GENERALIZED LEGENDRIAN RACKS: KNOT COLORING INVARIANTS, MEDIAL TENSORS, AND TABULATION

LỰC TA

ABSTRACT. Generalized Legendrian racks, also called GL-racks or bi-Legendrian racks, are a nonassociative algebraic structure based on the Legendrian Reidemeister moves. This article develops computational and categorical approaches to GL-racks as Legendrian link invariants and algebraic objects in their own right. First, we settle a conjecture that GL-racks can distinguish between Legendrian knots sharing the same classical invariants and Legendrian nonsimple knot type. Then, we study tensor products of GL-racks and exhibit a symmetric monoidal closed structure on the category of medial GL-racks. We also construct an isomorphism of algebraic theories between classical racks and GL-quandles. Finally, we implement algorithms that classify GL-racks of small orders up to isomorphism and compute GL-rack coloring numbers of Legendrian links. We tabulate isomorphism classes of GL-racks of orders 7 and lower.

1. INTRODUCTION

Generalized Legendrian racks, also called *GL-racks* or *bi-Legendrian racks*, are a nonassociative algebraic structure whose axioms encode the contact geometry of Legendrian links in $\mathbb{R}^3$ or S^3. Despite their recency, GL-racks can be traced back to algebraic structures called *kei*, which Takasaki [37] introduced in 1942 to study symmetric spaces; *quandles*, which Joyce [21] introduced in 1982 to study links in S^3 and conjugation in groups; and *racks*, which Fenn and Rourke [16] introduced in 1992 to study framed links in 3-manifolds. Kei, quandles, and racks have enjoyed significant study as link invariants in geometric topology and in their own rights in quantum algebra.

More recently, authors have introduced variants of racks suitable for studying Legendrian links. In 2017, Karmakar and Prathamesh [29] introduced rack invariants of Legendrian knots. In 2021, Ceniceros et al. [6] refined these invariants by introducing *Legendrian racks*. In 2023, Karmakar et al. [22] and Kimura [25] independently strengthened these constructions by introducing GL-racks.

In this article, we apply GL-racks to confirm several conjectures about Legendrian links, study the category GLR of GL-racks and their homomorphisms, and extend several known results about quandles to GL-racks. Along with a classification of GL-racks of orders $n \leq 7$ up to isomorphism detailed in Appendices A and B, the main results of this article are as follows.

Theorem 1.1. *The two Legendrian knots with underlying topological knot type* 6_2 *and classical invariants* $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$ *given in* [8] *are distinguishable by GL-rack coloring numbers.*

Theorem 1.2. *The two Legendrian knots with underlying topological knot type* 8_{10} *and classical invariants* $(\mathrm{tb}, \mathrm{rot}) = (-8, 3)$ *given in* [3] *are distinguishable by GL-rack coloring numbers.*

Theorem 1.3. *Let* R_1 *and* R_2 *be GL-racks. If* R_2 *is medial, then* $\mathrm{Hom}_{\mathsf{GLR}}(R_1, R_2)$ *has a canonical medial GL-rack structure. If in addition* R_2 *is a GL-quandle, then so is* $\mathrm{Hom}_{\mathsf{GLR}}(R_1, R_2)$*.*

Theorem 1.4. *The full subcategory of* GLR *whose objects are medial is symmetric monoidal closed.*

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Theorem 1.5. *The algebraic theories of racks and GL-quandles are canonically isomorphic. Moreover, the algebraic theories of medial racks and medial GL-quandles are canonically isomorphic.*

The structure of this article is as follows. In Section 2, we give an overview of the questions in Legendrian knot theory motivating the study of GL-racks and quandles. We define these algebraic structures abstractly, discuss free GL-racks and free quandles, and establish notation.

In Section 3, we discuss how to assign a GL-racks to Legendrian links, give several worked examples, and discuss related Legendrian link invariants. Then, we prove Theorems 1.1 and 1.2, which we state as Theorems 3.10 and 3.13 respectively. Our approach to Theorem 1.1 offers a simpler and more algebraic alternative to Dynnikov and Prasolov's [10, Proposition 2.3] proof of the corresponding conjecture of Chongchitmate and Ng [8]. Theorem 1.1 also settles a question posed by Kimura [25, Section 4], as we formulate in Corollary 3.11 and strengthen in Proposition A.2. Theorem 1.2 confirms a conjecture of Bhattacharyya et al. [3]. Finally, we generalize an empirical observation of Karmakar et al. [22, Remark 8.2], which we state as Theorem 3.15.

In Section 4, we define *medial* or *abelian* GL-racks and tensor products of GL-racks. Using Theorem 1.3, which we state as Theorem 4.4, we propose a medial GL-rack-valued invariant of Legendrian links. Then, we prove Theorem 1.4, which we state as Theorem 4.6. These results extend Crans and Nelson's [9, Theorems 3 and 12] analogous results for medial quandles.

In Section 5, we prove Theorem 1.5, which we state as Theorem 5.5.

In Section 6, we propose questions for further research on GL-racks based on our results.

In Appendix A, we describe algorithms that can classify GL-racks of orders $n \leq 11$ up to isomorphism, building upon the work of Vojtěchovský and Yang [40]. We provide implementations of these algorithms in `GAP` [17] and the data we were able to compute and enumerate for all $n \leq 7$. We also provide an algorithm that computes the GL-rack coloring number of any oriented Legendrian link with respect to all GL-racks of a given order $n \leq 7$.

In Appendix B, we tabulate all GL-racks of orders $2 \leq n \leq 4$ up to isomorphism.

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2. Preliminaries

In this section, we contextualize the study of GL-racks and establish relevant terminology. In particular, we discuss the crossing and cusp relations afforded by Legendrian Reidemeister moves, which motivate the axioms of GL-racks. After stating some preliminary results, we discuss several functors appearing in the literature on GL-racks.

2.1. **Motivations: Legendrian link invariants.**

In this subsection, we discuss how the development of Legendrian link invariants motivates the study of GL-racks. Although we establish the relevant concepts here, we also refer the reader to [35] for an accessible introduction to Legendrian knot theory. For a more formal contact-geometric treatment, we refer the reader to [14].

Definition 2.1. A *knot* is a smooth embedding of the circle S^1 into $\mathbb{R}^3$, and a *link* is a disjoint union of a finite number of knots. A link Λ is called *Legendrian* if it lies everywhere tangent to the *standard contact structure* $\ker(dz - y\,dx)$ on $\mathbb{R}^3$, which is depicted in Figure 1. A *front projection*

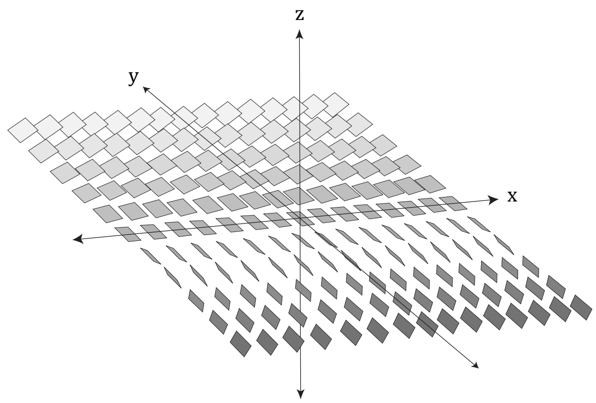

FIGURE 1. The standard contact structure on $\mathbb{R}^3$. Reprinted from [28, Figure 1].

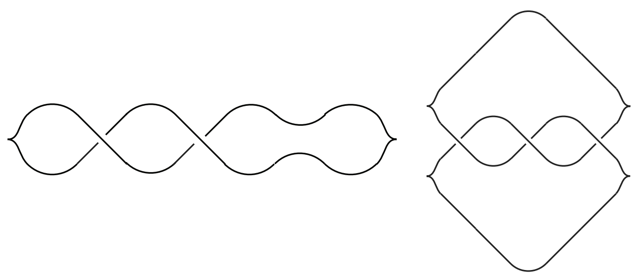

FIGURE 2. Unoriented front projections of a Legendrian unknot and a Legendrian trefoil. Adapted from [28, Figures 6 and 15].

or *front diagram* $D(\Lambda)$ is the projection of Λ onto the xz-plane. Finally, two Legendrian links are called *equivalent* or *Legendrian isotopic* if there exists a smooth ambient isotopy between them that preserves the condition of being Legendrian at every stage.

Note that Legendrian knots can also be studied as embeddings of S^1 into the 3-sphere S^3; see [14]. Since $\pi_1(\mathbb{R}^3 \setminus L) \cong \pi_1(S^3 \setminus L)$ for any link L, the choice of S^3 in place of $\mathbb{R}^3$ does not alter the results of this article. Also, we will denote the underlying *smooth link type* or *topological link type* of a Legendrian link Λ by L. For example, if Λ is one of the three Legendrian unknots depicted across Figures 2 and 3, then L denotes any unknot viewed up to ambient isotopy.

Central to contact geometry is the problem of distinguishing Legendrian links up to Legendrian isotopy. To this end, knot theorists typically study Legendrian links Λ through their front projections, which follow several restrictions thanks to the tangency condition on Λ. For one, at every crossing in $D(\Lambda)$, the strand with the more negative slope is always the overstrand. For two, $D(\Lambda)$ has cusps in place of vertical tangencies. Note that the numbers of crossings and cusps in a Legendrian front projection are finite due to smoothness. Moreover, $D(\Lambda)$ can be viewed as a *link diagram* of L, denoted by $D(L)$, by "ignoring" all cusps. For example, Figure 2 depicts unoriented front projections of a Legendrian unknot and a Legendrian trefoil, and Figure 3 depicts oriented front projections of two nonequivalent Legendrian unknots.

In fact, tangency to the standard contact structure implies that one can entirely recover the geometric structure of an oriented Legendrian link from its front projection; see [14]. A well-known result of Świątkowski [36, Theorem B] in 1992 states that two Legendrian links are Legendrian

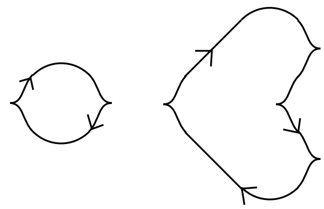

FIGURE 3. Front projections of nonequivalent oriented Legendrian unknots.

isotopic if and only if their front projections are related by a finite sequence of planar isotopies and the three *Legendrian Reidemeister moves* depicted in Figures 5–7.

Indeed, front projections also recover or characterize many *invariants* of Legendrian links, which are mathematical objects used to detect when two Legendrian links are nonequivalent. In particular, the *classical invariants* of a Legendrian link Λ, called the *Thurston-Bennequin number* and *rotation number* and denoted respectively by $\mathrm{tb}(\Lambda)$ and $\mathrm{rot}(\Lambda)$, can be defined as the integers

$$\mathrm{tb}(\Lambda) = P - N - \frac{1}{2}(D+U), \quad \mathrm{rot}(\Lambda) = \frac{1}{2}(D-U),$$

where P, N, D, and U are the numbers of positively oriented crossings, negatively oriented crossings, downward-oriented cusps, and upward-oriented cusps in $D(\Lambda)$, respectively.

It is well-known that two Legendrian links are equivalent only if their classical invariants are equal. An interesting challenge in distinguishing between Legendrian links is that the converse only holds within certain smooth link types, which are called *Legendrian simple*; see. e.g., [14, Section 5]. For example, Theorem 1.2 shows that the smooth knot type 8_{10} is Legendrian nonsimple.

Example 2.2. Let Λ_1 and Λ_2 be the oriented Legendrian unknots depicted on the left and right of Figure 3, respectively. Although Λ_1 and Λ_2 share the same underlying smooth knot type, they are not Legendrian isotopic because $\mathrm{tb}(\Lambda_1) = -1 \neq -2 = \mathrm{tb}(\Lambda_2)$ and $\mathrm{rot}(\Lambda_1) = 0 \neq 1 = \mathrm{rot}(\Lambda_2)$. Therefore, the two front projections in Figure 3 cannot be related by any sequence of Legendrian Reidemeister moves.

There are in fact infinitely many examples of nonequivalent Legendrian links with the same underlying smooth link type, including examples in Legendrian nonsimple link types that share the same classical invariants (see, e.g., [8]). The problem of distinguishing between Legendrian links has motivated various nonclassical invariants of Legendrian links, including the Chekanov-Eliashberg differential graded algebra and Legendrian contact homology (see [15]), decomposition and ruling invariants (see [14]), and the mosaic number (see [28,34]). Quantum algebraists have also developed rack-theoretic invariants of both Legendrian links and topological links. These include fundamental quandles and their Legendrian analogues (e.g., [6, 21, 23]), coloring numbers (e.g., [6, 23, 25]), cocycle invariants (e.g., [26]), and state-sum invariants (e.g., [22]), many of which have elegant enhancements and categorifications (e.g., [5, 7, 13]). These invariants motivate the study of GL-racks as a category.

The axioms of GL-racks are motivated by the *crossing* and *cusp relations* induced between strands of a Legendrian front projection modulo the relations afforded by the Legendrian Reidemeister moves. (Note that planar isotopies do not affect crossings or cusps, so they do not induce any such relations.) In Figure 4, (i) and (ii) depict crossing relations between strands in a Legendrian front projection, and (iii) and (iv) depict cusp relations. Note that `u` and `d` correspond to the relations induced by upward- and downward-oriented cusps, respectively. Figures 5–7 depict the crossing and cusp relations in one possible orientation of each of the three Legendrian Reidemeister

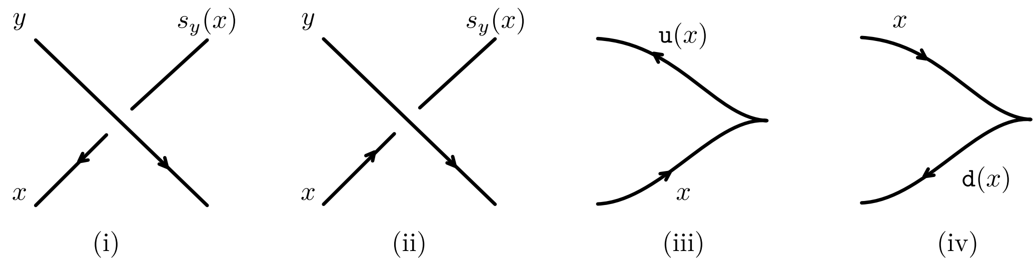

FIGURE 4. Crossing and cusp relations. Adapted from [23, Figure 4] under CC BY 4.0.

