# ON CATEGORIES OF GROUPS AND GENERALIZED LEGENDRIAN RACKS

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Abstract. In 2023, Karmakar et al. and Kimura introduced algebraic structures called GL-racks that can distinguish between Legendrian links. GL-racks generalize quandles, which Joyce introduced in 1980 to study the topological structure of knots and links. However, the non-associativity of GL-racks makes it cumbersome to understand their structure or detect their isomorphisms, motivating us to study them categorically and in relation to groups. Doing this allows us to generalize several results known for quandles to GL-racks in this paper. First, we show that the enveloping group of the GL-rack of a Legendrian link is isomorphic to the fundamental group of the link complement in $\mathbb{R}$. Then, we define medial GL-racks, introduce tensor products of GL-racks, and prove that the category of medial GL-quandles is symmetric monoidal closed. We also exhibit an adjunction between the category of GL-racks and their surjective homomorphisms and a category constructed from groups and conjugation-stable generating subsets. Finally, we introduce a category with an essentially surjective functor to the category of GL-racks and deduce a group-theoretic sufficient condition for any two GL-racks to be isomorphic.

## 1. Introduction

In 1980, Joyce [16] introduced algebraic structures called *quandles* as a means of capturing the topological structure of knots, links, and symmetric spaces. Since then, quandles and slightly more general algebraic objects called *racks* have enjoyed significant study as link invariants among geometric topologists and in their own right among quantum algebraists. Recently, various authors have constructed generalizations of racks and quandles to study *Legendrian links* in contact geometry (see, for example, [6, 18, 19, 22]). Moreover, defining homomorphisms of these algebraic structures yields categories like $\mathsf{GLR}$, whose objects are *generalized Legendrian racks* (also called *GL-racks* or *bi-Legendrian racks*).

In this paper, we study categories of GL-racks and their functorial relationships with certain categories constructed from groups and group homomorphisms, including $\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}$ and $\mathsf{GrpTup}$. This helps us better understand the structure of GL-racks and detect their isomorphisms, which can otherwise be cumbersome due to their non-associativity. This also allows us to generalize several results known for quandles to GL-racks. The main results of this paper are as follows.

**Theorem 1.1.** *Let $\Lambda \subset \mathbb{R}^3$ be an oriented Legendrian link, let $\mathrm{GLR}(\Lambda)$ be the GL-rack of $\Lambda$ as defined in Definition 3.2, and let $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda))$ be its enveloping group as defined in Definition 2.18. Then there exists a group isomorphism*

$$\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda)) \cong \pi_1(\mathbb{R}^3 \setminus \Lambda).$$

**Theorem 1.2.** *The category $\mathsf{GLQ}^{\mathrm{med}}$ of medial GL-quandles is symmetric monoidal closed. In particular, it is self-enriched.*

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**Theorem 1.3.** *Let* $\mathsf{GLR}_{\mathrm{surj}}$ *be the category of GL-racks and surjective GL-rack homomorphisms. Then there exists a functor* $\mathcal{F} : \mathsf{GLR}_{\mathrm{surj}} \to \mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}$ *that sends any GL-rack* $(X, s, \mathtt{u}, \mathtt{d})$ *to* $(\operatorname{Inn} X, s(X))$*. Furthermore,* $\mathcal{F}$ *has a right adjoint.*

**Theorem 1.4.** *There exists an essentially surjective functor* $\mathcal{F} : \mathsf{GrpTup} \twoheadrightarrow \mathsf{GLR}$*. This functor induces a group-theoretic sufficient condition for any two GL-racks to be isomorphic.*

In Section 2, we give an overview of the questions in Legendrian knot theory motivating the study of GL-racks and quandles. We proceed by defining these algebraic structures abstractly and introducing related groups, categories, and functors in the literature.

In Section 3, we discuss how to assign a GL-rack to an oriented Legendrian link, provide a worked example for Legendrian $(2, -q)$ torus knots with maximal Thurston-Bennequin and rotation numbers, and prove Theorem 1.1, which we state as Corollary 3.7. Originally conjectured in [18, Remark 8.2], this result generalizes a property of the *fundamental quandle* of a link [16, Section 15]. The left adjointness of the functor $\operatorname{Env}_{\mathsf{GLR}} : \mathsf{GLR} \to \mathsf{Grp}$ plays a pivotal role in our proof.

In Section 4, we define *medial* or *abelian* GL-racks and tensor products of GL-racks before proving Theorem 1.2, which we state as Theorem 4.4 and Corollary 4.5. This generalizes an analogous result for medial quandles [9, Theorem 12], whose proof we follow closely.

In Section 5, we prove Theorem 1.3, which we state as Theorem 5.4. Our construction of $\mathcal{F}$ employs the functor $\operatorname{Inn} : \mathsf{GLR}_{\mathrm{surj}} \to \mathsf{Grp}$, using the proof of [31, Theorem 1.1] as a model.

In Section 6, we introduce a category $\mathsf{GrpTup}$ whose objects are constructed from left cosets of groups. These objects were originally employed in [18, Theorem 5.2] to produce a homogeneous representation of any GL-rack. Then, we prove Theorem 1.4, the first part of which we state as Theorem 6.1 and the second part of which we discuss afterward.

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## 2. Definitions and notation

In this section, we provide motivations, definitions, and notation for the study of GL-racks. In particular, we discuss the crossing and cusp relations afforded by Legendrian Reidemeister moves, which motivate the axioms of GL-racks. After stating some preliminary results on GL-racks, we discuss several functors appearing in the literature on GL-racks and quandles.

### 2.1. Motivations: Legendrian knots and links.

In this subsection, we introduce Legendrian links and Legendrian Reidemeister moves. These give contact-geometric motivations for studying GL-racks and quandles.

**Definition 2.1.** A *knot* is the image of a smooth embedding of the circle $S^1$ into $\mathbb{R}^3$, and a *link* is a disjoint union of a finite number of knots. A link $\Lambda$ is called *Legendrian* if it lies everywhere tangent to the *standard contact structure* $\xi_{\mathrm{std}} := \ker(dz - y\,dx)$ on $\mathbb{R}^3$, which is depicted in Figure

1. That is, $T_x\Lambda \in \xi_{\text{std}}$ for all $x \in \Lambda$, where $T_x\Lambda$ denotes the tangent space of $\Lambda$ at $x$. A *front projection* or *front diagram* $D(\Lambda)$ is the projection of $\Lambda$ to the $xz$-plane.

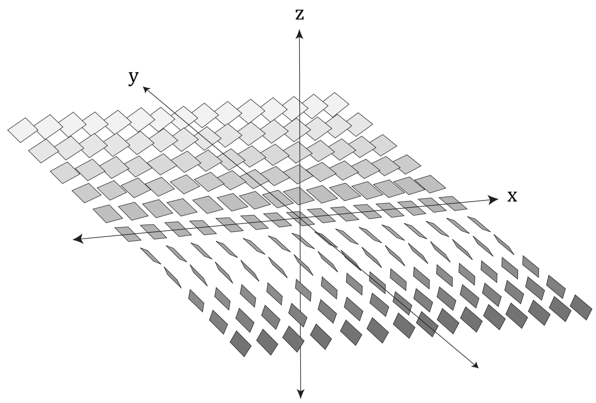


FIGURE 1. The standard contact structure $\xi_{\text{std}}$ on $\mathbb{R}^3$. Taken from [21].

When distinguishing Legendrian links from links that are not necessarily Legendrian, we will call the latter *smooth links* despite the fact that Legendrian links are themselves smooth. Also, we will denote the underlying smooth link of a Legendrian link $\Lambda$ by $L$. In particular, $D(\Lambda)$ can be viewed as a *link diagram* of $L$, denoted by $D(L)$, by "ignoring" all cusps.

Central to contact geometry is the question of when two Legendrian links can be smoothly deformed to each other in a way that preserves the condition of being Legendrian at every stage. This notion, called *Legendrian isotopy*, is formalized as follows.

**Definition 2.2.** Viewed as smooth embeddings of $S^1$ into $\mathbb{R}^3$, two Legendrian links $\Lambda_1, \Lambda_2$ are considered *Legendrian isotopic* if they exists a smooth homotopy $H : S^1 \times [0,1] \to \mathbb{R}^3$ such that $H \times \{0\} = \Lambda_1$, $H \times \{1\} = \Lambda_2$, and $H \times \{t\}$ is a Legendrian link for all $t \in [0,1]$.

The question of distinguishing between Legendrian links has motivated the development of numerous *invariants* of Legendrian links, called so because they are constructed to be invariant under Legendrian isotopy. Examples include the Thurston-Bennequin and rotation numbers [13], the Chekanov-Eliashberg differential graded algebra and associated polynomial invariants [13, 25], various (co)homology theories (see [15] for a list), and the mosaic number [21, 27]. For an accessible introduction to Legendrian knot theory, we refer the reader to [28]. For a more detailed survey of the field, we refer the reader to [13].

GL-racks and quandles have also been used to define algebro-combinatorial and cohomological invariants of both Legendrian links and smooth links. These include fundamental quandles and their Legendrian analogues [6, 16, 18], rack colorings [6, 18–20], cocycle invariants [7, 20], and state-sum invariants [4, 12], many of which have elegant categorifications (e.g., [5, 7, 8]). These invariants motivate the study of GL-racks and quandles as categories.

On the other hand, Legendrian links $\Lambda$ are typically studied through their front projections, which follow several restrictions thanks to the tangency condition on $\Lambda$. For one, at every crossing in $D(\Lambda)$, the strand with the more negative slope is always the overstrand. For two, $D(\Lambda)$ has cusps in place of vertical tangencies. For example, Figure 2 depicts front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot, and Figure 3 depicts front projections of two

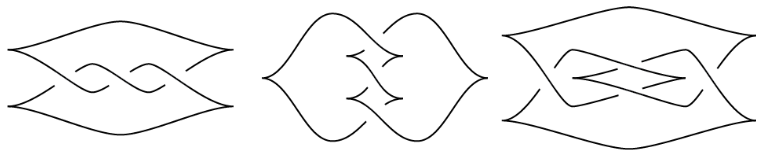

Figure 2. Unoriented front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot. Adapted from [13].

distinct oriented Legendrian unknots. Note that the numbers of crossings and cusps in a Legendrian front projection are finite due to smoothness.

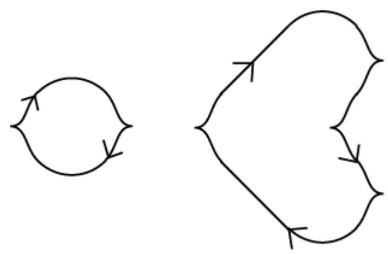

Figure 3. Front projections of distinct oriented Legendrian unknots. Adapted from [27].

