Computing the Mosaic Numbers of Legendrian Knots

Joint Mathematical Meetings 2025

Spectra Special Session I on Research by LGBTQ+ Mathematicians

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2024 Moravian University REU

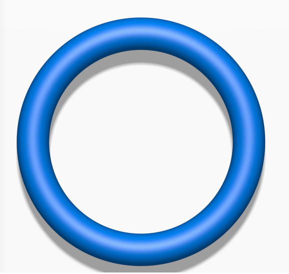
January 10, 2025

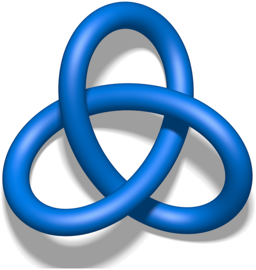
Outline

Smooth knots

Definition

A *knot* is a smooth simple closed curve in $\mathbb{R}^3$.

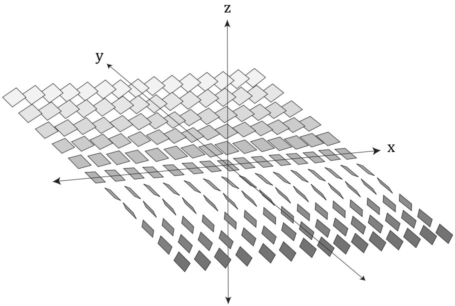




The standard contact structure

Definition

The **standard contact structure** on $\mathbb{R}^3$, denoted by ξ_{std}, is an assignment of a plane to each point (x, y, z) defined by $dz - y\,dx = 0$.



When $y = 0$, the planes are flat. When moving in the positive y-direction, the slopes grow more negative; when moving in the negative y-direction, the slopes grow more positive.

Legendrian knots

Definition

A smooth knot is called *Legendrian* if it lies everywhere tangent to ξ_{std}.

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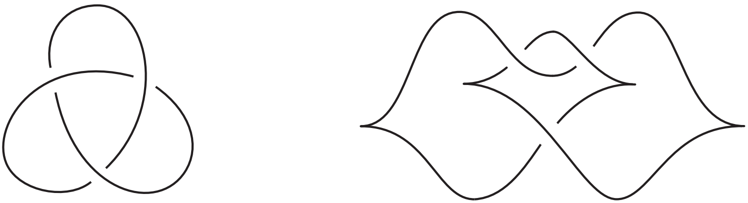

We often study Legendrian knots via their *front projections* onto the xz-plane, viewed from the negative y-axis.

- Have cusps rather than vertical tangencies
- Only have one type of crossing

Distinguishing between Legendrian knots

Definition (Legendrian isotopy)

Two Legendrian knots are equivalent iff one can be deformed into the other without cutting, self-intersecting, or losing tangency to ξ_{std}.

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Definition (Classical invariants)

Let Λ be a Legendrian knot. Using its front projection, define

$$\mathrm{tb}(\Lambda) := \#\text{pos. crossings} - \#\text{neg. crossings} - \frac{1}{2}\#\text{cusps}$$

$$\text{and}\quad \mathrm{rot}(\Lambda) := \frac{1}{2}(\#\text{downward cusps} - \#\text{upward cusps}).$$

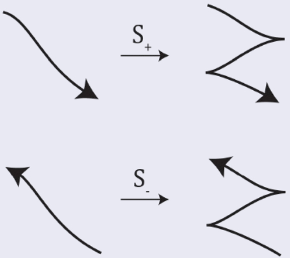
Some Legendrian knots, including unknots, are completely determined by their smooth knot type and classical invariants.

Stabilization

Fix an orientation of a Legendrian knot Λ.

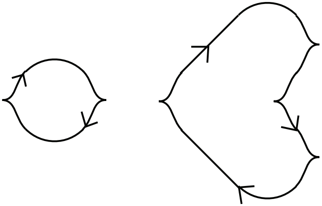
Definition

A **stabilization** transforms Λ without altering its underlying knot type by adding a "zig-zag" (i.e., two cusps) to a strand of the front projection:



Stabilization subtracts 1 from $tb(\Lambda)$ and adds ± 1 to $rot(\Lambda)$ (depending on orientation).

