ON THE CATEGORY OF GENERALIZED LEGENDRIAN RACKS AND ITS APPLICATIONS

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Abstract. In 2023, Karmakar et al. and Kimura introduced a nonassociative algebraic structure, called a *GL-rack*, that can distinguish between Legendrian links in $\mathbb{R}^3$. In this article, we study the category of GL-racks, allowing us to better understand their structure and prove several conjectures about Legendrian links. To motivate the study of GL-racks, we use GL-rack coloring numbers to give a short algebraic proof that the topological knot 6_2 has two distinct Legendrian representatives with $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$. Then, we confirm an empirical observation of Karmakar et al. that the enveloping group of the GL-rack of a Legendrian link Λ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus \Lambda)$. We also introduce tensor products of GL-racks that induce a symmetric monoidal closed structure on the full subcategory of medial GL-quandles. From an additional functorial construction, we obtain a group-theoretic sufficient condition for any two GL-racks or quandles to be isomorphic.

1. Introduction

In 1980, Joyce [17] introduced algebraic structures called *quandles* as a means of capturing the topological structure of knots, links, and symmetric spaces. Since then, quandles and slightly more general algebraic objects called *racks* have enjoyed significant study as link invariants among geometric topologists and in their own right among quantum algebraists. Recently, various authors have constructed generalizations of racks and quandles to study *Legendrian links* in contact geometry (e.g., [5, 19, 20, 24]). Defining homomorphisms of these algebraic structures yields categories like GLR, whose objects are called *GL-racks*, *generalized Legendrian racks*, or *bi-Legendrian racks.*

In this paper, we study the category GLR and its applications. This helps us understand the structure of GL-racks and detect their isomorphisms, which can otherwise be cumbersome due to their nonassociativity. This also allows us to prove several conjectures about Legendrian links and generalize several properties of quandles to GL-racks. The main results of this paper are as follows.

Theorem 1.1. *The two Legendrian knots with* $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$ *and underlying smooth knot type* 6_2 *given in* [8] *are distinguishable by GL-rack coloring numbers.*

Theorem 1.2. *Let* $\Lambda \subset \mathbb{R}^3$ *be an oriented Legendrian link, let* $\mathrm{GLR}(\Lambda)$ *be the GL-rack of* Λ *as defined in Definition 3.2, and let* $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda))$ *be its enveloping group as defined in Definition 2.19. Then there exists a group isomorphism*

$$\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda)) \cong \pi_1(\mathbb{R}^3 \setminus \Lambda).$$

Theorem 1.3. *The full subcategory of* GLR *whose objects are medial GL-quandles is symmetric monoidal closed. In particular, it is self-enriched.*

Inspired by the homogeneous representations of GL-racks constructed in [19, Section 5], we also construct a category GrpTup satisfying the following.

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Theorem 1.4. *There exists an essentially surjective functor $\mathcal{F} : \mathsf{GrpTup} \twoheadrightarrow \mathsf{GLR}$. This functor induces a group-theoretic sufficient condition for any two GL-racks or quandles to be isomorphic.*

In Section 2, we give an overview of the questions in Legendrian knot theory motivating the study of GL-racks and quandles. We proceed by defining these algebraic structures abstractly and introducing related groups, categories, and functors in the literature.

In Section 3, we discuss how to assign a GL-rack to an oriented Legendrian link, give several worked examples, and discuss related invariants of Legendrian links. Then, we prove Theorems 1.1 and 1.2, which we state as Theorem 3.7 and Corollary 3.10, respectively. Our approach to the former result offers a simpler algebraic alternative to Dynnikov and Prasolov's proof in [10, Proposition 2.3] and gives a positive answer to a question posed in [20, Section 4], which we state as Corollary 3.8. The latter result confirms an empirical observation from the original version of [19, Remark 8.2].

In Section 4, we define *medial* or *abelian* GL-racks and tensor products of GL-racks. After proposing a medial GL-quandle-valued invariant of Legendrian links, we prove Theorem 1.3, which we state as Theorem 4.4 and Corollary 4.5. This generalizes an analogous result for medial quandles [9, Theorem 12], whose proof we follow closely.

In Section 5, we define GrpTup using objects constructed from collections of left cosets of groups. Karmakar et al. originally employed these objects in [19, Theorem 5.2] to produce a homogeneous representation of any GL-rack. Then, we prove Theorem 1.4, the first part of which we state as Theorem 5.1 and the second part of which we discuss afterward.

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2. Definitions and notation

In this section, we provide motivations, definitions, and notation for the study of GL-racks. In particular, we discuss the crossing and cusp relations afforded by Legendrian Reidemeister moves, which motivate the axioms of GL-racks. After stating some preliminary results on GL-racks, we discuss several functors appearing in the literature on GL-racks and quandles.

2.1. **Motivations: Legendrian knots and links.**

In this subsection, we introduce Legendrian links and Legendrian Reidemeister moves. These give contact-geometric motivations for studying GL-racks and quandles. (For an accessible introduction to Legendrian knot theory, we refer the reader to [30]. For a more detailed survey of the field, we refer the reader to [14].)

Definition 2.1. A *knot* is the image of a smooth embedding of the circle S^1 into $\mathbb{R}^3$, and a *link* is a disjoint union of a finite number of knots. A link Λ is called *Legendrian* if it lies everywhere tangent to the *standard contact structure* $\xi_{\text{std}} := \ker(dz - y\,dx)$ on $\mathbb{R}^3$, which is depicted in Figure 1. That is, $T_x\Lambda \in \xi_{\text{std}}$ for all $x \in \Lambda$, where $T_x\Lambda$ denotes the tangent space of Λ at x. A *front projection* or *front diagram* $D(\Lambda)$ is the projection of Λ to the xz-plane.

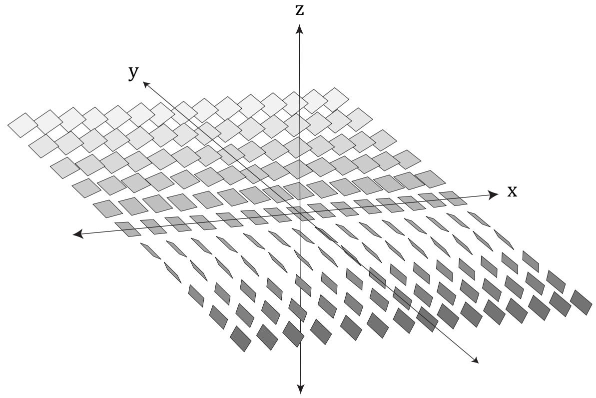

Figure 1. The standard contact structure ξ_{std} on $\mathbb{R}^3$. Taken from [23].

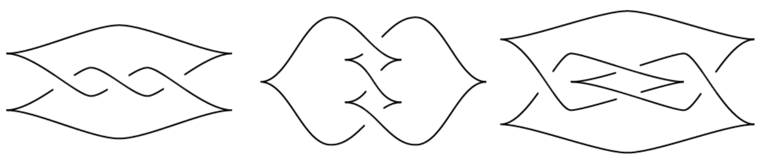

Figure 2. Unoriented front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot. Adapted from [14].

When distinguishing Legendrian links from links that are not necessarily Legendrian, we will call the latter *smooth links* even though Legendrian links are themselves smooth. Also, we will denote the underlying smooth link of a Legendrian link Λ by L. In particular, $D(\Lambda)$ can be viewed as a *link diagram* of L, denoted by $D(L)$, by "ignoring" all cusps in $D(\Lambda)$.

Central to contact geometry is the question of when two Legendrian links can be smoothly deformed to each other in a way that preserves the condition of being Legendrian at every stage. This notion, called *Legendrian isotopy*, is formalized as follows.

Definition 2.2. Viewed as smooth embeddings of S^1 into $\mathbb{R}^3$, two Legendrian links Λ_1, Λ_2 are considered *Legendrian isotopic* if they exists a smooth homotopy $H : S^1 \times [0,1] \to \mathbb{R}^3$ such that $H \times \{0\} = \Lambda_1$, $H \times \{1\} = \Lambda_2$, and $H \times \{t\}$ is a Legendrian link for all $t \in [0,1]$.

Legendrian links Λ are typically studied through their front projections, which follow several restrictions thanks to the tangency condition on Λ. For one, at every crossing in $D(\Lambda)$, the strand with the more negative slope is always the overstrand. For two, $D(\Lambda)$ has cusps in place of vertical tangencies. For example, Figure 2 depicts front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot, and Figure 3 depicts front projections of two distinct oriented Legendrian unknots. Note that the numbers of crossings and cusps in a Legendrian front projection are finite due to smoothness.

In fact, one can use the tangency condition to show that the geometric structure of an oriented Legendrian link Λ can be recovered entirely from its front projection $D(\Lambda)$ (cf. [14].) For example, two invariants called the *Thurston-Bennequin number* and *rotation number* of Λ, denoted respectively by $\text{tb}(\Lambda)$ and $\text{rot}(\Lambda)$, can be defined as the integers

$$\text{tb}(\Lambda) = P - N - \frac{1}{2}(D + U), \quad \text{rot}(\Lambda) = \frac{1}{2}(D - U),$$

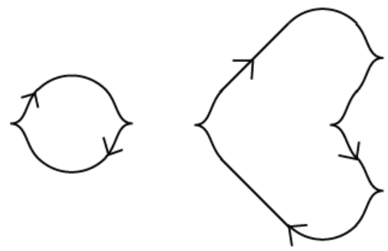

FIGURE 3. Front projections of distinct oriented Legendrian unknots. Adapted from [29].