In fact, one can use the tangency condition to show that the geometric structure of an oriented Legendrian link $\Lambda$ can be recovered entirely from its front projection $D(\Lambda)$ (cf. [13].) For example, two invariants called the *Thurston-Bennequin number* and *rotation number* of $\Lambda$, denoted respectively by $\mathrm{tb}(\Lambda)$ and $\mathrm{rot}(\Lambda)$, can be defined as the integers

$$\mathrm{tb}(\Lambda) = P - N - \frac{1}{2}(D+U), \quad \mathrm{rot}(\Lambda) = \frac{1}{2}(D-U),$$

where $P$, $N$, $D$, and $U$ are the numbers of positively oriented crossings, negatively oriented crossings, downward-oriented cusps, and upward-oriented cusps in $D(\Lambda)$, respectively. It is well-known that two Legendrian links are Legendrian isotopic only if their Thurston-Bennequin and rotation numbers are equal. Indeed, a celebrated theorem of Świątkowski in 1992 offers a method of comparing Legendrian links using only their front projections.

**Proposition 2.3.** [30, Theorem B] *Two Legendrian links are Legendrian isotopic if and only if their front projections are related by a finite sequence of planar isotopies and the three* Legendrian Reidemeister moves *depicted in Figure 4.*

**Example 2.4.** Let $\Lambda_L$ and $\Lambda_R$ be the oriented Legendrian unknots depicted on the left and right of Figure 3, respectively. Although $\Lambda_L$ and $\Lambda_R$ share the same underlying smooth knot type, they are not Legendrian isotopic because $\mathrm{tb}(\Lambda_L) = -1 \neq -2 = \mathrm{tb}(\Lambda_R)$ and $\mathrm{rot}(\Lambda_L) = 0 \neq 1 = \mathrm{rot}(\Lambda_R)$. Proposition 2.3 asserts that the two front projections in Figure 3 cannot be related by any sequence of the Legendrian Reidemeister moves in Figure 4.

There are in fact infinitely many examples of distinct Legendrian links having the same underlying smooth link type, making distinguishing between Legendrian links significantly more difficult than distinguishing between smooth links. This further motivates the study of GL-racks to develop stronger and more easily computable invariants of Legendrian links.

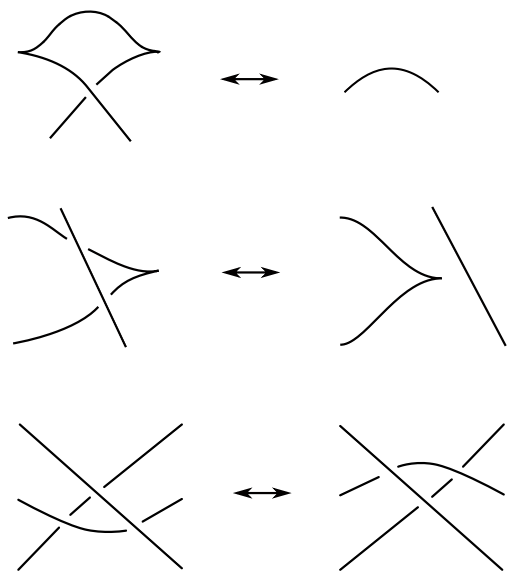

FIGURE 4. The three Legendrian Reidemeister moves. Taken from [18].

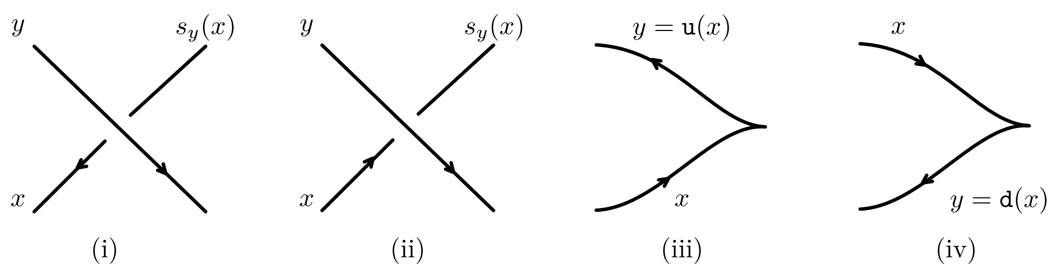


FIGURE 5. Crossing and cusp relations. Adapted from [18].

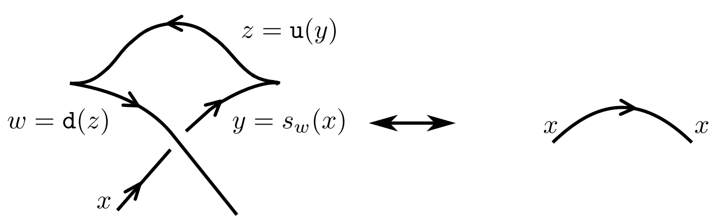


FIGURE 6. Crossing and cusp relations in one possible orientation of the first Legendrian Reidemeister move. Adapted from [18].

The axioms of GL-racks are motivated by the *crossing* and *cusp relations* induced between strands of an oriented Legendrian link modulo the relations afforded by the Legendrian Reidemeister moves. In Figure 5, (i) and (ii) depict crossing relations between strands in a Legendrian front projection, and (iii) and (iv) depict cusp relations. Note that $\mathtt{u}$ and $\mathtt{d}$ correspond to the relations induced by upward- and downward-oriented cusps, respectively. Figures 6-8 depict the crossing and cusp relations in one possible orientation of each of the three Legendrian Reidemeister moves. For a complete list of all possible orientations and their induced crossing and cusp relations, we refer the reader to [19, Figures 6-8].

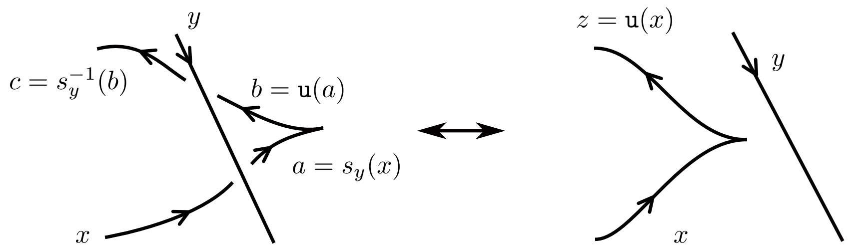


FIGURE 7. Crossing and cusp relations in one possible orientation of the second Legendrian Reidemeister move. Adapted from [18].

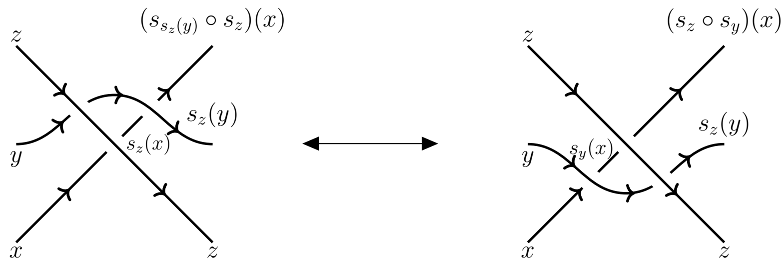


FIGURE 8. Crossing relations in one possible orientation of the third Legendrian Reidemeister move. Adapted from [6].

In Subsection 3.1, we detail how to assign a GL-rack to any oriented Legendrian link $\Lambda$ using the cusp and crossing relations in $D(\Lambda)$. This assignment is independent of the choice of front projection of $\Lambda$, making it an invariant of Legendrian links [18, Theorem 4.3].

2.2. **GL-racks and quandles.** In this subsection, we define GL-racks and quandles abstractly by translating the crossing and cusp relations in Subsection 2.1 into the language of *rack symmetries*. We defer discussing how one can assign these algebraic structures to Legendrian links to Subsection 3.1.

Although racks and quandles are often defined as sets $Q$ endowed with binary operations $\triangleright : Q \times Q \to Q$ and $\triangleright^{-1} : Q \times Q \to Q$ satisfying axioms corresponding to the crossing relations induced by the three Reidemeister moves, they may also be characterized in terms of bijections $s_q : Q \to Q$, defined for each element $q \in Q$, that satisfy the same axioms (cf. [11, Section 2; 16, Definition 1.1; 31, Definition 2.7]). These bijections are called *symmetries*. One may translate between the two conventions via the formulas $s_x(y) = y \triangleright x$ and $s_x^{-1}(y) = y \triangleright^{-1} x$. In this paper, we adopt the definitions using symmetries due to its convenience for categorical proofs. In particular, we have rewritten all crossing relations in Figures 5 through 8 in the notation of rack symmetries. We refer the reader to [24] for an accessible introduction to quandle theory in terms of $\triangleright$ and $\triangleright^{-1}$. We also refer the reader to [10] for a more comprehensive survey of modern algebraic research on racks and quandles.

The rack and quandle axioms encapsulate the crossing relations depicted in Figures 6-8.

**Definition 2.5.** Let $Q$ be a set, and let $s$ be a map from $Q$ to $\mathrm{Sym}(Q)$ defined by $x \mapsto s_x$. We call the pair $(Q, s)$ a *rack* or a *wrack* if, for all $x, y \in Q$, we have $s_x \circ s_y = s_{s_x(y)} \circ s_x$. We say that $s$ is a *rack structure* on $Q$, and we say that $s_x$ is the *symmetry at $x$*. If in addition $s_x(x) = x$ for all $x \in Q$, we say that $(Q, s)$ is a *quandle*, and we say that $s$ is a *quandle structure* on $Q$. Finally, we say that $(Q, s)$ is *faithful* if $s$ is injective.

**Example 2.6.** Let $G$ be a group, and let $\Omega$ be a union of conjugacy classes in $G$. Define $s : \Omega \to \mathrm{Sym}(\Omega)$ by
$$\alpha \mapsto s_\alpha := [\omega \mapsto \alpha\omega\alpha^{-1}].$$
Then $(\Omega, s)$ is a quandle called the *conjugation quandle*, and we denote it by $\mathrm{Conj}\,\Omega$.

The axioms of *GL-racks* encode not only the crossing relations but also the cusp relations given by the Legendrian Reidemeister moves in Figures 6-8. GL-racks were introduced independently in [17] and [19] to generalize the *Legendrian racks* introduced in [6] and [22]. Once again, we translate the original definition into the language of rack symmetries.

