Why knot? Algebraic coloring invariants of Legendrian knots

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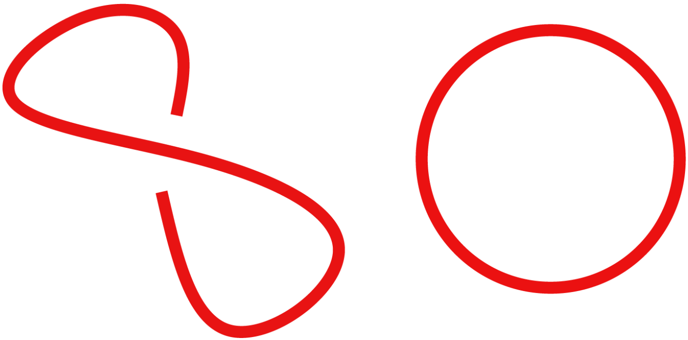
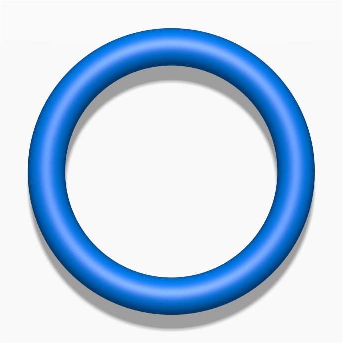
Outline

Let’s play a game. . . (1/2)

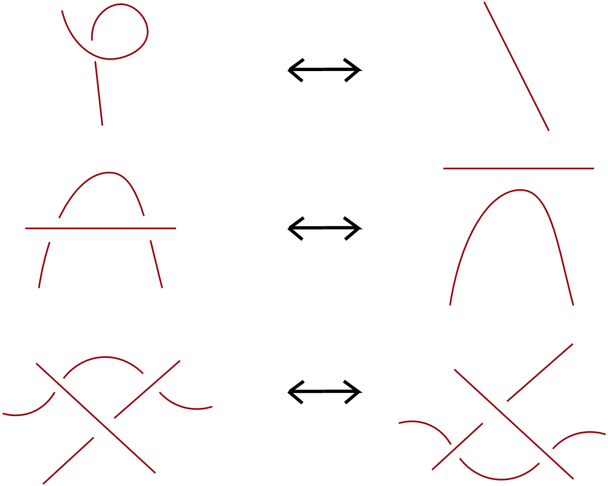


Let's play a game. . . (2/2)



Reidemeister moves

Two knots are equivalent if and only if they're related by the three *Reidemeister moves* (twist, poke, and slide).



(R1)

(R2)

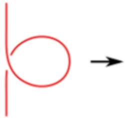
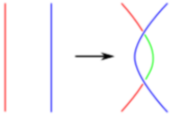
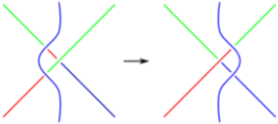
(R3)

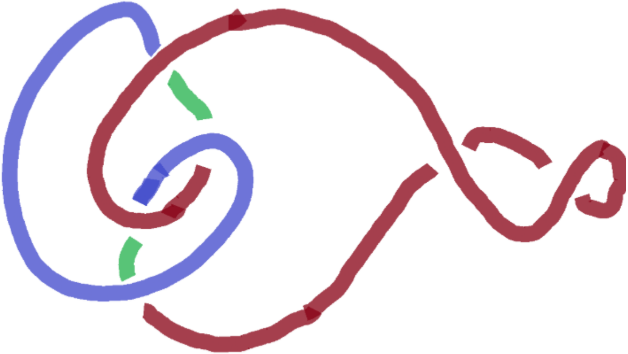
An early way to distinguish knots

A knot is *tricolorable* if we can color its strands in the following way:

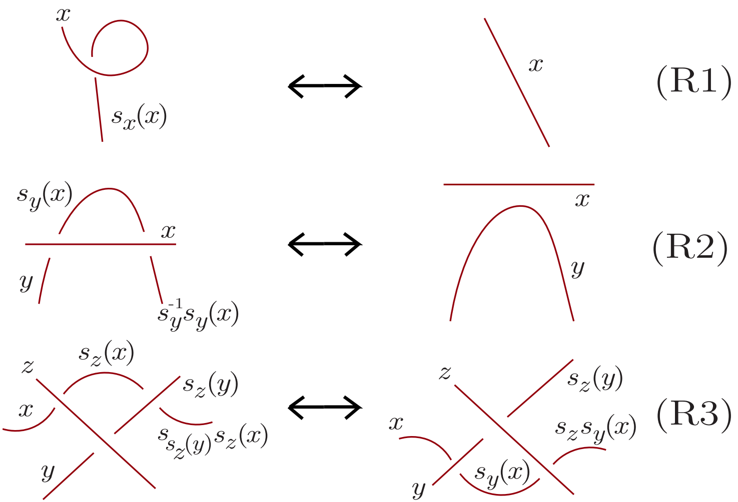
- We use either 2 or 3 colors.
- The three strands at each crossing either (a) all share the same color or (b) all have different colors.