Quick example



Example

Positively stabilizing the left unknot yields the right unknot, so

$$\mathrm{tb}(\Lambda_{\text{left}}) = -1 \neq -2 = \mathrm{tb}(\Lambda_{\text{right}}), \text{ and}$$
$$\mathrm{rot}(\Lambda_{\text{left}}) = 0 \neq 1 = \mathrm{rot}(\Lambda_{\text{right}}).$$

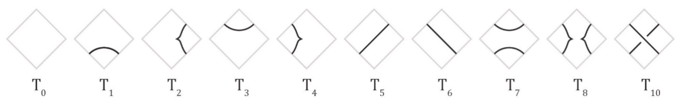
Hence, these unknots are not Legendrian isotopic, despite being isotopic as smooth knots.

Outline

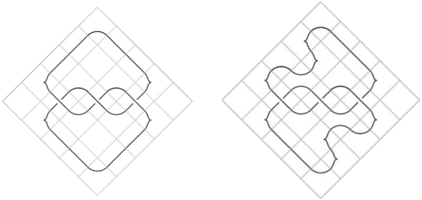
Legendrian knot mosaics

Definition

A **Legendrian *n*-mosaic** depicts a Legendrian front projection using the following tiles in an $n \times n$ grid:

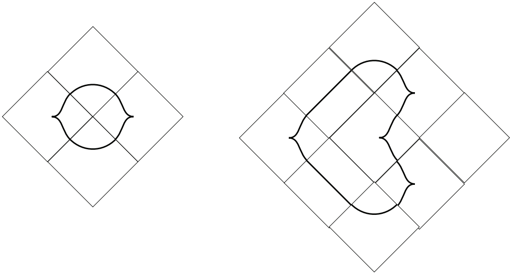


The **mosaic number** of Λ, denoted by $m(\Lambda)$, is the smallest possible size n of a Legendrian knot mosaic of Λ.



Research questions

Pezzimenti and Pandey (2022) showed that stabilization can increase or fix the mosaic number, like with the unknots we saw earlier.



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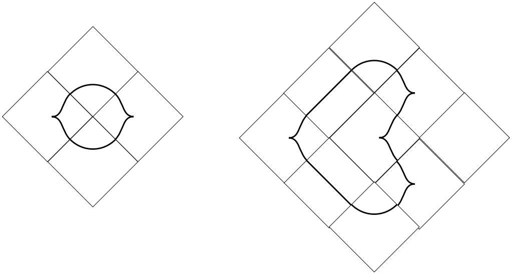
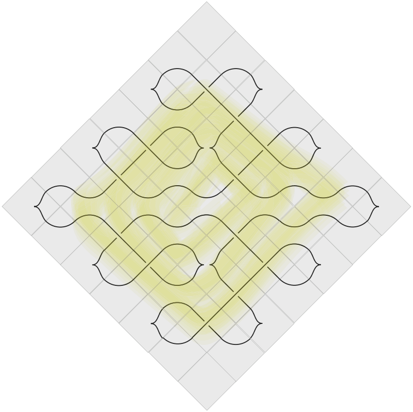


They left off with these questions:

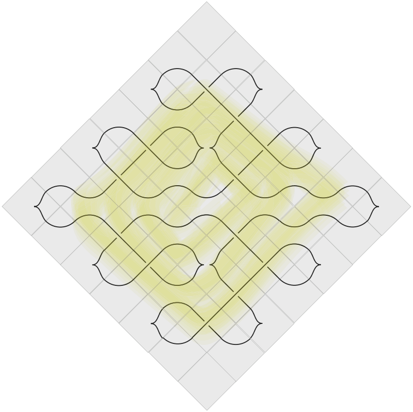
- Do there exist bounds on $m(\Lambda)$ in terms of $\mathrm{tb}(\Lambda)$ and $\mathrm{rot}(\Lambda)$? **Yes!**
- Can stabilizing a Legendrian knot reduce its mosaic number? **Yes!**
- Within a smooth knot type, is the minimal mosaic number always attained by a Legendrian representative with maximal Thurston-Bennequin invariant? **No!**

