

MATH 350 (Fall 2024): Midterm Review Session with Luc :)

Throughout these problems, let G be a group with identity element 1, let \mathbb{F} be a field, and let \mathbb{F}^\times be the multiplicative group of nonzero elements of \mathbb{F} .

Problem 1. Let $G = \mathbb{Z}/4 \times \mathbb{Z}/4$, and let H be the subgroup generated by $(2, 1)$. That is, H is the $(2, 1)$ -orbit of $(2, 1)$.

- (a) Write down all the cosets of H in G , each with a full list of its elements.
- (b) Argue that G/H is a group. What more familiar group is it isomorphic to?

Problem 2. Let $C \subset \mathbb{R}^3$ be a cube, and let G be the group of *rotational* symmetries of C (with composition as the group action), so that G acts on C by rotation.

- (a) Let F be a face of C . Describe the orbit of F and the stabilizer of F . Use this to compute the order of G .
- (b) Let e be an edge of C , and redo part (a) with e playing the role of F . Use this as an alternate way to compute the order of G .
- (c) Let v be a vertex of C , and redo part (a) with v playing the role of F . Use this as yet another way to compute the order of G . (*Hint: Let v' be the vertex diametrically opposite from v , and consider the axis containing both v and v' ...*)
- (d) *Insightful challenge:* Prove that $G \cong S_4$. (*Hint: Consider the 4 pairs of diametrically opposite vertices of C ...*)

Problem 3. Let G act on a set X . Let K be a group, and let $\psi : K \rightarrow G$ be a group homomorphism.

- (a) Fix $g \in G$. Show that the map $x \mapsto g \cdot x$ defines a bijection $X \xrightarrow{\sim} X$.
- (b) Let H be a subgroup of G , and let Y be any subset of X . Verify that the action of G on X induces an action of H on Y . Deduce that G induces an action of $K/\ker \psi$ on X .
- (c) Prove that the *pullback* $h \cdot x := \psi(h) \cdot x$ defines an action of K on X .
- (d) If ψ is injective, find a necessary and sufficient condition for the pullback action to be faithful.
- (e) If ψ is surjective, find a necessary and sufficient condition for the pullback action to be transitive.

Problem 4. Let V and W be vector spaces over \mathbb{F} , and let \mathbb{F} act on V and W by scalar multiplication.

- (a) Observe that V and W are abelian additive groups. Then, argue that a map $\psi : V \rightarrow W$ is a linear transformation if and only if it is both a morphism of \mathbb{F} -sets and a group homomorphism.
- (b) Find a necessary and sufficient condition for the action of \mathbb{F} on V to be faithful.
- (c) Find a necessary and sufficient condition for the action of \mathbb{F} on V to be transitive. Then, find a necessary and sufficient condition for the action of \mathbb{F} on $V \setminus \{0\}$ to be transitive.

Problem 5.

- (a) Suppose p is a prime number such that $p \mid |G|$, but $p^2 \nmid |G|$. Prove that the number of elements of order p in G is exactly $N_p(p - 1)$. (You probably used this or a similar result on HW7.)
- (b) Let G be a group of order $495 = 3^2 \cdot 5 \cdot 11$. Show that G is not simple.
- (c) Let G be a group of order $132 = 2^2 \cdot 3 \cdot 11$. Show that G is not simple.

Problem 6. Define the *quaternion group* as the subgroup $Q_8 := \{\pm 1, \pm i, \pm j, \pm h\}$ of $\text{GL}_2(\mathbb{C})$, where

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(In physics, these can be obtained from the *Pauli matrices* by multiplying by i .) Note that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

- (a) Show that every subgroup of Q_8 is normal. (*Hint: There's a very quick way to do this.*)
- (b) Deduce the following two sentences: If G is abelian, then all of its subgroups are normal. However, the converse is not necessarily true.
- (c) Give two different proofs that Q_8 is not isomorphic to $D_8 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^{-1} \rangle$.
- (d) Find a group G and a normal subgroup H such that G/H is not isomorphic to any subgroup of G .

Problem 7. True or false? If the statement is true, prove it (or write its name, if it's a named theorem or previous problem like "Sun Ze's theorem" or "HW5 #3(d).") If not, give a counterexample.

- (a) If $|G| = n$, then G is isomorphic to some subgroup of S_n . (*Try proving this one!*)
- (b) If $g \in G$ is the only element of G having order 2, then $g \in Z(G)$.
- (c) Let p be the smallest prime number dividing the order of G . If H is a subgroup of G such that $[G : H] = p$, then H is normal in G .
- (d) If p is a prime number dividing $|G|$, then G contains at least $p - 1$ distinct elements of order p .
- (e) If $|G| < \infty$ and $H \leq G$, then $|H|$ divides $|G|$.
- (f) If $|G| < \infty$ and $g \in G$, then $|g|$ divides $|G|$.
- (g) *Challenge:* If $H \cong K$ are isomorphic normal subgroups of G , then $G/H \cong G/K$.

Problem 8. Let p be a prime number, and fix $n, d \in \mathbb{Z}^+$. Let V be a d -dimensional vector space over \mathbb{Z}/p . (You can take for granted in this problem that \mathbb{Z}/p is a field iff p is prime.) Let G be a subgroup of $\text{GL}_d(\mathbb{Z}/p)$ such that $|G| = p^n$. Prove that there exists a nonzero vector $\mathbf{v} \in V$ such that $M\mathbf{v} = \mathbf{v}$ for all $M \in G$. (*Hint: This problem should remind you of a certain lemma from class.*)

Problem 9. This problem gives us an opportunity to practice using the the result in problem 2(a) in HW6. (This is actually a major result in group theory called the *first isomorphism theorem*.)

