Generalized Legendrian racks: Knot coloring invariants and algebraic classification

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Session: Abstract Algebra 1A

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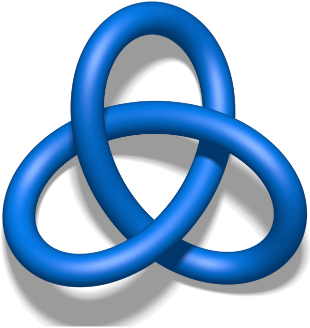
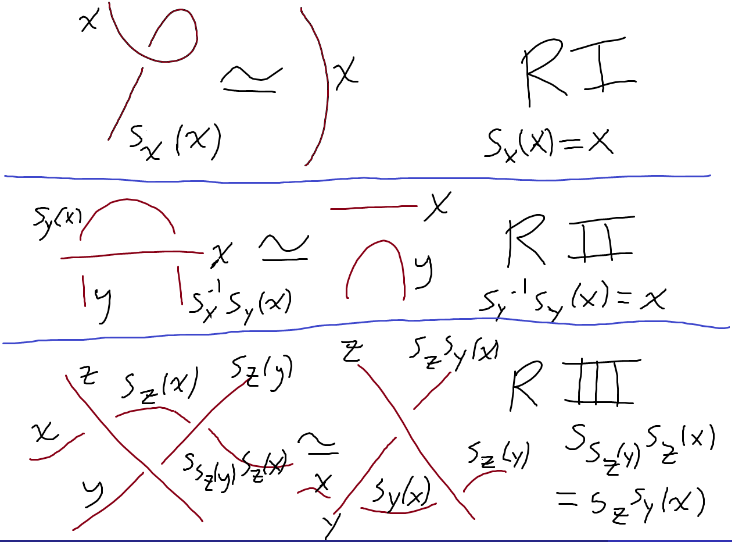
Outline

Motivation: Distinguishing knots



Reidemeister moves

Two knots are equivalent up to ambient isotopy if and only if they're related by a finite sequence of *Reidemeister moves*.



Racks and quandles

Definition

Let X be a set, let $s : X \to S_X$ be a map, and write $s_x := s(x)$.
We call the pair (X, s) a **rack** if

$$s_x s_y = s_{s_x(y)} s_x$$

for all $x, y \in X$. If in addition $s_x(x) = x$ for all $x \in X$, then we say that (X, s) is a **quandle**.

Example (Permutation racks)

Fix a permutation $\sigma \in S_X$, and define $s_x := \sigma$ for all $x \in X$. Then $(X, \sigma)_{\mathrm{perm}} := (X, s)$ is a rack, and it's a quandle if and only if $\sigma = \mathrm{id}_X$.

Example (Conjugation quandles)

Let X be a union of conjugacy classes in a group G, and let s_x be the conjugation map $y \mapsto xyx^{-1}$. Then $\operatorname{Conj} X := (X, s)$ is a quandle.

Rack homomorphisms

Loosely speaking, we can study knot colorings as maps between racks.

Definition

A map $\varphi : X \to Y$ is a **homomorphism** between racks (X, s) and (Y, t) if $\varphi s_x = t_{\varphi(x)}\varphi$ for all $x \in X$.

Example

Every group homomorphism $\varphi : G \to H$ is also a quandle homomorphism $\varphi : \operatorname{Conj} G \to \operatorname{Conj} H$ because

$$\varphi s_x(y) = \varphi(xyx^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1} = s_{\varphi(x)}\varphi(y)$$

for all $x, y \in G$.

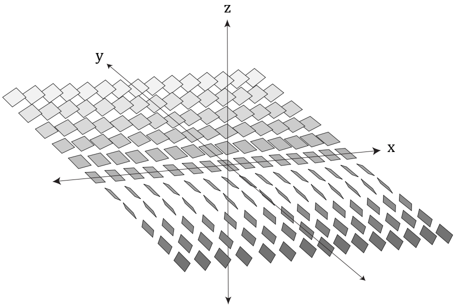
Example

Every rack $R = (X, s)$ has a canonical automorphism θ_R defined by $x \mapsto s_x(x)$. Also, R is a quandle if and only if $\theta_R = \mathrm{id}_X$.

