$-4 \leq (-0.096N^4 + \frac{1}{12}\Delta^2 t 0.0096^2 N^8)\Delta^2 t \leq 0$$

Solve the inequality then we have:

$$\Delta^2 t \leq 125/N^4$$

or

$$\Delta t \leq 5\sqrt{5}/N^2$$

Solution (e)

Now, we have the initial condition as

$$u_t(x, y, 0) = \sin(B\pi x)\sin(B\pi y)$$

Rewrite the code in a fix grid under 64*64, and choose various B value under 0.75s. Here is the plot.

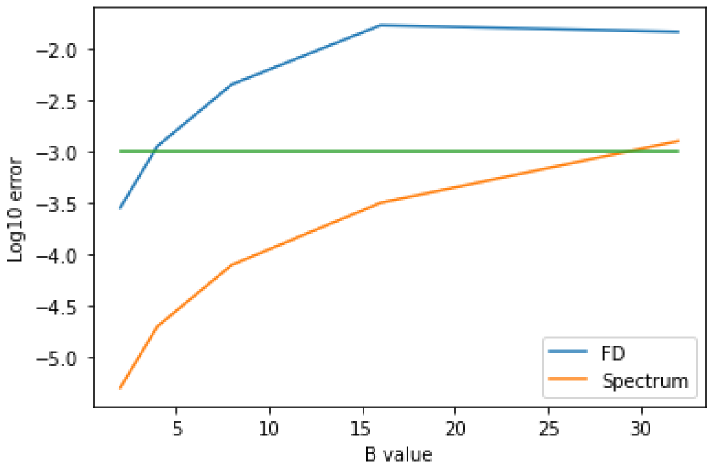

Figure 3. the log error plot for the varied B value for Finite Difference method (blue line) compared with spectrum method (orange line). The green line is the benchmark for 1e-3 accuracy.

The results of plot shows that spectrum method acts to the wave numbers significantly, as error decreases when the B value decrease. This obeys the tuition that large B values stand for larger wave fluctuation, leading to larger error of approximation of derivatives. The FD methods respond to the B value pretty much than Spectrum method, since it is evenly sampling grid that interpolation accuracy won't change at all. The spectrum has really tiny error for the small B values then increase dramatically as the B value increases.
For selection of B, $B = 5$ will be more the FD method to meet 1e-3 accuracy, while $B = 28$ can help Spectrum method achieve $1e - 3$ accuracy.

Solution D

We can apply Fourier Transform:

$$\hat{f}(\xi) = \int_R e^{-ix\xi} f(x)dx = \int_R \frac{-i}{\xi}(e^{-i\xi x})' f(x)dx$$

We can put the $f(x)$ into derivative:

$$= \frac{-i}{\xi} \int_R e^{-i\xi x} Df(x)$$

Where D is the differential operator. Since we have f is integrable and We can use smooth function to approximate and thus the derivative can still make sense.
Finally we can put the norm inside the integral and obtain our results by dominated convergence theorem:

$$|\hat{f}(\xi)| \leq \frac{1}{|\xi|} \int |Df| = ||f||_{TV}/|\xi|$$