**Definition 2.7.** A *generalized Legendrian rack*, or *GL-rack*, is a quadruple $(X, s, \mathtt{u}, \mathtt{d})$ in which $(X, s)$ is a rack and $\mathtt{u}, \mathtt{d} : X \to X$ are maps such that the following hold for all $x \in X$:

(L1) $(\mathtt{ud} \circ s_x)(x) = x = (\mathtt{du} \circ s_x)(x)$.
(L2) $\mathtt{u} \circ s_x = s_x \circ \mathtt{u}$ and $\mathtt{d} \circ s_x = s_x \circ \mathtt{d}$.
(L3) $s_{\mathtt{u}(x)} = s_x = s_{\mathtt{d}(x)}$.

We call the pair $(\mathtt{u}, \mathtt{d})$ a *GL-structure* on $(Q, s)$. If in addition $(X, s)$ is a quandle, we say that $(X, s, \mathtt{u}, \mathtt{d})$ is a *generalized Legendrian quandle*, or *GL-quandle*.

Due to the pair $(\mathtt{u}, \mathtt{d})$, GL-racks are also called *bi-Legendrian racks* (cf. [19]).

**Example 2.8.** [19, Example 3.6] Let $G$ be a group, let $z \in Z(G)$, and define $f : G \to G$ by $g \mapsto zg$. Then $(\mathrm{Conj}\,G, f, f^{-1})$ is a GL-quandle.

**Example 2.9.** Let $(X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack. If $\mathtt{u} = \mathtt{d}$, we say that the triple $(X, s, \mathtt{u})$ is a *Legendrian rack* (cf. [6, Definition 4; 18, Remark 3.3]). Thus, GL-racks generalize Legendrian racks. In particular, *Legendrian quandles* are precisely the GL-racks in which $\mathtt{u} = \mathtt{d}$ and $\mathtt{u}$ is an involution.

**Example 2.10.** [18, Example 3.4] Any GL-rack of the form $(X, s, \mathrm{id}_X, \mathrm{id}_X)$ is called a *trivial GL-rack*. In particular, any quandle $(Q, s)$ can be identified with the trivial GL-rack $(Q, s, \mathrm{id}_Q, \mathrm{id}_Q)$, cf. Lemma 2.16. In other words, GL-racks generalize quandles.

We define homomorphisms of these algebraic structures as follows.

**Definition 2.11.** Let $(X, s)$ and $(Y, t)$ be racks (resp. quandles). A map $\varphi : X \to Y$ is called a *rack* (resp. *quandle*) *homomorphism* if $\varphi \circ s_x = t_{\varphi(x)} \circ \varphi$ for all $x \in X$. If in addition $(\mathtt{u}_1, \mathtt{d}_2)$ and $(\mathtt{u}_2, \mathtt{d}_2)$ are GL-structures on $(X, s)$ and $(Y, t)$, we say that a $\varphi$ is also a *GL-rack* (resp. *GL-quandle*) *homomorphism* if $\varphi \circ \mathtt{u}_1 = \mathtt{u}_2 \circ \varphi$ and $\varphi \circ \mathtt{d}_1 = \mathtt{d}_2 \circ \varphi$.

Evidently, we have the following; the last statement is from [18, Proposition 3.2].

**Proposition 2.12.** *Let $(X, s)$ be a rack with maps $\mathtt{u}, \mathtt{d} : X \to X$ satisfying axioms (L1) and (L3) of Definition 2.7. Then $(X, s, \mathtt{u}, \mathtt{d})$ is a GL-rack if and only if $\mathtt{u}$ and $\mathtt{d}$ are endomorphisms of the underlying rack $(X, s)$. In this case, $\mathtt{u}$ and $\mathtt{d}$ are actually rack automorphisms.*

Axiom (L1) immediately yields the following.

**Proposition 2.13.** *Let $(X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack. Then the underlying rack $(X, s)$ is a quandle if and only if $\mathtt{ud} = \mathrm{id}_X = \mathtt{du}$, that is, $\mathtt{d} = \mathtt{u}^{-1}$ as rack automorphisms.*

One easy consequence of Propositions 2.12 and 2.13 is the following:

**Proposition 2.14.** *Let $(Q, s)$ be a quandle, and let $\mathtt{u}, \mathtt{d}$ be maps from $Q$ to $Q$. Then the following are equivalent:*

*(1) The pair $(\mathtt{u}, \mathtt{d})$ defines a GL-structure on $(Q, s)$. That is, $(Q, s, \mathtt{u}, \mathtt{d})$ is a GL-quandle.*

*(2)* $\mathtt{u}$ *and* $\mathtt{d}$ *are rack endomorphisms satisfying axiom (L3), and* $\mathtt{ud} = \mathrm{id}_Q = \mathtt{du}$.

Another easy consequence completely characterizes faithful GL-racks.

**Proposition 2.15.** *Let* $(X, s, \mathtt{u}, \mathtt{d})$ *be a GL-rack. Then* $(X, s)$ *is a faithful rack if and only if* $(X, s)$ *is a faithful quandle. In this case,* $\mathtt{u} = \mathrm{id}_X = \mathtt{d}$.

2.3. **Functors of interest in the literature.** In this subsection, we define several categories and functors appearing in the literature on GL-racks, quandles, and their relationships with groups. These functors will play crucial roles in proving the main theorems of Sections 3 and 5.

We will begin by defining some important categories. First, let $\mathsf{Qnd}$ be the category of quandles and quandle homomorphisms. Let $\mathsf{GLR}$ be the category of GL-racks and GL-rack homomorphisms, and let $\mathsf{GLR}_{\mathrm{surj}}$ be the category of GL-racks and surjective GL-rack homomorphisms. By Example 2.10 and Proposition 2.13, we have the following.

**Lemma 2.16.** *The correspondence* $(Q, s) \mapsto (Q, s, \mathrm{id}_Q, \mathrm{id}_Q)$ *defines a canonical isomorphism from* $\mathsf{Qnd}$ *to the full subcategory of* $\mathsf{GLR}$ *whose objects are trivial GL-racks.*

Next, let $\mathsf{Grp}$ denote the category of groups and group homomorphisms, and let $\mathsf{Grp}_{\mathrm{epi}}$ be the category of groups and surjective group homomorphisms. In Section 5, we will consider the related category $\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}$ constructed in [31, Definition 2.20]. The objects of $\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}$ are pairs $(G, \Omega)$ where $G$ is a group, $\Omega \subset G$ is a generating set of $G$, and $g\Omega g^{-1} \subset \Omega$ for all $g \in G$. We say that $\Omega$ is *conjugation-stable*. The morphisms in $\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}$ are group homomorphisms $\varphi : (G, \Omega) \to (H, \Gamma)$ such that $\varphi(\Omega) = \Gamma$. Note that $\varphi$ is surjective since $\Gamma$ generates $H$.

In the sense of universal algebra, $\mathsf{GLR}$ is definitionally an equational algebraic category, so it is complete and cocomplete. (For details, we refer the reader to [1, Corollary 1.2, Theorem 4.5].) In particular, we can express GL-racks in terms of generators and relations by imposing equivalence relations on *free GL-racks*, which we define as follows.

**Definition 2.17.** [18, Section 4] Let $X$ be a set. We define the *free GL-rack on* $X$, denoted by $\mathrm{FGLR}(X)$, as follows. If $X$ is empty, let $\mathrm{FGLR}(X)$ be the trivial GL-rack with one element. Otherwise, define the *universe of words generated by* $X$ to be the set $W(X)$ satisfying the following:

(1) $x \in W(X)$ for every $x \in X$.
(2) $s_y(x),\ s_y^{-1}(x),\ \mathtt{u}(x),\ \mathtt{d}(x) \in W(X)$ for all $x, y \in W(X)$.

Let $F(X)$ be the set of equivalence classes of elements of $W(X)$ modulo the equivalence relation generated by the following relations for all $x, y, z \in W(X)$:

(1) $s_y^{-1}(s_y(x))y \sim x \sim s_y(s_y^{-1}(x))$.
(2) $s_z(s_y(x)) \sim s_{s_z(y)}(s_z(x))$.
(3) $\mathtt{u}(\mathtt{d}(s_x(x))) \sim x \sim \mathtt{d}(\mathtt{u}(s_x(x)))$.
(4) $\mathtt{u}(s_y(x)) \sim s_y(\mathtt{u}(x))$ and $\mathtt{d}(s_y(x)) \sim s_y(\mathtt{d}(x))$.
(5) $s_{\mathtt{u}(y)}(x) \sim s_y(x)$ and $s_{\mathtt{d}(y)}(x) \sim s_y(x)$.

Thus, we have maps $s : F(X) \to \mathrm{Sym}(F(X))$ defined by $x \mapsto s_x := [y \mapsto s_x(y)]$ and $\mathtt{u}, \mathtt{d} : F(X) \to F(X)$ defined by $x \mapsto \mathtt{u}(x)$ and $x \mapsto \mathtt{d}(x)$. We define $\mathrm{FGLR}(X)$ to be the GL-rack $(F(X), s, \mathtt{u}, \mathtt{d})$. The *free quandle on* $X$ is defined similarly, except that we also impose the relations $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ and $s_x(x) \sim x \sim s_x^{-1}(x)$ for all $x \in W(X)$ on $F(X)$.

As one would expect, the functor $\mathsf{Set} \to \mathsf{GLR}$ defined by $X \mapsto \mathrm{FGLR}(X)$ is left adjoint to the forgetful functor $\mathsf{GLR} \to \mathsf{Set}$ [18, Proposition 4.2].

Another functor of interest in Section 3 assigns an *enveloping group* to any GL-rack.

**Definition 2.18.** [17, Section 8] Given a GL-rack $R = (X, s, \mathtt{u}, \mathtt{d})$, its *enveloping group* is

$$\mathrm{Env}_{\mathsf{GLR}}(R) := \left\langle e_x,\ x \in X \mid e_{s_x(y)} = e_x^{-1} e_y e_x,\ e_{\mathtt{u}(x)} = e_x,\ e_{\mathtt{d}(x)} = e_x,\ x, y \in X \right\rangle.$$

By taking $\mathtt{u} = \mathrm{id}_X = \mathtt{d}$, we can also define the enveloping group of a quandle $(Q, s)$ to be

$$\mathrm{Env}_{\mathsf{Qnd}}(Q, s) := \left\langle e_x,\ x \in Q \mid e_{s_x(y)} = e_x^{-1} e_y e_x,\ x, y \in Q \right\rangle.$$

Example 3.5 in Subsection 3.2 computes the enveloping groups of the GL-rack and fundamental quandle of a Legendrian $(2, -q)$ torus knot, which we define in Subsection 3.1.

The functor $\mathrm{Env}_{\mathsf{GLR}} : \mathsf{GLR} \to \mathsf{Grp}$ has a right adjoint that results from taking $\Omega = G$ in Example 2.6, as specified in the following result adapted from [17, Proposition 8.4].

