

# APPLICATIONS AND ENUMERATION OF THE CATEGORY OF GENERALIZED LEGENDRIAN RACKS

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**ABSTRACT.** In 2023, Karmakar et al. and Kimura introduced a nonassociative algebraic structure called a *GL-rack* that can distinguish between Legendrian links in  $\mathbb{R}^3$ . To help develop the theory of GL-racks, we apply them to prove several conjectures about Legendrian links, study the category of GL-racks and their homomorphisms, and provide a library classifying GL-racks of orders  $n \leq 7$  up to isomorphism. In particular, we give short algebraic proofs distinguishing unstabilized Legendrian representatives of knot types  $6_2$  and  $8_{10}$ . We also give a brief categorical proof that the enveloping group of the GL-rack of a Legendrian link  $\Lambda$  is isomorphic to  $\pi_1(\mathbb{R}^3 \setminus \Lambda)$ . Then, we propose a medial GL-rack-valued invariant of Legendrian links, and we introduce tensor products of GL-racks that induce a symmetric monoidal closed structure on the full subcategory of medial GL-racks. From an additional functorial construction, we obtain a group-theoretic sufficient condition for any two GL-racks or quandles to be isomorphic. Finally, we provide algorithms classifying finite GL-racks up to isomorphism and computing GL-rack coloring numbers of oriented Legendrian links.

## 1. INTRODUCTION

In 1980, Joyce [20] introduced algebraic structures called *quandles* as a means of capturing the topological structure of knots, links, and symmetric spaces. Since then, quandles and slightly more general algebraic objects called *racks* have enjoyed significant study as link invariants among geometric topologists and in their own right among quantum algebraists. Recently, various authors have constructed generalizations of racks and quandles to study *Legendrian links* in contact geometry (e.g., [6, 22, 23, 27]). Defining homomorphisms of these algebraic structures yields categories like GLR, whose objects are called *GL-racks*, *generalized Legendrian racks*, or *bi-Legendrian racks*.

In this paper, we study the category GLR and its applications. This helps us understand the structure of GL-racks and detect their isomorphisms, which can otherwise be cumbersome due to their nonassociativity. This also allows us to prove several conjectures about Legendrian links and generalize several properties of quandles to GL-racks. Along with a classification of GL-racks of order  $n \leq 7$  up to isomorphism in Appendix A, the main results of this paper are as follows.

**Theorem 1.1.** *The two Legendrian knots with  $(\text{tb}, \text{rot}) = (-7, 2)$  and underlying smooth knot type  $6_2$  given in [9] are distinguishable by GL-rack coloring numbers.*

**Theorem 1.2.** *The two Legendrian knots with  $(\text{tb}, \text{rot}) = (-8, 3)$  and underlying smooth knot type  $8_{10}$  given in [3] are distinguishable by GL-rack coloring numbers.*

**Theorem 1.3.** *Let  $\Lambda \subset \mathbb{R}^3$  be an oriented Legendrian link, let  $\mathcal{G}(\Lambda)$  be the GL-rack of  $\Lambda$  as defined in Definition 3.2, and let  $\text{Env}_{\text{GLR}}(\mathcal{G}(\Lambda))$  be its enveloping group as defined in Definition 2.16. Then there exists a group isomorphism*

$$\text{Env}_{\text{GLR}}(\mathcal{G}(\Lambda)) \cong \pi_1(\mathbb{R}^3 \setminus \Lambda).$$

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**Theorem 1.4.** *The full subcategory of GLR whose objects are medial GL-racks is symmetric monoidal closed. In particular, it is self-enriched.*

Inspired by the homogeneous representations of GL-racks constructed in [22, Section 5], we also construct a category  $\mathbf{GrpTup}$  satisfying the following.

**Theorem 1.5.** *There exists an essentially surjective functor  $\mathcal{F} : \mathbf{GrpTup} \rightarrow \mathbf{GLR}$ . This functor induces a group-theoretic sufficient condition for any two GL-racks or quandles to be isomorphic.*

In Section 2, we give an overview of the questions in Legendrian knot theory motivating the study of GL-racks and quandles. We proceed by defining these algebraic structures abstractly and introducing related groups, categories, and functors in the literature.

In Section 3, we discuss how to assign a GL-rack to an oriented Legendrian link, give several worked examples, and discuss related invariants of Legendrian links. Then, we prove Theorems 1.1, 1.2, and 1.3, which we state as Theorem 3.8, Theorem 3.11, and Corollary 3.13, respectively. Our approach to the first result offers a simpler algebraic alternative to Dynnikov and Prasolov’s proof in [11, Proposition 2.3] and gives a positive answer to a question posed in [23, Section 4], which we state as Corollary 3.9. The second result confirms a conjecture of Bhattacharyya et al. in [3], and the third result confirms an empirical observation from the original version of [22, Remark 8.2].

In Section 4, we define *medial* or *abelian* GL-racks and tensor products of GL-racks. After proposing a medial GL-rack-valued invariant of Legendrian links with suggestions for future research, we prove Theorem 1.4, which we state as Theorem 4.5 and Corollary 4.6. This generalizes Crans and Nelson’s analogous result for medial quandles in [10, Theorem 12].

In Section 5, we define  $\mathbf{GrpTup}$  using objects constructed from collections of left cosets of groups. Karmakar et al. originally employed these objects in [22, Theorem 5.2] to produce a homogeneous representation of any GL-rack. Then, we prove Theorem 1.5, the first part of which we state as Theorem 5.1 and the second part of which we discuss afterward.

In Appendix A, we describe an algorithm in `GAP` [19] that can classify all GL-racks of a given orders  $n \leq 11$ , building on the work of Vojtěchovský and Yang in [38]. However, we were only able to finish the computation for  $n \leq 7$ . After writing the isomorphism classes for all  $n \leq 4$ , we provide a link to our code and data for all  $n \leq 7$ . We also provide an algorithm that computes all homomorphisms from the GL-rack of a given oriented Legendrian link to all GL-racks of a given order between 1 and 7.

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## 2. DEFINITIONS AND NOTATION

In this section, we provide motivations, definitions, and notation for the study of GL-racks. In particular, we discuss the crossing and cusp relations afforded by Legendrian Reidemeister moves, which motivate the axioms of GL-racks. After stating some preliminary results on GL-racks, we discuss several functors appearing in the literature on GL-racks and quandles.



FIGURE 1. The standard contact structure  $\xi_{\text{std}}$  on  $\mathbb{R}^3$ . Taken from [26].

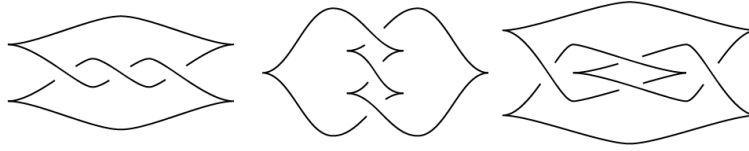


FIGURE 2. Unoriented front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot. Adapted from [16].

**2.1. Motivations: Legendrian knots and links.** In this subsection, we introduce Legendrian links and Legendrian Reidemeister moves. These give contact-geometric motivations for studying GL-racks and quandles. (For an accessible introduction to Legendrian knot theory, we refer the reader to [33]. For a more detailed survey of the field, we refer the reader to [16].)

**Definition 2.1.** A *knot* is a smooth embedding of the circle  $S^1$  into  $\mathbb{R}^3$ , and a *link* is a disjoint union of a finite number of knots. A link  $\Lambda$  is called *Legendrian* if it lies everywhere tangent to the *standard contact structure*  $\xi_{\text{std}} := \ker(dz - ydx)$  on  $\mathbb{R}^3$ , which is depicted in Figure 1. That is,  $T_x\Lambda \in \xi_{\text{std}}$  for all  $x \in \Lambda$ , where  $T_x\Lambda$  denotes the tangent space of  $\Lambda$  at  $x$ . A *front projection* or *front diagram*  $D(\Lambda)$  is the projection of  $\Lambda$  to the  $xz$ -plane. Finally, two Legendrian links are called *equivalent* or *Legendrian isotopic* if there exists a smooth homotopy between them that preserves the condition of being Legendrian at every stage.

When discussing Legendrian links alongside general links, we will call the latter *smooth links* or *topological links*. Also, we will denote the underlying smooth link of a Legendrian link  $\Lambda$  by  $L$ .

Central to contact geometry is the question of how to distinguish Legendrian links up to Legendrian isotopy. To this end, knot theorists typically study Legendrian links  $\Lambda$  through their front projections, which follow several restrictions thanks to the tangency condition on  $\Lambda$ . For one, at every crossing in  $D(\Lambda)$ , the strand with the more negative slope is always the overstrand. For two,  $D(\Lambda)$  has cusps in place of vertical tangencies. Note that the numbers of crossings and cusps in a Legendrian front projection are finite due to smoothness. Moreover,  $D(\Lambda)$  can be viewed as a *link diagram* of  $L$ , denoted by  $D(L)$ , by “ignoring” all cusps. For example, Figure 2 depicts unoriented front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot, and Figure 3 depicts oriented front projections of two distinct Legendrian unknots.

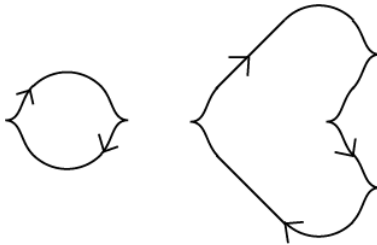


FIGURE 3. Front projections of distinct oriented Legendrian unknots. Adapted from [32].

In fact, one can use the tangency condition to show that the geometric structure of an oriented Legendrian link  $\Lambda$  can be recovered entirely from its front projection (see [16]). For example, two invariants called the *Thurston-Bennequin number* and *rotation number* of  $\Lambda$ , denoted respectively by  $\text{tb}(\Lambda)$  and  $\text{rot}(\Lambda)$ , can be defined as the integers

$$\text{tb}(\Lambda) = P - N - \frac{1}{2}(D + U), \quad \text{rot}(\Lambda) = \frac{1}{2}(D - U),$$

where  $P$ ,  $N$ ,  $D$ , and  $U$  are the numbers of positively oriented crossings, negatively oriented crossings, downward-oriented cusps, and upward-oriented cusps in  $D(\Lambda)$ , respectively. It is well-known that two Legendrian links are Legendrian isotopic only if their Thurston-Bennequin and rotation numbers are equal. Indeed, a celebrated theorem of Świątkowski in 1992 offers a method of comparing Legendrian links using only their front projections.