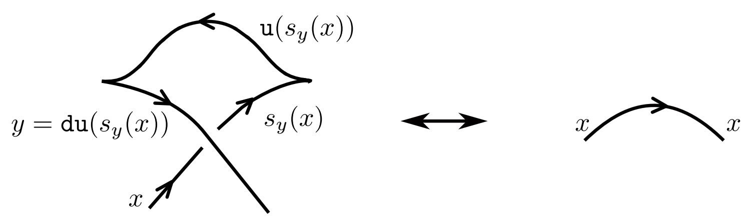

FIGURE 5. Crossing and cusp relations in one possible orientation of the first Legendrian Reidemeister move. Adapted from [23, Figure 5] under CC BY 4.0.

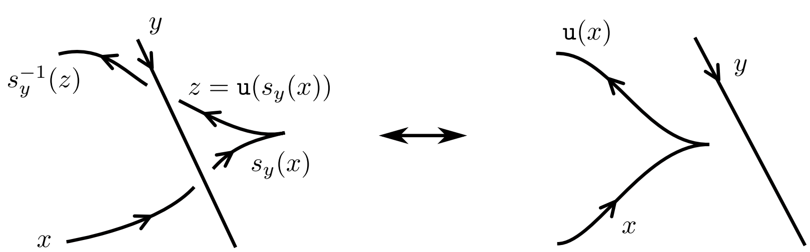

FIGURE 6. Crossing and cusp relations in one possible orientation of the second Legendrian Reidemeister move. Adapted from [23, Figure 6] under CC BY 4.0.

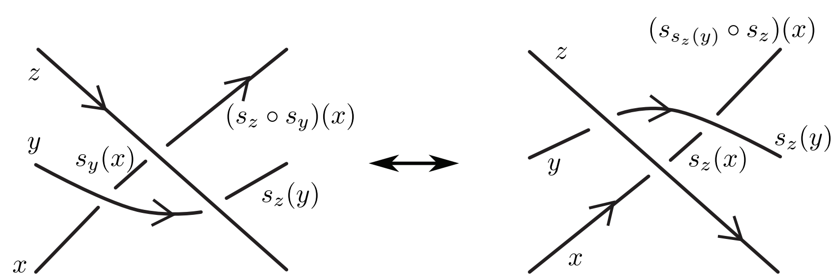

FIGURE 7. Crossing and cusp relations in one possible orientation of the third Legendrian Reidemeister move. Adapted from [23, Figure 3] under CC BY 4.0.

moves. For a complete list of all possible orientations and their induced crossing and cusp relations, we refer the reader to [25, Figures 6–8].

2.2. **GL-racks.** In this subsection, we define racks, quandles, and GL-racks abstractly by translating the crossing and cusp relations in Subsection 2.1 into the language of *rack symmetries*. Henceforth, we will denote the group of all bijections from a set X to itself by $\mathrm{Sym}(X)$.

Although racks and quandles are often defined as sets X endowed with two nonassociative, right-distributive binary operations $\triangleright$ and $\triangleright^{-1}$, they may also be characterized in terms of symmetries $s_x \in \mathrm{Sym}(X)$ assigned to each element $x \in X$; cf. [11, Definition 2.1; 21, Definition 1.1]. One may translate between the two conventions via the formulas $s_x(y) = y \triangleright x$ and $s_x^{-1}(y) = y \triangleright^{-1} x$. In this article, we adopt the definitions using symmetries due to their convenience for abstract proofs and exhaustive search algorithms; we have (re)written all crossing relations in Figures 4–7 in this notation. We refer the reader to [12, 32] for accessible introductions to quandle theory, [12, 33] for references on racks and quandles as they concern low-dimensional topology, and [11] for a survey of modern algebraic research on racks and quandles.

The rack and quandle axioms encapsulate the crossing relations depicted in Figures 5-7.

Definition 2.3. Let X be a set, and let $s : X \to \mathrm{Sym}(X)$ be a map defined by $x \mapsto s_x$. We call the pair (X, s) a *rack* or a *wrack* if, for all $x, y \in X$, we have $s_x \circ s_y = s_{s_x(y)} \circ s_x$. We say that s_x is the *symmetry at* x, and we say that $|X|$ is the *order* of (X, s). If in addition $s_x(x) = x$ for all $x \in X$, then we say that (X, s) is a *quandle*. Finally, if $Y \subset X$ and $s_y(z) \in Y$ for all $y, z \in Y$, then we say that $(Y, s|_Y)$ is a *subrack* of (X, s).

Example 2.4. [33, Example 2.13] Let X be a union of conjugacy classes in a group G, and define $s : X \to \mathrm{Sym}(X)$ by $x \mapsto s_x := [y \mapsto xyx^{-1}]$. Then (X, s) is a quandle called a *conjugation quandle* or *conjugacy quandle*, and we denote it by $\mathrm{Conj}(X)$.

Example 2.5. [12, Example 99] Let X be a set, and fix $\sigma \in \mathrm{Sym}(X)$. Define $s : X \to \mathrm{Sym}(X)$ by $x \mapsto \sigma$, so that $s_x(y) = \sigma(y)$ for all $x, y \in X$. Then (X, s) is a rack called a *permutation rack* or *constant action rack*, and we denote it by $(X, \sigma)_p$. Note that $(X, \sigma)_p$ is a quandle if and only if $\sigma = \mathrm{id}_X$.

In 2023, Karmakar et al. [22] and Kimura [25] independently introduced *GL-racks* to generalize the *Legendrian racks* introduced by Kulkarni and Prathamesh [29] in 2017 and refined by Ceniceros et al. [6] in 2021. The GL-rack axioms encode the crossing and cusp relations induced by the Legendrian Reidemeister moves in Figures 5-7. Once again, we translate the original definition into the language of rack symmetries.

Definition 2.6. [23, Definition 3.1] A *GL-rack*, also called a *generalized Legendrian rack* or a *bi-Legendrian rack*, is a quadruple $(X, s, \mathtt{u}, \mathtt{d})$ in which (X, s) is a rack, $\mathtt{u}, \mathtt{d} : X \to X$ are maps, and the following axioms hold for all $x \in X$:

- (L1) $(\mathtt{ud} \circ s_x)(x) = x = (\mathtt{du} \circ s_x)(x)$.
- (L2) $\mathtt{u} \circ s_x = s_x \circ \mathtt{u}$ and $\mathtt{d} \circ s_x = s_x \circ \mathtt{d}$.
- (L3) $s_{\mathtt{u}(x)} = s_x = s_{\mathtt{d}(x)}$.

We call the ordered pair $(\mathtt{u}, \mathtt{d})$ a *GL-structure* on (X, s). If in addition (X, s) is a quandle, we say that $(X, s, \mathtt{u}, \mathtt{d})$ is a *GL-quandle*.

Example 2.7. [25, Example 3.6] Let G be a group, let $z \in Z(G)$, and define $f : G \to G$ by $g \mapsto zg$. Then $(\mathrm{Conj}(G), f, f^{-1})$ is a GL-quandle.

Example 2.8. [25, Example 3.7] Let $(X, \sigma)_p$ be a permutation rack, and let $\mathtt{u}, \mathtt{d} : X \to X$ be maps. Then $(\mathtt{u}, \mathtt{d})$ defines a GL-structure on $(X, \sigma)_p$ if and only if $\mathtt{ud} = \sigma^{-1} = \mathtt{du}$. In this case, we say that $((X, \sigma)_p, \mathtt{u}, \mathtt{d})$ is a *permutation GL-rack* or *constant action GL-rack*, and we denote it by $(X, \sigma, \mathtt{u}, \mathtt{d})_p$.

Example 2.9. [23, Example 3.4] Any GL-rack of the form $(X, s, \mathrm{id}_X, \mathrm{id}_X)$ is called a *trivial GL-rack*. In particular, any quandle (Q, s) can be identified with the trivial GL-rack $(Q, s, \mathrm{id}_Q, \mathrm{id}_Q)$; cf. Lemma 2.13. In other words, GL-racks generalize quandles.

We define homomorphisms of these algebraic structures as follows.

Definition 2.10. Let (X, s) and (Y, t) be racks. A map $\varphi : X \to Y$ is called a *rack homomorphism* if $\varphi \circ s_x = t_{\varphi(x)} \circ \varphi$ for all $x \in X$. If in addition $(\mathtt{u}_1, \mathtt{d}_2)$ and $(\mathtt{u}_2, \mathtt{d}_2)$ are GL-structures on (X, s) and (Y, t), we say that φ is also a *GL-rack homomorphism* if $\varphi \circ \mathtt{u}_1 = \mathtt{u}_2 \circ \varphi$ and $\varphi \circ \mathtt{d}_1 = \mathtt{d}_2 \circ \varphi$. A *(GL-)rack isomorphism* is simply a bijective (GL-)rack homomorphism. If R is a GL-rack with underlying rack (X, s), we denote its group of rack automorphisms by $\mathrm{Aut}_{\mathsf{Rack}}(X, s)$ and the subgroup of GL-rack automorphisms by $\mathrm{Aut}_{\mathsf{GLR}}(R)$. In particular, $s_x \in \mathrm{Aut}_{\mathsf{GLR}}(R)$ for all $x \in X$.

Evidently, we have the following; the final sentence is from [23, Proposition 3.2].

Proposition 2.11. *Let (X, s) be a rack with maps $\mathtt{u}, \mathtt{d} : X \to X$ satisfying axioms (L1) and (L3) of Definition 2.6. Then $R := (X, s, \mathtt{u}, \mathtt{d})$ is a GL-rack if and only if $\mathtt{u}$ and $\mathtt{d}$ are endomorphisms of the underlying rack (X, s). In this case, we actually have $\mathtt{u}, \mathtt{d} \in \mathrm{Aut}_{\mathsf{GLR}}(R)$.*

Axiom (L1) immediately yields the following.

Proposition 2.12. *Let $(X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack. Then the underlying rack (X, s) is a quandle if and only if $\mathtt{ud} = \mathrm{id}_X = \mathtt{du}$, that is, $\mathtt{d} = \mathtt{u}^{-1}$ as GL-rack automorphisms.*

2.3. **Free GL-racks.** In this subsection, we define free GL-racks. These are defined similarly to free objects in other algebraic theories, like free groups and free modules.

We begin by defining several categories. Let Set be the category of sets with set maps. Let Rack be the category of racks with rack homomorphisms, and let Qnd be the full subcategory of Rack whose objects are quandles. Let GLR be the category of GL-racks with GL-rack homomorphisms.

Example 2.9 and Proposition 2.12 immediately imply the following lemma.

Lemma 2.13. *The correspondence $(Q, s) \mapsto (Q, s, \mathrm{id}_Q, \mathrm{id}_Q)$ defines a canonical isomorphism from Qnd to the full subcategory of GLR whose objects are trivial GL-racks.*

In the sense of universal algebra, GLR is an equational class, so it is complete and cocomplete; see [1, Corollary 1.2, Theorem 4.5]. In particular, we can give presentations of GL-racks in terms of generators and relations, similarly as we can with groups. These presentations are quotients of *free GL-racks*, which Karmakar et al. [22] introduced in 2023.

Definition 2.14. [23, Section 4] Let X be a set. We define the *free GL-rack on X*, denoted by $\mathrm{FGLR}(X)$, as follows. If $X = \emptyset$, let $\mathrm{FGLR}(X)$ be the trivial GL-rack with one element. Else, define the *universe of words generated by X* to be the set $W(X)$ such that $X \subset W(X)$ and $s_y(x)$, $s_y^{-1}(x)$, $\mathtt{u}(x)$, $\mathtt{d}(x) \in W(X)$ for all $x, y \in W(X)$. Let $F(X)$ be the set of equivalence classes of elements of $W(X)$ modulo the equivalence relation generated by the following relations for all $x, y, z \in W(X)$:

(1) $s_y^{-1}(s_y(x))y \sim x \sim s_y(s_y^{-1}(x))$.
(2) $s_z(s_y(x)) \sim s_{s_z(y)}(s_z(x))$.
(3) $\mathtt{u}(\mathtt{d}(s_x(x))) \sim x \sim \mathtt{d}(\mathtt{u}(s_x(x)))$.
(4) $\mathtt{u}(s_y(x)) \sim s_y(\mathtt{u}(x))$ and $\mathtt{d}(s_y(x)) \sim s_y(\mathtt{d}(x))$.
(5) $s_{\mathtt{u}(y)}(x) \sim s_y(x)$ and $s_{\mathtt{d}(y)}(x) \sim s_y(x)$.

Thus, we have maps $s : F(X) \to \mathrm{Sym}(F(X))$ defined by $x \mapsto s_x := [y \mapsto s_x(y)]$ and $\mathtt{u}, \mathtt{d} : F(X) \to F(X)$ defined by $x \mapsto \mathtt{u}(x)$ and $x \mapsto \mathtt{d}(x)$. We define $\mathrm{FGLR}(X)$ to be the GL-rack $(F(X), s, \mathtt{u}, \mathtt{d})$.

The *free quandle on* X is defined similarly; in the sense of Lemma 2.13, it is simply $\mathrm{FGLR}(X)$ modulo the relations $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in W(X)$.

Free GL-racks satisfy the universal property expected of free objects in an equational category; see [23, Proposition 4.2] for details. This universal property implies that the functor $\mathsf{Set} \to \mathsf{GLR}$ assigning $X \mapsto \mathrm{FGLR}(X)$ is left adjoint to the forgetful functor $\mathsf{GLR} \to \mathsf{Set}$.

3. On rack-theoretic invariants of Legendrian links

In this section, we study GL-racks as invariants of Legendrian links. We begin by defining the GL-rack of an oriented Legendrian link Λ and the fundamental quandle of its underlying smooth link L, both of which are invariant under Legendrian isotopy. After a few worked examples, we give short algebraic proofs of several conjectures relating to Legendrian links and their invariants.

3.1. The GL-rack of a Legendrian link.

In this subsection, we discuss how to assign GL-racks and coloring invariants to Legendrian links, as Karmakar et al. [22] introduced in 2023.