- (a) Fix $n \geq 3$, and let $k \in \mathbb{Z}^+$ be a divisor of n . Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$ be the dihedral group of the regular n -gon. If $k \geq 3$, show that $\langle r^k \rangle = \{r^k, r^{2k}, r^{3k}, \dots, r^{n-k}, r^n = 1\}$ is a normal subgroup of D_{2n} , and show that $D_{2n}/\langle r^k \rangle \cong D_{2k}$. What if $k \leq 2$?
- (b) For all $g \in G$, define the *conjugation map* $\varphi_g : G \rightarrow G$ by $x \mapsto gxg^{-1}$, and define the *inner automorphism group* of G as the set

$$\text{Inn}(G) := \{\varphi_g \mid g \in G\}.$$

Argue that $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$, the *automorphism group* of G . Then, find a subgroup $H \leq G$ such that $G/H \cong \text{Inn}(G)$. (*Hint: Hopefully, this problem reminds you of HW2.*)

- (c) Fix $n \in \mathbb{Z}^+$. Show that $\mathrm{SL}_n(\mathbb{F})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{F})$. What group is $\mathrm{GL}_n(\mathbb{F})/\mathrm{SL}_n(\mathbb{F})$ isomorphic to? (*Hint: How is $\mathrm{SL}_n(\mathbb{F})$ defined?*) From this, deduce the following sentence: If M and N are invertible $n \times n$ matrices, then they have the same determinant if and only if there exists a matrix S such that $M = SN$ and $\det(S) = 1$. (In fact, $S = MN^{-1}$.) Also, deduce that if \mathbb{F} is a finite field of order q , then $[\mathrm{GL}_n(\mathbb{F}) : \mathrm{SL}_n(\mathbb{F})] = q - 1$.
- (d) Let V be a finite-dimensional vector space over \mathbb{F} . Recall the following linear algebra construction: If W is a linear subspace of V , then we have a *quotient space* V/W whose dimension is $\dim(V/W) = \dim V - \dim W$. Using this fact, give a basis-free proof of the rank-nullity theorem. That is, let T be a linear transformation from V to some other vector space over \mathbb{F} , and show that $\dim(\mathrm{Im} T) + \dim(\ker T) = \dim V$ without ever writing the word “basis.”

Problem 10. Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$ be the dihedral group of the regular n -gon. For all $k \in \mathbb{Z}^+$, let $\langle r^k \rangle$ be the r^k -orbit of r^k . That is, $\langle r^k \rangle$ is the subgroup $\{r^k, r^{2k}, r^{3k}, \dots, r^{n-k}, r^n = 1\}$.

- (a) Deduce from Problem 9(a) that every subgroup of $\langle r \rangle$ is normal in D_{2n} .
- (b) Let p be an odd prime number such that $p \mid n$. Deduce the following (not necessarily in order):
 (i) D_{2n} has a unique Sylow p -subgroup P , (ii) P is cyclic, and (iii) P is normal. (*Hint: Recall from class that $N_p = 1$ if and only if there exists a normal Sylow p -subgroup.*)
- (c) Write an explicit list of all Sylow p -subgroups of D_{12} . Justify that your list is complete.

Problem 11. Let P be a Sylow p -subgroup of G , and let H be a subgroup of G that contains P . For all $g \in G$, show that $K := gPg^{-1}$ is a Sylow p -subgroup of gHg^{-1} . (*Hint: If you're stuck, then one of the results in Problem 9(b) might help.*)

Problem 12. Let S^1 denote the *unit circle*, considered as a closed curve in the complex plane:

$$S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\} = \{e^{i\theta} : \theta \in [0, 2\pi)\} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}^\times.$$

(You may have seen in other courses that for all $\theta \in \mathbb{R}$, the complex number $e^{i\theta} = \cos \theta + i \sin \theta$ is the point on S^1 obtained by starting at $1 = e^{i0}$ and rotating by θ radians about the origin. If you like, you can play around with the slider at [this link](#) to get a feel for how this works!)

- (a) Using the fact that adding or subtracting 2π radians to an angle doesn't alter the angle, argue that S^1 is a subgroup of \mathbb{C}^\times . Then, find a normal subgroup H of \mathbb{R} such that $\mathbb{R}/H \cong S^1$.
- (b) Define a nontrivial action of \mathbb{R} on S^1 . Is this action transitive? Is it faithful? If not, what is its kernel?
- (c) On an unrelated note, let G_1 and G_2 be groups with normal subgroups N_1 and N_2 , respectively. Consider the group homomorphisms $\phi : G_1 \times G_2 \rightarrow G_1/N_1$ defined by $(g_1, g_2) \mapsto g_1N_1$ and $\psi : G_1 \times G_2 \rightarrow G_2/N_2$ defined by $(g_1, g_2) \mapsto g_2N_2$. Use HW2 #4(a) and the first isomorphism theorem to show that $(G_1 \times G_2)/(N_1 \times N_2) \cong G_1/N_1 \times G_2/N_2$.
- (d) Deduce that the quotient group $\mathbb{R}^2/\mathbb{Z}^2$ is isomorphic to the *torus* $S^1 \times S^1$. (There's a nice geometric interpretation of these isomorphisms—if you're curious, ask me about it at office hours, and I'll draw it!)

You're doing great! Good luck on the midterm—I believe in you! :)