

## MATH 350 (Fall 2024): Midterm Review Session with Luc :)

Throughout these problems, let  $G$  be a group with identity element 1, let  $\mathbb{F}$  be a field, and let  $\mathbb{F}^\times$  be the multiplicative group of nonzero elements of  $\mathbb{F}$ .

**Problem 1.** Let  $G = \mathbb{Z}/4 \times \mathbb{Z}/4$ , and let  $H$  be the subgroup generated by  $(2, 1)$ . That is,  $H$  is the  $(2, 1)$ -orbit of  $(2, 1)$ .

- (a) Write down all the cosets of  $H$  in  $G$ , each with a full list of its elements.
- (b) Argue that  $G/H$  is a group. What more familiar group is it isomorphic to?

**Problem 2.** Let  $C \subset \mathbb{R}^3$  be a cube, and let  $G$  be the group of *rotational* symmetries of  $C$  (with composition as the group action), so that  $G$  acts on  $C$  by rotation.

- (a) Let  $F$  be a face of  $C$ . Describe the orbit of  $F$  and the stabilizer of  $F$ . Use this to compute the order of  $G$ .
- (b) Let  $e$  be an edge of  $C$ , and redo part (a) with  $e$  playing the role of  $F$ . Use this as an alternate way to compute the order of  $G$ .
- (c) Let  $v$  be a vertex of  $C$ , and redo part (a) with  $v$  playing the role of  $F$ . Use this as yet another way to compute the order of  $G$ . (*Hint: Let  $v'$  be the vertex diametrically opposite from  $v$ , and consider the axis containing both  $v$  and  $v'$ ...*)
- (d) *Insightful challenge:* Prove that  $G \cong S_4$ . (*Hint: Consider the 4 pairs of diametrically opposite vertices of  $C$ ...*)

**Problem 3.** Let  $G$  act on a set  $X$ . Let  $K$  be a group, and let  $\psi : K \rightarrow G$  be a group homomorphism.

- (a) Fix  $g \in G$ . Show that the map  $x \mapsto g \cdot x$  defines a bijection  $X \xrightarrow{\sim} X$ .
- (b) Let  $H$  be a subgroup of  $G$ , and let  $Y$  be any subset of  $X$ . Verify that the action of  $G$  on  $X$  induces an action of  $H$  on  $Y$ . Deduce that  $G$  induces an action of  $K/\ker \psi$  on  $X$ .
- (c) Prove that the *pullback*  $h \cdot x := \psi(h) \cdot x$  defines an action of  $K$  on  $X$ .
- (d) If  $\psi$  is injective, find a necessary and sufficient condition for the pullback action to be faithful.
- (e) If  $\psi$  is surjective, find a necessary and sufficient condition for the pullback action to be transitive.

**Problem 4.** Let  $V$  and  $W$  be vector spaces over  $\mathbb{F}$ , and let  $\mathbb{F}^\times$  act on  $V$  and  $W$  by scalar multiplication.

- (a) Observe that  $V$  and  $W$  are abelian additive groups. Then, argue that a map  $\psi : V \rightarrow W$  is a linear transformation if and only if it is both a morphism of  $\mathbb{F}^\times$ -sets and a group homomorphism.
- (b) Find a necessary and sufficient condition for the action of  $\mathbb{F}^\times$  on  $V$  to be faithful.
- (c) Find a necessary and sufficient condition for the action of  $\mathbb{F}^\times$  on  $V$  to be transitive. Then, find a necessary and sufficient condition for the action of  $\mathbb{F}^\times$  on  $V \setminus \{0\}$  to be transitive.

**Problem 5.**

- (a) Suppose  $p$  is a prime number such that  $p \mid |G|$ , but  $p^2 \nmid |G|$ . Prove that the number of elements of order  $p$  in  $G$  is exactly  $N_p(p - 1)$ . (You probably used this or a similar result on HW7.)
- (b) Let  $G$  be a group of order  $495 = 3^2 \cdot 5 \cdot 11$ . Show that  $G$  is not simple.
- (c) Let  $G$  be a group of order  $132 = 2^2 \cdot 3 \cdot 11$ . Show that  $G$  is not simple.