**Proposition 2.2.** [35, Theorem B] *Two Legendrian links are Legendrian isotopic if and only if their front projections are related by a finite sequence of planar isotopies<sup>1</sup> and the three Legendrian Reidemeister moves depicted in Figures 5-7.*

The axioms of GL-racks are motivated by the *crossing* and *cusp relations* induced between strands of an oriented Legendrian link modulo the relations afforded by the Legendrian Reidemeister moves. In Figure 4, (i) and (ii) depict crossing relations between strands in a Legendrian front projection, and (iii) and (iv) depict cusp relations. Note that  $\mathbf{u}$  and  $\mathbf{d}$  correspond to the relations induced by upward- and downward-oriented cusps, respectively. Figures 5-7 depict the crossing and cusp relations in one possible orientation of each of the three Legendrian Reidemeister moves. For a complete list of all possible orientations and their induced crossing and cusp relations, we refer the reader to [23, Figures 6-8].

**Example 2.3.** Let  $\Lambda_1$  and  $\Lambda_2$  be the oriented Legendrian unknots depicted on the left and right of Figure 3, respectively. Although  $\Lambda_1$  and  $\Lambda_2$  share the same underlying smooth knot type, they are not Legendrian isotopic because  $\text{tb}(\Lambda_1) = -1 \neq -2 = \text{tb}(\Lambda_2)$  and  $\text{rot}(\Lambda_1) = 0 \neq 1 = \text{rot}(\Lambda_2)$ . Proposition 2.2 asserts that the two front projections in Figure 3 cannot be related by any sequence of Legendrian Reidemeister moves.

There are in fact infinitely many examples of distinct Legendrian links having the same underlying smooth link type, making distinguishing between Legendrian links significantly more difficult than distinguishing between smooth links. This has motivated the development of numerous *invariants* of Legendrian links, called so because they are constructed to be invariant under Legendrian isotopy. Examples include the Thurston-Bennequin and rotation numbers [16], the

<sup>1</sup>A *planar isotopy* is a homotopy of two front projections, viewed as embeddings into the  $xz$ -plane, that preserves crossings, cusps and the lack of vertical tangencies. Thus, they do not induce any crossing or cusp relations.



FIGURE 4. Crossing and cusp relations. Adapted from [22].

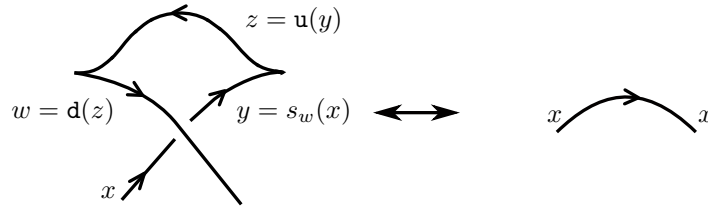


FIGURE 5. Crossing and cusp relations in one possible orientation of the first Legendrian Reidemeister move. Adapted from [22].

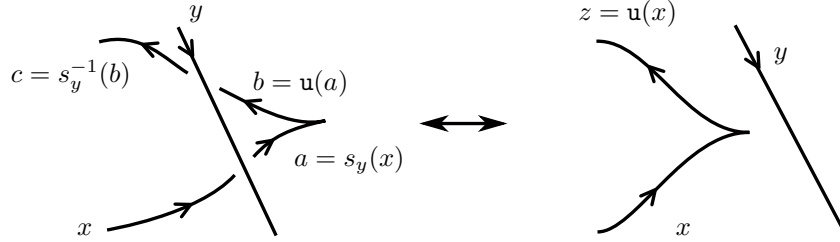


FIGURE 6. Crossing and cusp relations in one possible orientation of the second Legendrian Reidemeister move. Adapted from [22].

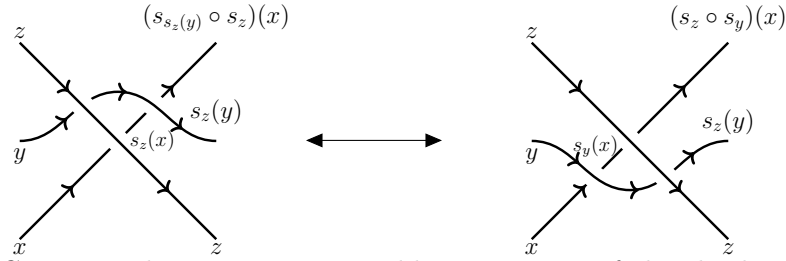


FIGURE 7. Crossing relations in one possible orientation of the third Legendrian Reidemeister move. Adapted from [6].

Chekanov-Eliashberg differential graded algebra and associated polynomial invariants [16, 30], various (co)homology theories (see [18] for a list), and the mosaic number [26, 32].

GL-racks and quandles have also been used to define algebro-combinatorial and cohomological invariants of both Legendrian links and smooth links. These include fundamental quandles and their Legendrian analogues [6, 20, 22],  $R$ -coloring numbers [6, 22, 23], cocycle invariants [7, 24], and state-sum invariants [4, 15], many of which have elegant categorifications and enhancements (e.g., [5, 7, 8, 14]). These invariants motivate the study of GL-racks as a category.

In Subsection 3.1, we discuss how to assign a GL-rack to any oriented Legendrian link  $\Lambda$  using the cusp and crossing relations in  $D(\Lambda)$ . This assignment is independent of the choice of front projection of  $\Lambda$ , making it an invariant of Legendrian links [22, Theorem 4.3].

**2.2. GL-racks and quandles.** In this subsection, we define GL-racks and quandles abstractly by translating the crossing and cusp relations in Subsection 2.1 into the language of *rack symmetries*. For any set  $X$ , we denote the set of all bijections from  $X$  to itself by  $\text{Sym}(X)$ .

Although racks and quandles are often defined as sets  $X$  endowed with binary operations  $\triangleright : X \times X \rightarrow X$  and  $\triangleright^{-1} : X \times X \rightarrow X$ , they may also be characterized in terms of symmetries  $s_x \in \text{Sym}(X)$  assigned to each element  $x \in X$ , cf. [13, Section 2; 20, Definition 1.1; 36, Definition 2.7]. One may translate between the two conventions via the formulas  $s_x(y) = y \triangleright x$  and  $s_x^{-1}(y) = y \triangleright^{-1} x$ . In this article, we adopt the definitions using symmetries due to their convenience for categorical proofs, and we have rewritten all crossing relations in Figures 4-7 in this notation. We refer the reader to [29] for an accessible introduction to quandle theory and [12] for a more comprehensive survey of modern algebraic research on racks and quandles.

The rack and quandle axioms encapsulate the crossing relations depicted in Figures 5-7.

**Definition 2.4.** Let  $X$  be a set, and let  $s : X \rightarrow \text{Sym}(X)$  be a map defined by  $x \mapsto s_x$ . We call the pair  $(X, s)$  a *rack* or a *wrack* if, for all  $x, y \in X$ , we have  $s_x \circ s_y = s_{s_x(y)} \circ s_x$ . We say that  $s_x$  is the *symmetry at  $x$* , and we say that the *order* of  $(X, s)$  is  $|X|$ . If in addition  $s_x(x) = x$  for all  $x \in X$ , then we say that  $(X, s)$  is a *quandle*. Finally, if  $Y \subset X$  and  $s_y(z) \in Y$  for all  $y, z \in Y$ , then we say that  $(Y, s|_Y)$  is a *subrack* of  $(X, s)$ .

**Example 2.5.** [36, Definition 2.11] Let  $\Omega$  be a union of conjugacy classes in a group  $G$ , and define  $s : \Omega \rightarrow \text{Sym}(\Omega)$  by

$$\alpha \mapsto s_\alpha := [\omega \mapsto \alpha \omega \alpha^{-1}].$$

Then  $(\Omega, s)$  is called a *conjugation quandle*, and we denote it by  $\text{Conj}(\Omega)$ .

**Example 2.6.** Let  $X$  be any set, and fix  $\sigma \in \text{Sym}(X)$ . Define  $s : X \rightarrow \text{Sym}(X)$  by  $x \mapsto \sigma$ , so that  $s_x(y) = \sigma(y)$  for all  $x, y \in X$ . Then  $(X, s)$  is called a *constant rack*, and we denote it by  $(X, \sigma)_c$ .

The axioms of *GL-racks* encode not only the crossing relations but also the cusp relations given by the Legendrian Reidemeister moves in Figures 5-7. GL-racks were introduced independently in [21] and [23] to generalize the *Legendrian racks* introduced in [6] and [27]. Once again, we translate the original definition into the language of rack symmetries.

**Definition 2.7.** [21, Definition 3.1] A *GL-rack*, also called a *generalized Legendrian rack* or a *bi-Legendrian rack*, is a quadruple  $(X, s, \mathbf{u}, \mathbf{d})$  in which  $(X, s)$  is a rack and  $\mathbf{u}, \mathbf{d} : X \rightarrow X$  are maps such that the following hold for all  $x \in X$ :

- (L1)  $(\mathbf{u} \circ s_x)(x) = x = (\mathbf{d} \circ s_x)(x)$ .
- (L2)  $\mathbf{u} \circ s_x = s_x \circ \mathbf{u}$  and  $\mathbf{d} \circ s_x = s_x \circ \mathbf{d}$ .
- (L3)  $s_{\mathbf{u}(x)} = s_x = s_{\mathbf{d}(x)}$ .

We call the pair  $(\mathbf{u}, \mathbf{d})$  a *GL-structure* on  $(Q, s)$ . If in addition  $(X, s)$  is a quandle, we say that  $(X, s, \mathbf{u}, \mathbf{d})$  is a *GL-quandle*.

**Example 2.8.** [23, Example 3.6] Let  $G$  be a group, let  $z \in Z(G)$ , and define  $f : G \rightarrow G$  by  $g \mapsto zg$ . Then  $(\text{Conj}(G), f, f^{-1})$  is a GL-quandle.

**Example 2.9.** [23, Example 3.7] Let  $(X, \sigma)_c$  be a constant rack, and let  $\mathbf{u}, \mathbf{d} : X \rightarrow X$  be maps. Then  $(\mathbf{u}, \mathbf{d})$  defines a GL-structure on  $(X, \sigma)_c$  if and only if  $\mathbf{u} \mathbf{d} = \mathbf{d} \mathbf{u} = \sigma^{-1}$ . In this case, we say that  $((X, \sigma)_c, \mathbf{u}, \mathbf{d})$  is a *constant GL-rack*, and we denote it by  $(X, \sigma, \mathbf{u}, \mathbf{d})_c$ .

**Example 2.10.** [22, Example 3.4] Any GL-rack of the form  $(X, s, \text{id}_X, \text{id}_X)$  is called a *trivial GL-rack*. In particular, any quandle  $(Q, s)$  can be identified with the trivial GL-rack  $(Q, s, \text{id}_Q, \text{id}_Q)$ , cf. Lemma 2.14. In other words, GL-racks generalize quandles.

We define homomorphisms of these algebraic structures as follows.

