# APPLICATIONS AND TABULATION OF THE CATEGORY OF GENERALIZED LEGENDRIAN RACKS

LỰC TA

Abstract. GL-racks are a type of nonassociative algebraic structure recently introduced to distinguish between Legendrian links in $\mathbb{R}^3$. To help develop the theory of GL-racks, we apply them to confirm several conjectures about Legendrian links, study the category of GL-racks and their homomorphisms, and classify GL-racks of orders $n \leq 7$ up to isomorphism. In particular, we give short algebraic proofs distinguishing unstabilized Legendrian representatives of knot types $6_2$ and $8_{10}$ using GL-rack coloring numbers. We also show that the GL-rack of a Legendrian link $\Lambda$ recovers $\pi_1(\mathbb{R}^3 \setminus \Lambda)$. Then, we endow hom-sets of medial GL-racks with a medial GL-rack structure, propose a medial GL-rack-valued invariant of Legendrian links, and exhibit a symmetric monoidal closed structure on the category of medial GL-racks. From an additional functorial construction, we obtain a group-theoretic sufficient condition for any two GL-racks or quandles to be isomorphic. Finally, we provide links to algorithms that tabulate medial and nonmedial GL-racks and GL-quandles of small orders up to isomorphism and compare colorings of oriented Legendrian links by GL-racks.

## 1. Introduction

In 1980, Joyce [22] introduced algebraic structures called *quandles* as a means of capturing the topological structure of knots, links, and symmetric spaces. Since then, quandles and slightly more general algebraic objects called *racks* have enjoyed significant study as link invariants among geometric topologists and in their own right among quantum algebraists. Recently, various authors have constructed generalizations of racks and quandles to study *Legendrian links* in contact geometry (e.g., [6, 24, 25, 29]). Defining homomorphisms of these algebraic structures yields categories like $\mathsf{GLR}$, whose objects are called *GL-racks*, *generalized Legendrian racks*, or *bi-Legendrian racks.*

In this paper, we study the category $\mathsf{GLR}$ and its applications. This helps us understand the structure of GL-racks and detect their isomorphisms, which can otherwise be cumbersome due to their nonassociativity. This also allows us to prove several conjectures about Legendrian links and generalize several properties of quandles to GL-racks. Along with a classification of GL-racks of orders $n \leq 7$ up to isomorphism in Appendix A, the main results of this paper are as follows.

**Theorem 1.1.** *The two Legendrian knots with* $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$ *and underlying smooth knot type* $6_2$ *given in* [9] *are distinguishable by GL-rack coloring numbers.*

**Theorem 1.2.** *The two Legendrian knots with* $(\mathrm{tb}, \mathrm{rot}) = (-8, 3)$ *and underlying smooth knot type* $8_{10}$ *given in* [3] *are distinguishable by GL-rack coloring numbers.*

**Theorem 1.3.** *Let* $\Lambda \subset \mathbb{R}^3$ *be an oriented Legendrian link, let* $\mathcal{G}(\Lambda)$ *be the GL-rack of* $\Lambda$ *as defined in Definition 3.2, and let* $\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda))$ *be its enveloping group as defined in Definition 2.16. Then there exists a group isomorphism*

$$\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda)) \cong \pi_1(\mathbb{R}^3 \setminus \Lambda).$$

*Mathematics Subject Classification* 2020. Primary 57K12; Secondary 18B99, 57K10, 57K33.

*Key words and phrases.* Bi-Legendrian rack, enumeration, GL-rack, link coloring, Legendrian knot, quandle, tabulation.

**Theorem 1.4.** *Let $R_1$ and $R_2$ be GL-racks. If $R_2$ is medial, then* $\mathrm{Hom}_{\mathsf{GLR}}(R_1, R_2)$ *has a canonical medial GL-rack structure. Moreover, if $R_2$ is a GL-quandle, then so is* $\mathrm{Hom}_{\mathsf{GLR}}(R_1, R_2)$.

**Theorem 1.5.** *The full subcategory of* $\mathsf{GLR}$ *whose objects are medial GL-racks is symmetric monoidal closed. In particular, it is self-enriched.*

Inspired by the homogeneous representations of GL-racks constructed in [24, Section 5], we also construct a category $\mathsf{GrpTup}$ satisfying the following.

**Theorem 1.6.** *There exists an essentially surjective functor* $\mathcal{F} : \mathsf{GrpTup} \twoheadrightarrow \mathsf{GLR}$. *This functor induces a group-theoretic sufficient condition for any two GL-racks or quandles to be isomorphic.*

In Section 2, we give an overview of the questions in Legendrian knot theory motivating the study of GL-racks and quandles. We proceed by defining these algebraic structures abstractly and introducing related groups, categories, and functors in the literature.

In Section 3, we discuss how to assign a GL-rack to an oriented Legendrian link, give several worked examples, and discuss related invariants of Legendrian links. Then, we prove Theorems 1.1, 1.2, and 1.3, which we state as Theorem 3.8, Theorem 3.11, and Corollary 3.13, respectively. Our approach to the first result offers a simpler algebraic alternative to Dynnikov and Prasolov's proof in [11, Proposition 2.3] and gives a positive answer to a question posed in [25, Section 4], which we state as Corollary 3.9. The second result confirms a conjecture of Bhattacharyya et al. in [3], and the third result confirms an empirical observation from the original version of [24, Remark 8.2].

In Section 4, we define *medial* or *abelian* GL-racks and tensor products of GL-racks. Using Theorem 1.4, which we state as Theorem 4.3, we propose a medial GL-rack-valued invariant of Legendrian links with suggestions for future research. Then, we prove Theorem 1.5, which we state as Theorem 4.5 and Corollary 4.6. These results extend Crans and Nelson's analogous results for medial quandles in [10, Theorems 3 and 12] to medial GL-racks.

In Section 5, we define $\mathsf{GrpTup}$ using objects constructed from collections of left cosets of groups. Karmakar et al. originally employed these objects in [24, Theorem 5.2] to produce a homogeneous representation of any GL-rack. Then, we prove Theorem 1.6, the first part of which we state as Theorem 5.1 and the second part of which we discuss afterward.

In Appendix A, we describe an algorithm in `GAP` [19] that can tabulate all isomorphism classes of GL-racks of a given order $1 \leq n \leq 11$, building on the work of Vojtěchovský and Yang in [41]. After presenting the data for all $n \leq 4$, we provide a link to our code and the data we were able to compute for all $n \leq 7$. The data includes explicit representatives of each isomorphism class and also classifies medial GL-racks, GL-quandles, and medial GL-quandles of orders $n \leq 7$ up to isomorphism. We also provide an algorithm that computes all homomorphisms from the GL-rack of a given oriented Legendrian link to all GL-racks of a given order $1 \leq n \leq 7$.

**Acknowledgments.** This article was written in partial fulfillment of the senior requirements for the mathematics major at Yale. I would like to thank Sam Raskin for serving as my senior thesis advisor even while on academic leave and as I dove into unfamiliar subjects. I also thank Patrick Devlin, Miki Havlíčková, Kati Hubley, Matthew King, Andrew Neitzke, and Carol Rutschman (Mojo) for their mentorship and support. I especially want to thank Samantha Pezzimenti for introducing me to mathematics research and communication, Legendrian knot theory, and the UnKnot V conference in 2024. Finally, I want to thank Jose Ceniceros and Peyton Wood for helpful discussions about quandles, knot coloring, and categorification between talks at UnKnot V.

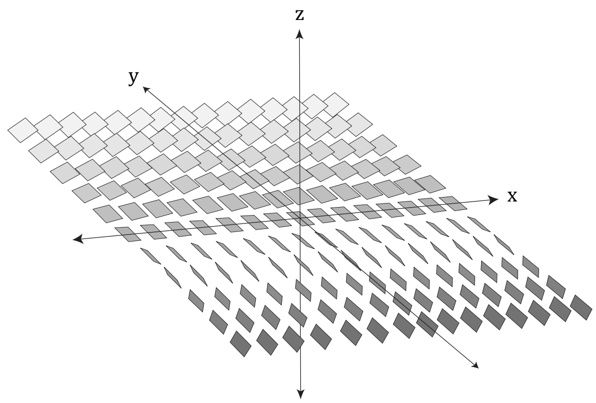


FIGURE 1. The standard contact structure $\xi_{\text{std}}$ on $\mathbb{R}^3$. Taken from [28].

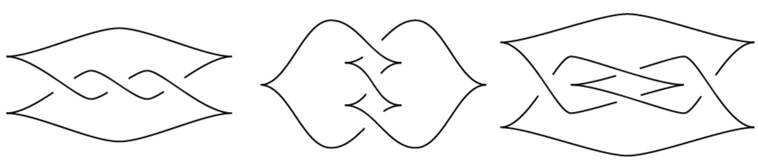

FIGURE 2. Unoriented front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot. Adapted from [16].

## 2. Preliminaries

In this section, we discuss motivations and establish pertinent definitions for the study of GL-racks. In particular, we discuss the crossing and cusp relations afforded by Legendrian Reidemeister moves, which motivate the axioms of GL-racks. After stating some preliminary results, we discuss several functors relating to the category of GL-racks.

### 2.1. **Motivations: Legendrian knots and links.**

In this subsection, we discuss how the development of Legendrian link invariants in contact geometry motivates the study of GL-racks. (For an accessible introduction to Legendrian knot theory, we refer the reader to [36]. For a more detailed survey of the field, we refer the reader to [16].)

**Definition 2.1.** A *knot* is a smooth embedding of the circle $S^1$ into $\mathbb{R}^3$, and a *link* is a disjoint union of a finite number of knots. A link $\Lambda$ is called *Legendrian* if it lies everywhere tangent to the *standard contact structure* $\xi_{\text{std}} := \ker(dz - y\,dx)$ on $\mathbb{R}^3$, which is depicted in Figure 1. That is, $T_x\Lambda \in \xi_{\text{std}}$ for all $x \in \Lambda$, where $T_x\Lambda$ denotes the tangent space of $\Lambda$ at $x$. A *front projection* or *front diagram* $D(\Lambda)$ is the projection of $\Lambda$ to the $xz$-plane. Finally, two Legendrian links are called *equivalent* or *Legendrian isotopic* if there exists a smooth homotopy between them that preserves the condition of being Legendrian at every stage.

When discussing Legendrian links alongside general links, we will call the latter *smooth links* or *topological links*. Also, we will denote the underlying smooth link of a Legendrian link $\Lambda$ by $L$.

Central to contact geometry is the question of how to distinguish Legendrian links up to Legendrian isotopy. To this end, knot theorists typically study Legendrian links $\Lambda$ through their front projections, which follow several restrictions thanks to the tangency condition on $\Lambda$. For one,

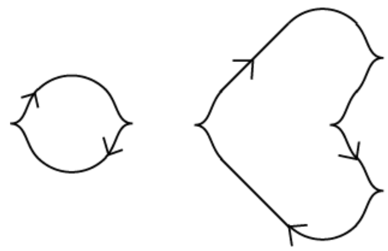

FIGURE 3. Front projections of distinct oriented Legendrian unknots. Adapted from [35].

at every crossing in $D(\Lambda)$, the strand with the more negative slope is always the overstrand. For two, $D(\Lambda)$ has cusps in place of vertical tangencies. Note that the numbers of crossings and cusps in a Legendrian front projection are finite due to smoothness. Moreover, $D(\Lambda)$ can be viewed as a *link diagram* of $L$, denoted by $D(L)$, by "ignoring" all cusps. For example, Figure 2 depicts unoriented front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot, and Figure 3 depicts oriented front projections of two distinct Legendrian unknots.

In fact, one can use the tangency condition to show that the geometric structure of an oriented Legendrian link $\Lambda$ can be recovered entirely from its front projection (see [16]). For example, two invariants called the *Thurston-Bennequin number* and *rotation number* of $\Lambda$, denoted respectively by $\mathrm{tb}(\Lambda)$ and $\mathrm{rot}(\Lambda)$, can be defined as the integers

$$\mathrm{tb}(\Lambda) = P - N - \frac{1}{2}(D + U), \quad \mathrm{rot}(\Lambda) = \frac{1}{2}(D - U),$$

where $P$, $N$, $D$, and $U$ are the numbers of positively oriented crossings, negatively oriented crossings, downward-oriented cusps, and upward-oriented cusps in $D(\Lambda)$, respectively. It is well-known that two Legendrian links are Legendrian isotopic only if their Thurston-Bennequin and rotation numbers are equal. Indeed, a celebrated theorem of Świątkowski in 1992 offers a method of comparing Legendrian links using only their front projections.

**Proposition 2.2.** [38, Theorem B] *Two Legendrian links are Legendrian isotopic if and only if their front projections are related by a finite sequence of planar isotopies and the three* Legendrian Reidemeister moves *depicted in Figures 5-7.*

The axioms of GL-racks are motivated by the *crossing* and *cusp relations* induced between strands of an oriented Legendrian link modulo the relations afforded by the Legendrian Reidemeister moves. (Note that planar isotopies do not affect crossings or cusps, so they do not induce any such relations.) In Figure 4, (i) and (ii) depict crossing relations between strands in a Legendrian front projection, and (iii) and (iv) depict cusp relations. Note that `u` and `d` correspond to the relations induced by upward- and downward-oriented cusps, respectively. Figures 5-7 depict the crossing and cusp relations in one possible orientation of each of the three Legendrian Reidemeister moves. For a complete list of all possible orientations and their induced crossing and cusp relations, we refer the reader to [25, Figures 6-8].

