

Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 9

Last Lecture:

- FEM a priori error analysis

Today's lecture:

- FEM L2 a priori error analysis
- The Generalised Galerkin Method
- Strang's Lemma
- FEM analysis including quadrature
- Convection-diffusion-reaction problems (intro)

Computer practical

- linear FEM for two-points bvp (end)
- higher-order FEM for two-points bvp

Analysis of FEM for elliptic problems

- $u \in V : A(u, v) = F(v) \quad \forall v \in V$
($V = H_0^1(\Omega)$)
- Galerkin / FEM
 - $V_h = \text{P.W. poly nomials } (k) \subseteq C^0(\Omega)$
 - $u_h \in V_h = V_h^k : A(u_h, v) = F(v) \quad \forall v \in V_h$
- if $u \in H^s(\Omega) \cap H_0^1(\Omega)$ for $s \geq 2$ actually
 - $|u - u_h|_{1/\Omega} \leq \frac{\delta}{2} C_I h^{l-1} |u|_{l,\Omega}$ can be relaxed & simplified
 - $s > 1 \quad d=2$
 - $s > 3/2 \quad d=3$

where $l = \min(k+1, s)$

Comments:

- also have "local" bound, that is where r.h.s. written as $\left(\sum_{T \in \mathcal{T}_h} h_T^{2(l-1)} \|u\|_{l,T}^2 \right)^{1/2}$
- bound should be interpreted as asymptotic:

$$\|u - u_h\|_{l,R} = O(h^{l-1}) \text{ as } h \rightarrow 0$$

L^2 bound

(Poisson for simplicity)

Recall elliptic regularity result for Poisson:

$$(P) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if $f \in L^2(\Omega)$, and also Ω is reg. enough
(of class C^2) or convex polygon, then

$$u \in H^2(\Omega) \quad \text{and} \quad \|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

Theorem (L^2 bound): let Ω smooth or convex polygon

If $u \in H^s(\Omega) \cap H_0^1(\Omega)$, $s \geq 2$ sol of (P),
and letting $u_h \in V_h^k$ corresp. FEM solution, then

$$\|u - u_h\|_{0,\Omega} \leq C h^l \|u\|_{l,\Omega} \quad l = \min(k+1, s)$$

Proof: by Aubin-Hitsche duality trick:

introduce auxiliary adjoint problem:

$$\begin{cases} -\Delta \phi = e_n := u - u_h & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

in weak form: $\phi \in H_0^1(\Omega)$:

$$R(\phi, v) = (\nabla \phi, \nabla v)_{L^2(\Omega)} = (e_n, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

note $\phi \in H^2(\Omega)$. Testing adjoint with e_n

$$= (e_n, e_n)_{L^2(\Omega)} = \|e_n\|_{0,\Omega}^2$$

$$\begin{aligned} &= (\nabla \phi, \nabla e_n) \\ &= (\nabla e_n, \nabla \phi) \quad \xrightarrow{\text{Galerkin orthogonality}} \end{aligned}$$

$$\begin{aligned} &= R(\phi, e_n) \\ &= R(e_n, \phi) \\ &= R(e_n, \phi - \phi_h) \end{aligned}$$

$$\leq \|e_n\|_{1,\Omega} \|\phi - \phi_h\|_{1,\Omega}$$

$$\text{for } \phi_h = I_h^k \phi$$

$$\leq C \|e_n\|_{1,\Omega} \|\phi\|_{2,\Omega}$$

$$= C h \|e_n\|_{1,\Omega} \|e_n\|_{2,\Omega}$$

result follows from H^1 -Seminorm
error bound. □

- Comment: extra power of h is global

Uniform norm bound

It is possible to prove

$$\|v - I_h^k v\|_2 \leq C h^2 \|v\|_{\infty} \quad (W^{1,\infty})$$

from which, if h small enough, $\{\mathcal{E}_h\}$ quasi-uniform

$$\|u - u_h\|_2 \leq C h^2 \log(1/h) \|u\|_{\infty}$$

(see Hervékamp 1984, Ciarlet, Bremer-Scott)

- + superconvergence $O(h^2)$ at the nodes if mesh is squared?

The Generalized Galerkin method

- example: in the implementation of FEM requires quadrature, that is FEM solves: find $u_h \in V_h^k$:

$$A_h(u_h, v) = F_h(v) \quad \forall v \in V_h^k$$

where A_h, F_h are approximations of A, F

this is a

VARIATIONAL CRIME

- replacement of A and/or F

- $V_h^k \not\subset V$

- approx of Ω so that $U_T \neq \Omega$
TEG_h

(e.g. when Ω is curved)

Def: If a variational crime is committed (any of the above) the problem

$$\text{find } u_h \in V_h^k: A_h(u_h, v) = F_h(v) \quad \forall v \in V_h^k$$

15. GENERALISED GALERKIN METHOD

Def (Truncation error):

$$T_h = \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{|T_h(v_h)|}{\|v_h\|_V}$$

where $T_h(v_h) = A_h(u, v_h) - F_h(v_h)$

(so now this makes sense)

- ① if $A_h = A$, $F_h = F$ then $T_h \equiv 0$
 (Gol. orth.) Full consistent à la lemma
- ② otherwise, gen. Galerkin method is
 CONSISTENT if $T_h \xrightarrow{h \rightarrow 0} 0$
 STRONG CONSISTENCY if T_h is of
 maximal expected rate
à Strong lemma

Analysis of Gen. Galerkin method when
 A_h, F_h replace A, F .