**Definition 2.11.** Let  $(X, s)$  and  $(Y, t)$  be racks. A map  $\varphi : X \rightarrow Y$  is called a *rack homomorphism* if  $\varphi \circ s_x = t_{\varphi(x)} \circ \varphi$  for all  $x \in X$ . If in addition  $(\mathbf{u}_1, \mathbf{d}_1)$  and  $(\mathbf{u}_2, \mathbf{d}_2)$  are GL-structures on  $(X, s)$  and  $(Y, t)$ , we say that a  $\varphi$  is also a *GL-rack homomorphism* if  $\varphi \circ \mathbf{u}_1 = \mathbf{u}_2 \circ \varphi$  and  $\varphi \circ \mathbf{d}_1 = \mathbf{d}_2 \circ \varphi$ . A *(GL-)rack isomorphism* is simply a bijective (GL-)rack homomorphism. If  $R$  is a GL-rack, we denote its group of GL-rack automorphisms by  $\text{Aut}(R)$ .

Evidently, we have the following; the final sentence is from [22, Proposition 3.2].

**Proposition 2.12.** *Let  $(X, s)$  be a rack with maps  $\mathbf{u}, \mathbf{d} : X \rightarrow X$  satisfying axioms (L1) and (L3) of Definition 2.7. Then  $R := (X, s, \mathbf{u}, \mathbf{d})$  is a GL-rack if and only if  $\mathbf{u}$  and  $\mathbf{d}$  are endomorphisms of the underlying rack  $(X, s)$ . In this case,  $\mathbf{u}\mathbf{d} = \mathbf{d}\mathbf{u}$ , and  $\mathbf{u}, \mathbf{d} \in \text{Aut}(R)$ .*

Axiom (L1) immediately yields the following.

**Proposition 2.13.** *Let  $(X, s, \mathbf{u}, \mathbf{d})$  be a GL-rack. Then the underlying rack  $(X, s)$  is a quandle if and only if  $\mathbf{u}\mathbf{d} = \text{id}_X = \mathbf{d}\mathbf{u}$ , that is,  $\mathbf{d} = \mathbf{u}^{-1}$  as GL-rack automorphisms.*

**2.3. Functors of interest in the literature.** In this subsection, we define several categories and functors appearing in the literature on GL-racks, quandles, and their relationships with groups.

We begin by defining several categories. Let **Set** and **Grp** be the categories of sets with functions and groups with group homomorphisms, respectively. Let **Rack** be the category of racks with rack homomorphisms, let **Qnd** be the full subcategory of **Rack** whose objects are quandles, and let **GLR** be the category of GL-racks with GL-rack homomorphisms. By Example 2.10 and Proposition 2.13, we have the following.

**Lemma 2.14.** *The correspondence  $(Q, s) \mapsto (Q, s, \text{id}_Q, \text{id}_Q)$  defines a canonical isomorphism from **Qnd** to the full subcategory of **GLR** whose objects are trivial GL-racks.*

In the sense of universal algebra, **GLR** is an equational algebraic category, so it is complete and cocomplete (see [1, Corollary 1.2, Theorem 4.5]). Thus, we can express GL-racks in terms of generators and relations using quotients of *free GL-racks*, which we define as follows.

**Definition 2.15.** [22, Section 4] Let  $X$  be a set. We define the *free GL-rack on  $X$* , denoted by  $\text{FGLR}(X)$ , as follows. If  $X = \emptyset$ , let  $\text{FGLR}(X)$  be the trivial GL-rack with one element. Else, define the *universe of words generated by  $X$*  to be the set  $W(X)$  such that  $X \subset W(X)$  and  $s_y(x), s_y^{-1}(x), \mathbf{u}(x), \mathbf{d}(x) \in W(X)$  for all  $x, y \in W(X)$ . Let  $F(X)$  be the set of equivalence classes of elements of  $W(X)$  modulo the equivalence relation generated by the following relations for all  $x, y, z \in W(X)$ :

- (1)  $s_y^{-1}(s_y(x))y \sim x \sim s_y(s_y^{-1}(x))$ .
- (2)  $s_z(s_y(x)) \sim s_{s_z(y)}(s_z(x))$ .
- (3)  $\mathbf{u}(\mathbf{d}(s_x(x))) \sim x \sim \mathbf{d}(\mathbf{u}(s_x(x)))$ .
- (4)  $\mathbf{u}(s_y(x)) \sim s_y(\mathbf{u}(x))$  and  $\mathbf{d}(s_y(x)) \sim s_y(\mathbf{d}(x))$ .
- (5)  $s_{\mathbf{u}(y)}(x) \sim s_y(x)$  and  $s_{\mathbf{d}(y)}(x) \sim s_y(x)$ .

Thus, we have maps  $s : F(X) \rightarrow \text{Sym}(F(X))$  defined by  $x \mapsto s_x := [y \mapsto s_x(y)]$  and  $\mathbf{u}, \mathbf{d} : F(X) \rightarrow F(X)$  defined by  $x \mapsto \mathbf{u}(x)$  and  $x \mapsto \mathbf{d}(x)$ . We define  $\text{FGLR}(X)$  to be the GL-rack  $(F(X), s, \mathbf{u}, \mathbf{d})$ . The *free quandle on  $X$*  is defined similarly; in the sense of Lemma 2.14, it is simply  $\text{FGLR}(X)$  modulo the relations  $\mathbf{u}(x) \sim x \sim \mathbf{d}(x)$  for all  $x \in W(X)$ .

To rephrase [22, Proposition 4.2], the functor  $\mathbf{Set} \rightarrow \mathbf{GLR}$  defined by  $X \mapsto \mathbf{FGLR}(X)$  is left adjoint to the forgetful functor  $\mathbf{GLR} \rightarrow \mathbf{Set}$ , as one might expect.

Another functor of interest in Section 3 assigns an *enveloping group* to any GL-rack.

**Definition 2.16.** [21, Section 8] Given a GL-rack  $R = (X, s, \mathbf{u}, \mathbf{d})$ , its *enveloping group* is

$$\mathbf{Env}_{\mathbf{GLR}}(R) := \langle e_x, x \in X \mid e_{s_x(y)} = e_x^{-1} e_y e_x, e_{\mathbf{u}(x)} = e_x, e_{\mathbf{d}(x)} = e_x, x, y \in X \rangle.$$

By taking  $\mathbf{u} = \text{id}_X = \mathbf{d}$ , we can also define the enveloping group of a quandle  $(Q, s)$  to be

$$\mathbf{Env}_{\mathbf{Qnd}}(Q, s) := \langle e_x, x \in Q \mid e_{s_x(y)} = e_x^{-1} e_y e_x, x, y \in Q \rangle.$$

The end of Example 3.6 in Subsection 3.2 computes the enveloping groups of the GL-rack and fundamental quandle of a Legendrian  $(2, -q)$  torus knot as defined in Subsection 3.1.

The functor  $\mathbf{Env}_{\mathbf{GLR}} : \mathbf{GLR} \rightarrow \mathbf{Grp}$  has a right adjoint that results from taking  $\Omega = G$  in Example 2.5, as specified in the following result adapted from [21, Proposition 8.4].

**Proposition 2.17.** *There exists a functor  $\mathbf{Env}_{\mathbf{GLR}} : \mathbf{GLR} \rightarrow \mathbf{Grp}$  that sends any GL-rack to its enveloping group and sends any GL-rack homomorphism  $\psi : (X, s, \mathbf{u}_1, \mathbf{d}_1) \rightarrow (Y, t, \mathbf{u}_2, \mathbf{d}_2)$  to the group homomorphism  $\tilde{\psi} : \mathbf{Env}_{\mathbf{GLR}}(X, s, \mathbf{u}_1, \mathbf{d}_1) \rightarrow \mathbf{Env}_{\mathbf{GLR}}(Y, t, \mathbf{u}_2, \mathbf{d}_2)$  defined by  $e_x \mapsto e_{\psi(x)}$  for all  $x \in X$ . Also,  $\mathbf{Env}_{\mathbf{GLR}}$  is left adjoint to a functor sending any group  $G$  to the GL-rack  $(\mathbf{Conj}(G), \text{id}_G, \text{id}_G)$ , which is isomorphic to the quandle  $\mathbf{Conj}(G)$  in the sense of Lemma 2.14.*

Thus, some authors denote the enveloping group of a GL-rack or quandle  $R$  by  $\mathbf{Adconj}(R)$  or  $\mathbf{As}(R)$  and call it the *associated group of  $R$* , cf. [20, Section 6; 31, Definition 2.19].

### 3. ON RACK-THEORETIC INVARIANTS OF LEGENDRIAN LINKS

In this section, we begin by defining the GL-rack of an oriented Legendrian link  $\Lambda$  and the fundamental quandle of its underlying smooth link  $L$ , both of which are invariant under Legendrian isotopy. After a few worked examples, we give short algebraic proofs of several conjectures relating to Legendrian links and their invariants.

**3.1. The GL-rack of a Legendrian link.** In this subsection, we discuss how to assign a GL-rack to a Legendrian link in a way invariant under Legendrian isotopy. We begin with several definitions.

**Definition 3.1.** Given a front projection  $D(\Lambda)$  of an oriented Legendrian link  $\Lambda$ , define a *cusped strand* of  $D(\Lambda)$  to be a connected segment in  $D(\Lambda)$  that either starts and ends at a crossing or (in the case that  $D(\Lambda)$  contains no crossings) ends where it started. Also, define an *uncusped strand* of  $D(\Lambda)$  to be a maximal (with respect to inclusion) connected subset of a cusped strand of  $D(\Lambda)$  that both starts and ends at either a crossing or a cusp.

**Definition 3.2.** [22, Section 4] Let  $\Lambda$  be an oriented Legendrian link with front projection  $D(\Lambda)$ , and let  $X_\Lambda$  be a set in one-to-one correspondence with the cusped strands of  $D(\Lambda)$ . At each cusp, label the neighboring uncusped strands using the cusp relations in Figure 4. Then, at each crossing, impose the corresponding crossing relation between uncusped strands in Figure 4 on  $\mathbf{FGLR}(X_\Lambda)$ . The *GL-rack of  $\Lambda$* , denoted by  $\mathcal{G}(\Lambda)$ , is defined to be the set of equivalence classes of elements of  $\mathbf{FGLR}(X_\Lambda)$  modulo the equivalence relation generated by these relations. If  $L$  is a smooth link with link diagram  $D(L)$ , then we define the *fundamental quandle of  $L$* , denoted by  $\mathcal{Q}(L)$ , in a similar manner. However, we use the free quandle on  $X_\Lambda$  in place of  $\mathbf{FGLR}(X_\Lambda)$ , and we do not employ any cusp relations.



For several examples of how to compute the GL-rack of a Legendrian link, see Subsection 3.2.

The assignment of  $\mathcal{G}(\Lambda)$  to  $\Lambda$  (resp.  $\mathcal{Q}(L)$  to  $L$ ) is independent of the choice of front projection  $D(\Lambda)$  (resp. link diagram  $D(L)$ ), as captured in the following result of Karmakar et al.

