Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 8

Last Lecture:

- Cea's lemma
- The Finite Element Method
- Error analysis for two-points BVPs
- FEM in higher-dimensions

Today's lecture:

- FEM in more dimensions
- Conditioning of the FEM system
- FEM a priori error analysis

Q4, chap4

Computer practical

- FEM for two-points bvps

From Lecture 7,

Considering FEM or order k on $(a, b) = \Omega$

for $-\Delta u + cu = f$, we arrived at:

$$|u - u_h|_{H^1(\Omega)} \leq C h^s |u|_{H^{s+1}(\Omega)} \qquad 1 \leq s \leq k, \quad u \in H^{s+1}(\Omega)$$

$$C = C(k)$$

k \ $u \in$	$H^{2 = s+1}$	$H^{3 = s+1}$	H^4	H^5	$\dots$
1	$\boxed{1}$	1	1	1	
2	1	$\boxed{2}$	2	2	
3	1	2	$\boxed{3}$	3	
4	1	2	3	$\boxed{4}$	
$\vdots$				"optimal"	

Remark: So called "hp"-bounds explicit both in mesh size (h) and polynomial degree (p) also available in litterature (much tougher to prove!) (?)

For example (Babuska - Suri '78), 2D

$$|u-u_h|_{H^1(\Omega)} \leq C \frac{h^{r-1}}{k^s} \|u\|_{H^{s+1}(\Omega)} \qquad u \in H^{s+1}$$

with $r = \min(s+1, k+1)$, C indep. of h, k, u,

assuming quasi-uniform meshes.

FEM in higher dimensions

For simplicity, fix $d=2$, $\Omega \subseteq \mathbb{R}^2$ polygonal domain, and consider b.v.p.:

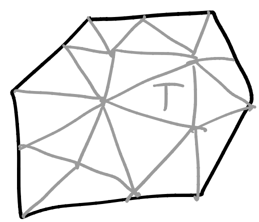
$$\begin{cases} \mathcal{L} u := -\nabla\cdot(a \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in weak form, find $u \in H^1_0(\Omega) =: V$:

$$\underbrace{\int_\Omega a \nabla u \cdot \nabla v}_{= \mathcal{A}(u,v)} = \underbrace{\int_\Omega f v}_{= F(v)} \qquad \forall v \in H^1_0(\Omega)$$

We want to apply a "practical" Galerkin method:

1. • Fix triangulations of Ω



$\mathcal{T}_h = \{ T \}$

T are triangles (simplices)

$\bar{\Omega} = \bigcup T$

all "elements" T are "congruent", that is they share either a vertex or an entire edge or their intersections are $\emptyset$.

$$h_T = \operatorname{diam}(T) \qquad \forall T \in \mathcal{T}_h$$

$$h = \max_T h_T$$

The vertices are called nodes of $\mathcal{T}_h$.

2. $$X_h = \left\{ v \in C^0(\Omega) : v_{|T} \in \mathbb{P}^k(T), \ \forall T \in \mathcal{T}_h \right\}$$

$$V \supseteq V_h = \left\{ v \in X_h : v_{|\partial\Omega} = 0 \right\}$$

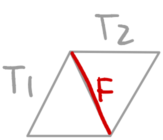
Proposition: $v \in H^1(\Omega) \iff$

$$\begin{cases} \bullet \ v_{|T} \in H^1(T) & \forall T \in \mathcal{T}_h \\ \bullet \ \text{if } F = T_1 \cap T_2 & (v_{|T_1})_{|F} = (v_{|T_2})_{|F} \end{cases}$$

$\exists \ T_1, T_2 \in \mathcal{T}_h$

F edge of $\mathcal{T}_h$

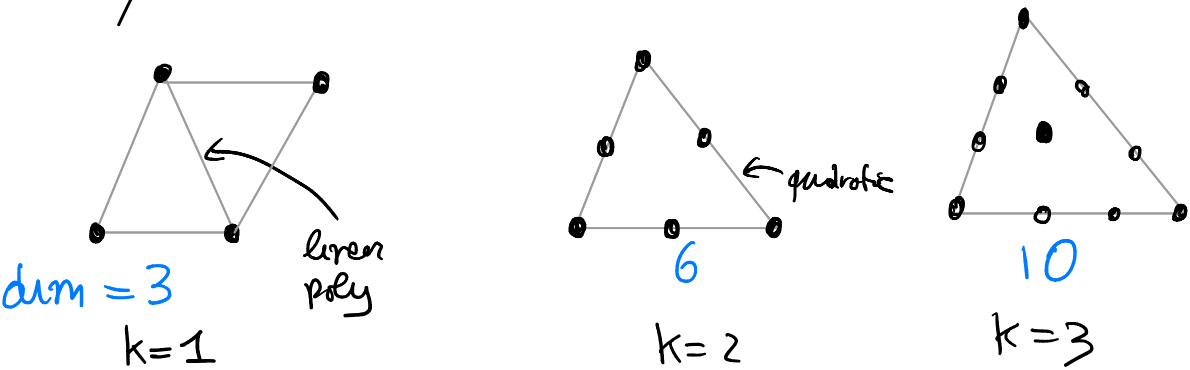
trace operator



Proof: exercise (apply def of weak deriv. to $D_j(v_{|T})$)

+ use Green's theorem.

