APPLICATIONS AND GENERALIZATIONS OF QUANTUM KNOT INVARIANTS TO LEGENDRIAN KNOTS

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ABSTRACT. We give a brief survey of applications and generalizations of skein-theoretic polynomial knot invariants to Legendrian knots.

1. INTRODUCTION

Motivated by quantum wavefronts in optics, the question of distinguishing between Legendrian knots up to Legendrian isotopy holds an important place in contact geometry. Since every smooth knot has infinitely many distinct Legendrian representatives, distinguishing between Legendrian knots is strictly harder than distinguishing between topological knots. This motivates the generalization and application of quantum invariants of smooth knots to the Legendrian setting.

Roughly, Legendrian knots are knots with certain restrictions on their tangent spaces, and Legendrian isotopy is a type of smooth deformation that preserves this restriction at every stage. This is formalized as the following.

Definition 1.1. A link $\Lambda \subset \mathbb{R}^3$ is called *Legendrian* if it lies everywhere tangent to the *standard contact structure* $\xi_{\text{std}} := \ker(dz - y\,dx)$ on $\mathbb{R}^3$, which is depicted in Figure 1. That is, $T_x\Lambda \in \xi_{\text{std}}$ for all $x \in \Lambda$, where $T_x\Lambda$ denotes the tangent space of Λ at x. A *front projection* or *front diagram* $D(\Lambda)$ is the projection of Λ to the xz-plane.

Definition 1.2. Viewed as smooth embeddings of S^1 into $\mathbb{R}^3$, two Legendrian links Λ_1, Λ_2 are considered *Legendrian isotopic* if they exists a smooth homotopy $H : S^1 \times [0,1] \to \mathbb{R}^3$ such that $H \times \{0\} = \Lambda_1$, $H \times \{1\} = \Lambda_2$, and $H \times \{t\}$ is a Legendrian link for all $t \in [0,1]$.

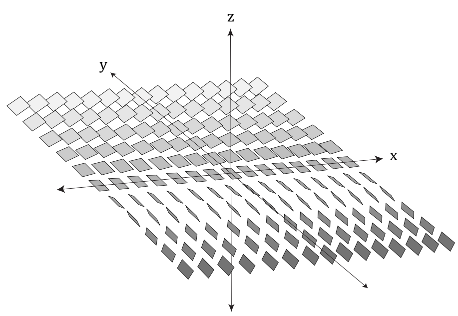


FIGURE 1. The standard contact structure ξ_{std} on $\mathbb{R}^3$. Taken from [3].

Legendrian links Λ are typically studied through their front projections, which follow several restrictions thanks to the tangency condition on Λ. For one, at every crossing in $D(\Lambda)$, the strand with the more negative slope is always the overstrand. For two, $D(\Lambda)$ has cusps in place of vertical tangencies. For example, Figure 2 depicts front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot, and Figure 3 depicts front projections of two distinct oriented Legendrian unknots. Note that the numbers of crossings and cusps in a Legendrian front projection are finite due to smoothness.

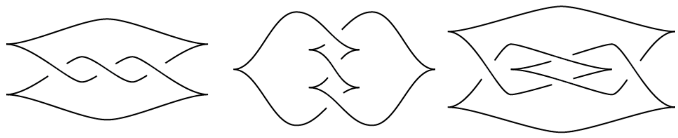


FIGURE 2. Unoriented front projections of two distinct Legendrian trefoils and a Legendrian figure-eight knot. Adapted from [1].

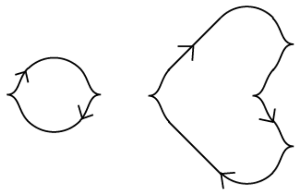


FIGURE 3. Front projections of distinct oriented Legendrian unknots. Adapted from [6].

Legendrian knots can be described by their so-called "classical invariants," which are integer-valued and defined contact-geometrically but can also be obtained from front projections.

Definition 1.3. Let Λ be an oriented Legendrian link with front projection $D(\Lambda)$. The *Thurston-Bennequin number* of Λ, denoted by $\mathrm{tb}(\Lambda)$ is defined as the linking number of Λ with Λ', where Λ' is obtained from Λ by a push-forward along a vector field transverse to ξ_{std}. Equivalently, if $\omega(D(\Lambda))$, $u(D(\Lambda))$ and $d(D(\Lambda))$ denote the writhe, number of upward-oriented cusps, and number of downward-oriented cusps of $D(\Lambda)$, respectively, then

$$\mathrm{tb}(\Lambda) = \omega(D(\Lambda)) - \frac{u(D(\Lambda)) + d(D(\Lambda))}{2}.$$

We can also define the *rotation number* of Λ in terms of its front projection:

$$\mathrm{rot}(\Lambda) := \frac{u(D(\Lambda)) - d(D(\Lambda))}{2}.$$

Equivalently, $\mathrm{rot}(\Lambda)$ is the degree of the map $S^1 \to S^1$ defined by

$$t \mapsto \frac{\pi(\frac{\partial \Lambda}{\partial t}(t))}{|\pi(\frac{\partial \Lambda}{\partial t}(t))|},$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}^2 \cong \mathbb{C}$ is the projection $(x, y, z) \mapsto (y, z)$.