**Example 2.3.** Let $\Lambda_1$ and $\Lambda_2$ be the oriented Legendrian unknots depicted on the left and right of Figure 3, respectively. Although $\Lambda_1$ and $\Lambda_2$ share the same underlying smooth knot type, they are not Legendrian isotopic because $\mathrm{tb}(\Lambda_1) = -1 \neq -2 = \mathrm{tb}(\Lambda_2)$ and $\mathrm{rot}(\Lambda_1) = 0 \neq 1 = \mathrm{rot}(\Lambda_2)$. Proposition 2.2 asserts that the two front projections in Figure 3 cannot be related by any sequence of Legendrian Reidemeister moves.

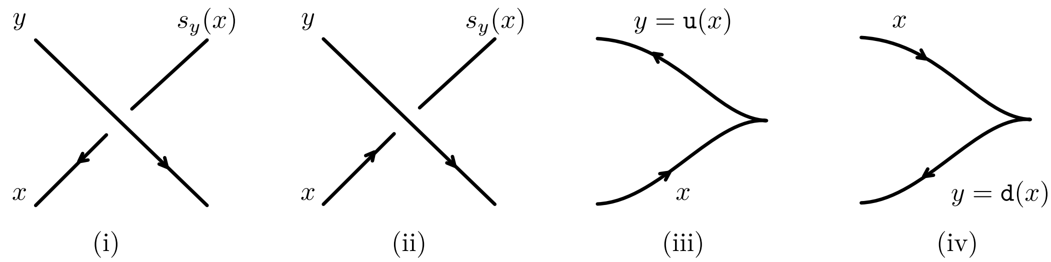


FIGURE 4. Crossing and cusp relations. Adapted from [24].

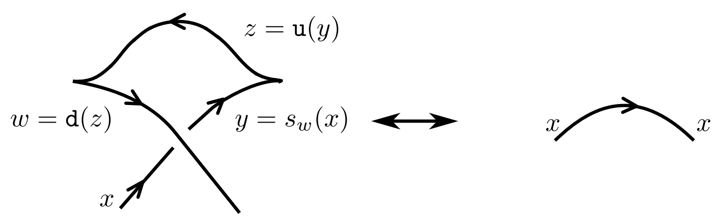


FIGURE 5. Crossing and cusp relations in one possible orientation of the first Legendrian Reidemeister move. Adapted from [24].

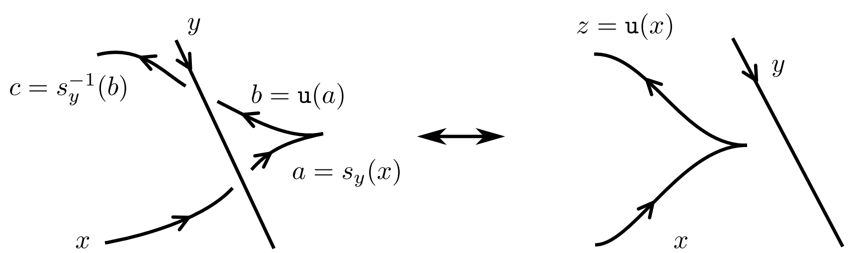


FIGURE 6. Crossing and cusp relations in one possible orientation of the second Legendrian Reidemeister move. Adapted from [24].

There are in fact infinitely many examples of distinct Legendrian links having the same underlying smooth link type, making distinguishing between Legendrian links significantly more difficult than distinguishing between smooth links. This has motivated the development of numerous *invariants* of Legendrian links, called so because they are constructed to be invariant under Legendrian isotopy. Examples include the Thurston-Bennequin and rotation numbers [16], the Chekanov-Eliashberg differential graded algebra and associated polynomial invariants [16,33], various (co)homology theories (see [18] for a list), and the mosaic number [28,35].

GL-racks and quandles have also been used to define algebro-combinatorial and cohomological invariants of both Legendrian links and smooth links. These include fundamental quandles and their Legendrian analogues [6,22,24], $R$-coloring numbers [6,24,25], cocycle invariants [7,26], and state-sum invariants [4,15], many of which have elegant categorifications and enhancements (e.g., [5,7,8,14]). These invariants motivate the study of GL-racks as a category.

In Subsection 3.1, we discuss how to assign a GL-rack to any oriented Legendrian link $\Lambda$ using the cusp and crossing relations in $D(\Lambda)$. This assignment is independent of the choice of front projection of $\Lambda$, making it an invariant of Legendrian links [24, Theorem 4.3].

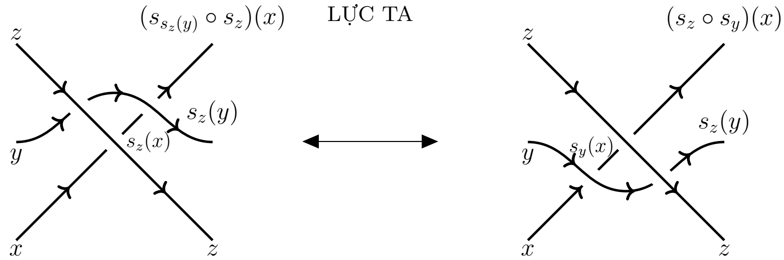


FIGURE 7. Crossing relations in one possible orientation of the third Legendrian Reidemeister move. Adapted from [6].

2.2. **GL-racks and quandles.** In this subsection, we define GL-racks and quandles abstractly by translating the crossing and cusp relations in Subsection 2.1 into the language of *rack symmetries*. For any set $X$, we denote the set of all bijections from $X$ to itself by $\mathrm{Sym}(X)$.

Although racks and quandles are often defined as sets $X$ endowed with binary operations $\triangleright : X \times X \to X$ and $\triangleright^{-1} : X \times X \to X$, they may also be characterized in terms of symmetries $s_x \in \mathrm{Sym}(X)$ assigned to each element $x \in X$, cf. [13, Section 2; 22, Definition 1.1; 39, Definition 2.7]. One may translate between the two conventions via the formulas $s_x(y) = y \triangleright x$ and $s_x^{-1}(y) = y \triangleright^{-1} x$. In this article, we adopt the definitions using symmetries due to their convenience for categorical proofs, and we have rewritten all crossing relations in Figures 4-7 in this notation. We refer the reader to [32] for an accessible introduction to quandle theory and [12] for a more comprehensive survey of modern algebraic research on racks and quandles.

The rack and quandle axioms encapsulate the crossing relations depicted in Figures 5-7.

**Definition 2.4.** Let $X$ be a set, and let $s : X \to \mathrm{Sym}(X)$ be a map defined by $x \mapsto s_x$. We call the pair $(X, s)$ a *rack* or a *wrack* if, for all $x, y \in X$, we have $s_x \circ s_y = s_{s_x(y)} \circ s_x$. We say that $s_x$ is the *symmetry at* $x$, and we say that $|X|$ is the *order* of $(X, s)$. If in addition $s_x(x) = x$ for all $x \in X$, then we say that $(X, s)$ is a *quandle*. Finally, if $Y \subset X$ and $s_y(z) \in Y$ for all $y, z \in Y$, then we say that $(Y, s|_Y)$ is a *subrack* of $(X, s)$.

**Example 2.5.** [39, Definition 2.11] Let $\Omega$ be a union of conjugacy classes in a group $G$, and define $s : \Omega \to \mathrm{Sym}(\Omega)$ by $\alpha \mapsto s_\alpha := [\omega \mapsto \alpha\omega\alpha^{-1}]$. Then $(\Omega, s)$ is called a *conjugation quandle*, and we denote it by $\mathrm{Conj}(\Omega)$.

**Example 2.6.** Let $X$ be any set, and fix $\sigma \in \mathrm{Sym}(X)$. Define $s : X \to \mathrm{Sym}(X)$ by $x \mapsto \sigma$, so that $s_x(y) = \sigma(y)$ for all $x, y \in X$. Then $(X, s)$ is called a *constant rack*, and we denote it by $(X, \sigma)_c$.

The axioms of *GL-racks* encode not only the crossing relations but also the cusp relations given by the Legendrian Reidemeister moves in Figures 5-7. GL-racks were introduced independently in [23] and [25] to generalize the *Legendrian racks* introduced in [6] and [29]. Once again, we translate the original definition into the language of rack symmetries.

**Definition 2.7.** [23, Definition 3.1] A *GL-rack*, also called a *generalized Legendrian rack* or a *bi-Legendrian rack*, is a quadruple $(X, s, \mathtt{u}, \mathtt{d})$ in which $(X, s)$ is a rack and $\mathtt{u}, \mathtt{d} : X \to X$ are maps such that the following hold for all $x \in X$:

(L1) $(\mathtt{ud} \circ s_x)(x) = x = (\mathtt{du} \circ s_x)(x)$.
(L2) $\mathtt{u} \circ s_x = s_x \circ \mathtt{u}$ and $\mathtt{d} \circ s_x = s_x \circ \mathtt{d}$.
(L3) $s_{\mathtt{u}(x)} = s_x = s_{\mathtt{d}(x)}$.

We call the pair $(\mathtt{u}, \mathtt{d})$ a *GL-structure* on $(Q, s)$. If in addition $(X, s)$ is a quandle, we say that $(X, s, \mathtt{u}, \mathtt{d})$ is a *GL-quandle*.

**Example 2.8.** [25, Example 3.6] Let $G$ be a group, let $z \in Z(G)$, and define $f : G \to G$ by $g \mapsto zg$. Then $(\mathrm{Conj}(G), f, f^{-1})$ is a GL-quandle.

**Example 2.9.** [25, Example 3.7] Let $(X,\sigma)_c$ be a constant rack, and let $\mathtt{u},\mathtt{d}: X \to X$ be maps. Then $(\mathtt{u},\mathtt{d})$ defines a GL-structure on $(X,\sigma)_c$ if and only if $\mathtt{ud} = \sigma^{-1} = \mathtt{du}$. In this case, we say that $((X,\sigma)_c,\mathtt{u},\mathtt{d})$ is a *constant GL-rack*, and we denote it by $(X,\sigma,\mathtt{u},\mathtt{d})_c$.

**Example 2.10.** [24, Example 3.4] Any GL-rack of the form $(X,s,\mathrm{id}_X,\mathrm{id}_X)$ is called a *trivial GL-rack*. In particular, any quandle $(Q,s)$ can be identified with the trivial GL-rack $(Q,s,\mathrm{id}_Q,\mathrm{id}_Q)$, cf. Lemma 2.14. In other words, GL-racks generalize quandles.

We define homomorphisms of these algebraic structures as follows.

**Definition 2.11.** Let $(X,s)$ and $(Y,t)$ be racks. A map $\varphi: X \to Y$ is called a *rack homomorphism* if $\varphi \circ s_x = t_{\varphi(x)} \circ \varphi$ for all $x \in X$. If in addition $(\mathtt{u}_1,\mathtt{d}_2)$ and $(\mathtt{u}_2,\mathtt{d}_2)$ are GL-structures on $(X,s)$ and $(Y,t)$, we say that a $\varphi$ is also a *GL-rack homomorphism* if $\varphi\circ\mathtt{u}_1 = \mathtt{u}_2\circ\varphi$ and $\varphi\circ\mathtt{d}_1 = \mathtt{d}_2\circ\varphi$. A *(GL-)rack isomorphism* is simply a bijective (GL-)rack homomorphism. If $R$ is a GL-rack, we denote its group of GL-rack automorphisms by $\mathrm{Aut}_{\mathsf{GLR}}(R)$.

Evidently, we have the following; the final sentence is from [24, Proposition 3.2].

**Proposition 2.12.** *Let $(X,s)$ be a rack with maps $\mathtt{u},\mathtt{d}: X\to X$ satisfying axioms (L1) and (L3) of Definition 2.7. Then $R := (X,s,\mathtt{u},\mathtt{d})$ is a GL-rack if and only if $\mathtt{u}$ and $\mathtt{d}$ are endomorphisms of the underlying rack $(X,s)$. In this case, we actually have $\mathtt{u},\mathtt{d}\in\mathrm{Aut}_{\mathsf{GLR}}(R)$.*

Axiom (L1) immediately yields the following.

**Proposition 2.13.** *Let $(X,s,\mathtt{u},\mathtt{d})$ be a GL-rack. Then the underlying rack $(X,s)$ is a quandle if and only if $\mathtt{ud} = \mathrm{id}_X = \mathtt{du}$, that is, $\mathtt{d} = \mathtt{u}^{-1}$ as GL-rack automorphisms.*

2.3. **Functors of interest in the literature.** In this subsection, we define several categories and functors appearing in the literature on GL-racks, quandles, and their relationships with groups.