Def: $A_h: V_h \times V_h \rightarrow \mathbb{R}$ is UNIFORMLY
 V_h -elliptic if $\exists \tilde{\lambda} > 0$:

$$A_h(v_h, v_h) \geq \tilde{\lambda} \|v_h\|_V^2 \quad \forall v_h \in V$$

Theorem (1st strong lemma):

If $F_h \in V'$ and A_h is uniformly V_h -elliptic
 then,

- $\exists! u_h$ sol. of gen. Galerkin method
- $\|u_h\|_V \leq \frac{1}{\tilde{\lambda}} \|F_h\|_{V'}$
- $\|u - u_h\|_V \leq \inf_{v_h \in V_h} \left[\left(1 + \frac{\gamma}{\tilde{\lambda}}\right) \|u - v_h\|_V + \frac{1}{\tilde{\lambda}} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|A_h(v_h, w_h) - A_h(v_h, v_h)|}{\|w_h\|_V} \right]$

$$+ \frac{1}{2} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} |F(w_h) - F_h(w_h)|$$

proof: $\|u - u_h\|_V \leq \|u - v_h\|_V + \|v_h - u_h\|_V$ $\forall v_h \in V_h$

v_h -ellipticity

$$\frac{1}{2} \|\tilde{e}_h\|_V^2 \leq A_h(u_h - v_h, \tilde{e}_h)$$

by Gen.Gol.
method

$$= F_h(\tilde{e}_h) - A_h(v_h, \tilde{e}_h)$$

$$\underbrace{A(u - v_h, \tilde{e}_h) - F(\tilde{e}_h) + A(v_h, \tilde{e}_h)}_{=0}$$

$$= A(u - v_h, v_h - u_h) + (A - A_h)(v_h, \tilde{e}_h) + (F_h - F)(\tilde{e}_h)$$

$$\leq \gamma \|u - v_h\|_V \|v_h - u_h\|_V + |(A - A_h)(v_h, \tilde{e}_h)| + |(F - F_h)(\tilde{e}_h)|$$

assure $\tilde{e}_h \neq 0$, and divide

$$\|\tilde{e}_h\|_V \leq \frac{\gamma}{2} \|u - v_h\|_V + \frac{1}{2} \left(\frac{|(A - A_h)(v_h, \tilde{e}_h)|}{\|\tilde{e}_h\|_V} + \frac{|(F - F_h)(\tilde{e}_h)|}{\|\tilde{e}_h\|_V} \right)$$

$$\leq \frac{\delta}{2} \|u - v_h\|_V + \frac{1}{2} \sup_{\substack{w \in V_h \\ w \neq 0}} \left(\frac{|(A - A_h)(v_h, w_h)|}{\|w_h\|_V} + \frac{|(F - F_h)(v_h)|}{\|w_h\|_V} \right)$$

+ take inf over v_h .

□

FEN: 1st term for $V_h = V_h^k$ is $O(h^k)$

if "Strong terms" are same order say
method is strongly consistent

EXAMPLE (quadrature): (Cartlet)

If on each element quadrature rule is exact on P_{2k-2} then

- A_h is unif. V_h -elliptic
- method remains strongly consistent and

$$\|u - u_h\|_{1, \Omega} \leq C h^{l-1} (\|u\|_{l, \Omega} + \|f\|_{l-1})$$

if $u \in H^s(\Omega)$

or $l = \min(k+1, s)$

CONVECTION /REACTION DOMINATED DIFFUSION PROBLEMS

Consider general elliptic problem (scalar diff.)

+
non-div. form
of convective term

$$\begin{cases} \Delta u = -\underbrace{\nabla \cdot (\alpha \nabla u)}_{\text{diffusion}} + \underbrace{b \cdot \nabla u}_{\text{convection}} + cu = f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$$

non-div. form

seen that if $c - \frac{1}{2} \operatorname{div} b \geq 0, \alpha \geq \lambda_0$

then A is coercive with $\lambda = \frac{\lambda_0}{1 + C_\Omega}$

C_Ω = Poincaré const

and continuous with $\gamma = \max \left\{ \max_{\alpha, \Omega} \max_i \|b_i\|_{\alpha, \Omega}, \max \|c\|_{\alpha, \Omega} \right\}$

$$\Rightarrow LM \quad \|u\| \leq \frac{1}{\lambda} \|f\|_{V'}$$

and for FEM $\xrightarrow{\text{Lax lemma}}$

$$\|u - u_h\| \leq \frac{\gamma}{\lambda} \inf_{v_h \in V_h} \|u - v_h\|$$

that is $\gamma \gg \lambda_0$

Question: if $\gamma \gg \lambda$ error could
be very large (proportional to γ/λ) !

(when α small w.r.t. b and/or c)

problem is convection/reaction
dominated !