**Problem 6.** Define the *quaternion group* as the subgroup  $Q_8 := \{\pm 1, \pm i, \pm j, \pm h\}$  of  $GL_2(\mathbb{C})$ , where

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(In physics, these can be obtained from the *Pauli matrices* by multiplying by  $i$ .) Note that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

- Show that every subgroup of  $Q_8$  is normal. (*Hint: There's a very quick way to do this.*)
- Deduce the following two sentences: If  $G$  is abelian, then all of its subgroups are normal. However, the converse is not necessarily true.
- Give two different short proofs that  $D_8 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^{-1} \rangle$ , the dihedral group of the square, is not isomorphic to  $Q_8$ .
- Find a group  $G$  and a normal subgroup  $H$  such that  $G/H$  is not isomorphic to any subgroup of  $G$ .

**Problem 7.** True or false? If the statement is true, prove it (or write its name, if it's a named theorem or previous problem like "Sun Ze's theorem" or "HW5 #3(d)"). If not, give a counterexample.

- If  $|G| = n$ , then  $G$  is isomorphic to some subgroup of  $S_n$ . (*Try proving this one!*)
- If  $g \in G$  is the only element of  $G$  having order 2, then  $g \in Z(G)$ .
- Let  $p$  be the smallest prime number dividing the order of  $G$ . If  $H$  is a subgroup of  $G$  such that  $[G : H] = p$ , then  $H$  is normal in  $G$ .
- If  $p$  is a prime number dividing  $|G|$ , then  $G$  contains at least  $p - 1$  distinct elements of order  $p$ .
- If  $|G| < \infty$  and  $H \leq G$ , then  $|H|$  divides  $|G|$ .
- If  $|G| < \infty$  and  $g \in G$ , then  $|g|$  divides  $|G|$ .
- Challenge:* If  $H \cong K$  are isomorphic normal subgroups of  $G$ , then  $G/H \cong G/K$ .

**Problem 8.** Let  $p$  be a prime number, and fix  $n, d \in \mathbb{Z}^+$ . Let  $V$  be a  $d$ -dimensional vector space over  $\mathbb{Z}/p$ . (You can take for granted in this problem that  $\mathbb{Z}/p$  is a field iff  $p$  is prime.) Let  $G$  be a subgroup of  $GL_d(\mathbb{Z}/p)$  such that  $|G| = p^n$ . Prove that there exists a nonzero vector  $\mathbf{v} \in V$  such that  $M\mathbf{v} = \mathbf{v}$  for all  $M \in G$ . (*Hint: This problem should remind you of a certain lemma from class.*)

**Problem 9.** This problem (along with Problems 3(b) and 12(a,c)) gives us an opportunity to practice using the the result in problem 2(a) in HW6. (This is actually a major result in group theory called the *first isomorphism theorem*.)

- Fix  $n \geq 3$ , and let  $k \in \mathbb{Z}^+$  be a divisor of  $n$ . Let  $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$  be the dihedral group of the regular  $n$ -gon. If  $k \geq 3$ , show that  $\langle r^k \rangle = \{r^k, r^{2k}, r^{3k}, \dots, r^{n-k}, r^n = 1\}$  is a normal subgroup of  $D_{2n}$ , and show that  $D_{2n}/\langle r^k \rangle \cong D_{2k}$ . What if  $k \leq 2$ ?
- For all  $g \in G$ , define the *conjugation map*  $\varphi_g : G \rightarrow G$  by  $x \mapsto gxg^{-1}$ , and define the *inner automorphism group* of  $G$  as the set

$$\text{Inn}(G) := \{\varphi_g \mid g \in G\}.$$

Argue that  $\text{Inn}(G)$  is a subgroup of  $\text{Aut}(G)$ , the *automorphism group* of  $G$ . Then, find a subgroup  $H \leq G$  such that  $G/H \cong \text{Inn}(G)$ . (*Hint: Hopefully, this problem reminds you of HW2.*)

