Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 13

Last Lecture:
- A posteriori error analysis

Today's lecture:
- Intro to parabolic problems
- Finite Difference methods for parabolic problems

see Larsson - Thomée

my notes (Morton-Mayers)

Computer practical:
- Intro to FEniCSx
- FEM implementation in 1D

Parabolic problems

A parabolic PDE is of the form

$$u_t + \mathcal{L} u = f$$

with $\mathcal{L}$ a linear elliptic operator of 2nd order in $\mathbb{R}^d$.

Prototypes operators:

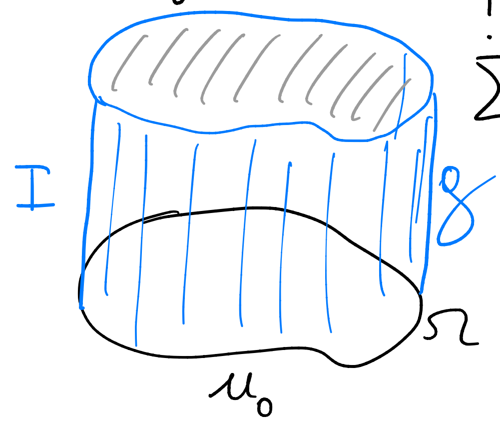
- Heat eq: $\mathcal{L} u = -\Delta u$
- diffusion-convection-reaction in divergence form: $\mathcal{L} u = -\nabla\cdot(a \nabla u) + \nabla\cdot(\underline{b} u) + c u$

Prototype problem: $\begin{cases} u_t + \mathcal{L} u = f & \text{on } \Omega \times I \\ u(x,0) = u_0(x) & \text{on } \Omega \\ u(t,x) = g(t,x) & \text{on } \partial\Omega \times I \end{cases}$ Cauchy–Dirichlet pbm

$\partial\Omega \times I =: \Sigma$ "parabolic boundary"

$\Omega \subset \mathbb{R}^d$

$I = (0, T]$



with $g(0,x) = u_0(x)$

A priori bounds / max principle

- fix $\mathcal{L} = -\Delta u$ (heat eq.)
- $g = 0$
- call $E(t) u_0$ = solution operator of homog. problem ($f \equiv 0$)

Then, $u(\cdot, t) = u(t)$

$$= E(t) u_0 + \int_0^t E(t-s) f(s)\, ds$$

superposition of homog. sol.

Duhamel principle

$$\Rightarrow \|u(t)\|_{0,\Omega} \leq \|u_0\|_{0,\Omega} + \int_0^t \|f(s)\|_{0,\Omega}\, ds$$

apriori bound (stability)

Maximum Principle:

Let u: $u_t - \Delta u \leq 0$ in $\Omega \times I$. Then u attains max on parabolic boundary Σ or at $t=0$.

If
$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times I \\ u(x,0) = u_0 & \text{in } \Omega \times \{0\} \\ u = g & \text{on } \Sigma \end{cases}$$

then
$$\max |u| \leq \max\left\{ \max_{\Sigma} |g|, \max_{\Omega} |u_0| \right\} + C \max_{\Omega \times I} |f|$$

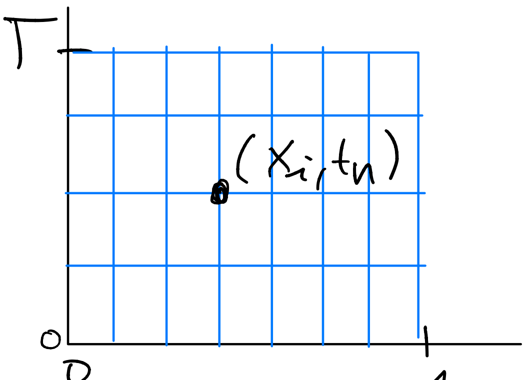
FINITE DIFFERENCE METHODS

$d = 1$, Heat problem in 1D:

$$\begin{cases} u_t = a\, u_{xx} & \text{in } \overset{\Omega}{\overset{\shortparallel}{(0,1)}} \times I \\ u(x,0) = u_0(x) & \text{in } (0,1) \\ u(0,t) = u(1,t) = 0 \end{cases}$$

