Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA (first 40h)

LECTURE 1

House keeping:

- 32 45mins lectures + 8 45mins practicals (2 OR 3 per session)
- Where: all in room A-133 @SISSA
  Online ? …

- Website:
  https://www.math.sissa.it/course/phd-course-master-course/numerical-solution-pdes-0

- Course material:
  https://github.com/andreacangiani/NSPDE-ANA2023

IDEA OF COURSE:
- mainly FEM but present the fundamental ideas/methods and the interlink between different approaches.
- explore different problems/methods to see that fundamental ideas are ubiquitous
- understand that appropriate numerical methods depend on the problem
-practicals in python using google colab

This course is conceived to be linked to the other numerical analysis courses:
- Heltai+Rozza Applied Mathematics: an Introduction to Scientific Computing by Numerical Analysis (1st semester)
- Heltai's Advanced topics on the analysis of Finite Element Methods (starting March) -> saddle point problems (first 8h complete ANA)
- Cangiani's Advanced FEM Techniques (starting May)

ASSESSMENT: oral exam

BIBLIOGRAPHY:

- Quarteroni Numerical Models for Differential Problems. Springer, 2017.
- Larsson & Thomee Partial Differential Equations with Numerical Methods. Springer, 2009.
- Morton & Mayers Numerical Solution of Partial Differential Equations. Cambridge, 1994.
- Quarteroni & Valli Numerical Approximation of Partial Differential Equations. Springer, 1994.

Introduction (Q. 3.2)

super-simple problem (ODE): find $u \in C^2([0,1])$:

$$\begin{cases} -u''(x) = f(x) & \text{in } (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

1D Poisson b.v.p. (strong)

Discretise by introducing

grid $0 = x_0 < x_1 < \dots < x_N = 1$

e.g. $x_i = x_0 + h i$ $\quad h = 1/N \quad$ h = disc. parameter

Finite Difference (FD) approach:

look for values $U = \{U_i\}_{i=0}^{N} \approx u(x_i)$

approximate $u'(x) \approx \dfrac{u(x+h/2) - u(x-h/2)}{h}$

$$= \delta_h(x) \quad \text{centred difference}$$

$$\overset{!}{=} u'(x) + O(h^2)$$

$$u''(x) \approx \delta_h^2(x) = \delta_h(\delta_h(x))$$

$$= \delta_h\left(\frac{u(x+h/2) - u(x-h/2)}{h}\right)$$

$$= \frac{1}{h}\left[\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}\right]$$

$$= \frac{u(x+h)-2u(x)+u(x-h)}{h^2}$$

$$= u''(x) + O(h^2)$$

FD method: find $U = \{U_i\}_{i=0}^{M}$:

$-u'' = f$ $\quad\begin{cases} U_0 = 0 \\ -\dfrac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = f_i = f(x_i) \qquad i = 1, \dots, M-1 \\ U_M = 0 \end{cases}$

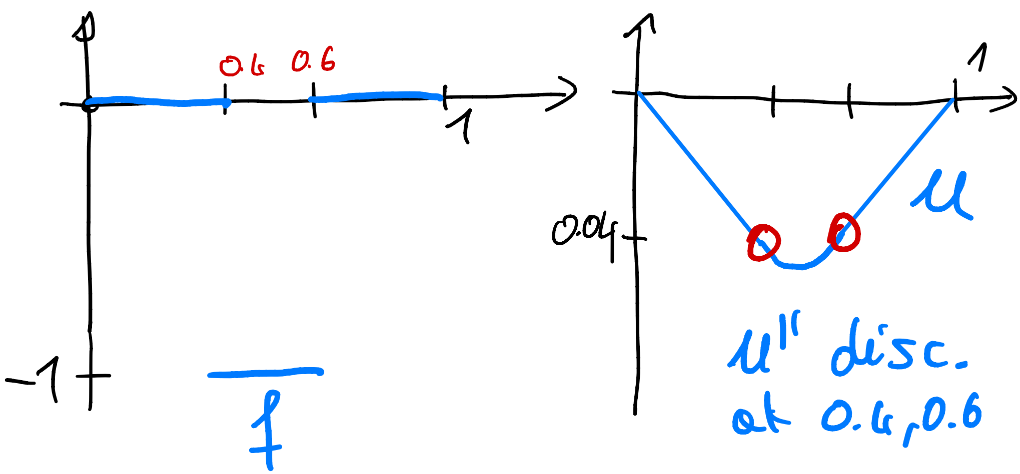
Note: ① rather than finding an approx solution (function), we are approximating values of u at given points (analysis?)

