Math 370, homework 1

due Thursday, January 23, at 10pm ET

Solution late by less than two hours will receive full credit. The submission site will close at 11:59pm.

Be sure to read the homework notes attached in the posted assignment, so that you know what is permitted and expected on Math 370 assignments (and to read about an extra point that each problem has assigned to proof writing).

The comments in italics are not a part of the homework, they're just notes about the problems, why they're important, sometimes I write hints in there too.

Extra note: The Stewart problems are out of the 5th (new) edition. I will make a note where these differ from the 4th, whenever I notice a difference.

Topics

Chapter 1: This chapter is mostly for fun, we did some of it this week, and will do some later on.

Chapter 2: You are not responsible for the proof of FTA (yet), though you should read it if you haven't seen it, just for yourself. We will do a Galois Theory proof later during the semester.

Chapter 3: I've skipped around a little, because I like to use reduction mod p to prove Gauss's Lemma. $K[x]$ first, as written. Then reduction mod p, Eisenstein criterion, and Gauss's Lemma.

Problems

First an algebraic proof that a polynomial gives the zero evaluation map on $\mathbb{C}$ only if all of its coefficients are zero (which seems obvious, but it takes some doing if you don't want to use analysis). The conclusion is that the map $\mathbb{C}[t] \longrightarrow \{\text{functions} : \mathbb{C} \to \mathbb{C}\}$ we discussed in class is injective. Beware of using lecture notes: we assumed this result in class, and 2.6 is included here to justify the assumption (so you cannot use results from lecture to prove it).

Extra Stewart 2.5 (same in 4th and 5th edition). *This is for practice only - do think about it, but you don't have to turn it in.*

1. Stewart 2.6 (same in 4th and 5th edition)

Extra Stewart 3.1, 3.2, 3.3 *Again, practice only.*

2. Stewart 3.4, plus factor the polynomials into irreducibles (with proof). (same in 4th and 5th edition)

Polynomials over finite fields, as promised.

3. Determine the monic irreducible polynomials of degrees 1, 2, and 3 in $\mathbb{F}_3[x]$.
   You must explicitly list these polynomials as part of your solution. Remember to show your work.

4. What is the number of monic irreducible polynomials of degree 4 in $\mathbb{F}_3[x]$?
   Remember that answers must come with proofs, or, in this case, work shown to justify it.
   For those of you less familiar with combinatorics, I posted a short video in the media library, with a combinatorial trick that can help you with the calculation. You can use the trick without proving that it works (just so long as you have justified to yourself that it works).

The next two problems are fun facts, for practice with Chapter 3 ideas. Use those ideas for the proofs (no extra number theory or other topics are permitted for these two problems).

5. *This problem is to be done without any aid: no books, notes, computers, internet, or anything else. You may collaborate with your classmates. (It's a fun puzzle to work out - don't spoil it by looking up the answer. Feel free to come work on it during one of the ULA sessions though!)*

   From the Chinese remainder theorem, we know the following: If $(m, n) = 1$, then there is a ring isomorphism
   $$\varphi : \mathbb{Z}/mn\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

   The map from left to right is $\varphi(z) = (z, z)$ (where the first z is viewed mod mn, and the next two mod m and n, respectively).

   Note that in this situation, there are integers a, b such that $am + bn = 1$.

   Write an explicit formula for $\varphi^{-1}(x, y)$ in terms of a, b, m, n, x, y.

   Then work out the formula in terms of x, y in the case of $m = 99, n = 40$. (*Write out how you calculated this. You may use a calculator to add, subtract, and multiply integers; no other aid is permitted.*)

6. Prove that two integer polynomials are relatively prime in $\mathbb{Q}[x]$ if and only if the ideal they generate in $\mathbb{Z}[x]$ contains a nonzero integer.

7. Prove that $f = x^4 - 3x + 45 \in \mathbb{Z}[x]$ is irreducible. Credit will be awarded for correctness as well as efficiency of the solution.
   In other words, you should make as much use as you can of the things we've shown in class; unnecessary computations will cost credit in this question.