The standard contact structure

Definition

The **standard contact structure** on $\mathbb{R}^3$, denoted by ξ_{std}, is an assignment of a plane to each point (x, y, z) defined by $dz - y\,dx = 0$.

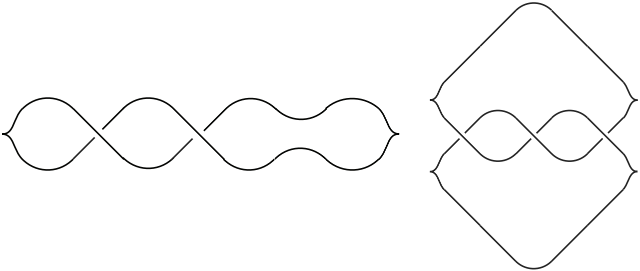


When $y = 0$, the planes are flat. When moving in the positive y-direction, the slopes grow more negative; when moving in the negative y-direction, the slopes grow more positive.

Legendrian knots

Definition

A smooth knot is called **Legendrian** if it lies everywhere tangent to ξ_{std}.

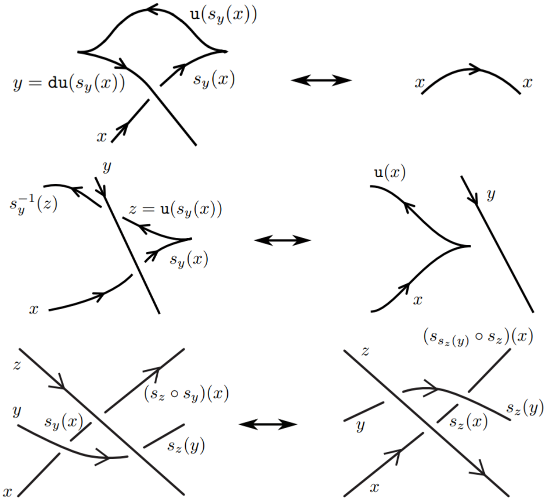


We often study Legendrian knots via their *front projections* onto the *xz*-plane, viewed from the negative *y*-axis.

- Cusps instead of vertical tangencies
- Only one type of crossing

Distinguishing between Legendrian knots

Two Legendrian knots are equivalent up to Legendrian isotopy if and only if they're related by the *Legendrian Reidemeister moves*.



Outline

GL-racks and their homomorphisms

Definition

A **GL-structure** on a rack $R = (X, s)$ is a rack automorphism $\mathrm{u} \in \operatorname{Aut} R$ such that $\mathrm{u}s_x = s_x\mathrm{u}$ for all $x \in X$. We call (R, u) a **GL-rack**.

Example (Permutation GL-racks)

GL-structures on permutation racks $(X, \sigma)_{\mathrm{perm}}$ are permutations $\mathrm{u} \in S_X$ such that $\mathrm{u}\sigma = \sigma\mathrm{u}$.

Example (Conjugation GL-quandles)

Given a group G and a central element $z \in Z(G)$, multiplication by z is a GL-structure on $\operatorname{Conj} G$.

Definition

A **GL-rack homomorphism** is a rack homomorphism that commutes with/intertwines GL-structures.

Coloring Legendrian knots (1/2)

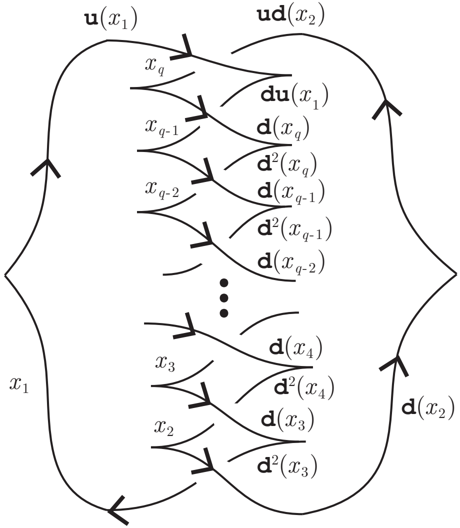


Figure: The Legendrian $(2, -q)$-torus knot Λ with maximal classical invariants.

