Ways to prove infsup

1.6.2023

1) Direct method
2) Fortin's trick
3) Macro element techniques

1): $\forall q_h \in Q_h \quad \exists v_h \in V_h$ s.t.

- $b(v_h, q_h) = \|q_h\|_Q^2$ • $\|v_h\|_V \le c \|q_h\|_Q$

$$\Rightarrow \sup_{v_h \in V_h} \inf_{q_h \in Q_h} \frac{b(v_h, q_h)}{\|v_h\| \|q_h\|} \geq \frac{1}{c}$$

2) $\boxed{\text{FORTIN'S TRICK}}$

If $\exists \ \Pi \in \mathcal{L}(V, V_h)$, s.t. $\exists c_\Pi$ s.t.

1) $\|\Pi v\| \le c_\Pi \|v\| \qquad \forall v \in V$
2) $b(\Pi v - v, q_h) = 0 \qquad \forall q_h \in Q_h$
3) Continuous INF SUP is satisfied (with constant β)

$\Rightarrow$ Discrete infsup is satisfied with constant $\dfrac{\beta}{c_\Pi}$

Proof:

$$\underbrace{\sup_{v_h \in V_h} b(v_h, q_h) \geq \sup_{v \in V} b(\Pi v, q_h)}_{\text{true by def of sup}} \underbrace{=}_{\text{by (2)}} \sup_{v \in V} b(v, q_h)$$

$$= \sup_{v \in V} \underbrace{b(v, q_h) \geq \beta \|q_h\|}_{\text{true by 3}}$$

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\| \, \|q_h\|} \geq \frac{\beta}{C_\Pi}$$

3) $\boxed{\text{MACRO ELEMENT TECHNIQUE}}$

i) split domain in Macroelements (patches) $\underline{M}$ (finit union of adjacent elements)

ii) M is isomorphic to $\hat{M}$ if $\exists$ a map F_M, continuous invertible, s.t. $F(\hat{M}) = M$

$F_M |_{K_J}$ is affine for $\forall K_T \subset M$

iii) Macrospace $V_{0,M} := \{ v \in V_h ,\ v|_{\Omega \setminus M} = 0 \}$

$Q_{0M} := \{ q \in Q_h ,\ q|_{\Omega \setminus M} = 0,\ (q, 1) = 0 \}$

iv) $K_M := \ker B_h^T$ on M

$= \{ p \in Q_{0M} \text{ s.t. } B_h^T p = 0 \}$

INF-SUP$_h$ is satisfied if

1) $V_h - P_H^0$ is satisfied

2) $K_M = \{0\}$

Stokes on triangles

if inf sup is satisfied

$$\|u-u_h\|_V + \|p-p_h\|_Q \leq c\left(\|u-v_h\|_V + \|p-q_h\|_Q\right) \quad \forall v_h, q_h \in V_h \times Q_h$$

$$V_h := P_c^k \qquad Q_h = P_d^\ell$$

$$V := H_0^1(\Omega) \qquad Q := L_0^2(\Omega)$$

usually : $k = \ell + 1$

because $\|u - I_h u\|_{m,\Omega} \leq C h^{k-m+1} |u|_{k+1}$ $\quad m \leq k$

$$\|p - I_h p\|_{m,\Omega} \leq C h^{\ell-m+1} |p|_{\ell+1} \qquad m \leq \ell$$

$\Rightarrow$ with $k = \ell + 1$ $\quad \|u - I_h u\|_1 \sim h^k$

$$\|p - I_h p\|_0 \sim h^k$$

Ideally : $\boxed{\left(P_c^k\right)^{\dim} - P_d^{k-1}}$

For example: $\left(P_{c}^1\right)^{\dim} - P_d^0$

- $\dim = 2 \quad \rightsquigarrow \quad \dim(V_h) = 2 N_V$ $\quad$ # of vertices

$\dim(Q_h) = N_T$ $\quad$ # of triangles.

Euler Law $\quad N_{V_i} :=$ # of internal vertices.

$N_{V_b} :=$ # of boundary vertices

$$\boxed{N_T = 2 N_{V_i} + N_{V_b} - 2}$$

$N_T > 4 \Rightarrow \boxed{2N_V < N_T}$

recall INFSUP_h implies that B_h is full rank

If $\dim(V_h) < \dim(Q_h) \Rightarrow B_h$ is NOT full rank

$\boxed{\text{Locking}}$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \qquad Bu = 0 \Rightarrow u = 0$$

$\boxed{(P_c^2)^{\dim} - P_d^0}$??

