Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 10

Last Lecture:
- FEM a priori error analysis

Today's lecture:
- FEM L2 a priori error analysis
- The Generalised Galerkin Method
- Strang's Lemma
- FEM analysis including quadrature
- Convection-diffusion-reaction problems (intro)

Computer practical
- linear FEM for two-points bvps (end)
- higher-order FEM for two-points bvps

Analysis of FEM for elliptic problems

- $u \in V : \mathcal{A}(u,v) = F(v) \quad \forall v \in V$

  $(V = H^1_0(\Omega))$

- Galerkin / FEM

  - V_h = p.w. polynomials $(k) \subseteq C^0(\Omega)$

  - $u_h \in V_h = V_h^k : \mathcal{A}(u_h, v) = F(v) \quad \forall v \in V_h$

- if $u \in H^s(\Omega) \cap H^1_0(\Omega)$ for $s \geq 2$

  $$|u - u_h|_{1,\Omega} \leq \frac{\gamma}{\alpha} C_I h^{\ell-1} |u|_{\ell,\Omega}$$

actually can be relaxed a bit
$s > 1 \quad d=2$
$s > 3/2 \quad d=3$

where $\ell = \min(k+1, s)$

comments:

- also have "local" bound, that is where r.h.s. written as $\left(\sum_{T \in \mathcal{T}_h} h_T^{2(\ell-1)} |u|^2_{\ell,\Omega} \right)^{1/2}$
- bound should be interpreted as asymptotic:

$$|u - u_h|_{1,\Omega} = \mathcal{O}(h^{\ell-1}) \text{ as } h \to 0$$

L^2 bound

(Poisson for simplicity)

Recall elliptic regularity result for Poisson:

$$(P) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if $f \in L^2(\Omega)$, and also $\partial\Omega$ is reg. enough (of class $\mathcal{C}^2$) or convex polygon, then

$$u \in H^2(\Omega) \quad \text{and} \quad \|u\|_{2,\Omega} \leq C \|f\|_{0,\Omega}$$

Theorem (L^2 bound): Let Ω smooth or convex polygon. If $u \in H^s(\Omega) \cap H^1_0(\Omega)$, $s \geq 2$ sol of (P), and letting $u_h \in V_h^k$ corresp. FEM solution, then

$$\|u - u_h\|_{0,\Omega} \leq C h^{\ell} |u|_{\ell,\Omega} \qquad \ell = \min(k+1, s)$$

Proof: by Aubin-Nitsche duality trick:

introduce auxiliary adjoint problem:

$$\begin{cases} -\Delta \phi = e_h := u - u_h & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

in weak form: $\phi \in H_0^1(\Omega)$:

$$a(\phi, v) := (\nabla \phi, \nabla v)_{L^2(\Omega)} = (e_h, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

note $\phi \in H^2(\Omega)$. Testing adjoint with e_h

$$\|e_h\|_{0,\Omega}^{\cancel{2}} = (e_h, e_h)_{L^2(\Omega)}$$

$$= (\nabla \phi, \nabla e_h) \qquad = a(\phi, e_h)$$

Galerkin orthogonality

$$= (\nabla e_h, \nabla \phi) = (\nabla e_h, \nabla \phi - \nabla \phi_h) \qquad = a(e_h, \phi) = a(e_h, \phi - \phi_h)$$

$$(\nabla e_h, \nabla \phi_h) = 0$$

$$\leq |e_h|_{1,\Omega} \, |\phi - \phi_h|_{1,\Omega}$$

for $\phi_h = I_h^k \phi$

$$\leq C \, |e_h|_{1,\Omega} \, h \, |\phi|_{2,\Omega}$$

$$= C h \, |e_h|_{1,\Omega} \, \cancel{\|e_h\|_{0,\Omega}}$$

result follows from H^1-seminorm error bound. $\square$

- comment: extra power of h is global

Uniform norm bound

It is possible to prove

$$\|v - I_h^k v\|_{\tau} \leq c h^2 \|v\|_{\tau^2} \qquad (W^{2,\infty})$$

from which, if h small enough, $\{\tau_h\}$ quasi-uniform

$$\|u - u_h\|_{\tau} \leq C h^2 \log(1/h) \|u\|_{\tau^2}$$

(see Heverkamp 1984, Ciarlet, Brenner-Scott)

- + superconvergence $O(h^2)$ at the nodes if mesh is squeezed?

The Generalised Galerkin method

- example: in the implementation of FEM requires quadrature, that is FEM solves: find $u_h \in V_h^k$:

$$a_h(u_h, v) = F_h(v) \qquad \forall v \in V_h^k$$

where a_h, F_h are approximations of a, F

this is a

VARIATIONAL CRIME

- replacement of a and or F
- $V_h^k \not\subset V$
- approx of Ω so that $\bigcup_{T \in \mathcal{T}_h} T \neq \Omega$

(e.g. when Ω is curved)

Def: If a variational crime is committed (any of the above) the problem

find $u_h \in V_h^k$: $a_h(u_h, v) = F_h(v) \quad \forall v \in V_h^k$

is a GENERALISED GALERKIN METHOD

Def (Truncation error):

$$T_h = \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{|T_h(v_h)|}{\|v_h\|_V}$$

where $T_h(v_h) = a_h(u, v_h) - F_h(v_h)$

(assuming this makes sense)

- if $a_h = a$, $F_h = F$ then $T_h \equiv 0$
  (Gal. orth.) Full consistent $\Longrightarrow$ Céa lemma
- otherwise, gen. Galerkin method is
  CONSISTENT if $T_h \xrightarrow{h \to 0} 0$

  STRONG CONSISTENCY if T_h is of maximal expected rate
  $\longrightarrow$ Strong lemma

analysis of Gen. Galerkin method when $\mathcal{A}_h, F_h$ replace $\mathcal{A}, F$.