$a = a(x,t)$

- Fix grid: $N_x, N_t \in \mathbb{N}$, $h = \frac{1}{N_x}$, $k = \frac{1}{N_t}$
(uniform)



$$U_i^n \approx u(x_i, t_n)$$

Explicit (forward) Euler (EE) method

$$u_t(x,t) \approx \delta_{k,+}^{t}\, u(x,t) = \frac{u(x,t+k) - u(x,t)}{k}$$

$$u_{xx}(x,t) \approx \left(\delta_h^x\right)^2 u(x,t) = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

$$\begin{cases} U_i^0 = u(x_i) \\[2mm] \dfrac{U_i^{n+1}-U_i^n}{k} - a_i^n \dfrac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0 \qquad \forall\, n = 0, \dots, N_t - 1 \\[2mm] U_0^{n+1} = 0 = U_{N_x}^{n+1} \end{cases}$$

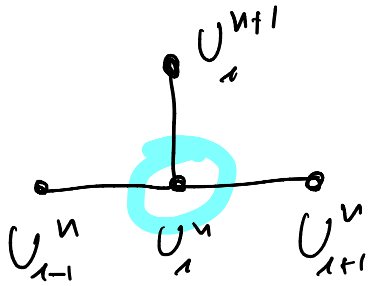
$a_i^n = a(x_i, t_n)$

$$\Longleftrightarrow \boxed{U_i^{n+1} = a_i^n \mu\, U_{i+1}^n + a_i^n (1-2\mu)\, U_i^n + a_i^n \mu\, U_{i-1}^n}$$

