Why knot? Algebraic coloring invariants of Legendrian knots

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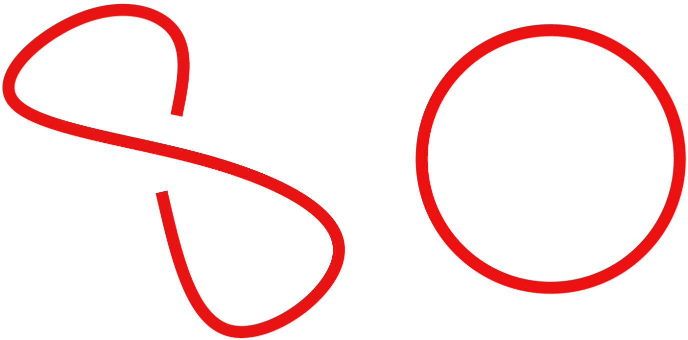
Outline

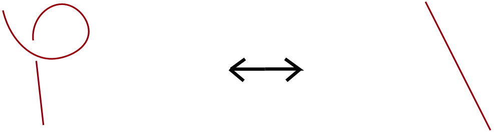
Let's play a game. . . (1/2)



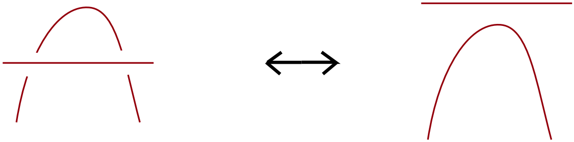
Let's play a game. . . (2/2)

Reidemeister moves

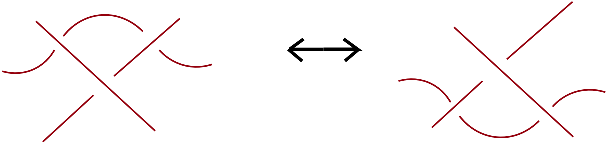
Two knots are equivalent if and only if they're related by the three *Reidemeister moves* (twist, poke, and slide).



(R1)



(R2)



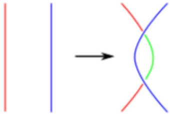
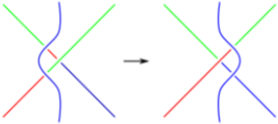
(R3)

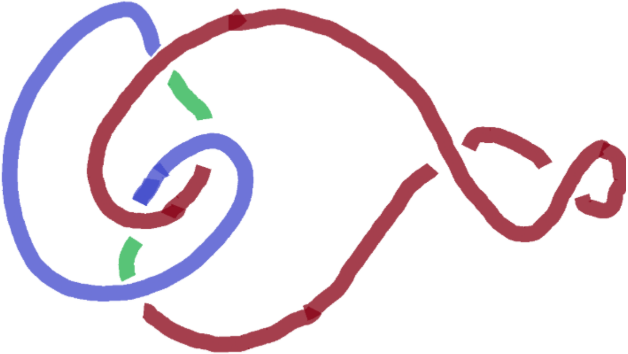
An early way to distinguish knots

A knot is *tricolorable* if we can color its strands in the following way:

- We use exactly 3 colors (no more, no fewer).
- The three strands at each crossing either (a) all share the same color or (b) all have different colors.

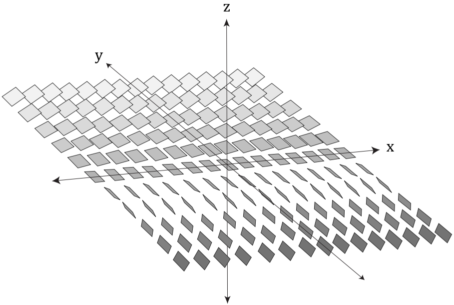
Why is tricolorability a “knot invariant”?

Reidemeister Move I is tricolorable.	Reidemeister Move II is tricolorable.	Reidemeister Move III is tricolorable.
		



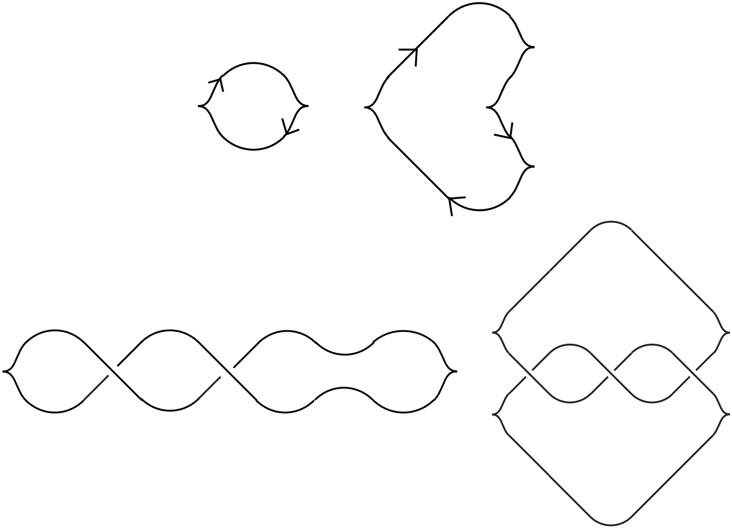
The standard contact structure

Legendrian knots are knots with a certain restriction on their shape, determined by something called the *standard contact structure*:



- When $y = 0$, the planes are flat.
- When moving in the positive y-direction, the slopes grow more negative.
- When moving in the negative y-direction, the slopes grow more positive.

Legendrian knot diagrams

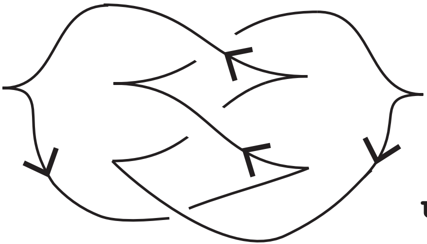


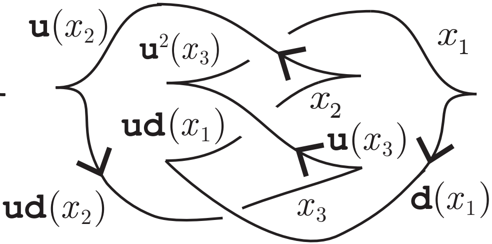
Outline

GL-racks: Gadgets that color Legendrian knots

- We can use a gadget called a **GL-rack** to color Legendrian knots (hence distinguishing them).
- GL-racks are expansions of *racks* and *quandles*, which are gadgets that color non-Legendrian knots.
- We can assign a GL-rack $\mathcal{G}(\Lambda)$ to every Legendrian knot Λ.
  - Colorings of Λ must satisfy certain criteria imposed by $\mathcal{G}(\Lambda)$.
- To distinguish Legendrian knots, it suffices to distinguish their GL-racks. And that's when we get to "color"!

Example of $\mathcal{G}(\Lambda)$

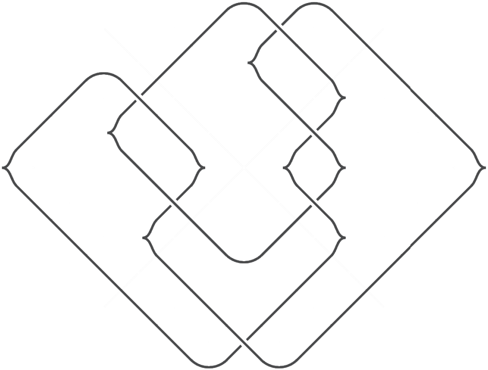


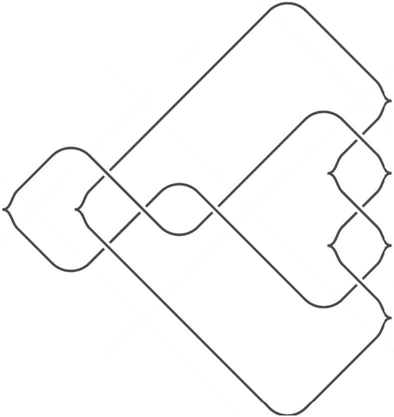


Outline

Solving an old conjecture (1/2)

- Like before, we construct GL-racks $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$.
- These GL-racks satisfy certain criteria determined by cusps, strands, and crossings.





Solving an old conjecture (2/2)

Theorem

Λ_1 *and* Λ_2 *are distinct.*

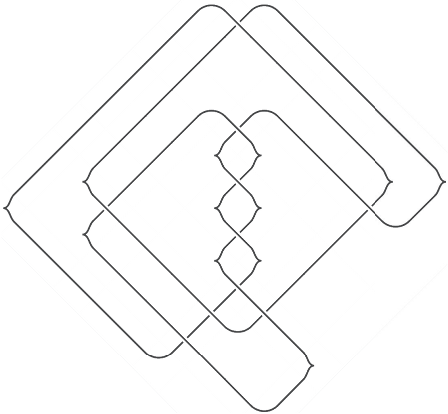
Proof sketch.