2. Results

In the past two decades, researchers have found interesting relationships between these classical invariants and various quantum invariants, including the Jones polynomial $J(\Lambda, A)$ and the more general HOMFLY polynomial $P(\Lambda)$.

Theorem 2.1. [2, Theorem 2.5] *Let Λ be an oriented Legendrian link, and let n be the lowest degree in A of $J(\Lambda, A) \pmod 2$. Then $\mathrm{tb}(\Lambda) < n$.*

Corollary 2.2. *If Λ is a Legendrian left-handed trefoil, then $\mathrm{tb}(\Lambda) \leq -6$.*

The bound in Corollary 2.2 is well-known to be sharp. Theorem 2.1 is proven by reducing to the case that Λ is a knot and using HOMFLY polynomials.

One interesting application of the HOMFLY polynomial in the context of Legendrian knot theory relates to a well-known inequality of Bennequin.

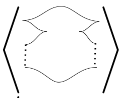
Proposition 2.3 (Bennequin's inequality)**.** *Let Λ be an oriented Legendrian knot with genus $g(\Lambda)$. Then*

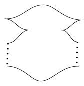
$$\mathrm{tb}(\Lambda) + |\operatorname{rot}(\Lambda)| \leq g(\Lambda) - 1.$$

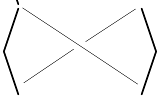
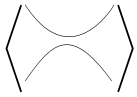
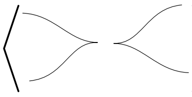
Indeed, by considering the difference between the maximum and minimum degrees of $P(\Lambda)$, Stoimenow in [7] gave a skein-theoretic proof of the following.

Theorem 2.4. [7, Corollary 1] *Bennequin's inequality becomes arbitrarily unsharp on any sequence of Legendrian representatives of distinct mirrored homogeneous knots.*

Recently, Kulkarni and Yadav in [4] introduced a skein-theoretic polynomial invariant for Legendrian knots, which we will denote by $J_L(\Lambda, A, r)$, that generalizes the usual Jones polynomial. Their construction begins with the following bracket relations:

(1) $\Big\langle$  $\Big\rangle = -A^2r^{-1} - A^{-2}r$, for any unknot with no crossings.

(2) $\Big\langle D(\Lambda) \sqcup$  $\Big\rangle = (-A^2r^{-1} - A^{-2}r)\langle D(\Lambda)\rangle$.

(3) $\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1}r \Big\langle$  $\Big\rangle$.

Given a front projection, we iteratively resolve crossings in the usual way and apply the above bracket relations at each step. This yields a polynomial $\langle D(\Lambda)\rangle$ in two variables, which we will call the *Legendrian Kauffman bracket.* While the Legendrian Kauffman bracket is not invariant under Legendrian isotopy, the following invariant is.

Theorem 2.5. [4, Theorem 1.1] *Given an oriented Legendrian link Λ with front projection $D(\Lambda)$, define the* Legendrian Jones polynomial *to be*

$$J_L(\Lambda, A, r) = (-A)^{-3\omega(D(\Lambda))} r^{\frac{c(D(\Lambda))}{2} - \ell(D(\Lambda))} \langle D(\Lambda)\rangle,$$

where $\ell(D(\Lambda))$ denotes the number of left-handed crossings in $D(\Lambda)$. Then $J_L(\Lambda, A, r)$ is an invariant of Legendrian knots, and $J_L(\Lambda, A, 1) = J(\Lambda, A)$.

The authors also introduce *Legendrian Khonanov homology* and show it to be an invariant of Legendrian knots. Fascinatingly, the graded Euler characteristic of the Khovanov homology groups recovers the coefficients of $J_L(\Lambda, A, r)$ [4, Theorem 1.3]. This categorifies the Legendrian Jones polynomial in a way reminiscent to the Poincaré polynomial of a topological space.

There is also an interesting version of colored Kauffman polynomials for Legendrian knots that takes into account the oriented cusps and so-called "switches" in Legendrian front projections [5]. Although the details of this construction are too involved to list here, an interesting property of this *ungraded n-colored ruling polynomial*, denoted by $R_{n,\Lambda}(q)$, relates to the representation theory of the Legendrian contact homology differential graded algebra $(\mathcal{A}(\Lambda), \partial)$ over finite fields of characteristic 2.

Theorem 2.6. [5, Theorem 1.1] *Let Λ be a Legendrian knot, fix $n \geq 1$, and let $\mathbb{F}_q$ be a finite field of order q and characteristic 2. Let $\mathrm{Rep}(\Lambda, \mathbb{F}_q^n)$ denote the number of 1-graded total n-dimensional representations of the Legendrian contact homology DGA of Λ on $(\mathbb{F}_q^n, d)$, where d is a linear map such that $d^2 = 0$. Then*

$$\mathrm{Rep}(\Lambda, \mathbb{F}_q^n) = R_{n,\Lambda}(q) = K_n(\Lambda, a, q)|_{a^{-1}=0},$$

where $K_n(\Lambda, a, q)$ denotes the n-colored Kauffman polynomial for framed knots. In particular, the value of $\mathrm{Rep}(\Lambda, \mathbb{F}_q^n)$ depends only on the underlying framed knot type of Λ.

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