<u>Def</u>: $\mathcal{A}_h : V_h \times V_h \longrightarrow \mathbb{R}$ is UNIFORMLY V_h-elliptic if $\exists\ \tilde{\alpha} > 0$:

$$\mathcal{A}_h(v_h, v_h) \geq \tilde{\alpha}\ \|v_h\|_V^2 \quad \forall v_h \in V$$

Theorem (1st strong lemma):

If $F_h \in V'$ and $\mathcal{A}_h$ is <u>uniformly V_h-elliptic</u> then,

- $\exists!\ u_h$ sol. of gen. Galerkin method
- $\|u_h\|_V \leq \frac{1}{\tilde{\alpha}} \|F_h\|_{V'}$

well-posedness

- $$\|u - u_h\|_V \leq \inf_{v_h \in V_h} \left[\left(1 + \frac{\gamma}{\tilde{\alpha}}\right) \|u - v_h\|_V + \frac{1}{\tilde{\alpha}} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|\mathcal{A}(v_h, w_h) - \mathcal{A}_h(v_h, w_h)|}{\|w_h\|_V} \right]$$

$$+ \frac{1}{\tilde{\alpha}} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|F(w_h) - F_h(w_h)|}{\|w_h\|_V}$$

proof: $\|u-u_h\|_V \leq \|u-v_h\| + \|\underbrace{v_h - u_h}_{\tilde{e}_h}\| \quad \forall v_h \in V_h$

$$\tilde{\alpha}\, \|\tilde{e}_h\|_V^2 \overset{V_h\text{-ellipticity}}{\leq} a_h(u_h - v_h, \tilde{e}_h)$$

by Gen. Gal. method
$$= F_h(\tilde{e}_h) - a_h(v_h, \tilde{e}_h) \underbrace{- a(u - v_h, \tilde{e}_h) - F(\tilde{e}_h) + a(v_h, \tilde{e}_h)}_{=0}$$

$$= a(u - v_h, v_h - u_h) + (a - a_h)(v_h, \tilde{e}_h) + (F_h - F)(\tilde{e}_h)$$

$$\leq \gamma \|u - v_h\|_V \|v_h - u_h\|_V + |(a - a_h)(v_h, \tilde{e}_h)| + |(F - F_h)(\tilde{e}_h)|$$

assume $\tilde{e}_h \neq 0$, and divide

$$\|\tilde{e}_h\|_V \leq \frac{\gamma}{\tilde{\alpha}} \|u - v_h\|_V + \frac{1}{\tilde{\alpha}} \left(\frac{|(a - a_h)(v_h, \tilde{e}_h)|}{\|\tilde{e}_h\|_V} + \frac{|(F - F_h)(\tilde{e}_h)|}{\|\tilde{e}_h\|_V} \right)$$

$$\leq \frac{\gamma}{\tilde{\alpha}} \|u - v_h\|_V + \frac{1}{\tilde{\alpha}} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \left(\frac{|(\mathcal{A} - \mathcal{A}_h)(v_h, w_h)|}{\|w_h\|_V} + \frac{|(F - F_h)(w_h)|}{\|w_h\|_V} \right)$$

$\square$

+ take inf over v_h.

FEM: 1^{st} term for $V_h = V_h^k$ is $O(h^k)$

if "strong terms" are same order say

method is strongly consistent

EXAMPLE (quadrature): (Ciarlet)

If on each element quadrature rule is exact on $\mathbb{P}_{2k-2}$ then

- $\mathcal{A}_h$ is unif. V_h-elliptic
- method remains strongly consistent and

$$|u - u_h|_{1,\Omega} \leq c h^{\ell-1} \left(|u|_{\ell,\Omega} + \|f\|_{\ell-1} \right)$$

if $u \in H^s(\Omega)$

and $\ell = \min(k+1, s)$

CONVECTION / REACTION DOMINATED DIFFUSION PROBLEMS

Consider general elliptic problem (scalar diff.

+

non-div. form

of convective term)

$$\begin{cases} \mathcal{L}u = \underbrace{-\nabla\cdot(a\nabla u)}_{\text{diffusion}} + \underbrace{\underbrace{\underline{b}\cdot\nabla u}_{\text{convection}}}_{\text{non-div. form}} + \underbrace{cu}_{\text{reaction}} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

seen that if $c - \frac{1}{2}\operatorname{div}\underline{b} \geq 0$, $a \geq \alpha_0$

then $\mathcal{A}$ is coercive with $\alpha = \dfrac{\alpha_0}{1 + C_\Omega}$

C_Ω = Poincare const

and continuous with $\gamma = \max\{\|a\|_{\infty,\Omega}, \max_i \|b_i\|_{\infty,\Omega}, \max \|c\|_{\infty,\Omega}\}$

$\Rightarrow$ LM $\|u\| \leq \frac{1}{\alpha}\|F\|_{V'}$

and for FEM

$$\|u - u_h\|_V \underset{\text{Céa lemma}}{\leq} \frac{\gamma}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V$$

Question: if $\gamma \gg \alpha$ (that is $\gamma \gg \alpha_0$) error could be very large (proportional to γ/α!

(when a small w.r.t. b and/or c)

problem is convection/reaction dominated!