Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 6

Last Lecture:

- Basic notions of functional analysis
- Intro to weak formulations

Today's lecture:

- Weak formulations:
  - different boundary conditions
  - general elliptic differential operators
  - well-posedness by Lax-Milgram Lemma
- The method of Galerkin

Computer practical

- FD for more general two-points BVPs

(Q. Chap. 3)

Examples on weak formulations

E1: Poisson with Neumann boundary conditions (b.c)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ \dfrac{\partial u}{\partial \bar{n}} = g & \text{on } \partial\Omega \end{cases}$$

$\bar{n}$ is unit outward pointing normal at $\partial\Omega$

Consider $V = H^1(\Omega)$, test, integrate:

$\forall v \in V,$

$$(f, v) = (-\Delta u, v) \qquad (u,v) := \int_\Omega u v$$

$$\overset{\text{parts}}{=} (\nabla u, \nabla v) - \int_{\partial\Omega} \frac{\partial u}{\partial \underline{n}} v$$

$$\overset{\text{(use b.c.)}}{=} (\nabla u, \nabla v) - \int_{\partial\Omega} g v$$

$\Rightarrow$ weak form. : find $u \in H^1(\Omega)$:

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v}_{\overset{!!}{=} a(u,v)} = \underbrace{\int_\Omega f v + \int_{\partial\Omega} g v}_{F(v)} \quad \forall v \in H^1(\Omega)$$

note : Neumann cond. is inforced "naturally" within the weak form (NATURAL b.c.) as opposed to the Dirichlet condition which we force into the space (ESSENTIAL b.c.)

E2: Poisson with non-homogeneous Dirichlet b.c.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases}$$

Question: V ? / $g \in$?

<u>Trace Theorem</u>: Let $\Omega \subset \mathbb{R}^d$ bounded, open, Lipshitz or polygonal.

Then, $\exists\ \gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$

"trace operator" (extension of trace for continuous functions)

"space of traces"

of traces which is linear and continuous. that is

(1) $\gamma_0 v = v|_{\partial\Omega}$ $\quad \forall v \in H^1(\Omega) \cap C^0(\bar{\Omega})$

(2) $\exists\ C^* > 0 : \|\gamma_0 v\|_{L^2(\partial\Omega)} \leq C^* \|v\|_{H^1(\Omega)}$ $\quad \forall v \in H^1(\Omega)$

continuity

From Trace theorem,

given $g \in H^{1/2}(\partial\Omega)$, let

$\tilde{g} \in H^1(\Omega) : \gamma_0(\tilde{g}) = g$

and write: Find $\hat{u} \in H^1_0(\Omega)$:

$$-\Delta \hat{u} = f + \Delta \tilde{g} \quad \text{in } \Omega$$

and $u = \hat{u} + \tilde{g}$.

Weak formulation:

Find $\hat{u} \in H^1_0(\Omega)$: $\displaystyle\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v - \int_\Omega \nabla \tilde{g} \cdot \nabla v$

$\forall v \in H^1_0(\Omega)$

General Elliptic Operator case

Consider homog. Dirichlet b.v.p:

$$\begin{cases} Lu = -\sum_{i,j=1}^{d} D_i\big(\underbrace{a_{ij}(x)}_{A(x)} D_j u\big) + \sum_{i=1}^{d} D_i(b_i u) + cu = f \quad \underline{b} = (b_1, \dots, b_d) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega . \end{cases}$$

- Test and integrate by parts :

$$\underbrace{\int_\Omega A \nabla u \cdot \nabla v - \int_\Omega u\, \underline{b} \cdot \nabla v + \int_\Omega c u v}_{a(u,v)}$$

$$\underbrace{-\int_{\partial\Omega} (A\nabla u \cdot \underline{n}) v}_{=0} + \underbrace{\int_{\partial\Omega} \underline{b}\cdot\underline{n}\, u v}_{=0} = \underbrace{\int_\Omega f v}_{F(v)} \qquad \forall v \in H_0^1(\Omega) = V$$

weak form: Find $u \in V = H_0^1(\Omega)$: $a(u,v) = F(v)$ $\forall v \in V$.

(WP)

examples of other b.c.