**Proposition 2.19.** *There exists a functor* $\mathrm{Env}_{\mathsf{GLR}} : \mathsf{GLR} \to \mathsf{Grp}$ *that sends any GL-rack to its enveloping group and sends any GL-rack homomorphism* $\psi : (X, s, \mathtt{u}_1, \mathtt{d}_1) \to (Y, t, \mathtt{u}_1, \mathtt{d}_2)$ *to the group homomorphism* $\widetilde{\psi} : \mathrm{Env}_{\mathsf{GLR}}(X, s, \mathtt{u}_1, \mathtt{d}_1) \to \mathrm{Env}_{\mathsf{GLR}}(Y, t, \mathtt{u}_1, \mathtt{d}_2)$ *defined by* $e_x \mapsto e_{\psi(x)}$ *for all* $x \in X$*. Also,* $\mathrm{Env}_{\mathsf{GLR}}$ *is left adjoint to a functor sending any group* $G$ *to the GL-rack* $(\mathrm{Conj}\, G, \mathrm{id}_G, \mathrm{id}_G)$*, which is isomorphic to the quandle* $\mathrm{Conj}\, G$ *in the sense of Lemma 2.16.*

Thus, some authors denote the enveloping group of a GL-rack or quandle $R$ by $\mathrm{Adconj}\, R$ or $\mathrm{As}\, R$ and call it the *associated group of* $R$ (cf. [16, Section 6; 26, Definition 2.19]).

In Section 5, we will also consider the *inner automorphism group* of a GL-rack.

**Definition 2.20.** Let $(X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack. The *inner automorphism group* of $(X, s, \mathtt{u}, \mathtt{d})$, denoted by $\mathrm{Inn}\, X$, is the group generated by the set $s(X)$ under function composition.

We will also employ a generalization of [31, Lemma 2.16] for GL-racks. Since no proof is given in the original article, we provide one below.

**Lemma 2.21.** *If* $(X, s, \mathtt{u}, \mathtt{d})$ *is a GL-rack, then* $(\mathrm{Inn}\, X, s(X)) \in \mathrm{Ob}(\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}})$.

*Proof.* Certainly, $\mathrm{Inn}(X)$ is a group. We need to show that $g \circ s_x \circ g^{-1} \in s(X)$ for all $g \in \mathrm{Inn}(X)$ and $s_x \in s(X)$. Since $\mathrm{Inn}(X)$ is generated by $s(X)$, we know $g$ has the form $s_{y_1} \circ \cdots \circ s_{y_n}$ for some $n \in \mathbb{N}$. To that end, we proceed by induction on $n$. When $n = 1$, the fact that $(X, s)$ is a rack implies

$$s_y \circ s_x \circ s_y^{-1} = s_{s_x(y)} \circ s_y \circ s_y^{-1} = s_{s_x(y)},$$

which is contained in $s(X)$ since $s_x(y) \in X$. The inductive step for $n > 1$ is similar. □

Finally, we employ a version of [3, Theorem 3.1] for GL-racks. Since the proof of the original statement for quandles does not employ any quandle axioms not also satisfied by GL-racks, the proof is identical to the one in the original article.

**Proposition 2.22.** *There exists a functor* $\mathrm{Inn} : \mathsf{GLR}_{\mathrm{surj}} \to \mathsf{Grp}_{\mathrm{epi}}$ *that sends any GL-rack* $R$ *to* $\mathrm{Inn}(R)$ *and sends any surjective GL-rack homomorphism* $f : (X, s, \mathtt{u}_1, \mathtt{d}_1) \twoheadrightarrow (Y, t, \mathtt{u}_2, \mathtt{d}_2)$ *to the surjective group homomorphism* $\mathrm{Inn}\, f : \mathrm{Inn}(X) \twoheadrightarrow \mathrm{Inn}(Y)$ *defined by*

$$s_{x_1}^{\varepsilon_1} \circ \cdots \circ s_{x_n}^{\varepsilon_n} \mapsto t_{f(x_1)}^{\varepsilon_1} \circ \cdots \circ t_{f(x_n)}^{\varepsilon_n},$$

*where* $\varepsilon_i \in \{1, -1\}$ *for all* $1 \le i \le n$.

## 3. On $\mathrm{Env}_{\mathsf{GLR}}$ and invariants of Legendrian links

In this section, we begin by defining the GL-rack of an oriented Legendrian link $\Lambda$ and the fundamental quandle of its underlying smooth link $L$, both of which are invariant under Legendrian isotopy. Then, we prove that the enveloping groups of these invariants are isomorphic, implying that the former is isomorphic to the fundamental group of $\mathbb{R}^3 \setminus L$ (which is called the *knot group* when $L$ is a knot). This generalizes the analogous result for the fundamental quandle [16, Section 15] and proves a conjecture in [17, Remark 8.2].

### 3.1. **The GL-rack of a Legendrian link.**

In this subsection, we discuss how to assign a GL-rack to a Legendrian link in a way invariant under Legendrian isotopy. We begin with several definitions.

**Definition 3.1.** Given an oriented smooth link $L$ and a link diagram $D(L)$ of $L$, define a *strand* of $D(L)$ to be a connected segment in $D(L)$ that either starts and ends at a crossing or (in the case that $D(L)$ contains no crossings) ends where it started.

**Definition 3.2.** [18, Section 4] Let $\Lambda$ be an oriented Legendrian link with front projection $D(\Lambda)$. Let $X_\Lambda$ be a set in one-to-one correspondence with the strands of $D(\Lambda)$. At each crossing (resp. cusp), we impose the corresponding crossing (resp. cusp) relation in Figure 5 on $\mathrm{FGLR}(X_\Lambda)$. The *GL-rack of* $\Lambda$, denoted by $\mathrm{GLR}(\Lambda)$, is defined to be the set of equivalence classes of elements of $\mathrm{FGLR}(X_\Lambda)$ modulo the equivalence relation generated by these crossing and cusp relations. If $L$ is a smooth link with link diagram $D(L)$, then we define the *fundamental quandle of* $L$, denoted by $Q(L)$, in a similar manner. However, we use the free quandle generated by $X_L$ in place of $\mathrm{FGLR}(X_\Lambda)$, and we do not impose any cusp relations.

In Example 3.5 in Subsection 3.2, we compute $\mathrm{GLR}(\Lambda)$ and $Q(L)$ when $L$ is a $(2, -q)$ torus knot and $\Lambda$ is the Legendrian representative of $L$ having maximal Thurston-Bennequin and rotation numbers. We also compute the corresponding enveloping groups in the case that $q = 3$.

The assignment of $\mathrm{FGLR}(\Lambda)$ to $\Lambda$ (resp. $Q(L)$ to $L$) is independent of the choice of front projection $D(\Lambda)$ (resp. $D(L)$), as captured in the following result of Karmakar et al.

**Proposition 3.3.** [18, Theorem 4.3] *If two oriented Legendrian front projections are related by a finite sequence of Legendrian Reidemeister moves, then their induced GL-racks are isomorphic. Hence, the GL-rack of a Legendrian link is invariant under Legendrian isotopy.*

This is a consequence of Proposition 2.3 and the fact that the GL-rack axioms capture the crossing and cusp relations induced by the Legendrian Reidemeister moves.

Definition 3.2 helps us interpret of the structure of finitely presented GL-racks geometrically. By Example 2.9, imposing the equivalence relation $\mathtt{u}(x) \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$ onto $\mathrm{GLR}(\Lambda) = (F(X_\Lambda), s, \mathtt{u}, \mathtt{d})$ yields a Legendrian rack. Geometrically, this amounts to "ignoring" the orientations of cusps in $D(\Lambda)$. Imposing the equivalence relation $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$ onto $\mathrm{GLR}(\Lambda)$ yields a quandle in the sense of Lemma 2.16. Geometrically, this amounts to "ignoring" all cusps in $D(\Lambda)$ and viewing $D(\Lambda)$ only as a diagram of the underlying smooth link $L$. This recovers $Q(L)$ from $\mathrm{GLR}(\Lambda)$, yielding the following result.

**Lemma 3.4.** *Let $\Lambda$ be an oriented Legendrian link, and let $L$ be its underlying smooth link. With $X_\Lambda$ and $X_L$ as defined in Definition 3.2, impose an equivalence relation onto* $\mathrm{GLR}(\Lambda)$ *defined by $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$. Then, the resulting GL-rack is canonically isomorphic to* $(Q(L), \mathrm{id}_{X_L}, \mathrm{id}_{X_L})$*, which is isomorphic to $Q(L)$ in the sense of Lemma 2.16.*

3.2. **Example calculation of** $\mathrm{GLR}(\Lambda)$ **and** $Q(L)$**.** To supplement the discussion of GL-racks of oriented Legendrian links and fundamental quandles of oriented smooth links in Subsection 3.1, we compute these invariants for the Legendrian $(2,-q)$-torus knot with maximal Thurston-Bennequin and rotation numbers. We also compute the enveloping groups of these invariants in the case that $q=3$.

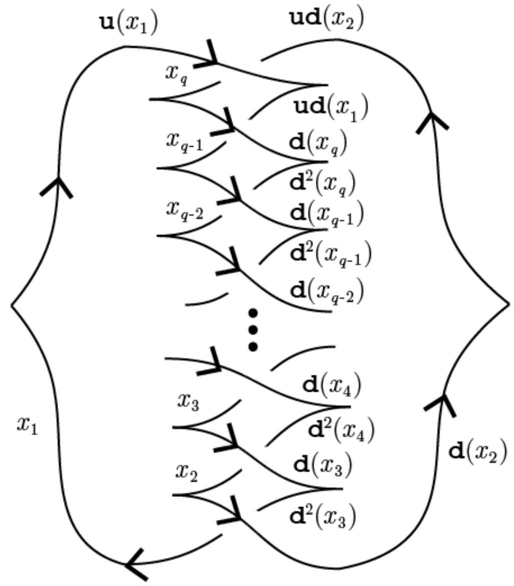


FIGURE 9. Front projection $D(\Lambda)$ and induced cusp relations of the Legendrian $(2,-q)$-torus knot $\Lambda$ with maximal Thurston-Bennequin and rotation numbers.