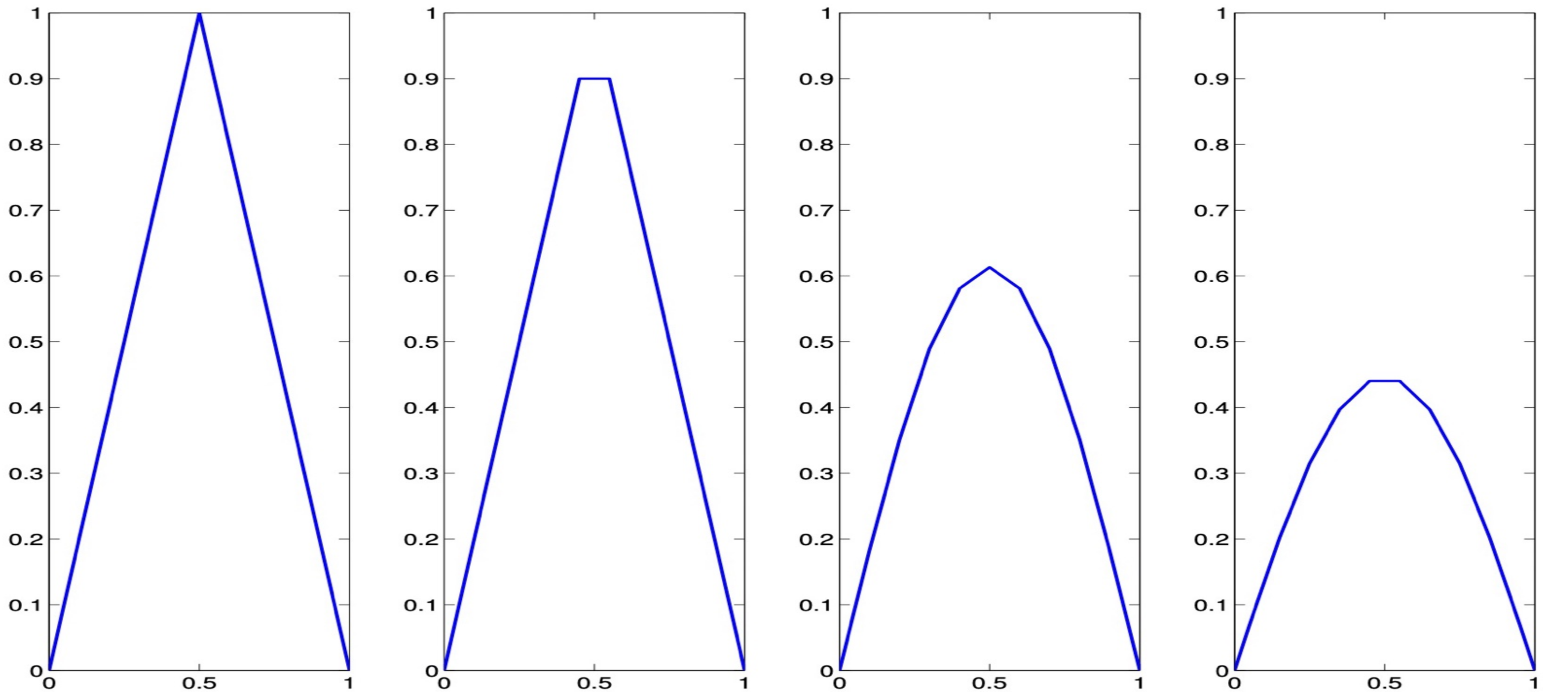
$\mu = k/h^2$ "curent number" of scheme

$\Updownarrow$

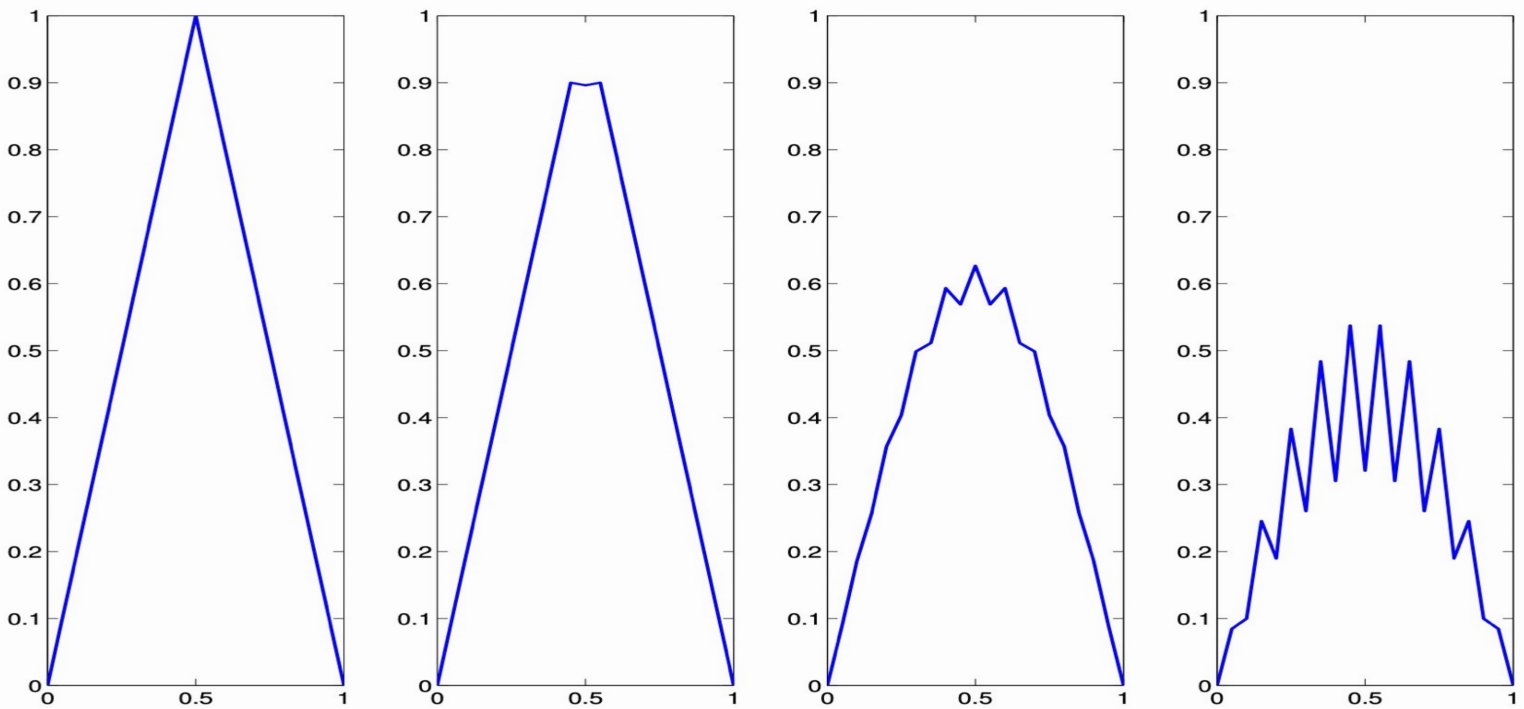
Stencil:



h=0.05, k=0.00125 giving mu=0.5



n=0 1 25 50



h=0.05, k=0.0013 giving mu=0.52

analysis of EE

- $U^n = (U_0^n, \ldots, U_{M_x}^n)$

$$\|U^n\|_{\infty,h} = \max_i |U_i^n|$$

- Discrete sol. op. E_k or $E_k U^n = U^{n+1}$

$$(\Rightarrow \quad U^n = E_k^n U_0)$$

- $e_i^n = u_i^n - U_i^n \qquad u_i^n := u(x_i, t_n)$

- Truncation (local disc.) error

$$T(x,t) := \delta_{k,+}^t u(x,t) - a\,(\delta_h^x)^2 u(x,t)$$

$$T_i^n := T(x_i, t_n), \quad T_M = \max |T_i^n|$$

Consistency: $\displaystyle |T_i^n| \le \frac{k}{2} M_{tt} + \frac{h^2}{12} \bar{a} M_{xxxx}$

$$= \frac{k}{2}\left(M_{tt} + \frac{\mu}{6} \bar{a} M_{xxxx}\right)$$

where $\displaystyle \bar{a} = \max_{\Omega \times I} |a(x,t)|$

$$M_{tt} = \max |u_{tt}|, \quad M_{xxxx} = \max |u_{xxxx}|$$

Proof: $\frac{u_i^{n+1} - u_i^n}{k} = (u_t)_i^n + \frac{k}{2} u_{tt}(x_i, \rho_n)$

$\delta_{k,+}^t u =$

$\rho_n \in (t_n, t_{n+1})$

$$(\delta_h^x)^2 u_i^n = (u_{xx})_i^n + \frac{h^2}{24}\left(u_{xxxx}(\xi_i, t_n) + u_{xxxx}(\zeta_i, t_n)\right)$$

$\xi_i \in [x_{i-1}, x_i]$

$\zeta_i \in [x_i, x_{i+1}]$

take difference and apply PDE:

$$T_i^n = \underbrace{(u_t)_i^n - a_i^n (u_{xx})_i^n}_{=0}$$

$+$ reminder terms

<u>Stability</u>: The EE method is stable $\Longleftrightarrow$

$a\mu \leq 1/2$ and $\|U^n\|_{\infty,h} \leq \|U^0\|_{\infty,h}$

<u>Proof</u>: (1) suff.