We begin by defining several categories. Let $\mathsf{Set}$ and $\mathsf{Grp}$ be the categories of sets with functions and groups with group homomorphisms, respectively. Let $\mathsf{Rack}$ be the category of racks with rack homomorphisms, let $\mathsf{Qnd}$ be the full subcategory of $\mathsf{Rack}$ whose objects are quandles, and let $\mathsf{GLR}$ be the category of GL-racks with GL-rack homomorphisms. By Example 2.10 and Proposition 2.13, we have the following.

**Lemma 2.14.** *The correspondence $(Q,s)\mapsto(Q,s,\mathrm{id}_Q,\mathrm{id}_Q)$ defines a canonical isomorphism from $\mathsf{Qnd}$ to the full subcategory of $\mathsf{GLR}$ whose objects are trivial GL-racks.*

In the sense of universal algebra, $\mathsf{GLR}$ is an equational algebraic category, so it is complete and cocomplete (see [1, Corollary 1.2, Theorem 4.5]). Thus, we can express GL-racks in terms of generators and relations using quotients of *free GL-racks*, which we define as follows.

**Definition 2.15.** [24, Section 4] Let $X$ be a set. We define the *free GL-rack on $X$*, denoted by $\mathrm{FGLR}(X)$, as follows. If $X=\emptyset$, let $\mathrm{FGLR}(X)$ be the trivial GL-rack with one element. Else, define the *universe of words generated by $X$* to be the set $W(X)$ such that $X\subset W(X)$ and $s_y(x)$, $s_y^{-1}(x)$, $\mathtt{u}(x)$, $\mathtt{d}(x)\in W(X)$ for all $x,y\in W(X)$. Let $F(X)$ be the set of equivalence classes of elements of $W(X)$ modulo the equivalence relation generated by the following relations for all $x,y,z\in W(X)$:

(1) $s_y^{-1}(s_y(x))y \sim x \sim s_y(s_y^{-1}(x))$.
(2) $s_z(s_y(x)) \sim s_{s_z(y)}(s_z(x))$.
(3) $\mathtt{u}(\mathtt{d}(s_x(x))) \sim x \sim \mathtt{d}(\mathtt{u}(s_x(x)))$.
(4) $\mathtt{u}(s_y(x)) \sim s_y(\mathtt{u}(x))$ and $\mathtt{d}(s_y(x)) \sim s_y(\mathtt{d}(x))$.
(5) $s_{\mathtt{u}(y)}(x) \sim s_y(x)$ and $s_{\mathtt{d}(y)}(x) \sim s_y(x)$.

Thus, we have maps $s : F(X) \to \mathrm{Sym}(F(X))$ defined by $x \mapsto s_x := [y \mapsto s_x(y)]$ and $\mathtt{u}, \mathtt{d} : F(X) \to F(X)$ defined by $x \mapsto \mathtt{u}(x)$ and $x \mapsto \mathtt{d}(x)$. We define $\mathrm{FGLR}(X)$ to be the GL-rack $(F(X), s, \mathtt{u}, \mathtt{d})$. The *free quandle on* $X$ is defined similarly; in the sense of Lemma 2.14, it is simply $\mathrm{FGLR}(X)$ modulo the relations $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in W(X)$.

To rephrase [24, Proposition 4.2], the functor $\mathsf{Set} \to \mathsf{GLR}$ defined by $X \mapsto \mathrm{FGLR}(X)$ is left adjoint to the forgetful functor $\mathsf{GLR} \to \mathsf{Set}$, as one might expect.

Another functor of interest in Section 3 assigns an *enveloping group* to any GL-rack.

**Definition 2.16.** [23, Section 8] Given a GL-rack $R = (X, s, \mathtt{u}, \mathtt{d})$, its *enveloping group* is

$$\mathrm{Env}_{\mathsf{GLR}}(R) := \langle e_x,\ x \in X \mid e_{s_x(y)} = e_x^{-1} e_y e_x,\ e_{\mathtt{u}(x)} = e_x,\ e_{\mathtt{d}(x)} = e_x,\ x, y \in X \rangle.$$

By taking $\mathtt{u} = \mathrm{id}_X = \mathtt{d}$, we can also define the enveloping group of a quandle $(Q, s)$ to be

$$\mathrm{Env}_{\mathsf{Qnd}}(Q, s) := \langle e_x,\ x \in Q \mid e_{s_x(y)} = e_x^{-1} e_y e_x,\ x, y \in Q \rangle.$$

The end of Example 3.6 in Subsection 3.2 computes the enveloping groups of the GL-rack and fundamental quandle of a Legendrian $(2, -q)$ torus knot as defined in Subsection 3.1.

The functor $\mathrm{Env}_{\mathsf{GLR}} : \mathsf{GLR} \to \mathsf{Grp}$ has a right adjoint that results from taking $\Omega = G$ in Example 2.5, as specified in the following result adapted from [23, Proposition 8.4].

**Proposition 2.17.** *There exists a functor* $\mathrm{Env}_{\mathsf{GLR}} : \mathsf{GLR} \to \mathsf{Grp}$ *that sends any GL-rack to its enveloping group and sends any GL-rack homomorphism* $\psi : (X, s, \mathtt{u}_1, \mathtt{d}_1) \to (Y, t, \mathtt{u}_1, \mathtt{d}_2)$ *to the group homomorphism* $\widetilde{\psi} : \mathrm{Env}_{\mathsf{GLR}}(X, s, \mathtt{u}_1, \mathtt{d}_1) \to \mathrm{Env}_{\mathsf{GLR}}(Y, t, \mathtt{u}_1, \mathtt{d}_2)$ *defined by* $e_x \mapsto e_{\psi(x)}$ *for all* $x \in X$. *Also,* $\mathrm{Env}_{\mathsf{GLR}}$ *is left adjoint to a functor sending any group* $G$ *to the GL-rack* $(\mathrm{Conj}(G), \mathrm{id}_G, \mathrm{id}_G)$, *which is isomorphic to the quandle* $\mathrm{Conj}(G)$ *in the sense of Lemma 2.14.*

Thus, some authors denote the enveloping group of a GL-rack or quandle $R$ by $\mathrm{Adconj}(R)$ or $\mathrm{As}(R)$ and call it the *associated group of* $R$, cf. [22, Section 6; 34, Definition 2.19].

## 3. On rack-theoretic invariants of Legendrian links

In this section, we begin by defining the GL-rack of an oriented Legendrian link $\Lambda$ and the fundamental quandle of its underlying smooth link $L$, both of which are invariant under Legendrian isotopy. After a few worked examples, we give short algebraic proofs of several conjectures relating to Legendrian links and their invariants.

### 3.1. The GL-rack of a Legendrian link.

In this subsection, we discuss how to assign a GL-rack to a Legendrian link in a way invariant under Legendrian isotopy. We begin with several definitions.

**Definition 3.1.** Given a front projection $D(\Lambda)$ of an oriented Legendrian link $\Lambda$, define a *cusped strand* of $D(\Lambda)$ to be a connected segment in $D(\Lambda)$ that either starts and ends at a crossing or (in the case that $D(\Lambda)$ contains no crossings) ends where it started. Also, define an *uncusped strand* of $D(\Lambda)$ to be a maximal (with respect to inclusion) connected subset of a cusped strand of $D(\Lambda)$ that both starts and ends at either a crossing or a cusp.

**Definition 3.2.** [24, Section 4] Let $\Lambda$ be an oriented Legendrian link with front projection $D(\Lambda)$, and let $X_\Lambda$ be a set in one-to-one correspondence with the cusped strands of $D(\Lambda)$. At each cusp, label the neighboring uncusped strands using the cusp relations in Figure 4. Then, at each crossing, impose the corresponding crossing relation between uncusped strands in Figure 4 on $\mathrm{FGLR}(X_\Lambda)$. The *GL-rack of* $\Lambda$, denoted by $\mathcal{G}(\Lambda)$, is defined to be the set of equivalence classes of elements of $\mathrm{FGLR}(X_\Lambda)$ modulo the equivalence relation generated by these relations. If $L$ is a smooth link with link diagram $D(L)$, then we define the *fundamental quandle of* $L$, denoted by $\mathcal{Q}(L)$, in a similar manner. However, we use the free quandle on $X_\Lambda$ in place of $\mathrm{FGLR}(X_\Lambda)$, and we do not employ any cusp relations.

For several examples of how to compute the GL-rack of a Legendrian link, see Subsection 3.2.

The assignment of $\mathcal{G}(\Lambda)$ to $\Lambda$ (resp. $\mathcal{Q}(L)$ to $L$) is independent of the choice of front projection $D(\Lambda)$ (resp. link diagram $D(L)$), as captured in the following result of Karmakar et al.

**Proposition 3.3.** [24, Theorem 4.3] *If two oriented Legendrian front projections are related by a finite sequence of Legendrian Reidemeister moves, then their induced GL-racks are isomorphic. Hence, the GL-rack of a Legendrian link is invariant under Legendrian isotopy.*

This is a consequence of Proposition 2.2 and the fact that the GL-rack axioms capture the crossing and cusp relations induced by the Legendrian Reidemeister moves. In turn, Proposition 3.3 implies that the *GL-rack coloring number* of $\Lambda$ with respect to a fixed GL-rack, as defined below, is invariant under Legendrian isotopy (see [6, 24, 25]).

**Definition 3.4.** Let $R$ be a GL-rack. The *$R$-coloring number* of an oriented Legendrian link $\Lambda$, denoted by $\mathrm{Col}(\Lambda, R)$, is defined to be the cardinality of the hom-set $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), R)$.

Kulkarni and Prathamesh in [29, Main Theorem 2], Kimura in [25, Theorem 4.1], and Karmakar et al. in [24, Theorem 4.6] each used $R$-coloring numbers to distinguish between infinitely many Legendrian unknots. Karmakar et al. also used $R$-coloring numbers to distinguish between infinitely many Legendrian trefoils in [24, Theorem 4.7], and Ceniceros et al. in [6, Example 16] used them to distinguish between connected sums of Legendrian trefoils. That said, there also exist distinct Legendrian knots having GL-racks (see [26, Subsection 4.2]), so neither $\mathcal{G}(\Lambda)$ nor $R$-coloring numbers are complete Legendrian knot invariants. Nevertheless, we will use the latter in Subsection 3.2 to distinguish between unstabilized Legendrian representatives of knot types $6_2$ and $8_{10}$, which cannot be done using the graded ruling invariant or linearized contact homology.

Given $\Lambda$, note that imposing the equivalence relation $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$ onto $\mathcal{G}(\Lambda)$ yields a quandle in the sense of Lemma 2.14. Geometrically, this amounts to "ignoring" all cusps in $D(\Lambda)$ and viewing $D(\Lambda)$ only as a diagram of the underlying smooth link $L$. This recovers $\mathcal{Q}(L)$ from $\mathcal{G}(\Lambda)$, yielding the following observation.

**Lemma 3.5.** [26, Remark 23] *Let $\Lambda$ be an oriented Legendrian link, and let $L$ be its underlying smooth link. After imposing an equivalence relation onto $\mathcal{G}(\Lambda)$ defined by $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$, the resulting GL-rack is canonically isomorphic to $\mathcal{Q}(L)$ in the sense of Lemma 2.14.*

3.2. **Example calculations and applications.** In this section, we give several examples of how to compute the GL-rack of an oriented Legendrian knot. This allows us to give relatively brief algebraic proofs of conjectures in [9] and [3] about Legendrian $6_2$ and $8_{10}$ knots, respectively.

**Example 3.6.** Let $q \geq 3$ be an odd integer, let $L$ be a $(2, -q)$-torus knot, and let $\Lambda$ be the Legendrian representative of $L$ having maximal Thurston-Bennequin and rotation numbers. (By [17, Theorem 4.3], $\Lambda$ is the unique such Legendrian representative up to Legendrian isotopy.) In this example, we compute $\mathcal{G}(\Lambda)$, $\mathcal{Q}(L)$, $\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda))$, and $\mathrm{Env}_{\mathsf{Qnd}}(\mathcal{Q}(L))$ using the front projection $D(\Lambda)$ in Figure 8. Starting at any crossing (which, in Figure 8, we arbitrarily choose to be the bottommost crossing), traverse $D(\Lambda)$ along its depicted orientation. By recording the induced cusp and crossing relations using Figure 4, we compute that $\mathcal{G}(\Lambda)$ is the free GL-rack on the set $X_\Lambda = \{x_1, \dots, x_q\}$ modulo the crossing relations

$$s_{\mathtt{u}(x_1)}(x_q) = \mathtt{ud}(x_2),\ s_{\mathtt{d}(x_q)}(x_{q-1}) = \mathtt{ud}(x_1),\ \text{and } s_{\mathtt{d}(x_{i-1})}(x_{i-2}) = \mathtt{d}^2(x_i) \text{ for all } 3 \leq i \leq q.$$

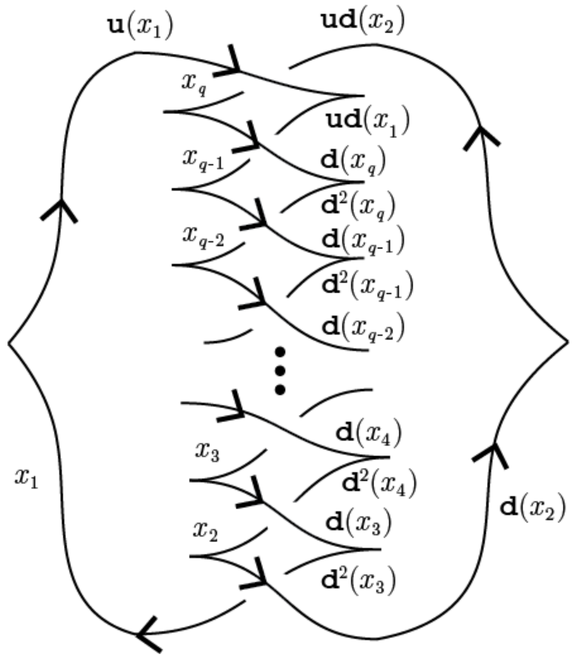


FIGURE 8. Front projection $D(\Lambda)$ and induced cusp relations of the Legendrian $(2,-q)$-torus knot $\Lambda$ with maximal Thurston-Bennequin and rotation numbers.