Definition 3.1. Given a front projection $D(\Lambda)$ of an oriented Legendrian link Λ, define a *cusped strand* of $D(\Lambda)$ to be a maximal (with respect to inclusion) connected segment in $D(\Lambda)$. Also, define an *uncusped strand* of $D(\Lambda)$ to be a maximal (with respect to inclusion) connected subset of a cusped strand of $D(\Lambda)$ that both starts and ends at either a crossing or a cusp.

Definition 3.2. [23, Section 4] Let Λ be an oriented Legendrian link with front projection $D(\Lambda)$, and let X_Λ be a set in bijection with the cusped strands of $D(\Lambda)$. At each cusp, label the neighboring uncusped strands using the cusp relations in Figure 4. Then, at each crossing, impose the corresponding crossing relation between uncusped strands in Figure 4 on $\mathrm{FGLR}(X_\Lambda)$. The *GL-rack of* Λ, denoted by $\mathcal{G}(\Lambda)$, is defined to be the set of equivalence classes of elements of $\mathrm{FGLR}(X_\Lambda)$ modulo the equivalence relation generated by these relations. If L is a smooth link with link diagram $D(L)$, then we define the *fundamental quandle of* L, denoted by $\mathcal{Q}(L)$, in a similar way. However, we use the free quandle on X_Λ in place of $\mathrm{FGLR}(X_\Lambda)$, and we forgo any cusp relations.

While the original formulation defines X_Λ using uncusped strands, the cusp relations make this definition equivalent to Definition 3.2. For examples that compute $\mathcal{G}(\Lambda)$, see Subsection 3.2.

In 2023, Karmakar et al. [23, Theorem 4.3] proved that the assignment of $\mathcal{G}(\Lambda)$ to Λ (resp. $\mathcal{Q}(L)$ to L) is independent of the choice of front projection $D(\Lambda)$ (resp. link diagram $D(L)$). This is a consequence of how the GL-rack axioms encode the crossing and cusp relations induced by the Legendrian Reidemeister moves. As a result, the *GL-rack coloring number* of Λ with respect to a fixed GL-rack, as defined below, is invariant under Legendrian isotopy; see, e.g., [6, 23, 25].

Definition 3.3. Let R be a GL-rack. The *GL-rack coloring number* of an oriented Legendrian link Λ by R, denoted by $\mathrm{Col}(\Lambda, R)$, is defined to be the cardinality of the hom-set $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), R)$.

Kulkarni and Prathamesh [29, Main Theorem 2], Kimura [25, Theorem 4.1], and Karmakar et al. [23, Theorem 4.6] each used rack-theoretic coloring invariants to distinguish between infinitely many Legendrian unknots. Karmakar et al. [23, Theorem 4.7] also used GL-rack coloring numbers to distinguish between infinitely many Legendrian trefoils, and Ceniceros et al. [6, Example 16] used them to distinguish between connected sums of Legendrian trefoils.

That said, there also exist nonequivalent Legendrian knots with isomorphic GL-racks (see [26, Examples 21–24]), so neither $\mathcal{G}(\Lambda)$ nor $\mathrm{Col}(\Lambda, R)$ are complete Legendrian knot invariants. Nevertheless, we will use the latter in Subsection 3.2 to distinguish between unstabilized Legendrian 6_2 and 8_{10} knots, which cannot be done using the linearized contact homology or the ruling invariant.

At the time of writing, this is to our knowledge the first use of GL-racks to distinguish between Legendrian knots sharing the same classical invariants and Legendrian nonsimple smooth knot type.

Given a Legendrian link Λ, note that imposing the equivalence relation $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$ onto $\mathcal{G}(\Lambda)$ yields a quandle in the sense of Lemma 2.13. Geometrically, imposing this equivalence relation amounts to "ignoring" all cusps in $D(\Lambda)$ and viewing $D(\Lambda)$ only as a diagram of the underlying smooth link L. This recovers $\mathcal{Q}(L)$ from $\mathcal{G}(\Lambda)$, yielding the following observation.

Lemma 3.4. [26, Remark 23] *Let Λ be an oriented Legendrian link, and let L be its underlying smooth link. After imposing an equivalence relation onto $\mathcal{G}(\Lambda)$ defined by $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$, the resulting GL-rack is canonically isomorphic to $\mathcal{Q}(L)$ in the sense of Lemma 2.13.*

Finally, we consider a canonical functor from GLR to Grp, the category of groups with group homomorphisms. This functor assigns an *enveloping group* to each GL-rack, which Karmakar et al. [22] studied in 2023.

Definition 3.5. [22, Section 8] Given a GL-rack $R = (X, s, \mathtt{u}, \mathtt{d})$, its *enveloping group* is

$$\mathrm{Env}_{\mathsf{GLR}}(R) := \langle e_x,\ x \in X \mid e_{s_x(y)} = e_x^{-1} e_y e_x,\ e_{\mathtt{u}(x)} = e_x,\ e_{\mathtt{d}(x)} = e_x,\ x, y \in X \rangle.$$

By taking $\mathtt{u} = \mathrm{id}_X = \mathtt{d}$, we can also define the enveloping group of a quandle (Q, s) to be

$$\mathrm{Env}_{\mathsf{Qnd}}(Q, s) := \langle e_x,\ x \in Q \mid e_{s_x(y)} = e_x^{-1} e_y e_x,\ x, y \in Q \rangle.$$

Proposition 3.6. [22, Proposition 8.4] *There exists a functor $\mathrm{Env}_{\mathsf{GLR}} : \mathsf{GLR} \to \mathsf{Grp}$ that sends a GL-rack to its enveloping group and sends any GL-rack homomorphism $\psi : (X, s, \mathtt{u}_1, \mathtt{d}_1) \to (Y, t, \mathtt{u}_2, \mathtt{d}_2)$ to the group homomorphism $\widetilde{\psi} : \mathrm{Env}_{\mathsf{GLR}}(X, s, \mathtt{u}_1, \mathtt{d}_1) \to \mathrm{Env}_{\mathsf{GLR}}(Y, t, \mathtt{u}_2, \mathtt{d}_2)$ defined by $e_x \mapsto e_{\psi(x)}$ for all $x \in X$. Also, $\mathrm{Env}_{\mathsf{GLR}}$ is left adjoint to a functor defined by $G \mapsto (\mathrm{Conj}(G), \mathrm{id}_G, \mathrm{id}_G)$.*

Thus, some authors denote the enveloping group of a GL-rack or quandle R by $\mathrm{Adconj}(R)$ or $\mathrm{As}(R)$ and call it the *associated group of R*; see, e.g., [21, Section 6; 33, Definition 2.19]. For an example of how to compute the enveloping group of a GL-rack of a Legendrian link, see Example 3.7 in Subsection 3.2. We generalize this example in Subsection 3.3.

3.2. **Example calculations and applications.** In this section, we give several examples of how to compute the GL-rack of an oriented Legendrian knot. This allows us to give relatively brief algebraic proofs of conjectures in [8] and [3] about Legendrian 6_2 and 8_{10} knots, respectively.

Example 3.7. Let $q \geq 3$ be an odd integer, let L be a $(2, -q)$-torus knot, and let Λ be the Legendrian representative of L having maximal Thurston-Bennequin and rotation numbers. (This choice is well-defined since all torus knot types are Legendrian simple; see [14, Subsection 5.2].) In this example, we compute $\mathcal{G}(\Lambda)$, $\mathcal{Q}(L)$, $\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda))$, and $\mathrm{Env}_{\mathsf{Qnd}}(\mathcal{Q}(L))$ using the front projection $D(\Lambda)$ in Figure 8. Starting at any crossing (which, in Figure 8, we arbitrarily choose to be the bottommost crossing), traverse $D(\Lambda)$ along its depicted orientation. By recording the induced cusp and crossing relations using Figure 4, we compute that $\mathcal{G}(\Lambda)$ is the free GL-rack on the set $X_\Lambda = \{x_1, \dots, x_q\}$ modulo the crossing relations

$$s_{\mathtt{u}(x_1)}(x_q) = \mathtt{ud}(x_2),\ s_{\mathtt{d}(x_q)}(x_{q-1}) = \mathtt{ud}(x_1),\ \text{and } s_{\mathtt{d}(x_{i-1})}(x_{i-2}) = \mathtt{d}^2(x_i) \text{ for all } 3 \leq i \leq q.$$

Using GL-rack axiom (L3), we can simplify these crossing relations to

$$s_{x_1}(x_q) = \mathtt{ud}(x_2),\ s_{x_q}(x_{q-1}) = \mathtt{ud}(x_1),\ \text{and } s_{x_{i-1}}(x_{i-2}) = \mathtt{d}^2(x_i) \text{ for all } 3 \leq i \leq q.$$

Now that we have a presentation of $\mathcal{G}(\Lambda)$, let us compute $\mathcal{Q}(L)$. To do this, we could traverse $D(\Lambda)$ again while ignoring all cusps and only considering crossing relations. In view of Lemma 3.4,

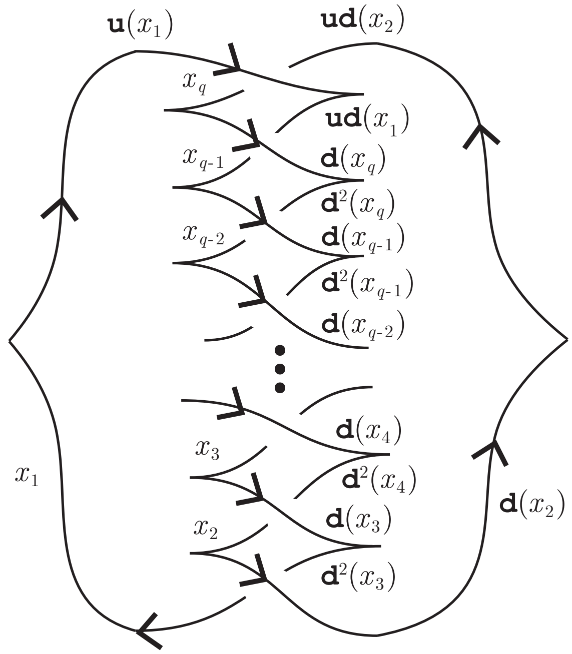

FIGURE 8. Front projection $D(\Lambda)$ and induced cusp relations of the Legendrian $(2,-q)$-torus knot Λ with maximal classical invariants.

we could equivalently impose the relations $\mathtt{u}(x_i) = x_i = \mathtt{d}(x_i)$ for all $1 \leq i \leq q$ onto $\mathcal{G}(\Lambda)$. Either method shows that $\mathcal{Q}(L)$ is the free quandle on X_Λ modulo the crossing relations

$$s_{x_1}(x_q) = x_2,\ s_{x_q}(x_{q-1}) = x_1,\ \text{and}\ s_{x_{i-1}}(x_{i-2}) = x_i \text{ for all } 3 \leq i \leq q.$$

Indeed, if we invert each symmetry in the relations of $\mathcal{Q}(L)$, then we recover the fundamental quandle of the mirror image of L computed in [2, Remark 3], as predicted by [38, Section 1].

If $q = 3$, then L is a left-handed trefoil, and the crossing relations show that $\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda))$ and $\mathrm{Env}_{\mathsf{Qnd}}(\mathcal{Q}(L))$ are both isomorphic to the group

$$\begin{aligned}&\langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{s_{x_1}(x_3)} = e_{x_1}^{-1} e_{x_3} e_{x_1},\ e_{s_{x_2}(x_1)} = e_{x_2}^{-1} e_{x_1} e_{x_2},\ e_{s_{x_3}(x_2)} = e_{x_3}^{-1} e_{x_2} e_{x_3}\rangle \\ &= \langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{x_2} = e_{x_1}^{-1} e_{x_3} e_{x_1},\ e_{x_3} = e_{x_2}^{-1} e_{x_1} e_{x_2},\ e_{x_1} = e_{x_3}^{-1} e_{x_2} e_{x_3}\rangle.\end{aligned}$$

Note that this is precisely the Wirtinger presentation of the knot group $\pi_1(\mathbb{R}^3 \setminus L) \cong \langle x, y \mid x^2 = y^3 \rangle$ of the trefoil; see, e.g., [12, Example 81]. Subsection 3.3 will generalize this observation.

Example 3.8. Let Λ_1 and Λ_2 be the oriented Legendrian knots on the left and right of Figure 9, respectively. In this example, we compute $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$ in preparation for a proof that Λ_1 and Λ_2 are not Legendrian isotopic. We note from [8] that Λ_1 and Λ_2 are both Legendrian representatives of the topological knot 6_2 with classical invariants $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$.

Let us begin with Λ_1. Traverse $D(\Lambda_1)$ using its given orientation while labeling all uncusped strands as in Figure 4. By writing down the induced crossing relations as in Figure 4, we find that

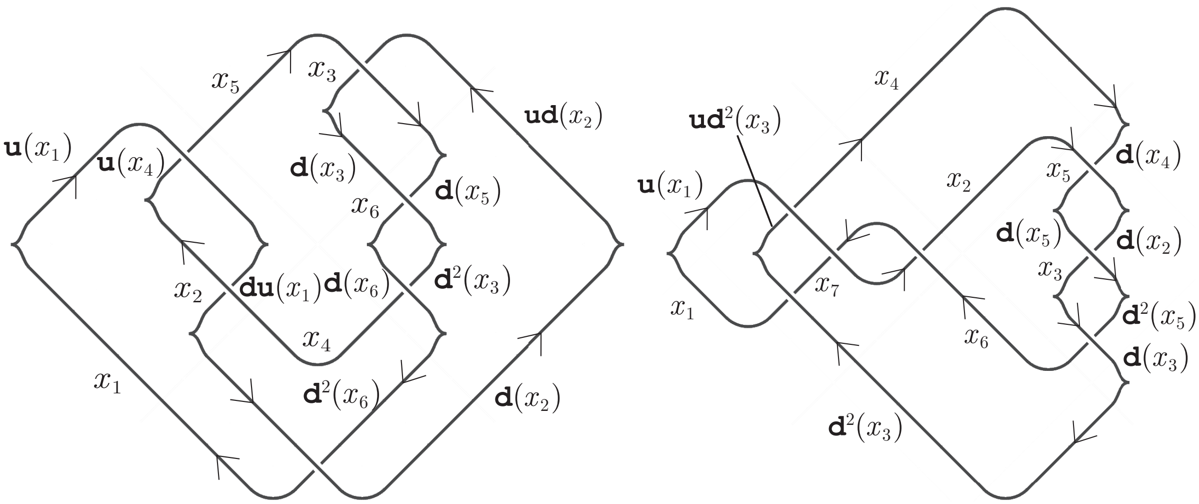

FIGURE 9. Front projections of the two Legendrian representatives of the topological knot 6_2 with $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$ given in [8]. Created using [27]; cf. [28].

