Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 11

Last Lecture:

- FEM L2 a priori error analysis
- The Generalised Galerkin Method
- Strang's Lemma
- FEM analysis including quadrature
- Convection-diffusion-reaction problems (intro)

Today's lecture:

- Convection-diffusion-reaction problems
- The Streamline diffusion method

Quarteron Chapter 13

Computer practical

- higher-order FEM for two-points bvps

Recall, for

$$\begin{cases} \mathcal{L}u = -\nabla\cdot(a\nabla u) + \underline{b}\cdot\nabla u + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\|u-u_h\|_V \leq \frac{\gamma}{\alpha} \inf_{v_h\in V_h} \|u-v_h\|_V$$

for $\alpha \ll \gamma$ error allowed to be large... seen in practice!

1D case, convection-diffusion

$a, b > 0$ constants

$$\begin{cases} -a u'' + b u' = 0 & \text{in } \Omega = (0,1) \\ u(0) = 0 \ ; \ u(1) = 1 \end{cases}$$

fix lifting function, e.g. $g(x) = x$ and

set $\tilde{u} = u - \tilde{g}$, solve

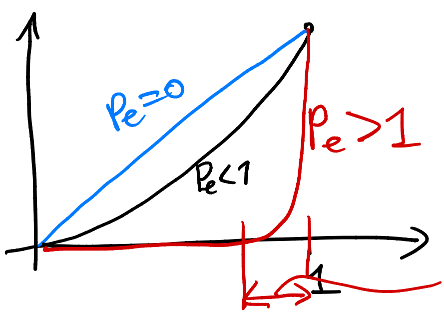
find $\tilde{u} \in H^1_0(\Omega) =: V$: $\mathcal{A}(\tilde{u}, v) = -\mathcal{A}(g, v)$ $\forall v \in V$

with $\mathcal{A}(u,v) = a(u', v') + b(u', v)$

$$\mathcal{A}(g,v) = \underbrace{a(g', v')}_{=0} + b(g', v) = b \int_0^1 v(x)\,dx .$$

Pèclet Number $\quad P_e := \dfrac{bL}{2a} \qquad L = |\Omega| \stackrel{!}{=} 1$

exact solution: $u(x) = \dfrac{e^{b/a\, x} - 1}{e^{b/a} - 1}$



$\mathcal{O}(a/b)$

"boundary layer"

"singularly perturbed" problem or

- for $a=0$ problem changes type,

$$\begin{cases} b u' = 0 \\ u(0) = 0 \end{cases} \qquad u \equiv 0$$

- $a \ll 1$ problem is "close" to reduced problem but adjusts to right boundary value on right end point

- "singular" or gradient becomes unbounded at $x \to 1$ or $a \to 0$

Linear FEM / FD discretisation

$h = 1/N$ uniform grid size

$$A \tilde{U} = F \qquad \tilde{U} = (U_1, \dots, U_{N-1}) \quad U_N = 1$$

$$A = \text{tridiag}\left(-\frac{a}{h} - \frac{b}{2},\ \frac{2a}{h},\ -\frac{a}{h} + \frac{b}{2}\right)$$

$$F = \left(0, \dots\dots,\ \frac{a}{h} - \frac{b}{2}\right)$$

Mesh Pèclet number: $Pe_h = \frac{bh}{2a}$

+ divide by a/h

$$A = \text{tridiag}\left(-1-Pe_h,\ 2,\ -1+Pe_h\right)$$

$$F = \left(0, \dots\dots, 1-Pe_h\right)$$

$$\boxed{U_j = \frac{1-\left(\frac{1+Pe_h}{1-Pe_h}\right)^j}{1-\left(\frac{1+Pe_h}{1-Pe_h}\right)^N}} \qquad j=1,\dots,N-1$$

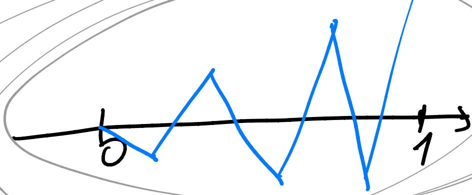
(extends to $j=0$, $j=N$)

$Pe_h \neq 1$

if $Pe_h = 1 \Rightarrow \begin{cases} U_j = 0 & j=0,1,\dots,N-1 \\ U_N = 1 \end{cases}$