Using axiom (L3) of a GL-rack, we can simplify these crossing relations to

$$s_{x_1}(x_q) = \mathtt{ud}(x_2),\ s_{x_q}(x_{q-1}) = \mathtt{ud}(x_1),\ \text{and } s_{x_{i-1}}(x_{i-2}) = \mathtt{d}^2(x_i) \text{ for all } 3 \le i \le q.$$

Now that we have a presentation of $\mathcal{G}(\Lambda)$, let us compute $\mathcal{Q}(L)$. To do this, we could traverse $D(\Lambda)$ again while ignoring all cusps and only considering crossing relations. In view of Lemma 3.5, we could equivalently impose the relations $\mathtt{u}(x_i) = x_i = \mathtt{d}(x_i)$ for all $1 \le i \le q$ onto $\mathcal{G}(\Lambda)$. Either method shows that $\mathcal{Q}(L)$ is the free quandle on $X_\Lambda$ modulo the crossing relations

$$s_{x_1}(x_q) = x_2,\ s_{x_q}(x_{q-1}) = x_1,\ \text{and } s_{x_{i-1}}(x_{i-2}) = x_i \text{ for all } 3 \le i \le q.$$

Indeed, if we invert each symmetry in the relations of $\mathcal{Q}(L)$, then we recover the fundamental quandle of the mirror image of $L$ computed in [2, Remark 3], as predicted by [40, Section 1].

If $q = 3$, then $L$ is a left-handed trefoil, and the crossing relations show that $\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda))$ and $\mathrm{Env}_{\mathsf{Qnd}}(\mathcal{Q}(L))$ are both isomorphic to the group

$$\begin{aligned}&\langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{s_{x_1}(x_3)} = e_{x_1}^{-1} e_{x_3} e_{x_1},\ e_{s_{x_2}(x_1)} = e_{x_2}^{-1} e_{x_1} e_{x_2},\ e_{s_{x_3}(x_2)} = e_{x_3}^{-1} e_{x_2} e_{x_3}\rangle \\ &\quad = \langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{x_1} e_{x_2} = e_{x_3} e_{x_1},\ e_{x_2} e_{x_3} = e_{x_1} e_{x_2},\ e_{x_3} e_{x_1} = e_{x_2} e_{x_3}\rangle .\end{aligned}$$

Note that this is precisely the Wirtinger presentation of the knot group $\pi_1(\mathbb{R}^3 \setminus L) \cong \langle x, y \mid x^2 = y^3\rangle$ of $L$ (see [37, Subsection 4.2.5]). Subsection 3.3 will generalize this observation.

**Example 3.7.** Let $\Lambda_1$ and $\Lambda_2$ be the oriented Legendrian knots on the left and right of Figure 9, respectively. In this example, we compute $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$ in preparation for a proof that $\Lambda_1$ and $\Lambda_2$ are not Legendrian isotopic. We note from [9] that $\Lambda_1$ and $\Lambda_2$ are both Legendrian representatives of the topological knot $6_2$ that satisfy $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$.

Let us begin with $\Lambda_1$. Traverse $D(\Lambda_1)$ using its given orientation while labeling all uncusped strands as in Figure 4. By writing down the induced crossing relations as in Figure 4, we find that

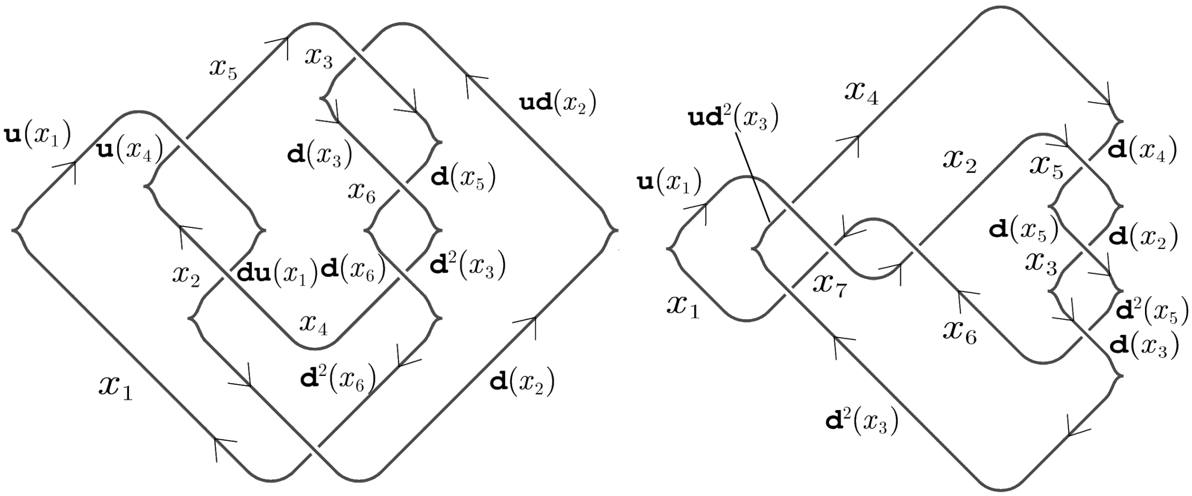


FIGURE 9. Front projections of the two Legendrian representatives of the topological knot $6_2$ with $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$ given in [9]. Created using [27], cf. [28].

$\mathcal{G}(\Lambda_1)$ is the free GL-rack on the set $X_{\Lambda_1} = \{x_1, \ldots, x_6\}$ modulo the following crossing relations:

$$\mathcal{G}(\Lambda_1)\begin{cases} s_{\mathtt{u}(x_1)}(\mathtt{u}(x_4)) = x_5 \iff s_{x_1}(\mathtt{u}(x_4)) = x_5, \\ s_{x_4}(\mathtt{du}(x_1)) = x_2, \\ s_{\mathtt{d}(x_2)}(x_1) = \mathtt{d}^2(x_6) \iff s_{x_2}(x_1) = \mathtt{d}^2(x_6), \\ s_{x_5}(x_3) = \mathtt{ud}(x_2), \\ s_{\mathtt{d}(x_3)}(x_6) = \mathtt{d}(x_5) \iff s_{x_3}(x_6) = \mathtt{d}(x_5), \\ s_{\mathtt{d}(x_6)}(x_4) = \mathtt{d}^2(x_3) \iff s_{x_6}(x_4) = \mathtt{d}^2(x_3). \end{cases} \tag{1}$$

Note that we have simplified the first, third, fifth, and sixth relations using GL-rack axiom (L3).

Similarly, we compute $\mathcal{G}(\Lambda_2)$ to be the free GL-rack on the set $X_{\Lambda_1} = \{x_1, \ldots, x_7\}$ modulo the following crossing relations:

$$\mathcal{G}(\Lambda_2)\begin{cases} s_{x_1}(\mathtt{ud}^2(x_3)) = x_4, & s_{x_5}(x_3) = \mathtt{d}(x_2), \\ s_{x_1}(x_6) = x_7, & s_{x_3}(x_6) = \mathtt{d}^2(x_5), \\ s_{x_6}(x_2) = \mathtt{u}(x_1), & s_{x_3}(x_7) = x_1, \\ s_{x_2}(x_5) = \mathtt{d}(x_4). \end{cases} \tag{2}$$

We can use these calculations to prove a conjecture of Chongchitmate and Ng in [9] that $\Lambda_1$ and $\Lambda_2$ in the previous example are not Legendrian isotopic. In [11, Proposition 2.3], Dynnikov and Prasolov proved this conjecture using impressive topological and combinatorial machinery. At the time of writing, theirs is the only proof of which we are aware, as $\Lambda_1$ and $\Lambda_2$ cannot be distinguished using the graded ruling invariant or linearized contact homology, cf. [9]. However, $R$-coloring numbers offer a simpler algebraic alternative. In the following, we denote the symmetric group on $n$ letters by $S_n$.

**Theorem 3.8.** *The two oriented Legendrian knots in Figure 9 are not Legendrian isotopic; in fact, they are distinguishable using coloring numbers with respect to a constant GL-rack of order 3.*

*Proof.* As before, let $\Lambda_1$ and $\Lambda_2$ be the oriented Legendrian knots on the left and right of Figure 9, respectively. Let $Y = \{1, 2, 3\}$. In cycle notation, let $\sigma \in S_3$ be the permutation $(123)$. In the notation of Example 2.9, let $R := (Y, \sigma, \sigma^{-1}, \mathrm{id}_Y)_c$, so that $R$ is the eleventh GL-rack in Table 3. We will show that $\mathrm{Col}(\Lambda_2, R) > \mathrm{Col}(\Lambda_1, R)$. To that end, let $A$ denote the underlying set of $\mathcal{G}(\Lambda_2)$ as presented in Example 3.7, and define $\varphi_1, \varphi_2, \varphi_3 : A \to Y$ by the following:

$$\varphi_1(x_i) := \begin{cases} 1 & \text{if } i \in \{1,3,4\}, \\ 2 & \text{if } i \in \{2,6\}, \\ 3 & \text{if } i \in \{5,7\}. \end{cases} \quad \varphi_2(x_i) := \begin{cases} 2 & \text{if } i \in \{1,3,4\}, \\ 3 & \text{if } i \in \{2,6\}, \\ 1 & \text{if } i \in \{5,7\}. \end{cases} \quad \varphi_3(x_i) := \begin{cases} 3 & \text{if } i \in \{1,3,4\}, \\ 1 & \text{if } i \in \{2,6\}, \\ 2 & \text{if } i \in \{5,7\}. \end{cases}$$

Using the relations in (2), it is straightforward to verify that $\varphi_1$, $\varphi_2$, and $\varphi_3$ define GL-rack homomorphisms from $\mathcal{G}(\Lambda_2)$ to $R$. Hence, $\mathrm{Col}(\Lambda_2, R) \geq 3$. (In fact, using a similar method as in the remainder of this proof, one can show that this bound is actually an equality.)

On the other hand, we claim that $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda_1), R) = \emptyset$. Let $B$ denote the underlying set of $\mathcal{G}(\Lambda_1)$ as presented in Example 3.7, and suppose to the contrary that some map $\varphi : B \to Y$ defines a GL-rack homomorphism from $\mathcal{G}(\Lambda_1)$ to $R$ with $\varphi(x_i) = y_i$. Since $\varphi$ is a GL-rack homomorphism, the relations in (1) must hold when we replace each $x_i$ with $y_i$, each $s_{x_i}$ with $\sigma$, each $\mathtt{u}$ with $\sigma^{-1}$, and each $\mathtt{d}$ with $\mathrm{id}_Y$. These substitutions yield the following system of equations in $R$:

$$R \begin{cases} (\sigma \circ \sigma^{-1})(y_4) = y_5 \iff y_4 = y_5, \\ (\sigma \circ \sigma^{-1})(y_1) = y_2 \iff y_1 = y_2, \\ \sigma(y_1) = y_6, \\ \sigma(y_3) = \sigma^{-1}(y_2) \iff y_3 = \sigma(y_2), \\ \sigma(y_6) = y_5, \\ \sigma(y_4) = y_3. \end{cases} \tag{3}$$

Here, we have used the fact that $\sigma^3 = \mathrm{id}_Y$ to rewrite the fourth equality. We now deduce that

$$\sigma(y_2) = y_3 = \sigma(y_4) = \sigma(y_5) = \sigma^2(y_6) = \sigma^3(y_1) = y_1 = y_2.$$

However, $\sigma$ has no fixed points in $Y$, so the system of equations in (3) has no solutions in $R$. Hence, $\varphi$ cannot exist. □

Incidentally, Theorem 3.8 gives a positive answer to a question posed by Kimura in [25, Section 4], as we state below.