**Proposition 3.3.** [22, Theorem 4.3] *If two oriented Legendrian front projections are related by a finite sequence of Legendrian Reidemeister moves, then their induced GL-racks are isomorphic. Hence, the GL-rack of a Legendrian link is invariant under Legendrian isotopy.*

This is a consequence of Proposition 2.2 and the fact that the GL-rack axioms capture the crossing and cusp relations induced by the Legendrian Reidemeister moves. In turn, Proposition 3.3 implies that the *GL-rack coloring number* of  $\Lambda$  with respect to a fixed GL-rack, as defined below, is invariant under Legendrian isotopy (see [6, 22, 23]).

**Definition 3.4.** Let  $R$  be a GL-rack. The  *$R$ -coloring number* of an oriented Legendrian link  $\Lambda$ , denoted by  $\text{Col}(\Lambda, R)$ , is defined to be the cardinality of the hom-set  $\text{Hom}_{\text{GLR}}(\mathcal{G}(\Lambda), R)$ .

Kulkarni and Prathamesh in [27, Main Theorem 2], Kimura in [23, Theorem 4.1], and Karmakar et al. in [22, Theorem 4.6] each used  $R$ -coloring numbers to distinguish between infinitely many Legendrian unknots. Karmakar et al. also used  $R$ -coloring numbers to distinguish between infinitely many Legendrian trefoils in [22, Theorem 4.7], and Cenicerós et al. in [6, Example 16] used them to distinguish between connected sums of Legendrian trefoils. That said, there also exist distinct Legendrian knots having GL-racks (see [24, Subsection 4.2]), so neither  $\mathcal{G}(\Lambda)$  nor  $R$ -coloring numbers are complete Legendrian knot invariants. Nevertheless, we will use the latter in Subsection 3.2 to distinguish between unstabilized Legendrian representatives of knot types  $6_2$  and  $8_{10}$ , which cannot be done using the graded ruling invariant or linearized contact homology.

Given  $\Lambda$ , note that imposing the equivalence relation  $\mathbf{u}(x) \sim x \sim \mathbf{d}(x)$  for all  $x \in X_\Lambda$  onto  $\mathcal{G}(\Lambda)$  yields a quandle in the sense of Lemma 2.14. Geometrically, this amounts to “ignoring” all cusps in  $D(\Lambda)$  and viewing  $D(\Lambda)$  only as a diagram of the underlying smooth link  $L$ . This recovers  $\mathcal{Q}(L)$  from  $\mathcal{G}(\Lambda)$ , yielding the following observation.

**Lemma 3.5.** [24, Remark 23] *Let  $\Lambda$  be an oriented Legendrian link, and let  $L$  be its underlying smooth link. After imposing an equivalence relation onto  $\mathcal{G}(\Lambda)$  defined by  $\mathbf{u}(x) \sim x \sim \mathbf{d}(x)$  for all  $x \in X_\Lambda$ , the resulting GL-rack is canonically isomorphic to  $\mathcal{Q}(L)$  in the sense of Lemma 2.14.*

**3.2. Example calculations and applications.** In this section, we give several examples of how to compute the GL-rack of an oriented Legendrian knot. This allows us to give relatively brief algebraic proofs of conjectures in [9] and [3] about Legendrian  $6_2$  and  $8_{10}$  knots, respectively.

**Example 3.6.** Let  $q \geq 3$  be an odd integer, let  $L$  be a  $(2, -q)$ -torus knot, and let  $\Lambda$  be the Legendrian representative of  $L$  having maximal Thurston-Bennequin and rotation numbers. (By [17, Theorem 4.3],  $\Lambda$  is the unique such Legendrian representative up to Legendrian isotopy.) In this example, we compute  $\mathcal{G}(\Lambda)$ ,  $\mathcal{Q}(L)$ ,  $\text{Env}_{\text{GLR}}(\mathcal{G}(\Lambda))$ , and  $\text{Env}_{\text{Qnd}}(\mathcal{Q}(L))$  using the front projection  $D(\Lambda)$  in Figure 8. Starting at any crossing (which, in Figure 8, we arbitrarily choose to be the bottommost crossing), traverse  $D(\Lambda)$  along its depicted orientation. By recording the induced cusp and crossing relations using Figure 4, we compute that  $\mathcal{G}(\Lambda)$  is the free GL-rack on the set  $X_\Lambda = \{x_1, \dots, x_q\}$  modulo the crossing relations

$$s_{\mathbf{u}(x_1)}(x_q) = \mathbf{ud}(x_2), \quad s_{\mathbf{d}(x_q)}(x_{q-1}) = \mathbf{ud}(x_1), \quad \text{and} \quad s_{\mathbf{d}(x_{i-1})}(x_{i-2}) = \mathbf{d}^2(x_i) \text{ for all } 3 \leq i \leq q.$$

Using axiom (L3) of a GL-rack, we can simplify these crossing relations to

$$s_{x_1}(x_q) = \mathbf{ud}(x_2), \quad s_{x_q}(x_{q-1}) = \mathbf{ud}(x_1), \quad \text{and} \quad s_{x_{i-1}}(x_{i-2}) = \mathbf{d}^2(x_i) \text{ for all } 3 \leq i \leq q.$$

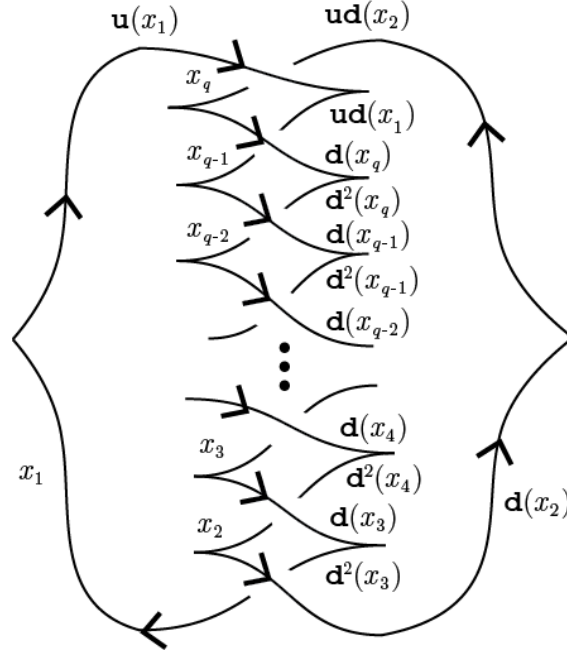


FIGURE 8. Front projection  $D(\Lambda)$  and induced cusp relations of the Legendrian  $(2, -q)$ -torus knot  $\Lambda$  with maximal Thurston-Bennequin and rotation numbers.

Now that we have a presentation of  $\mathcal{G}(\Lambda)$ , let us compute  $\mathcal{Q}(L)$ . To do this, we could traverse  $D(\Lambda)$  again while ignoring all cusps and only considering crossing relations. In view of Lemma 3.5, we could equivalently impose the relations  $u(x_i) = x_i = d(x_i)$  for all  $1 \leq i \leq q$  onto  $\mathcal{G}(\Lambda)$ . Either method shows that  $\mathcal{Q}(L)$  is the free quandle on  $X_\Lambda$  modulo the crossing relations

$$s_{x_1}(x_q) = x_2, \quad s_{x_q}(x_{q-1}) = x_1, \quad \text{and} \quad s_{x_{i-1}}(x_{i-2}) = x_i \text{ for all } 3 \leq i \leq q.$$

Indeed, if we invert each symmetry in the relations of  $\mathcal{Q}(L)$ , then we recover the fundamental quandle of the mirror image of  $L$  computed in [2, Remark 3], as predicted by [37, Section 1].

If  $q = 3$ , then  $L$  is a left-handed trefoil, and the crossing relations show that  $\text{Env}_{\text{GLR}}(\mathcal{G}(\Lambda))$  and  $\text{Env}_{\text{Qnd}}(\mathcal{Q}(L))$  are both isomorphic to the group

$$\begin{aligned} \langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{s_{x_1}(x_3)} &= e_{x_1}^{-1} e_{x_3} e_{x_1}, \quad e_{s_{x_2}(x_1)} = e_{x_2}^{-1} e_{x_1} e_{x_2}, \quad e_{s_{x_3}(x_2)} = e_{x_3}^{-1} e_{x_2} e_{x_3} \rangle \\ &= \langle e_{x_1}, e_{x_2}, e_{x_3} \mid e_{x_1} e_{x_2} = e_{x_3} e_{x_1}, \quad e_{x_2} e_{x_3} = e_{x_1} e_{x_2}, \quad e_{x_3} e_{x_1} = e_{x_2} e_{x_3} \rangle. \end{aligned}$$

Note that this is precisely the Wirtinger presentation of the knot group  $\pi_1(\mathbb{R}^3 \setminus L) \cong \langle x, y \mid x^2 = y^3 \rangle$  of  $L$  (see [34, Subsection 4.2.5]). Subsection 3.3 will generalize this observation.

**Example 3.7.** Let  $\Lambda_1$  and  $\Lambda_2$  be the oriented Legendrian knots on the left and right of Figure 9, respectively. In this example, we compute  $\mathcal{G}(\Lambda_1)$  and  $\mathcal{G}(\Lambda_2)$  in preparation for a proof that  $\Lambda_1$  and  $\Lambda_2$  are not Legendrian isotopic. We note from [9] that  $\Lambda_1$  and  $\Lambda_2$  are Legendrian representatives of the topological knot  $6_2$ , and  $(\text{tb}(\Lambda_1), \text{rot}(\Lambda_1)) = (-7, 2) = (\text{tb}(\Lambda_2), \text{rot}(\Lambda_2))$ .

Let us begin with  $\Lambda_1$ . Traverse  $D(\Lambda_1)$  using its given orientation while labeling all uncusped strands as in Figure 4. By writing down the induced crossing relations as in Figure 4, we find that

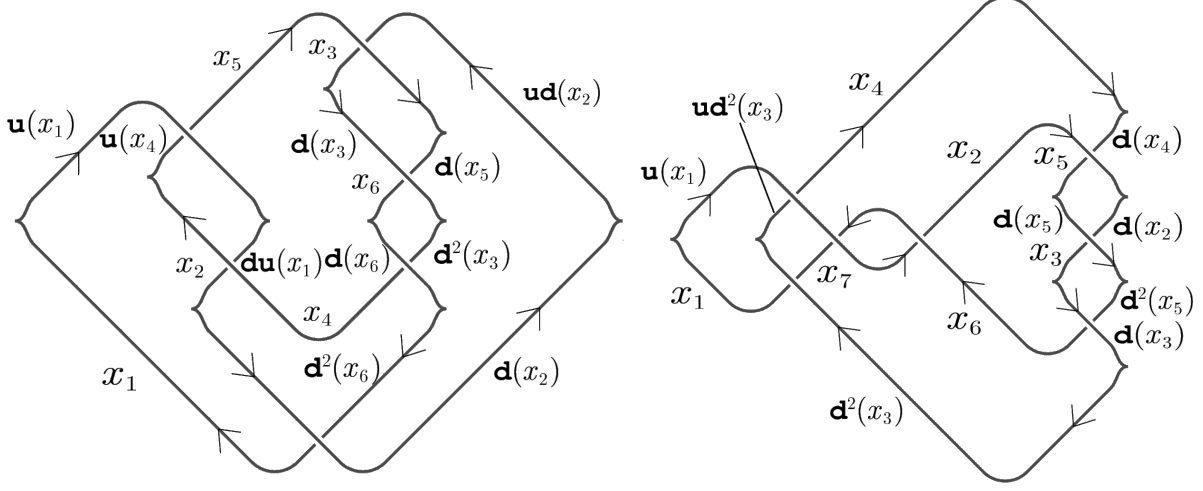


FIGURE 9. Front projections of the two Legendrian representatives of the topological knot  $6_2$  with  $(\text{tb}, \text{rot}) = (-7, 2)$  given in [9]. Created using [25], cf. [26].