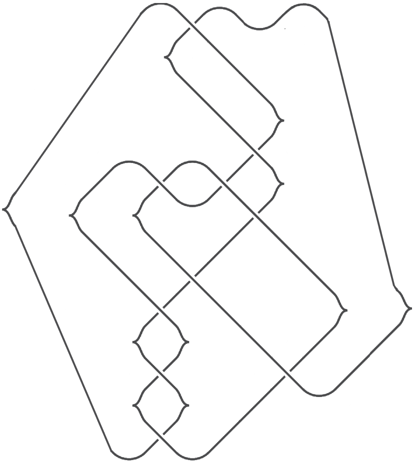
1. Using a computer search, we look for ways to color Λ_1 and Λ_2 with GL-racks.
2. The computer finds a GL-rack L satisfying the following:
   - There are 3 ways to color Λ_1 by L under the criteria of $\mathcal{G}(\Lambda_1)$.
   - However, there aren't any ways to color Λ_2 by L like this!
3. Since their coloring numbers are different, Λ_1 and Λ_2 are distinct.

$\square$

Solving recent conjectures (1/2)

Similarly, we can distinguish these two Legendrian knots. . .

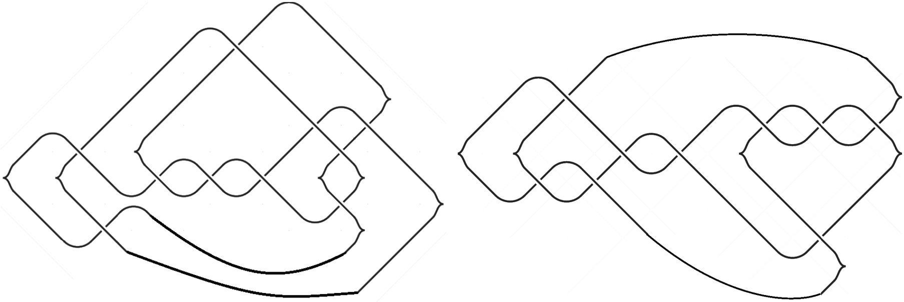




Solving recent conjectures (2/2)

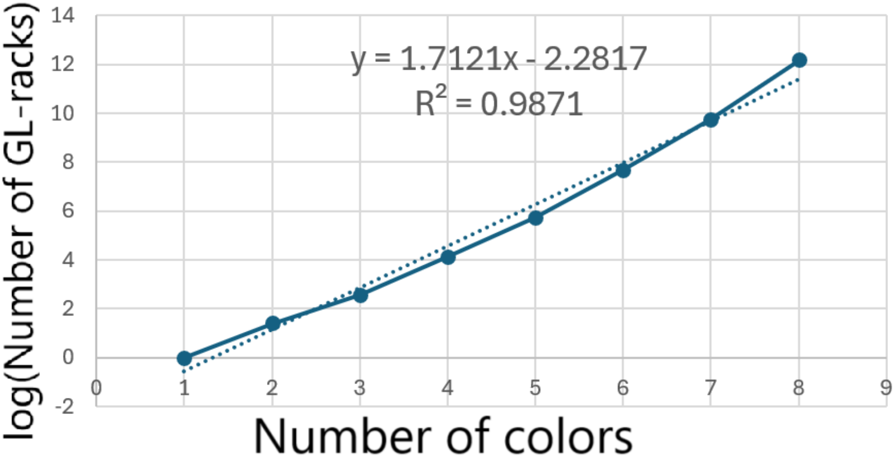
. . . and these two Legendrian knots.

This solves several open questions posed in 2023!



Outline

Enumeration (1/2)



Enumeration (2/2)

- *Quandles* are a specific type of rack. They were invented 10 years before racks were.
- We can count the algebraic gadgets we've seen today based on how many "colors" they contain:

Number of colors	0	1	2	3	4	5	6	7	8
GL-racks	1	1	4	13	62	308	2132	17268	189373
GL-quandles	1	1	2	6	19	74	353	2080	16023
Racks	1	1	2	6	19	74	353	2080	16023
Quandles	1	1	1	3	7	22	73	298	1581

Outline

Algebraic results: A bird's-eye view

A brief overview of some even more abstract findings...

- There's a natural one-to-one correspondence between racks and GL-quandles.
  - Cool because racks and GL-quandles were developed independently!
- We can use *symmetries* to...
  - classify several infinite families of GL-racks.
  - better understand how different GL-racks work.
  - understand how GL-racks interact with one another.
- Certain GL-racks allow for other methods of distinguishing Legendrian knots.
  - These methods may be more effective than coloring in certain cases.
  - They also have a natural relationship to algebraic gadgets called *tensor products*.

Outline

Thank you!

arXiv > math > arXiv:2504.12671

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[Submitted on 17 Apr 2025]

Generalized Legendrian racks: Classification, tensors, and knot coloring invariants

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