$\mathcal{G}(\Lambda_1)$ is the free GL-rack on the set $X_{\Lambda_1} = \{x_1, \dots, x_6\}$ modulo the following crossing relations:

$$\mathcal{G}(\Lambda_1)\begin{cases} s_{\mathtt{u}(x_1)}(\mathtt{u}(x_4)) = x_5 \iff s_{x_1}(\mathtt{u}(x_4)) = x_5, \\ s_{x_4}(\mathtt{du}(x_1)) = x_2, \\ s_{\mathtt{d}(x_2)}(x_1) = \mathtt{d}^2(x_6) \iff s_{x_2}(x_1) = \mathtt{d}^2(x_6), \\ s_{x_5}(x_3) = \mathtt{ud}(x_2), \\ s_{\mathtt{d}(x_3)}(x_6) = \mathtt{d}(x_5) \iff s_{x_3}(x_6) = \mathtt{d}(x_5), \\ s_{\mathtt{d}(x_6)}(x_4) = \mathtt{d}^2(x_3) \iff s_{x_6}(x_4) = \mathtt{d}^2(x_3). \end{cases} \tag{1}$$

Note that we have simplified the first, third, fifth, and sixth relations using GL-rack axiom (L3).

Similarly, we compute $\mathcal{G}(\Lambda_2)$ to be the free GL-rack on the set $X_{\Lambda_2} = \{x_1, \dots, x_7\}$ modulo the following crossing relations:

$$\mathcal{G}(\Lambda_2)\begin{cases} s_{x_1}(\mathtt{ud}^2(x_3)) = x_4, & s_{x_5}(x_3) = \mathtt{d}(x_2), \\ s_{x_1}(x_6) = x_7, & s_{x_3}(x_6) = \mathtt{d}^2(x_5), \\ s_{x_6}(x_2) = \mathtt{u}(x_1), & s_{x_3}(x_7) = x_1, \\ s_{x_2}(x_5) = \mathtt{d}(x_4). \end{cases} \tag{2}$$

In Theorem 3.10, we use the above calculations to prove a conjecture of Chongchitmate and Ng [8] that Λ_1 and Λ_2 in Example 3.8 are not Legendrian isotopic. In 2021, Dynnikov and Prasolov [10, Proposition 2.3] proved this conjecture using impressive topological and combinatorial machinery. At the time of writing, theirs is the only proof of which we are aware. Indeed, Λ_1 and Λ_2 cannot be distinguished using the classical invariants, linearized contact homology, or the ruling invariant; see [8]. However, GL-rack coloring numbers offer a simpler and more algebraic alternative.

In the remainder of this article, we will denote the symmetric group on n letters by S_n. In the remainder of this subsection, we will also employ the following characterization of GL-rack homomorphisms; cf. [26, Remark 12]. By using [23, Proposition 4.1], one proves Lemma 3.9 similarly to the analogous result for group homomorphisms, so we omit the proof.

Lemma 3.9. *Let $R_0 = (X, s, \mathtt{u}_1, \mathtt{d}_1)$ and $R = (Y, t, \mathtt{u}_2, \mathtt{d}_2)$ be GL-racks. Given a presentation for R_0 (see Subsection 2.3), a map $\varphi : X \to Y$ defines a GL-rack homomorphism from R_0 to R if and only if the images of the generators of R_0 under φ satisfy the relations of R_0 when s, $\mathtt{u}_1$, and $\mathtt{d}_1$ are replaced with t, $\mathtt{u}_2$, and $\mathtt{d}_2$, respectively.*

Theorem 3.10. *The two oriented Legendrian knots in Figure 9 are not Legendrian isotopic; they are distinguishable using coloring numbers with respect to a permutation GL-rack of order 3.*

Proof. As before, let Λ_1 and Λ_2 be the oriented Legendrian knots on the left and right of Figure 9, respectively. Let $Y := \{1, 2, 3\}$. In cycle notation, let $\sigma \in S_3$ be the permutation (123). In the notation of Example 2.8, let $R := (Y, \sigma, \sigma^{-1}, \mathrm{id}_Y)_p$, so that R is the 11th GL-rack in Table 3. We will show that $\mathrm{Col}(\Lambda_2, R) > \mathrm{Col}(\Lambda_1, R)$. To that end, let W denote the underlying set of $\mathcal{G}(\Lambda_2)$ as presented in Example 3.8, and define $\varphi : W \to Y$ by

$$\varphi(x_i) := \begin{cases} 1 & \text{if } i \in \{1, 3, 4\}, \\ 2 & \text{if } i \in \{2, 6\}, \\ 3 & \text{if } i \in \{5, 7\}. \end{cases}$$

Using Lemma 3.9 and the relations in (2), it is straightforward to verify that φ, $\sigma \circ \varphi$, and $\sigma^2 \circ \varphi$ all define GL-rack homomorphisms from $\mathcal{G}(\Lambda_2)$ to R. Hence, $\mathrm{Col}(\Lambda_2, R) \geq 3$. (In fact, using a similar method as in the remainder of this proof, one can show that this bound is actually an equality.)

On the other hand, we claim that $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda_1), R) = \emptyset$. Let X denote the underlying set of $\mathcal{G}(\Lambda_1)$ as presented in Example 3.8, and suppose to the contrary that some map $\varphi : X \to Y$ defines a GL-rack homomorphism from $\mathcal{G}(\Lambda_1)$ to R. By Lemma 3.9, the elements $y_i := \varphi(x_i) \in Y$ satisfy the following system of equations in R:

$$R \begin{cases} (\sigma \circ \sigma^{-1})(y_4) = y_5 \iff y_4 = y_5, \\ (\sigma \circ \sigma^{-1})(y_1) = y_2 \iff y_1 = y_2, \\ \sigma(y_1) = y_6, \\ \sigma(y_3) = \sigma^{-1}(y_2) \iff y_3 = \sigma(y_2), \\ \sigma(y_6) = y_5, \\ \sigma(y_4) = y_3. \end{cases} \tag{3}$$

Here, we have used the fact that $\sigma^3 = \mathrm{id}_Y$ to rewrite the fourth equality. We now deduce that

$$\sigma(y_2) = y_3 = \sigma(y_4) = \sigma(y_5) = \sigma^2(y_6) = \sigma^3(y_1) = y_1 = y_2.$$

However, σ has no fixed points in Y. Thus, the system of equations in (3) has no solutions in R, which is a contradiction. Hence, φ cannot exist. ☐

Incidentally, Theorem 3.10 gives a positive answer to a question Kimura [25, Section 4] posed in 2023, as we state below; cf. [25, Theorem 4.3].

Corollary 3.11. *GL-rack coloring numbers by nonquandle GL-racks are not generally unable to distinguish between nonequivalent Legendrian knots sharing the same underlying smooth knot type and classical invariants.*

Similarly, GL-rack coloring numbers distinguish between the Legendrian representatives of the topological knot 8_{10} with $(\mathrm{tb}, \mathrm{rot}) = (-8, 3)$ in Figure 10. This proves a conjecture of Bhattacharyya et al. [3]. At the time of writing, we are unaware of any other proofs of this conjecture.

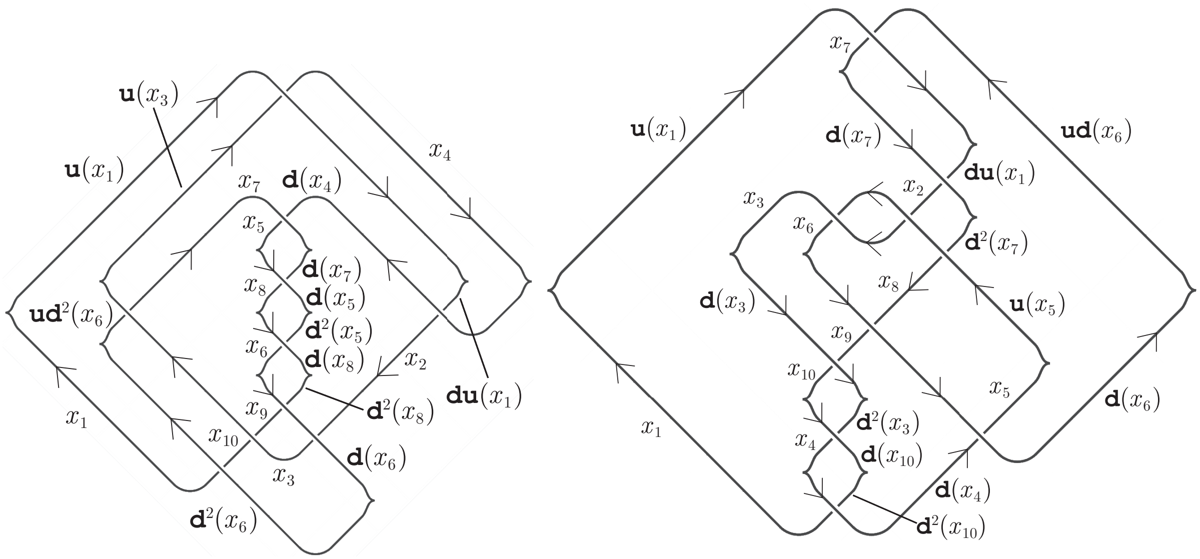

FIGURE 10. Front projections of the two Legendrian representatives of the topological knot 8_{10} with $(\mathrm{tb}, \mathrm{rot}) = (-8, 3)$ given in [3]. Created using [27]; cf. [28].

Example 3.12. Let Λ_1 and Λ_2 be the oriented Legendrian knots on the left and right of Figure 10, respectively. We compute that $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$ are the free GL-racks on the set $\{x_1, \dots, x_{10}\}$ modulo the following crossing relations:

$$\mathcal{G}(\Lambda_1)\begin{cases} s_{x_1}(\mathtt{u}(x_3)) = x_4, & s_{x_4}(\mathtt{du}(x_1)) = x_2, \\ s_{x_7}(x_5) = \mathtt{d}(x_4), & s_{x_5}(x_8) = \mathtt{d}(x_7), \\ s_{x_8}(x_6) = \mathtt{d}^2(x_5), & s_{x_6}(x_9) = \mathtt{d}(x_6), \\ s_{x_6}(x_3) = x_2, & s_{x_6}(x_{10}) = x_1, \\ s_{x_3}(x_7) = \mathtt{ud}^2(x_6), & s_{x_3}(x_9) = x_{10}. \end{cases} \quad \mathcal{G}(\Lambda_2)\begin{cases} s_{x_1}(x_7) = \mathtt{ud}(x_6), & s_{x_7}(x_2) = \mathtt{du}(x_1), \\ s_{x_5}(x_2) = x_3, & s_{x_3}(\mathtt{u}(x_5)) = x_6, \\ s_{x_3}(x_{10}) = x_9, & s_{x_{10}}(x_4) = \mathtt{d}^2(x_3), \\ s_4(x_1) = \mathtt{d}^2(x_{10}), & s_{x_6}(\mathtt{d}(x_4)) = x_5, \\ s_{x_5}(\mathtt{d}^2(x_7)) = x_8, & s_{x_6}(x_9) = x_8. \end{cases}$$

Theorem 3.13. *The two oriented Legendrian knots in Figure 10 are not Legendrian isotopic; they are distinguishable using coloring numbers with respect to a permutation GL-rack of order 2.*

Proof. Once again, let Λ_1 and Λ_2 be the oriented Legendrian knots on the left and right of Figure 10, respectively. Let $Y := \{1, 2\}$. In cycle notation, let $\sigma \in S_2$ be the permutation (12). In the notation of Example 2.8, let $R := (Y, \sigma, \sigma, \mathrm{id}_Y)_p$, so that R is the fourth GL-rack in Table 2. We will show that $\mathrm{Col}(\Lambda_1, R) > \mathrm{Col}(\Lambda_2, R)$. To that end, let W denote the underlying set of $\mathcal{G}(\Lambda_1)$ as presented in Example 3.12, and define $\varphi : W \to Y$ by

$$\varphi(x_i) := \begin{cases} 1 & \text{if } i \in \{1, 2, 5, 8, 9\}, \\ 2 & \text{if } i \in \{3, 4, 6, 7, 10\}. \end{cases}$$

Using Lemma 3.9 and the relations of $\mathcal{G}(\Lambda_1)$ in Example 3.12, it is straightforward to verify that φ and $\sigma \circ \varphi$ both define GL-rack homomorphisms from $\mathcal{G}(\Lambda_1)$ to R. Hence, $\mathrm{Col}(\Lambda_1, R) \geq 2$ (which is actually an equality).

On the other hand, we claim that $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda_2), R) = \emptyset$. Let X denote the underlying set of $\mathcal{G}(\Lambda_2)$ as presented in Example 3.12, and suppose to the contrary that some map $\varphi : B \to Y$ defines a GL-rack homomorphism from $\mathcal{G}(\Lambda_2)$ to R. By Lemma 3.9, the elements $y_i := \varphi(x_i) \in Y$

satisfy the following system of equations in R:

$$R\begin{cases}\sigma(y_7)=\sigma(y_6), & \sigma(y_2)=\sigma(x_1),\\ \sigma(y_2)=y_3, & \sigma^2(y_5)=y_6,\\ \sigma(y_{10})=y_9, & \sigma(y_4)=y_3,\\ \sigma(y_1)=y_{10}, & \sigma(y_4)=y_5,\\ \sigma(y_7)=y_8, & \sigma(y_9)=y_8.\end{cases}$$

Since $\sigma^2=\mathrm{id}_Y$, we can also rewrite the equalities $\sigma(y_7)=\sigma(x_6)$, $\sigma^2(y_5)=y_6$, and $\sigma(y_{10})=y_9$ as $y_7=y_6$, $y_5=y_6$, and $y_{10}=\sigma(y_9)$, respectively. Therefore, we have

$$y_7=y_6=y_5=\sigma(y_4)=y_3=\sigma(y_2)=\sigma(y_1)=y_{10}=\sigma(y_9)=y_8=\sigma(y_7),$$

which is impossible since σ has no fixed points in Y. Hence, φ cannot exist. $\square$

We selected the permutations GL-racks and GL-rack homomorphisms used to prove Theorems 3.10 and 3.13 using exhaustive computer searches in `GAP` [17]. See Subsection A.3 for details. To help complete the atlas of Legendrian knots, we encourage the reader to the download the program linked in Appendix A and tackle even more of the conjectures in [3, 8] in this fashion.