Outline

Lower bound on $m(\Lambda)$, part 1/3

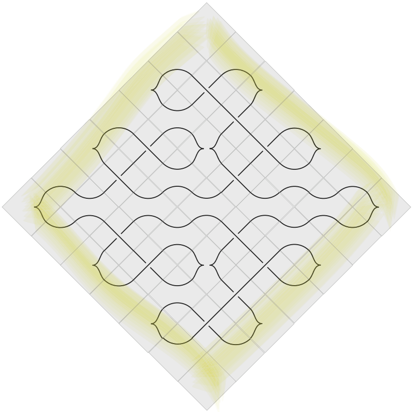


Lower bound on $m(\Lambda)$, part 1/3



$$\mathrm{tb}^*(\text{inner tiles}) \geq -(n-2)^2$$

Lower bound on $m(\Lambda)$, part 2/3



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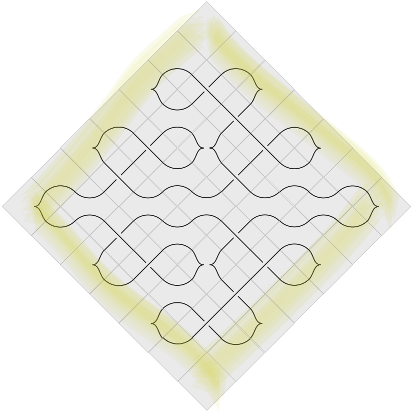

$$\mathrm{tb}^*(\text{outer tiles}) \geq -(n-1)$$
$$\implies \mathrm{tb}(\Lambda) \geq -(n-2)^2 - (n-1)$$

Lower bound on $m(\Lambda)$, part 3/3

Theorem

If $\operatorname{tb}(\Lambda) < 0$, *then*

$$m(\Lambda) \geq \left\lceil \sqrt{-\operatorname{tb}(\Lambda) - \frac{3}{4}} + \frac{3}{2} \right\rceil.$$

Lower bound on $m(\Lambda)$, part 3/3

Theorem

If $\mathrm{tb}(\Lambda) < 0$, *then*

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Side note: We also found a weaker bound that works when $\mathrm{tb}(\Lambda) \geq 0$.

Theorem

If Λ *is a Legendrian knot with* $4|\mathrm{rot}(\Lambda)| + \mathrm{tb}(\Lambda) \geq 0$, *then*

$$m(\Lambda) \geq \left\lceil \sqrt{4|\mathrm{rot}(\Lambda)| + \mathrm{tb}(\Lambda)} \right\rceil.$$

Sharpness: The crab bucket sequence

The first of these lower bounds is attained by Legendrian representatives of infinitely many distinct smooth knot types! Consider the sequences of odd and even *crab buckets*:

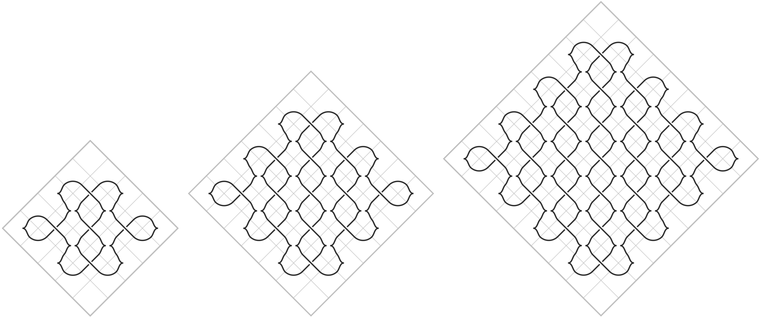


Figure: The first three odd crab buckets: β_5, β_7, and β_9.

Both sequences attain the first lower bound on the previous slide.

Odd crab buckets

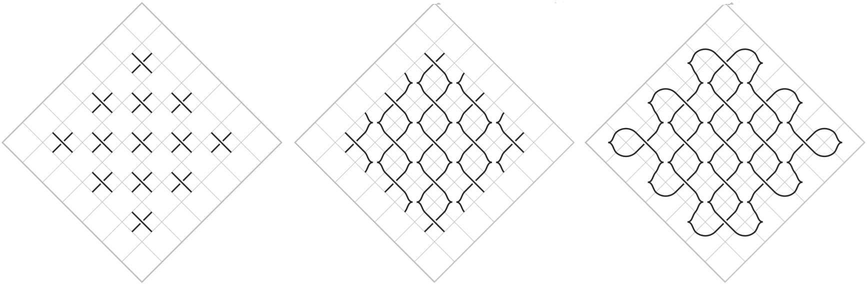


Figure: The construction of β_7.