**Example 3.5.** Let $q\geq 3$ be an odd integer, let $L$ be a $(2,-q)$-torus knot, and let $\Lambda$ be the Legendrian representative of $L$ having maximal Thurston-Bennequin and rotation numbers. (By [14, Theorem 4.3], $\Lambda$ is the unique such Legendrian representative up to Legendrian isotopy.) In this example, we compute $\mathrm{GLR}(\Lambda)$, $Q(L)$, $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda))$, and $\mathrm{Env}_{\mathsf{Qnd}}(Q(L))$ using the front projection $D(\Lambda)$ in Figure 9. Note that $\mathrm{tb}(\Lambda)=-2q$ and $\mathrm{rot}(\Lambda)=q-2$ as predicted by [14, Theorems 4.1 and 4.4], and $D(\Lambda)$ contains $q$ strands. Traversing $D(\Lambda)$ along its depicted orientation and recording the induced cusp and crossing relations using Figure 5, we compute that $\mathrm{GLR}(\Lambda)$ is the free GL-rack on the set $X_\Lambda=\{x_1,\dots,x_q\}$ modulo the cusp relations

$$x_1=\mathtt{u}(x_1)=\mathtt{ud}(x_1),\ x_2=\mathtt{d}(x_2)=\mathtt{ud}(x_2),\ \text{and } x_i=\mathtt{d}(x_i)=\mathtt{d}^2(x_i) \text{ for all } 3\leq i\leq q$$

and the crossing relations

$$s_{\mathtt{u}(x_1)}(x_q)=\mathtt{ud}(x_2),\ s_{\mathtt{d}(x_q)}(x_{q-1})=\mathtt{ud}(x_1),\ \text{and } s_{\mathtt{d}(x_{i-1})}(x_{i-2})=\mathtt{d}^2(x_i) \text{ for all } 3\leq i\leq q.$$

In light of Proposition 2.12, applying $\mathtt{u}^{-1}$ shows that the first cusp relation is equivalent to $x_1=\mathtt{u}(x_1)=\mathtt{d}(x_1)$. Also, the second cusp relation is equivalent to $x_2=\mathtt{d}(x_2)=\mathtt{u}(x_2)$, and for all $3\leq i\leq q$, the $i$th cusp relation is equivalent to $x_i=\mathtt{d}(x_i)$. In particular, we have $\mathtt{d}=\mathrm{id}_{X_\Lambda}$, so $\mathrm{GLR}(\Lambda)$ can be presented more simply as $\mathrm{FGLR}(X_\Lambda)$ modulo the cusp relations

$$x_1=\mathtt{u}(x_1),\ x_2=\mathtt{u}(x_2),\ \text{and } x_i=\mathtt{d}(x_i) \text{ for all } 1\leq i\leq q$$

and the crossing relations

$$s_{x_1}(x_q)=x_2,\ s_{x_q}(x_{q-1})=x_1,\ \text{and } s_{x_{i-1}}(x_{i-2})=x_i \text{ for all } 3\leq i\leq q. \tag{1}$$

To compute $Q(L)$, one could traverse $D(\Lambda)$ again while ignoring all cusps and only considering crossing relations. In view of Lemma 3.4, we can equivalently impose the relations $\mathtt{u}(x_i) = x_i = \mathtt{d}(x_i)$ for all $1 \leq i \leq q$ onto $\mathrm{GLR}(\Lambda)$. Either method shows that $Q(L)$ is the free quandle on $X_\Lambda$ modulo the crossing relations in (1). (Indeed, if we invert each symmetry in the relations of $Q(L)$, then we recover the fundamental quandle of the mirror image of $L$ as computed in [2, Remark 3].) If $q = 3$, then $L$ is a left-handed trefoil, and the crossing relations show that $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda))$ and $\mathrm{Env}_{\mathsf{Qnd}}(Q(L))$ are both isomorphic to the group

$$\begin{aligned}&\left\langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{s_{x_1}(x_3)} = e_{x_1}^{-1} e_{x_3} e_{x_1}, e_{s_{x_2}(x_1)} = e_{x_2}^{-1} e_{x_1} e_{x_2}, e_{s_{x_3}(x_2)} = e_{x_3}^{-1} e_{x_2} e_{x_3} \right\rangle \\ &\qquad = \left\langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{x_1} e_{x_2} = e_{x_3} e_{x_1}, e_{x_2} e_{x_3} = e_{x_1} e_{x_2}, e_{x_3} e_{x_1} = e_{x_2} e_{x_3} \right\rangle.\end{aligned}$$

This is precisely the Wirtinger presentation of the knot group $\pi_1(\mathbb{R}^3 \setminus L) \cong \left\langle x, y \mid x^2 = y^3 \right\rangle$ of $L$ (cf. [29, Subsection 4.2.5]), as predicted by the results of the next subsection.

3.3. **Isomorphism of** $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda))$ **and** $\pi_1(\mathbb{R}^3 \setminus L)$**.** We now prove the main results of this section.

**Theorem 3.6.** *Let $R = (X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack, and let $R' = (X/\sim, s^*, \mathrm{id}_{X/\sim}, \mathrm{id}_{X/\sim})$ be the GL-rack obtained by imposing an equivalence relation $\sim$ onto $R$ defined by $\mathtt{u}(x) = x = \mathtt{d}(x)$ for all $x \in X$. Then $\mathrm{Env}_{\mathsf{GLR}}(R)$ and $\mathrm{Env}_{\mathsf{GLR}}(R')$ are isomorphic as groups. In particular, $\mathrm{Env}_{\mathsf{GLR}}(R)$ and $\mathrm{Env}_{\mathsf{Qnd}}(X/\sim, s^*)$ are isomorphic as groups.*

*Proof.* By definition, $R'$ is the quotient object of the equivalence relation $\sim$ on $R$ in $\mathsf{GLR}$. In other words, $R$ is the colimit of the following diagram in $\mathsf{GLR}$:

$$R \underset{\mathrm{id}_X}{\overset{\mathtt{u}}{\rightrightarrows}} R \underset{\mathrm{id}_X}{\overset{\mathtt{d}}{\rightrightarrows}} R$$

Recall that left adjoint functors preserve colimits. It follows from Proposition 2.19 that $\mathrm{Env}_{\mathsf{GLR}}(R')$ is the colimit of the following diagram in $\mathsf{Grp}$:

$$\mathrm{Env}_{\mathsf{GLR}}(R) \underset{\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)}{\overset{\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})}{\rightrightarrows}} \mathrm{Env}_{\mathsf{GLR}}(R) \underset{\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)}{\overset{\mathrm{Env}_{\mathsf{GLR}}(\mathtt{d})}{\rightrightarrows}} \mathrm{Env}_{\mathsf{GLR}}(R)$$

By Proposition 2.19, the group homomorphism $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})$ is defined by $e_x \mapsto e_{\mathtt{u}(x)}$ for all $x \in X$, but $e_x = e_{\mathtt{u}(x)}$ in $\mathrm{Env}_{\mathsf{GLR}}(R)$. Thus, $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})$ is the identity map. Similarly, $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{d})$ and $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)$ are the identity maps, so we have a group isomorphism

$$\mathrm{Env}_{\mathsf{GLR}}(R) \cong \mathrm{Env}_{\mathsf{GLR}}(R').$$

By Lemma 2.16, the right-hand side is isomorphic to $\mathrm{Env}_{\mathsf{Qnd}}(X/\sim, s^*)$, so we are done. ☐

**Corollary 3.7.** *Let $\Lambda \subset \mathbb{R}^3$ be an oriented Legendrian link, and let $L$ denote its underlying smooth link. Then there exists a group isomorphism*

$$\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda)) \cong \pi_1(\mathbb{R}^3 \setminus L).$$

*Proof.* Taking $R = \mathrm{GLR}(\Lambda)$ in Theorem 3.6, we have $R' \cong (Q(L), \mathrm{id}_{X_L}, \mathrm{id}_{X_L})$ by Lemma 3.4. By Theorem 3.6, it suffices to show that $\mathrm{Env}_{\mathsf{Qnd}}(Q(L))$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus L)$. But this is shown in [16, Section 15] using the Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus L)$. ☐

# 4. $\mathsf{GLQ}^{\mathrm{med}}$ IS SYMMETRIC MONOIDAL CLOSED

In this section, we define medial GL-racks and introduce tensor products of GL-racks. We also consider the full subcategory $\mathsf{GLQ}^{\mathrm{med}}$ of $\mathsf{GLR}$ whose objects are medial GL-quandles, and we show that $\mathsf{GLQ}^{\mathrm{med}}$ is symmetric monoidal closed. This generalizes the analogous result for the category of medial quandles [9, Theorem 12], whose proof we follow closely.

4.1. **Hom-sets of medial GL-quandles are also medial GL-quandles.** In this subsection, we define medial GL-racks and show that hom-sets between medial GL-quandles are themselves medial GL-quandles.

**Definition 4.1.** A rack (resp. quandle) $(X,s)$ is called *medial* or *abelian* if, for all $x,y,z\in X$, we have an equality of symmetries

$$s_{s_x(z)}\circ s_y=s_{s_x(y)}\circ s_z.$$

If in addition $(\mathtt{u},\mathtt{d})$ defines a GL-structure on $(X,s)$, then we call $(X,s,\mathtt{u},\mathtt{d})$ a *medial* or *abelian* GL-rack (resp. GL-quandle). Note that this definition is not synonymous with a rack (resp. quandle) being *commutative*, a condition requiring $s_x(y)=s_y(x)$ for all $x,y\in X$.

Let $\mathsf{GLQ}$ be the full subcategory of $\mathsf{GLR}$ whose objects are GL-quandles. By Proposition 2.13, any GL-structure on a quandle $(Q,s)$ has the form $(\mathtt{u},\mathtt{u}^{-1})$. Thus, we can write any object in $\mathsf{GLQ}$ simply as a triple $(Q,s,\mathtt{u})$. We will do so hereafter for the sake of brevity.