$\mathcal{G}(\Lambda_1)$  is the free GL-rack on the set  $X_{\Lambda_1} = \{x_1, \dots, x_6\}$  modulo the following crossing relations:

$$(1) \quad \mathcal{G}(\Lambda_1) \begin{cases} s_{u(x_1)}(u(x_4)) = x_5 \iff s_{x_1}(u(x_4)) = x_5, \\ s_{x_4}(du(x_1)) = x_2, \\ s_{d(x_2)}(x_1) = d^2(x_6) \iff s_{x_2}(x_1) = d^2(x_6), \\ s_{x_5}(x_3) = ud(x_2), \\ s_{d(x_3)}(x_6) = d(x_5) \iff s_{x_3}(x_6) = d(x_5), \\ s_{d(x_6)}(x_4) = d^2(x_3) \iff s_{x_6}(x_4) = d^2(x_3). \end{cases}$$

Note that we have simplified the first, third, fifth, and sixth relations using GL-rack axiom (L3).

Similarly, we compute  $\mathcal{G}(\Lambda_2)$  to be the free GL-rack on the set  $X_{\Lambda_2} = \{x_1, \dots, x_7\}$  modulo the following crossing relations:

$$(2) \quad \mathcal{G}(\Lambda_2) \begin{cases} s_{x_1}(ud^2(x_3)) = x_4, & s_{x_5}(x_3) = d(x_2), \\ s_{x_1}(x_6) = x_7, & s_{x_3}(x_6) = d^2(x_5), \\ s_{x_6}(x_2) = u(x_1), & s_{x_3}(x_7) = x_1, \\ s_{x_2}(x_5) = d(x_4). \end{cases}$$

We can use these calculations to prove a conjecture of Chongchitmate and Ng in [9] that  $\Lambda_1$  and  $\Lambda_2$  in the previous example are not Legendrian isotopic. In [11, Proposition 2.3], Dynnikov and Prasolov proved this conjecture using impressive topological and combinatorial machinery. At the time of writing, theirs is the only proof of which we are aware, as  $\Lambda_1$  and  $\Lambda_2$  cannot be distinguished using the graded ruling invariant or linearized contact homology, cf. [9]. However,  $R$ -coloring numbers offer a simpler algebraic alternative. In the following, we denote the symmetric group on  $n$  letters by  $S_n$ .

**Theorem 3.8.** *The two oriented Legendrian knots in Figure 9 are not Legendrian isotopic; in fact, they are distinguishable using coloring numbers with respect to a constant GL-rack of order 3.*

*Proof.* As before, let  $\Lambda_1$  and  $\Lambda_2$  be the oriented Legendrian knots on the left and right of Figure 9, respectively. Let  $Y = \{1, 2, 3\}$ . In cycle notation, let  $\sigma \in S_3$  be the permutation (123), and

note that  $\sigma^3 = \text{id}_Y$ . In the notation of Example 2.9, let  $R = (Y, \sigma, \sigma^{-1}, \text{id}_Y)_c$ . We will show that  $\text{Col}(\Lambda_2, R) > \text{Col}(\Lambda_1, R)$ . To that end, let  $A$  denote the underlying set of  $\mathcal{G}(\Lambda_2)$  as presented in Example 3.7, and define  $\varphi_1, \varphi_2, \varphi_3 : A \rightarrow Y$  by the following:

$$\varphi_1(x_i) := \begin{cases} 1 & \text{if } i \in \{1, 3, 4\}, \\ 2 & \text{if } i \in \{2, 6\}, \\ 3 & \text{if } i \in \{5, 7\}. \end{cases} \quad \varphi_2(x_i) := \begin{cases} 2 & \text{if } i \in \{1, 3, 4\}, \\ 3 & \text{if } i \in \{2, 6\}, \\ 1 & \text{if } i \in \{5, 7\}. \end{cases} \quad \varphi_3(x_i) := \begin{cases} 3 & \text{if } i \in \{1, 3, 4\}, \\ 1 & \text{if } i \in \{2, 6\}, \\ 2 & \text{if } i \in \{5, 7\}. \end{cases}$$

Using the relations in (2), it is straightforward to verify that  $\varphi_1, \varphi_2$ , and  $\varphi_3$  define GL-rack homomorphisms from  $\mathcal{G}(\Lambda_2)$  to  $R$ . Hence,  $\text{Col}(\Lambda_2, R) \geq 3$ . (In fact, using a similar method as in the remainder of this proof, one can show that this bound is actually an equality.)

On the other hand, we claim that  $\text{Hom}_{\text{GLR}}(\mathcal{G}(\Lambda_1), R) = \emptyset$ . Let  $B$  denote the underlying set of  $\mathcal{G}(\Lambda_1)$  as presented in Example 3.7, and suppose to the contrary that some map  $\varphi : B \rightarrow Y$  defines a GL-rack homomorphism from  $\mathcal{G}(\Lambda_1)$  to  $R$  with  $\varphi(x_i) = y_i$ . Since  $\varphi$  is a GL-rack homomorphism, the relations in (1) must hold when we replace each  $x_i$  with  $y_i$ , each  $s_{x_i}$  with  $\sigma$ , each  $u$  with  $\sigma^{-1}$ , and each  $d$  with  $\text{id}_Y$ . These substitutions yield the following system of equations in  $R$ :

$$(3) \quad R \begin{cases} (\sigma \circ \sigma^{-1})(y_4) = y_5 \iff y_4 = y_5, \\ (\sigma \circ \sigma^{-1})(y_1) = y_2 \iff y_1 = y_2, \\ \sigma(y_1) = y_6, \\ \sigma(y_3) = \sigma^{-1}(y_2) \iff \sigma^3(y_3) = (\sigma^2 \circ \sigma^{-1})(y_2) \iff y_3 = \sigma(y_2), \\ \sigma(y_6) = y_5, \\ \sigma(y_4) = y_3. \end{cases}$$

Here, we have used the fact that  $\sigma^3 = \text{id}_Y$  to rewrite the fourth equality. We now deduce that

$$\sigma(y_2) = y_3 = \sigma(y_4) = \sigma(y_5) = \sigma^2(y_6) = \sigma^3(y_1) = y_1 = y_2.$$

However,  $\sigma$  has no fixed points in  $Y$ , so the system of equations in (3) has no solutions in  $R$ . Hence,  $\varphi$  cannot exist.  $\square$

Incidentally, Theorem 3.8 gives a positive answer to a question posed by Kimura in [23, Section 4], as we state below.

**Corollary 3.9.** *Although  $R$ -coloring numbers cannot distinguish between Legendrian knots having the same underlying smooth knot type, Thurston-Bennequin number, and rotation number if  $R$  is a GL-quandle by [23, Theorem 4.3], the same is not generally true if  $R$  is a nonquandle GL-rack.*

Similarly,  $R$ -coloring numbers distinguish between the Legendrian representatives of the topological knot  $8_{10}$  with  $(\text{tb}, \text{rot}) = (-8, 3)$  in Figure 10. This proves a conjecture of Bhattacharyya et al. in [3]. At the time of writing, we are unaware of any other proofs of this conjecture; similarly to the Legendrian  $6_2$  knots in Example 3.7, the graded ruling invariant and linearized contact homology cannot distinguish between these knots [3].

**Example 3.10.** Let  $\Lambda_1$  and  $\Lambda_2$  be the oriented Legendrian knots on the left and right of Figure 10, respectively. We compute that  $\mathcal{G}(\Lambda_1)$  and  $\mathcal{G}(\Lambda_2)$  are the free GL-racks on the set  $\{x_1, \dots, x_{10}\}$

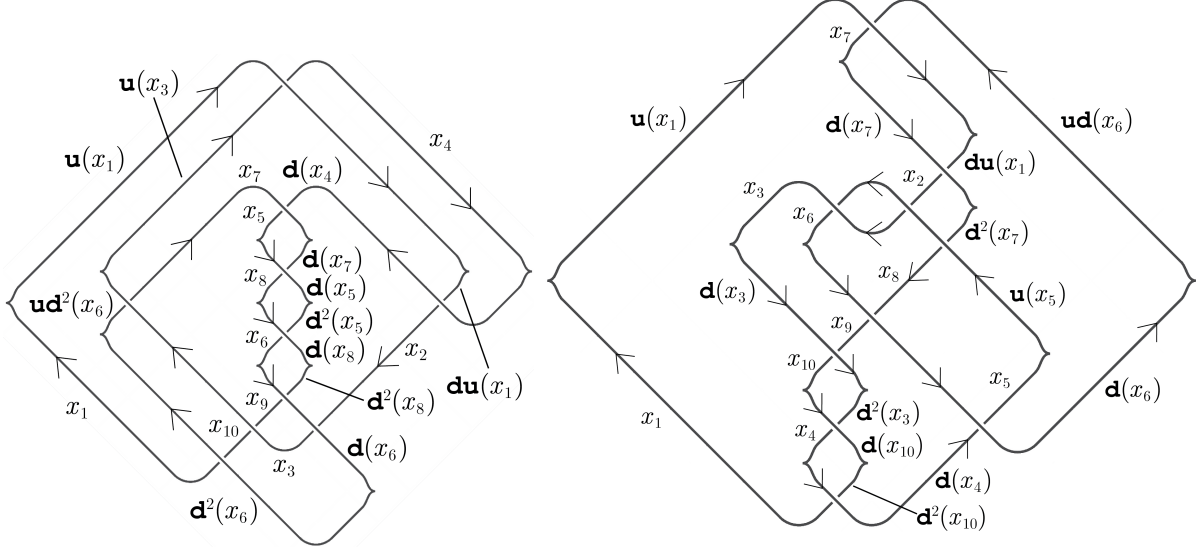


FIGURE 10. Front projections of the two Legendrian representatives of the topological knot  $8_{10}$  with  $(tb, rot) = (-8, 3)$  given in [3]. Created using [25], cf. [26].