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{\int \operatorname{div} v_h \, q_h}{\|v_h\|_1 \, \|q_h\|_0} \geq \beta_h \geq \beta_0 > 0$$

$$P^0 \Rightarrow \sup_{v_h \in V_h} \frac{\int_K \operatorname{div} v_h}{\|v_h\|} \geq \beta_h \qquad \forall K \in \mathcal{T}$$

we build $\Pi : V \to V_h$ s.t.

$$b(v - \Pi v, q_h) = 0 \qquad \forall v \in V$$

$$\int_K \operatorname{div}(v - \Pi v) = 0 \qquad \forall v \in H^1(K)$$

can we build Π s.t. $\|\Pi v\| \leq c\|v\|$ and

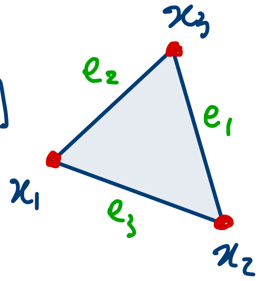
$$\int_K \operatorname{div} v = \int_K \operatorname{div} \Pi v \quad ??$$

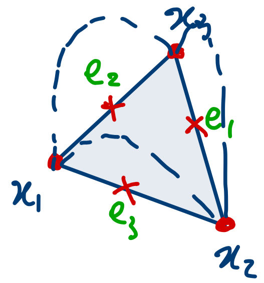
$$\Rightarrow \int_{\partial K} (v - \Pi v) \cdot n = 0$$

Morally: we split in two parts:

1. $\Pi v(x_i) =$ "$v(x_i)$" $\quad i \in [1,2,3]$

   $\int_{e_J} \Pi v = \int_{e_J} v \quad J \in [1,2,3]$



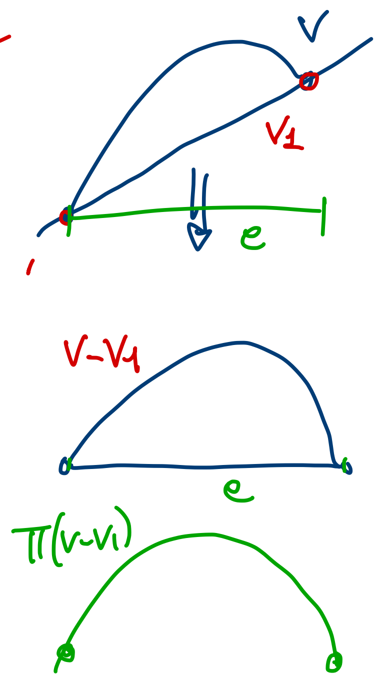


$\mathcal{C}\ell : V \to P^1_c$

$\|\mathcal{C}\ell\, v\| \leq c\|v\|$ Clement

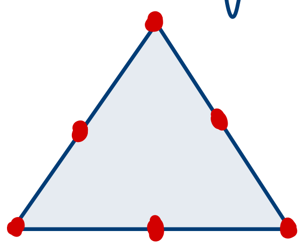
$\Pi := \Pi_2(I - \mathcal{C}\ell) + \mathcal{C}\ell$

$\|\Pi_2 v\| \leq c\|v\|_0 + h\,|v|_1$



$\int_e \Pi(v - v_1) = 0$

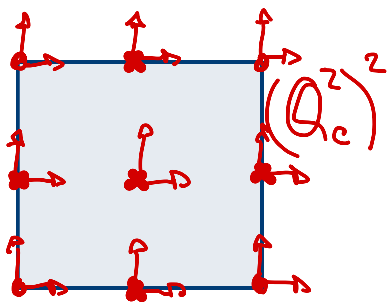
Π satisfies Fortin's trick

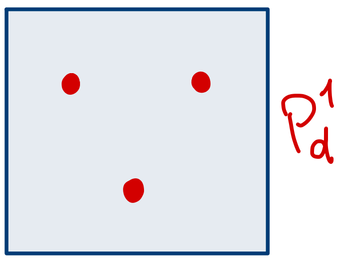


$= \text{span}\{x, y, 1, x^2, y^2, xy\}$

— $P^2_c - P^0_d$ —

Quadrilateral elements $\quad \boxed{Q^2_c - P^0_d}$





P^k = span of monomials of order at most k.

$P^1(2D) := \text{span}\{x, y, 1\}$

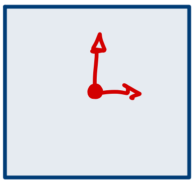
$Q^2 = \text{span}\{x, y, 1, xy, x^2, y^2, x^2y, y^2x, x^2y^2\}$

$Q^1 = \text{span}\{x, y, 1, xy\}$

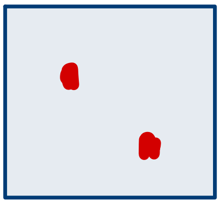
Q^k = span of monomials of order at most k in each coord axis.

Macroelement to a single element K: $\boxed{Q_e^k - P_d^{k-1}}$

$Q_e^2 - P_d^1$



$$V_{0,M} := \left(\text{span}\,\{\text{bubble}\}\right)^2$$



$$Q_{0M} := \text{span}\,\{x-\bar{x},\ y-\bar{y}\}$$

$$K_M = \{0\}$$

$$\tilde{v}_i := \{b_x,\ b_y\}$$

$$\tilde{q}_\alpha := \{x-\bar{x},\ y-\bar{y}\}$$

$$B_{q,M} := b(\tilde{v}_i,\ \tilde{q}_\alpha)$$

$$B_{q,M} := \begin{pmatrix} \int_T \operatorname{div} \tilde{v}_1\ \tilde{q}_1 & \int_T \operatorname{div} \tilde{v}_1\ \tilde{q}_2 \\ \int_T \operatorname{div} \tilde{v}_2\ q_1 & \int_T \operatorname{div} \tilde{v}_2\ \tilde{q}_2 \end{pmatrix} = \begin{pmatrix} \int_T b_1 & 0 \\ 0 & \int_T b_2 \end{pmatrix}$$

<u>Invertible</u>