Coloring Legendrian knots (2/2)

For all GL-racks $(R, \mathtt{u})$, define $\mathtt{d} \in \operatorname{Aut} R$ to be $\theta_R^{-1}\mathtt{u}^{-1}$.

Example

In the previous figure, $\mathcal{G}(\Lambda)$ is the free GL-rack on the set $\{x_1, \dots, x_q\}$ modulo the congruence generated by the relations

$$s_{\mathtt{u}(x_1)}(x_q) = \mathtt{ud}(x_2), s_{\mathtt{d}(x_q)}(x_{q-1}) = \mathtt{du}(x_1), s_{\mathtt{d}(x_{i-1})}(x_{i-2}) = \mathtt{d}^2(x_i) \forall 3 \leq i \leq q.$$

To distinguish Λ from some other Legendrian knot Λ', it suffices to show that $\mathcal{G}(\Lambda)$ and $\mathcal{G}(\Lambda')$ are nonisomorphic.

To do *that*, it suffices to find a GL-rack L such that

$$|\operatorname{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), L)| \neq |\operatorname{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda'), L)|,$$

an inequality of **coloring invariants**.

Outline

Legendrian 6_2 knots (1/3)

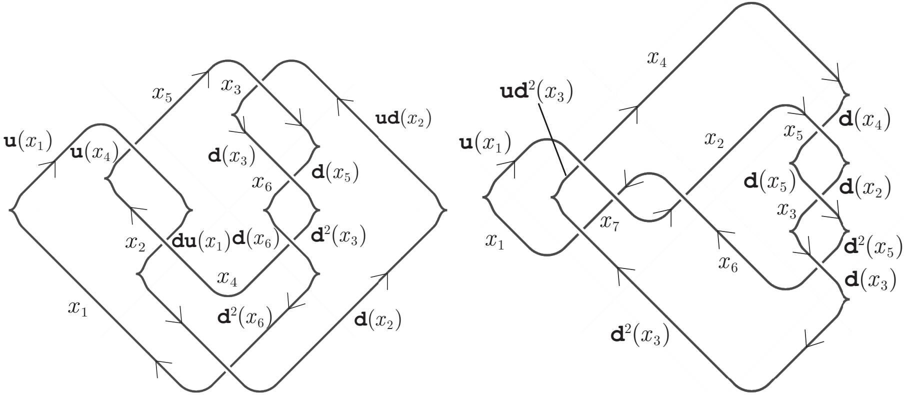


Figure: Legendrian 6_2 knots with classical invariants $(\mathrm{tb}, \mathrm{rot}) = (-7, 2)$.

Legendrian 6_2 knots (2/3)

Let Λ_1 and Λ_2 be the knots on the left and right of the previous figure, respectively. Then $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$ are free GL-racks modulo congruence generated by the relations

$$\mathcal{G}(\Lambda_1)\begin{cases} s_{x_1}\mathtt{u}(x_4) = x_5, & s_{x_4}\mathtt{du}(x_1) = x_2, \\ s_{x_2}(x_1) = \mathtt{d}^2(x_6), & s_{x_5}(x_3) = \mathtt{ud}(x_2), \\ s_{x_3}(x_6) = \mathtt{d}(x_5), & s_{x_6}(x_4) = \mathtt{d}^2(x_3). \end{cases}$$

$$\mathcal{G}(\Lambda_2)\begin{cases} s_{x_1}\mathtt{ud}^2(x_3) = x_4, & s_{x_5}(x_3) = \mathtt{d}(x_2), \\ s_{x_1}(x_6) = x_7, & s_{x_3}(x_6) = \mathtt{d}^2(x_5), \\ s_{x_6}(x_2) = \mathtt{u}(x_1), & s_{x_3}(x_7) = x_1. \\ s_{x_2}(x_5) = \mathtt{d}(x_4), & \end{cases}$$

Legendrian 6_2 knots (3/3)

Theorem

Λ_1 and Λ_2 are nonequivalent.

