MATH 350 (Fall 2024): Midterm Review Session with Luc :)

Throughout these problems, let G be a group with identity element 1, let $\mathbb{F}$ be a field, and let $\mathbb{F}^\times$ be the multiplicative group of nonzero elements of $\mathbb{F}$.

Problem 1. Let $G = \mathbb{Z}/4 \times \mathbb{Z}/4$, and let H be the subgroup generated by $(2,1)$. That is, H is the $(2,1)$-orbit of $(2,1)$.

(a) Write down all the cosets of H in G, each with a full list of its elements.

(b) Is G/H a group? If so, what more familiar group is it isomorphic to?

Problem 2. Let $C \subset \mathbb{R}^3$ be a cube, and let G be the group of *rotational* symmetries of C (with composition as the group action), so that G acts on C by rotation.

(a) Let F be a face of C. Describe the orbit of F and the stabilizer of F. Use this to compute the order of G.

(b) Let e be an edge of C, and redo part (a) with e playing the role of F. Use this as an alternate way to compute the order of G.

(c) Let v be a vertex of C, and redo part (a) with v playing the role of F. Use this as yet another way to compute the order of G. *(Hint: Let v' be the vertex diametrically opposite from v, and consider the axis containing both v and v'...)*

(d) *Insightful challenge:* Prove that $G \cong S_4$. *(Hint: Consider the 4 pairs of diametrically opposite vertices of C...)*

Problem 3. Let G act on a set X. Let K be a group, and let $\psi : K \to G$ be a group homomorphism.

(a) Fix $g \in G$. Show that the map $x \mapsto g \cdot x$ defines a bijection $X \xrightarrow{\sim} X$.

(b) Let H be a subgroup of G, and let Y be any subset of X. Verify that the action of G on X induces an action of H on Y. Deduce that G induces an action of $K/\ker\psi$ on X.

(c) Prove that the *pullback* $h \cdot x := \psi(h) \cdot x$ defines an action of K on X.

(d) If ψ is injective, find a necessary and sufficient condition for the pullback action to be faithful.

(e) If ψ is surjective, find a necessary and sufficient condition for the pullback action to be transitive.

Problem 4. Let V and W be vector spaces over $\mathbb{F}$, and let $\mathbb{F}^\times$ act on V and W by scalar multiplication.

(a) Observe that V and W are abelian additive groups. Then, argue that a map $\psi : V \to W$ is a linear transformation if and only if it is both a morphism of $\mathbb{F}^\times$-sets and a group homomorphism.

(b) Find a necessary and sufficient condition for the action of $\mathbb{F}^\times$ on V to be faithful.

(c) Find a necessary and sufficient condition for the action of $\mathbb{F}^\times$ on V to be transitive. Then, find a necessary and sufficient condition for the action of $\mathbb{F}^\times$ on $V \setminus \{\mathbf{0}\}$ to be transitive.

Problem 5.

(a) Suppose p is a prime number such that $p \mid |G|$, but $p^2 \nmid |G|$. Prove that the number of elements of order p in G is exactly $N_p(p-1)$. (You probably used this or a similar result on HW7.)

(b) Let G be a group of order $495 = 3^2 \cdot 5 \cdot 11$. Show that G is not simple.

(c) Let G be a group of order $132 = 2^2 \cdot 3 \cdot 11$. Show that G is not simple.

Problem 6. Define the *quaternion group* as the subgroup $Q_8 := \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{h}\}$ of $\mathrm{GL}_2(\mathbb{C})$, where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(In physics, these can be obtained from the *Pauli matrices* by multiplying by i.) Note that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

(a) Show that every subgroup of Q_8 is normal. *(Hint: There's a very quick way to do this.)*

(b) Deduce the following two sentences: If G is abelian, then all of its subgroups are normal. However, the converse is not necessarily true.

(c) Give two different short proofs that $D_8 = \langle r, s \mid r^4 = s^2 = 1,\ rs = sr^{-1}\rangle$, the dihedral group of the square, is not isomorphic to Q_8.

(d) Find a group G and a normal subgroup H where G/H isn't isomorphic to any subgroup of G.

(e) Prove that Q_8 cannot be written as a nontrivial semidirect product.

Problem 7. True or false? If the statement is true, prove it (or write its name, if it's a named theorem or previous problem like "Sun Ze's theorem" or "HW5 #3(d)"). If not, give a counterexample.

(a) If $|G| = n$, then G is isomorphic to some subgroup of S_n. *(Try proving this one!)*

(b) If $g \in G$ is the only element of G having order 2, then $g \in Z(G)$.

(c) Let p be the smallest prime number dividing the order of G. If H is a subgroup of G such that $[G : H] = p$, then H is normal in G.

(d) If p is a prime number dividing $|G|$, then G contains at least $p - 1$ distinct elements of order p.

(e) If $|G| < \infty$ and $H \leq G$, then $|H|$ divides $|G|$.

(f) If $|G| < \infty$ and $g \in G$, then $|g|$ divides $|G|$.

(g) *Challenge:* If $H \cong K$ are isomorphic normal subgroups of G, then $G/H \cong G/K$.

Problem 8. Let p be a prime number, and fix $n, d \in \mathbb{Z}^+$. Let V be a d-dimensional vector space over $\mathbb{Z}/p$. (You can take for granted in this problem that $\mathbb{Z}/p$ is a field iff p is prime.) Let G be a subgroup of $\mathrm{GL}_d(\mathbb{Z}/p)$ such that $|G| = p^n$. Prove that there exists a nonzero vector $\mathbf{v} \in V$ such that $M\mathbf{v} = \mathbf{v}$ for all $M \in G$. *(Hint: This problem should remind you of a certain lemma from class.)*

Problem 9. This problem (along with Problems 3(b) and 12(a,c)) gives us an opportunity to practice using the the result in problem 2(a) in HW6. (This is actually a major result in group theory called the *first isomorphism theorem*.)