Note, this works nicely:



$$\dim \mathbb{P}_k(\mathbb{R}^{\overset{2}{\overset{\shortparallel}{d}}}) = \binom{d+k}{k} = \frac{(\overset{2}{d}+k)!}{k!\ \underset{2}{d}!}$$

The points $x_j,\ j=1,\dots,\dim \mathbb{P}_k$ in figure define degrees of freedom (DOF) for local polynomial space $\Rightarrow$

(3) can construct "Lagrange" bases

$$\varphi_i \in V_h : \quad \varphi_i(x_j) = \delta_{ij}$$

$\{x_j\}$ Lagrange nodes

Def (Lagrange Finite element "à la Ciarlet"):

(T, Σ, P) where

- T = domain (triangle)
- P = finite dim. space of functions over T, with basis $\{\varphi_i\}$ $(\mathbb{P}_k(T))$
- Σ = set of unisolvent functionals (DOF):

$$\Sigma = \{\gamma_j\}_{j=1}^{\dim P} : \gamma_j : P \to \mathbb{R}$$

(nodal values)

such that $\gamma_j(\varphi_i) = \delta_{ij}$

FEM ASSEMBLY

Implementation of: $u_h \in V_h$:

$$a(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h$$

By considering the basis φ_j, write

$$u_h \in \sum_j U_j \varphi_j$$

and test with φ_i, $\forall i$:

find $U = \{U_j\}_{j=1}^{\dim V_h}$:

$$\sum_j U_j \underbrace{\int_\Omega a \, \nabla\varphi_j \cdot \nabla\varphi_i}_{A_{ij}} = \underbrace{\int_\Omega f \varphi_i}_{F_i} \quad \forall i = 1, \dots, \dim V_h$$

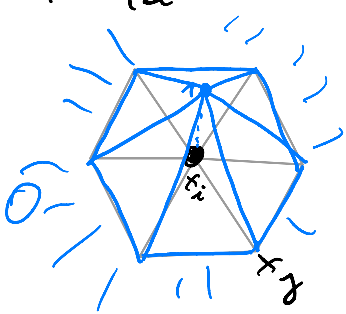
$\Rightarrow$ algebraic form $AU = F$

- Assembly $\equiv$ setting up of A and F
- solve, i.e. $U = A^{-1}F$.

<u>Assembly</u>: issue computation of $\int_\Omega \overset{a(x)}{\overset{\parallel}{a}} \nabla\varphi_j \cdot \nabla\varphi_i$

note: support of φ_i is localised

for example, $k=1$



$\varphi_i(x_i) = 1$

$\varphi_i(x_j) = 0 \quad \forall j \neq i$

"pyramid"

$\Rightarrow A$ is sparse

In practice, we split:

$$A_{ij} = \sum \underbrace{\int_T a \nabla\varphi_j \cdot \nabla\varphi_i}_{\neq 0 \text{ only if } \operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j \neq \emptyset}$$

hence, FEM code follows the steps:

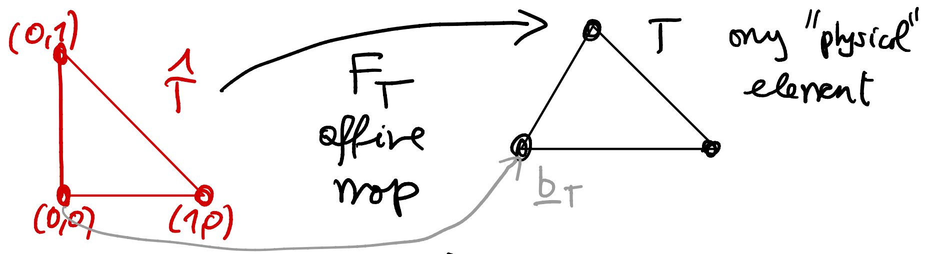
① loop over $T \in \mathcal{T}_h$

② $\forall T$, compute $\int_T$ by appropriate quadrature rule

Actually, we can use quadrature on a

REFERENCE TRIANGLE (ELEMENT)

• def $(\hat{T}, \hat{\Sigma}, \hat{P})$ reference FE



• $T = F_T(\hat{T})$

• $P = \{ p : T \to \mathbb{R} : p \circ F_T \in \hat{P} \}$

• $\Sigma = F_T(\hat{\Sigma})$

$$F_T(\hat{x}) = \underset{\substack{\uparrow \\ \text{invertible} \\ \text{matrix}}}{B_T} \, \hat{x} + \underset{\substack{\uparrow \\ \text{position} \\ \text{of a node of } T}}{b_T}$$

$$J_T = \text{Jacobian of } F_T = B_T$$

Then $\int_T a(x) \nabla\varphi_j \cdot \nabla\varphi_i \, dx = \int_{\hat{T}} a(F_T(\hat{x})) \, J_T^{-T} \hat{\nabla}\hat{\varphi}_j \cdot J_T^{-T} \hat{\nabla}\hat{\varphi}_i \, \underbrace{\det(J_T)}_{= \det B_T} \, d\hat{x}$

this requires

- evaluation of basis functions (and their gradients) on reference element once and for all
- plus eval. of coeffs on the mapped quadrature points for each T.