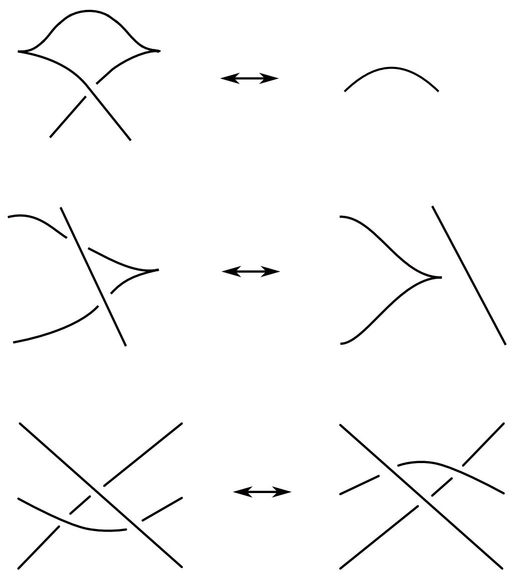

FIGURE 4. The three Legendrian Reidemeister moves. Taken from [19].

where P, N, D, and U are the numbers of positively oriented crossings, negatively oriented crossings, downward-oriented cusps, and upward-oriented cusps in $D(\Lambda)$, respectively. It is well-known that two Legendrian links are Legendrian isotopic only if their Thurston-Bennequin and rotation numbers are equal. Indeed, a celebrated theorem of Świątkowski in 1992 offers a method of comparing Legendrian links using only their front projections.

Proposition 2.3. [32, Theorem B] *Two Legendrian links are Legendrian isotopic if and only if their front projections are related by a finite sequence of planar isotopies and the three* Legendrian Reidemeister moves *depicted in Figure 4.*

Example 2.4. Let Λ_L and Λ_R be the oriented Legendrian unknots depicted on the left and right of Figure 3, respectively. Although Λ_L and Λ_R share the same underlying smooth knot type, they are not Legendrian isotopic because $\mathrm{tb}(\Lambda_L) = -1 \neq -2 = \mathrm{tb}(\Lambda_R)$ and $\mathrm{rot}(\Lambda_L) = 0 \neq 1 = \mathrm{rot}(\Lambda_R)$. Proposition 2.3 asserts that the two front projections in Figure 3 cannot be related by any sequence of the Legendrian Reidemeister moves in Figure 4.

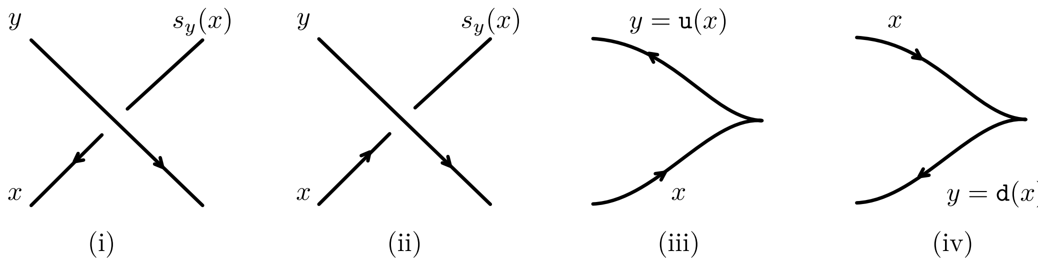

FIGURE 5. Crossing and cusp relations. Adapted from [19].

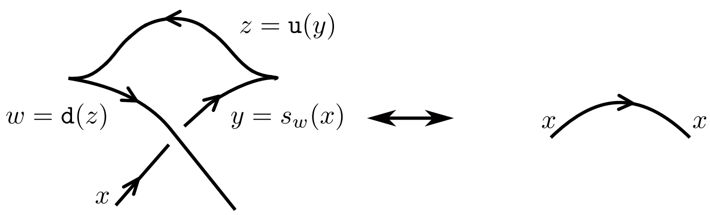

FIGURE 6. Crossing and cusp relations in one possible orientation of the first Legendrian Reidemeister move. Adapted from [19].

There are in fact infinitely many examples of distinct Legendrian links having the same underlying smooth link type, making distinguishing between Legendrian links significantly more difficult than distinguishing between smooth links. This has motivated the development of numerous *invariants* of Legendrian links, called so because they are constructed to be invariant under Legendrian isotopy. Examples include the Thurston-Bennequin and rotation numbers [14], the Chekanov-Eliashberg differential graded algebra and associated polynomial invariants [14, 27], various (co)homology theories (see [16] for a list), and the mosaic number [23, 29].

GL-racks and quandles have also been used to define algebro-combinatorial and cohomological invariants of both Legendrian links and smooth links. These include fundamental quandles and their Legendrian analogues [5, 17, 19], R-coloring numbers [5, 19, 20], cocycle invariants [6, 21], and state-sum invariants [3, 13], many of which have elegant categorifications (e.g., [4, 6, 7]). These invariants motivate the study of GL-racks and quandles as categories.

The axioms of GL-racks are motivated by the *crossing* and *cusp relations* induced between strands of an oriented Legendrian link modulo the relations afforded by the Legendrian Reidemeister moves. In Figure 5, (i) and (ii) depict crossing relations between strands in a Legendrian front projection, and (iii) and (iv) depict cusp relations. Note that $\mathtt{u}$ and $\mathtt{d}$ correspond to the relations induced by upward- and downward-oriented cusps, respectively. Figures 6-8 depict the crossing and cusp relations in one possible orientation of each of the three Legendrian Reidemeister moves. For a complete list of all possible orientations and their induced crossing and cusp relations, we refer the reader to [20, Figures 6-8].

In Subsection 3.1, we detail how to assign a GL-rack to any oriented Legendrian link Λ using the cusp and crossing relations in $D(\Lambda)$. This assignment is independent of the choice of front projection of Λ, making it an invariant of Legendrian links [19, Theorem 4.3].

2.2. **GL-racks and quandles.** In this subsection, we define GL-racks and quandles abstractly by translating the crossing and cusp relations in Subsection 2.1 into the language of *rack symmetries*.

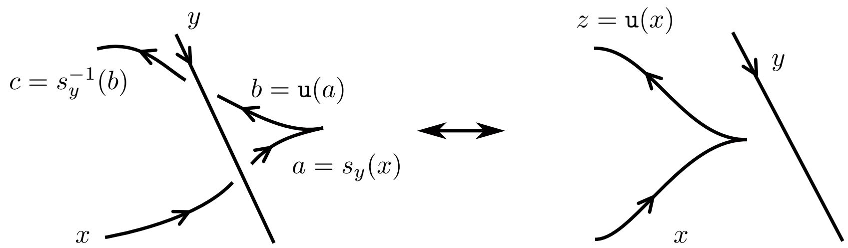

FIGURE 7. Crossing and cusp relations in one possible orientation of the second Legendrian Reidemeister move. Adapted from [19].

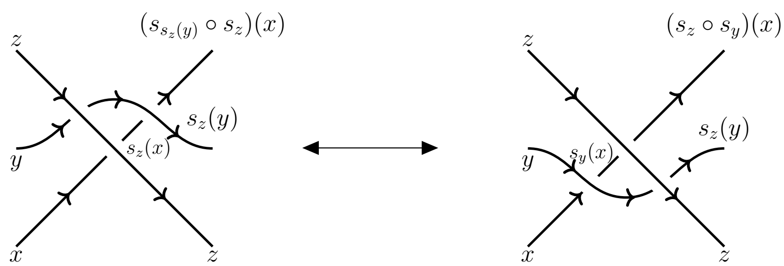

FIGURE 8. Crossing relations in one possible orientation of the third Legendrian Reidemeister move. Adapted from [5].

We defer discussing how one can assign these algebraic structures to Legendrian links to Subsection 3.1.

Although racks and quandles are often defined as sets X endowed with binary operations $\triangleright : X \times X \to X$ and $\triangleright^{-1} : X \times X \to X$ satisfying axioms corresponding to the crossing relations induced by the three Reidemeister moves, they may also be characterized in terms of bijections $s_x : X \to X$, one for each element $x \in X$, that satisfy the same axioms (cf. [12, Section 2; 17, Definition 1.1; 33, Definition 2.7]). Such a bijection is called the *symmetry at* x. One may translate between the two conventions via the formulas $s_x(y) = y \triangleright x$ and $s_x^{-1}(y) = y \triangleright^{-1} x$. In this article, we adopt the definitions using symmetries due to their convenience for categorical proofs. In particular, we have rewritten all crossing relations in Figures 5 through 8 in the notation of rack symmetries. We refer the reader to [26] for an accessible introduction to quandle theory in terms of $\triangleright$ and $\triangleright^{-1}$. We also refer the reader to [11] for a more comprehensive survey of modern algebraic research on racks and quandles.

The rack and quandle axioms encapsulate the crossing relations depicted in Figures 6-8.

Definition 2.5. Let X be a set, and let s be a map from X to $\operatorname{Sym} X$ defined by $x \mapsto s_x$. We call the pair (X, s) a *rack* or a *wrack* if, for all $x, y \in X$, we have $s_x \circ s_y = s_{s_x(y)} \circ s_x$. We say that s is a *rack structure* on X, and we say that s_x is the *symmetry at* x. If in addition $s_x(x) = x$ for all $x \in X$, we say that (X, s) is a *quandle*, and we say that s is a *quandle structure* on X. In this case, if $Q \subset X$ and $s_x(y) \in Q$ for all $x, y \in Q$, then we say that $(Q, s|_Q)$ is a *subquandle* of (X, s). Finally, we say that a rack (X, s) is *faithful* if s is injective.