Proof.

In cycle notation, let $\sigma \in S_3$ be the permutation (123). Consider the permutation GL-rack

$$L := ((\{1,2,3\}, \sigma)_{\mathrm{perm}}, \sigma^{-1}).$$

Then we have a GL-rack homomorphism $\psi : \mathcal{G}(\Lambda_1) \to L$ defined by

$$\psi(x_i) := \begin{cases} 1 & \text{if } i \in \{1,3,4\}, \\ 2 & \text{if } i \in \{2,6\}, \\ 3 & \text{if } i \in \{5,7\}, \end{cases}$$

but a straightforward computation yields $|\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda_2), L)| = 0$. $\square$

Similarly, one can show that these Legendrian knots are nonequivalent...

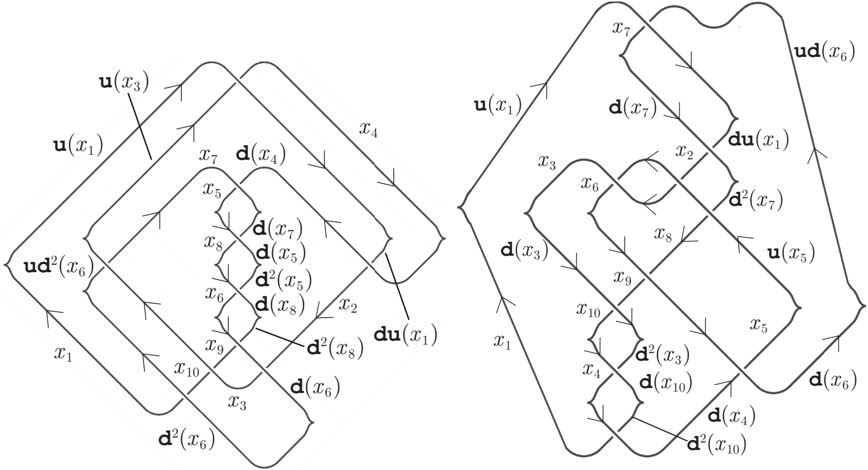


Figure: Legendrian 8_{10} knots with classical invariants $(\mathrm{tb}, \mathrm{rot}) = (-8, 3)$.

. . . and that these Legendrian knots are also nonequivalent.

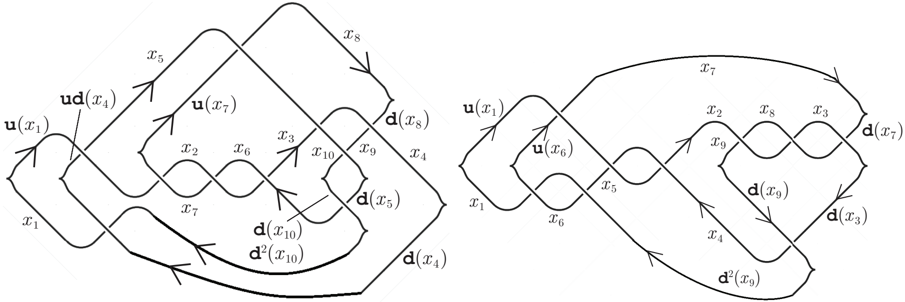


Figure: Legendrian 8_{13} knots with classical invariants $(\mathrm{tb}, \mathrm{rot}) = (-6, 1)$.

Settling conjectures

This completes the classification of Legendrian 8_{13} knots; the two on the previous slide were the only ones not distinguished yet.

