Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 15

Last Lecture:
- Implicit FD methods
- Weak form of parabolic problems
- Semidiscrete in space FEM

Today's lecture:
- A priori analysis of semidiscrete in space FEM
- Fully discrete FD_FEM: the theta-method
- Stability and convergence
- Higher-order time-stepping

Quarteroni
Larsson-Thomee

Computer practical:
- Computing errors
- FEM implementation in 2D

Given $\mathcal{T}_h$ mesh on $\Omega \subset \mathbb{R}^d$, $V_h^k \subset V \ (= H_0^1(\Omega))$

FE space of order k.

Semidiscrete in space FEM:

$\forall t \in [0,T]$, find $u_h(t) \in V_h^k$:

$$\begin{cases} \left(\frac{d}{dt} u_h(t), v\right) + a(u_h(t), v) = F(t, v) & \forall v \in V_h^k \\ u_h(0) = u_{0,h} & \text{approx of } u_0 \ (\text{e.g. interpolant, projection}) \end{cases}$$

$$\mathcal{A}(u,v) = (\nabla u, \nabla v) \qquad \text{Heat eq.}$$

$$\mathcal{A}(u,v) = (\underset{\substack{\uparrow \\ \text{matrix}}}{a}\nabla u, \nabla v) + (\nabla\cdot(\underline{b}u), v) + (cu, v)$$

$\mathcal{A}$ is (weakly) <u>coercive</u> + continuous

Def (elliptic (Ritz) projection): $\forall v \in V$ the ELLIPTIC PROJ. $(R_h v)$ is given as the unique sol. of

$$\mathcal{A}(R_h v, v_h) = \underbrace{\mathcal{A}(v, v_h)}_{F(v_h)} \qquad \forall v_h \in V_h^k$$

we know that if Ω is convex (required by L^2 estimate)

then $\|v - R_h v\|_0 + h\,|v - R_h v|_1 \leq C h^{k+1} \|v\|_{k+1}$ \quad $\mathcal{T}_h$ shape regular

Idea: exploit ellip proj by splitting error:

$$u_h(t) - u(t) = \underbrace{u_h(t) - R_h u(t)}_{\vartheta} + \underbrace{R_h u(t) - u(t)}_{\rho}$$

elliptic \qquad parabolic component

$$\|\rho\|_0 \leq C h^{k+1} \|u(t)\|_{k+1} \leq C h^{k+1}\left(\|u_0\|_{k+1} + \int_0^t \left\|\frac{du}{dt}(\tau)\right\|_{k+1} d\tau\right)$$

$$u(t) = u_0 + \int_0^t \frac{du}{dt}(\tau)\, d\tau$$

Estimation of ϑ: energy argument.

starting from error eq. for ϑ

$$(\vartheta_t, v_h) + a(\vartheta, v_h) = \underbrace{\left(\frac{d}{dt} u_h(t), v_h\right) + a(u_h, v_h)}_{\text{FEM scheme}} - \left(\frac{d}{dt} R_h u(t), v_h\right) - \underbrace{a(R_h u(t), v_h)}_{\text{by def ellipt. proj}} \quad \forall v_h$$

$$= \underbrace{(f, v_h) - (R_h u_t(t), v_h) - a(u(t), v_h)}_{u \text{ is weak sol.}}$$

$$= (u_t(t), v_h) - (R_h u_t(t), v_h)$$

$$= (u_t(t) - R_h u_t(t), v_h)$$

$$(\vartheta_t, v_h) + a(\vartheta, v_h) = -(\rho_t, v_h) \qquad \forall v_h \in V_h.$$

Test error eq. with $v_h = \vartheta$

$\frac{1}{2}\frac{d}{dt}\|\vartheta\|_0^2$ coercivity CS

$$\frac{1}{2}\frac{d}{dt}\|\vartheta\|_0^2 + \alpha \|\vartheta\|_1^2 \leq \|\rho_t\|_0 \, \|\vartheta\|_0$$

In particular,

$$\left(\frac{d}{dt}\|\vartheta\|_0\right) \cancel{\|\vartheta\|_0} \leq \|\rho_t\|_0 \, \cancel{\|\vartheta\|_0}$$

integrate in $(0, t)$

$$\|\vartheta(t)\|_0 \leq \|\underbrace{\vartheta(0)}_{u_{0,h} - R_h u_0}\|_0 + \int_0^t \|\rho_t\|_0$$

which is bounded exactly as before

$\Rightarrow$

Theorem:

- Ω convex
- $\mathcal{T}_h$ shape regular
- a continuous + coercive
- $u_0 \in H^{k+1}(\Omega)$, $k \geq 1$
- $\frac{\partial u}{\partial t} \in L^1(I; H^{k+1}(\Omega))$
- $V_h^h \subset V$ standard FE

Then $\forall t \in [0,T]$,

$$\|u(t) - u_h(t)\|_0 \leq \|u_0 - u_{0,h}\|_0 + C h^{k+1} \left(\|u_0\|_{k+1} + \int_0^t \left\| \frac{\partial u}{\partial t}(\tau) \right\|_{k+1} d\tau \right)$$