Observation

For all odd $n \geq 5$, β_n is the Legendrian connected sum of torus knots

$$(2,3)\#(2,5)\#\ldots\#(2,n-2)\#(2,n-4)\#\ldots\#(2,3).$$

Even crab buckets

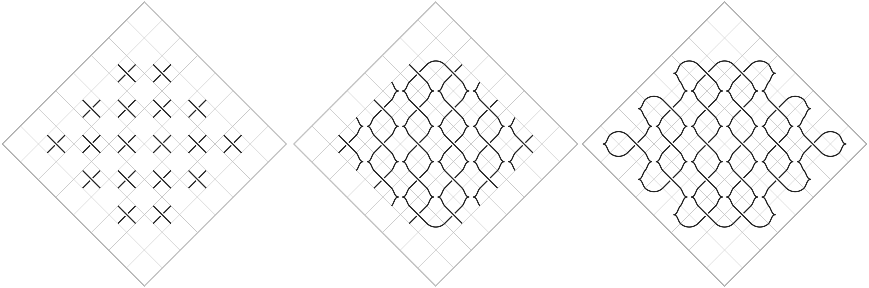


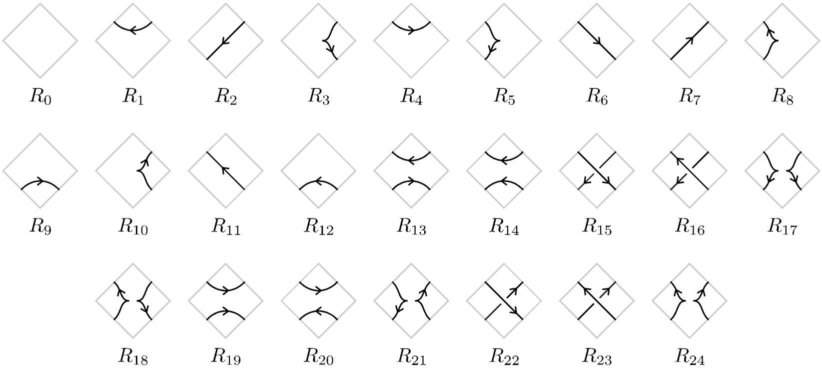
Figure: The construction of β_8.

Observation

For all even $n \geq 6$, β_n is the Legendrian connected sum of torus knots

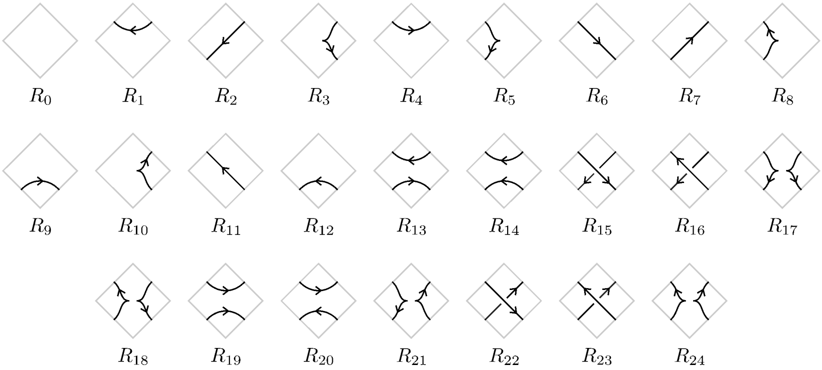
$$(2,3)\#(2,5)\#\ldots\#(2,n-3)\#(2,n-3)\#(2,n-5)\#\ldots\#(2,3).$$

Linear algebraic perspective

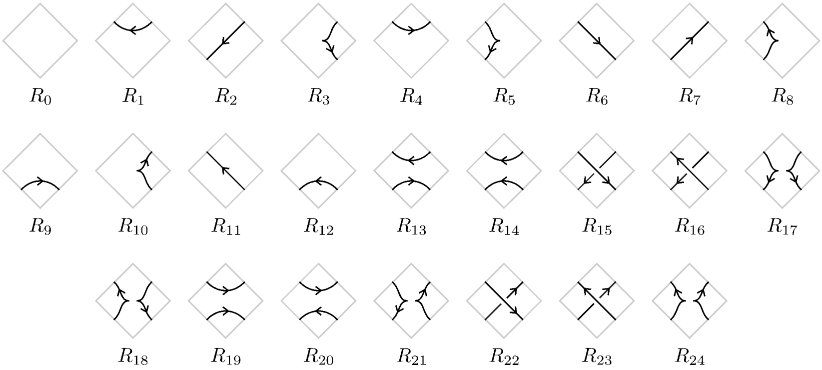


- We can assign five integers to each oriented tile t, including $\mathrm{tb}^*(t)$ and $\mathrm{rot}^*(t)$.