modulo the following crossing relations:

$$\mathcal{G}(\Lambda_1) \begin{cases} s_{x_1}(u(x_3)) = x_4, & s_{x_4}(du(x_1)) = x_2, \\ s_{x_7}(x_5) = d(x_4), & s_{x_5}(x_8) = d(x_7), \\ s_{x_8}(x_6) = d^2(x_5), & s_{x_6}(x_9) = d(x_6), \\ s_{x_6}(x_3) = x_2, & s_{x_6}(x_{10}) = x_1, \\ s_{x_3}(x_7) = ud^2(x_6), & s_{x_3}(x_9) = x_{10}. \end{cases} \quad \mathcal{G}(\Lambda_2) \begin{cases} s_{x_1}(x_7) = ud(x_6), & s_{x_7}(x_2) = du(x_1), \\ s_{x_5}(x_2) = x_3, & s_{x_3}(u(x_5)) = x_6, \\ s_{x_3}(x_{10}) = x_9, & s_{x_{10}}(x_4) = d^2(x_3), \\ s_4(x_1) = d^2(x_{10}), & s_{x_6}(d(x_4)) = x_5, \\ s_{x_5}(d^2(x_7)) = x_8, & s_{x_6}(x_9) = x_8. \end{cases}$$

**Theorem 3.11.** *The two oriented Legendrian knots in Figure 10 are not Legendrian isotopic; in fact, they are distinguishable using coloring numbers with respect to a constant GL-rack of order 2.*

*Proof.* Once again, let  $\Lambda_1$  and  $\Lambda_2$  be the oriented Legendrian knots on the left and right of Figure 10, respectively. Let  $Y = \{1, 2\}$ . In cycle notation, let  $\sigma \in S_2$  be the permutation (12). In the notation of Example 2.9, let  $R = (Y, \sigma, \sigma, id_Y)_c$ . We will show that  $Col(\Lambda_1, R) > Col(\Lambda_2, R)$ . To that end, let  $A$  denote the underlying set of  $\mathcal{G}(\Lambda_1)$  as presented in Example 3.10, and define  $\varphi_1, \varphi_2 : A \rightarrow Y$  by the following:

$$\varphi_1(x_i) := \begin{cases} 1 & \text{if } i \in \{1, 2, 5, 8, 9\}, \\ 2 & \text{if } i \in \{3, 4, 6, 7, 10\}. \end{cases} \quad \varphi_2(x_i) := \begin{cases} 2 & \text{if } i \in \{1, 2, 5, 8, 9\}, \\ 1 & \text{if } i \in \{3, 4, 6, 7, 10\}. \end{cases}$$

Using the relations of  $\mathcal{G}(\Lambda_1)$  in Example 3.10, it is straightforward to verify that  $\varphi_1$  and  $\varphi_2$  define GL-rack homomorphisms from  $\mathcal{G}(\Lambda_1)$  to  $R$ . Hence,  $Col(\Lambda_1, R) \geq 2$  (which is actually an equality).

On the other hand, we claim that  $Hom_{GLR}(\mathcal{G}(\Lambda_2), R) = \emptyset$ . Let  $B$  denote the underlying set of  $\mathcal{G}(\Lambda_2)$  as presented in Example 3.10, and suppose to the contrary that some map  $\varphi : B \rightarrow Y$  defines a GL-rack homomorphism from  $\mathcal{G}(\Lambda_2)$  to  $R$  with  $\varphi(x_i) = y_i$ . Just like before, the relations

of  $\mathcal{G}(\Lambda_2)$  in Example 3.10 yield the following system of equations in  $R$ :

$$R \begin{cases} \sigma(y_7) = \sigma(y_6), & \sigma(y_2) = \sigma(x_1), \\ \sigma(y_2) = y_3, & \sigma^2(y_5) = y_6, \\ \sigma(y_{10}) = y_9, & \sigma(y_4) = y_3, \\ \sigma(y_1) = y_{10}, & \sigma(y_4) = y_5, \\ \sigma(y_7) = y_8, & \sigma(y_9) = y_8. \end{cases}$$

Since  $\sigma^2 = \text{id}_Y$ , we can also rewrite the equalities  $\sigma(y_7) = \sigma(x_6)$ ,  $\sigma^2(y_5) = y_6$ , and  $\sigma(y_{10}) = y_9$  as  $y_7 = y_6$ ,  $y_5 = y_6$ , and  $y_{10} = \sigma(y_9)$ , respectively. Therefore, we have

$$y_7 = y_6 = y_5 = \sigma(y_4) = y_3 = \sigma(y_2) = \sigma(y_1) = y_{10} = \sigma(y_9) = y_8 = \sigma(y_7),$$

which is impossible since  $\sigma$  has no fixed points in  $Y$ . Hence,  $\varphi$  cannot exist.  $\square$

We chose the constant GL-racks and coloring maps used to prove Theorems 3.8 and 3.11 through exhaustive searches conducted in **GAP** [19]. For details, see Subsection A.2.

**3.3. Isomorphism of  $\text{Env}_{\text{GLR}}(\mathcal{G}(\Lambda))$  and  $\pi_1(\mathbb{R}^3 \setminus L)$ .** We now prove an empirical observation of Karmakar et al. in the original version of [21, Remark 8.2], which we state as Corollary 3.13. Although the authors have removed this remark from subsequent releases of the article, we are unaware of any other proofs of this statement at the time of writing. To allow for a short categorical proof, we begin by proving a more abstract version of the conjecture.

**Theorem 3.12.** *Let  $R = (X, s, \mathbf{u}, \mathbf{d})$  be a GL-rack, and let  $R' = (X/\sim, s^*, \text{id}_{X/\sim}, \text{id}_{X/\sim})$  be the GL-rack obtained by imposing an equivalence relation  $\sim$  onto  $R$  defined by  $\mathbf{u}(x) = x = \mathbf{d}(x)$  for all  $x \in X$ . Then  $\text{Env}_{\text{GLR}}(R)$  and  $\text{Env}_{\text{GLR}}(R')$  are isomorphic as groups. In particular,  $\text{Env}_{\text{GLR}}(R)$  and  $\text{Env}_{\text{Qnd}}(X/\sim, s^*)$  are isomorphic as groups.*

*Proof.* By definition,  $R'$  is the quotient object of the equivalence relation  $\sim$  on  $R$  in **GLR**. In other words,  $R$  is the colimit of the following diagram in **GLR**:

$$R \begin{array}{c} \xrightarrow{\mathbf{u}} \\ \text{id}_X \end{array} R \begin{array}{c} \xrightarrow{\mathbf{d}} \\ \text{id}_X \end{array} R$$

Recall that left adjoint functors preserve colimits. It follows from Proposition 2.17 that  $\text{Env}_{\text{GLR}}(R')$  is the colimit of the following diagram in **Grp**:

$$\text{Env}_{\text{GLR}}(R) \begin{array}{c} \xrightarrow{\text{Env}_{\text{GLR}}(\mathbf{u})} \\ \text{Env}_{\text{GLR}}(\text{id}_X) \end{array} \text{Env}_{\text{GLR}}(R) \begin{array}{c} \xrightarrow{\text{Env}_{\text{GLR}}(\mathbf{d})} \\ \text{Env}_{\text{GLR}}(\text{id}_X) \end{array} \text{Env}_{\text{GLR}}(R)$$

By Proposition 2.17, the group homomorphism  $\text{Env}_{\text{GLR}}(\mathbf{u})$  is defined by  $e_x \mapsto e_{\mathbf{u}(x)}$  for all  $x \in X$ , but  $e_x = e_{\mathbf{u}(x)}$  in  $\text{Env}_{\text{GLR}}(R)$ . Thus,  $\text{Env}_{\text{GLR}}(\mathbf{u})$  is the identity map. Similarly,  $\text{Env}_{\text{GLR}}(\mathbf{d})$  and  $\text{Env}_{\text{GLR}}(\text{id}_X)$  are the identity maps, so we have a group isomorphism

$$\text{Env}_{\text{GLR}}(R) \cong \text{Env}_{\text{GLR}}(R').$$

By Lemma 2.14, the right-hand side is isomorphic to  $\text{Env}_{\text{Qnd}}(X/\sim, s^*)$ , so we are done.  $\square$

**Corollary 3.13.** *Let  $\Lambda \subset \mathbb{R}^3$  be an oriented Legendrian link, and let  $L$  denote its underlying smooth link. Then there exists a group isomorphism*

$$\text{Env}_{\text{GLR}}(\mathcal{G}(\Lambda)) \cong \pi_1(\mathbb{R}^3 \setminus L).$$

*Proof.* Taking  $R = \mathcal{G}(\Lambda)$  in Theorem 3.12, we have  $R' \cong (\mathcal{Q}(L), \text{id}_{X_L}, \text{id}_{X_L})$  by Lemma 3.5. By Theorem 3.12, it suffices to show that  $\text{Env}_{\text{Qnd}}(\mathcal{Q}(L))$  is isomorphic to  $\pi_1(\mathbb{R}^3 \setminus L)$ . But Joyce showed this in [20, Section 15] using the Wirtinger presentation of  $\pi_1(\mathbb{R}^3 \setminus L)$ .  $\square$

#### 4. ON MEDIAL GL-RACKS, HOM-SETS, AND TENSOR PRODUCTS

In this section, we define medial GL-racks, propose a medial GL-rack-valued invariant of Legendrian links with questions for future research, and introduce tensor products of GL-racks that make the category of medial GL-rack symmetric monoidal closed. This generalizes a similar result for the category of medial quandles [10, Theorem 12], whose proof we follow closely.

**4.1. Hom-sets of medial GL-racks are also medial GL-racks.** In this subsection, we define medial GL-racks, introduce a medial GL-rack structure on any hom-set from a GL-rack to a medial GL-rack, and propose a medial GL-rack-valued invariant of Legendrian links.

**Definition 4.1.** A rack  $(X, s)$  is called *medial* or *abelian* if, for all  $x, y, z \in X$ , we have

$$s_{s_x(z)} \circ s_y = s_{s_x(y)} \circ s_z.$$

If in addition  $(u, d)$  defines a GL-structure on  $(X, s)$ , then we call  $(X, s, u, d)$  a *medial* or *abelian* GL-rack. (Note that this definition is not synonymous with the condition that  $(X, s)$  is *commutative*, which states that  $s_x(y) = s_y(x)$  for all  $x, y \in X$ .)

Since  $R$ -coloring numbers are not complete invariants of Legendrian links (see [23, Theorem 4.3]), it would be desirable to construct *enhancements* or *refinements* of  $R$ -coloring numbers that encode more information about Legendrian front projections. To this end, we will show that if  $M$  is a medial GL-rack, then  $\text{Hom}_{\text{GLR}}(\mathcal{G}(\Lambda), M)$  can also be endowed with a medial GL-rack structure.