**Proposition 4.2.** *Let $R_1=(X,s^X,\mathtt{u}_1)$ and $R_2=(Y,s^Y,\mathtt{u}_2)$ be GL-quandles, and suppose $R_2$ is medial. Let $H:=\mathrm{Hom}_{\mathsf{GLQ}}(R_1,R_2)$, and define $\mathtt{u}:H\to H$ by $\mathtt{u}(f):=\mathtt{u}_2\circ f=f\circ\mathtt{u}_1$, so that $\mathtt{u}$ has a two-sided inverse $\mathtt{u}^{-1}$ given by $\mathtt{u}^{-1}(f)=\mathtt{u}_2^{-1}\circ f=f\circ\mathtt{u}_1^{-1}$. Also, define $s:H\to\mathrm{Sym}(H)$ by $s_g(f):=[q\mapsto(s^Y_{g(q)}\circ f)(q)]$. Then $(H,s,\mathtt{u})$ is a medial GL-quandle.*

*Proof.* Since $(X,s^X)$ and $(Y,s^Y)$ are quandles and the latter is medial, it follows from [9, Theorem 3] that $(H,s)$ is a medial quandle. Clearly, $\mathtt{u}\mathtt{u}^{-1}=\mathrm{id}_H=\mathtt{u}^{-1}\mathtt{u}$. By Proposition 2.14, to show that $(\mathtt{u},\mathtt{u}^{-1})$ defines a GL-structure on $(H,s)$, it remains to show that $\mathtt{u}$ and $\mathtt{u}^{-1}$ are rack endomorphisms satisfying GL-rack axiom (L3). So, fix $f,g\in H$. We have

$$\begin{aligned}(\mathtt{u}\circ s_g)(f)&=[q\mapsto(\mathtt{u}_2\circ s^Y_{g(q)}\circ f)(q)]\\&=[q\mapsto(s^Y_{\mathtt{u}_2(g(q))}\circ\mathtt{u}_2\circ f)(q)]&&\text{by applying Proposition 2.12 to }\mathtt{u}_2\\&=s_{\mathtt{u}_2\circ g}(\mathtt{u}_2\circ f)\\&=(s_{\mathtt{u}(g)}\circ\mathtt{u})(f),\end{aligned}$$

as desired. Similarly, $(\mathtt{u}^{-1}\circ s_g)(f)=(s_{\mathtt{u}^{-1}(g)}\circ\mathtt{u}^{-1})(f)$, showing that $\mathtt{u}$ and $\mathtt{u}^{-1}$ are rack endomorphisms. To verify GL-rack axiom (L3), apply this axiom to $\mathtt{u}_2$ to get

$$s_{\mathtt{u}(g)}(f)=[q\mapsto(s^Y_{\mathtt{u}_2(g(q))}\circ f)(q)]=[q\mapsto(s^Y_{g(q)}\circ f)(q)]=s_g(f),$$

as desired. Similarly, $s_{\mathtt{u}^{-1}(g)}(f)=s_g(f)$. This completes the proof. $\square$

4.2. **Tensor products of GL-racks.** Let $\mathsf{GLQ}^{\mathrm{med}}$ be the full subcategory of $\mathsf{GLQ}$ whose objects are medial. In this subsection, we introduce tensor products in $\mathsf{GLR}$ and show that $\mathsf{GLQ}^{\mathrm{med}}$ is symmetric monoidal closed.

We define tensor products of GL-racks similarly to [9, Subsection 8.1].

**Definition 4.3.** If $R_1=(X,s^X,\mathtt{u}_1,\mathtt{d}_1)$ and $R_2=(Y,s^Y,\mathtt{u}_2,\mathtt{d}_2)$ are GL-racks, then we define their *tensor product* to be the free GL-rack $\mathrm{FGLR}(X\times Y)$ modulo the following relations for all $x,x_1,x_2\in X$ and $y,y_1,y_2\in Y$:

(1) $s_{(x,y_2)}(x,y_1) = (x, s^Y_{y_2}(y_1))$.
(2) $s_{(x_2,y)}(x_1,y) = (s^X_{x_2}(x_1), y)$.
(3) $\mathtt{u}(x,y) = (\mathtt{u}_1(x), y) = (x, \mathtt{u}_2(y))$.
(4) $\mathtt{d}(x,y) = (\mathtt{d}_1(x), y) = (x, \mathtt{d}_2(y))$.

We denote this GL-rack simply by $R_1 \otimes R_2$.

Note that if $R_1$ and $R_2$ are GL-quandles, then so is $R_1 \otimes R_2$. In light of Lemma 2.16, tensor products of GL-racks generalize tensor products of quandles (cf. [9, Subsection 8.1]).

Tensor products of GL-racks are of particular interest to the development of (co)homology theories for GL-racks and their induced Legendrian link invariants (cf. [4, 7, 17, 18, 20]). Indeed, the next two results show that tensor products of medial GL-quandles satisfy the expected universal property and internal hom-tensor adjunction.

**Theorem 4.4.** $\mathsf{GLQ}^{\mathrm{med}}$ *is symmetric monoidal closed under the tensor product* $\otimes$ *in Definition 4.3 and closed structure* $\mathrm{Hom}_{\mathsf{GLQ}^{\mathrm{med}}}(-,-)$ *in Proposition 4.2.*

*Proof.* The unit object in $\mathsf{GLQ}^{\mathrm{med}}$ is the trivial GL-quandle with one element. Using this fact, it is straightforward to verify that $\mathsf{GLQ}^{\mathrm{med}}$ is monoidal and symmetric. Now, recall that $\mathsf{GLQ}^{\mathrm{med}}$ is an equational category. By the main theorem of [23], to show that $\mathsf{GLQ}^{\mathrm{med}}$ is closed, it suffices to show in the sense of universal algebra that $\mathsf{GLQ}^{\mathrm{med}}$ is commutative as an algebraic theory (cf. [23, Section 6]). Indeed, for any medial GL-quandle $(X, s, \mathtt{u})$ and for all elements $x_{11}, x_{12}, x_{21}, x_{22} \in X$, we have the following equalities:

$$\begin{cases} (\mathtt{u} \circ s_{x_{12}})(x_{11}) = (s_{\mathtt{u}(x_{12})} \circ \mathtt{u})(x_{11}) & \text{by Proposition 2.12,} \\ (\mathtt{u}^{-1} \circ s_{x_{12}})(x_{11}) = (s_{\mathtt{u}^{-1}(x_{12})} \circ \mathtt{u}^{-1})(x_{11}) & \text{by Proposition 2.12.} \\ (s_{s_{x_{22}}(x_{21})} \circ s_{x_{12}})(x_{11}) = (s_{s_{x_{22}}(x_{12})} \circ s_{x_{21}})(x_{11}) & \text{since } (X,s) \text{ is medial.} \end{cases}$$

Together with the tautologies $(\mathtt{u} \circ \mathtt{u}^{-1})(x_{11}) = (\mathtt{u}^{-1} \circ \mathtt{u})(x_{11})$, $\mathtt{u}^2(x_{11}) = \mathtt{u}^2(x_{11})$, and $(\mathtt{u}^{-1})^2(x_{11}) = (\mathtt{u}^{-1})^2(x_{11})$, these equalities show that $\mathsf{GLQ}^{\mathrm{med}}$ forms a commutative algebraic theory. This completes the proof. $\square$

In light of Proposition 4.2, we immediately deduce the following.

**Corollary 4.5.** $\mathsf{GLQ}^{\mathrm{med}}$ *is self-enriched. Explicitly, if $A$, $B$, and $C$ are medial GL-quandles, then there exists a natural isomorphism of medial GL-quandles*

$$\mathrm{Hom}_{\mathsf{GLQ}^{\mathrm{med}}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathsf{GLQ}^{\mathrm{med}}}(A, \mathrm{Hom}_{\mathsf{GLQ}^{\mathrm{med}}}(B, C)).$$

## 5. Adjunction between $\mathsf{GLR}_{\mathrm{surj}}$ and $\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}$

In [31, Theorem 1.1], Tada exhibits an equivalence of categories between the full subcategory of $\mathsf{Qnd}_{\mathrm{surj}}$ whose objects are faithful quandles and $\mathsf{Grp}^{\mathrm{gcf}}_{\mathrm{epi}}$, the full subcategory of $\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}$ whose objects are pairs $(G, \Omega)$ such that the conjugation action of $G$ on $\Omega$ is faithful. (It follows from Proposition 2.15 that $\mathsf{Grp}^{\mathrm{gcf}}_{\mathrm{surj}}$ is equivalent to the full subcategory of $\mathsf{GLR}$ whose objects are faithful GL-racks.) In this section, we extend Tada's constructions to exhibit an adjunction between $\mathsf{GLR}_{\mathrm{surj}}$ and $\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}$. In its use of the Inn functor, this adjunction contrasts with the adjunction between $\mathsf{GLR}$ and $\mathsf{Grp}$ in Proposition 2.19.

5.1. **Construction of $\mathcal{F}$ and $\mathcal{G}$.** In this subsection, we define functors $\mathcal{F} : \mathsf{GLR}_{\mathrm{surj}} \to \mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}$ and $\mathcal{G} : \mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}} \to \mathsf{GLR}_{\mathrm{surj}}$. We begin with $\mathcal{F}$. For any GL-rack $(X, s, \mathtt{u}, \mathtt{d})$, define $\mathcal{F}(X, s, \mathtt{u}, \mathtt{d}) := (\mathrm{Inn}(X), s(X))$. Also, for any surjective homomorphism of GL-racks $\varphi : (X, s, \mathtt{u}_1, \mathtt{d}_2) \twoheadrightarrow (Y, t, \mathtt{u}_1, \mathtt{d}_1)$, define $\mathcal{F}\varphi : (\mathrm{Inn}(X), s(X)) \to (\mathrm{Inn}(Y), t(Y))$ by $s_x \mapsto t_{\varphi(x)}$.

**Proposition 5.1.** *$\mathcal{F}$ is a functor.*

*Proof.* Since $s(X)$ generates $\mathrm{Inn}(X)$, it is easy to see that if $\varphi : (X, s, \mathtt{u}_1, \mathtt{d}_1) \twoheadrightarrow (Y, t, \mathtt{u}_2, \mathtt{d}_2)$ is a surjective GL-rack homomorphism, then $\mathcal{F}\varphi = \mathrm{Inn}\,\varphi$, where $\mathrm{Inn}\,\varphi$ is defined as in Proposition 2.22. Thanks to this proposition, $\mathcal{F}\varphi$ is a surjective group homomorphism. Since $\varphi(X) = Y$ as sets, we also have $(\mathcal{F}\varphi)(s(X)) = t(Y)$. Altogether, this shows that $\mathcal{F}\varphi \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\mathrm{Inn}(X), s(X), (\mathrm{Inn}(Y), t(Y)]$, as desired. Clearly, $\mathcal{F}\,\mathrm{id}_Q = \mathrm{id}_{\mathrm{Inn}(Q)}$. It is also evident that if $\psi : (Y, t, \mathtt{u}_2, \mathtt{d}_2) \twoheadrightarrow (Z, r, \mathtt{u}_3, \mathtt{d}_3)$ is a surjective GL-rack homomorphism, then we have $\mathcal{F}(\psi \circ \varphi) = [s_x \mapsto v_{\psi(\varphi(x))}] = \mathcal{F}\psi \circ \mathcal{F}\varphi$. By Lemma 2.21, we are done. $\square$

Next, we define $\mathcal{G}$. For any object $(G, \Omega) \in \mathrm{Ob}(\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}})$, let $\mathcal{G}(G, \Omega) := (\mathrm{Conj}\,\Omega, \mathrm{id}_\Omega, \mathrm{id}_\Omega)$. Given $\psi \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}((G, \Omega), (H, \Gamma))$, define $\mathcal{G}\psi : (\mathrm{Conj}\,\Omega, \mathrm{id}_\Omega, \mathrm{id}_\Omega) \to (\mathrm{Conj}\,\Gamma, \mathrm{id}_\Gamma, \mathrm{id}_\Gamma)$ by $\omega \mapsto \psi(\omega)$.