3.3. **Isomorphism of $\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda))$ and $\pi_1(\mathbb{R}^3\setminus L)$.** We now prove an empirical observation of Karmakar et al. in the original version of [22, Remark 8.2], which we state as Theorem 3.15. Although the original observation is absent from subsequent versions of the article, we are unaware of any other proofs at the time of writing. We begin with a more abstract lemma.

Lemma 3.14. *Let $R=(X,s,\mathtt{u},\mathtt{d})$ be a GL-rack, and let R' be the GL-rack obtained by imposing an equivalence relation $\sim$ onto R defined by $\mathtt{u}(x)=x=\mathtt{d}(x)$ for all $x\in X$. Then, in Grp, we have $\mathrm{Env}_{\mathsf{GLR}}(R)\cong\mathrm{Env}_{\mathsf{GLR}}(R')$. In particular, $\mathrm{Env}_{\mathsf{GLR}}(R)\cong\mathrm{Env}_{\mathsf{Qnd}}(X/\sim,s^*)$.*

Proof. By definition, R' is the quotient object of the equivalence relation $\sim$ on R in GLR. In other words, R' is the colimit of the following diagram in GLR:

$$R \underset{\mathrm{id}_X}{\overset{\mathtt{u}}{\rightrightarrows}} R \underset{\mathrm{id}_X}{\overset{\mathtt{d}}{\rightrightarrows}} R$$

Recall that left adjoint functors preserve colimits. It follows from Proposition 3.6 that $\mathrm{Env}_{\mathsf{GLR}}(R')$ is the colimit of the following diagram in Grp:

$$\mathrm{Env}_{\mathsf{GLR}}(R) \underset{\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)}{\overset{\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})}{\rightrightarrows}} \mathrm{Env}_{\mathsf{GLR}}(R) \underset{\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)}{\overset{\mathrm{Env}_{\mathsf{GLR}}(\mathtt{d})}{\rightrightarrows}} \mathrm{Env}_{\mathsf{GLR}}(R)$$

By Proposition 3.6, the group homomorphism $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})$ is defined by $e_x\mapsto e_{\mathtt{u}(x)}$ for all $x\in X$, but $e_x=e_{\mathtt{u}(x)}$ in $\mathrm{Env}_{\mathsf{GLR}}(R)$. Thus, $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})$ is the identity map. Similarly, $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{d})$ and $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)$ are the identity maps, so we have a group isomorphism $\mathrm{Env}_{\mathsf{GLR}}(R)\cong\mathrm{Env}_{\mathsf{GLR}}(R')$. By Lemma 2.13, we have $\mathrm{Env}_{\mathsf{GLR}}(R')\cong\mathrm{Env}_{\mathsf{Qnd}}(X/\sim,s^*)$, which completes the proof. $\square$

Theorem 3.15. *Let $\Lambda\subset\mathbb{R}^3$ be an oriented Legendrian link, and let L denote its underlying smooth link. Then there exists a group isomorphism*

$$\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda))\cong\pi_1(\mathbb{R}^3\setminus L).$$

Proof. In the setting of Lemma 3.14, take $R:=\mathcal{G}(\Lambda)$. Then, in GLR, we have $R'\cong(\mathcal{Q}(L),\mathrm{id}_{X_L},\mathrm{id}_{X_L})$ by Lemma 3.4. By Lemma 3.14, it suffices to show that $\mathrm{Env}_{\mathsf{Qnd}}(\mathcal{Q}(L))\cong\pi_1(\mathbb{R}^3\setminus L)$ in Grp. Indeed, Joyce [21, Section 15] showed this using the Wirtinger presentation of $\pi_1(\mathbb{R}^3\setminus L)$. $\square$

4. On medial GL-racks, hom-sets, and tensor products

We now turn our attention to the algebraic theory of GL-racks. In this section, we define medial GL-racks, propose a medial GL-rack-valued invariant of Legendrian links, and introduce tensor products of GL-racks that make the category of medial GL-racks symmetric monoidal closed. This extends several results of Crans and Nelson [9] in 2014; we follow their approach closely.

4.1. **Hom-sets of medial GL-racks are also medial GL-racks.**

In this subsection, we define medial GL-racks, introduce a canonical medial GL-rack structure on any hom-set from a GL-rack to a medial GL-rack, and propose a medial GL-rack-valued invariant of Legendrian links.

Definition 4.1. [18, Section 3] A rack (X, s) is called *medial* or *abelian* if the map $X \times X \to X$ defined by $(x, y) \mapsto s_y(x)$ is a rack homomorphism. Explicitly, this means that

$$s_{s_x(z)} \circ s_y = s_{s_x(y)} \circ s_z$$

for all $x, y, z \in X$. (Note that this definition is not synonymous with the condition that (X, s) is *commutative*, which requires that $s_x(y) = s_y(x)$ for all $x, y \in X$.) If in addition $(\mathtt{u}, \mathtt{d})$ defines a GL-structure on (X, s), then we say that the GL-rack $(X, s, \mathtt{u}, \mathtt{d})$ is *medial* or *abelian*.

Example 4.2. Every permutation rack is medial, so every permutation GL-rack is medial.

Example 4.3. The tables in Appendix B state that, up to isomorphism, there is exactly one nonmedial GL-rack of order 4 or lower. This GL-rack, which is the second-to-last GL-rack listed in Table 4, is defined as follows. Let $X := \{1, 2, 3, 4\}$. In cycle notation, define $s : X \to S_4$ by $i \mapsto s_i$ with $s_1 := \mathrm{id}_X$, $s_2 := (34)$, $s_3 := (24)$, and $s_4 := (23)$. Then $(X, s, \mathrm{id}_X, \mathrm{id}_X)$ is a trivial GL-rack, and it is nonmedial because, for example,

$$s_{s_1(3)} \circ s_2 = (24)(34) \neq (34)(24) = s_{s_1(2)} \circ s_3,$$

where the permutations are composed from right to left.

Let $\mathsf{GLR}^{\mathrm{med}}$ be the full subcategory of GLR whose objects are medial. In this article, we adopt the term "medial" over "abelian" since neither GLR nor $\mathsf{GLR}^{\mathrm{med}}$ is an abelian (or even additive) category. Indeed, neither category has a zero object; the initial object in either category is the GL-rack of order 0, but terminal objects in either category are trivial GL-racks of order 1.

That said, the relationship of (GL-)racks to medial (GL-)racks is similar to that of groups to abelian groups; a group A is abelian if and only if its group operation $A \times A \to A$ is a group homomorphism. Below, we analogize the fact that $\mathrm{Hom}_{\mathsf{Grp}}(G, A)$ is an abelian group if A is abelian.

Theorem 4.4. *Let $R_1 := (X, s^X, \mathtt{u}_1, \mathtt{d}_1)$ and $R_2 := (Y, s^Y, \mathtt{u}_2, \mathtt{d}_2)$ be GL-racks, and suppose that R_2 is medial. Let $H := \mathrm{Hom}_{\mathsf{GLR}}(R_1, R_2)$, and define $s : H \to \mathrm{Sym}(H)$ by $g \mapsto s_g$, where $s_g(f) := [x \mapsto (s^Y_{g(x)} \circ f)(x)]$. Define $\mathtt{u} : H \to H$ by $f \mapsto \mathtt{u}_2 \circ f$, and define $\mathtt{d} : H \to H$ by $f \mapsto \mathtt{d}_2 \circ f$. Then $R := (H, s, \mathtt{u}, \mathtt{d})$ is a medial GL-rack. If R_1 or R_2 is also a GL-quandle, then so is R.*

Proof. First, we will show that (H, s) is a medial rack. Let $\widetilde{H} := \mathrm{Hom}_{\mathsf{Rack}}((X, s^X), (Y, s^Y))$, and extend s to a function $\widetilde{s} : \widetilde{H} \to \mathrm{Sym}(\widetilde{H})$ using the same definition of s. In 2021, Grøsfjeld [18, Proposition 3.3] showed that $(\widetilde{H}, \widetilde{s})$ is a medial rack and, if (X, s^X) or (Y, s^Y) is a quandle, also a quandle. To show the same for (H, s), it suffices to show that (H, s) is a subrack of $(\widetilde{H}, \widetilde{s})$.

To that end, fix $f, g \in H$. Then $s_g(f) \in H$ because

$$\begin{aligned}
\mathtt{u}_2 \circ s_g(f) &= [x \mapsto (\mathtt{u}_2 \circ s^Y_{g(x)} \circ f)(x)] \\
&= [x \mapsto (s^Y_{g(x)} \circ \mathtt{u}_2 \circ f)(x)] && \text{by GL-rack axiom (L2)} \\
&= [x \mapsto (s^Y_{g(x)} \circ f \circ \mathtt{u}_1)(x)] && \text{since } f \in H \\
&= s_g(f) \circ \mathtt{u}_1
\end{aligned}$$

and, similarly, $\mathtt{d}_2 \circ s_g(f) = s_g(f) \circ \mathtt{d}_1$. Thus, (H, s) is a subrack of $(\widetilde{H}, \widetilde{s})$, as desired.

To show that R is a GL-rack, it remains to show that $(\mathtt{u}, \mathtt{d})$ defines a GL-structure on (H, s). Fix $f, g \in H$. Since f is a GL-rack homomorphism and R_1 satisfies GL-rack axiom (L1), we have

$$\begin{aligned}
(\mathtt{ud} \circ s_f)(f) &= [x \mapsto (\mathtt{u}_2 \circ \mathtt{d}_2 \circ s^Y_{f(x)} \circ f)(x)] \\
&= [x \mapsto (\mathtt{u}_2 \circ \mathtt{d}_2 \circ f \circ s^X_x)(x)] \\
&= [x \mapsto f((\mathtt{u}_1 \circ \mathtt{d}_1 \circ s^X_x)(x))] \\
&= [x \mapsto f(x)] = f
\end{aligned}$$

and, similarly, $(\mathtt{du} \circ s_f)(f) = f$. So, R satisfies axiom (L1). Since R_2 satisfies axiom (L2), we have

$$(\mathtt{u} \circ s_g)(f) = [x \mapsto (\mathtt{u}_2 \circ s^Y_{g(x)} \circ f)(x)] = [x \mapsto (s^Y_{g(x)} \circ \mathtt{u}_2 \circ f)(x)] = s_g(\mathtt{u}_2 \circ f) = (s_g \circ \mathtt{u})(f)$$

and, similarly, $(\mathtt{d} \circ s_g)(f) = (s_g \circ \mathtt{d})(f)$. So, R satisfies axiom (L2). Finally, since R_2 satisfies axiom (L3), we have

$$s_{\mathtt{u}(g)}(f) = [x \mapsto (s^Y_{\mathtt{u}_2(g(x))} \circ f)(x)] = [x \mapsto (s^Y_{g(x)} \circ f)(x)] = s_g(f)$$

and, similarly, $s_{\mathtt{d}(g)}(f) = s_g(f)$. Hence, R satisfies axiom (L3), and the proof is complete. □

Theorem 4.4 implies that for a fixed medial GL-rack M and for any oriented Legendrian link Λ, the isomorphism class of $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), M)$ as a medial GL-rack is an invariant of Λ. This is a Legendrian analogue of the medial quandle-valued invariant of smooth links that Crans and Nelson introduced in [9, Section 6]. Moreover, $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), M)$ is an *enhancement* or *refinement* of the coloring number $\mathrm{Col}(\Lambda, M)$ defined in Subsection 3.1. This is significant since GL-rack coloring numbers are not complete invariants of Legendrian links; see [25, Theorem 4.3]. In Section 6, we propose questions for future research on $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), M)$.

4.2. **Tensor products of GL-racks.** In this subsection, we define tensor products that induce symmetric monoidal structures on GLR and $\mathsf{GLR}^{\mathrm{med}}$. In the latter, we show that this structure is compatible with the closed structure given by Theorem 4.4. (Note that GLR and $\mathsf{GLR}^{\mathrm{med}}$ also have Cartesian monoidal structures given by the categorical product, i.e., the Cartesian product $\times$.)

Definition 4.5. If $R_1 = (X, s^X, \mathtt{u}_1, \mathtt{d}_1)$ and $R_2 = (Y, s^Y, \mathtt{u}_2, \mathtt{d}_2)$ are GL-racks, then we define their *tensor product*, denoted by $R_1 \otimes R_2$, to be the free GL-rack $\mathrm{FGLR}(X \times Y)$ modulo the following relations for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$:

(1) $s_{(x,y_2)}(x, y_1) = (x, s^Y_{y_2}(y_1))$.
(2) $s_{(x_2,y)}(x_1, y) = (s^X_{x_2}(x_1), y)$.
(3) $\mathtt{u}(x, y) = (\mathtt{u}_1(x), y) = (x, \mathtt{u}_2(y))$.
(4) $\mathtt{d}(x, y) = (\mathtt{d}_1(x), y) = (x, \mathtt{d}_2(y))$.

We also define the *medial tensor product* of R_1 and R_2, denoted by $R_1 \otimes_{\mathrm{med}} R_2$, to be $R_1 \otimes R_2$ modulo the following relations for all $x_1, y_1, z_1, a \in X$ and $x_2, y_2, z_2, b \in Y$:

$$(s_{s_{(x_1,x_2)}(z_1,z_2)} \circ s_{(y_1,y_2)})(a, b) = (s_{s_{(x_1,x_2)}(y_1,y_2)} \circ s_{(z_1,z_2)})(a, b).$$

Note that if R_1 or R_2 is a GL-quandle, then so is $R_1 \otimes R_2$. By Lemma 2.13, Definition 4.5 recovers Crans and Nelson's [9, Subsection 8.1] tensor products of medial quandles. The following result shows that medial tensor products of medial GL-racks also satisfy a universal property and internal hom-tensor adjunction similar to those of tensor products of modules.