**Corollary 3.9.** *Although $R$-coloring numbers cannot distinguish between Legendrian knots having the same underlying smooth knot type, Thurston-Bennequin number, and rotation number if $R$ is a GL-quandle by* [25, Theorem 4.3]*, the same is not generally true if $R$ is a nonquandle GL-rack.*

Similarly, $R$-coloring numbers distinguish between the Legendrian representatives of the topological knot $8_{10}$ with $(\mathrm{tb}, \mathrm{rot}) = (-8, 3)$ in Figure 10. This proves a conjecture of Bhattacharyya et al. in [3]. At the time of writing, we are unaware of any other proofs of this conjecture.

**Example 3.10.** Let $\Lambda_1$ and $\Lambda_2$ be the oriented Legendrian knots on the left and right of Figure 10, respectively. We compute that $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$ are the free GL-racks on the set $\{x_1, \dots, x_{10}\}$ modulo the following crossing relations:

$$\mathcal{G}(\Lambda_1) \begin{cases} s_{x_1}(\mathtt{u}(x_3)) = x_4, & s_{x_4}(\mathtt{du}(x_1)) = x_2, \\ s_{x_7}(x_5) = \mathtt{d}(x_4), & s_{x_5}(x_8) = \mathtt{d}(x_7), \\ s_{x_8}(x_6) = \mathtt{d}^2(x_5), & s_{x_6}(x_9) = \mathtt{d}(x_6), \\ s_{x_6}(x_3) = x_2, & s_{x_6}(x_{10}) = x_1, \\ s_{x_3}(x_7) = \mathtt{ud}^2(x_6), & s_{x_3}(x_9) = x_{10}. \end{cases} \quad \mathcal{G}(\Lambda_2) \begin{cases} s_{x_1}(x_7) = \mathtt{ud}(x_6), & s_{x_7}(x_2) = \mathtt{du}(x_1), \\ s_{x_5}(x_2) = x_3, & s_{x_3}(\mathtt{u}(x_5)) = x_6, \\ s_{x_3}(x_{10}) = x_9, & s_{x_{10}}(x_4) = \mathtt{d}^2(x_3), \\ s_4(x_1) = \mathtt{d}^2(x_{10}), & s_{x_6}(\mathtt{d}(x_4)) = x_5, \\ s_{x_5}(\mathtt{d}^2(x_7)) = x_8, & s_{x_6}(x_9) = x_8. \end{cases}$$

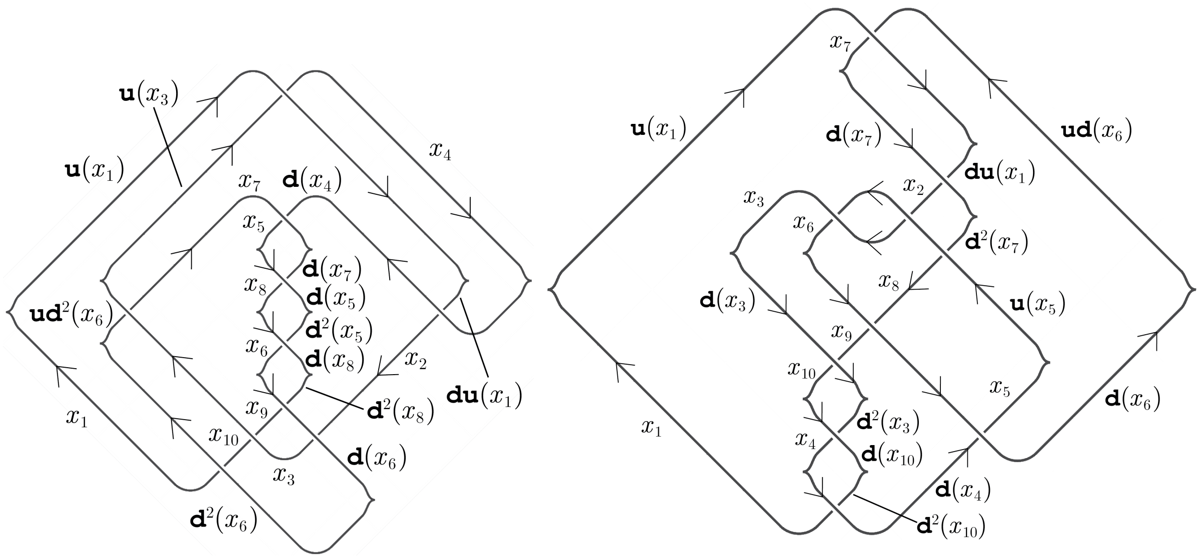


FIGURE 10. Front projections of the two Legendrian representatives of the topological knot $8_{10}$ with $(\mathrm{tb}, \mathrm{rot}) = (-8, 3)$ given in [3]. Created using [27], cf. [28].

**Theorem 3.11.** *The two oriented Legendrian knots in Figure 10 are not Legendrian isotopic; in fact, they are distinguishable using coloring numbers with respect to a constant GL-rack of order 2.*

*Proof.* Once again, let $\Lambda_1$ and $\Lambda_2$ be the oriented Legendrian knots on the left and right of Figure 10, respectively. Let $Y = \{1, 2\}$. In cycle notation, let $\sigma \in S_2$ be the permutation $(12)$. In the notation of Example 2.9, let $R = (Y, \sigma, \sigma, \mathrm{id}_Y)_c$, so that $R$ is the fourth GL-rack in Table 2. We will show that $\mathrm{Col}(\Lambda_1, R) > \mathrm{Col}(\Lambda_2, R)$. To that end, let $A$ denote the underlying set of $\mathcal{G}(\Lambda_1)$ as presented in Example 3.10, and define $\varphi_1, \varphi_2 : A \to Y$ by the following:

$$\varphi_1(x_i) := \begin{cases} 1 & \text{if } i \in \{1, 2, 5, 8, 9\}, \\ 2 & \text{if } i \in \{3, 4, 6, 7, 10\}. \end{cases} \quad \varphi_2(x_i) := \begin{cases} 2 & \text{if } i \in \{1, 2, 5, 8, 9\}, \\ 1 & \text{if } i \in \{3, 4, 6, 7, 10\}. \end{cases}$$

Using the relations of $\mathcal{G}(\Lambda_1)$ in Example 3.10, it is straightforward to verify that $\varphi_1$ and $\varphi_2$ define GL-rack homomorphisms from $\mathcal{G}(\Lambda_1)$ to $R$. Hence, $\mathrm{Col}(\Lambda_1, R) \geq 2$ (which is actually an equality).

On the other hand, we claim that $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda_2), R) = \emptyset$. Let $B$ denote the underlying set of $\mathcal{G}(\Lambda_2)$ as presented in Example 3.10, and suppose to the contrary that some map $\varphi : B \to Y$ defines a GL-rack homomorphism from $\mathcal{G}(\Lambda_2)$ to $R$ with $\varphi(x_i) = y_i$. Just like before, the relations of $\mathcal{G}(\Lambda_2)$ in Example 3.10 yield the following system of equations in $R$:

$$R \begin{cases} \sigma(y_7) = \sigma(y_6), & \sigma(y_2) = \sigma(x_1), \\ \sigma(y_2) = y_3, & \sigma^2(y_5) = y_6, \\ \sigma(y_{10}) = y_9, & \sigma(y_4) = y_3, \\ \sigma(y_1) = y_{10}, & \sigma(y_4) = y_5, \\ \sigma(y_7) = y_8, & \sigma(y_9) = y_8. \end{cases}$$

Since $\sigma^2 = \mathrm{id}_Y$, we can also rewrite the equalities $\sigma(y_7) = \sigma(x_6)$, $\sigma^2(y_5) = y_6$, and $\sigma(y_{10}) = y_9$ as $y_7 = y_6$, $y_5 = y_6$, and $y_{10} = \sigma(y_9)$, respectively. Therefore, we have

$$y_7 = y_6 = y_5 = \sigma(y_4) = y_3 = \sigma(y_2) = \sigma(y_1) = y_{10} = \sigma(y_9) = y_8 = \sigma(y_7),$$

which is impossible since $\sigma$ has no fixed points in $Y$. Hence, $\varphi$ cannot exist. $\square$

We selected the constant GL-racks and coloring maps used to prove Theorems 3.8 and 3.11 using exhaustive computer searches in `GAP` [19]. See Subsection A.2 for details. To help complete the atlas of Legendrian knots, we encourage the reader to the download the program linked in Appendix A and tackle even more of the conjectures in [9] and [3] in this fashion.

3.3. **Isomorphism of $\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda))$ and $\pi_1(\mathbb{R}^3 \setminus L)$.** We now prove an empirical observation of Karmakar et al. in the original version of [23, Remark 8.2], which we state as Corollary 3.13. Although the authors have removed this remark from subsequent releases of the article, we are unaware of any other proofs at the time of writing. To allow for a short categorical proof, we begin by proving a more abstract version of the observation.

**Theorem 3.12.** *Let $R = (X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack, and let $R' = (X/\sim, s^*, \mathrm{id}_{X/\sim}, \mathrm{id}_{X/\sim})$ be the GL-rack obtained by imposing an equivalence relation $\sim$ onto $R$ defined by $\mathtt{u}(x) = x = \mathtt{d}(x)$ for all $x \in X$. Then $\mathrm{Env}_{\mathsf{GLR}}(R)$ and $\mathrm{Env}_{\mathsf{GLR}}(R')$ are isomorphic as groups. In particular, $\mathrm{Env}_{\mathsf{GLR}}(R)$ and $\mathrm{Env}_{\mathsf{Qnd}}(X/\sim, s^*)$ are isomorphic as groups.*

*Proof.* By definition, $R'$ is the quotient object of the equivalence relation $\sim$ on $R$ in $\mathsf{GLR}$. In other words, $R'$ is the colimit of the following diagram in $\mathsf{GLR}$:

$$R \underset{\mathrm{id}_X}{\overset{\mathtt{u}}{\rightrightarrows}} R \underset{\mathrm{id}_X}{\overset{\mathtt{d}}{\rightrightarrows}} R$$

Recall that left adjoint functors preserve colimits. It follows from Proposition 2.17 that $\mathrm{Env}_{\mathsf{GLR}}(R')$ is the colimit of the following diagram in $\mathsf{Grp}$:

$$\mathrm{Env}_{\mathsf{GLR}}(R) \underset{\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)}{\overset{\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})}{\rightrightarrows}} \mathrm{Env}_{\mathsf{GLR}}(R) \underset{\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)}{\overset{\mathrm{Env}_{\mathsf{GLR}}(\mathtt{d})}{\rightrightarrows}} \mathrm{Env}_{\mathsf{GLR}}(R)$$

By Proposition 2.17, the group homomorphism $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})$ is defined by $e_x \mapsto e_{\mathtt{u}(x)}$ for all $x \in X$, but $e_x = e_{\mathtt{u}(x)}$ in $\mathrm{Env}_{\mathsf{GLR}}(R)$. Thus, $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})$ is the identity map. Similarly, $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{d})$ and $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)$ are the identity maps, so we have a group isomorphism

$$\mathrm{Env}_{\mathsf{GLR}}(R) \cong \mathrm{Env}_{\mathsf{GLR}}(R').$$

By Lemma 2.14, the right-hand side is isomorphic to $\mathrm{Env}_{\mathsf{Qnd}}(X/\sim, s^*)$, so we are done. $\square$

**Corollary 3.13.** *Let $\Lambda \subset \mathbb{R}^3$ be an oriented Legendrian link, and let $L$ denote its underlying smooth link. Then there exists a group isomorphism*

$$\mathrm{Env}_{\mathsf{GLR}}(\mathcal{G}(\Lambda)) \cong \pi_1(\mathbb{R}^3 \setminus L).$$

*Proof.* Taking $R = \mathcal{G}(\Lambda)$ in Theorem 3.12, we have $R' \cong (\mathcal{Q}(L), \mathrm{id}_{X_L}, \mathrm{id}_{X_L})$ by Lemma 3.5. By Theorem 3.12, it suffices to show that $\mathrm{Env}_{\mathsf{Qnd}}(\mathcal{Q}(L))$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus L)$. But Joyce showed this in [22, Section 15] using the Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus L)$. $\square$

## 4. On medial GL-racks, hom-sets, and tensor products

In this section, we define medial GL-racks, propose a medial GL-rack-valued invariant of Legendrian links with questions for future research, and introduce tensor products of GL-racks that make the category of medial GL-rack symmetric monoidal closed. This generalizes Crans and Nelson's similar results for the category of medial quandles in [10]; we follow their approach closely.

4.1. **Hom-sets of medial GL-racks are also medial GL-racks.** In this subsection, we define medial GL-racks, introduce a medial GL-rack structure on any hom-set from a GL-rack to a medial GL-rack, and propose a medial GL-rack-valued invariant of Legendrian links.

**Definition 4.1.** A rack $(X, s)$ is called *medial* or *abelian* if, for all $x, y, z \in X$, we have

$$s_{s_x(z)} \circ s_y = s_{s_x(y)} \circ s_z.$$

If in addition $(\mathtt{u}, \mathtt{d})$ defines a GL-structure on $(X, s)$, then we call $(X, s, \mathtt{u}, \mathtt{d})$ a *medial* or *abelian* GL-rack. (Note that this definition is not synonymous with the condition that $(X, s)$ is *commutative*, which states that $s_x(y) = s_y(x)$ for all $x, y \in X$.)