(a) Fix $n \geq 3$, and let $k \in \mathbb{Z}^+$ be a divisor of n. Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1}\rangle$ be the dihedral group of the regular n-gon. If $k \geq 3$, show that $\langle r^k \rangle = \{r^k, r^{2k}, r^{3k}, \ldots, r^{n-k}, r^n = 1\}$ is a normal subgroup of D_{2n}, and show that $D_{2n}/\langle r^k\rangle \cong D_{2k}$. What if $k \leq 2$?

(b) For all $g \in G$, define the *conjugation* map $\varphi_g : G \to G$ by $x \mapsto gxg^{-1}$, and define the *inner automorphism group* of G as the set

$$\mathrm{Inn}(G) := \{\varphi_g \mid g \in G\}.$$

Argue that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$, the *automorphism group* of G. Then, find a subgroup $H \leq G$ such that $G/H \cong \mathrm{Inn}(G)$. *(Hint: Hopefully, this problem reminds you of HW2.)*

(c) Fix $n \in \mathbb{Z}^+$. Show that $\mathrm{SL}_n(\mathbb{F})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{F})$. What group is $\mathrm{GL}_n(\mathbb{F})/\mathrm{SL}_n(\mathbb{F})$ isomorphic to? *(Hint: How is $\mathrm{SL}_n(\mathbb{F})$ defined?)* From this, deduce the following sentence: If M and N are invertible $n \times n$ matrices, then they have the same determinant if and only if there exists a matrix S such that $M = SN$ and $\det(S) = 1$. (In fact, $S = MN^{-1}$.) Also, deduce that if $\mathbb{F}$ is a finite field of order q, then $[\mathrm{GL}_n(\mathbb{F}) : \mathrm{SL}_n(\mathbb{F})] = q - 1$.

(d) Let V be a finite-dimensional vector space over $\mathbb{F}$. Recall the following linear algebra construction: If W is a linear subspace of V, then we have a *quotient space* V/W whose dimension is $\dim(V/W) = \dim V - \dim W$. Using this fact, give a basis-free proof of the rank-nullity theorem. That is, let T be a linear transformation from V to some other vector space over $\mathbb{F}$, and show that $\dim(\operatorname{Im} T) + \dim(\ker T) = \dim V$ without ever writing the word "basis."

(e) Write $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$. What is $\mathbb{C}/\mathbb{R}$ isomorphic to? (There's a nice geometric way of understanding this isomorphism—if you're curious, ask me at office hours, and I'll draw it!)

Problem 10. Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle$ be the dihedral group of the regular n-gon.

(a) Deduce from Problem 9(a) that every subgroup of $\langle r \rangle = \{r, r^2, r^3, \ldots, r^{n-1}, r^n = 1\}$ is normal in D_{2n}.

(b) Let p be an odd prime number such that $p \mid n$. Deduce the following (not necessarily in order): (i) D_{2n} has a unique Sylow p-subgroup P, (ii), P is cyclic, and (iii) P is normal. *(Hint: Recall from class that $N_p = 1$ if and only if there exists a normal Sylow p-subgroup.)*

(c) Write an explicit list of all Sylow p-subgroups of D_{12}. Justify that your list is complete.

Problem 11. Let P be a Sylow p-subgroup of G, and let H be a subgroup of G that contains P. For all $g \in G$, show that gPg^{-1} is a Sylow p-subgroup of gHg^{-1}. *(Hint: If you're stuck, then one of the results in Problem 9(b) might help.)*

Problem 12. Let S^1 denote the *unit circle*, considered as a closed curve in the complex plane:

$$S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\} = \{e^{i\theta} : \theta \in [0, 2\pi)\} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}^\times.$$

(You may have seen in other courses that for all $\theta \in \mathbb{R}$, the complex number $e^{i\theta} = \cos\theta + i\sin\theta$ is the point on S^1 obtained by starting at $1 = e^{i0}$ and rotating by θ radians about the origin. If you like, you can play around with the slider at this link to get a feel for how this works! Note that adding or subtracting 2π radians to an angle doesn't alter the angle, hence the second equality.)

(a) Argue that S^1 is a subgroup of $\mathbb{C}^\times$. Then, find a normal subgroup H of $\mathbb{R}$ such that $\mathbb{R}/H \cong S^1$.

(b) Define a nontrivial action of $\mathbb{R}$ on S^1. Is this action transitive? Is it faithful? If not, what is its kernel?

(c) On an unrelated note, let G_1 and G_2 be groups with normal subgroups N_1 and N_2, respectively. Consider the group homomorphisms $\phi : G_1 \times G_2 \to G_1/N_1$ defined by $(g_1, g_2) \mapsto g_1 N_1$ and $\psi : G_1 \times G_2 \to G_2/N_2$ defined by $(g_1, g_2) \mapsto g_2 N_2$. Use HW2 #4(a) and the first isomorphism theorem to show that $(G_1 \times G_2)/(N_1 \times N_2) \cong G_1/N_1 \times G_2/N_2$.

(d) Deduce that the quotient group $\mathbb{R}^2/\mathbb{Z}^2$ is isomorphic to the *torus* $S^1 \times S^1$. (There's a nice geometric interpretation of these isomorphisms—if you're curious, ask me about it at office hours, and I'll draw it!)

You're doing great! Good luck on the midterm—I believe in you! :)