Linear algebraic perspective



- We can assign five integers to each oriented tile t, including $\text{tb}^*(t)$ and $\text{rot}^*(t)$.
- Then, we can encode them into a matrix viewed as a linear map $\mathbb{Z}^{25} \to \mathbb{Z}^5$. After some Fourier-Motzkin elimination...

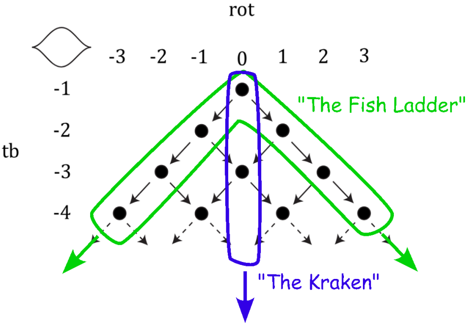


- We can assign five integers to each oriented tile t, including $\mathrm{tb}^*(t)$ and $\mathrm{rot}^*(t)$.
- Then, we can encode them into a matrix viewed as a linear map $\mathbb{Z}^{25} \to \mathbb{Z}^5$. After some Fourier-Motzkin elimination…
- …we get the same bounds as before or slightly worse.

Outline

On Legendrian unknots

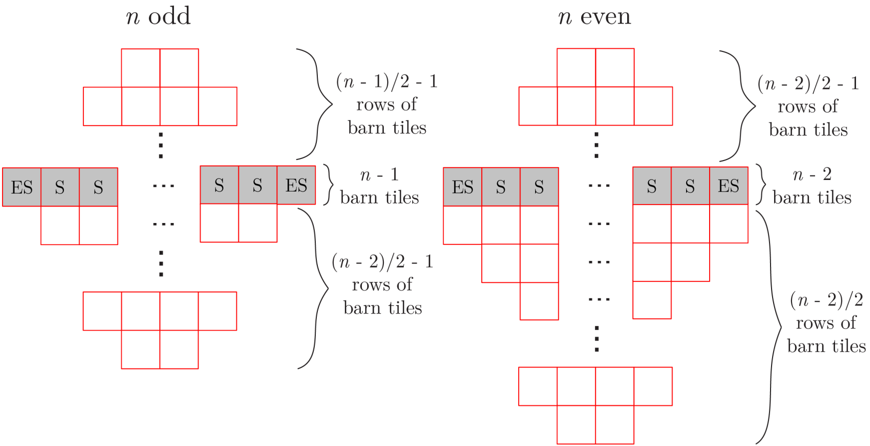
To get upper bounds on $m(\Lambda)$, it suffices to produce a valid mosaic for Λ. Pezzimenti and Pandey (2022) constructed mosaics for infinitely many Legendrian unknots, called the *Kraken sequence*, with $\text{rot} = 0$. Inspired, we initially constructed a sequence of unknots, called *fish ladders*, having maximal $|\,\text{rot}\,|$ (the boundaries of the *mountain diagram* shown below).



But what about all the other Legendrian unknots?

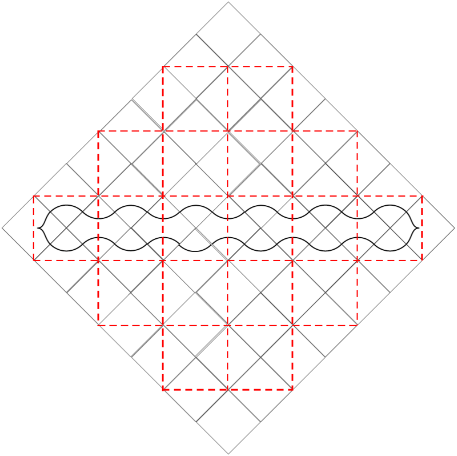
Barn tiles

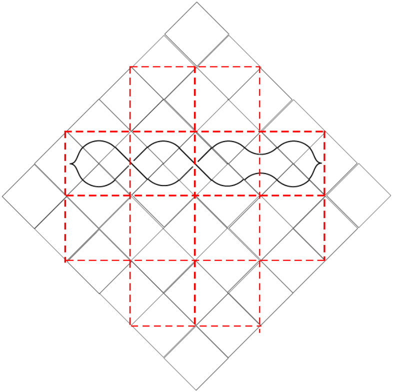
For any Legendrian unknot Λ_U, we can construct a mosaic for Λ_U using *barn tiles*, which are bounded by four mosaic tiles.



