Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 2

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Last Lecture:

- Three different points of view: FD, FV, and FEM

Today's lecture: + See also notes online

1 • Poisson problem in strong form: Well-posedness
2 • Numerical methods and the Lax-Richtmyer equivalence theorem
3 • Finite Difference discretisation: consistency, stability, convergence

1 Poisson problem strong form:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subseteq \mathbb{R}^d \\ u = g & \text{on } \partial\Omega \end{cases}$$

boundary of $\Omega$

**Def**: An harmonic function in $\Omega$ is a function $u : \Omega \to \mathbb{R}$ : $\begin{cases} u \in C^2(\Omega) \\ \Delta u = 0 \end{cases}$

Theorem (mean-value formula):
[see Evans, p. 25]

$u$ is harmonic in $\Omega$ iff

$$u(x) = ⨍_{\partial B_r(x)} u \, dS \qquad \forall x \in \Omega$$

and we also have

$$u(x) = ⨍_{B_r(x)} u \, dV$$

for each ball $B_r(x) \subset \Omega$.

Theorem (Strong Maximum Principle)

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ harmonic in $\Omega$, Then

(i) $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$

(ii) If $\Omega$ connected and $\exists\, x_0 \in \Omega$ s.t.

$$u(x_0) = \max_{\overline{\Omega}} u$$

then $u \equiv \text{const}$ in $\Omega$

<u>Proof</u>: (ii) suppose $x_0 \in \Omega$:

$$u(x_0) = M := \max_{\overline{\Omega}} u$$

then

$$M = u(x_0) = \fint_{B_{x_0}(r)} u \, dV$$

$$\leq M$$

$$\Rightarrow u \equiv M \quad \text{in } B_{x_0}(r)$$

Hence $\{ x \in \Omega : u(x) = M \} = S$

- open
- relatively closed in $\Omega$

$\Rightarrow S = \Omega$ when $\Omega$ is connected.

(i) immediate from (ii)

Corollary: (positiveness)

If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \partial\Omega \end{cases}$$

with $\Omega$ connected, then

- If $g \geq 0 \Rightarrow u \geq 0$ in $\Omega$
- If $\begin{cases} g \geq 0 \\ g(x) > 0 \ \exists x \end{cases} \Rightarrow u > 0$ in $\Omega$

Corollary (uniqueness)

Let $g \in C(\partial\Omega)$, $f \in C(\Omega)$.

Then there exist at most 1 solution of the Poisson pbm:

$$(P) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Proof: ex

Theorem: Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ sol of $(P)$. Then,

$$\|u\|_{C(\bar{\Omega})} \leq \|g\|_{C(\partial\Omega)} + C\|\underset{\overset{\|}{\Delta u}}{f}\|_{C(\Omega)}$$

$$\left( \|w\|_{C(\omega)} := \sup_{x\in\omega} |w(x)| \right)$$

Corollary: if $u, \tilde{u}$ are sol. of (P) with data $f, g$ and $\tilde{f}, \tilde{g}$, respectively, then

$$\|u-\tilde{u}\| \le \|g-\tilde{g}\|_{C(\partial\Omega)} + C\|f-\tilde{f}\|_{C(\Omega)}.$$

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Numerical discretisations (Q. 1.2)

Let $P(u,g)=0$ the PDE problem $\begin{cases} u = u(g) & \text{the solution} \\ g & \text{data} \end{cases}$

$$P_H(u_H, g_H) = 0 \quad \text{the}$$

approximate "numerical" PDE problem, w.r.t. $M$ = disc. parameter, e.g. dimension of discrete problem

We say that the numerical method is

- convergent if $\| u - u_M \| \to 0$ as $M \to +\infty$

  some appropriate norm

- $\boxed{\text{consistent}}$ if

$$P_M(u, g) \to 0 \text{ as } M \to \infty$$

(equivalently, $P_H(u, g) - P(u, g) \to 0$ as $H \to \infty$

strongly consistent if

$$P_H(u, g) = 0$$

- $\boxed{\text{stable}}$ if

$\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0 :$ if $\| \delta g_H \| < \delta$

$\Rightarrow \| \delta u_H \| \leq \varepsilon \quad \forall H$

where $u_H + \delta u_H$ is the numerical solution of the perturbed problem

$$P_H(u_H + \delta u_H, g_H + \delta g_H) = 0$$

**Theorem:** (Lax-Richtmyer equivalence).

A discretization $P_\pi(u_\pi, g_\pi)=0$ to a PDE problem $P(u, g)=0$ is convergente $\iff$ it is consistent + stable

note: analogue to Dahlquist equivalence

Consider Poisson in 1D:

$$\begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

and its FD discretisation on

$$x_i = ih \qquad i = 0, 1, \dots, N \quad \text{with } h = 1/N$$

$$\begin{cases} -\delta_h^2 U_i = f_i & i = 1, \dots, N-1 \\ U_0 = 0 \\ U_N = 0 \end{cases}$$

$$\delta_h^2 u(x) = \delta_h\left(\delta_h u(x)\right) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

Lemma: if $u \in C^4([0,1])$, $h > 0$, $x \in [h, 1-h]$

$$u''(x) - \delta_h^2 u(x) = \frac{h^2}{24}\left(u^{(IV)}(\xi_1) + u^{(IV)}(\xi_2)\right)$$