recall the scheme:

$$U_i^{n+1} = a_i^n \mu\, U_{i+1}^n + (1 - a_i^n \mu)\, U_i^n + a_i^n \mu\, U_{i-1}^n$$

$$\Rightarrow \left|U_i^{n+1}\right| \le a_i^n \mu \left|U_{i+1}^n\right| + \left|1 - a_i^n \mu\right| \left|U_i^n\right| + a_i^n \mu \left|U_{i-1}^n\right|$$

if $a_i^n \mu \le 1/2$ $\Rightarrow$ all coeffs are positive and add up to 1:

$$= a_i^n \mu \left|U_{i+1}^n\right| + (1 - a_i^n \mu) \left|U_i^n\right| + a_i^n \mu \left|U_{i-1}^n\right|$$

$$\le \|U^n\|_{\infty,h} \le \cdots \le \|U^0\|_{\infty,h}$$

(2) necc.

let $U_i^0 = (-1)^i \varepsilon$, $\varepsilon > 0$

$$\Rightarrow \|U^0\|_{\infty,h} = \varepsilon .$$

and apply scheme:

$$U_i^1 = a_i^1 \mu\, (-1)^{i+1} + (1 - 2\mu a_i^1)(-1)^i + a_i^1 \mu\, (-1)^{i-1}$$

$$= (-1)^i (1 - 4 a_i^1 \mu)\, \varepsilon$$

$$\Rightarrow U_i^n = (1 - 4 a_i^1 \mu) \cdots (1 - 4 a_i^n \mu)\, (-1)^i \varepsilon$$

if $a = a(x)$ then

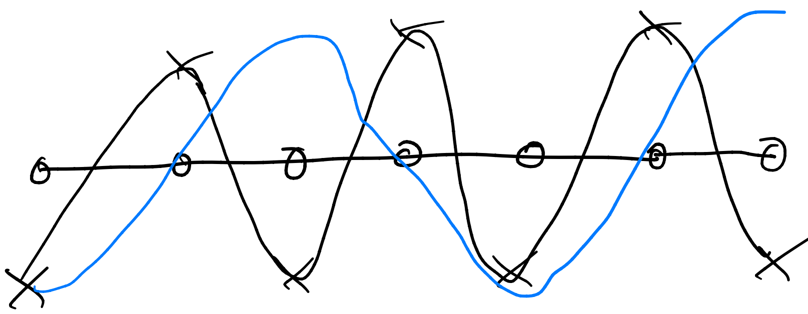
$$\stackrel{!}{=} (1 - 4 a_1)^n (-1)^i \varepsilon$$

if $a_1^n \mu > 1/2$ $\quad \|U^n\|_{\infty,h} \xrightarrow{h \to 0} \infty$ for any ε fixed $\square$

(LT)

Remarks

- sawtooth type initial cond is the highest mode scheme can carry



and is the one most likely amplified.

- due to roundoff errors, instability expected always.

- a closer look to stability

($\Rightarrow$ Von Neumann stability analysis)

motivation:

$$u(t,x) = \sum_{j=1}^{\infty} \hat{u}_j^0 \, e^{-\lambda_j t} \underbrace{\varphi_j(x)}_{}$$

countable modes carried by exact sol

similarly,

$$U_i^n = \sum_{-\infty}^{\infty} a_j \, \lambda(j)^n e^{ij(ih)}$$

(in fact this reduces to finite sum

$\Rightarrow$ makes sense to study

$$U_i^0 = e^{ij(ih)} \qquad (a \equiv 0)$$

$$\Rightarrow U_i^1 = \mu e^{ij(i+1)h} + (1-2\mu) e^{ij(ih)} + \mu e^{ij(i-1)h}$$

$$= e^{ijih} \left(\underbrace{\mu e^{ijh} + 1 - 2\mu + \mu e^{ijh}}_{= \mu \left(e^{\frac{ijh}{2}} - e^{-\frac{ijh}{2}} \right)^2} \right)$$

$$U_i^1 = e^{i\gamma ih} \underbrace{\left(1 - 4\mu \sin^2\left(\frac{\gamma h}{2}\right)\right)}_{\lambda(\gamma)}$$