Since $R$-coloring numbers are not complete invariants of Legendrian links (see [25, Theorem 4.3]), it would be desirable to construct *enhancements* or *refinements* of $R$-coloring numbers that encode more information about front projections. To this end, we will show that if $M$ is a medial GL-rack, then $\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), M)$ also enjoys a canonical medial GL-rack structure.

**Lemma 4.2.** *Let $(X, s)$ be a rack, and let $(Y, t)$ be a medial rack. Let $\widetilde{H} := \mathrm{Hom}_{\mathsf{Rack}}((X, s^X), (Y, s^Y))$, and define $\widetilde{s} : \widetilde{H} \to \mathrm{Sym}(\widetilde{H})$ by $g \mapsto s_g$, where $s_g(f) := [x \mapsto (s^Y_{g(x)} \circ f)(x)]$. Then, $\widetilde{R} := (\widetilde{H}, \widetilde{s})$ is a medial rack. If in addition $(Y, s^Y)$ is a quandle, then so is $\widetilde{R}$.*

*Proof.* In [10, Theorem 3], Crans and Nelson proved that $\widetilde{R}$ is a medial quandle under the additional assumptions that $(X, s^X)$ and $(Y, s^Y)$ are quandles. However, they never used the quandle axiom that $s^X_x(x) = x$ for all $x \in X$. Moreover, their proof that $\widetilde{R}$ satisfies the nonquandle rack axioms and mediality did not use the quandle axiom that $s^Y_y(y) = y$ for all resp. $y \in Y$; the authors only used this axiom to show that $\widetilde{R}$ also satisfies it. □

**Theorem 4.3.** *In the setting of Lemma 4.2, suppose in addition that $R_1 := (X, s^X, \mathtt{u}_1, \mathtt{d}_1)$ and $R_2 := (Y, s^Y, \mathtt{u}_2, \mathtt{d}_2)$ are GL-racks, so that $R_2$ is medial. Let $H := \mathrm{Hom}_{\mathsf{GLR}}(R_1, R_2) \subset \widetilde{H}$, let $s := \widetilde{s}|_H$, and let $R := (H, s)$. Define $\mathtt{u} : H \to H$ by $f \mapsto \mathtt{u}_2 \circ f$, and define $\mathtt{d} : H \to H$ by $f \mapsto \mathtt{d}_2 \circ f$. Then, $(R, \mathtt{u}, \mathtt{d})$ is a medial GL-rack. If $R_2$ is also a GL-quandle, then so is $(R, \mathtt{u}, \mathtt{d})$.*

*Proof.* In the notation of Lemma 4.2, we know $\widetilde{R}$ is a medial rack. To show that $R$ is a medial rack, it will suffice to show that $R$ is a subrack of $\widetilde{R}$. To that end, fix $f, g \in H$. Then, we have $s_g(f) \in H$ because

$$\begin{aligned}
\mathtt{u}_2 \circ s_g(f) &= [x \mapsto (\mathtt{u}_2 \circ s^Y_{g(x)} \circ f)(x)] & \\
&= [x \mapsto (s^Y_{(\mathtt{u}_2 \circ g)(x)} \circ \mathtt{u}_2 \circ f)(x)] & \text{by Proposition 2.12} \\
&= [x \mapsto (s^Y_{(g \circ \mathtt{u}_1)(x)} \circ f \circ \mathtt{u}_1)(x)] & \text{since } f, g \in H \\
&= s_g(f) \circ \mathtt{u}_1 &
\end{aligned}$$

and, similarly, $\mathtt{d}_2 \circ s_g(f) = s_g(f) \circ \mathtt{d}_1$. Thus, $R$ is a subrack of $\widetilde{R}$, so $R$ is medial. In particular, if $R_2$ is a GL-quandle, then Lemma 4.2 implies that $R$ is a subquandle of $\widetilde{R}$.

It remains to show that $(\mathtt{u}, \mathtt{d})$ satisfies the three GL-rack axioms. Fix $f, g \in H$. Since $f$ is a GL-rack homomorphism and $(\mathtt{u}_1, \mathtt{d}_1)$ satisfies GL-rack axiom (L1), we have

$$\begin{aligned}
(\mathtt{ud} \circ s_f)(f) &= [x \mapsto (\mathtt{u}_2 \circ \mathtt{d}_2 \circ s^Y_{f(x)} \circ f)(x)] \\
&= [x \mapsto (\mathtt{u}_2 \circ \mathtt{d}_2 \circ f \circ s^X_x)(x)] \\
&= [x \mapsto f((\mathtt{u}_1 \circ \mathtt{d}_1 \circ s^X_x)(x))] \\
&= [x \mapsto f(x)] = f
\end{aligned}$$

and, similarly, $(\mathtt{du} \circ s_f)(f) = f$. So, $(\mathtt{u}, \mathtt{d})$ satisfies axiom (L1). Since $(\mathtt{u}_2, \mathtt{d}_2)$ satisfies axiom (L2), we also have

$$(\mathtt{u} \circ s_g)(f) = [x \mapsto (\mathtt{u}_2 \circ s^Y_{g(x)} \circ f)(x)] = [x \mapsto (s^Y_{g(x)} \circ \mathtt{u}_2 \circ f)(x)] = s_g(\mathtt{u}_2 \circ f) = (s_g \circ \mathtt{u})(f)$$

and, similarly, $(\mathtt{d} \circ s_g)(f) = (s_g \circ \mathtt{d})(f)$. So, $(\mathtt{u}, \mathtt{d})$ satisfies axiom (L2). Finally, since $(\mathtt{u}_2, \mathtt{d}_2)$ satisfies axiom (L3), we have

$$s_{\mathtt{u}(g)}(f) = [x \mapsto (s^Y_{\mathtt{u}_2(g(x))} \circ f)(x)] = [x \mapsto (s^Y_{g(x)} \circ f)(x)] = s_g(f)$$

and, similarly, $s_{\mathtt{d}(g)}(f) = s_g(f)$. Hence, $(\mathtt{u}, \mathtt{d})$ satisfies axiom (L3), and the proof is complete. □

Proposition 3.3 and Theorem 4.3 imply that, for a fixed medial GL-rack $M$ and for any oriented Legendrian link $\Lambda$, the isomorphism class of $\mathcal{H}(\Lambda, M) := \mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), M)$ as a medial GL-rack is an invariant of $\Lambda$. In light of Lemmas 3.5 and 2.14, $\mathcal{H}(\Lambda, M)$ generalizes the medial quandle-valued invariant $\mathrm{Hom}_{\mathsf{Qnd}}(\mathcal{Q}(L), Q)$ of smooth links from [10, Section 6], where $Q$ is a medial quandle.

This medial GL-rack-valued invariant of Legendrian links raises intriguing questions for future research. For example, do there exist Legendrian links $\Lambda_1$ and $\Lambda_2$ and a medial GL-rack $M$ such that $\mathrm{Col}(\Lambda_1, M) = \mathrm{Col}(\Lambda_2, M)$ but $\mathcal{H}(\Lambda_1, M) \ncong \mathcal{H}(\Lambda_2, M)$ as medial GL-racks? In the spirit of Corollary 3.9, do there exist such Legendrian links that also share the same underlying smooth link type, Thurston-Bennequin number, and rotation number? Moreover, Elhamdadi et al. recently introduced proper enhancements of $\mathrm{Hom}_{\mathsf{Qnd}}(\mathcal{Q}(L), Q)$ by considering $k$-algebra homomorphisms between quandle rings and colorings of smooth links by idempotents of quandle rings in [14, Theorems 4.2 and 5.1]. Do similar proper enhancements of $\mathcal{H}(\Lambda, M)$ also exist?

4.2. **Tensor products of GL-racks.** Let $\mathsf{GLR}^{\mathrm{med}}$ be the full subcategory of $\mathsf{GLR}$ whose objects are medial. To help us understand the structure of $\mathsf{GLR}^{\mathrm{med}}$, we introduce tensor products in $\mathsf{GLR}$ that make $\mathsf{GLR}^{\mathrm{med}}$ symmetric monoidal closed.

We define tensor products of GL-racks in such a way that, by way of Lemma 2.14, they generalize tensor products of quandles as defined by Crans and Nelson in [10, Subsection 8.1].

**Definition 4.4.** If $R_1 = (X, s^X, \mathtt{u}_1, \mathtt{d}_1)$ and $R_2 = (Y, s^Y, \mathtt{u}_2, \mathtt{d}_2)$ are GL-racks, then we define their *tensor product*, denoted by $R_1 \otimes R_2$, to be the free GL-rack $\mathrm{FGLR}(X \times Y)$ modulo the following relations for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$:

(1) $s_{(x,y_2)}(x, y_1) = (x, s^Y_{y_2}(y_1))$.
(2) $s_{(x_2,y)}(x_1, y) = (s^X_{x_2}(x_1), y)$.
(3) $\mathtt{u}(x, y) = (\mathtt{u}_1(x), y) = (x, \mathtt{u}_2(y))$.
(4) $\mathtt{d}(x, y) = (\mathtt{d}_1(x), y) = (x, \mathtt{d}_2(y))$.

The next two results show that tensor products of medial GL-racks satisfy the expected universal property and internal hom-tensor adjunction.

**Theorem 4.5.** *The category* $\mathsf{GLR}^{\mathrm{med}}$ *is symmetric monoidal closed with respect to the closed structure* $\mathrm{Hom}_{\mathsf{GLR}^{\mathrm{med}}}(-,-)$ *in Proposition 4.3 and the tensor product* $\otimes$ *in Definition 4.4.*

*Proof.* The unit object in $\mathsf{GLR}^{\mathrm{med}}$ is the trivial GL-rack with one element. Using this fact, it is straightforward to verify that $\mathsf{GLR}^{\mathrm{med}}$ is monoidal and symmetric. On the other hand, $\mathsf{GLR}^{\mathrm{med}}$ is defined as an equational algebraic category. Thus, the main theorem of Linton in [30] states that our claim is true if and only if, in the sense of universal algebra, $\mathsf{GLR}^{\mathrm{med}}$ is commutative as an algebraic theory (see [30, Section 6], cf. [10, Subsection 8.1]). Indeed, for any medial GL-rack

$(X, s, \mathtt{u}, \mathtt{d})$ and for all elements $x_{11}, x_{12}, x_{21}, x_{22} \in X$, we have the following equalities:

$$\begin{cases} (\mathtt{u} \circ s_{x_{12}})(x_{11}) = (s_{\mathtt{u}(x_{12})} \circ \mathtt{u})(x_{11}) & \text{by Proposition 2.12,} \\ (\mathtt{d} \circ s_{x_{12}})(x_{11}) = (s_{\mathtt{d}(x_{12})} \circ \mathtt{d})(x_{11}) & \text{by Proposition 2.12,} \\ (\mathtt{u} \circ \mathtt{d})(x_{11}) = (\mathtt{d} \circ \mathtt{u})(x_{11}) & \text{by Proposition 2.12,} \\ (s_{s_{x_{22}}(x_{21})} \circ s_{x_{12}})(x_{11}) = (s_{s_{x_{22}}(x_{12})} \circ s_{x_{21}})(x_{11}) & \text{since } (X, s) \text{ is medial.} \end{cases}$$

Together with the tautologies $\mathtt{u}^2(x_{11}) = \mathtt{u}^2(x_{11})$ and $\mathtt{d}^2(x_{11}) = \mathtt{d}^2(x_{11})$, these equalities show that $\mathsf{GLR}^{\mathrm{med}}$ forms a commutative algebraic theory. This completes the proof. □

In light of Proposition 4.3, we immediately deduce the following.

**Corollary 4.6.** $\mathsf{GLR}^{\mathrm{med}}$ *is self-enriched. Explicitly, if $A$, $B$, and $C$ are medial GL-racks, then there exists a natural isomorphism of medial GL-racks*

$$\mathrm{Hom}_{\mathsf{GLR}^{\mathrm{med}}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathsf{GLR}^{\mathrm{med}}}(A, \mathrm{Hom}_{\mathsf{GLR}^{\mathrm{med}}}(B, C)).$$

## 5. A sufficient condition for GL-rack isomorphisms

In [24, Section 5], Karmakar et al. constructed a homogeneous representation for any GL-rack $R$ from the orbits of $R$ under the action of $\mathrm{Aut}_{\mathsf{GLR}}(R)$. In this section, we adapt this construction into a category $\mathsf{GrpTup}$ with an essentially surjective functor $\mathcal{F} : \mathsf{GrpTup} \twoheadrightarrow \mathsf{GLR}$. This functor induces a group-theoretic sufficient condition for any two GL-racks or quandles to be isomorphic.

### 5.1. Construction of $\mathsf{GrpTup}$.

In this subsection, we introduce a category $\mathsf{GrpTup}$ with a functorial relationship to $\mathsf{GLR}$.