(2) What is the right "functional" setting

Strong form (in C^2) not adequate in general

example:

* take $f(x) = -\chi_{[0.4,0.6]}(x)$ ← characteristic function

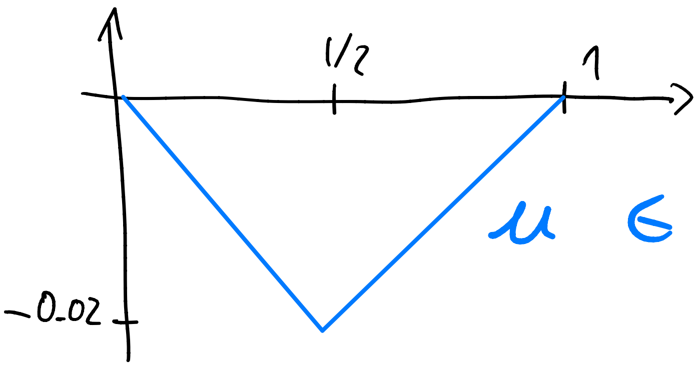


u'' disc. at 0.4, 0.6

$u \in C^1([0,1])$

Dirac Delta

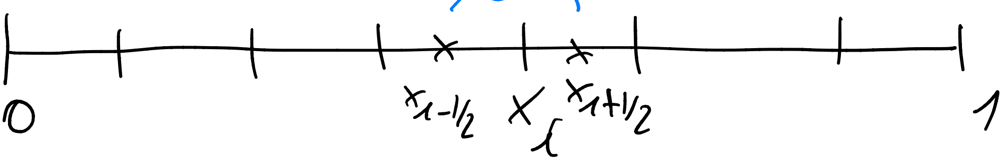
* $f(x) = -\delta_{1/2}(x) = \begin{cases} 1 & \text{if } x = 1/2 \\ 0 & \text{o.w.} \end{cases}$



$u \in C^0([0,1])$

Finite Volume approach:

control volume I_i



FV: integrate the eq. $-u'' = f$ in I_i

$$= \int_{x_{i-1/2}}^{x_{i+1/2}} u'' = \int_{x_{i-1/2}}^{x_{i+1/2}} f$$

$$= u'(x_{i+1/2}) - u'(x_{i-1/2})$$

FD

$$\approx \frac{u(x_{i+1}) - u(x_i)}{h} - \frac{u(x_i) - u(x_{i-1})}{h}$$

$$= \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h}$$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f(x)\,dx \simeq h\, f(x_i)$$

by rectangle rule

$$\Rightarrow \quad -\frac{U_{i+1} - 2U_i + U_{i-1}}{h} = h f_i$$

Dirichlet Principle: the string minimises a potential energy

$$J(v) = \frac{1}{2}\int_0^1 (v')^2\,dx - \int_0^1 f v$$

among all functions v such that $v(0)=0=v(1)$.

note: energy principle does not require 2nd derivative of sol.

Variational problem

If $-u'' = f$ then $\forall v$ appropriate "test function" from some space V with $v(0) = 0 = v(1)$

"variations"

$$-u''v = fv$$

$$-\int_0^1 u''v = \int_0^1 fv$$

$$\Downarrow \text{ by parts}$$

$$(u,v)_1 = \int_0^1 u'v' = \int_0^1 fv \qquad (*)$$

It remains to fix V

- We could take

$$V = \{ v \in C^1([0,1]) : v(0) = 0 = v(1) \}$$

then $(*)$ would make sense

However

(1) with $f = \delta_{1/2}$ $\Rightarrow u \notin C^1([0,1])$

(2) even if $f \in C^0([0,1])$

$\exists!$ of solutions in V is

<u>not guaranteed</u> !