Along the way, we answered an open question of Kimura (2023):

Corollary

GL-rack coloring invariants can distinguish Legendrian knots not distingushable by classical (or even homological) invariants.

Outline

Classifying small GL-racks

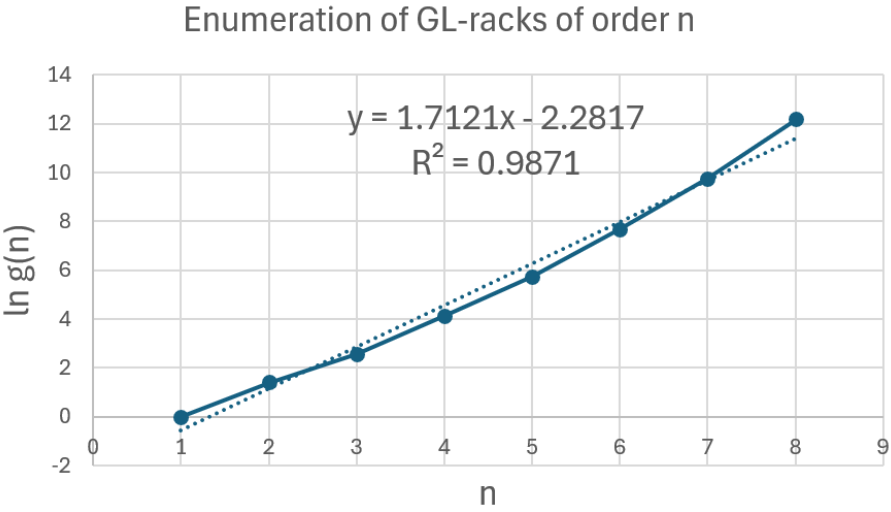
Using the computer algebra system `GAP`, Vojtěchovský and Yang (2019) classified racks up to order 11.

We can use this data to classify GL-racks of any order $n \leq 11$.

1. For all racks R with underlying set $X := \{1, \ldots, n\}$, exhaustively search for GL-structures $\mathtt{u} \in S_n$ on R.
2. Search for GL-rack isomorphisms between the GL-racks found in the previous step.
3. Throw out isomorphic copies.

Similarly, we can exhaustively search for Legendrian knot colorings.

Results (1/2)


Enumeration of GL-racks of order n
y = 1.7121x - 2.2817
R² = 0.9871
ln g(n)
n
14
12
10
8
6
4
2
0
-2
0
1
2
3
4
5
6
7
8
9

Results (2/2)

n	0	1	2	3	4	5	6	7	8
$g(n)$	1	1	4	13	62	308	2132	17268	189373
$g_q(n)$	1	1	2	6	19	74	353	2080	16023
$r(n)$	1	1	2	6	19	74	353	2080	16023
$r_q(n)$	1	1	1	3	7	22	73	298	1581

Table: The numbers of GL-racks $g(n)$ and GL-quandles $g^m(n)$ versus racks $r(n)$ and quandles $r_q(n)$ of order n.

There appears to be a one-to-one correspondence between racks and GL-quandles. As it turns out, this correspondence generalizes in a natural way. . .

Outline

Categorical background

Recall:

- A **category** is a collection of *objects* and *morphisms* that satisfy identity, composition, and associativity laws.
- A **functor** is a structure-preserving map between two categories.
- A functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if there's another functor $G : \mathcal{D} \to \mathcal{C}$ such that FG and GF are *naturally isomorphic* to the identity functors $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$, respectively.

Example (Not important to know for our purposes)

The category of representations of a group G on a field $\mathbb{F}$ is equivalent (actually isomorphic) to the category of left $\mathbb{F}[G]$-modules.

Construction of functors

Let Rack be the category of racks, and let GLQ be the category of GL-quandles. Let's construct functors $F : \mathrm{Rack} \to \mathrm{GLQ}$ and $G : \mathrm{GLQ} \to \mathrm{Rack}$. They'll act trivially on (homo)morphisms.