**Lemma 4.2.** *Let  $(X, s)$  be a rack, and let  $(Y, t)$  be a medial rack. Let  $\tilde{H} := \text{Hom}_{\text{Rack}}((X, s), (Y, t))$ , and define  $\tilde{s} : \tilde{H} \rightarrow \text{Sym}(\tilde{H})$  by  $g \mapsto s_g$ , where  $s_g(f) := [x \mapsto (s_{g(x)}^Y \circ f)(x)]$ . Then,  $\tilde{R} := (\tilde{H}, \tilde{s})$  is a medial rack. If in addition  $(Y, t)$  is a quandle, then so is  $\tilde{R}$ .*

*Proof.* In [10, Theorem 3], Crans and Nelson proved that  $\tilde{R}$  is a medial quandle under the additional assumptions that  $(X, s)$  and  $(Y, t)$  are quandles. However, they never used the quandle axiom that  $s_x^X(x) = x$  for all  $x \in X$ . Moreover, their proof that  $\tilde{R}$  satisfies the nonquandle rack axioms and mediality did not use the quandle axiom that  $s_y^Y(y) = y$  for all resp.  $y \in Y$ ; the authors only used this axiom to show that  $\tilde{R}$  also satisfies it.  $\square$

**Theorem 4.3.** *In the setting of Lemma 4.2, suppose in addition that  $R_1 := (X, s^X, u_1, d_1)$  and  $R_2 := (Y, s^Y, u_2, d_2)$  are GL-racks, so that  $R_2$  is medial. Let  $H := \text{Hom}_{\text{GLR}}(R_1, R_2) \subset \tilde{H}$ , let  $s := \tilde{s}|_H$ , and let  $R := (H, s)$ . Define  $u : H \rightarrow H$  by  $f \mapsto u_2 \circ f$ , and define  $d : H \rightarrow H$  by  $f \mapsto d_2 \circ f$ . Then,  $(R, u, d)$  is a medial GL-rack. If  $R_2$  is also a GL-quandle, then so is  $(R, u, d)$ .*

*Proof.* In the notation of Lemma 4.2, we know  $\tilde{R}$  is a medial rack. To show that  $R$  is a medial rack, it will suffice to show that  $R$  is a subrack of  $\tilde{R}$ . To that end, fix  $f, g \in H$ . Then, we have  $s_g(f) \in H$  because

$$\begin{aligned} u_2 \circ s_g(f) &= [x \mapsto (u_2 \circ s_{g(x)}^Y \circ f)(x)] \\ &= [x \mapsto (s_{(u_2 \circ g)(x)}^Y \circ u_2 \circ f)(x)] && \text{by Proposition 2.12} \\ &= [x \mapsto (s_{(g \circ u_1)(x)}^Y \circ f \circ u_1)(x)] && \text{since } f, g \in H \\ &= s_g(f) \circ u_1 \end{aligned}$$

and, similarly,  $\mathbf{d}_2 \circ s_g(f) = s_g(f) \circ \mathbf{d}_1$ . Thus,  $R$  is a medial subrack of  $\tilde{R}$ . In particular, Lemma 4.2 provides the final statement of the theorem. We will now show that  $(\mathbf{u}, \mathbf{d})$  satisfies the three GL-rack axioms. Fix  $f, g \in H$ . Since  $f$  is a GL-rack homomorphism and  $(\mathbf{u}_1, \mathbf{d}_1)$  satisfies GL-rack axiom (L1), we have

$$\begin{aligned} (\mathbf{u}\mathbf{d} \circ s_f)(f) &= [x \mapsto (\mathbf{u}_2 \circ \mathbf{d}_2 \circ s_{f(x)}^Y \circ f)(x)] \\ &= [x \mapsto (\mathbf{u}_2 \circ \mathbf{d}_2 \circ f \circ s_x^X)(x)] \\ &= [x \mapsto (f \circ \mathbf{u}_1 \circ \mathbf{d}_1 \circ s_x^X)(x)] \\ &= [x \mapsto (f \circ \text{id}_X)(x)] = f \end{aligned}$$

and, similarly,  $(\mathbf{d}\mathbf{u} \circ s_f)(f) = f$ . So,  $(\mathbf{u}, \mathbf{d})$  satisfies axiom (L1). Since  $(\mathbf{u}_2, \mathbf{d}_2)$  satisfies axiom (L2), we also have

$$(\mathbf{u} \circ s_g)(f) = [x \mapsto (\mathbf{u}_2 \circ s_{g(x)}^Y \circ f)(x)] = [x \mapsto (s_{g(x)}^Y \circ \mathbf{u}_2 \circ f)(x)] = s_g(\mathbf{u}_2 \circ f) = (s_g \circ \mathbf{u})(f)$$

and, similarly,  $(\mathbf{d} \circ s_g)(f) = (s_g \circ \mathbf{d})(f)$ . So,  $(\mathbf{u}, \mathbf{d})$  satisfies axiom (L2). Finally, since  $(\mathbf{u}_2, \mathbf{d}_2)$  satisfies axiom (L3), we have

$$s_{\mathbf{u}(g)}(f) = [x \mapsto (s_{\mathbf{u}_2(g(x))}^Y \circ f)(x)] = [x \mapsto (s_{g(x)}^Y \circ f)(x)] = s_g(f)$$

and, similarly,  $s_{\mathbf{d}(g)}(f) = s_g(f)$ . Hence,  $(\mathbf{u}, \mathbf{d})$  satisfies axiom (L3), and the proof is complete.  $\square$

Together, Propositions 3.3 and 4.3 imply that, for a fixed medial GL-rack  $M$  and for any oriented Legendrian link  $\Lambda$ , the medial GL-quandle  $\mathcal{H}(\Lambda, M) := \text{Hom}_{\text{GLR}}(\mathcal{G}(\Lambda), M)$  is an invariant of  $\Lambda$ . In light of Lemmas 3.5 and 2.14,  $\mathcal{H}(\Lambda, M)$  generalizes the medial quandle-valued invariant  $\text{Hom}_{\text{Qnd}}(\mathcal{Q}(L), Q)$  of smooth links presented in recent work of Elhamdadi et al. [14, Section 4], where  $Q$  is a medial quandle.

This medial GL-rack-valued invariant of Legendrian links raises intriguing questions for future research. For example, do there exist Legendrian links  $\Lambda_1$  and  $\Lambda_2$  and a medial GL-rack  $M$  such that  $\text{Col}(\Lambda_1, M) = \text{Col}(\Lambda_2, M)$  but  $\mathcal{H}(\Lambda_1, M) \not\cong \mathcal{H}(\Lambda_2, M)$  as GL-racks? In the spirit of Corollary 3.9, do there exist such Legendrian links that also share the same underlying smooth link type, Thurston-Bennequin number, and rotation number? Moreover, Elhamdadi et al. recently introduced proper enhancements of  $\text{Hom}_{\text{Qnd}}(\mathcal{Q}(L), Q)$  by considering  $k$ -algebra homomorphisms between quandle rings and colorings of smooth links by idempotents of quandle rings [14, Theorems 4.2 and 5.1]. Do similar proper enhancements of  $\mathcal{H}(\Lambda, M)$  also exist?

**4.2. Tensor products of GL-racks.** Let  $\text{GLR}^{\text{med}}$  be the full subcategory of GLR whose objects are medial. In this subsection, we introduce tensor products in GLR that induce a symmetric monoidal closed structure on GLR.

We define tensor products of GL-racks in such a way that, by way of Lemma 2.14, they generalize tensor products of quandles as defined by Crans and Nelson in [10, Subsection 8.1]

**Definition 4.4.** If  $R_1 = (X, s^X, \mathbf{u}_1, \mathbf{d}_1)$  and  $R_2 = (Y, s^Y, \mathbf{u}_2, \mathbf{d}_2)$  are GL-racks, then we define their *tensor product*, denoted by  $R_1 \otimes R_2$ , to be the free GL-rack  $\text{FGLR}(X \times Y)$  modulo the following relations for all  $x, x_1, x_2 \in X$  and  $y, y_1, y_2 \in Y$ :

- (1)  $s_{(x, y_2)}(x, y_1) = (x, s_{y_2}^Y(y_1))$ .
- (2)  $s_{(x_2, y)}(x_1, y) = (s_{x_2}^X(x_1), y)$ .
- (3)  $\mathbf{u}(x, y) = (\mathbf{u}_1(x), y) = (x, \mathbf{u}_2(y))$ .
- (4)  $\mathbf{d}(x, y) = (\mathbf{d}_1(x), y) = (x, \mathbf{d}_2(y))$ .



Tensor products of GL-racks may be of particular interest to the development of (co)homology theories for GL-racks and their induced Legendrian link invariants (e.g., [4, 21, 24]). Indeed, the next two results show that tensor products of medial GL-racks satisfy the expected universal property and internal hom-tensor adjunction.

**Theorem 4.5.** *The category  $\text{GLR}^{\text{med}}$  is symmetric monoidal closed with respect to the closed structure  $\text{Hom}_{\text{GLR}^{\text{med}}}(-, -)$  in Proposition 4.3 and the tensor product  $\otimes$  in Definition 4.4.*

*Proof.* The unit object in  $\text{GLR}^{\text{med}}$  is the trivial GL-rack with one element. Using this fact, it is straightforward to verify that  $\text{GLR}^{\text{med}}$  is monoidal and symmetric. On the other hand,  $\text{GLR}^{\text{med}}$  is defined as an equational algebraic category. Thus, the main theorem of [28] states that  $\text{GLR}^{\text{med}}$  is closed if and only if, in the sense of universal algebra,  $\text{GLR}^{\text{med}}$  is commutative as an algebraic theory (see [28, Section 6]). Indeed, for any medial GL-quandle  $(X, s, u)$  and for all elements  $x_{11}, x_{12}, x_{21}, x_{22} \in X$ , we have the following equalities:

$$\begin{cases} (u \circ s_{x_{12}})(x_{11}) = (s_{u(x_{12})} \circ u)(x_{11}) & \text{by Proposition 2.12,} \\ (u^{-1} \circ s_{x_{12}})(x_{11}) = (s_{u^{-1}(x_{12})} \circ u^{-1})(x_{11}) & \text{by Proposition 2.12,} \\ (s_{s_{x_{22}}(x_{21})} \circ s_{x_{12}})(x_{11}) = (s_{s_{x_{22}}(x_{12})} \circ s_{x_{21}})(x_{11}) & \text{since } (X, s) \text{ is medial.} \end{cases}$$

Together with the tautologies  $(u \circ u^{-1})(x_{11}) = (u^{-1} \circ u)(x_{11})$ ,  $u^2(x_{11}) = u^2(x_{11})$ , and  $(u^{-1})^2(x_{11}) = (u^{-1})^2(x_{11})$ , these equalities show that  $\text{GLR}^{\text{med}}$  forms a commutative algebraic theory. This completes the proof.  $\square$

In light of Proposition 4.3, we immediately deduce the following.