- Neumann b.c., let $\frac{\partial u}{\partial n_L} := A\nabla u \cdot n - (b\cdot n) u$ ← conormal derivative

$$\begin{cases} Lu = f \\ \frac{\partial u}{\partial n_L} = g \end{cases}$$

→ weak form: $u \in H^1(\Omega)$: $a(u,v) = F(v) + \int_\Omega g v$ $\forall v \in H^1(\Omega)$

- Propose and derive weak form for
  - $\frac{\partial u}{\partial n_L} + \mu u = g$ ROBIN b.c.
  - $\begin{cases} u = g_D & \text{on } \partial\Omega_D \\ \frac{\partial u}{\partial n_L} = g_N & \text{on } \partial\Omega_N \end{cases}$ MIXED b.c.

    where $\partial\Omega_D \cup \partial\Omega_N = \partial\Omega$

Analysis of (WP)

Def: $\mathcal{A}$ is coercive in V if

$\exists\, \alpha > 0 : \mathcal{A}(v,v) \geq \alpha \|v\|_V \quad \forall v \in V$

and continuous if

$\exists\, \gamma > 0 : \mathcal{A}(w,v) \leq \gamma \|w\|_V \|v\|_V \quad \forall w,v \in V$

- elliptic, by def, means coercive in 2nd order term

A sufficient cond. for coercivity:

$$\mathcal{A}(v,v) = \int_\Omega A \nabla v \cdot \nabla v + \int_\Omega v\, \underline{b} \cdot \nabla v + \int_\Omega c\, v^2$$

by assumption of ellipticity $\quad \geq \qquad\qquad\qquad\qquad \|$

$$\alpha_0 \|\nabla v\|_{L^2(\Omega)} \qquad \frac{1}{2}\int_\Omega \underline{b}\cdot\nabla v^2 = -\frac{1}{2}\int_\Omega (\nabla\cdot\underline{b})\, v^2$$

$$\geq \alpha_0 \|\nabla v\|_{L^2(\Omega)} + \int_\Omega \left(c - \frac{1}{2}\nabla\cdot\underline{b}\right) v^2$$

assume $\quad c - \frac{1}{2}\nabla\cdot\underline{b} \geq 0$

$$\Rightarrow \qquad \geq \alpha_0 \|\nabla v\|_{L^2(\Omega)}$$

by Poincarè $\quad \|v\|^2_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|^2_{L^2(\Omega)}$

$$\Rightarrow \qquad \geq \frac{\alpha_0}{C_\Omega} \|v\|^2_{L^2(\Omega)}$$

$$\Rightarrow \begin{cases} \frac{1}{\alpha_0} A(v,v) \geq \|\nabla v\|^2_{L^2(\Omega)} \\ \frac{C_\Omega}{\alpha_0} A(v,v) \geq \|v\|^2_{L^2(\Omega)} \end{cases}$$

add them up $\Rightarrow$

$$\Omega(v,v) \geq \frac{\alpha_0}{1+C_\Omega} \|v\|^2_{H^1(\Omega)}$$

$$\|v\|^2_{H^1(\Omega)} = \int_\Omega |\nabla v|^2 + \int_\Omega v^2$$

hence $c - \frac{1}{2}\nabla\cdot \underline{b} \geq 0$ is sufficient (but not necessary!) for coercivity!

<u>Recall</u> symmetric case we had well posedness by Riesz and equivalence with minimization problem: find $u \in V$:

$$J(u) \leq J(v) \quad \forall v \in V,$$

$$J(v) := \frac{1}{2} \underbrace{A}_{\text{bilinear, coerc, cont}} (v, v) - \underbrace{F}_{\in V'}(v)$$

general case, need proper generalisation

Theorem: Let $(V,(\cdot,\cdot))$ be a Hilbert space $\left(\|v\|=(v,v)^{1/2}\right)$

- $A(\cdot,\cdot)$ bilinear form with
  - continuous : $\exists \gamma>0: A(w,v)\le \gamma\|w\|\,\|v\|$
  - coercive : $\exists \alpha>0: A(v,v)\ge \alpha\|v\|^2$
- $F(\cdot)$ linear functional
  - continuous : $\exists C_0>0: |F(v)|\le C_0\|v\|$

Then $\exists!\, u\in V: A(u,v)=F(v)\ \forall v\in V$

and we have the energy estimate

$$\|u\|\le \frac{1}{\alpha}\|F\|_{V'}$$

Proof (LT Appendix)

based on Riesz + Closed range Th.

STEP 1: rewrite problem:

- Or $F \in V' \Rightarrow \exists!\, r \in V : F(v) = (r, v)$

Riesz

- $\forall w,\ \mathcal{A}(w, \cdot) \in V'$

$\Rightarrow \exists!\, A(w) \in V : \mathcal{A}(w, v) = (A(w), v)$

plus, the dependence $w \longrightarrow A(w)$ linear + bounded

Rewrite problem:

$\forall r \in V,\ \exists!\, u \in V : (A(u), v) = (r, v) \quad \forall v \in V$

$$\Updownarrow$$

$$A(u) = r$$

$$\Updownarrow$$

$A : V \to V$ is bijective

Proof A is bijective:

using coercivity of a:

$\forall v \in V$, $\alpha \|v\|^2 \leq a(v,v)$

$= (A(v), v) \leq \|Av\| \|v\|$

$$\Rightarrow \boxed{\alpha \|v\| \leq \|Av\|} \quad (*)$$

- A is injective

if $Av = 0$ then by $(*)$ $v = 0$

$(\Rightarrow \operatorname{Ker} A = \{0\})$

- A is surjective $\iff$

$$\begin{cases} R(A) = \{w \in V : w = A(v)\ \exists v \in V\} \text{ is closed} \\ R(A)^{\perp} = \{0\} \end{cases}$$

$\Downarrow$

$\forall w \in V,\ w = \underset{R(A)}{\underset{\cap}{z}} + \underset{R(A)^{\perp}}{\underset{\cap}{w_0}}$

- $R(A)$ is closed. Take $\{A(v_n)\}_n \longrightarrow w \in V$

thesis: $w \in R(A)$

sequence $\{A(v_n)\}_n$ is Cauchy $\Rightarrow$

$$\Rightarrow \| A(v_n) - A(v_m) \| \longrightarrow 0$$
$$\overset{!}{=} \| A(v_n - v_m) \|$$

by (*) $\geq \alpha \|v_n - v_m\|$ that is $\{v_n\}$ is Cauchy. But V is Hilbert $\Rightarrow$

$v_n \to v$ and $\boxed{Av = w}$ i.e. $w \in R(A)$

- $R(A)^{\perp} = \{0\}$: suppose $w \in R(A)^{\perp}$:

$\forall v \in V \Rightarrow (w, A(v)) = 0$
$\overset{!}{=} (A(v), w) = a(v, w)$

with $v := w$ gives

$$0 = \mathcal{A}(w, w) \geq \alpha \|w\|^2$$

$$\Rightarrow w = 0$$

Proof of stability bound:

test WP with $v = u$, gives

$$\alpha \|u\|^2 \underset{\text{coerc.}}{\leq} \mathcal{A}(u, u) = F(u) \Rightarrow \|u\| \leq \frac{1}{\alpha} \frac{F(u)}{\|u\|}$$

$$\leq \frac{1}{\alpha} \|F\|_{V'}$$

$\square$

The Galerkin Method (Q Chap. 4)

Idea: restrict and solve WP in finite-dim subspaces, say $V_n < V$:

($n = \dim V_n$)

GM: Given $V_n < V$, find $u_n \in V_n$:

$$\mathcal{A}(u_n, v) = F(v) \quad \forall v \in V_n$$

Justification: suppose H is separable, then we can define $V_n = \text{span}\{\varphi_1, \dots, \varphi_n\}$

Reasonable to expect that

$$u_n \xrightarrow{n \to +\infty} u$$

For instance, symmetric case, the problem in V_n is minimisation over a smaller space

(Ritz-Galerkin method)

Lax-Milgram applies also in V_n, so $\forall V_n$, $\exists!\, u_n$ solution of (GM) and $\|u_n\|_V \le \frac{1}{\alpha} \|F\|_{V'}$. STABILITY

Algebric view-point

- given $\{\varphi_i\}_{i=1}^n$ basis of V_n

GM is equivalent to:

1) $u_n(x) = \sum_{j=1}^{n} U_j \varphi_j(x)$

2) find $U = \{U_j\}_{j=1}^n \in \mathbb{R}^n$:

$$\sum_{j=1}^{n} U_j \underbrace{a(\varphi_j, \varphi_i)}_{A} = \underbrace{F(\varphi_i)}_{F} \quad \forall i = 1, \dots, n$$

$$\Updownarrow$$

$$\boxed{AU = F}$$

note: (proof ex or look Quarteroni)

- a coerive $\Rightarrow$ A pos def $\Rightarrow$ invertible $\Downarrow$ $\exists!$ sol
- a symm $\Rightarrow$ A sym.

In sym case,

$$\text{GM} \iff AU = F$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$J(u_h) \le J(v_h) \quad \forall v_h \in V_h$ $\qquad$ U minimizes $\phi(V) = \frac{1}{2} V^T A V - V^T F$

Questions: Galerkin sol converges to sol?
How good is it?

CONSISTENCY :

Galerkin orthogonality: $a(u-u_n, v)=0 \quad \forall v \in V_n$ STRONG CONSISTENCY

proof : $\begin{cases} a(u_n, v) = F(v) \quad \forall v \in V_n & \text{by GM} \\ a(u, v) = F(v) \quad \forall v \in V_n & \text{by WP} \\ \quad \text{(as true in } V) & \end{cases}$

and take difference

Lemma of Cèa (quasi-optimality property):
Under assumptions of LM, and for $V_n < V$
given $u \in V$ sol WP, $u_n \in V_n$ sol GM, we have

$$\|u-u_n\|_V \leq \frac{\gamma}{\alpha} \inf_{v_n \in V_n} \|u-v_n\|$$

cont coerc
constants

Convergence follows if

$$\lim_{n \to +\infty} \inf_{v_n \in V_n} \|v-v_n\| = 0$$