Example 2.6. [33, Definition 2.11] Let Ω be a union of conjugacy classes in a group G, and define $s : \Omega \to \operatorname{Sym} \Omega$ by

$$\alpha \mapsto s_\alpha := [\omega \mapsto \alpha\omega\alpha^{-1}].$$

Then (Ω, s) is a quandle called the *conjugation quandle*, and we denote it by $\operatorname{Conj} \Omega$.

The axioms of *GL-racks* encode not only the crossing relations but also the cusp relations given by the Legendrian Reidemeister moves in Figures 6-8. GL-racks were introduced independently in

[18] and [20] to generalize the *Legendrian racks* introduced in [5] and [24]. Once again, we translate the original definition into the language of rack symmetries.

Definition 2.7. A *generalized Legendrian rack*, or *GL-rack*, is a quadruple $(X, s, \mathtt{u}, \mathtt{d})$ in which (X, s) is a rack and $\mathtt{u}, \mathtt{d} : X \to X$ are maps such that the following hold for all $x \in X$:

(L1) $(\mathtt{ud} \circ s_x)(x) = x = (\mathtt{du} \circ s_x)(x)$.
(L2) $\mathtt{u} \circ s_x = s_x \circ \mathtt{u}$ and $\mathtt{d} \circ s_x = s_x \circ \mathtt{d}$.
(L3) $s_{\mathtt{u}(x)} = s_x = s_{\mathtt{d}(x)}$.

We call the pair $(\mathtt{u}, \mathtt{d})$ a *GL-structure* on (Q, s). If in addition (X, s) is a quandle, we say that $(X, s, \mathtt{u}, \mathtt{d})$ is a *generalized Legendrian quandle*, or *GL-quandle*.

Due to the pair $(\mathtt{u}, \mathtt{d})$, GL-racks are also called *bi-Legendrian racks* (cf. [20]).

Example 2.8. [20, Example 3.6] Let G be a group, let $z \in Z(G)$, and define $f : G \to G$ by $g \mapsto zg$. Then $(\operatorname{Conj} G, f, f^{-1})$ is a GL-quandle.

Example 2.9. Let $R = (X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack. If $\mathtt{u} = \mathtt{d}$, we say that R is a *Legendrian rack* (cf. [5, Definition 4; 19, Remark 3.3]). If in addition (X, s) is a quandle, we say that R is a *Legendrian quandle*. Thus, GL-racks generalize Legendrian racks.

Example 2.10. [5, Example 12] Let $X = \{1, 2, 3, 4\}$. In cycle notation, let $\sigma \in S_4$ be the permutation (123), and let $\mathtt{u} = \mathtt{d} = \sigma$. Also, define $s : X \to S_4$ by $x \mapsto \sigma$ for all $x \in X$, so that $s_x(y) = \sigma(y)$ for all $x, y \in X$. Then $(X, s, \mathtt{u}, \mathtt{d})$ is a (generalized) Legendrian rack but not a Legendrian quandle.

Example 2.11. [19, Example 3.4] Any GL-rack of the form $(X, s, \mathrm{id}_X, \mathrm{id}_X)$ is called a *trivial GL-rack*. In particular, any quandle (Q, s) can be identified with the trivial GL-rack $(Q, s, \mathrm{id}_Q, \mathrm{id}_Q)$, cf. Lemma 2.17. In other words, GL-racks generalize quandles.

We define homomorphisms of these algebraic structures as follows.

Definition 2.12. Let (X, s) and (Y, t) be racks. A map $\varphi : X \to Y$ is called a *rack homomorphism* if $\varphi \circ s_x = t_{\varphi(x)} \circ \varphi$ for all $x \in X$. If in addition $(\mathtt{u}_1, \mathtt{d}_2)$ and $(\mathtt{u}_2, \mathtt{d}_2)$ are GL-structures on (X, s) and (Y, t), we say that a φ is also a *GL-rack homomorphism* if $\varphi \circ \mathtt{u}_1 = \mathtt{u}_2 \circ \varphi$ and $\varphi \circ \mathtt{d}_1 = \mathtt{d}_2 \circ \varphi$. A *(GL-)rack isomorphism* is simply a bijective (GL-)rack homomorphism. If R is a GL-rack, we denote its group of automorphisms by $\operatorname{Aut} R$.

Evidently, we have the following; the last statement is from [19, Proposition 3.2].

Proposition 2.13. *Let (X, s) be a rack with maps $\mathtt{u}, \mathtt{d} : X \to X$ satisfying axioms (L1) and (L3) of Definition 2.7. Then $(X, s, \mathtt{u}, \mathtt{d})$ is a GL-rack if and only if $\mathtt{u}$ and $\mathtt{d}$ are endomorphisms of the underlying rack (X, s). In this case, $\mathtt{u}$ and $\mathtt{d}$ are actually rack automorphisms.*

Axiom (L1) immediately yields the following.

Proposition 2.14. *Let $(X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack. Then the underlying rack (X, s) is a quandle if and only if $\mathtt{ud} = \mathrm{id}_X = \mathtt{du}$, that is, $\mathtt{d} = \mathtt{u}^{-1}$ as rack automorphisms.*

One easy consequence of Propositions 2.13 and 2.14 is the following.

Proposition 2.15. *Let (Q, s) be a quandle, and let $\mathtt{u}, \mathtt{d}$ be maps from Q to Q. Then the following are equivalent:*

(1) The pair $(\mathtt{u}, \mathtt{d})$ defines a GL-structure on (Q, s). That is, $(Q, s, \mathtt{u}, \mathtt{d})$ is a GL-quandle.
(2) $\mathtt{u}$ and $\mathtt{d}$ are rack endomorphisms satisfying axiom (L3), and $\mathtt{ud} = \mathrm{id}_Q = \mathtt{du}$.

Another easy consequence completely characterizes faithful GL-racks.

Proposition 2.16. *Let $(X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack. Then (X, s) is a faithful rack if and only if (X, s) is a faithful quandle. In this case, $\mathtt{u} = \mathrm{id}_X = \mathtt{d}$.*

2.3. **Functors of interest in the literature.** In this subsection, we define several categories and functors appearing in the literature on GL-racks, quandles, and their relationships with groups.

We begin by defining several categories. Let Set and Grp be the categories of sets and groups, respectively. Let Rack be the category of racks, and let Qnd be the full subcategory of Rack whose objects are quandles. Similarly, let GLR be the category of GL-racks, and let GLQ be the full subcategory of GLR whose objects are GL-quandles. By Example 2.11 and Proposition 2.14, we have the following.

Lemma 2.17. *The correspondence $(Q,s)\mapsto(Q,s,\mathrm{id}_Q,\mathrm{id}_Q)$ defines a canonical isomorphism from Qnd to the full subcategory of GLR whose objects are trivial GL-racks.*

In the sense of universal algebra, GLR is an equational algebraic category, so it is complete and cocomplete (cf. [1, Corollary 1.2, Theorem 4.5]). Thus, we can express GL-racks in terms of generators and relations using quotients of *free GL-racks*, which we define as follows.

Definition 2.18. [19, Section 4] Let X be a set. We define the *free GL-rack on X*, denoted by $\mathrm{FGLR}(X)$, as follows. If $X=\emptyset$, let $\mathrm{FGLR}(X)$ be the trivial GL-rack with one element. Else, define the *universe of words generated by X* to be the set $W(X)$ such that $X\subset W(X)$ and $s_y(x)$, $s_y^{-1}(x)$, $\mathtt{u}(x)$, $\mathtt{d}(x)\in W(X)$ for all $x,y\in W(X)$. Let $F(X)$ be the set of equivalence classes of elements of $W(X)$ modulo the equivalence relation generated by the following relations for all $x,y,z\in W(X)$:

(1) $s_y^{-1}(s_y(x))y\sim x\sim s_y(s_y^{-1}(x))$.
(2) $s_z(s_y(x))\sim s_{s_z(y)}(s_z(x))$.
(3) $\mathtt{u}(\mathtt{d}(s_x(x)))\sim x\sim \mathtt{d}(\mathtt{u}(s_x(x)))$.
(4) $\mathtt{u}(s_y(x))\sim s_y(\mathtt{u}(x))$ and $\mathtt{d}(s_y(x))\sim s_y(\mathtt{d}(x))$.
(5) $s_{\mathtt{u}(y)}(x)\sim s_y(x)$ and $s_{\mathtt{d}(y)}(x)\sim s_y(x)$.

Thus, we have maps $s:F(X)\to\operatorname{Sym}F(X)$ defined by $x\mapsto s_x:=[y\mapsto s_x(y)]$ and $\mathtt{u},\mathtt{d}:F(X)\to F(X)$ defined by $x\mapsto\mathtt{u}(x)$ and $x\mapsto\mathtt{d}(x)$. We define $\mathrm{FGLR}(X)$ to be the GL-rack $(F(X),s,\mathtt{u},\mathtt{d})$. The *free quandle on X* is defined similarly; in the sense of Lemma 2.17, it is simply $\mathrm{FGLR}(X)$ modulo the relations $\mathtt{u}(x)\sim x\sim\mathtt{d}(x)$ for all $x\in W(X)$.

To rephrase [19, Proposition 4.2], the functor $\mathsf{Set}\to\mathsf{GLR}$ defined by $X\mapsto\mathrm{FGLR}(X)$ is left adjoint to the forgetful functor $\mathsf{GLR}\to\mathsf{Set}$, as one might expect.

Another functor of interest in Section 3 assigns an *enveloping group* to any GL-rack.