Comments

- also true for $\sup_{t \in I}$ $\Rightarrow$ this is an $L^\infty - L^2$ estimate
- we can also derive $L^2 - H^1$ estimate

$$\frac{1}{2} \frac{d}{dt} \|v\|_0^2 + \alpha \|v\|_1^2 \leq \|\rho_t\|_0 \quad \|v\|_0$$

$\forall \varepsilon > 0$ Young's $\leq \varepsilon \|\rho_t\|_0^2 + \frac{1}{4\varepsilon} \|v\|_0^2$

$\varepsilon = \frac{1}{2\alpha}$ $= \frac{1}{2\alpha} \|\rho_t\|_0^2 + \frac{\alpha}{2} \|v\|_0^2$

$$\Rightarrow \cancel{\frac{1}{2}} \frac{d}{dt} \|v\|_0^2 + \frac{\alpha}{\cancel{2}} \|v\|_1^2 \leq \frac{1}{\cancel{2}\alpha} \|\rho_t\|_0^2$$

integ. in time

$$\cancel{\|v\|_0^2} + \alpha \int_0^t \|v\|_1^2 \, d\tau \leq \|v(0)\|_0^2 + \frac{1}{\alpha} \int_0^t \|\rho_t\|_0^2$$

$L^2 - H^1$

FULLY DISCRETE METHODS

- FD in time + FEM in space

ϑ- method

V_h^h FEM space, $t_n = n \Delta t$ time nodes

$\Delta t = T/N_t$ time step

$\forall n = 0, \dots, N_t - 1$, find $u_h^{n+1} \in V_h^k$:

$$(\vartheta) \begin{cases} \frac{1}{\Delta t} (u_h^{n+1} - u_h^n, v_h) + a(\vartheta u_h^{n+1} + (1-\vartheta) u_h^n, v_h) = \underbrace{(\vartheta f(t_{n+1}) + (1-\vartheta) f(t_n)}_{f_n^\vartheta}, v_h) \quad \forall v_h \in V_h^k \\ u_h^0 = u_{0,h} \in V_h^k \end{cases}$$

$$v_h^\vartheta = \vartheta v^{n+1} + (1-\vartheta) v^n$$

$$\bar{\partial}_t u_h^n = \delta_{\Delta t}^t u_h^{n+1/2} = \frac{u_h^{n+1} - u_h^n}{\Delta t}$$

rewrite scheme:

$$(\bar{\partial}_t u_h^n, v_h) + \mathcal{A}(u_\vartheta^n, v_h) = (f_\vartheta^n, v_h)$$

e.g. $\begin{cases} \vartheta = 0 & \text{EE + FEM} \\ \vartheta = 1 & \text{IE + FEM} \\ \vartheta = 1/2 & \text{CN + FEM} \end{cases}$

Stability $L^\infty - L^2$

EE $(\vartheta = 0)$

test with $v_h = u_h^n$ mult. by Δt

$$(u_h^{n+1} - u_h^n, u_h^n) + \Delta t \mathcal{A}(\underbrace{u_h^n, u_h^n}_{\geq \alpha \|u_h^n\|_1}) = \Delta t (f^n, u_h^n)$$

u_h

mumble mumble ...

IE $(\vartheta = 1)$

test with $v_h = u_h^{n+1}$

$$(u_h^{n+1} - u_h^n, u_h^{n+1}) + \Delta t \mathcal{A}(u_h^{n+1}, u_h^{n+1}) = \Delta t (f^{n+1}, u_h^{n+1})$$

$$\|u_h^{n+1}\|_0^2 - \|u_h^n\|_0 \|u_h^{n+1}\| + \Delta t \|u_h^{n+1}\|_1^2 \leq \Delta t \|f^{n+1}\|_0 \|u_h^{n+1}\|_0$$

$$\Rightarrow \|u_h^{n+1}\|_0^{\cancel{2}} - \|u_h^n\|_0 \cancel{\|u_h\|_0^{n+1}} \le \Delta t \|f^{n+1}\|_0 \cancel{\|u_h^{n+1}\|_0}$$

$$\Rightarrow \quad \|u_h^{n+1}\|_0 \le \|u_h^n\|_0 + \Delta t \|f^{n+1}\|_0$$

$$\boxed{\|u_h^n\|_0 \le \cdots \le \|u_{h,0}\|_0 + \Delta t \sum_{j=1}^{n} \|f^j\|_0}$$

Stability (unconditional !)

