MATH 370 (Spring 2025): Weekend Session 2 (with Lực Ta)

This week's session is on semidirect products of groups! Yay! :) Let's begin by recalling a result from last Sunday's session. But first, remember to sign in, using either the QR code or this link.

Theorem ("Big homomorphism theorem," DF pp. 38–39). *Let G be a group with presentation $G = \langle s_1, \ldots, s_m \mid r_1, \ldots, r_n \rangle$. Recall that the s_i's are called generators of G, and the r_j's are equations in terms of the s_i's called relations.*

Let H be another group, and let $\varphi : G \to H$ be a map of sets. Then φ is a group homomorphism if and only if the r_j's are still satisfied after we replace each s_i with $\varphi(s_i) \in H$. In other words, a group homomorphism $\varphi : G \to H$ is determined completely by where it sends the generators s_i, and the images $\varphi(s_i) \in H$ of those generators must satisfy the relations r_j when viewed as equations in H.

Problem 1. Let's warm up with a few group homomorphism calculations.

(a) Find all homomorphisms from Z_3 to itself. Which ones are automorphisms?

(b) Find all homomorphisms from Z_2 to itself. Which ones are automorphisms?

(c) Find all homomorphisms from Z_8 to Z_6.

(d) Find all homomorphisms from $D_3 = \langle r, s \mid r^3 = 1 = s^2,\ srs = r^{-1} \rangle$ to Z_2.

(e) Find all homomorphisms from Z_6 to itself. Which ones are automorphisms?

Definition (DF p. 176). Let H and K be groups, and let $\varphi : K \to \mathrm{Aut}(H)$ be a group homomorphism, so that K acts on H by $k \cdot h := (\varphi(k))(h)$. Define the following multiplication operation on the set $H \times K$:

$$(h_1, k_1)(h_2, k_2) := (h_1(k_1 \cdot h_2), k_1 k_2).$$

This multiplication turns the set $H \times K$ into a group $G := H \rtimes_\varphi K$, which we call the *(externak) semidirect product* of H and K with respect to φ. Note that $H \trianglelefteq G$ and $H \cap K = 1$. Also, conjugation in G of $h \in H$ by $k \in K$ is given by $khk^{-1} = k \cdot h = (\varphi(k))(h)$.

Problem 2. Let's work through a few examples. As a consequence, we'll obtain a classification of groups of order 6.

(a) Verify that $\mathrm{Aut}(Z_3) \cong Z_2$ and $\mathrm{Aut}(Z_2) = 1$.

(b) Find all semidirect products $Z_3 \rtimes Z_2$. Are any of them isomorphic?

(c) Find all semidirect products $Z_2 \rtimes Z_3$.

(d) Use your answers to parts (b) and (c) to classify all groups of order 6 up to isomorphism. Do the groups you found have more familiar names?

Proposition (DF p. 177). *In the setting of the previous definition, the following are equivalent:*

1. *G is isomorphic to the direct product $H \times K$.*
2. *$\varphi : K \to \mathrm{Aut}(H)$ is the trivial homomorphism.*
3. *K acts trivially on H. That is, $k \cdot h = h$ for all $k \in K$ and $h \in H$.*
4. *$K \trianglelefteq G$.*

Lemma (DF p. 136). *For all n, we have $\mathrm{Aut}(Z_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, which is a group of order $\varphi(n)$, where $\varphi : \mathbb{Z}^+ \to \mathbb{Z}^+$ denotes Euler's totient function. In particular, if p is an odd prime and $k \geq 1$, let $q := \varphi(p^k) = p^{k-1}(p-1)$. Then $\mathrm{Aut}(Z_{p^k}) \cong Z_q$ consists precisely of the maps $[x \mapsto x^m] : Z_{p^k} \to Z_{p^k}$ for which $1 \leq m < p^k$ and $\gcd(p^k, m) = 1$.*

(We won't need it for today, but for a more general characterization of $(\mathbb{Z}/n\mathbb{Z})^\times$, check out DF p. 314 or Problem 7 of my Math 350 exam review problems.)

Problem 3. Let's practice using the results we've seen so far.

(a) Find all semidirect products $Z_7 \rtimes Z_8$. Are any isomorphic?

(b) Find all semidirect products $Z_3 \rtimes D_3$.

(c) Find all semidirect products $Z_9 \rtimes Z_6$ in which the action of Z_6 on Z_9 is faithful. What about the semidirect products in which the action is transitive?

Theorem ("Identification theorem," DF p. 180)**.** *Let G be any group, and let H and K be subgroups of G such that*

1. $H \lhd G$,
2. $H \cap K = 1$, *and*
3. $|G| = |H||K|$.

Let $\varphi : K \to \mathrm{Aut}(H)$ be the conjugation map $k \mapsto [h \mapsto khk^{-1}]$. Then G is isomorphic to the (internal) semidirect product $H \rtimes_\varphi K$.

Problem 4. Let's apply our results to dihedral groups.

(a) Write $D_3 = \langle r, s \mid r^3 = 1 = s^2,\ srs = r^{-1}\rangle$ as a nontrivial internal semidirect product $H \rtimes K$. (Here, "nontrivial" means that $1 < |H|, |K| < |D_3| = 6$.) Justify your answer.

(b) Generalize your result from part (a) to D_n for any positive integer n.

(c) Now, write D_n as an external semidirect product of two cyclic groups $Z_h \rtimes Z_k$ (you get to choose what h and k are). Remember to describe the action of Z_k on Z_h.

Problem 5. Consider the *quaternion group* $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where multiplication is defined as follows:

$$ij = k,\ jk = i,\ ki = j,\ i^2 = j^2 = k^2 = -1, \quad \text{and } (-1)a = -a = a(-1) \text{ for all } a \in Q_8.$$

(a) Describe the subgroups of Q_8. Which ones of them are normal? (Even though Q_8 is nonabelian, is there a shortcut for showing that some of these subgroups are normal?)

(b) Can Q_8 be written as a nontrivial (internal) semidirect product $H \rtimes_\varphi K$? (Here, "nontrivial" means that $1 < |H|, |K| < |Q_8| = 8$.) If so, describe the action. If not, explain why.

You're doing great! :)