Definition 2.19. [18, Section 8] Given a GL-rack $R=(X,s,\mathtt{u},\mathtt{d})$, its *enveloping group* is

$$\mathrm{Env}_{\mathsf{GLR}}(R):=\left\langle e_x,\ x\in X\mid e_{s_x(y)}=e_x^{-1}e_ye_x,\ e_{\mathtt{u}(x)}=e_x,\ e_{\mathtt{d}(x)}=e_x,\ x,y\in X\right\rangle.$$

By taking $\mathtt{u}=\mathrm{id}_X=\mathtt{d}$, we can also define the enveloping group of a quandle (Q,s) to be

$$\mathrm{Env}_{\mathsf{Qnd}}(Q,s):=\left\langle e_x,\ x\in Q\mid e_{s_x(y)}=e_x^{-1}e_ye_x,\ x,y\in Q\right\rangle.$$

The end of Example 3.5 in Subsection 3.2 computes the enveloping groups of the GL-rack and fundamental quandle of a Legendrian $(2,-q)$ torus knot as defined in Subsection 3.1.

The functor $\mathrm{Env}_{\mathsf{GLR}}:\mathsf{GLR}\to\mathsf{Grp}$ has a right adjoint that results from taking $\Omega=G$ in Example 2.6, as specified in the following result adapted from [18, Proposition 8.4].

Proposition 2.20. *There exists a functor $\mathrm{Env}_{\mathsf{GLR}}:\mathsf{GLR}\to\mathsf{Grp}$ that sends any GL-rack to its enveloping group and sends any GL-rack homomorphism $\psi:(X,s,\mathtt{u}_1,\mathtt{d}_1)\to(Y,t,\mathtt{u}_1,\mathtt{d}_2)$ to the group homomorphism $\widetilde{\psi}:\mathrm{Env}_{\mathsf{GLR}}(X,s,\mathtt{u}_1,\mathtt{d}_1)\to\mathrm{Env}_{\mathsf{GLR}}(Y,t,\mathtt{u}_1,\mathtt{d}_2)$ defined by $e_x\mapsto e_{\psi(x)}$ for all $x\in X$.*

Also, $\mathrm{Env}_{\mathsf{GLR}}$ *is left adjoint to a functor sending any group* G *to the GL-rack* $(\mathrm{Conj}\, G, \mathrm{id}_G, \mathrm{id}_G)$*, which is isomorphic to the quandle* $\mathrm{Conj}\, G$ *in the sense of Lemma 2.17.*

Thus, some authors denote the enveloping group of a GL-rack or quandle R by $\mathrm{Adconj}\, R$ or $\mathrm{As}\, R$ and call it the *associated group of* R (cf. [17, Section 6; 28, Definition 2.19]).

3. On rack-theoretic invariants of Legendrian links

In this section, we begin by defining the GL-rack of an oriented Legendrian link Λ and the fundamental quandle of its underlying smooth link L, both of which are invariant under Legendrian isotopy. After a few worked examples, we give short algebraic proofs of several conjectures relating to Legendrian links and their invariants.

3.1. The GL-rack of a Legendrian link.

In this subsection, we discuss how to assign a GL-rack to a Legendrian link in a way invariant under Legendrian isotopy. We begin with several definitions.

Definition 3.1. Given a front projection $D(\Lambda)$ of an oriented Legendrian link Λ, define a *cusped strand* of $D(\Lambda)$ to be a connected segment in $D(\Lambda)$ that either starts and ends at a crossing or (in the case that $D(\Lambda)$ contains no crossings) ends where it started. Also, define an *uncusped strand* of $D(\Lambda)$ to be a maximal (with respect to inclusion) connected subset of a cusped strand of $D(\Lambda)$ that both starts and ends at either a crossing or a cusp. Also,

Thus, each cusped strand may be partitioned into uncusped strands.

Definition 3.2. [19, Section 4] Let Λ be an oriented Legendrian link with front projection $D(\Lambda)$, and let X_Λ be a set in one-to-one correspondence with the cusped strands of $D(\Lambda)$. At each cusp, label the neighboring uncusped strands using the cusp relations in Figure 5. Then, at each crossing, impose the corresponding crossing relation between uncusped strands in Figure 5 on $\mathrm{FGLR}(X_\Lambda)$. The *GL-rack of* Λ, denoted by $\mathrm{GLR}(\Lambda)$, is defined to be the set of equivalence classes of elements of $\mathrm{FGLR}(X_\Lambda)$ modulo the equivalence relation generated by these relations. If L is a smooth link with link diagram $D(L)$, then we define the *fundamental quandle of* L, denoted by $Q(L)$, in a similar manner. However, we use the free quandle on X_Λ in place of $\mathrm{FGLR}(X_\Lambda)$, and we do not employ any cusp relations.

For several examples of how to compute the GL-rack of a Legendrian link, see Subsection 3.2.

The assignment of $\mathrm{FGLR}(\Lambda)$ to Λ (resp. $Q(L)$ to L) is independent of the choice of front projection $D(\Lambda)$ (resp. $D(L)$), as captured in the following result of Karmakar et al.

Proposition 3.3. [19, Theorem 4.3] *If two oriented Legendrian front projections are related by a finite sequence of Legendrian Reidemeister moves, then their induced GL-racks are isomorphic. Hence, the GL-rack of a Legendrian link is invariant under Legendrian isotopy.*

This is a consequence of Proposition 2.3 and the fact that the GL-rack axioms capture the crossing and cusp relations induced by the Legendrian Reidemeister moves.

Definition 3.2 helps us interpret the structure of finitely presented GL-racks geometrically. By Example 2.9, imposing the equivalence relation $\mathtt{u}(x) \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$ onto $\mathrm{GLR}(\Lambda) = (F(X_\Lambda), s, \mathtt{u}, \mathtt{d})$ yields a Legendrian rack. Geometrically, this amounts to "ignoring" the orientations of cusps in $D(\Lambda)$. Imposing the equivalence relation $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$ onto $\mathrm{GLR}(\Lambda)$ yields a quandle in the sense of Lemma 2.17. Geometrically, this amounts to "ignoring" all cusps in $D(\Lambda)$ and viewing $D(\Lambda)$ only as a diagram of the underlying smooth link L. This recovers $Q(L)$ from $\mathrm{GLR}(\Lambda)$, yielding the following result.

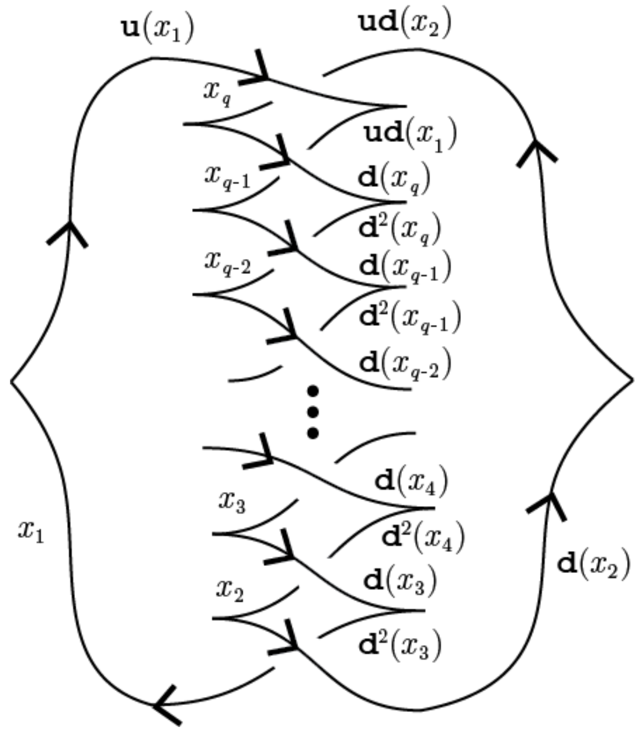

FIGURE 9. Front projection $D(\Lambda)$ and induced cusp relations of the Legendrian $(2,-q)$-torus knot Λ with maximal Thurston-Bennequin and rotation numbers.

Lemma 3.4. [21, Remark 23] *Let Λ be an oriented Legendrian link with front projection $D(\Lambda)$, and let L be its underlying smooth link. With $X_\Lambda = X_L$ as defined in Definition 3.2 using $D(\Lambda)$ (where X_L is defined by viewing $D(\Lambda)$ as a link diagram of L), impose an equivalence relation onto* $\mathrm{GLR}(\Lambda)$ *defined by $\mathtt{u}(x) \sim x \sim \mathtt{d}(x)$ for all $x \in X_\Lambda$. Then, the resulting GL-rack is canonically isomorphic to $(Q(L), \mathrm{id}_{X_L}, \mathrm{id}_{X_L})$, which is isomorphic to $Q(L)$ in the sense of Lemma 2.17.*

3.2. **Example calculations of** $\mathrm{GLR}(\Lambda)$ **and** $Q(L)$**.** In this section, we give examples of how to compute the GL-rack and fundamental quandle of an oriented Legendrian link. This allows us to give a relatively brief algebraic proof of a conjecture in [8] about Legendrian 6_2 knots.