Indeed, defining scalar product

$$(u,v)_1 := \int_0^1 u'v'$$

$$(\Rightarrow |v|_1 := (v,v))$$

V with $(\cdot,\cdot)_1$ is <u>not</u>

<u>complete</u> !

Def: $L^2(0,1) = \left\{ v : (0,1) \to \mathbb{R} : \left(\int_0^1 |v|^2 dx \right)^{1/2} =: \|v\|_2 < +\infty \right\}$

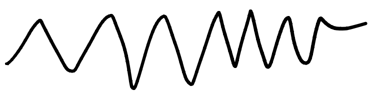
$H^1(0,1) = \{ v \in L^2(0,1) : v' \in L^2(0,1) \}$

$H^1_0(0,1) = \{ v \in H^1(0,1) : v(0) = 0 = v(1) \}$

Th: $(H^1_0, (\cdot,\cdot)_1)$ is a Hilbert space

(Inner product space complete)

- $H^1(0,1) \not\subset C^1(0,1)$ (only C^0)



- It is the closure of C^1
- Right ambient space for Poisson:

Theorem: Let $(V, (\cdot,\cdot)_V)$ Hilbert

$\|v\|_V = [(v,v)_V]^{1/2}$

$\mathcal{A} : V \times V \longrightarrow \mathbb{R}$

$\forall u, v \in V$

- continuous $\exists \gamma > 0 : \mathcal{A}(u,v) \le \gamma \|u\|_V \|v\|_V$

- coercive: $\exists \alpha_0 > 0 : \mathcal{A}(v,v) \geq \alpha_0 \|v\|_V^2 \quad \forall v \in V$
- symetric $\mathcal{A}(u,v) = \mathcal{A}(v,u)$
- bilinear

ex: $\mathcal{A}(u,v) = \int_0^1 u'v'$

$F : V \longrightarrow \mathbb{R}$ continuous linear form

$F(v) = \int_0^1 f v$

Then, $\exists!\, u \in V : \mathcal{A}(u,v) = F(v) \quad \forall v \in V$

Weak problem (WP)

$$\exists!\, u \in H_0^1(0,1) : (u,v)_1 = \int_0^1 f v \quad \forall v \in V$$

Theorem: u solves the WP if and only iff u is the solution of the potential energy minimiz- problem:

$$\text{Find } u \in V: \ J(u) \leq J(v) \quad \forall v \in V$$

where $J(v) := \frac{1}{2} a(v,v) - F(v)$

proof: "$\Rightarrow$" u sol. of (WP)

$\forall w \in V \quad J(u+w)$

$= \frac{1}{2} a(u+w, u+w) - F(u+w)$

$$= \frac{1}{2} A(u,u) + \underbrace{\frac{1}{2} A(u,w) + \frac{1}{2} A(w,u)}_{= A(u,w)}$$

linear

$$+ \frac{1}{2} A(w,w) - F(u) - F(w)$$

$= 0$ (VP)

$$= J(u) + \frac{1}{2} A(w,w) \geq J(u)$$

≥ 0 (coercivity)

$$\Rightarrow J(u) \leq J(u+w)$$

"$\Leftarrow$" if u minimiser

take $t \in \mathbb{R}$ $\frac{d}{dt} J(u+tv)\Big|_{t=0} = 0$

$$J(u+tv) = \frac{1}{2} A(u,u) + \frac{t^2}{2} A(v,v) + t\, A(u,v) - t\, F(v) - F(u)$$

$$0 = \frac{d}{dt} J(u+tv) = a(u,v) - F(v)$$

$$\forall v \in V$$

FEM approach: restrict the weak (miniviz.) problem to a appropriate finite dim subspace!