Proposition

The following defines a functor $F : \mathrm{Rack} \to \mathrm{GLQ}$. Given a rack $R = (X, s)$, define $\widetilde{s} : X \to S_X$ by $x \mapsto \widetilde{s}_x$, where $\widetilde{s}_x := \theta_R^{-1} s_x$. Then $F(R) := (X, \widetilde{s}, \theta_R)$ is a GL-quandle.

Proposition

The following defines a functor $G : \mathrm{GLQ} \to \mathrm{Rack}$. Given a GL-rack $L = (X, s, \mathrm{u})$, define $\hat{s} : X \to S_X$ by $x \mapsto \hat{s}_x$, where $\hat{s}_x := \mathrm{u} s_x$. Then $G(L) := (X, \hat{s})$ is a rack.

Result

Theorem

The categories Rack *and* GLQ *are equivalent—in fact, isomorphic.*

Proof.

For all GL-quandles $L = (X, s, \mathrm{u})$, $FG(L) = (X, \widetilde{\hat{s}}, \theta_{G(L)}) = L$ because

$$\widetilde{\hat{s}}_y(x) = \theta_{G(L)}^{-1}\hat{s}_y(x) = \hat{s}_y\theta_{G(L)}^{-1}(x) = \mathrm{u}s_y\hat{s}_x^{-1}(x) = s_y\mathrm{u}\mathrm{u}^{-1}s_x^{-1}(x) = s_y(x)$$

$$\text{and } \theta_{G(L)}(x) = \hat{s}_x(x) = \mathrm{u}s_x(x) = \mathrm{u}(x).$$

For all racks $R = (X, s)$, $GF(R) = (X, \hat{\widetilde{s}}) = R$ because

$$\hat{\widetilde{s}}_x = \theta_R\widetilde{s}_x = \theta_R\theta_R^{-1}s_x = s_x.$$

Hence, $FG = \mathbf{1}_{\mathrm{GLQ}}$ and $GF = \mathbf{1}_{\mathrm{Rack}}$. □

So! For all $n \in \mathbb{N}$, $r(n) = g_q(n)$, generalizing our computational results.

Outline

The group of GL-structures

In 2023, Karmakar et al. posed the following question: given a rack R, what is the set U_R of all possible GL-structures on R?

To answer this question, we restate our (simplified but equivalent) definition of a GL-rack in group-theoretic language:

Definition

The **inner automorphism group** of a rack $R = (X, s)$, denoted by $\operatorname{Inn} R$, is the (normal) subgroup of $\operatorname{Aut} R$ generated by the s_x's.

Theorem

For all racks R, U_R is the centralizer

$$U_R = C_{\operatorname{Aut} R}(\operatorname{Inn} R) \trianglelefteq \operatorname{Aut} R.$$

Moreover, $(R, \mathrm{u}_1) \cong (R, \mathrm{u}_2)$ if and only if u_1 and u_2 are conjugate in $\operatorname{Aut} R$.

Classifying results

Let's use this characterization to classify GL-structures on some infinite families of racks.

Corollary (Permutation GL-racks)

Let $P = (X, \sigma)_{\mathrm{perm}}$ be a permutation rack. Then

$$U_P = C_{S_X}(\sigma) = \operatorname{Aut} P,$$

and $U_P/\sim$ is the set of conjugacy classes of $C_{S_X}(\sigma)$.

Corollary (Conjugation GL-quandles)

Let G be a group, and let $Q := \operatorname{Conj} G$. If G is abelian, then $U_Q = S_G$, and $U_Q/\sim$ is the set of conjugacy classes of S_G. On the other hand, if G is centerless, then $U_Q = \{\mathrm{id}_G\}$.

Takasaki GL-kei (1/2)

Definition

Let A be an abelian additive group. Define $s_b(a) := 2b - a$ for all $a, b \in A$. Then $T(A) := (A, s)$ is a quandle called a **Takasaki kei**.