Example 3.5. Let $q \geq 3$ be an odd integer, let L be a $(2,-q)$-torus knot, and let Λ be the Legendrian representative of L having maximal Thurston-Bennequin and rotation numbers. (By [15, Theorem 4.3], Λ is the unique such Legendrian representative up to Legendrian isotopy.) In this example, we compute $\mathrm{GLR}(\Lambda)$, $Q(L)$, $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda))$, and $\mathrm{Env}_{\mathsf{Qnd}}(Q(L))$ using the front projection $D(\Lambda)$ in Figure 9. Note that $\mathrm{tb}(\Lambda) = -2q$ and $\mathrm{rot}(\Lambda) = q-2$ as predicted by [15, Theorems 4.1 and 4.4], and $D(\Lambda)$ contains q strands. Traversing $D(\Lambda)$ along its depicted orientation and recording the induced cusp and crossing relations using Figure 5, we compute that $\mathrm{GLR}(\Lambda)$ is the free GL-rack on the set $X_\Lambda = \{x_1, \dots, x_q\}$ modulo the crossing relations

$$s_{\mathtt{u}(x_1)}(x_q) = \mathtt{ud}(x_2),\ s_{\mathtt{d}(x_q)}(x_{q-1}) = \mathtt{ud}(x_1),\ \text{and } s_{\mathtt{d}(x_{i-1})}(x_{i-2}) = \mathtt{d}^2(x_i) \text{ for all } 3 \leq i \leq q.$$

By axiom (L3) of a GL-rack, these crossing relations can be simplified to

$$s_{x_1}(x_q) = \mathtt{ud}(x_2),\ s_{x_q}(x_{q-1}) = \mathtt{ud}(x_1),\ \text{and } s_{x_{i-1}}(x_{i-2}) = \mathtt{d}^2(x_i) \text{ for all } 3 \leq i \leq q.$$

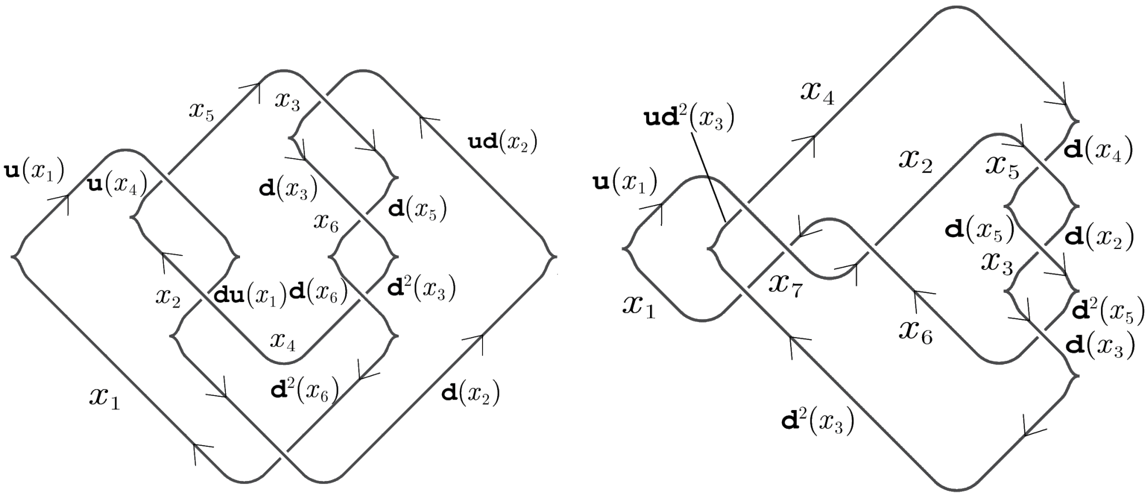

FIGURE 10. Front projections of the two Legendrian representatives of the topological knot 6_2 with $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$ given in [8]. Created using [22] (cf. [23]).

Now that we have a presentation of $\mathrm{GLR}(\Lambda)$, let us compute $Q(L)$. To do this, we could traverse $D(\Lambda)$ again while ignoring all cusps and only considering crossing relations. In view of Lemma 3.4, we could equivalently impose the relations $\mathtt{u}(x_i) = x_i = \mathtt{d}(x_i)$ for all $1 \le i \le q$ onto $\mathrm{GLR}(\Lambda)$. Either method shows that $Q(L)$ is the free quandle on X_Λ modulo the crossing relations

$$s_{x_1}(x_q) = x_2,\ s_{x_q}(x_{q-1}) = x_1,\ \text{and } s_{x_{i-1}}(x_{i-2}) = x_i \text{ for all } 3 \le i \le q.$$

Indeed, if we invert each symmetry in the relations of $Q(L)$, then we recover the fundamental quandle of the mirror image of L computed in [2, Remark 3], as predicted by [34, Section 1].

If $q = 3$, then L is a left-handed trefoil, and the crossing relations show that $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda))$ and $\mathrm{Env}_{\mathsf{Qnd}}(Q(L))$ are both isomorphic to the group

$$\begin{aligned}&\left\langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{s_{x_1}(x_3)} = e_{x_1}^{-1} e_{x_3} e_{x_1}, e_{s_{x_2}(x_1)} = e_{x_2}^{-1} e_{x_1} e_{x_2}, e_{s_{x_3}(x_2)} = e_{x_3}^{-1} e_{x_2} e_{x_3} \right\rangle \\ &\quad = \langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{x_1} e_{x_2} = e_{x_3} e_{x_1}, e_{x_2} e_{x_3} = e_{x_1} e_{x_2}, e_{x_3} e_{x_1} = e_{x_2} e_{x_3} \rangle .\end{aligned}$$

As predicted by the results of the next subsection, this is precisely the Wirtinger presentation of the knot group $\pi_1(\mathbb{R}^3 \setminus L) \cong \langle x, y \mid x^2 = y^3 \rangle$ of L (cf. [31, Subsection 4.2.5]).

Example 3.6. Let Λ_1 and Λ_2 be the oriented Legendrian knots on the left and right of Figure 10, respectively. In this example, we compute $\mathrm{GLR}(\Lambda_1)$ and $\mathrm{GLR}(\Lambda_2)$ in preparation for a proof that Λ_1 and Λ_2 are not Legendrian isotopic. We note from [8] that Λ_1 and Λ_2 are both Legendrian representatives of the topological knot 6_2, and $(\mathrm{tb}(\Lambda_1), \mathrm{rot}(\Lambda_1)) = (-7, 2) = (\mathrm{tb}(\Lambda_2), \mathrm{rot}(\Lambda_2))$. Moreover, these knots cannot be distinguished using the graded ruling invariant or linearized contact homology.

Let us begin with Λ_1. Starting at an arbitrarily chosen crossing (which, in this example, we choose to be the bottommost crossing), traverse $D(\Lambda_1)$ using its given orientation while labeling all uncusped strands as in Figure 5. By writing down the induced crossing relations as in Figure 5, we find that $\mathrm{GLR}(\Lambda_1)$ is the free GL-rack on the set $X_{\Lambda_1} = \{x_1, \ldots, x_6\}$ modulo the following crossing relations:

$$\text{(1)} \qquad \mathrm{GLR}(\Lambda_1)\begin{cases} s_{\mathtt{u}(x_1)}(\mathtt{u}(x_4)) = x_5 \iff s_{x_1}(\mathtt{u}(x_4)) = x_5, \\ s_{x_4}(\mathtt{du}(x_1)) = x_2, \\ s_{\mathtt{d}(x_2)}(x_1) = \mathtt{d}^2(x_6) \iff s_{x_2}(x_1) = \mathtt{d}^2(x_6), \\ s_{x_5}(x_3) = \mathtt{ud}(x_2), \\ s_{\mathtt{d}(x_3)}(x_6) = \mathtt{d}(x_5) \iff s_{x_3}(x_6) = \mathtt{d}(x_5), \\ s_{\mathtt{d}(x_6)}(x_4) = \mathtt{d}^2(x_3) \iff s_{x_6}(x_4) = \mathtt{d}^2(x_3), \end{cases}$$

where we have simplified the first, third, fifth, and sixth relations using GL-rack axiom (L3).

Similarly, we compute $\mathrm{GLR}(\Lambda_2)$ to be the free GL-rack on the set $X_{\Lambda_1} = \{x_1, \dots, x_7\}$ modulo the following crossing relations:

$$\text{(2)} \qquad \mathrm{GLR}(\Lambda_2)\begin{cases} s_{x_1}(\mathtt{ud}^2(x_3)) = x_4, \\ s_{x_1}(x_6) = x_7, \\ s_{x_6}(x_2) = \mathtt{u}(x_1), \\ s_{x_2}(x_5) = \mathtt{d}(x_4), \\ s_{x_5}(x_3) = \mathtt{d}(x_2), \\ s_{x_3}(x_6) = \mathtt{d}^2(x_5), \\ s_{x_3}(x_7) = x_1. \end{cases}$$

We can use these calculations to prove a conjecture of Chongchitmate and Ng in [8] that Λ_1 and Λ_2 in the previous example are not Legendrian isotopic. In [10, Proposition 2.3], Dynnikov and Prasolov proved this conjecture using magnificent topological and combinatorial machinery. At the time of writing, this was the only proof we could find of the conjecture. As a simpler algebraic alternative, we propose a GL-rack-theoretic approach. We employ the *R-coloring number* of an oriented Legendrian link Λ with respect to a fixed GL-rack R, which is defined to be the cardinality of the hom-set $\mathrm{Hom}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda), R)$. Due to Proposition 3.3, the R-coloring number is a Legendrian link invariant (cf. [5, 19, 20]).