**Proposition 5.2.** *$\mathcal{G}$ is a functor.*

*Proof.* Let $(G, \Omega) \in \mathrm{Ob}(\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}})$. Certainly, $\mathcal{G}(G, \Omega) \in \mathrm{Ob}(\mathsf{Qnd}_{\mathrm{surj}})$, and $\mathcal{G}\,\mathrm{id}_{(G,\Omega)} = \mathrm{id}_{\mathrm{Conj}\,\Omega}$. Let $\psi \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(G, \Omega), (H, \Gamma)]$. Since $\psi$ is a group homomorphism and the rack structures on $\mathcal{G}(\Omega)$ and $\mathcal{G}(\Gamma)$ are given by conjugation, verifying that $\mathcal{G}\psi$ is a GL-rack homomorphism is straightforward. Since $\psi(\Omega) = \Gamma$, it follows that $(\mathcal{G}\psi)(\mathrm{Conj}\,\Omega) = \mathrm{Conj}\,\Gamma$ as sets, so $\mathcal{G}\psi$ is surjective. Thus, $\mathcal{G}\psi \in \mathrm{Hom}_{\mathsf{GLR}_{\mathrm{surj}}}[(\mathrm{Conj}\,\Omega, \mathrm{id}_\Omega, \mathrm{id}_\Omega), (\mathrm{Conj}\,\Gamma, \mathrm{id}_\Gamma, \mathrm{id}_\Gamma)]$. Finally, if $\varphi \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(H, \Gamma), (K, \Delta)]$, then $\mathcal{G}(\varphi \circ \psi) = [\omega \mapsto \varphi(\psi(\omega))] = \mathcal{G}\varphi \circ \mathcal{G}\psi$, as desired. $\square$

5.2. **Adjointness of $\mathcal{F}$ and $\mathcal{G}$.** In this subsection, we prove that $\mathcal{F}$ is left adjoint to $\mathcal{G}$. To begin the proof, for any objects $(X, s, \mathtt{u}, \mathtt{d}) \in \mathrm{Ob}(\mathsf{GLR}_{\mathrm{surj}})$ and $(G, \Omega) \in \mathrm{Ob}(\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}})$, we will define a bijection of hom-sets

$$\eta_{X_{s,\mathtt{u},\mathtt{d}} G_\Omega} : \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\mathrm{Inn}\,X, s(X)), (G, \Omega)] \to \mathrm{Hom}_{\mathsf{GLR}_{\mathrm{surj}}}[(X, s, \mathtt{u}, \mathtt{d}), (\mathrm{Conj}\,\Omega, \mathrm{id}_\Omega, \mathrm{id}_\Omega)].$$

Given any morphism $f \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\mathrm{Inn}\,X, s(X)), (G, \Omega)]$, we have $f(s(X)) = \Omega$. Since $\Omega$ is the underlying set of $\mathrm{Conj}\,\Omega$, the map $[x \mapsto f(s_x)]$ from $(X, s)$ to $\mathrm{Conj}\,\Omega$ is surjective. So, we can define $\eta_{X_{s,\mathtt{u},\mathtt{d}} G_\Omega}$ by

$$f \mapsto [x \mapsto f(s_x)].$$

**Proposition 5.3.** *$\eta_{X_{s,\mathtt{u},\mathtt{d}} G_\Omega}$ as defined above is a bijection from $\mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\mathrm{Inn}\,X, s(X)), (G, \Omega)]$ to $\mathrm{Hom}_{\mathsf{GLR}_{\mathrm{surj}}}[(X, s, \mathtt{u}, \mathtt{d}), (\mathrm{Conj}\,\Omega, \mathrm{id}_\Omega, \mathrm{id}_\Omega)]$.*

*Proof.* We have already shown that $[x \mapsto f(s_x)]$ is surjective. To show that this map is a GL-rack homomorphism, fix $y \in X$, and recall that the rack structure on $\mathrm{Conj}\,\Omega$ is given by the map

$\alpha \mapsto [\omega \mapsto \alpha\omega\alpha^{-1}]$. Taking $\alpha = f(s_y)$, we have

$$\begin{aligned}
[\omega \mapsto f(s_y)\omega f(s_y)^{-1}] \circ [x \mapsto f(s_x)] &= [x \mapsto f(s_y)f(s_x)f(s_y)^{-1}] \\
&= [x \mapsto f(s_y \circ s_x \circ s_y^{-1})] \\
&= [x \mapsto f(s_{s_y(x)} \circ s_y \circ s_y^{-1})] \qquad \text{since } (X, s) \text{ is a rack} \\
&= [x \mapsto f(s_{s_y(x)})] \\
&= [x \mapsto f(s_x)] \circ [x \mapsto s_y(x)] \\
&= [x \mapsto f(s_x)] \circ s_y,
\end{aligned}$$

where we have used the fact that $f$ is a group homomorphism. This shows that $[x \mapsto f(s_x)]$ is a rack homomorphism. Moreover, axiom (L3) of GL-racks implies

$$[x \mapsto f(s_x)] \circ \mathtt{u} = [x \mapsto f(s_{\mathtt{u}(x)})] = [x \mapsto f(s_x)] = \mathrm{id}_\Omega \circ [x \mapsto f(s_x)].$$

Similarly, $[x \mapsto f(s_x)] \circ \mathtt{d} = \mathrm{id}_\Omega \circ [x \mapsto f(s_x)]$. Thus, $[x \mapsto f(s_x)]$ is a GL-rack homomorphism.

Since $s(X)$ generates $\operatorname{Inn} X$, any morphism $f \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\operatorname{Inn} X, s(X)), (G, \Omega)]$ is completely determined by its values on $s(X)$, so $\eta_{X_{s,\mathtt{u},\mathtt{d}} G_\Omega}$ is injective. To show surjectivity, take a morphism $g \in \mathrm{Hom}_{\mathsf{GLR}_{\mathrm{surj}}}[(X, s, \mathtt{u}, \mathtt{d}), (\operatorname{Conj} \Omega, \mathrm{id}_\Omega, \mathrm{id}_\Omega)]$. To define a group homomorphism $f : \operatorname{Inn} X \to G$, it suffices to define $f$ on $s(X)$, so define $f$ by $s_x \mapsto g(x)$ for all $x \in X$. Extending $f$ to the rest of $\operatorname{Inn} X$ via $f(s_x \circ s_y) = f(s_x)f(s_y)$ yields a group homomorphism. Since $g$ is surjective and $\operatorname{Conj} \Omega = \Omega$ as sets, we have $f(s(X)) = g(X) = \Omega$ by the definition of $f$. Since $\Omega$ generates $G$, it follows that $f$ is surjective. Altogether, this shows $f \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\operatorname{Inn} X, s(X)), (G, \Omega)]$. Since $\eta_{X_{s,\mathtt{u},\mathtt{d}} G_\Omega}(f) = g$, the proof is complete. ☐

It remains to show that the collection $\eta$ of all such bijections is natural.

**Theorem 5.4.** *$\mathcal{F}$ is left adjoint to $\mathcal{G}$.*

*Proof.* First, let $(X, s, \mathtt{u}_1, \mathtt{d}_1), (Y, t, \mathtt{u}_2, \mathtt{d}_2) \in \mathrm{Ob}(\mathsf{GLR}_{\mathrm{surj}})$, let $(G, \Omega) \in \mathrm{Ob}(\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}})$, and fix a morphism $\varphi \in \mathrm{Hom}_{\mathsf{GLR}_{\mathrm{surj}}}[(Y, t, \mathtt{u}_2, \mathtt{d}_2), (X, s, \mathtt{u}_1, \mathtt{d}_1)]$. We need to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\operatorname{Inn} X, s(X)), (G, \Omega)] \arrow[r, "\eta_{X_{s,\mathtt{u}_1,\mathtt{d}_1} G_\Omega}"] \arrow[d, "-\circ \mathcal{F}\varphi"'] & \mathrm{Hom}_{\mathsf{GLR}_{\mathrm{surj}}}[(X, s, \mathtt{u}_1, \mathtt{d}_1), (\operatorname{Conj} \Omega, \mathrm{id}_\Omega, \mathrm{id}_\Omega)] \arrow[d, "-\circ\varphi"] \\
\mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\operatorname{Inn} Y, t(Y)), (G, \Omega)] \arrow[r, "\eta_{Y_{t,\mathtt{u}_2,\mathtt{d}_2} G_\Omega}"] & \mathrm{Hom}_{\mathsf{GLR}_{\mathrm{surj}}}[(Y, t, \mathtt{u}_2, \mathtt{d}_2), (\operatorname{Conj} \Omega, \mathrm{id}_\Omega, \mathrm{id}_\Omega)]
\end{tikzcd}$$

Indeed, for all $f \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\operatorname{Inn} X, s(X)), (G, \Omega)]$, we have

$$\eta_{Y_{t,\mathtt{u}_2,\mathtt{d}_2} G_\Omega}(f \circ \mathcal{F}\varphi) = \eta_{Y_{t,\mathtt{u}_2,\mathtt{d}_2} G_\Omega}(f \circ [t_y \mapsto s_{\varphi(y)}]) = \eta_{Y_{t,\mathtt{u}_2,\mathtt{d}_2} G_\Omega}[t_y \mapsto f(s_{\varphi(y)})] = [y \mapsto f(s_{\varphi(y)})],$$

and

$$\eta_{X_{s,\mathtt{u}_1,\mathtt{d}_1} G_\Omega}(f) \circ \varphi = [x \mapsto f(s_x)] \circ [y \mapsto \varphi(y)] = [y \mapsto f(s_{\varphi(y)})] = \eta_{Y_{t,\mathtt{u}_2,\mathtt{d}_2} G_\Omega}(f \circ \mathcal{F}\varphi),$$

as desired. Next, let $(H,\Gamma) \in \mathrm{Ob}(\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}})$, and fix a morphism $\psi \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(G,\Omega),(H,\Gamma)]$. We need to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\mathrm{Inn}\,X, s(X)),(G,\Omega)] & \xrightarrow{\eta_{X_{s,\mathtt{u}_1,\mathtt{d}_1}\,G_\Omega}} & \mathrm{Hom}_{\mathsf{GLR}_{\mathrm{surj}}}[(X,s,\mathtt{u}_1,\mathtt{d}_1),(\mathrm{Conj}\,\Omega,\mathrm{id}_\Omega,\mathrm{id}_\Omega)] \\
{\scriptstyle \psi\circ -}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{G}\psi\circ -} \\
\mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\mathrm{Inn}\,X, s(X)),(H,\Gamma)] & \xrightarrow{\eta_{X_{s,\mathtt{u}_1,\mathtt{d}_1}\,H_\Gamma}} & \mathrm{Hom}_{\mathsf{GLR}_{\mathrm{surj}}}[(X,s,\mathtt{u}_1,\mathtt{d}_1),(\mathrm{Conj}\,\Gamma,\mathrm{id}_\Gamma,\mathrm{id}_\Gamma)]
\end{array}$$