Galerkin Method

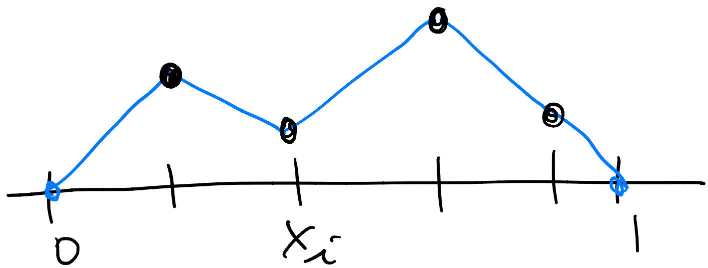
example: linear (Courant) FE on same mesh as before

$$x_i = x_0 + ih \qquad i = 1, \dots, N$$
$$x_0 = 0$$

$$I_i = [x_{i-1}, x_i)$$

polynomials of degree ≤ 1

$$V_h = \left\{ v \in H_0^1(0,1) : \begin{array}{l} v|_{I_i} \in \mathbb{P}^1(I_i) \\ v(0) = 0 = v(1) \end{array} \right\}$$



$\cap$

$C^0([0,1])$

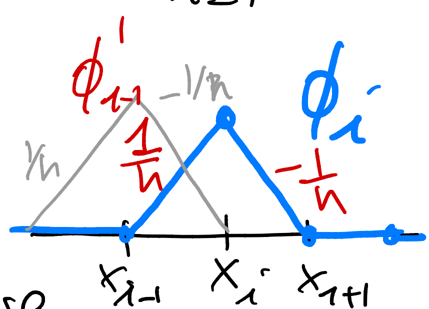
FEM: Find $u_h \in V_h$: $a(u_h, v_h) = F(v_h)$ $\forall v_h \in V_h$

well posed for some reason as before!

Algebric form of FEM

Note $V_h = \text{span}\{\phi_i\}_{i=1}^{N-1}$:

$$\phi_i = \begin{cases} \dfrac{x - x_{i-1}}{h} & x \in I_i \\ \dfrac{x_{i+1} - x}{h} & x \in I_{i+1} \\ 0 & \text{otherwise} \end{cases}$$



hat function

$$u(x) = \sum_{j=1}^{M-1} U_j \, \phi_j(x)$$

$$\rightarrow U = \{U_j\}_{j=1}^{M-1}$$ degrees of freedom DOF

Weak

Problem is linear $\Rightarrow$ sufficient to test with ϕ_i, $i = 1, \dots, M-1$:

FEM: Find $U \in \mathbb{R}^{M-1}$:

$$\sum_{j=1}^{M-1} U_j \int_0^1 \phi_j' \, \phi_i' \, dx = \int_0^1 f \phi_i \, dx \quad \forall i = 1, \dots, M-1$$

i-th equation:

$$\Bigg(\int_{x_{i-1}}^{x_i} \underbrace{\phi_{i-1}'}_{=-\frac{1}{h}}\ \underbrace{\phi_i'}_{=\frac{1}{h}}\Bigg) U_{i-1} + \Bigg(\int_{x_{i-1}}^{x_{i+1}} \overbrace{\phi_i'\ \phi_i'}^{=\pm\frac{1}{h}\ \pm\frac{1}{h}}\Bigg) U_i + \Bigg(\int_{x_i}^{x_{i+1}} \overbrace{\phi_i'\ \phi_{i+1}'}^{=-1/h\ \ =1/h}\Bigg)$$

$(j=i-1)$ $\qquad$ $(j=i)$ $\qquad$ $(j=i+1)$

$$= \underbrace{\int_{x_{i-1}}^{x_i} f\phi_i + \int_{x_i}^{x_{i+1}} f\phi_i}_{\text{trapezoidal rule}}$$

$$-\frac{1}{h}U_{i-1} + \frac{2}{h}U_i - \frac{1}{h}U_i \approx h\,\frac{f(x_{i-1})\overset{=0}{\phi_i(x_{i-1})} + f(x_i)\overset{=1}{\phi_i(x_i)}}{2} + h\,\frac{f(x_i)\overset{=1}{\phi_i(x_i)} + f(x_{i+1})\overset{=0}{\phi_i(x_{i+1})}}{2}$$

$$= h\, f(x_i)$$

$\equiv$ FD $\equiv$ FV

- FEM analysis is usually easier (based of functional analysis setting)
- Weak formulation allows for less smooth (H^1 instead of C^2) solutions
- (FEM easier to generalise to more complex domains in $\mathbb{R}^d$, $d > 1$)