Why is tricolorability a "knot invariant"?

Reidemeister Move I is tricolorable.	Reidemeister Move II is tricolorable.	Reidemeister Move III is tricolorable.
		



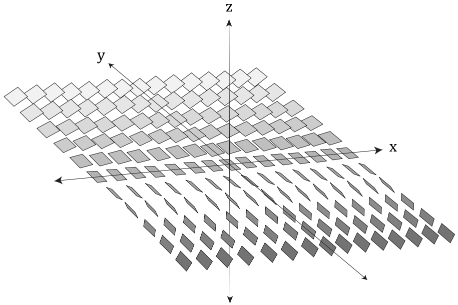
Coloring knots on steroids



Using the *relations* that Reidemeister moves induce on the strands of a knot, we can color knots with abstract objects called **racks** and **quandles**.

The standard contact structure

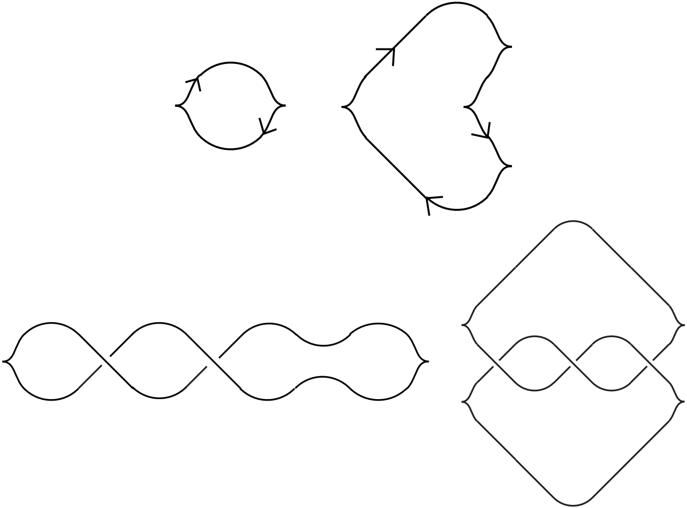
Legendrian knots are knots with a certain restriction on their shape, determined by something called the *standard contact structure*:



- When $y = 0$, the planes are flat.
- When moving in the positive y-direction, the slopes grow more negative.
- When moving in the negative y-direction, the slopes grow more positive.

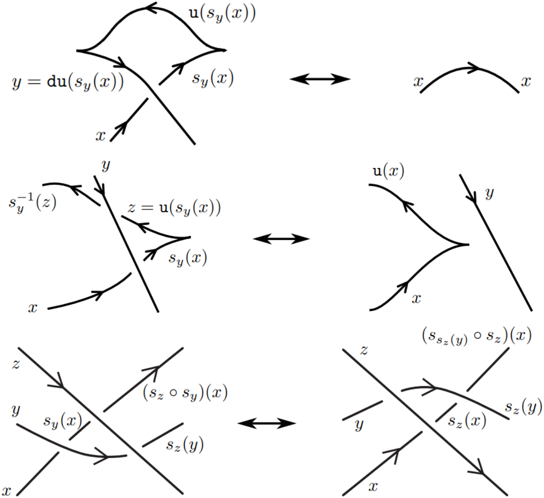
Legendrian knots

We often study Legendrian knots via their *front projections*, which have *cusps* instead of vertical segments.



Distinguishing Legendrian knots

The cusps impose restrictions on the *Legendrian Reidemeister moves*.

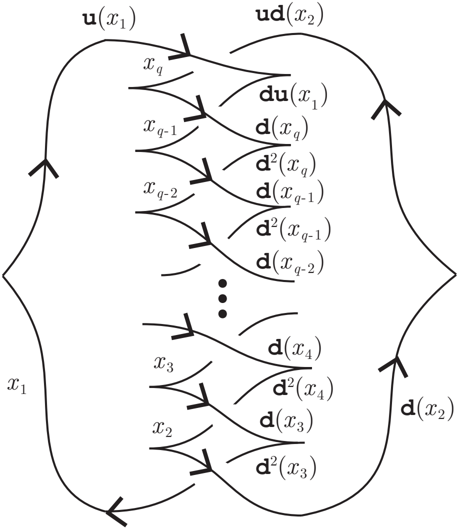


Outline

The idea behind GL-racks

- We'd like to distinguish Legendrian knots by coloring them.
- Since the cusps add extra structure to Legendrian knots, we also need to add extra structure to racks.
  - We'll give each rack a structure $\mathtt{u}$ corresponding to up-cusps.
  - We can also attach a structure $\mathtt{d}$ corresponding to down-cusps...
- We obtain an algebraic gadget called a **GL-rack**.
- We can assign a GL-rack $\mathcal{G}(\Lambda)$ to every Legendrian knot Λ.
- To distinguish Legendrian knots, it suffices to distinguish their GL-racks. And that's when we get to "color"!