- (c) Fix  $n \in \mathbb{Z}^+$ . Show that  $\mathrm{SL}_n(\mathbb{F})$  is a normal subgroup of  $\mathrm{GL}_n(\mathbb{F})$ . What group is  $\mathrm{GL}_n(\mathbb{F})/\mathrm{SL}_n(\mathbb{F})$  isomorphic to? (*Hint: How is  $\mathrm{SL}_n(\mathbb{F})$  defined?*) From this, deduce the following sentence: If  $M$  and  $N$  are invertible  $n \times n$  matrices, then they have the same determinant if and only if there exists a matrix  $S$  such that  $M = SN$  and  $\det(S) = 1$ . (In fact,  $S = MN^{-1}$ .) Also, deduce that if  $\mathbb{F}$  is a finite field of order  $q$ , then  $[\mathrm{GL}_n(\mathbb{F}) : \mathrm{SL}_n(\mathbb{F})] = q - 1$ .
- (d) Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}$ . Recall the following linear algebra construction: If  $W$  is a linear subspace of  $V$ , then we have a *quotient space*  $V/W$  whose dimension is  $\dim(V/W) = \dim V - \dim W$ . Using this fact, give a basis-free proof of the rank-nullity theorem. That is, let  $T$  be a linear transformation from  $V$  to some other vector space over  $\mathbb{F}$ , and show that  $\dim(\mathrm{Im} T) + \dim(\ker T) = \dim V$  without ever writing the word “basis.”

**Problem 10.** Let  $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$  be the dihedral group of the regular  $n$ -gon.

- (a) Deduce from Problem 9(a) that every subgroup of  $\langle r \rangle = \{r, r^2, r^3, \dots, r^{n-1}, r^n = 1\}$  is normal in  $D_{2n}$ .
- (b) Let  $p$  be an odd prime number such that  $p \mid n$ . Deduce the following (not necessarily in order):  
 (i)  $D_{2n}$  has a unique Sylow  $p$ -subgroup  $P$ , (ii)  $P$  is cyclic, and (iii)  $P$  is normal. (*Hint: Recall from class that  $N_p = 1$  if and only if there exists a normal Sylow  $p$ -subgroup.*)
- (c) Write an explicit list of all Sylow  $p$ -subgroups of  $D_{12}$ . Justify that your list is complete.

**Problem 11.** Let  $P$  be a Sylow  $p$ -subgroup of  $G$ , and let  $H$  be a subgroup of  $G$  that contains  $P$ . For all  $g \in G$ , show that  $gPg^{-1}$  is a Sylow  $p$ -subgroup of  $gHg^{-1}$ . (*Hint: If you're stuck, then one of the results in Problem 9(b) might help.*)

**Problem 12.** Let  $S^1$  denote the *unit circle*, considered as a closed curve in the complex plane:

$$S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\} = \{e^{i\theta} : \theta \in [0, 2\pi)\} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}^\times.$$

(You may have seen in other courses that for all  $\theta \in \mathbb{R}$ , the complex number  $e^{i\theta} = \cos \theta + i \sin \theta$  is the point on  $S^1$  obtained by starting at  $1 = e^{i0}$  and rotating by  $\theta$  radians about the origin. If you like, you can play around with the slider at [this link](#) to get a feel for how this works!)

- (a) Using the fact that adding or subtracting  $2\pi$  radians to an angle doesn't alter the angle, argue that  $S^1$  is a subgroup of  $\mathbb{C}^\times$ . Then, find a normal subgroup  $H$  of  $\mathbb{R}$  such that  $\mathbb{R}/H \cong S^1$ .
- (b) Define a nontrivial action of  $\mathbb{R}$  on  $S^1$ . Is this action transitive? Is it faithful? If not, what is its kernel?
- (c) On an unrelated note, let  $G_1$  and  $G_2$  be groups with normal subgroups  $N_1$  and  $N_2$ , respectively. Consider the group homomorphisms  $\phi : G_1 \times G_2 \rightarrow G_1/N_1$  defined by  $(g_1, g_2) \mapsto g_1N_1$  and  $\psi : G_1 \times G_2 \rightarrow G_2/N_2$  defined by  $(g_1, g_2) \mapsto g_2N_2$ . Use HW2 #4(a) and the first isomorphism theorem to show that  $(G_1 \times G_2)/(N_1 \times N_2) \cong G_1/N_1 \times G_2/N_2$ .
- (d) Deduce that the quotient group  $\mathbb{R}^2/\mathbb{Z}^2$  is isomorphic to the *torus*  $S^1 \times S^1$ . (There's a nice geometric interpretation of these isomorphisms—if you're curious, ask me about it at office hours, and I'll draw it!)

**You're doing great! Good luck on the midterm—I believe in you! :)**