Indeed, for all $f \in \mathrm{Hom}_{\mathsf{Grp}^{\mathrm{gc}}_{\mathrm{surj}}}[(\mathrm{Inn}\,X, s(X)),(G,\Omega)]$, we have

$$\mathcal{G}\psi \circ \eta_{X_{s,\mathtt{u}_1,\mathtt{d}_1}\,G_\Omega}(f) = [\omega \mapsto \psi(\omega)] \circ [x \mapsto f(s_x)] = [x \mapsto \psi(f(s_x))],$$

and

$$\eta_{X_{s,\mathtt{u}_1,\mathtt{d}_1}\,H_\Gamma}(\psi \circ f) = [x \mapsto \psi(f(s_x))] = \mathcal{G}\psi \circ \eta_{X_{s,\mathtt{u}_1,\mathtt{d}_1}\,G_\Omega}(f),$$

as desired. This completes the proof of the theorem. □

## 6. On $\mathsf{GrpTup}$ and GL-rack isomorphisms

In [18, Section 5], Karmakar et al. construct a homogeneous representation for any GL-rack $R$ from the orbits of $R$ under the action of its automorphism group. In this section, we introduce morphisms that make these group-theoretic constructions into a category $\mathsf{GrpTup}$ with an essentially surjective functor $\mathcal{F} : \mathsf{GrpTup} \twoheadrightarrow \mathsf{GLR}$. This functor induces a group-theoretic sufficient condition for two GL-racks to be isomorphic.

**6.1. Construction of $\mathsf{GrpTup}$.** In this subsection, we introduce a category $\mathsf{GrpTup}$ with a functorial relationship to $\mathsf{GLR}$.

To define the objects in $\mathsf{GrpTup}$, we adapt the construction of [18, Proposition 5.1]. Let the objects in $\mathsf{GrpTup}$ be all sextuples $(I, \bigsqcup_{i\in I} G/H_i, Z_I, Q_I, R_I, \tau)$ satisfying the following:

(1) $I$ is an indexing set, $G$ is a group, and the sets $Z_I = \{z_i^G \mid i \in I\}$, $Q_I = \{q_i^G \mid i \in I\}$, and $R_I = \{r_i^G \mid i \in I\}$ are subsets of $G$ indexed by $I$.
(2) $\{H_i \mid i \in I\}$ is a family of subgroups of $G$ such that $H_i \le C_G(z_i^G)$ for all $i \in I$.
(3) $\tau : I \to I$ is a bijection such that the following hold for all $i \in I$:
 (a) $q_i^G \in N_G(H_{\tau(i)})$.
 (b) $r_i^G \in N_G(H_{\tau^{-1}(i)})$.
 (c) $z_i^G q_i^G r_{\tau(i)}^G,\ z_i^G r_i^G q_{\tau^{-1}(i)}^G \in H_i$.
 (d) $z_i^G q_i^G (z_{\tau(i)}^G)^{-1} = q_i^G$.
 (e) $z_i^G r_i^G (z_{\tau^{-1}(i)}^G)^{-1} = r_i^G$.

For the sake of brevity, we will denote such an object as $\widetilde{G}$ when there is no room for confusion. For an opposing object in $\mathsf{GrpTup}$, we will write $\widetilde{K} = (J, \bigsqcup_{j\in J} K/L_j, Z_J, Q_J, R_J, \pi)$.

Now, we define the morphisms in $\mathsf{GrpTup}$. Given any two objects $\widetilde{G}, \widetilde{K}$ in $\mathsf{GrpTup}$, let $\mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$ be the set of all triples $\varphi := (\varphi_1, \varphi_2, \varphi_3)$ satisfying the following:

(1) $\varphi_1 : G \to K$ is a group homomorphism.
(2) $\varphi_2 : \bigsqcup_{i\in I} G/H_i \to \bigsqcup_{j\in J} K/L_j$ and $\varphi_3 : I \to J$ are morphisms in $\mathsf{Set}$.
(3) $\pi \circ \varphi_3 = \varphi_3 \circ \tau$.
(4) For all $i \in I$ and $g \in G$, we have $\varphi_1(z_i^G) = z_{\varphi_3(i)}^K$, $\varphi_1(q_i^G) = q_{\varphi_3(i)}^K$, $\varphi_1(r_i^G) = r_{\varphi_3(i)}^K$, and $\varphi_2(gH_i) = \varphi_1(g)L_{\varphi_3(i)}$.

Define the composition of morphisms in $\mathsf{GrpTup}$ by $\psi \circ \varphi := (\psi_1 \circ \varphi_1, \psi_2 \circ \varphi_2, \psi_3 \circ \varphi_3)$. Also, define the identity morphism $\mathrm{id}^{\widetilde{G}} : \widetilde{G} \xrightarrow{\sim} \widetilde{G}$ by letting $\mathrm{id}_1^{\widetilde{G}}$, $\mathrm{id}_2^{\widetilde{G}}$, and $\mathrm{id}_3^{\widetilde{G}}$ as defined above be identity maps. Associativity and unit laws are immediate. Hence, $\mathsf{GrpTup}$ is a category.

6.2. **Construction of $\mathcal{F} : \mathsf{GrpTup} \to \mathsf{GLR}$.** In this subsection, we construct an essentially surjective functor $\mathcal{F} : \mathsf{GrpTup} \twoheadrightarrow \mathsf{GLR}$. Then, we discuss a group-theoretic sufficient condition for two GL-racks to be isomorphic.

By [18, Proposition 5.1], given any object $\widetilde{G}$ in $\mathsf{GrpTup}$, the set $X := \bigsqcup_{i \in I} G/H_i$ admits a GL-rack structure where $s^X : X \to \mathrm{Sym}(X)$ and $\mathtt{u}_G, \mathtt{d}_G : X \to X$ are defined by

$$s^X_{yH_j}(xH_i) := y z_j^G y^{-1} x H_i, \quad \mathtt{u}_G(xH_i) := x q_i^G H_{\tau(i)}, \quad \text{and} \quad \mathtt{d}_G(xH_i) := x r_i^G H_{\tau^{-1}(i)}.$$

So, we can define a functor $\mathcal{F} : \mathsf{GrpTup} \to \mathsf{GLR}$ by sending any object $\widetilde{G}$ in $\mathsf{GrpTup}$ to the GL-rack $(X, s^X, \mathtt{u}_G, \mathtt{d}_G)$ and sending any morphism $\varphi \in \mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$ to $\varphi_2$.

**Theorem 6.1.** *$\mathcal{F}$ is an essentially surjective functor.*

*Proof.* Essential surjectivity is precisely the statement of [18, Theorem 5.2]. Certainly, $\mathcal{F}$ preserves identity morphisms and composition of morphisms. To complete the proof of functoriality, it remains to show that if $\varphi \in \mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$, then $\mathcal{F}\varphi : \mathcal{F}(\widetilde{G}) \to \mathcal{F}(\widetilde{K})$ is a GL-rack homomorphism. Write $\mathcal{F}(\widetilde{G}) = (X, s^X, \mathtt{u}_G, \mathtt{d}_G)$ and $\mathcal{F}(\widetilde{K}) = (Y, s^Y, \mathtt{u}_K, \mathtt{d}_K)$, and fix $gH_a \in X$. Since $\varphi_1$ is a group homomorphism, we have

$$\begin{aligned}
\mathcal{F}\varphi \circ s^X_{gH_a} &= [xH_i \mapsto \varphi_2(g z_a^G g^{-1} x H_i)] \\
&= [xH_i \mapsto \varphi_1(g z_a^G g^{-1} x) L_{\varphi_3(i)}] \\
&= [xH_i \mapsto \varphi_1(g)\varphi_1(z_a^G)\varphi_1(g^{-1})\varphi_1(x) L_{\varphi_3(i)}] \\
&= [xH_i \mapsto \varphi_1(g) z^K_{\varphi_3(a)} \varphi_1(g)^{-1} \varphi_2(xH_i)] \\
&= s^Y_{\varphi_1(g)L_{\varphi_3(a)}} \circ \varphi_2 = s^Y_{\varphi_2(gH_a)} \circ \varphi_2 = s^Y_{\mathcal{F}\varphi(gH_a)} \circ \mathcal{F}\varphi,
\end{aligned}$$

so $\mathcal{F}\varphi$ is a rack homomorphism. Moreover, we have

$$\begin{aligned}
\mathcal{F}\varphi \circ \mathtt{u}_G &= [xH_i \mapsto \varphi_2(x q_i^G H_{\tau(i)})] \\
&= [xH_i \mapsto \varphi_1(x)\varphi_1(q_i^G) L_{\varphi_3(\tau(i))}] \\
&= [xH_i \mapsto \varphi_1(x) q^K_{\varphi_3(i)} L_{\pi(\varphi_3(i))}] \\
&= [yL_j \mapsto y q_j^K L_{\pi(j)}] \circ [xH_i \mapsto \varphi_1(x) L_{\varphi_3(i)}] \\
&= \mathtt{u}_K \circ [xH_i \mapsto \varphi_2(xH_i)] = \mathtt{u}_K \circ \mathcal{F}\varphi,
\end{aligned}$$

as desired. Similarly, $\mathcal{F}\varphi \circ \mathtt{d}_G = \mathtt{d}_K \circ \mathcal{F}\varphi$, so $\mathcal{F}\varphi$ is a GL-rack homomorphism. □

This result gives us a group-theoretic way to show that two GL-racks $R_1$ and $R_2$ are isomorphic. The proof of [18, Theorem 5.2] describes a procedure to construct objects $\widetilde{G}$ and $\widetilde{K}$ in $\mathsf{GrpTup}$ such that $R_1 \cong \mathcal{F}(\widetilde{G})$ and $R_2 \cong \mathcal{F}(\widetilde{K})$ in $\mathsf{GLR}$. To show that $R_1 \cong R_2$ in $\mathsf{GLR}$, it suffices to find a morphism $\varphi \in \mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$ such that $\varphi_2$ is bijective, since then $\mathcal{F}(\varphi) : \mathcal{F}(\widetilde{G}) \xrightarrow{\sim} \mathcal{F}(\widetilde{H})$ will also be bijective and, hence, an isomorphism of GL-racks.

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