Example calculation of $\mathcal{G}(\Lambda)$



$$
\begin{cases}
s_{\mathtt{u}(x_1)}(x_q) = \mathtt{ud}(x_2), \\
s_{\mathtt{d}(x_q)}(x_{q-1}) = \mathtt{du}(x_1), \\
\text{and } s_{\mathtt{d}(x_{i-1})}(x_{i-2}) = \mathtt{d}^2(x_i) \\
\quad \text{for all } 3 \le i \le q.
\end{cases}
$$

Figure: The Legendrian $(2, -q)$-torus knot Λ with maximal classical invariants.

Outline

Legendrian 6_2 knots (1/2)

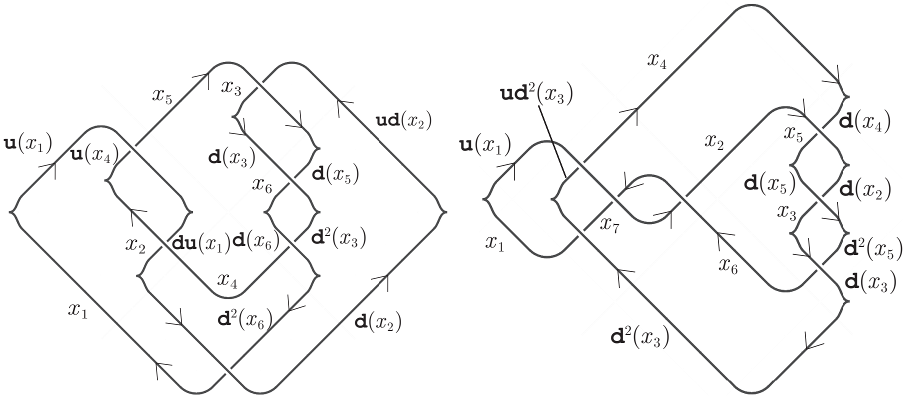


Figure: Legendrian 6_2 knots Λ_1 and Λ_2 with classical invariants $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$.

- Like before, we construct GL-racks $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$.
- These GL-racks satisfy certain relations. These relations are determined by the cusps, strands, and crossings shown above.

Legendrian 6_2 knots (2/2)

Theorem

Λ_1 and Λ_2 are nonequivalent.

Proof sketch.

- Using a computer search, we find a certain GL-rack L containing three "colors."
- Like $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$, L also satisfies certain relations.
- There are three ways to color Λ_1 by L in ways that satisfy the relations of $\mathcal{G}(\Lambda_1)$ of L.
- However, there aren't any ways to color Λ_2 by L like this!
- Since their coloring numbers are different, Λ_1 and Λ_2 are nonequivalent.

□

Legendrian 8_{10} knots

Similarly, one can show that these Legendrian knots are nonequivalent...

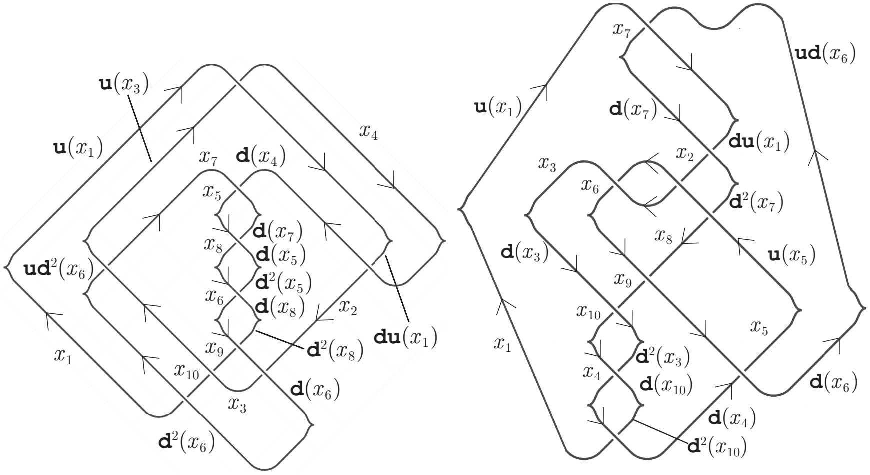


Figure: Legendrian 8_{10} knots with classical invariants $(\mathrm{tb}, \mathrm{rot}) = (-8, 3)$.