Soil setups

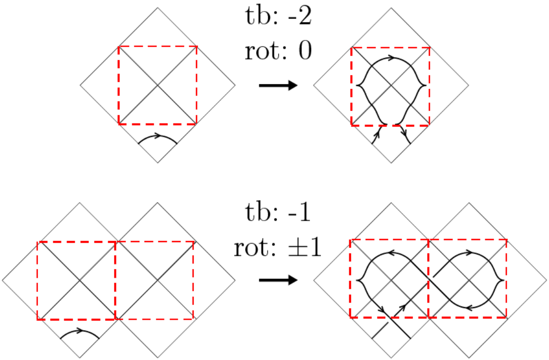
Our construction begins with a *soil setup*, which can look like either of these (possibly with more crossings, called *soil crossings*):





Krakens and fishes

Then, we perform moves on barn tiles above and below the soil, based on the Kraken and fish ladder constructions.



The numbers of mosaic tiles, soil crossings, Kraken moves, and fish moves we use are determined by $\mathrm{tb}(\Lambda_U)$ and $\mathrm{rot}(\Lambda_U)$.

Example when $\mathrm{rot}(\Lambda_U) \neq 0$

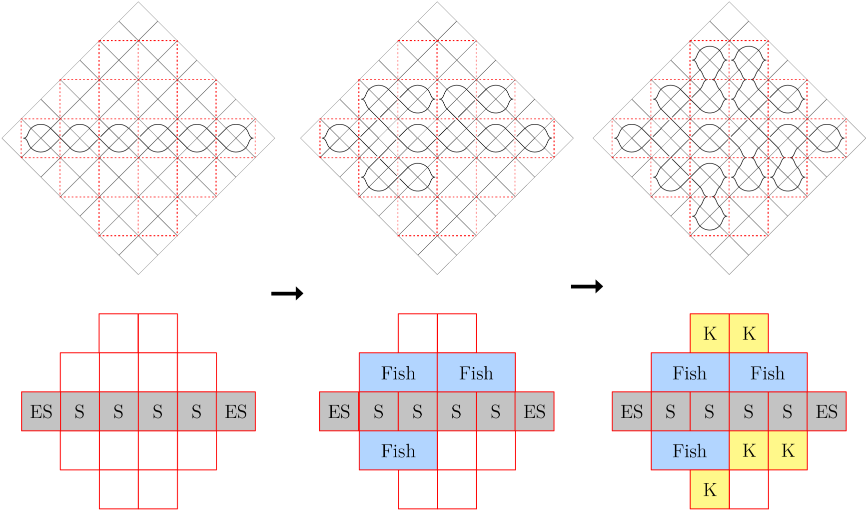


Figure: Construction when $\mathrm{tb}(\Lambda_U) = -19$ and $r(\Lambda_U) = \pm 4$.

Example when $\text{rot}(\Lambda_U) = 0$

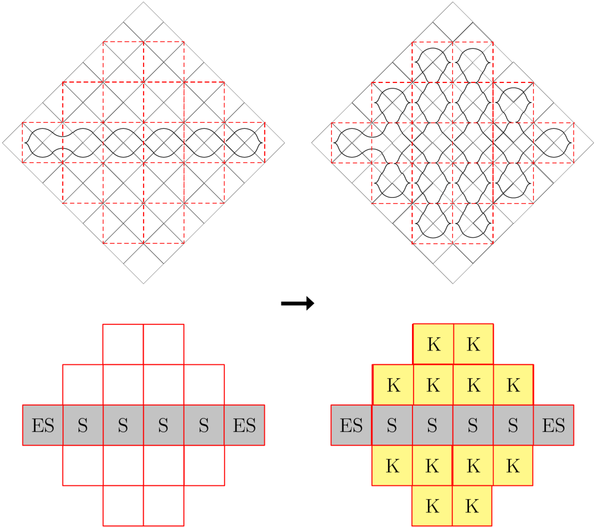


Figure: Construction when $\text{tb}(\Lambda_U) = -29$ and $r(\Lambda_U) = 0$.

Results

Working backwards from this algorithmic construction yields the following.

Theorem

If Λ_U is a Legendrian unknot with $\operatorname{rot}(\Lambda_U) \neq 0$*, then*

$$m(\Lambda_U) \leq \left\lceil \sqrt{3|\operatorname{rot}(\Lambda_U)| - \operatorname{tb}(\Lambda_U) - \frac{11}{4}} + \frac{3}{2} \right\rceil.$$

If instead $\operatorname{rot}(\Lambda_U) = 0$*, then*

$$m(\Lambda_U) \leq \left\lceil \sqrt{-\operatorname{tb}(\Lambda_U) + \frac{5}{4}} + \frac{3}{2} \right\rceil.$$

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The upshot: Using all of our bounds thus far, we were able to compute the exact mosaic numbers of 141 distinct Legendrian unknots!