Theorem 3.7. *The two oriented Legendrian knots in Figure 10 are not Legendrian isotopic; in fact, they are distinguishable using R-coloring numbers.*

Proof. As before, let Λ_1 and Λ_2 be the oriented Legendrian knots on the left and right of Figure 10, respectively. By Proposition 3.3, it suffices to show that $\mathrm{GLR}(\Lambda_1)$ and $\mathrm{GLR}(\Lambda_2)$ are nonisomorphic. To that end, let $R = (Y, t, f, f)$ be the Legendrian rack in Example 2.10. We will show that $|\mathrm{Hom}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda_1), R)| > |\mathrm{Hom}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda_2), R)|$. Recall that $t_y(z) = f(z) = \sigma(z)$ for all $y, z \in X$, where $\sigma \in S_4$ is the permutation (123) in cycle notation. Now, let A denote the underlying set of $\mathrm{GLR}(\Lambda_1)$, define $\varphi_4 : A \to Y$ by $x_i \mapsto 4$ for all $1 \le i \le 6$, and define $\varphi_1, \varphi_2, \varphi_3 : A \to X$ by the following:

$$\varphi_1(x_i) := \begin{cases} 1 & \text{if } i \in \{1,2,4\}, \\ 2 & \text{if } i = 3, \\ 3 & \text{if } i \in \{5,6\}. \end{cases} \qquad \varphi_2(x_i) := \begin{cases} 2 & \text{if } i \in \{1,2,4\}, \\ 3 & \text{if } i = 3, \\ 1 & \text{if } i \in \{5,6\}. \end{cases} \qquad \varphi_3(x_i) := \begin{cases} 3 & \text{if } i \in \{1,2,4\}, \\ 1 & \text{if } i = 3, \\ 2 & \text{if } i \in \{5,6\}. \end{cases}$$

Using the relations in (1), it is straightforward to verify that φ_1, φ_2, φ_3, and φ_4 define GL-rack homomorphisms from $\mathrm{GLR}(\Lambda_1)$ to R. Hence, $|\mathrm{Hom}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda_1), R)| \ge 4$. (In fact, using a similar method as in the remainder of this proof, one can show that this bound is actually an equality.)

On the other hand, we claim that $|\operatorname{Hom}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda_2), R)| = 1$. Consider the presentation of $\mathrm{GLR}(\Lambda_2)$ in (2), and let B denote the underlying set of $\mathrm{GLR}(\Lambda_2)$. We will show that if $\varphi : B \to Y$ defines a GL-rack homomorphism from $\mathrm{GLR}(\Lambda_2)$ to R with $\varphi(x_i) = y_i$, then we must have $y_i = 4$ for all $1 \le i \le 6$. (Note that the map sending each x_i to 4 is indeed a GL-rack homomorphism.) Since φ is a GL-rack homomorphism, the relations in (2) must hold when we replace each x_i with y_i, each s_{x_i} with $t_{y_i} = \sigma$, and each $\mathtt{u}$ and $\mathtt{d}$ with $f = \sigma$. Therefore, we have the following system of equations:

$$\begin{cases} \sigma^4(y_3) = y_4, \\ \sigma(y_6) = y_7, \\ \sigma(y_2) = \sigma(y_1), \\ \sigma(y_5) = \sigma(y_4), \\ \sigma(y_3) = \sigma(y_2), \\ \sigma(y_6) = \sigma^2(y_5), \\ \sigma(y_7) = y_1. \end{cases}$$

Since σ is injective and $\sigma^3 = \mathrm{id}_X$, these seven equations simplify to the following:

$$\begin{cases} \sigma(y_3) = y_4 = y_5, \\ y_2 = y_1 = y_3 = \sigma(y_7), \\ \sigma(y_6) = y_7, \\ y_6 = \sigma(y_5). \end{cases} \tag{3}$$

It follows that

$$y_5 = \sigma(y_3) = \sigma^2(y_7) = \sigma^3(y_6) = y_6 = \sigma(y_5).$$

Since the only fixed point of σ is 4, it follows that $y_5 = 4$. By taking $y_5 = 4$ in (3), we immediately deduce that $y_i = 4$ for all $1 \le i \le 7$, as desired. □

Incidentally, Theorem 3.7 also gives a positive answer to a question posed by Kimura in [20, Section 4], which we state below. This results from the fact that the GL-rack R used in the proof is not a GL-quandle.

Corollary 3.8. *Although R-coloring numbers cannot distinguish between Legendrian knots having the same underlying smooth knot type, Thurston-Bennequin number, and rotation number if R is a GL-quandle* [20, Theorem 4.3]*, the same is not generally true if R is a nonquandle GL-rack.*

3.3. **Isomorphism of $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda))$ and $\pi_1(\mathbb{R}^3 \setminus L)$.** We now prove an empirical observation of Karmakar et al. in the original version of [18, Remark 8.2], which we state as Corollary 3.10. Although the authors have removed this remark from subsequent releases of the article, we are unaware of any other proofs of this statement at the time of writing. To allow for a short categorical proof, we begin by proving a more abstract version of the conjecture.

Theorem 3.9. *Let $R = (X, s, \mathtt{u}, \mathtt{d})$ be a GL-rack, and let $R' = (X/\sim, s^*, \mathrm{id}_{X/\sim}, \mathrm{id}_{X/\sim})$ be the GL-rack obtained by imposing an equivalence relation $\sim$ onto R defined by $\mathtt{u}(x) = x = \mathtt{d}(x)$ for all $x \in X$. Then $\mathrm{Env}_{\mathsf{GLR}}(R)$ and $\mathrm{Env}_{\mathsf{GLR}}(R')$ are isomorphic as groups. In particular, $\mathrm{Env}_{\mathsf{GLR}}(R)$ and $\mathrm{Env}_{\mathsf{Qnd}}(X/\sim, s^*)$ are isomorphic as groups.*

Proof. By definition, R' is the quotient object of the equivalence relation $\sim$ on R in GLR. In other words, R is the colimit of the following diagram in GLR:

$$R \underset{\mathrm{id}_X}{\overset{\mathtt{u}}{\rightrightarrows}} R \underset{\mathrm{id}_X}{\overset{\mathtt{d}}{\rightrightarrows}} R$$

Recall that left adjoint functors preserve colimits. It follows from Proposition 2.20 that $\mathrm{Env}_{\mathsf{GLR}}(R')$ is the colimit of the following diagram in Grp:

$$\mathrm{Env}_{\mathsf{GLR}}(R) \underset{\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)}{\overset{\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})}{\rightrightarrows}} \mathrm{Env}_{\mathsf{GLR}}(R) \underset{\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)}{\overset{\mathrm{Env}_{\mathsf{GLR}}(\mathtt{d})}{\rightrightarrows}} \mathrm{Env}_{\mathsf{GLR}}(R)$$

By Proposition 2.20, the group homomorphism $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})$ is defined by $e_x \mapsto e_{\mathtt{u}(x)}$ for all $x \in X$, but $e_x = e_{\mathtt{u}(x)}$ in $\mathrm{Env}_{\mathsf{GLR}}(R)$. Thus, $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{u})$ is the identity map. Similarly, $\mathrm{Env}_{\mathsf{GLR}}(\mathtt{d})$ and $\mathrm{Env}_{\mathsf{GLR}}(\mathrm{id}_X)$ are the identity maps, so we have a group isomorphism

$$\mathrm{Env}_{\mathsf{GLR}}(R) \cong \mathrm{Env}_{\mathsf{GLR}}(R').$$

By Lemma 2.17, the right-hand side is isomorphic to $\mathrm{Env}_{\mathsf{Qnd}}(X/\sim, s^*)$, so we are done. □

Corollary 3.10. *Let $\Lambda \subset \mathbb{R}^3$ be an oriented Legendrian link, and let L denote its underlying smooth link. Then there exists a group isomorphism*

$$\mathrm{Env}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda)) \cong \pi_1(\mathbb{R}^3 \setminus L).$$

Proof. Taking $R = \mathrm{GLR}(\Lambda)$ in Theorem 3.9, we have $R' \cong (Q(L), \mathrm{id}_{X_L}, \mathrm{id}_{X_L})$ by Lemma 3.4. By Theorem 3.9, it suffices to show that $\mathrm{Env}_{\mathsf{Qnd}}(Q(L))$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus L)$. But this is shown in [17, Section 15] using the Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus L)$. □

4. $\mathsf{GLQ}^{\mathrm{med}}$ IS SYMMETRIC MONOIDAL CLOSED

In this section, we define medial GL-racks and introduce tensor products of GL-racks that make the category of medial GL-quandles symmetric monoidal closed. This generalizes a similar result for the category of medial quandles [9, Theorem 12], whose proof we follow closely.

4.1. **Hom-sets of medial GL-quandles are also medial GL-quandles.** In this subsection, we define medial GL-racks, introduce a medial GL-quandle structure on any hom-set from a GL-rack to a medial GL-quandle, and propose a medial GL-quandle-valued invariant of Legendrian links.

Definition 4.1. A rack (X, s) is called *medial* or *abelian* if, for all $x, y, z \in X$, we have

$$s_{s_x(z)} \circ s_y = s_{s_x(y)} \circ s_z.$$

If in addition $(\mathtt{u}, \mathtt{d})$ defines a GL-structure on (X, s), then we call $(X, s, \mathtt{u}, \mathtt{d})$ a *medial* or *abelian* GL-rack. (Note that this definition is not synonymous with the condition that (X, s) is *commutative*, which states that $s_x(y) = s_y(x)$ for all $x, y \in X$.)

Let GLQ be the full subcategory of GLR whose objects are GL-quandles. By Proposition 2.14, any GL-structure on a quandle (Q, s) has the form $(\mathtt{u}, \mathtt{u}^{-1})$. Thus, we can write any object in GLQ simply as a triple $(Q, s, \mathtt{u})$. We will do so hereafter for the sake of brevity.