CN ($\vartheta = 1/2$)

test with $v_h = u_h^{n+1} + u_h^n$

$$(u_h^{n+1} - u_h^n, u_h^{n+1} + u_h^n) + \Delta t\, \Omega\left(\frac{u_h^{n+1} + u_h^n}{2}, u_h^{n+1} + u_h^n\right)$$

$$= \Delta t\left(f^n_{1/2}, u_h^{n+1} + u_h^n\right)$$

$$\Rightarrow \|u_h^{n+1}\|_0^2 - \|u_h^n\|_0^2 = \left(\|u_h^{n+1}\|_0 - \|u_h^n\|_0\right)\cancel{\left(\|u_h^{n+1}\|_0 + \|u_h^n\|_0\right)}$$

$$\le \Delta t \|f^n_{1/2}\|_0 \cancel{\|u_h^{n+1} + u_h^n\|_0}$$

$$\Rightarrow \|u_h^{n+1}\|_0 \le \|u_h^n\|_0 + \Delta t \|f^n_{1/2}\|$$

uncond. stab.

$$\Rightarrow \|u_h^n\|_0 \le \|u_{h,0}\|_0 + \Delta t \sum_{j=0}^{n} \|f^j\|_0$$

general stability result: $\to$ Quarteroni-Valli

Theorem (ϑ-method stability)

if $\quad \Delta t\left(1+C_I^2 h^{-2}\right) < \dfrac{2\alpha}{(1-2\vartheta)\gamma^2}$

EE $\Rightarrow \vartheta=0$

when $\quad \vartheta < 1/2$

(where C_I : $|v_h|_1 \leq C_I h^{-1} \|v_h\|_0$ in V_h^k (inverse estimate

then $\quad \|u_h^n\| \leq C_\vartheta \left(\|u_{0,h}\| + \sup_{t\in I} \|f(t)\|_0\right)$

Theorem (convergence): Under all assumptions as before, u_0, f, u suff. regular,

$\forall n \geq 1$

$$\underbrace{\max_n \|u(t_n)-u_h^n\|_0^2}_{L^\infty-L^2} + \underbrace{2\alpha \Delta t \sum_{m=1}^{n} \|u(t^m)-u_h^m\|_1^2}_{L^2-H^1}$$

$$\leq C(u_0,f,u)\left(h^{2k} + \begin{cases} \Delta t^2 & \vartheta \neq 1/2 \\ \Delta t^4 & \vartheta = 1/2 \end{cases}\right)$$

Hi

Higher-order time stepping

2^{nd} order BDF (backward difference formula)

$$\bar{\delta}_2 u^{n+1} = \bar{\delta} u^{n+1} + \frac{1}{2}\Delta t\, \bar{\delta}^2 u^{n+1}$$
$$= \frac{u^{n+1}-u^n}{\Delta t} + \frac{1}{2}\Delta t\, \frac{u^{n+1}-2u^n+u^{n-1}}{\Delta t^2}$$
$$= \frac{1}{\Delta t}\left(\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}\right)$$

(one sided formula, see lecture 2 or 3)

$$\bar{\delta}_2 u(t_{n+1}) = u_t(t_{n+1}) + \mathcal{O}(\Delta t^2)$$

issue: needs 2 previous time steps!
→ needs to be initialised:

BDF2-FEM

- $u_h^0 = u_{0,h} \in V_h^k$
- → $(\bar{\delta} u_h^1, v_h) + a(u_h^0, v_h) = (f^0, v_h)$ only 1-step method
- $(\bar{D} u_h^{n+1}, v_h) + a(u_h^{n+1}, v_h) = (f^{n+1}, v_h)$

$n = 2, \dots, M_t - 1$

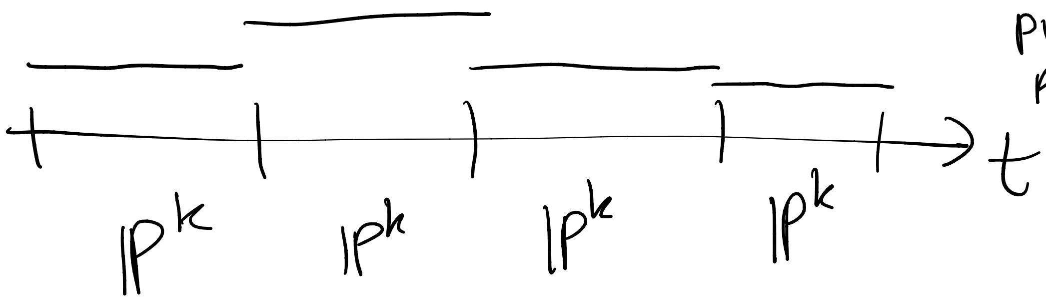
Theorem (LT): Under usual assumptions, $\vartheta = 0$

$$\|u(t_n) - u_h^n\|_0 \le c h^{k+1}\left(\|u_0\|_{k+1} + \int_0^{t_n} |u_t|_{k+1}\right) + c\Delta t \int_0^{\Delta t} \|u_{tt}\| + c\Delta t^2 \int_0^{t_n} \|u_{ttt}\|$$

- what about FEM in time as well?

natural environment in time $L^2(\Omega)$

$\Rightarrow$ discontinuous Galerkin methods



piecewise polynomials but discontinuous !

$k \geq 0$

$k = 1 \Rightarrow$ IE + FEM !

time !