$Pe_h < 1$ $\quad$ $>0 \Rightarrow U_j$ monotone

$Pe_h > 1$ $\quad$ $<0 \Rightarrow U_j$ oscillates



NOTE: eventually, as $h \to 0$ $Pe_h < 1$

But, in practice, say

$$\begin{cases} a = 10^{-4} - 10^{-6} \\ b = 1 \end{cases}$$ need $h \simeq 10^{-4} - 10^{-6}$

	DOF
	$10^4 - 10^6$
2D	10^{12}
3D	10^{18}

cures are required

- adapt mesh

- modify scheme

Our scheme, written as FD scheme is

$$\begin{cases} -a \dfrac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + b \underbrace{\dfrac{U_{j+1} - U_{j-1}}{2h}}_{} = 0 & j = 1, \dots, N-1 \\ U_0 = 0 \ ; \ U_N = 1 \end{cases}$$

centred scheme also for convection

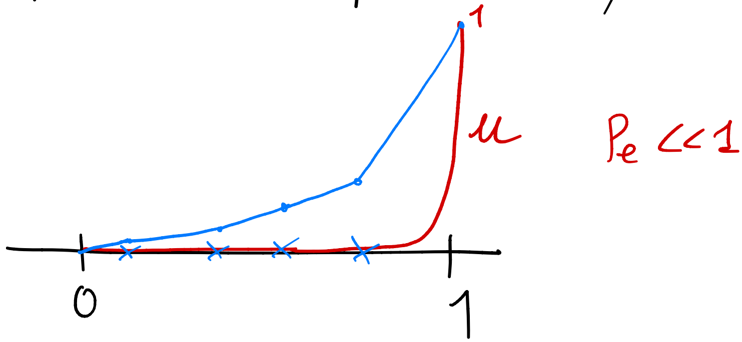
use instead UPWIND SCHEME

$$-a\,\frac{U_{j+1}-2U_j+U_{j-1}}{h^2}+b\,\frac{U_j-U_{j-1}}{h}=0$$

$$\rightarrow\quad U_j=\frac{(1+2\,Pe_h)^j-1}{(1+2\,Pe)^N-1}$$

monotonic increasing $\forall\, Pe_h$!

BUT $\quad |u(x_j)-U_j| = O(h)$



monotone but "too" diffused !

$$\underbrace{b\,\frac{U_j-U_{j-1}}{h}}_{\text{UPWIND}}=b\,\frac{U-\frac{1}{2}U_{j-1}-\frac{1}{2}U_{j-1}+\frac{1}{2}U_{j+1}-\frac{1}{2}U_{j+1}}{h}$$

$$=\underbrace{b\,\frac{U_{j+1}-U_{j-1}}{2h}}_{\text{CENTRAL}}-\underbrace{\frac{bh}{2}\,\frac{U_{j+1}-2U_j+U_{j-1}}{h^2}}_{\text{CENTRAL for } -\frac{bh}{2}u''}$$

upwind for $-au''+bu'=0$ $\quad\equiv\quad$ centred $-\underbrace{\Big(a+\underbrace{\tfrac{bh}{2}}_{\text{artificial diffusion}}\Big)}_{a_h :}u''+bu'=0$

$$Pe_h = \frac{bh}{2a_h} = \frac{Pe}{1+Pe} < 1 \quad \text{always}$$

Note: amount of artificial diffusion makes num. sol. monotone but low order!

In general, can define the artificial diff. function

$$a_h = a\left(1+\phi(Pe_h)\right)$$

- upwind $\phi(t) = t$
- Scharfetter-Gummel (exponential fitting)

$$\begin{cases} \phi(t) = t-1+B(2t) \\ B(t) = \begin{cases} \dfrac{t}{e^t-1} & \text{if } t>0 \\ 1 & t=0 \end{cases} \end{cases} \quad \text{(Bernoulli)}$$

is $\mathcal{O}(h^2)$, and $f \equiv$ const

it gives nodally exact num. solution

<u>analysis</u>: given a replaced by a_h

$\rightarrow$ variational crime ! $\rightarrow$ $\uparrow$

strong 1st lemma:

$$\|u - u_h\|_{1,\Omega} \leq \inf_{v_h \in V_h} \left\{ \left(1 + \frac{\gamma}{\tilde{\alpha}}\right) \|u - v_h\|_{1,\Omega} + \frac{1}{\tilde{\alpha}} \sup_{w_h \in V_h} \frac{|\Omega(v_h, w_h) - \Omega_h(v_h, w_h)|}{\|w_h\|_{1,\Omega}}\right.$$

a_h bilinear form with artificial diffusion

(quarteroni)

$$\leq \frac{c h^k}{a(1+\phi(\mathbb{P}_e))} \|u\|_{H^{k+1}(\Omega)} + \underbrace{\frac{\phi(\mathbb{P}_e)}{1+\phi(\mathbb{P}_e)}}_{} \|u\|_{1,\Omega}$$

upwind : $\phi(P_e) = O(h) \rightarrow$ method is $O(h)$

S. G. : $\phi(P_e) = O(h^2) \rightarrow$ " " $O(h^2)$

Towards strongly consistent methods

$\Omega \in \mathbb{R}^d$, $\begin{cases} \mathcal{L}u := -a\Delta u + \underline{b}\cdot\nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

"Idea 0": $\mathcal{L} - h\bar{b}\Delta$ $\qquad \bar{b} = \|\underline{b}\|_{\infty,\Omega}$

artificial diff. method

$$A_h(u,v) = A(u,v) + B_h(u,v)$$

$$B_h(u,v) = \bar{b}\, h\, (\nabla u, \nabla v)$$

Truncation error :

$$T(v_h) = A_h(u, v_h) - F(v_h)$$

$$= A_h(u, v_h) - A(u, v_h)$$

$$\stackrel{!}{=} B_h(u, v_h)$$

$$\stackrel{!}{=} \bar{\sigma}\, h\, (\nabla u, \nabla v_h)$$

→ Strong: $\|u - u_h\|_1 = O(h^k) + h\,\bar{\sigma}\,\|u\|_1$

ISSUE: adding artificial diffusion in all directions!

example: take 2D pbm: $\begin{cases} -a\Delta u + u_x = f & \Omega \\ u = 0 & \partial\Omega \end{cases}$

<u>IDEA 1</u>: add diffusion only along the streamlines

$$\mathcal{L}_h u := \mathcal{L}u - \bar{\sigma} h\, \nabla \cdot \left((\underline{b} \cdot \nabla u)\, \underline{b} \right)$$

$$\mathcal{A}_h(u, v) = \mathcal{A}(u, v) + B_h(u, v)$$

$$B_h(u, v) = \bar{\sigma} h\, (\underline{b} \cdot \nabla u,\ \underline{b} \cdot \nabla v)$$

still $O(h)$ consistency

IDEA 2: streamline-diffusion method

or SUPG [Hughes-Brooks 1979]

add streamline diff without destroying consistency:

$$\int_\Omega \left(-\varepsilon \Delta u + \underline{b} \cdot \nabla u\right)(\underline{b} \cdot \nabla v) = \int_\Omega f v \quad \forall v \in V$$

add

$$B_h := \sum_T \delta_h^T \int_T \left(-\varepsilon \Delta u_h + \underline{b} \cdot \nabla u_h\right)(\underline{b} \cdot \nabla v_h)$$

and consistently add $\sum_T \delta_h^T \int_T f \, (\underline{b} \cdot \nabla v_h)$

to right-hand side

$\delta_h^T \approx h$ streamline diffusion parameter

We have

$$\|u - u_h\|_{SD} = \mathcal{O}(h^k)$$

$$\|v\|_{SD} = \Big(a_0 |v|_1^2 + \underbrace{\sum_{T \in \mathcal{T}_h} \tau_T \|\underline{b} \cdot \nabla v\|_{L^2(T)}^2}_{\text{additional streamline diffusion direction control}} \Big)^{1/2}$$

if $\tau_h = \begin{cases} \delta_0 \, h_T & \text{if } Pe_{h_T} > 1 \ \text{(conv-dom)} \\ \delta_1 \, h_T^2/a & \text{if } Pe_{h_T} \le 1 \ \text{(diff-dom} \end{cases}$

$\delta_0, \delta_1 > 0$ big enough