Outline

Enumerating Legendrian link mosaics

Oh, Hong, Lee, and Lee (2015) used *state matrices* to compute the number of $m \times n$ classical link mosaics. We adapted their proof to enumerate $m \times n$ Legendrian link mosaics.

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Theorem

Let $m, n \in \mathbb{Z}^+$. If $m = 1$ or $n = 1$, then the total number $D_L^{(m,n)}$ of all $m \times n$ Legendrian link mosaics is 1. Otherwise,

$$D_L^{(m,n)} = 2||(X_{m-2} + O_{m-2})^{n-2}||,$$

where $||M||$ denotes the sum of all entries of a matrix M, and X_{m-2} and O_{m-2} are $2^{m-2} \times 2^{m-2}$ matrices defined recursively by

$$X_{k+1} := \begin{bmatrix} X_k & O_k \\ O_k & X_k \end{bmatrix} \text{ and } O_{k+1} := \begin{bmatrix} O_k & X_k \\ X_k & 3O_k \end{bmatrix}$$

for $k = 0, 1, \ldots, m-3$, with 1×1 matrices $X_0, O_0 := \begin{bmatrix} 1 \end{bmatrix}$.

Number of Legendrian link n-mosaics

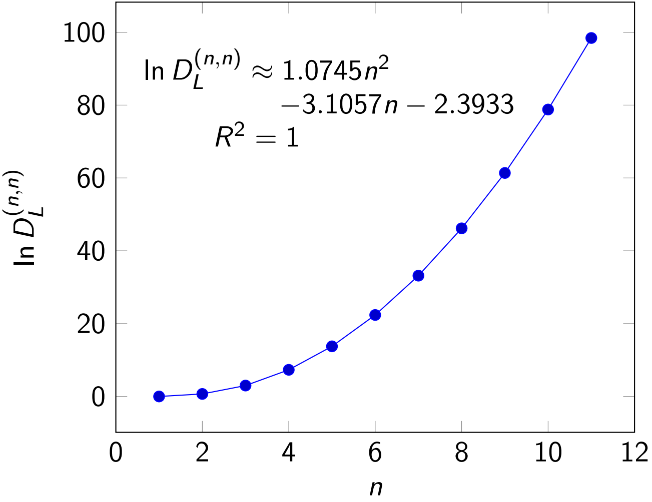


Figure: Quadratic exponential growth of the number $D_L^{(n,n)}$ of suitably connected Legendrian n-mosaics.

Legendrian n-mosaics vs. classical n-mosaics

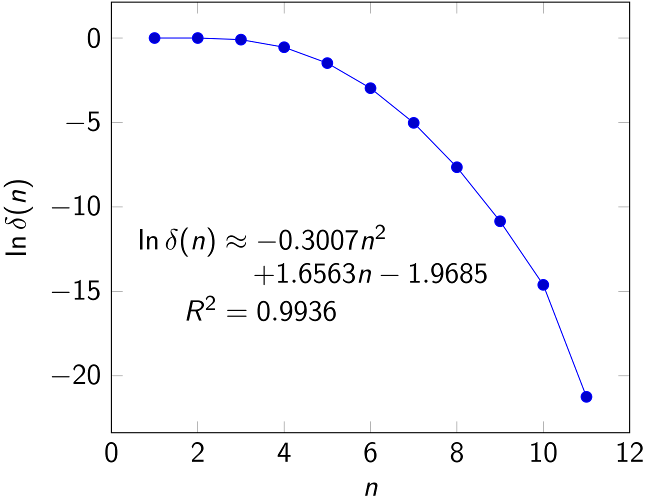
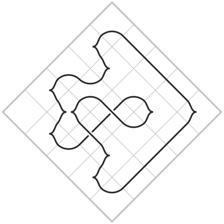


Figure: Negative quadratic exponential growth of the ratios $\delta(n)$ between the number of Legendrian link n-mosaics and the number of classical ones.