Proposition 4.2. *Let $R = (X, s^X, \mathtt{u}_1, \mathtt{d}_1)$ be a GL-rack, and let $Q = (Y, s^Y, \mathtt{u}_2)$ be a medial GL-quandle. Let $H := \mathrm{Hom}_{\mathsf{GLR}}(R, Q)$, and define $\mathtt{u} : H \to H$ by $f \mapsto \mathtt{u}_2 \circ f$, so that $\mathtt{u}$ has a two-sided inverse $\mathtt{u}^{-1}$ given by $f \mapsto \mathtt{u}_2^{-1} \circ f$. Also, define $s : H \to \mathrm{Sym}\, H$ by $g \mapsto s_g$, where $s_g(f) := [x \mapsto (s^Y_{g(x)} \circ f)(x)]$. Then $(H, s, \mathtt{u})$ is a medial GL-quandle.*

Proof. Let $\widetilde{H} := \mathrm{Hom}_{\mathsf{Rack}}((X, s^X), (Y, s^Y))$, and define $\widetilde{s} : \widetilde{H} \to \mathrm{Sym}\, \widetilde{H}$ the same way that s is defined. Since (Y, s^Y) is a medial quandle, [9, Theorem 3] states that $(\widetilde{H}, \widetilde{s})$ is a medial quandle.[1]

[1] Although the original authors assume that (X, s^X) is a quandle, the quandle axiom that $s^X_x(x) = x$ for all $x \in X$ goes unused in their proof. Hence, their result holds even when (X, s^X) is a nonquandle rack.

Therefore, to show that (H,s) is a medial quandle, it suffices to show that (H,s) is a subquandle of $(\widetilde{H},\widetilde{s})$. To that end, fix $f,g\in H$. Then, we have $s_g(f)\in H$ because

$$\begin{aligned}
\mathtt{u}_2\circ s_g(f) &= [x\mapsto(\mathtt{u}_2\circ s^Y_{g(x)}\circ f)(x)] & \\
&= [x\mapsto(s^Y_{(\mathtt{u}_2\circ g)(x)}\circ \mathtt{u}_2\circ f)(x)] & \text{by Proposition 2.13}\\
&= [x\mapsto(s^Y_{(g\circ \mathtt{u}_1)(x)}\circ f\circ \mathtt{u}_1)(x)] & \text{since } f,g\in H\\
&= s_g(f)\circ \mathtt{u}_1 &
\end{aligned}$$

and, similarly, $\mathtt{u}_2^{-1}\circ s_g(f)=s_g(f)\circ \mathtt{d}_1$. Thus, (H,s) is a medial subquandle of $(\widetilde{H},\widetilde{s})$. Clearly, $\mathtt{u}\mathtt{u}^{-1}=\mathrm{id}_H=\mathtt{u}^{-1}\mathtt{u}$. By Proposition 2.15, to complete the proof that $(\mathtt{u},\mathtt{u}^{-1})$ defines a GL-structure on (H,s), it remains to show that $\mathtt{u}$ and $\mathtt{u}^{-1}$ are rack endomorphisms satisfying GL-rack axiom (L3). Since $(\mathtt{u}\circ s_g)(f)=\mathtt{u}(s_g(f))=\mathtt{u}_2\circ s_g(f)$, the above calculation shows that

$$(\mathtt{u}\circ s_g)(f)=[x\mapsto(s^Y_{(\mathtt{u}_2\circ g)(x)}\circ \mathtt{u}_2\circ f)(x)]=s_{\mathtt{u}_2\circ g}(\mathtt{u}_2\circ f)=(s_{\mathtt{u}(g)}\circ \mathtt{u})(f)$$

and, similarly, $(\mathtt{u}^{-1}\circ s_g)(f)=(s_{\mathtt{u}^{-1}(g)}\circ \mathtt{u}^{-1})(f)$. Hence, $\mathtt{u}$ and $\mathtt{u}^{-1}$ are rack endomorphisms. To verify GL-rack axiom (L3), apply this axiom to $\mathtt{u}_2$ to obtain that

$$s_{\mathtt{u}(g)}(f)=[x\mapsto(s^Y_{\mathtt{u}_2(g(x))}\circ f)(x)]=[x\mapsto(s^Y_{g(x)}\circ f)(x)]=s_g(f)$$

and, similarly, $s_{\mathtt{u}^{-1}(g)}(f)=s_g(f)$. This completes the proof. □

Note that Proposition 4.2 does not require for R to be medial or a GL-quandle. As a result, Proposition 4.2 could result in strengthened versions of Q-coloring invariants of Legendrian links when Q is a medial GL-quandle. For example, do there exist (distinct) Legendrian links Λ_1 and Λ_2 and a medial GL-quandle Q such that $\mathrm{Hom}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda_1),Q)$ and $\mathrm{Hom}_{\mathsf{GLR}}(\mathrm{GLR}(\Lambda_2),Q)$ have equal cardinalities but are nonisomorphic as medial GL-quandles? In view of [20, Theorem 4.3], it would be especially interesting to ask this question with the additional conditions that Λ_1 and Λ_2 share the same underlying smooth link type, Thurston-Bennequin number, and rotation number.

4.2. **Tensor products of GL-racks.** Let $\mathsf{GLQ}^{\mathrm{med}}$ be the full subcategory of GLQ whose objects are medial. In this subsection, we introduce tensor products in GLR and show that $\mathsf{GLQ}^{\mathrm{med}}$ is symmetric monoidal closed.

We define tensor products of GL-racks similarly to those of quandles in [9].

Definition 4.3. If $R_1=(X,s^X,\mathtt{u}_1,\mathtt{d}_1)$ and $R_2=(Y,s^Y,\mathtt{u}_2,\mathtt{d}_2)$ are GL-racks, then we define their *tensor product*, denoted by $R_1\otimes R_2$, to be the free GL-rack $\mathrm{FGLR}(X\times Y)$ modulo the following relations for all $x,x_1,x_2\in X$ and $y,y_1,y_2\in Y$:

(1) $s_{(x,y_2)}(x,y_1)=(x,s^Y_{y_2}(y_1))$.
(2) $s_{(x_2,y)}(x_1,y)=(s^X_{x_2}(x_1),y)$.
(3) $\mathtt{u}(x,y)=(\mathtt{u}_1(x),y)=(x,\mathtt{u}_2(y))$.
(4) $\mathtt{d}(x,y)=(\mathtt{d}_1(x),y)=(x,\mathtt{d}_2(y))$.

Note that if R_1 and R_2 are GL-quandles, then so is $R_1\otimes R_2$. In light of Lemma 2.17, tensor products of GL-racks generalize tensor products of quandles (cf. [9, Subsection 8.1]).

Tensor products of GL-racks are of particular interest to the development of (co)homology theories for GL-racks and their induced Legendrian link invariants (cf. [3,6,18,19,21]). Indeed, the next two results show that tensor products of medial GL-quandles satisfy the expected universal property and internal hom-tensor adjunction.

Theorem 4.4. *The category* $\mathsf{GLQ}^{\mathrm{med}}$ *is symmetric monoidal closed with respect to the closed structure* $\mathrm{Hom}_{\mathsf{GLQ}^{\mathrm{med}}}(-,-)$ *in Proposition 4.2 and the tensor product* $\otimes$ *in Definition 4.3.*

Proof. The unit object in $\mathsf{GLQ}^{\mathrm{med}}$ is the trivial GL-quandle with one element. Using this fact, it is straightforward to verify that $\mathsf{GLQ}^{\mathrm{med}}$ is monoidal and symmetric. On the other hand, $\mathsf{GLQ}^{\mathrm{med}}$ is defined as an equational algebraic category. Thus, the main theorem of [25] states that $\mathsf{GLQ}^{\mathrm{med}}$ is closed if and only if, in the sense of universal algebra, $\mathsf{GLQ}^{\mathrm{med}}$ is commutative as an algebraic theory (cf. [25, Section 6]). Indeed, for any medial GL-quandle $(X, s, \mathtt{u})$ and for all elements $x_{11}, x_{12}, x_{21}, x_{22} \in X$, we have the following equalities:

$$\begin{cases} (\mathtt{u} \circ s_{x_{12}})(x_{11}) = (s_{\mathtt{u}(x_{12})} \circ \mathtt{u})(x_{11}) & \text{by Proposition 2.13,} \\ (\mathtt{u}^{-1} \circ s_{x_{12}})(x_{11}) = (s_{\mathtt{u}^{-1}(x_{12})} \circ \mathtt{u}^{-1})(x_{11}) & \text{by Proposition 2.13,} \\ (s_{s_{x_{22}}(x_{21})} \circ s_{x_{12}})(x_{11}) = (s_{s_{x_{22}}(x_{12})} \circ s_{x_{21}})(x_{11}) & \text{since } (X,s) \text{ is medial.} \end{cases}$$

Together with the tautologies $(\mathtt{u} \circ \mathtt{u}^{-1})(x_{11}) = (\mathtt{u}^{-1} \circ \mathtt{u})(x_{11})$, $\mathtt{u}^2(x_{11}) = \mathtt{u}^2(x_{11})$, and $(\mathtt{u}^{-1})^2(x_{11}) = (\mathtt{u}^{-1})^2(x_{11})$, these equalities show that $\mathsf{GLQ}^{\mathrm{med}}$ forms a commutative algebraic theory. This completes the proof. □

In light of Proposition 4.2, we immediately deduce the following.