**Corollary 4.6.**  *$\text{GLR}^{\text{med}}$  is self-enriched. Explicitly, if  $A$ ,  $B$ , and  $C$  are medial GL-racks, then there exists a natural isomorphism of medial GL-racks*

$$\text{Hom}_{\text{GLR}^{\text{med}}}(A \otimes B, C) \cong \text{Hom}_{\text{GLR}^{\text{med}}}(A, \text{Hom}_{\text{GLR}^{\text{med}}}(B, C)).$$

## 5. A SUFFICIENT CONDITION FOR GL-RACK ISOMORPHISMS

In [22, Section 5], Karmakar et al. constructed a homogeneous representation for any GL-rack  $R$  from the orbits of  $R$  under the action of  $\text{Aut}(R)$ . In this section, we adapt this construction into a category  $\text{GrpTup}$  with an essentially surjective functor  $\mathcal{F} : \text{GrpTup} \rightarrow \text{GLR}$ . This functor induces a group-theoretic sufficient condition for any two GL-racks or quandles to be isomorphic.

**5.1. Construction of  $\text{GrpTup}$ .** In this subsection, we introduce a category  $\text{GrpTup}$  with a functorial relationship to  $\text{GLR}$ .

To define the objects in  $\text{GrpTup}$ , we adapt the construction of [22, Proposition 5.1]. Let the objects in  $\text{GrpTup}$  be all sextuples  $(I, \bigsqcup_{i \in I} G/H_i, Z_I, Q_I, R_I, \tau)$  satisfying the following:

- (1)  $I$  is an indexing set,  $G$  is a group, and  $Z_I = \{z_i^G \mid i \in I\}$ ,  $Q_I = \{q_i^G \mid i \in I\}$ , and  $R_I = \{r_i^G \mid i \in I\}$  are multisets indexed by  $I$  whose elements lie in  $G$ .
- (2)  $\{H_i \mid i \in I\}$  is a family of subgroups of  $G$  such that  $H_i \leq C_G(z_i^G)$  for all  $i \in I$ .
- (3)  $\tau : I \rightarrow I$  is a bijection such that the following hold for all  $i \in I$ :
  - (a)  $q_i^G \in N_G(H_{\tau(i)})$ .
  - (b)  $r_i^G \in N_G(H_{\tau^{-1}(i)})$ .
  - (c)  $z_i^G q_i^G r_{\tau(i)}^G, z_i^G r_i^G q_{\tau^{-1}(i)}^G \in H_i$ .
  - (d)  $z_i^G q_i^G (z_{\tau(i)}^G)^{-1} = q_i^G$ .
  - (e)  $z_i^G r_i^G (z_{\tau^{-1}(i)}^G)^{-1} = r_i^G$ .

For the sake of brevity, we will denote such an object as  $\tilde{G}$  when there is no room for confusion. For an opposing object in  $\mathbf{GrpTup}$ , we will write  $\tilde{K} := (J, \bigsqcup_{j \in J} K/L_j, Z_J, Q_J, R_J, \pi)$ .

Now, we define the morphisms in  $\mathbf{GrpTup}$ . Given any two objects  $\tilde{G}, \tilde{K}$  in  $\mathbf{GrpTup}$ , let  $\text{Hom}_{\mathbf{GrpTup}}(\tilde{G}, \tilde{K})$  be the set of all triples  $\varphi := (\varphi_1, \varphi_2, \varphi_3)$  satisfying the following:

- (1)  $\varphi_1 : G \rightarrow K$  is a group homomorphism.
- (2)  $\varphi_2 : \bigsqcup_{i \in I} G/H_i \rightarrow \bigsqcup_{j \in J} K/L_j$  and  $\varphi_3 : I \rightarrow J$  are morphisms in  $\mathbf{Set}$ .
- (3)  $\pi \circ \varphi_3 = \varphi_3 \circ \tau$ .
- (4) For all  $i \in I$  and  $g \in G$ , we have  $\varphi_1(z_i^G) = z_{\varphi_3(i)}^K$ ,  $\varphi_1(q_i^G) = q_{\varphi_3(i)}^K$ ,  $\varphi_1(r_i^G) = r_{\varphi_3(i)}^K$ , and  $\varphi_2(gH_i) = \varphi_1(g)L_{\varphi_3(i)}$ .

Define the composition of morphisms in  $\mathbf{GrpTup}$  by  $\psi \circ \varphi := (\psi_1 \circ \varphi_1, \psi_2 \circ \varphi_2, \psi_3 \circ \varphi_3)$ . Also, define the identity morphism  $\text{id}_{\tilde{G}} : \tilde{G} \xrightarrow{\sim} \tilde{G}$  by letting  $\text{id}_1^{\tilde{G}}$ ,  $\text{id}_2^{\tilde{G}}$ , and  $\text{id}_3^{\tilde{G}}$  as defined above be identity maps. Associativity and unit laws are immediate. Hence,  $\mathbf{GrpTup}$  is a category.

**5.2. Construction of  $\mathcal{F} : \mathbf{GrpTup} \rightarrow \mathbf{GLR}$ .** In this subsection, we construct an essentially surjective functor  $\mathcal{F} : \mathbf{GrpTup} \rightarrow \mathbf{GLR}$ . Then, we discuss a group-theoretic sufficient condition for two GL-racks to be isomorphic.

By [22, Proposition 5.1], given any object  $\tilde{G}$  in  $\mathbf{GrpTup}$ , the set  $X := \bigsqcup_{i \in I} G/H_i$  admits a GL-rack structure in which  $s^X : X \rightarrow \text{Sym}(X)$  and  $\mathbf{u}_G, \mathbf{d}_G : X \rightarrow X$  are defined by

$$s^X(yH_j) := s_{yH_j}^X := [xH_i \mapsto yz_j^G y^{-1}xH_i], \quad \mathbf{u}_G(xH_i) := xq_i^G H_{\tau(i)}, \quad \text{and} \quad \mathbf{d}_G(xH_i) := xr_i^G H_{\tau^{-1}(i)}.$$

So, we can define a functor  $\mathcal{F} : \mathbf{GrpTup} \rightarrow \mathbf{GLR}$  by sending any object  $\tilde{G}$  in  $\mathbf{GrpTup}$  to the GL-rack  $(X, s^X, \mathbf{u}_G, \mathbf{d}_G)$  and sending any morphism  $\varphi \in \text{Hom}_{\mathbf{GrpTup}}(\tilde{G}, \tilde{K})$  to  $\varphi_2$ .

**Theorem 5.1.**  $\mathcal{F}$  is an essentially surjective functor.

*Proof.* Essential surjectivity is precisely the statement of [22, Theorem 5.2]. Certainly,  $\mathcal{F}$  preserves identity morphisms and composition of morphisms. To complete the proof of functoriality, it remains to show that if  $\varphi \in \text{Hom}_{\mathbf{GrpTup}}(\tilde{G}, \tilde{K})$ , then  $\mathcal{F}\varphi : \mathcal{F}(\tilde{G}) \rightarrow \mathcal{F}(\tilde{K})$  is a GL-rack homomorphism. Write  $\mathcal{F}(\tilde{G}) = (X, s^X, \mathbf{u}_G, \mathbf{d}_G)$  and  $\mathcal{F}(\tilde{K}) = (Y, s^Y, \mathbf{u}_K, \mathbf{d}_K)$ , and fix  $gH_a \in X$ . Since  $\varphi_1$  is a group homomorphism, we have

$$\begin{aligned} \mathcal{F}\varphi \circ s_{gH_a}^X &= [xH_i \mapsto \varphi_2(gz_a^G g^{-1}xH_i)] \\ &= [xH_i \mapsto \varphi_1(gz_a^G g^{-1}x)L_{\varphi_3(i)}] \\ &= [xH_i \mapsto \varphi_1(g)\varphi_1(z_a^G)\varphi_1(g^{-1})\varphi_1(x)L_{\varphi_3(i)}] \\ &= [xH_i \mapsto \varphi_1(g)z_{\varphi_3(a)}^K\varphi_1(g)^{-1}\varphi_2(xH_i)] \\ &= s_{\varphi_1(g)L_{\varphi_3(a)}}^Y \circ \varphi_2 = s_{\varphi_2(gH_a)}^Y \circ \varphi_2 = s_{\mathcal{F}\varphi(gH_a)}^Y \circ \mathcal{F}\varphi, \end{aligned}$$

so  $\mathcal{F}\varphi$  is a rack homomorphism. Moreover, we have

$$\begin{aligned} \mathcal{F}\varphi \circ \mathbf{u}_G &= [xH_i \mapsto \varphi_2(xq_i^G H_{\tau(i)})] \\ &= [xH_i \mapsto \varphi_1(x)\varphi_1(q_i^G)L_{\varphi_3(\tau(i))}] \\ &= [xH_i \mapsto \varphi_1(x)q_{\varphi_3(i)}^K L_{\pi(\varphi_3(i))}] \\ &= [yL_j \mapsto yq_j^K L_{\pi(j)}] \circ [xH_i \mapsto \varphi_1(x)L_{\varphi_3(i)}] \\ &= \mathbf{u}_K \circ [xH_i \mapsto \varphi_2(xH_i)] = \mathbf{u}_K \circ \mathcal{F}\varphi, \end{aligned}$$

as desired. Similarly,  $\mathcal{F}\varphi \circ \mathbf{d}_G = \mathbf{d}_K \circ \mathcal{F}\varphi$ , so  $\mathcal{F}\varphi$  is a GL-rack homomorphism.  $\square$

This result gives us a group-theoretic way to show that two GL-racks  $R_1$  and  $R_2$  are isomorphic. The proof of [22, Theorem 5.2] describes a procedure to construct objects  $\tilde{G}$  and  $\tilde{K}$  in  $\mathbf{GrpTup}$  such that  $R_1 \cong \mathcal{F}(\tilde{G})$  and  $R_2 \cong \mathcal{F}(\tilde{K})$  in GLR. To show that  $R_1 \cong R_2$  in GLR, it suffices to find a morphism  $\varphi \in \mathbf{Hom}_{\mathbf{GrpTup}}(\tilde{G}, \tilde{K})$  such that  $\varphi_2$  is bijective, since then  $\mathcal{F}(\varphi) : \mathcal{F}(\tilde{G}) \xrightarrow{\sim} \mathcal{F}(\tilde{K})$  will also be bijective and, hence, an isomorphism of GL-racks.

Finally, let  $\mathbf{GrpTrip}$  be the full subcategory of  $\mathbf{GrpTup}$  consisting of objects  $\tilde{G}$  for which  $\tau = \text{id}_I$ , and  $Q_I$  and  $R_I$  are multisets only containing copies of  $1_G$ . By Lemma 2.14 and [20, Theorem 7.2],  $\mathcal{F}$  induces an essentially surjective functor  $\mathbf{GrpTrip} \rightarrow \mathbf{Qnd}$ . Hence, our above discussion specializes to a sufficient condition for any two quandles to be isomorphic.

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## APPENDIX A. EXHAUSTIVE SEARCHES WITH GL-RACKS

(Writing in progress...)

A.1. **Enumeration of GL-racks up to isomorphism.**

A.2. **Exhaustive searches for  $R$ -coloring numbers.**