To define the objects in $\mathsf{GrpTup}$, we adapt a construction of Karmakar et al. in [24, Proposition 5.1]. Given any group $G$ with a subgroup $H$, let $G/H$ denote the set of left cosets of $H$ in $G$. Now, let the objects in $\mathsf{GrpTup}$ be all sextuples $(I, \bigsqcup_{i\in I} G/H_i, Z_I, Q_I, R_I, \tau)$ satisfying the following:

(1) $I$ is an indexing set, $G$ is a group, and $Z_I = \{z_i^G \mid i \in I\}$, $Q_I = \{q_i^G \mid i \in I\}$, and $R_I = \{r_i^G \mid i \in I\}$ are multisets indexed by $I$ whose elements lie in $G$.
(2) $\{H_i \mid i \in I\}$ is a family of subgroups of $G$ such that $H_i \leq C_G(z_i^G)$ for all $i \in I$.
(3) $\tau : I \to I$ is a bijection such that the following hold for all $i \in I$:
  (a) $q_i^G \in N_G(H_{\tau(i)})$.
  (b) $r_i^G \in N_G(H_{\tau^{-1}(i)})$.
  (c) $z_i^G q_i^G r_{\tau(i)}^G,\ z_i^G r_i^G q_{\tau^{-1}(i)}^G \in H_i$.
  (d) $z_i^G q_i^G (z_{\tau(i)}^G)^{-1} = q_i^G$.
  (e) $z_i^G r_i^G (z_{\tau^{-1}(i)}^G)^{-1} = r_i^G$.

For the sake of brevity, we will denote such an object as $\widetilde{G}$ when there is no room for confusion. For an opposing object in $\mathsf{GrpTup}$, we will write $\widetilde{K} := (J, \bigsqcup_{j\in J} K/L_j, Z_J, Q_J, R_J, \pi)$.

Now, we define the morphisms in $\mathsf{GrpTup}$. Given any two objects $\widetilde{G}, \widetilde{K}$ in $\mathsf{GrpTup}$, let $\mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$ be the set of all triples $\varphi := (\varphi_1, \varphi_2, \varphi_3)$ satisfying the following:

(1) $\varphi_1 : G \to K$ is a group homomorphism.
(2) $\varphi_2 : \bigsqcup_{i\in I} G/H_i \to \bigsqcup_{j\in J} K/L_j$ and $\varphi_3 : I \to J$ are morphisms in $\mathsf{Set}$.
(3) $\pi \circ \varphi_3 = \varphi_3 \circ \tau$.
(4) For all $i \in I$ and $g \in G$, we have $\varphi_1(z_i^G) = z_{\varphi_3(i)}^K$, $\varphi_1(q_i^G) = q_{\varphi_3(i)}^K$, $\varphi_1(r_i^G) = r_{\varphi_3(i)}^K$, and $\varphi_2(gH_i) = \varphi_1(g)L_{\varphi_3(i)}$.

Define the composition of morphisms in $\mathsf{GrpTup}$ by $\psi \circ \varphi := (\psi_1 \circ \varphi_1, \psi_2 \circ \varphi_2, \psi_3 \circ \varphi_3)$. Also, define the identity morphism $\mathrm{id}^{\widetilde{G}} : \widetilde{G} \xrightarrow{\sim} \widetilde{G}$ by letting $\mathrm{id}_1^{\widetilde{G}}$, $\mathrm{id}_2^{\widetilde{G}}$, and $\mathrm{id}_3^{\widetilde{G}}$ as defined above be identity maps. Associativity and unit laws are immediate. Hence, $\mathsf{GrpTup}$ is a category.

5.2. **Construction of $\mathcal{F} : \mathsf{GrpTup} \to \mathsf{GLR}$.** In this subsection, we construct an essentially surjective functor $\mathcal{F} : \mathsf{GrpTup} \twoheadrightarrow \mathsf{GLR}$. Then, we discuss a group-theoretic sufficient condition for two GL-racks to be isomorphic.

By [24, Proposition 5.1], given any object $\widetilde{G}$ in $\mathsf{GrpTup}$, the set $X := \bigsqcup_{i \in I} G/H_i$ admits a GL-rack structure in which $s^X : X \to \mathrm{Sym}(X)$ and $\mathtt{u}_G, \mathtt{d}_G : X \to X$ are defined by

$$s^X(yH_j) := s^X_{yH_j} := [xH_i \mapsto yz_j^G y^{-1} xH_i], \ \mathtt{u}_G(xH_i) := xq_i^G H_{\tau(i)}, \text{ and } \mathtt{d}_G(xH_i) := xr_i^G H_{\tau^{-1}(i)}.$$

So, we can define a functor $\mathcal{F} : \mathsf{GrpTup} \to \mathsf{GLR}$ by sending any object $\widetilde{G}$ in $\mathsf{GrpTup}$ to the GL-rack $(X, s^X, \mathtt{u}_G, \mathtt{d}_G)$ and sending any morphism $\varphi \in \mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$ to $\varphi_2$.

**Theorem 5.1.** *$\mathcal{F}$ is an essentially surjective functor.*

*Proof.* Essential surjectivity is precisely the statement of [24, Theorem 5.2]. Certainly, $\mathcal{F}$ preserves identity morphisms and composition of morphisms. To complete the proof of functoriality, it remains to show that if $\varphi \in \mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$, then $\mathcal{F}\varphi : \mathcal{F}(\widetilde{G}) \to \mathcal{F}(\widetilde{K})$ is a GL-rack homomorphism. Write $\mathcal{F}(\widetilde{G}) = (X, s^X, \mathtt{u}_G, \mathtt{d}_G)$ and $\mathcal{F}(\widetilde{K}) = (Y, s^Y, \mathtt{u}_K, \mathtt{d}_K)$, and fix $gH_a \in X$. Since $\varphi_1$ is a group homomorphism, we have

$$\begin{aligned}
\mathcal{F}\varphi \circ s^X_{gH_a} &= [xH_i \mapsto \varphi_2(gz_a^G g^{-1} xH_i)] \\
&= [xH_i \mapsto \varphi_1(gz_a^G g^{-1} x) L_{\varphi_3(i)}] \\
&= [xH_i \mapsto \varphi_1(g)\varphi_1(z_a^G)\varphi_1(g^{-1})\varphi_1(x) L_{\varphi_3(i)}] \\
&= [xH_i \mapsto \varphi_1(g) z^K_{\varphi_3(a)} \varphi_1(g)^{-1} \varphi_2(xH_i)] \\
&= s^Y_{\varphi_1(g)L_{\varphi_3(a)}} \circ \varphi_2 = s^Y_{\varphi_2(gH_a)} \circ \varphi_2 = s^Y_{\mathcal{F}\varphi(gH_a)} \circ \mathcal{F}\varphi,
\end{aligned}$$

so $\mathcal{F}\varphi$ is a rack homomorphism. Moreover, we have

$$\begin{aligned}
\mathcal{F}\varphi \circ \mathtt{u}_G &= [xH_i \mapsto \varphi_2(xq_i^G H_{\tau(i)})] \\
&= [xH_i \mapsto \varphi_1(x)\varphi_1(q_i^G) L_{\varphi_3(\tau(i))}] \\
&= [xH_i \mapsto \varphi_1(x) q^K_{\varphi_3(i)} L_{\pi(\varphi_3(i))}] \\
&= [yL_j \mapsto yq_j^K L_{\pi(j)}] \circ [xH_i \mapsto \varphi_1(x) L_{\varphi_3(i)}] \\
&= \mathtt{u}_K \circ [xH_i \mapsto \varphi_2(xH_i)] = \mathtt{u}_K \circ \mathcal{F}\varphi
\end{aligned}$$

and, similarly, $\mathcal{F}\varphi \circ \mathtt{d}_G = \mathtt{d}_K \circ \mathcal{F}\varphi$. Hence, $\mathcal{F}\varphi$ is a GL-rack homomorphism. $\square$

This result gives us a group-theoretic way to show that two GL-racks $R_1$ and $R_2$ are isomorphic. In the proof of [24, Theorem 5.2], Karmakar et al. describe a procedure to construct objects $\widetilde{G}$ and $\widetilde{K}$ in $\mathsf{GrpTup}$ such that $R_1 \cong \mathcal{F}(\widetilde{G})$ and $R_2 \cong \mathcal{F}(\widetilde{K})$ in $\mathsf{GLR}$. Their construction uses the decomposition of the underlying sets of $R_1$ and $R_2$ into orbits of $\mathrm{Aut}_{\mathsf{GLR}}(R_1)$ and $\mathrm{Aut}_{\mathsf{GLR}}(R_2)$, respectively. Then, to show that $R_1 \cong R_2$ in $\mathsf{GLR}$, it suffices to find a morphism $\varphi \in \mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$ such that $\varphi_2$ is bijective, since then $\mathcal{F}(\varphi) : \mathcal{F}(\widetilde{G}) \xrightarrow{\sim} \mathcal{F}(\widetilde{H})$ will also be bijective and, hence, an isomorphism of GL-racks.

Finally, let $\mathsf{GrpTrip}$ be the full subcategory of $\mathsf{GrpTup}$ consisting of objects $\widetilde{G}$ for which $\tau = \mathrm{id}_I$, and $Q_I$ and $R_I$ are multisets only containing copies of $1_G$. By Lemma 2.14 and [22, Theorem 7.2],

$\mathcal{F}$ induces an essentially surjective functor $\mathsf{GrpTrip} \twoheadrightarrow \mathsf{Qnd}$. Hence, our above discussion specializes to a sufficient condition for any two quandles to be isomorphic.

## References


[1] J. Adámek, J. Rosický, and E. M. Vitale, *Algebraic theories*, Cambridge Tracts in Mathematics, vol. 184, Cambridge University Press, Cambridge, 2011. A categorical introduction to general algebra, With a foreword by F. W. Lawvere. MR2757312

[2] Jagdeep Basi and Carmen Caprau, *Quandle coloring quivers of* $(p,2)$*-torus links*, J. Knot Theory Ramifications **31** (2022), no. 9, Paper No. 2250057, 14. MR4475496

[3] Nilangshu Bhattacharyya, Cyrus Cox, Justin Murray, Adithyan Pandikkadan, Shea Vela-Vick, and Angela Wu, *Legendrian knot atlas: List of* $(\mathrm{tb},\mathrm{r})$ *for non-destabilizable Legendrian knots representing knot type* $8_{10}$. `https://www.math.lsu.edu/~knotatlas/legendrian/8_10-knot.html`. Accessed: 2024-12-28.

[4] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford, and Masahico Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 3947–3989. MR1990571

[5] A. Cattabriga and E. Horvat, *Knot quandle decompositions*, Mediterr. J. Math. **17** (2020), no. 3, Paper No. 98, 22. MR4105750

[6] Jose Ceniceros, Mohamed Elhamdadi, and Sam Nelson, *Legendrian rack invariants of Legendrian knots*, Commun. Korean Math. Soc. **36** (2021), no. 3, 623–639. MR4292403

[7] Karina Cho and Sam Nelson, *Quandle cocycle quivers*, Topology Appl. **268** (2019), 106908, 10. MR4018585

[8] ———, *Quandle coloring quivers*, J. Knot Theory Ramifications **28** (2019), no. 1, 1950001, 12. MR3910948

[9] Wutichai Chongchitmate and Lenhard Ng, *An atlas of Legendrian knots*, Exp. Math. **22** (2013), no. 1, 26–37. MR3038780

[10] Alissa S. Crans and Sam Nelson, *Hom quandles*, J. Knot Theory Ramifications **23** (2014), no. 2, 1450010, 18. MR3197054

[11] Ivan Dynnikov and Maxim Prasolov, *Rectangular diagrams of surfaces: distinguishing Legendrian knots*, J. Topol. **14** (2021), no. 3, 701–860. MR4286839

[12] Mohamed Elhamdadi, *A survey of racks and quandles: some recent developments*, Algebra Colloq. **27** (2020), no. 3, 509–522. MR4141628

[13] Mohamed Elhamdadi, Jennifer Macquarrie, and Ricardo Restrepo, *Automorphism groups of quandles*, J. Algebra Appl. **11** (2012), no. 1, 1250008, 9. MR2900878

[14] Mohamed Elhamdadi, Brandon Nunez, and Mahender Singh, *Enhancements of link colorings via idempotents of quandle rings*, J. Pure Appl. Algebra **227** (2023), no. 10, Paper No. 107400, 16. MR4579329

[15] Mohamed Elhamdadi and Dipali Swain, *State sum invariants of knots from idempotents in quandle rings*, 2024. Preprint.

[16] John B. Etnyre, *Legendrian and transversal knots*, Handbook of knot theory, 2005, pp. 105–185. MR2179261

[17] John B. Etnyre and Ko Honda, *Knots and contact geometry. I. Torus knots and the figure eight knot*, J. Symplectic Geom. **1** (2001), no. 1, 63–120. MR1959579

[18] John B. Etnyre and Lenhard L. Ng, *Legendrian contact homology in* $\mathbb{R}^3$, Surveys in differential geometry 2020. Surveys in 3-manifold topology and geometry, [2022] ©2022, pp. 103–161. MR4479751

[19] *GAP – Groups, Algorithms, and Programming, Version 4.14.0*, The GAP Group, 2024.