ON CONDITIONING OF FE MATRIX

(Q: how difficult it is to solve a FE system?)

Def: For A matrix, given $\|\cdot\|$ a matrix norm

$$\chi(A) := \|A\| \, \|A^{-1}\| \quad \text{condition number of } A$$

In particular, let $\|A\| = \|A\|_2 = \sup_{|x|=1} |Ax|$

euclidean norm

$|x| = \sqrt{x^T x}$

<u>Def</u>: Spectral condition number is

$$\chi_{sp}(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$$

Prop: If $A = A^T > 0$ then $\chi_{sp}(A) = \chi(A)$
for $\|\cdot\| = \|\cdot\|_2$

We say A is well conditioned if χ_{sp} is small.

<u>Prop</u>: If $\mathcal{T}_n$ is quasi-uniform
$\left(\exists \rho > 0 : \min_{T \in \mathcal{T}_n} h_T \geq \rho h \right)$
then $\chi_{sp}(A) = \mathcal{O}(h^{-2})$

requires :

Lemma: Let $\mathcal{T}_h$ quasiuniform family of triangulations. $\exists\ C_1, C_2 > 0$: $\forall v_h \in V_h$, let $\underline{v} = \{v_i\}^{\in \mathbb{R}^N}$: $v_h = \sum \underset{\uparrow}{v_i} \varphi_i$, then

$$C_1 h^{d(=2)} |\underline{v}|^2 \leq \|v_h\|^2_{L^2(\Omega)} \leq C_2 h^d |\underline{v}|^2$$

Rayleigh quotient: $R(\underline{v}) := \dfrac{(A\underline{v}, \underline{v})}{|\underline{v}|^2} = \dfrac{a(v_h, v_h)}{|\underline{v}|}$

$(A\underline{v}, \underline{v}) = \underline{v}^T A \underline{v}$

We have
$$\begin{cases} \max\limits_{\underline{v} \neq 0} R(\underline{v}) = \lambda_{max}(A) \\ \min\limits_{\underline{v} \neq 0} R(\underline{v}) = \lambda_{min}(A) \end{cases}$$

proof of the proposition

$$\frac{(A\underline{v}, \underline{v})}{|\underline{v}|^2} = \frac{a(v, v)}{|\underline{v}|^2} \overset{\text{continuity of } a}{\leq} \gamma \frac{\|v_h\|^2_{H^1(\Omega)}}{|\underline{v}|^2} = \circledast$$

coercivity of Ω
$$\geq \alpha \frac{\|v_h\|^2_{H^1(\Omega)}}{|\underline{v}|^2} = (**)$$

$$(**) \geq \alpha \frac{\|v_h\|^2_{L^2(\Omega)}}{|\underline{v}|^2} \underset{\text{lemma}}{\geq} C_1 \alpha h^d$$

$(*)$ need to estimate from above $\frac{\|v_h\|^2_{H^1(\Omega)}}{|\underline{v}|^2}$

Prop (inverse inequality): Let $\mathcal{T}_h$ be quasi-uniform family of triangulations. Then $\exists\ C_I > 0$: $\forall v_h \in V_h$

$$\underbrace{|v_h|^2_{H^1(\Omega)}}_{\|\nabla v_h\|^2_{L^2(\Omega)}} \leq C_I h^{-2} \|v_h\|_0^2$$

(true locally)

(result not true in V !)

$$(*) = \gamma \frac{\|v_h\|^2_{H^1(\Omega)}}{|\underline{v}|^2} = \gamma \frac{\|v_h\|^2_{L^2(\Omega)} + |v_h|^2_{H^1(\Omega)}}{|\underline{v}|^2}$$

inverse ineq.

$$\leq \gamma \frac{(1 + C_I h^{-2}) \|v_h\|^2_{L^2(\Omega)}}{|\underline{v}|^2}$$

lemma

$$\leq \gamma C_2 h^d (1 + C_I h^{-2})$$

$$\Rightarrow \alpha C_1 h^d \leq \overbrace{\frac{(A \underline{v}, \underline{v})}{|\underline{v}|^2}}^{= R(\underline{v})} \leq \gamma C_2 h^d (1 + C_I h^{-2})$$

$$\Rightarrow \quad \lambda_{min}(A) \leq \lambda_{max}(A)$$

$$\Rightarrow \chi_{sp}(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)} \leq \frac{\gamma}{\alpha} \frac{C_2}{C_1} (1 + C_I h^{-2})$$

$$= \mathcal{O}(h^{-2})$$

$\square$