Taylor

error identity

$$\xi_1 \in [x-h, x], \quad \xi_2 \in [x, x+h]$$

error bound

$\delta_h^2$ is $O(h^2)$

$$|u''(x) - \delta_h^2 u(x)| \le \frac{h^2}{12} \|u^{(IV)}\|_{C[0,1]}$$

# A priori analysis

Def: For $\mathcal{L}_h$ a finite difference operator

$$T(x) = \mathcal{L}u(x) - \mathcal{L}_h u(x)$$

↑ diff. operator

$$\left(\text{e.g. } \mathcal{L}u = -u'' \, ; \, \mathcal{L}_h = \delta_h^2 \right)$$

is the TRUNCATION ERROR

$$\left( (\cancel{f} - \mathcal{L}_h u) - (\cancel{f} - \mathcal{L}u) = \mathcal{L}u - \mathcal{L}_h u \right)$$

$$P_H(u,f) - P(u,f)$$

$$= f(x) - \mathcal{L}_h u(x)$$

$$T(x) = f(x) - \delta_h u(x)$$

$$= f(x) - \delta_h^2 u(x)$$

$$= \cancel{f(x)} - \left[ \cancel{u''(x)} + \frac{h^2}{24}\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right) \right]$$

$$\Rightarrow \quad |T(x)| \leq \frac{h^2}{12} \, \| u^{(4)} \|_{C(0,1)}$$

thus $T(x) \xrightarrow{h \to 0} 0$ and it is $O(h^2)$.

<u>Note</u>: consistency does <u>not</u> imply that the discrete solution converges. Stability is also required. See Richtmyer and Morton 1967 for examples of disc. solutions which get further and further from the

exact solution as $h \to 0$ !

Stability comes from

Theorem (discrete max principle – DMP)

Let $V = \{V_i\}_{i=0}^{M}$.

If $\mathcal{L}_h V_i \le 0 \quad \forall i = 1, \dots, M-1$,

then $\max_i V_i = \max\{V_0, V_M\}$

Proof: by contradiction, let

$$V_n = \max_{1 \le i \le M-1} V_i \ , \quad V_n > V_0$$

or / and

$$V_n > M_M$$

then

$$0 \leq -h^2 \delta_h V_n = V_{n+1} - 2V_n + V_{n-1}$$

hypothesis

$V_n$ max

$$\Rightarrow \quad V_n \leq \frac{V_{n+1} + V_{n-1}}{2} \leq V_n$$

$$\Rightarrow \quad V_n = \frac{V_{n+1} + V_{n-1}}{2} \quad \Rightarrow V_n = V_{n-1} = V_{n+1}$$

repetition of argument $\Rightarrow V_n = V_i \ \forall i$

contradicting $V_n > V_0$ or/and $V_H$ ↯

Lemma (stability): Let $\delta_n = \delta_n^2$,

then for <u>any</u> $V = \{V_i\}_{i=0}^{H}$,

$$|V|_\infty \leq \max\{|V_0|, |V_H|\} + C|\delta_h V|_\infty$$

$= \max_i |V|$

Proof: by "comparison function"

$w(x) = x - x^2$ , $W_i := w(x_i)$

$$\delta_h W_i = \delta_h w(\overset{=ih}{x_i})$$

$$- \frac{\cancel{(i}\cancel{+1)}h - 2\cancel{ih} + \cancel{(i}\cancel{-1)}h}{h^2} + \frac{(i+1)^2 h^2 - 2 i^2 h^2 + (i-1)^2 h^2}{h^2}$$

$= 2$

So, defining

$$\tilde{V}_i^{\pm} := \pm V_i - \frac{1}{2} |\delta_h V|_\infty W_i$$

we have

$$\delta_h \tilde{V}_i^{\pm} = \pm \delta_h V_i - \frac{1}{2} |\delta_h V|_\infty \underbrace{\delta_h W_i}_{=2}$$

$$|\mathcal{L}_h V|_\infty$$

$$\le 0$$

$$\text{(DMP)} \Rightarrow \tilde{V}_i^{\pm} \le \max\{\pm V_0, \pm V_M\}$$

$$= \pm V_i - \frac{1}{2} |\mathcal{L}_h V|_\infty W_i$$

$$\Rightarrow \pm V_i \le \max\{|V_0|, |V_M|\} + |\mathcal{L}_h V|_\infty$$

$\square$

$$|V_i|$$

<u>Theorem</u>: (error estimate)

$\forall h > 0$, $\exists!$ discrete

solution $U$ of the FD scheme and

$$\max_i |u(x_i) - U_i| \le C h^2 \|u^{(4)}\|_{C(0,1)}$$

Proof: - uniqueness by contradiction

- existence by problem being on finite dim

- convergence: $e_i := u(x_i) - U_i$

(1) stability $|e_i| \le \max\{|e_0|, |e_N|\} + C\|\delta_h e\|_\infty$

$$= C\|\delta_h e\|_\infty$$

because $e_0 = e_N = 0$

(2) $\delta_h e_i = \delta_h u(x_i) - \delta_h U_i$

by FD $= \mathcal{L}_h u(x_i) - f_i$

by PDE $\overset{|}{=} \mathcal{L}_h u(x_i) - \mathcal{L} u(x_i)$

$\overset{|}{=} T(x_i)$

consistency

$\Rightarrow |\mathcal{L}_h e_i| \leq c\, h^2 |u|_{C^4(0,1)}$