Outline

Our algorithm

- Representing each tile as a base 10 digit, we can encode $n \times n$ mosaics as n^2-digit numbers by reading from left to right, top to bottom.



$$\Leftrightarrow \begin{bmatrix} 0 & 2 & 1 & 2 & 1 \\ 2 & 8 & 7 & 4 & 6 \\ 3 & 9 & 9 & 1 & 6 \\ 2 & 4 & 3 & 4 & 6 \\ 3 & 5 & 5 & 5 & 4 \end{bmatrix} \Leftrightarrow 0212128746399162434635554$$

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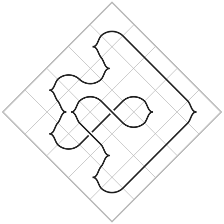

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- Starting at 0, we simply "count up" in a way that guarantees we list every suitably connected mosaic.
- To determine the smooth knot type of the resulting mosaics, we used the HOMFLY–PT polynomial.

Results

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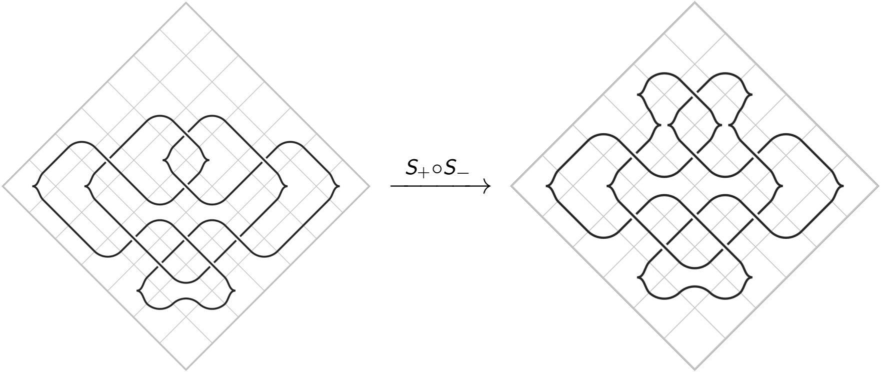
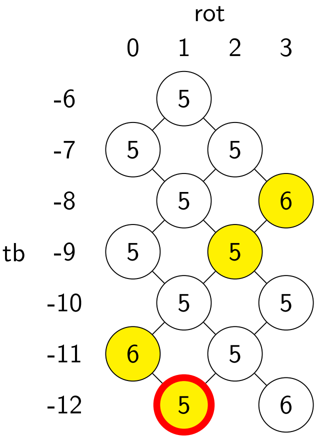
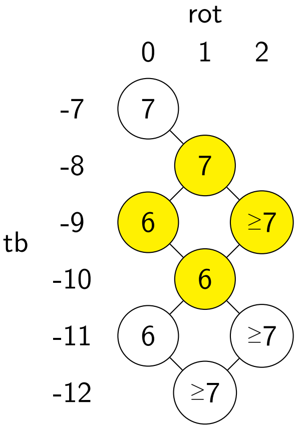
    Yes—the knot type 8_1!



Notable censuses

Below are the (abridged) censuses for negative trefoils and 8_1.





Outline

Crab buckets and destabilization

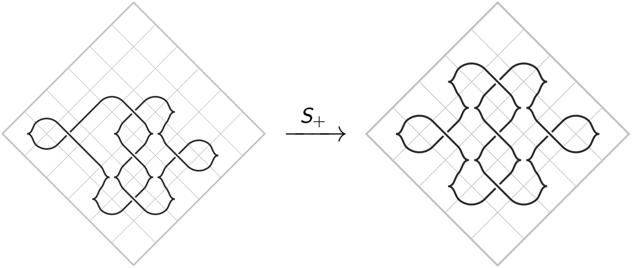
Question

Are there *infinitely* many cases where stabilization decreases $m(\Lambda)$?

Conjecture

Every odd crab bucket β_n, oriented so that $\mathrm{rot}(\beta_n) = 1$, cannot be negatively destabilized without increasing its mosaic number.

The computer search shows this is true for β_5, which is a negative trefoil:



More future questions

- Infinitely many examples like 8_1?
- Bounding by other invariants
- Extending barn tiles to Legendrian nontrivial knots
- Random Legendrian knot mosaics

Acknowledgment
This research was done at Moravian University as part of the Research Challenges of Computational Methods in Discrete Mathematics REU; it was funded by the National Science Foundation (MPS-2150299).

Disclaimer
Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Thank you!

Contact: `luc.ta@yale.edu` | `luc-ta.github.io`

arXiv > math > arXiv:2410.08064

Mathematics > Geometric Topology

[Submitted on 10 Oct 2024]

Bounds on the mosaic number of Legendrian Knots

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