[20] Richard Henderson, Todd Macedo, and Sam Nelson, *Symbolic computation with finite quandles*, J. Symbolic Comput. **41** (2006), no. 7, 811–817. MR2232202

[21] Přemysl Jedlička, Agata Pilitowska, David Stanovský, and Anna Zamojska-Dzienio, *The structure of medial quandles*, J. Algebra **443** (2015), 300–334. MR3400403

[22] David Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), no. 1, 37–65. MR638121

[23] Biswadeep Karmakar, Deepanshi Saraf, and Mahender Singh, *Cohomology of generalised Legendrian racks and state-sum invariants of Legendrian links*, 2023. Preprint.

[24] ———, *Generalised Legendrian racks of Legendrian links*, 2024. Preprint.

[25] Naoki Kimura, *Bi-Legendrian rack colorings of Legendrian knots*, J. Knot Theory Ramifications **32** (2023), no. 4, Paper No. 2350029, 16. MR4586264

[26] ———, *Rack coloring invariants of Legendrian knots*, 2024. Thesis (Ph.D.)–Waseda University Graduate School of Fundamental Science and Engineering.

[27] Margaret Kipe, *Legendrian-Knot-Mosaics*, 2024. https://github.com/margekk/Legendrian-Knot-Mosaics. Accessed: 2024-12-28.

[28] Margaret Kipe, Samantha Pezzimenti, Leif Schaumann, Luc Ta, and Wing Hong Tony Wong, *Bounds on the mosaic number of Legendrian knots*, 2024. Preprint.

[29] Dheeraj Kulkarni and T. V. H. Prathamesh, *On rack invariants of Legendrian knots*, 2017. Preprint.

[30] F. E. J. Linton, *Autonomous equational categories*, J. Math. Mech. **15** (1966), 637–642. MR190205

[31] James McCarron, *Sequence A181770 in the On-line Encyclopedia of Integer Sequences*, 2010. https://oeis.org/A181770. Accessed: 2024-12-30.

[32] Sam Nelson, *What is … a quandle?*, Notices Amer. Math. Soc. **63** (2016), no. 4, 378–380. MR3444659

[33] Lenhard Lee Ng, *Invariants of Legendrian links*, ProQuest LLC, Ann Arbor, MI, 2001. Thesis (Ph.D.)–Massachusetts Institute of Technology. MR2717015

[34] Takefumi Nosaka, *Quandles and topological pairs*, SpringerBriefs in Mathematics, Springer, Singapore, 2017. Symmetry, knots, and cohomology. MR3729413

[35] Samantha Pezzimenti and Abhinav Pandey, *Geography of Legendrian knot mosaics*, J. Knot Theory Ramifications **31** (2022), no. 1, Paper No. 2250002, 22. MR4411812

[36] Joshua M. Sabloff, *What is . . . a Legendrian knot?*, Notices Amer. Math. Soc. **56** (2009), no. 10, 1282–1284. MR2572757

[37] John C. Stillwell, *Classical topology and combinatorial group theory*, Graduate Texts in Mathematics, vol. 72, Springer-Verlag, New York-Berlin, 1980. MR602149

[38] Jacek Świątkowski, *On the isotopy of Legendrian knots*, Ann. Global Anal. Geom. **10** (1992), no. 3, 195–207. MR1186009

[39] Yasuki Tada, *On categories of faithful quandles with surjective or injective quandle homomorphisms*, Hiroshima Math. J. **54** (2024), no. 1, 61–86. MR4728697

[40] Lorenzo Traldi, *Reorienting quandle orbits*, 2024. Preprint.

[41] Petr Vojtěchovský and Seung Yeop Yang, *Enumeration of racks and quandles up to isomorphism*, Math. Comp. **88** (2019), no. 319, 2523–2540. MR3957904

Department of Mathematics, Yale University, 219 Prospect St, Floors 7-9, New Haven, CT 06511.
*Email address*: luc.ta@yale.edu

## Appendix A. Exhaustive searches with GL-racks

*(Note: Writing in progress and GitHub link to come...)*

A.1. **Computation of small GL-racks.** In Table 1, we enumerate the number of GL-racks, medial GL-racks, GL-quandles, and medial GL-quandles of orders $n \leq 7$ up to isomorphism. For comparison, we also list the corresponding numbers for classical racks and quandles. The former numbers were computed using the code linked above, while the latter numbers $r(n)$, $r^m(n)$, $q(n)$, and $q^m(n)$ were originally computed by McCarron in [31], Vojtěchovský and Yang in [41], Henderson et al. in [20], and Jedlička et al. in [21], respectively. It appears that each of $g(n)$, $g^m(n)$, $g_q(n)$, and $g_q^m(n)$ in Table 1 grows exponentially and at a much faster rate than its counterpart for classical racks.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $g(n)$ | 1 | 4 | 13 | 59 | 285 | 1811 | 13725 |
| $g^m(n)$ | 1 | 4 | 13 | 58 | 275 | 1767 | 13419 |
| $g_q(n)$ | 1 | 2 | 6 | 17 | 64 | 277 | 1393 |
| $g_q^m(n)$ | 1 | 2 | 6 | 16 | 58 | 254 | 1291 |
| $r(n)$ | 1 | 2 | 6 | 19 | 74 | 353 | 2080 |
| $r^m(n)$ | 1 | 2 | 6 | 18 | 68 | 329 | 1965 |
| $q(n)$ | 1 | 1 | 3 | 7 | 22 | 73 | 298 |
| $q^m(n)$ | 1 | 1 | 3 | 6 | 18 | 58 | 251 |

Table 1. The number of GL-racks $g(n)$, medial GL-racks $g^m(n)$, GL-quandles $g_q(n)$, and medial GL-quandles $g_q^m(n)$ of order $n$ up to isomorphism, compared against the corresponding number of racks $r(n)$, medial racks $r^m(n)$, quandles $q(n)$, and medial quandles $q^m(n)$. The data from the bottom four rows is taken from [41, Table 1].

For explicit representatives of each GL-rack isomorphism class counted in Table A.1, see Subsection A.3 for those of orders $2 \leq n \leq 4$ and the data linked above for those of orders $5 \leq n \leq 7$.

We now describe the exhaustive search algorithm in `GAP` [19] that we used to compute these isomorphism classes. *More to come soon...*

A.2. **Exhaustive searches for $R$-coloring numbers.** *Coming soon...*

A.3. **Tabulation of GL-racks of orders 2, 3, and 4.** Tables 2, 3, and 4 tabulate all isomorphism classes of GL-racks having orders 2, 3, and 4, respectively. We write each bijection $s_i$, $\mathtt{u}$, and $\mathtt{d}$ as either the identity map of the set $\{1, \dots, n\}$, which we denote by id, or a nonidentity element of $S_n$ in cycle notation, where $n$ is the order of the GL-rack.

Note that, up to isomorphism, the only GL-rack of order 1 is the trivial GL-rack with one element. We were also able to compute all isomorphism classes of GL-racks having orders 5, 6, and 7; we list one representative of each in the data linked above.

| $[s_1, s_2]$ | $[\mathtt{u}, \mathtt{d}]$ | GL-quandle? | Medial? |
|---|---|---|---|
| $[\mathrm{id}, \mathrm{id}]$ | $[\mathrm{id}, \mathrm{id}]$, $[(12), (12)]$ | Yes | Yes |
| $[(12), (12)]$ | $[\mathrm{id}, (12)]$, $[(12), \mathrm{id}]$ | No | Yes |

Table 2. The four isomorphism classes of GL-racks of order 2.

| $[s_1, s_2, s_3]$ | $[\mathtt{u}, \mathtt{d}]$ | GL-quandle? | Medial? |
|---|---|---|---|
| $[\mathrm{id}, \mathrm{id}, \mathrm{id}]$ | $[\mathrm{id}, \mathrm{id}]$, $[(23), (23)]$, $[(132), (123)]$ | Yes | Yes |
| $[\mathrm{id}, (23), (23)]$ | $[\mathrm{id}, (23)]$, $[(23), \mathrm{id}]$ | No | Yes |
| $[(23), \mathrm{id}, \mathrm{id}]$ | $[\mathrm{id}, \mathrm{id}]$, $[(23), (23)]$ | Yes | Yes |
| $[(23), (23), (23)]$ | $[\mathrm{id}, (23)]$, $[(23), \mathrm{id}]$ | No | Yes |
| $[(123), (123), (123)]$ | $[\mathrm{id}, (132)]$, $[(132), \mathrm{id}]$, $[(123), (123)]$ | No | Yes |
| $[(23), (13), (12)]$ | $[\mathrm{id}, \mathrm{id}]$ | Yes | Yes |

Table 3. The 13 isomorphism classes of GL-racks of order 3.

| $[s_1, s_2, s_3, s_4]$ | $[\mathtt{u}, \mathtt{d}]$ | GL-quandle? | Medial? |
|---|---|---|---|
| $[\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}]$ | $[\mathrm{id}, \mathrm{id}]$, $[(34), (34)]$, $[(243), (234)]$, $[(1432), (1234)]$ | Yes | Yes |
| $[\mathrm{id}, (13)(24), \mathrm{id}, (13)(24)]$ | $[\mathrm{id}, (24)]$, $[(24), \mathrm{id}]$, $[(13), (13)(24)]$, $[(13)(24), (13)]$ | No | Yes |

| | | | |
|---|---|---|---|
| $[(13)(24),(13)(24),(13)(24),(13)(24)]$ | $[\mathrm{id},(13)(24)]$,<br>$[(24),(13)]$,<br>$[(1432)(1432)]$,<br>$[(14)(23),(12)(34)]$,<br>$[(13)(24),\mathrm{id}]$ | No | Yes |
| $[\mathrm{id},\mathrm{id},(34),(34)]$ | $[\mathrm{id},(34)]$,<br>$[(34),\mathrm{id}]$,<br>$[(12),(12)(24)$,<br>$[(12)(34),(12)]$ | No | Yes |
| $[\mathrm{id},(34),\mathrm{id},\mathrm{id}]$ | $[\mathrm{id},\mathrm{id}]$,<br>$[(34),(34)]$ | Yes | Yes |
| $[\mathrm{id},(34),(34),(34)]$ | $[\mathrm{id},(34)]$,<br>$[(34),\mathrm{id}]$ | No | Yes |
| $[(34),(34),\mathrm{id},\mathrm{id}]$ | $[\mathrm{id},\mathrm{id}]$,<br>$[(34),(34)]$,<br>$[(12)(34),(12)(34)]$ | Yes | Yes |
| $[(34),(34),(34),(34)]$ | $[\mathrm{id},(34)]$,<br>$[(34),\mathrm{id}]$,<br>$[(12),(12)(34)]$,<br>$[(12)(34),(12)]$ | No | Yes |
| $[\mathrm{id},(234),(234),(234)]$ | $[\mathrm{id},(243)]$,<br>$[(243),\mathrm{id}]$,<br>$[(234),(234)]$ | No | Yes |
| $[(234),\mathrm{id},\mathrm{id},\mathrm{id}]$ | $[\mathrm{id},\mathrm{id}]$,<br>$[(243),(234)]$ | Yes | Yes |
| $[(234),(234),(234),(234)]$ | $[\mathrm{id},(243)]$,<br>$[(243),\mathrm{id}]$,<br>$[(234),(234)]$ | No | Yes |
| $[(234),(243),(243),(243)]$ | $[\mathrm{id},(234)]$,<br>$[(243),(243)]$,<br>$[(234),\mathrm{id}]$ | No | Yes |
| $[(34),(34),(12),(12)]$ | $[\mathrm{id},\mathrm{id}]$,<br>$[(34),(34)]$,<br>$[(12)(34),(12)(34)]$ | Yes | Yes |

| | | | |
|---|---|---|---|
| $[(34),(34),(12)(34),(12)(34)]$ | $[\mathrm{id},(34)]$, $[(34),\mathrm{id}]$, $[(12),(12)(34)]$, $[(12)(34),(12)]$ | No | Yes |
| $[(12),(12),(34),(34)]$ | $[\mathrm{id},(12)(34)]$, $[(34),(12)]$, $[(12)(34),\mathrm{id}]$ | No | Yes |
| $[(12),(12),(12)(34),(12)(34)]$ | $[\mathrm{id},(12)(34)]$, $[(34),(12)]$, $[(12)(34),\mathrm{id}]$ | No | Yes |
| $[(1324),(1324),(1324),(1324)]$ | $[\mathrm{id},(1423)]$, $[(1423),\mathrm{id}]$, $[(1324),(12)(34)]$, $[(12)(34),(1324)]$ | No | Yes |
| $[\mathrm{id},(34),(24),(23)]$ | $[\mathrm{id},\mathrm{id}]$ | Yes | No |
| $[(234),(143),(124),(132)]$ | $[\mathrm{id},\mathrm{id}]$ | Yes | Yes |

Table 4: The 59 isomorphism classes of GL-racks of order 4.