Theorem 4.6. *The category* $\mathsf{GLR}^{\mathrm{med}}$ *is symmetric monoidal closed with respect to the closed structure* $\mathrm{Hom}_{\mathsf{GLR}^{\mathrm{med}}}(-,-)$ *from Theorem 4.4 and the medial tensor product* $\otimes_{\mathrm{med}}$*.*

Proof. The unit object in $\mathsf{GLR}^{\mathrm{med}}$ is the trivial GL-rack with one element. Using this fact, it is straightforward to verify that $\mathsf{GLR}^{\mathrm{med}}$ is monoidal and symmetric. On the other hand, $\mathsf{GLR}^{\mathrm{med}}$ is defined as an equational algebraic category, and $\otimes_{\mathrm{med}}$ is precisely the tensor product constructed in [30, Section 4]. Thus, the main theorem of Linton in [30] states that our claim is true if and only if, in the sense of universal algebra, $\mathsf{GLR}^{\mathrm{med}}$ is commutative as an algebraic theory; see [30, Section 6] and cf. [9, Subsection 8.1]. Indeed, for any medial GL-rack $(X, s, \mathtt{u}, \mathtt{d})$ and for all elements $x_{11}, x_{12}, x_{21}, x_{22} \in X$, we have the following equalities:

$$\begin{cases} (\mathtt{u} \circ s_{x_{12}})(x_{11}) = (s_{\mathtt{u}(x_{12})} \circ \mathtt{u})(x_{11}) & \text{by Proposition 2.11,} \\ (\mathtt{d} \circ s_{x_{12}})(x_{11}) = (s_{\mathtt{d}(x_{12})} \circ \mathtt{d})(x_{11}) & \text{by Proposition 2.11,} \\ (\mathtt{u} \circ \mathtt{d})(x_{11}) = (\mathtt{d} \circ \mathtt{u})(x_{11}) & \text{by Proposition 2.11,} \\ (s_{s_{x_{22}}(x_{21})} \circ s_{x_{12}})(x_{11}) = (s_{s_{x_{22}}(x_{12})} \circ s_{x_{21}})(x_{11}) & \text{since } (X,s) \text{ is medial.} \end{cases}$$

Together with the tautologies $\mathtt{u}^2(x_{11}) = \mathtt{u}^2(x_{11})$ and $\mathtt{d}^2(x_{11}) = \mathtt{d}^2(x_{11})$, these equalities show that $\mathsf{GLR}^{\mathrm{med}}$ forms a commutative algebraic theory. This completes the proof. $\square$

Tensor products of GL-racks would be interesting to study more closely in future research. Moreover, the results in this section open interesting avenues for categorical research on GL-racks. In Section 6, we propose questions for further work in these directions.

5. Equivalence of racks and GL-quandles

In this section, we show that the algebraic theories of racks and GL-quandles are isomorphic in a way that preserves mediality. This generalizes the one-to-one correspondences observed in Appendix A.1 in a natural way, and it provides a sense in which the algebraic theory of GL-racks generalizes that of racks. In what follows, let GLQ be the full subcategory of GLR whose elements are GL-quandles. Note that, like Rack and GLR, GLQ is an equational algebraic category.

We will begin by defining a functor $\mathcal{F} : \mathsf{Rack} \to \mathsf{GLR}$. First, we define how $\mathcal{F}$ acts on objects.

Proposition 5.1. *Given a rack* (X, s)*, define* $\widetilde{\mathtt{u}} : X \to X$ *by* $x \mapsto s_x(x)$*, define* $\widetilde{\mathtt{u}}^{-1} : X \to X$ *by* $x \mapsto s_x^{-1}(x)$*, and define* $\widetilde{s} : X \to \mathrm{Sym}(X)$ *by* $x \mapsto \widetilde{s}_x$*, where* $\widetilde{s}_x := \widetilde{\mathtt{u}}^{-1} \circ s_x$*. Then* $\mathcal{F}(X, s) := (X, \widetilde{s}, \widetilde{\mathtt{u}}, \widetilde{\mathtt{u}}^{-1})$ *is a GL-quandle.*

Proof. By Proposition 2.12, it suffices to show that (i) $\widetilde{\mathtt{u}}$ and $\widetilde{\mathtt{u}}^{-1}$ are two-sided inverses, (ii) $(X, \widetilde{s})$ is a rack, and (iii) $\mathcal{F}(X, s)$ satisfies the three GL-rack axioms. To that end, we will employ a well-known construction called a *dual rack*. Namely, define $s' : X \to \mathrm{Sym}(X)$ by $x \mapsto s_x^{-1}$. Since (X, s) is a rack, the pair (X, s') is also a rack because

$$s^{-1}_{s_x^{-1}(y)} \circ s_x^{-1} = (s_x \circ s_{s_x^{-1}(y)})^{-1} = (s_{s_x(s_x^{-1}(y))} \circ s_x)^{-1} = (s_y \circ s_x)^{-1} = s_x^{-1} \circ s_y^{-1}$$

for all $x, y \in X$. Note that the group $\mathrm{Aut}_{\mathsf{Rack}}(X, s')$ contains $(s_x^{-1})^{-1} = s_x$ for all $x \in X$.

To prove (i), fix $x \in X$. Since $s_x \in \mathrm{Aut}_{\mathsf{Rack}}(X, s')$, we have

$$\widetilde{\mathtt{u}}^{-1}\widetilde{\mathtt{u}}(x) = (s^{-1}_{s_x(x)} \circ s_x)(x) = (s_x \circ s_x^{-1})(x) = x.$$

Dually, we also have $\widetilde{\mathtt{u}}\widetilde{\mathtt{u}}^{-1}(x) = x$. Since $x \in X$ was arbitrary, this proves (i). This reduces axiom (L1) to the claim that $\widetilde{s}_x(x) = x$, which once again follows from the fact that $s_x \in \mathrm{Aut}_{\mathsf{Rack}}(X, s')$:

$$\widetilde{s}_x(x) = \widetilde{\mathtt{u}}^{-1} \circ s_x(x) = (s^{-1}_{s_x(x)} \circ s_x)(x) = (s_x \circ s^{-1}_x)(x) = x.$$

Next, we will show that the quadruple $X' := (X, s, \widetilde{\mathtt{u}}, \widetilde{\mathtt{u}}^{-1})$ satisfies axioms (L2) and (L3). By using this fact along with (i), one straightforwardly verifies (ii) and the rest of (iii). To that end, fix $x, y \in X$. Since $s_x \in \mathrm{Aut}_{\mathsf{Rack}}(X, s')$, we have

$$(\widetilde{\mathtt{u}}^{-1} \circ s_x)(y) = (s^{-1}_{s_x(y)} \circ s_x)(y) = (s_x \circ s^{-1}_y)(y) = (s_x \circ \widetilde{\mathtt{u}}^{-1})(y).$$

We see more easily that $(\widetilde{\mathtt{u}} \circ s_x)(y) = s_x(y)$ as well. Therefore, X' satisfies axiom (L2). Since $s^{-1}_x \in \mathrm{Aut}_{\mathsf{Rack}}(X, s)$, we also have

$$s_{\widetilde{\mathtt{u}}^{-1}(x)} = s_{s^{-1}_x(x)} = s_{s^{-1}_x(x)} \circ s^{-1}_x \circ s_x = s^{-1}_x \circ s_x \circ s_x = s_x.$$

We see more easily that $s_{\widetilde{\mathtt{u}}(x)} = s_x$ as well. Hence, X' satisfies axiom (L3). $\square$

We now define how $\mathcal{F}$ acts on morphisms.

Proposition 5.2. *Given a rack homomorphism $f \in \mathrm{Hom}_{\mathsf{Rack}}((X, s^X), (Y, s^Y))$, let $\mathcal{F}f : X \to Y$ be the same set map as $f : X \to Y$. Then $\mathcal{F}f \in \mathrm{Hom}_{\mathsf{GLQ}}(\mathcal{F}(X, s^X), \mathcal{F}(Y, s^Y))$, and this definition makes $\mathcal{F}$ into a covariant functor.*

Proof. Certainly, $\mathcal{F}$ preserves the identity morphism and composition of morphisms, so we only need to verify that $\mathcal{F}f$ is a GL-rack homomorphism. To that end, write $\mathcal{F}(X, s^X) = (X, \widetilde{s}^X, \widetilde{\mathtt{u}}_X, \widetilde{\mathtt{u}}^{-1}_X)$ and $\mathcal{F}(Y, s^Y) = (Y, \widetilde{s}^Y, \widetilde{\mathtt{u}}_Y, \widetilde{\mathtt{u}}^{-1}_Y)$, and fix $x \in X$. Since $f \in \mathrm{Hom}_{\mathsf{Rack}}((X, s^X), (Y, s^Y))$, we have

$$(\mathcal{F}f \circ \widetilde{\mathtt{u}}_X)(x) = (f \circ s^X_x)(x) = (s^Y_{f(x)} \circ f)(x) = \widetilde{\mathtt{u}}_Y(f(x)) = (\widetilde{\mathtt{u}}_Y \circ \mathcal{F}f)(x),$$

as desired. Similarly, $(f \circ \widetilde{\mathtt{u}}^{-1}_X)(x) = (\widetilde{\mathtt{u}}^{-1}_Y \circ f)(x)$, so we have

$$\mathcal{F}f \circ \widetilde{s}^X_x = f \circ \widetilde{\mathtt{u}}^{-1}_X \circ s^X_x = \widetilde{\mathtt{u}}^{-1}_Y \circ f \circ s^X_x = \widetilde{\mathtt{u}}^{-1}_Y \circ s^Y_{f(x)} \circ f = \widetilde{s}^Y_{f(x)} \circ \mathcal{F}f.$$

Hence, $\mathcal{F}f$ is a GL-rack homomorphism from $\mathcal{F}(X, s^X)$ to $\mathcal{F}(Y, s^Y)$. $\square$

Now, we will define a functor $\mathcal{G} : \mathsf{GLQ} \to \mathsf{Rack}$ as the restriction of a functor $\widetilde{\mathcal{G}} : \mathsf{GLR} \to \mathsf{Rack}$ to GLQ. Given a GL-quandle $(X, s, \mathtt{u}, \mathtt{u}^{-1})$, let $\mathcal{G}(X, s, \mathtt{u}, \mathtt{u}^{-1})$ be the rack $(X, \hat{s})$ constructed below.

Proposition 5.3. *Given a GL-rack $(X, s, \mathtt{u}, \mathtt{d})$, define $\hat{s} : X \to \mathrm{Sym}(X)$ by $x \mapsto \hat{s}_x$, where $\hat{s}_x := \mathtt{u} \circ s_x$. Then $\widetilde{\mathcal{G}}(X, s, \mathtt{u}, \mathtt{d}) := (X, \hat{s})$ is a rack.*

Proof. Fix $x, y \in X$. Then GL-rack axioms (L2) and (L3) imply that

$$\hat{s}_{\hat{s}_x(y)} \circ \hat{s}_x = \hat{s}_{(\mathtt{u} \circ s_x)(y)} \circ \mathtt{u} \circ s_x = \mathtt{u} \circ s_{\mathtt{u} \circ s_x(y)} \circ \mathtt{u} \circ s_x = \mathtt{u}^2 \circ s_{s_x(y)} \circ s_x.$$

Since (X, s) is a rack, it follows that

$$\hat{s}_{\hat{s}_x(y)} \circ \hat{s}_x = \mathtt{u}^2 \circ s_x \circ s_y = \mathtt{u} \circ s_x \circ \mathtt{u} \circ s_y = \hat{s}_x \circ \hat{s}_y,$$

where in the second equality we have once again applied axiom (L2). $\square$

We now define how $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ act on morphisms.

Proposition 5.4. *Let $R_1 := (X, s^X, \mathtt{u}_X, \mathtt{d}_X)$ and $R_2 := (Y, s^Y, \mathtt{u}_Y, \mathtt{d}_Y)$ be GL-racks. Given a rack homomorphism $g \in \mathrm{Hom}_{\mathsf{GLR}}(R_1, R_2)$, let $\widetilde{\mathcal{G}}g : X \to Y$ be the same set map as $g : X \to Y$. Then $\widetilde{\mathcal{G}}g \in \mathrm{Hom}_{\mathsf{Rack}}(\widetilde{\mathcal{G}}(R_1), \widetilde{\mathcal{G}}(R_2))$. This definition makes $\widetilde{\mathcal{G}}$ (and, hence, $\mathcal{G}$) into a covariant functor.*

Proof. Certainly, $\widetilde{\mathcal{G}}$ preserves the identity morphism and composition of morphisms, so we only need to verify that $\widetilde{\mathcal{G}}g$ is a rack homomorphism. To that end, write $\widetilde{\mathcal{G}}(R_1) = (X, \hat{s}^X)$ and $\widetilde{\mathcal{G}}(R_2) = (Y, \hat{s}^Y)$, and fix $x \in X$. Since g is a GL-rack homomorphism, we have

$$\widetilde{\mathcal{G}}g \circ \hat{s}^X_x = g \circ \mathtt{u}_X \circ s^X_x = \mathtt{u}_Y \circ g \circ s^X_x = \mathtt{u}_Y \circ s^Y_{g(x)} \circ g = \hat{s}^Y_{g(x)} \circ \widetilde{\mathcal{G}}g,$$

so $\widetilde{\mathcal{G}}g$ is a rack homomorphism from $\widetilde{\mathcal{G}}(R_1)$ to $\widetilde{\mathcal{G}}(R_2)$. $\square$

Having defined $\mathcal{F}$ and $\mathcal{G}$, we are now ready to prove the main theorem of this section.

Theorem 5.5. *The functors $\mathcal{F}$ and $\mathcal{G}$ are isomorphisms of algebraic theories* $\mathsf{Rack} \cong \mathsf{GLQ}$*, and they preserve mediality.*

Proof. We will show that $\mathcal{F}$ and $\mathcal{G}$ are isomorphisms of categories. From there, it is straightforward to verify that $\mathcal{F}$ and $\mathcal{G}$ preserve finite products, making them isomorphisms of algebraic theories. In particular, since $\mathcal{F}$ and $\mathcal{G}$ fix morphisms (as set maps), the final claim of the theorem follows easily from combining Proposition 2.11 with Definition 4.1.