Definition

For all $n \in \mathbb{N}$, the Takasaki kei $R_n := T(\mathbb{Z}/n\mathbb{Z})$ is a **dihedral quandle**.

Takasaki GL-kei (2/2)

Corollary

If A is an abelian additive group without 2-torsion, then the only GL-structure on the Takasaki kei $T(A)$ is id_A. In particular, for all odd n, we have $U_{R_n} = \{\mathrm{id}_{\mathbb{Z}/n\mathbb{Z}}\}$.

Proof sketch.

For all such Takasaki kei, the automorphism group G and inner automorphism group H are known to be certain semidirect products. Use this classification to show that

$$U_{T(A)} = C_G(H) \subseteq \{(0, \psi) \in G : \psi|_{2A} = \mathrm{id}_{2A}\} = \{\mathrm{id}_A\}.$$

$\square$

Dihedral GL-quandles

The previous result has infinitely many counterexamples when A has 2-torsion!

Theorem

For all even $n \geq 2$, we have

$$U_{R_n} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } 4 \nmid n, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } 4 \mid n. \end{cases}$$

If $4 \nmid n$, then the two GL-structures in U_{R_n} yield nonisomorphic GL-quandles. If $4 \mid n$, there's exactly one isomorphic pair.

Proof sketch.

Once again, $\operatorname{Aut} R_n$ and $\operatorname{Inn} R_n$ are certain semidirect products. Compute how conjugation works in these semidirect products to compute the centralizer of $\operatorname{Inn} R_n$ in $\operatorname{Aut} R_n$. □

Automorphism groups

Proposition

For all GL-racks $(R, \mathtt{u})$*,*

$$\mathrm{Aut}_{\mathrm{GLR}}(R, \mathtt{u}) = C_{\mathrm{Aut}\,R}(\mathtt{u}).$$

Corollary

For all dihedral GL-quandles $(R_n, \mathtt{u})$*,*

$$\mathrm{Aut}_{\mathrm{GLR}}(R_n, \mathtt{u}) \cong \mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^{\times}$$

except for a certain GL-structure $\mathtt{u}'$ *on* R_n *when* $4 \mid n$*, in which case*

$$\mathrm{Aut}_{\mathrm{GLR}}(R_n, \mathtt{u}') \cong 2\mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^{\times}.$$

Proposition

The automorphism group of the free GL-rack on one element is $\mathbb{Z}^2$*.*

Categorical background

Recall: The **center** of a category $\mathcal{C}$ is the commutative monoid $Z(\mathcal{C})$ of natural endomorphisms of the identity functor $\mathbf{1}_{\mathcal{C}}$.

$$\begin{tikzcd} X \arrow[r, "\eta_{R_1}"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "\eta_{R_2}"'] & Y \end{tikzcd}$$

Essentially, elements of $Z(\mathcal{C})$ are collections η of morphisms in $\mathcal{C}$ that commute with all other morphisms φ in $\mathcal{C}$.

Example (Not important to know for our purposes)

Let A-mod be the category of modules over a ring A. Then the categorical center $Z(A\text{-mod})$ is isomorphic to the ring-theoretic center $Z(A)$ of A.

Results

We can use the previous Proposition to compute the centers of the category GLR of GL-racks and various full subcategories.

Theorem

Let θ be the collection of θ_R's for all racks R, and let u be the collection of all GL-structures on racks. We have the following:

1. *The center $Z(\mathrm{GLR})$ is the free abelian group $\langle \theta, \mathrm{u} \rangle \cong \mathbb{Z}^2$.*
2. *The centers of GLQ and the category of Legendrian racks are each the free group $\langle \mathrm{u} \rangle \cong \mathbb{Z}$.*
3. *The center of the category of Legendrian quandles is the group $\langle \mathrm{u} \mid \mathrm{u}^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof sketch.

Each of these categories is *strongly generated* by the free object F on one element. Show that the center is determined by F. Then, relate each group in the claim to the automorphism group of F. $\square$

Outline

Thank you!

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(Preprint expected within the next month on arXiv)