$$\Rightarrow U_i^n = \lambda(\gamma)^n e^{i\gamma ih}$$

stability: $|\lambda(\gamma)| \leq 1$

$$\Leftrightarrow \quad -1 \leq 1 - 4\mu \sin^2\left(\frac{\gamma h}{2}\right) \leq 1$$

$$\Leftrightarrow \quad 0 \leq 4\mu \sin^2\left(\frac{\gamma h}{2}\right) \leq 2$$

$$\Leftrightarrow \quad 0 \leq \mu \sin^2\left(\frac{\gamma h}{2}\right) \leq \frac{1}{2}$$

$$\Leftrightarrow \quad \mu \leq 1/2$$

same condition as before !

Convergence: If $a\mu \leq 1/2$, then ($\mu = k/h^2$)

$$\|e^n\|_{\infty,h} \leq T\left(\frac{k}{2} M_{tt} + \frac{h^2}{12} M_{xxxx}\right)$$

Proof: apply scheme + def. of trunc. error:

$$e_i^{n+1} - a_i^n\left(\mu e_{i+1}^n + \underbrace{(1-2\mu)}_{\geq 0} e_i^n + \mu e_{i-1}^n\right) = k T_i^n$$

$n = 0, \dots, M_t - 1$

$i = 1, \dots, M_x - 1$

Using $a\mu \leq 1/2$,

$$|e_i^{n+1}| \leq a_i^n \mu |e_{i+1}^n| + (1-2\mu a_i^n)|e_i^n| + a_i^n \mu |e_{i-1}^n| + k|T_i^n|$$

$\leq \|e\|_{\infty,h}$

$$\Rightarrow \|e^{n+1}\|_{\infty,h} \leq \|e^n\|_{\infty,h} + k\, T_M$$

$$\leq \dots \leq \underbrace{\|e^0\|_{\infty,h}}_{=0} + (n+1) k\, T_M$$

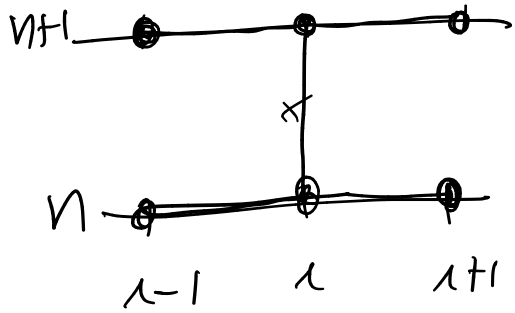
in particular,

$$\|e^n\|_{\infty,h} \leq T\, T_M$$

result follows from consistency bound (truncation error bound).

- if μ kept fixed, EE is $O(k)$

Implicit methods – ϑ method



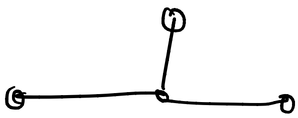
$$\delta_k^t U_j^{n+1/2} = \frac{U_j^{n+1} - U_j^n}{k}$$

$$= \vartheta \left(\delta_h^x\right)^2 U_j^{n+1} + (1-\vartheta)\left(\delta_h^x\right)^2 U_j^n$$

$\vartheta \in [0, 1]$

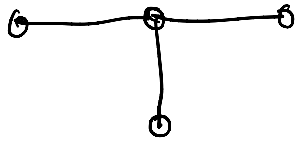
$\vartheta = 0$

EE



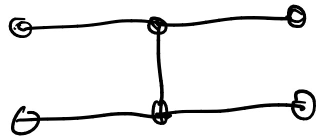
$\vartheta = 1$

IE



$\vartheta = 1/2$

CN



Truncation error

ϑ-method $\mathcal{O}(k, h^2)$

CN $(\vartheta = 1/2)$ $\mathcal{O}(k^2, h^2)$

Stability (von Neumann)

$\mu(1 - 2\vartheta) \leq 1/2$

$(a \equiv 1)$

so stability is conditionne

iif $\vartheta < 1/2$