To show that $\mathcal{F}$ and $\mathcal{G}$ are isomorphisms of categories, we only need to show that $\mathcal{GF}$ and $\mathcal{FG}$ fix the objects in the appropriate categories; $\mathcal{GF}$ and $\mathcal{FG}$ clearly fix the morphisms. To that end, let (X, s) be a rack. To see that $\mathcal{GF}(X, s) = (X, s)$, note that $\mathcal{GF}(X, s) = (X, \hat{\tilde{s}})$, where

$$\hat{\tilde{s}}_x = \tilde{\mathtt{u}} \circ \tilde{s}_x = \tilde{\mathtt{u}} \circ \tilde{\mathtt{u}}^{-1} \circ s_x = s_x$$

for all $x \in X$. That is, $\hat{\tilde{s}} = s$, so $\mathcal{GF} = 1_{\mathsf{Rack}}$, as desired. Next, let Q be a GL-quandle. By Proposition 2.12, Q has the form $(X, s, \mathtt{u}, \mathtt{u}^{-1})$. Write $\mathcal{FG}(Q) = (X, \tilde{\hat{s}}, \tilde{\mathtt{u}}, \tilde{\mathtt{u}}^{-1})$; we need to show that $\mathcal{FG}(Q) = Q$. To that end, fix $x \in X$. Since (X, s) is a quandle, we have

$$\tilde{\mathtt{u}}(x) = \hat{s}_x(x) = (\mathtt{u} \circ s_x)(x) = \mathtt{u}(x),$$

so $\tilde{\mathtt{u}} = \mathtt{u}$ and, by the uniqueness of two-sided inverses, $\tilde{\mathtt{u}}^{-1} = \mathtt{u}^{-1}$. Therefore,

$$\tilde{\hat{s}}_x = \tilde{\mathtt{u}}^{-1} \circ \hat{s}_x = \mathtt{u}^{-1} \circ \hat{s}_x = \mathtt{u}^{-1} \circ \mathtt{u} \circ s_x = s_x,$$

so $\tilde{\hat{s}} = s$. This shows that $\mathcal{GF} = 1_{\mathsf{GLQ}}$, completing the proof. $\square$

6. Further questions

In conclusion, we propose questions for future research on GL-racks, listed roughly in the order of their corresponding sections in this article. Let Λ be an oriented Legendrian link, and let R_1 and R_2 be GL-racks.

(1) Can GL-racks distinguish between any more of the conjecturally nonequivalent Legendrian knots listed in [3, 8]?
(2) Following our discussion at the end of Subsection 4.1, do there exist a smooth knot type L, two Legendrian representatives Λ_1 and Λ_2 of L, and a medial GL-rack M such that $\mathrm{Col}(\Lambda_1, M) = \mathrm{Col}(\Lambda_2, M)$ but $\mathrm{Hom}_{\mathsf{GLR}}(\Lambda_1, M) \ncong \mathrm{Hom}_{\mathsf{GLR}}(\Lambda_2, M)$ in $\mathsf{GLR}^{\mathrm{med}}$? In the spirit of Corollary 3.11, do there exist such Λ_1 and Λ_2 that share the same classical invariants?
(3) Let $\mathbb{F}$ be a field, and let M be a medial GL-rack. In 2023, Elhamdadi et al. [13, Theorems 4.2 and 5.1] properly enhanced medial quandle-valued invariants of smooth links using $\mathbb{F}$-algebra homomorphisms between quandle rings and colorings of smooth links by idempotents of quandle rings. Do similar proper enhancements of $\mathrm{Hom}_{\mathsf{GLR}}(\Lambda, M)$ exist?
(4) Are $\mathrm{Col}(\Lambda, R_1 \otimes R_2)$, $\mathrm{Col}(\Lambda, R_1 \otimes_{\mathrm{med}} R_2)$, $\mathrm{Col}(\Lambda, R_1 \times R_2)$ $\mathrm{Col}(\Lambda, R_1)$, and $\mathrm{Col}(\Lambda, R_2)$ related to each other?
(5) In light of Karmakar et al.'s [22, cf. 23] GL-rack cohomology theory, are the cohomologies of $R_1 \otimes R_2$, $R_1 \otimes_{\mathrm{med}} R_2$, $R_1 \times R_2$, and R_1 and R_2 related? Do Künneth formulas exist?

(6) Theorem 4.6 implies that $\mathsf{GLR}^{\mathrm{med}}$ enriches over itself; see, e.g., [24, Section 1.6]. Can we use enriched category theory to better understand the structure of medial GL-racks or create stronger invariants of Legendrian links?
(7) The categorical product, i.e., the Cartesian product $\times$, defines a Cartesian monoidal structure on GLR. Thus, we can consider group objects in GLR. How are these group objects characterized, and do they admit stronger invariants of Legendrian links?
(8) The tensor product $\otimes$, medial tensor product $\otimes_{\mathrm{med}}$, and Cartesian product $\times$ all define symmetric monoidal structures on GLR and $\mathsf{GLR}^{\mathrm{med}}$. Thus, we can consider the categories of monoid objects in GLR or $\mathsf{GLR}^{\mathrm{med}}$ with respect to $\otimes$, $\otimes_{\mathrm{med}}$, or $\times$. Can we better characterize these monoid objects, and do they admit stronger invariants of Legendrian links?
(9) Adding to the previous question, we can also consider left-module and right-module objects over monoid objects in GLR and $\mathsf{GLR}^{\mathrm{med}}$, which enjoy extra structure when $\mathsf{GLR}^{\mathrm{med}}$ is equipped with its symmetric monoidal closed structure from Theorem 4.6; see [4, Definition 4.1.7]. Can we better characterize these module objects, and do they admit stronger invariants of Legendrian links? Are they related to the Beck modules over GL-racks introduced by Karmakar et al. [23]?
(10) Can the categorical results in Section 4 and the algorithms in Appendix A be extended to the categories of 4-Legendrian racks and 4-Legendrian biracks introduced by Kimura [26]?

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Department of Mathematics, Yale University, New Haven, Connecticut 06511
Email address: `luc.ta@yale.edu`

Appendix A. Exhaustive search algorithms

The remainder of this article focuses on computational results and approaches to studying and applying GL-racks. In this section, we enumerate GL-racks, medial GL-racks, GL-quandles, and medial GL-quandles of orders $n \leq 7$ up to isomorphism and describe the algorithms we used to do so. An implementation of these algorithms in `GAP` [17] and the raw data we collected are available at the following GitHub repository: *(Link coming soon!)*

A.1. **Enumeration of small GL-racks.** In Table 1, we enumerate the number of GL-racks, medial GL-racks, GL-quandles, and medial GL-quandles of orders $n \leq 7$ up to isomorphism. For comparison, we also list the corresponding numbers for classical racks and quandles. We obtained the numbers $g(n)$ from Algorithm 2 and $g^m(n)$, $g_q(n)$, and $g_q^m(n)$ from Algorithm 3. Meanwhile, the numbers $r(n)$, $r^m(n)$, $r_q(n)$, and $r_q^m(n)$ were originally computed by McCarron [31] in 2010, Vojtěchovský and Yang [40] in 2019, Henderson et al. [19] in 2006, and Jedlička et al. [20] in 2015, respectively. It appears that each of $g(n)$, $g^m(n)$, $g_q(n)$, and $g_q^m(n)$ in Table 1 grows exponentially and at a much faster rate than its counterpart for classical racks.

n	0	1	2	3	4	5	6	7
$g(n)$	1	1	4	13	62	308	2132	17268
$g^m(n)$	1	1	4	13	61	298	2087	16941
$g_q(n)$	1	1	2	6	19	74	353	2080
$g_q^m(n)$	1	1	2	6	18	68	329	1965
$r(n)$	1	1	2	6	19	74	353	2080
$r^m(n)$	1	1	2	6	18	68	329	1965
$r_q(n)$	1	1	1	3	7	22	73	298
$r_q^m(n)$	1	1	1	3	6	18	58	251

Table 1. The number of GL-racks $g(n)$, medial GL-racks $g^m(n)$, GL-quandles $g_q(n)$, and medial GL-quandles $g_q^m(n)$ of order n up to isomorphism, compared against the corresponding number of racks $r(n)$, medial racks $r^m(n)$, quandles $r_q(n)$, and medial quandles $r_q^m(n)$.

For explicit representatives of each GL-rack isomorphism class counted in Table 1, see Appendix B for those of orders $2 \leq n \leq 4$ and the data linked above for those of orders $5 \leq n \leq 7$. The unique GL-rack isomorphism classes of orders 0 and 1 correspond to the initial and terminal objects in GLR, respectively.

Note in Table 1 that $g_q(n) = r(n)$ and $g_q^m(n) = r^m(n)$ for all $n \leq 7$. This observation was the original motivation for Theorem 5.5, which generalizes these one-to-one correspondences in a natural way.

A.2. **Classification of small GL-racks.** We now describe the exhaustive search algorithms in `GAP` [17] that we used to compute these isomorphism classes. We build upon the work of Vojtěchovský and Yang [40] in 2019, who compiled a library of representatives of all isomorphism classes of racks of orders $n \leq 11$ [39] in 2018. In what follows, let $\mathcal{R}_n$ denote Vojtěchovský and Yang's list of racks of order n, and let S_n denote the symmetric group on n letters. We will also denote any GL-rack $(X, s, \mathtt{u}, \mathtt{d})$ as a list $[(X, s), \mathtt{u}, \mathtt{d}]$ containing the elements (X, s), $\mathtt{u}$, and $\mathtt{d}$.

Algorithm 1: $\texttt{isGLR}((X, s), \texttt{u}, \texttt{d})$ verifies whether $(\texttt{u}, \texttt{d}) \in S_n \times S_n$ defines a GL-structure on a given finite rack (X, s), given the condition that $\texttt{ud} = \texttt{du}$.

Data: Rack (X, s) with $X = \{1, \ldots, n\}$ and bijections $\texttt{u}, \texttt{d} \in S_n$ such that $\texttt{ud} = \texttt{du}$
Result: Whether $[(X, s), \texttt{u}, \texttt{d}]$ is a GL-rack
begin
  foreach *x in X* **do**
    if $(\texttt{ud} \circ s_x)(x) \neq x$ **then** return false;
    if $\texttt{u} \circ s_x \neq s_x \circ \texttt{u}$ *or* $\texttt{d} \circ s_x \neq s_x \circ \texttt{d}$ **then** return false;
    if $s_x \neq s_{\texttt{u}(x)}$ *or* $s_x \neq s_{\texttt{d}(x)}$ **then** return false;
  return true;

Algorithm 1, called $\texttt{isGLR}((X, s), \texttt{u}, \texttt{d})$, tests whether two maps $\texttt{u}, \texttt{d} : X \to X$ define a GL-structure on a rack (X, s) with $X = \{1, \ldots, n\}$, given the necessary conditions that $\texttt{u}, \texttt{d} \in S_n$ and $\texttt{ud} = \texttt{du}$; see Proposition 2.11. The test is simply a verification of the three GL-rack axioms.

Algorithm 2: Classification of GL-racks of a given order $1 \leq n \leq 11$ up to isomorphism.

Data: List $\mathcal{R}_n$ of racks with underlying set $X = \{1, \ldots, n\}$ from the library of Vojtěchovský and Yang [39] with $1 \leq n \leq 11$
Result: List $\mathsf{isoClasses}$ of all isomorphism classes of GL-racks of order n with no repeats
begin
  $\mathsf{isoClasses} \leftarrow \emptyset$;
  foreach *pair* $((X, s), \texttt{u})$ *in* $\mathcal{R}_n \times S_n$ **do**
    foreach *bijection* $\texttt{d}$ *in the centralizer* $C_{S_n}(\texttt{u})$ **do**
      if $\texttt{isGLR}((X, s), \texttt{u}, \texttt{d})$ **then**
        $\mathsf{seen} \leftarrow$ false;
        foreach *GL-rack* $[(X, s'), \texttt{u}_2, \texttt{d}_2]$ *in* $\mathsf{isoClasses}$ *such that* $s' = s$ **do**
          if $\texttt{u}$ *and* $\texttt{u}_2$ *or* $\texttt{d}$ *and* $\texttt{d}_2$ *are not conjugate in* S_n **then** continue;
          foreach *bijection* φ *in* S_n **do**
            if *φ defines a GL-rack homomorphism* $[(X, s), \texttt{u}, \texttt{d}] \to [(X, s'), \texttt{u}_2, \texttt{d}_2]$ **then**
              $\mathsf{seen} \leftarrow$ true;
              break;
          if seen **then** break;
        if seen *is false* **then** Add($\mathsf{isoClasses}$, $[(X, s), \texttt{u}, \texttt{d}]$);

Algorithm 2 uses $\mathcal{R}_n$ to create a list $\mathsf{IsoClasses}$ with exactly one representative of each GL-rack isomorphism class with underlying set is $X = \{1, \ldots, n\}$. First, the algorithm runs $\texttt{isGLR}((X, s), \texttt{u}, \texttt{d})$ for each rack $(X, s) \in \mathcal{R}_n$ and each pair of bijections $\texttt{u}, \texttt{d} \in S_n$ such that $\texttt{ud} = \texttt{du}$; cf. Proposition 2.11. If $\texttt{isGLR}((X, s), \texttt{u}, \texttt{d})$ returns $\texttt{true}$, then to ensure $\mathsf{IsoClasses}$ contains no isomorphic elements, the algorithm searches for a bijection $\varphi \in S_n$ that defines a GL-rack homomorphism (hence an isomorphism) from $[(X, s), \texttt{u}, \texttt{d}]$ to any previously encountered GL-rack $[(X, s'), \texttt{u}_2, \texttt{d}_2] \in \mathsf{IsoClasses}$. This is true only if $\varphi \circ \texttt{u} \circ \varphi^{-1} = \texttt{u}_2$ and $\varphi \circ \texttt{d} \circ \varphi^{-1} = \texttt{d}_2$, so it suffices to only consider GL-racks

$[(X, s'), \mathtt{u}_2, \mathtt{d}_2]$ such that $\mathtt{u}$ and $\mathtt{u}_2$ are conjugate in S_n and $\mathtt{d}$ and $\mathtt{d}_2$ are conjugate in S_n. Moreover, any GL-rack isomorphism is also an isomorphism of the underlying racks. Since the racks in $\mathcal{R}_n$ are pairwise nonisomorphic, it suffices to only consider GL-racks $[(X, s'), \mathtt{u}_2, \mathtt{d}_2]$ satisfying $s = s'$.

On the author's personal computer, running our implementation of Algorithm 2 in `GAP` with $n = 7$ took 11269047 milliseconds, or slightly over three hours. We did not complete the computation for $n = 8$, which would appear to require longer than a month on the author's computer.