Corollary 4.5. $\mathsf{GLQ}^{\mathrm{med}}$ *is self-enriched. Explicitly, if A, B, and C are medial GL-quandles, then there exists a natural isomorphism of medial GL-quandles*

$$\mathrm{Hom}_{\mathsf{GLQ}^{\mathrm{med}}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathsf{GLQ}^{\mathrm{med}}}(A, \mathrm{Hom}_{\mathsf{GLQ}^{\mathrm{med}}}(B, C)).$$

5. A sufficient condition for GL-rack isomorphisms

In [19, Section 5], Karmakar et al. construct a homogeneous representation for any GL-rack R from the orbits of R under the action of $\mathrm{Aut}\, R$. In this section, we adapt this construction into a category GrpTup with an essentially surjective functor $\mathcal{F} : \mathsf{GrpTup} \twoheadrightarrow \mathsf{GLR}$. This functor induces a group-theoretic sufficient condition for any two GL-racks or quandles to be isomorphic.

5.1. Construction of GrpTup.

In this subsection, we introduce a category GrpTup with a functorial relationship to GLR.

To define the objects in GrpTup, we adapt the construction of [19, Proposition 5.1]. Let the objects in GrpTup be all sextuples $(I, \bigsqcup_{i \in I} G/H_i, Z_I, Q_I, R_I, \tau)$ satisfying the following:

1. I is an indexing set, G is a group, and $Z_I = \{z_i^G \mid i \in I\}$, $Q_I = \{q_i^G \mid i \in I\}$, and $R_I = \{r_i^G \mid i \in I\}$ are multisets indexed by I whose elements lie in G.
2. $\{H_i \mid i \in I\}$ is a family of subgroups of G such that $H_i \leq C_G(z_i^G)$ for all $i \in I$.
3. $\tau : I \to I$ is a bijection such that the following hold for all $i \in I$:
   - (a) $q_i^G \in N_G(H_{\tau(i)})$.
   - (b) $r_i^G \in N_G(H_{\tau^{-1}(i)})$.
   - (c) $z_i^G q_i^G r_{\tau(i)}^G,\ z_i^G r_i^G q_{\tau^{-1}(i)}^G \in H_i$.
   - (d) $z_i^G q_i^G (z_{\tau(i)}^G)^{-1} = q_i^G$.
   - (e) $z_i^G r_i^G (z_{\tau^{-1}(i)}^G)^{-1} = r_i^G$.

For the sake of brevity, we will denote such an object as $\widetilde{G}$ when there is no room for confusion. For an opposing object in GrpTup, we will write $\widetilde{K} := (J, \bigsqcup_{j \in J} K/L_j, Z_J, Q_J, R_J, \pi)$.

Now, we define the morphisms in GrpTup. Given any two objects $\widetilde{G}, \widetilde{K}$ in GrpTup, let $\mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$ be the set of all triples $\varphi := (\varphi_1, \varphi_2, \varphi_3)$ satisfying the following:

(1) $\varphi_1 : G \to K$ is a group homomorphism.
(2) $\varphi_2 : \bigsqcup_{i\in I} G/H_i \to \bigsqcup_{j\in J} K/L_j$ and $\varphi_3 : I \to J$ are morphisms in Set.
(3) $\pi \circ \varphi_3 = \varphi_3 \circ \tau$.
(4) For all $i \in I$ and $g \in G$, we have $\varphi_1(z_i^G) = z^K_{\varphi_3(i)}$, $\varphi_1(q_i^G) = q^K_{\varphi_3(i)}$, $\varphi_1(r_i^G) = r^K_{\varphi_3(i)}$, and $\varphi_2(gH_i) = \varphi_1(g)L_{\varphi_3(i)}$.

Define the composition of morphisms in GrpTup by $\psi \circ \varphi := (\psi_1 \circ \varphi_1, \psi_2 \circ \varphi_2, \psi_3 \circ \varphi_3)$. Also, define the identity morphism $\mathrm{id}^{\widetilde{G}} : \widetilde{G} \xrightarrow{\sim} \widetilde{G}$ by letting $\mathrm{id}_1^{\widetilde{G}}$, $\mathrm{id}_2^{\widetilde{G}}$, and $\mathrm{id}_3^{\widetilde{G}}$ as defined above be identity maps. Associativity and unit laws are immediate. Hence, GrpTup is a category.

5.2. **Construction of $\mathcal{F} : \mathsf{GrpTup} \to \mathsf{GLR}$.** In this subsection, we construct an essentially surjective functor $\mathcal{F} : \mathsf{GrpTup} \twoheadrightarrow \mathsf{GLR}$. Then, we discuss a group-theoretic sufficient condition for two GL-racks to be isomorphic.

By [19, Proposition 5.1], given any object $\widetilde{G}$ in GrpTup, the set $X := \bigsqcup_{i\in I} G/H_i$ admits a GL-rack structure in which $s^X : X \to \operatorname{Sym} X$ and $\mathtt{u}_G, \mathtt{d}_G : X \to X$ are defined by

$$s^X(yH_j) := s^X_{yH_j} := [xH_i \mapsto yz_j^G y^{-1}xH_i],\ \mathtt{u}_G(xH_i) := xq_i^G H_{\tau(i)},\ \text{and}\ \mathtt{d}_G(xH_i) := xr_i^G H_{\tau^{-1}(i)}.$$

So, we can define a functor $\mathcal{F} : \mathsf{GrpTup} \to \mathsf{GLR}$ by sending any object $\widetilde{G}$ in GrpTup to the GL-rack $(X, s^X, \mathtt{u}_G, \mathtt{d}_G)$ and sending any morphism $\varphi \in \operatorname{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$ to φ_2.

Theorem 5.1. *$\mathcal{F}$ is an essentially surjective functor.*

Proof. Essential surjectivity is precisely the statement of [19, Theorem 5.2]. Certainly, $\mathcal{F}$ preserves identity morphisms and composition of morphisms. To complete the proof of functoriality, it remains to show that if $\varphi \in \operatorname{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$, then $\mathcal{F}\varphi : \mathcal{F}(\widetilde{G}) \to \mathcal{F}(\widetilde{K})$ is a GL-rack homomorphism. Write $\mathcal{F}(\widetilde{G}) = (X, s^X, \mathtt{u}_G, \mathtt{d}_G)$ and $\mathcal{F}(\widetilde{K}) = (Y, s^Y, \mathtt{u}_K, \mathtt{d}_K)$, and fix $gH_a \in X$. Since φ_1 is a group homomorphism, we have

$$\begin{aligned}
\mathcal{F}\varphi \circ s^X_{gH_a} &= [xH_i \mapsto \varphi_2(gz_a^G g^{-1}xH_i)] \\
&= [xH_i \mapsto \varphi_1(gz_a^G g^{-1}x)L_{\varphi_3(i)}] \\
&= [xH_i \mapsto \varphi_1(g)\varphi_1(z_a^G)\varphi_1(g^{-1})\varphi_1(x)L_{\varphi_3(i)}] \\
&= [xH_i \mapsto \varphi_1(g)z^K_{\varphi_3(a)}\varphi_1(g)^{-1}\varphi_2(xH_i)] \\
&= s^Y_{\varphi_1(g)L_{\varphi_3(a)}} \circ \varphi_2 = s^Y_{\varphi_2(gH_a)} \circ \varphi_2 = s^Y_{\mathcal{F}\varphi(gH_a)} \circ \mathcal{F}\varphi,
\end{aligned}$$

so $\mathcal{F}\varphi$ is a rack homomorphism. Moreover, we have

$$\begin{aligned}
\mathcal{F}\varphi \circ \mathtt{u}_G &= [xH_i \mapsto \varphi_2(xq_i^G H_{\tau(i)})] \\
&= [xH_i \mapsto \varphi_1(x)\varphi_1(q_i^G)L_{\varphi_3(\tau(i))}] \\
&= [xH_i \mapsto \varphi_1(x)q^K_{\varphi_3(i)}L_{\pi(\varphi_3(i))}] \\
&= [yL_j \mapsto yq_j^K L_{\pi(j)}] \circ [xH_i \mapsto \varphi_1(x)L_{\varphi_3(i)}] \\
&= \mathtt{u}_K \circ [xH_i \mapsto \varphi_2(xH_i)] = \mathtt{u}_K \circ \mathcal{F}\varphi,
\end{aligned}$$

as desired. Similarly, $\mathcal{F}\varphi \circ \mathtt{d}_G = \mathtt{d}_K \circ \mathcal{F}\varphi$, so $\mathcal{F}\varphi$ is a GL-rack homomorphism. $\square$

This result gives us a group-theoretic way to show that two GL-racks R_1 and R_2 are isomorphic. The proof of [19, Theorem 5.2] describes a procedure to construct objects $\widetilde{G}$ and $\widetilde{K}$ in GrpTup such that $R_1 \cong \mathcal{F}(\widetilde{G})$ and $R_2 \cong \mathcal{F}(\widetilde{K})$ in GLR. To show that $R_1 \cong R_2$ in GLR, it suffices to find a morphism $\varphi \in \mathrm{Hom}_{\mathsf{GrpTup}}(\widetilde{G}, \widetilde{K})$ such that φ_2 is bijective, since then $\mathcal{F}(\varphi) : \mathcal{F}(\widetilde{G}) \xrightarrow{\sim} \mathcal{F}(\widetilde{H})$ will also be bijective and, hence, an isomorphism of GL-racks.

Finally, let $\mathsf{GrpTrip}$ be the full subcategory of GrpTup consisting of objects $\widetilde{G}$ for which $\tau = \mathrm{id}_I$, and Q_I and R_I are multisets only containing copies of 1_G. By Lemma 2.17 and [17, Theorem 7.2], $\mathcal{F}$ induces an essentially surjective functor $\mathsf{GrpTrip} \twoheadrightarrow \mathsf{Qnd}$. Hence, our above discussion specializes to a sufficient condition for any two quandles to be isomorphic.

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