Legendrian 8_{13} knots

...and that these Legendrian knots are also nonequivalent.

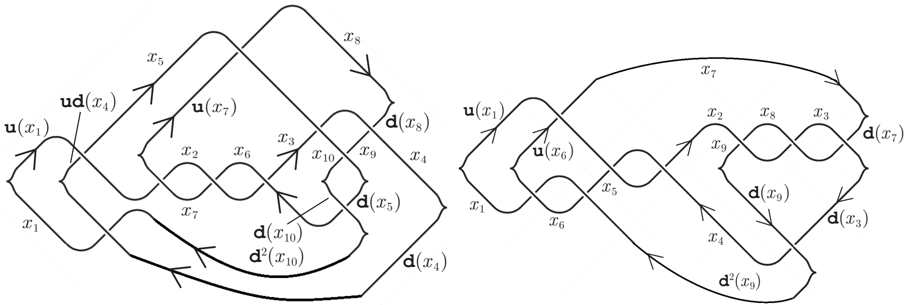


Figure: Legendrian 8_{13} knots with classical invariants $(\mathrm{tb}, \mathrm{rot}) = (-6, 1)$.

Settling conjectures

This completes the classification of Legendrian 8_{13} knots; the two on the previous slide were the only ones not distinguished yet.

Along the way, we answered an open question of Kimura (2023):

Corollary

GL-rack coloring numbers can distinguish Legendrian knots with the same classical invariants.

Outline

Classifying small GL-racks

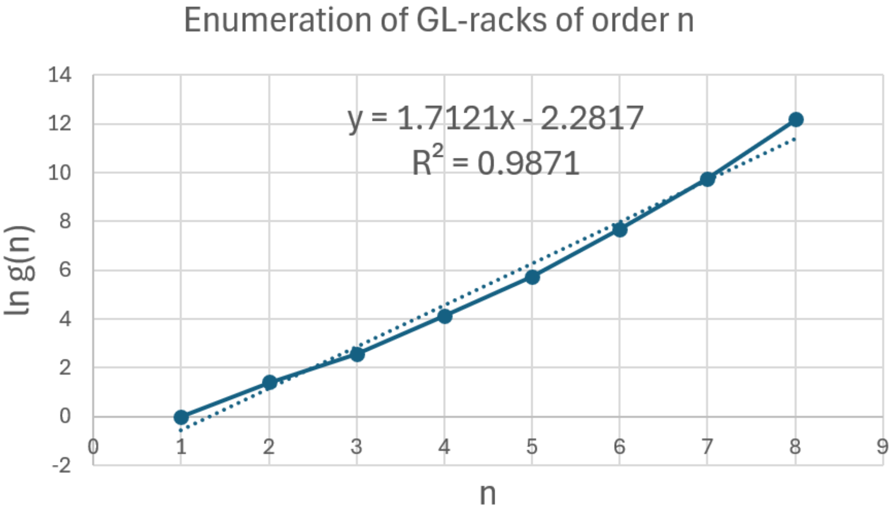
Using the computer algebra system `GAP`, Vojtěchovskỳ and Yang (2019) classified racks containing up to 11 "colors."

We can use this data to classify GL-racks containing ≤ 11 "colors."

1. For all racks R with n colors, exhaustively search for GL-structures $\mathtt{u}$ on R.
2. Check whether the resulting GL-racks are "the same" as any of the others.
3. Throw out duplicate GL-racks.

Similarly, we can exhaustively search for Legendrian knot colorings.

Enumeration (1/2)



Enumeration (2/2)

n	0	1	2	3	4	5	6	7	8
GL-racks	1	1	4	13	62	308	2132	17268	189373
GL-quandles	1	1	2	6	19	74	353	2080	16023
Racks	1	1	2	6	19	74	353	2080	16023
Quandles	1	1	1	3	7	22	73	298	1581

Table: The numbers of GL-racks and GL-quandles versus racks and quandles containing n "colors."

There appears to be a one-to-one correspondence between racks and GL-quandles. Turns out, this is true for *all* n...

Outline

Algebraic results: A bird's-eye view

This is where the research becomes a lot harder to describe in non-technical terms, but for a brief overview...

- There's a natural one-to-one correspondence between racks and GL-quandles.
- We can use *symmetries* to classify several infinite families of GL-racks.
- We can also study the symmetries of GL-racks themselves.
- We can even study the symmetries shared *across all* GL-racks.
- Certain GL-racks lead to an even stronger Legendrian knot invariant than coloring numbers.
- This stronger invariant has a natural relationship to algebraic gadgets called *tensor products*.

Outline

Thank you!

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