Algorithm 3: Classification of medial GL-racks, all GL-quandles, and medial GL-quandles of order n up to isomorphism, given a classification of GL-racks of order n.

```
Data: List isoClasses of isomorphism classes of GL-racks with underlying set
      X = {1, ..., n} returned by Algorithm 2
Result: Lists M_n, Q_n, and I_n of isomorphism classes of medial GL-racks, all
        GL-quandles, and medial GL-quandles with underlying set X, respectively
begin
    M_n, Q_n, I_n ← ∅;
    foreach GL-rack R = [(X, s), u, d] in isoClasses do
        isMedial ← true;
        foreach ordered triple (x, y, z) in X^3 do
            if s_{s_x(z)} ∘ s_y ≠ s_{s_x(y)} ∘ s_z then
                isMedial ← false;
                break;
        if isMedial then Add(M_n, R);
        if d = u^{-1} then
            Add(Q_n, R);
            if isMedial then Add(I_n, R);
```

Finally, Algorithm 3 tests whether or not each GL-rack in the output of Algorithm 2 is medial or a GL-quandle. To test for mediality, the algorithm simply verifies Definition 4.1. By Proposition 2.12, to test whether a GL-rack $[(X, s), \mathtt{u}, \mathtt{d}]$ is a GL-quandle, it suffices to verify whether $\mathtt{d} = \mathtt{u}^{-1}$.

On the author's personal computer, running our implementation of Algorithm 3 in `GAP` with $n = 7$ took about 2718094 milliseconds, or about 45 minutes.

A.3. **Exhaustive searches for GL-rack coloring numbers.** We now describe Algorithm 4, which computes all colorings of the GL-rack of an oriented Legendrian link Λ by each GL-rack $R = [(Y, s), \mathtt{u}, \mathtt{d}]$ in the list isoClasses computed by Algorithm 2. Before running Algorithm 4, the user must input a presentation of $\mathcal{G}(\Lambda)$ in terms of crossing relations between elements of $\mathrm{FGLR}(X_\Lambda)$; see Subsection 3.2 for examples of such presentations. By Lemma 3.9, it suffices for the algorithm to search for all valid solutions in R to the inputted crossing relations.

In particular, if $R = [(Y, s), \mathtt{u}, \mathtt{d}]$ is a GL-rack of order $n \leq 11$, then $\mathrm{Col}(\Lambda, R)$ is simply the number of lists in solutions produced by Algorithm 4 whose first three list elements are s, $\mathtt{u}$, and $\mathtt{d}$. To distinguish between two oriented Legendrian links Λ_1 and Λ_2, it suffices to run Algorithm 4 twice, once inputting $\mathcal{G}(\Lambda_1)$ and again inputting $\mathcal{G}(\Lambda_2)$, and find a GL-rack R in isoClasses such that $\mathrm{Col}(\Lambda_1, R) \neq \mathrm{Col}(\Lambda_2, R)$. For example, running Algorithm 4 with $n = 3$ and $n = 2$ is how we determined which GL-racks and homomorphisms to use in our proofs of Theorems 3.10 and 3.13, respectively. Running the algorithm with $n = 5$ also gave us the following example.

Algorithm 4: Computation of colorings of an oriented Legendrian link Λ by GL-racks of a given order $1 \leq n \leq 11$ as computed by Algorithm 2.

Data: List isoClasses of isomorphism classes of GL-racks with underlying set $Y = \{1, \dots, n\}$ from Algorithm 2 and a presentation of $\mathcal{G}(\Lambda) = [(X, s^\Lambda), \mathtt{u}^\Lambda, \mathtt{d}^\Lambda)]$

Result: List solutions whose elements are lists $[s, \mathtt{u}, \mathtt{d}, \mathbf{y}]$ such that the mapping $x_i \mapsto y_i$ defines a GL-rack homomorphism $\mathcal{G}(\Lambda) \to [(Y, s), \mathtt{u}, \mathtt{d}]$

begin
- $m \leftarrow |X_\Lambda|$;
- solutions $\leftarrow \emptyset$;
- **foreach** *GL-rack* $[(Y, s), \mathtt{u}, \mathtt{d}]$ *in* isoClasses **do**
  - **foreach** *ordered m-tuple* $\mathbf{y} \leftarrow (y_1, \dots, y_m)$ *in* Y^m **do**
    - **if** *all crossing relations are satisfied after replacing each* $x_i \in X_\Lambda$, s^Λ, $\mathtt{u}^\Lambda$, *and* $\mathtt{d}^\Lambda$ *with* y_i, s, $\mathtt{u}$, *and* $\mathtt{d}$, *respectively* **then** Add(solutions, $[s, \mathtt{u}, \mathtt{d}, \mathbf{y}]$);

```
Finding all colorings of knot 1 by GL-rack 222 of 308…
Finding all colorings of knot 2 by GL-rack 222 of 308…
[ [ (1,2,3,4,5), … (1,2,3,4,5) ], (1,3,5,2,4), (1,3,5,2,4), 1, 2, 3, 5, 1, 4, 5 ]
[ [ (1,2,3,4,5), … (1,2,3,4,5) ], (1,3,5,2,4), (1,3,5,2,4), 2, 3, 4, 1, 2, 5, 1 ]
[ [ (1,2,3,4,5), … (1,2,3,4,5) ], (1,3,5,2,4), (1,3,5,2,4), 3, 4, 5, 2, 3, 1, 2 ]
[ [ (1,2,3,4,5), … (1,2,3,4,5) ], (1,3,5,2,4), (1,3,5,2,4), 4, 5, 1, 3, 4, 2, 3 ]
[ [ (1,2,3,4,5), … (1,2,3,4,5) ], (1,3,5,2,4), (1,3,5,2,4), 5, 1, 2, 4, 5, 3, 4 ]
Number of colorings of knot 1 by GL-rack 222 of 308: 0
Number of colorings of knot 2 by GL-rack 222 of 308: 5
Since their GL-rack coloring numbers are distinct, these knots are not Legendrian
isotopic.
```

FIGURE 11. Excerpt from the output of our `GAP` implementation of Algorithm 4 with $n = 5$. Here, knots 1 and 2 are the Legendrian knots in Figure 9, while GL-rack 222 of 308 is the Legendrian rack defined in Example A.1.

Example A.1. In this example, we use Algorithm 4 to once again distinguish the Legendrian 6_2 knots Λ_1 and Λ_2 on the left and right of Figure 9, respectively. This time, we use the 222nd GL-rack R of order 5 listed in the data linked above, which is a Legendrian rack as introduced by Ceniceros et al. [6] in 2021. Let $Y := \{1, 2, 3, 4, 5\}$. In cycle notation, define $\sigma, f \in S_5$ by $\sigma := (12345)$ and $f := (13524)$. In the notation of Example 2.8, let $R := (Y, \sigma, f, f)_p$.

We input the relations of $\mathcal{G}(\Lambda_1)$ in (1) and then those of $\mathcal{G}(\Lambda_2)$ in (2) into our `GAP` implentation of Algorithm 4. After running the program with $n = 5$, the program outputs the text in Figure 11 upon reaching isoClasses[222] $= R$. The output states that $\mathrm{Col}(\Lambda_1, R) = 0 \neq 5 = \mathrm{Col}(\Lambda_2, R)$, and the images of the generators $(x_1, \dots, x_7)$ of $\mathcal{G}(\Lambda_2)$ under each element of $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda_2), R)$ are given by the orbit of $(1, 2, 3, 5, 1, 4, 5) \in Y^7$ under the action of the subgroup $\langle \sigma \rangle \leq S_5$ on Y^7.

Example A.1 also yields the following analogue of Corollary 3.11 for Legendrian racks.

Proposition A.2. *There exist Legendrian knots sharing the same topological knot type and classical invariants that are distinguished by the Legendrian rack coloring numbers of* [6, Proposition 1].

Appendix B. Tabulation of GL-racks of orders 2, 3, and 4

Tables 2, 3, and 4 tabulate all isomorphism classes of GL-racks having orders 2, 3, and 4, respectively, computed using Algorithm 2. Let X be the set $\{1, \ldots, n\}$, where n is the order of the GL-rack, and let $\mathrm{id} : X \to X$ denote the identity map. In the tables, we write each bijection s_i, $\mathtt{u}$, and $\mathtt{d}$ as either id or a nonidentity element of S_n in cycle notation, with permutations composed from right to left. The number of GL-racks of each order is given by the number of entries in the second column of each table. These entries denote all valid GL-structures $[\mathtt{u}, \mathtt{d}]$ up to isomorphism on the rack (X, s), where s is given by the corresponding entry in the first column. For example, the permutation GL-rack of order 3 with $s_1 = s_2 = s_3 = (123)$ and GL-structure $[\mathtt{u}, \mathtt{d}] = [(132), \mathrm{id}]$, which we used to prove Theorem 3.10, appears as the 11th entry in Table 3.

$[s_1, s_2]$	$[\mathtt{u}, \mathtt{d}]$	GL-quandle?	Medial?
$[\mathrm{id}, \mathrm{id}]$	$[\mathrm{id}, \mathrm{id}]$, $[(12), (12)]$	Yes	Yes
$[(12), (12)]$	$[\mathrm{id}, (12)]$, $[(12), \mathrm{id}]$	No	Yes

Table 2. The four isomorphism classes of GL-racks of order 2.

$[s_1, s_2, s_3]$	$[\mathtt{u}, \mathtt{d}]$	GL-quandle?	Medial?
$[\mathrm{id}, \mathrm{id}, \mathrm{id}]$	$[\mathrm{id}, \mathrm{id}]$, $[(23), (23)]$, $[(132), (123)]$	Yes	Yes
$[\mathrm{id}, (23), (23)]$	$[\mathrm{id}, (23)]$, $[(23), \mathrm{id}]$	No	Yes
$[(23), \mathrm{id}, \mathrm{id}]$	$[\mathrm{id}, \mathrm{id}]$, $[(23), (23)]$	Yes	Yes
$[(23), (23), (23)]$	$[\mathrm{id}, (23)]$, $[(23), \mathrm{id}]$	No	Yes
$[(123), (123), (123)]$	$[\mathrm{id}, (132)]$, $[(132), \mathrm{id}]$, $[(123), (123)]$	No	Yes
$[(23), (13), (12)]$	$[\mathrm{id}, \mathrm{id}]$	Yes	Yes

Table 3. The 13 isomorphism classes of GL-racks of order 3.

$[s_1, s_2, s_3, s_4]$	$[\mathtt{u}, \mathtt{d}]$	GL-quandle?	Medial?
$[\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}]$	$[\mathrm{id}, \mathrm{id}]$, $[(34), (34)]$, $[(243), (234)]$, $[(1432), (1234)]$, $[(14)(23), (14)(23)]$	Yes	Yes
$[\mathrm{id}, (13)(24), \mathrm{id}, (13)(24)]$	$[\mathrm{id}, (24)]$, $[(24), \mathrm{id}]$, $[(13), (13)(24)]$, $[(13)(24), (13)]$	No	Yes
$[(13)(24), (13)(24), (13)(24), (13)(24)]$	$[\mathrm{id}, (13)(24)]$, $[(24), (13)]$, $[(1432)(1432)]$, $[(14)(23), (12)(34)]$, $[(13)(24), \mathrm{id}]$	No	Yes
$[\mathrm{id}, \mathrm{id}, (34), (34)]$	$[\mathrm{id}, (34)]$, $[(34), \mathrm{id}]$, $[(12), (12)(24)$, $[(12)(34), (12)]$	No	Yes
$[\mathrm{id}, (34), \mathrm{id}, \mathrm{id}]$	$[\mathrm{id}, \mathrm{id}]$, $[(34), (34)]$	Yes	Yes
$[\mathrm{id}, (34), (34), (34)]$	$[\mathrm{id}, (34)]$, $[(34), \mathrm{id}]$	No	Yes
$[(34), (34), \mathrm{id}, \mathrm{id}]$	$[\mathrm{id}, \mathrm{id}]$, $[(34), (34)]$, $[(12), (12)]$, $[(12)(34), (12)(34)]$	Yes	Yes
$[(34), (34), (34), (34)]$	$[\mathrm{id}, (34)]$, $[(34), \mathrm{id}]$, $[(12), (12)(34)]$, $[(12)(34), (12)]$	No	Yes
$[\mathrm{id}, (234), (234), (234)]$	$[\mathrm{id}, (243)]$, $[(243), \mathrm{id}]$, $[(234), (234)]$	No	Yes
$[(234), \mathrm{id}, \mathrm{id}, \mathrm{id}]$	$[\mathrm{id}, \mathrm{id}]$, $[(243), (234)]$, $[(234), (243)]$	Yes	Yes

$[(234),(234),(234),(234)]$	$[\mathrm{id},(243)]$, $[(243),\mathrm{id}]$, $[(234),(234)]$	No	Yes
$[(234),(243),(243),(243)]$	$[\mathrm{id},(234)]$, $[(243),(243)]$, $[(234),\mathrm{id}]$	No	Yes
$[(34),(34),(12),(12)]$	$[\mathrm{id},\mathrm{id}]$, $[(34),(34)]$, $[(12)(34),(12)(34)]$	Yes	Yes
$[(34),(34),(12)(34),(12)(34)]$	$[\mathrm{id},(34)]$, $[(34),\mathrm{id}]$, $[(12),(12)(34)]$, $[(12)(34),(12)]$	No	Yes
$[(12),(12),(34),(34)]$	$[\mathrm{id},(12)(34)]$, $[(34),(12)]$, $[(12)(34),\mathrm{id}]$	No	Yes
$[(12),(12),(12)(34),(12)(34)]$	$[\mathrm{id},(12)(34)]$, $[(34),(12)]$, $[(12),(34)]$, $[(12)(34),\mathrm{id}]$	No	Yes
$[(1324),(1324),(1324),(1324)]$	$[\mathrm{id},(1423)]$, $[(1423),\mathrm{id}]$, $[(12)(34),(1324)]$, $[(1324),(12)(34)]$	No	Yes
$[\mathrm{id},(34),(24),(23)]$	$[\mathrm{id},\mathrm{id}]$	Yes	No
$[(234),(143),(124),(132)]$	$[\mathrm{id},\mathrm{id}]$	Yes	Yes

Table 4: The 62 isomorphism classes of GL-racks of order 4.