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Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 4

Last Lecture:

- Divided differences
- Elliptic PDEs
- General DMP result
- Applications: more general two-points BVPs, multi-D problems

Today's lecture:

- FD on non-uniform meshes
- FD on general domains
- Basic notions of functional analysis
- Intro to weak formulations

Computer practical

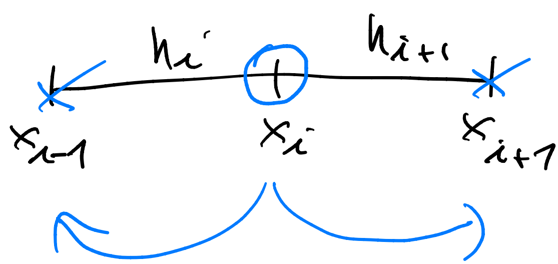
- Divided differences and FD for two-points BVPs

(see online notes Morton-Rogers)

$$\begin{cases} -u'' = f & \text{on } (a,b) \\ u(a) = 0 = u(b) \end{cases}$$

fix h_i, $i = 1, \dots, N$ such that $\sum_i h_i = b - a$

$$x_i := x_{i-1} + h_i$$



Method of indetermined coefficients:

determine coeffs $\alpha, \beta, \gamma \in \mathbb{R}$ in

$$L := \alpha u_{i+1} + \beta u_i + \gamma u_{i-1}$$

so as to obtain as high as possible approx of $u''(x_i) =: u_i''$ $(u_i = u(x_i))$

expand by x_i

$$L = \alpha\left(u_i + h_{i+1} u_i' + \frac{h_{i+1}^2}{2} u_i'' + \frac{h_{i+1}^3}{6} u'''(\xi_{i+1})\right) \qquad \xi_{i+1} \in [x_i, x_{i+1}]$$

if $h_{i+1} = h_i \Rightarrow$ || and gain 1 order

$$+ \beta u_i$$

$$+ \gamma\left(u_i - h_i u_i' + \frac{h_i^2}{2} u_i'' - \frac{h_i^3}{6} u'''(\xi_i)\right) \qquad \xi_i \in [x_{i-1}, x_i]$$

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha h_{i+1} = \gamma h_i \\ \alpha \frac{h_{i+1}^2}{2} + \gamma \frac{h_i^2}{2} = 1 \end{cases} \Rightarrow \begin{cases} \beta = -\alpha - \gamma \\ \alpha = \frac{h_i}{h_{i+1}} \gamma \\ \frac{h_i h_{i+1}}{2} \gamma + \frac{h_i^2}{2} \gamma = 1 \end{cases}$$

$$
\begin{cases}
\beta = -\dfrac{2\,\cancel{(h_i + h_{i+1})}}{h_i h_{i+1}\,\cancel{(h_i + h_{i+1})}} \\[2ex]
\alpha = \dfrac{2}{h_{i+1}(h_i + h_{i+1})} \\[2ex]
\gamma = \dfrac{2}{h_i(h_i + h_{i+1})}
\end{cases}
$$

FD scheme: Find $U = \{U_i\}_{i=0}^{N}$:

$$
\begin{cases}
U_0 = 0 \\[1ex]
-\left(\dfrac{2}{h_{i+1}(h_i + h_{i+1})} U_{i+1} - \dfrac{2}{h_i h_{i+1}} U_i + \dfrac{2}{h_i(h_i + h_{i+1})}\right) = f_i \\[2ex]
U_N = 0
\end{cases}
\qquad i = 1, \dots, N-1
$$

- <u>Truncation error</u>

$$
|T_i| \le \frac{2}{3} h_{max} \|u'''\|_C + \frac{1}{12} \frac{h_{max}^3}{h_{min}} \|u^{IV}\|_C
$$

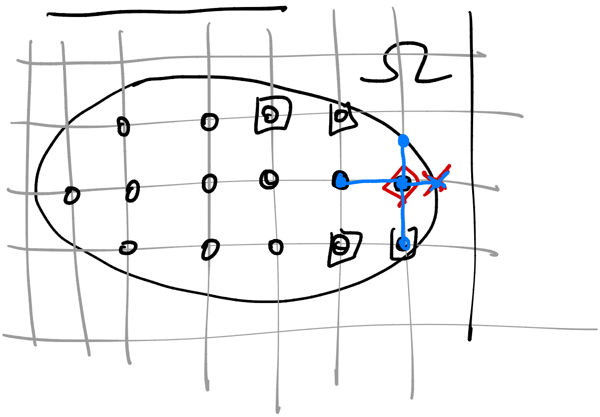
- error is $O(h_{max})$

- but need $\frac{h_{max}^3}{h_{min}} \to 0$ for convergence

FD for general domains

$\Omega \subset \mathbb{R}^2$ open, bounded domain, convex

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$



Ω_h = internal nodes with only internal neighboring nodes

ω_h = set of internal nodes with external neighbors

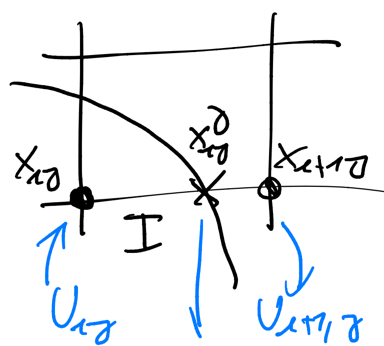
Shortley-Weller scheme: use the nonuniform grid scheme at near-boundary points ω_h

- Truncation error is $\begin{cases} \text{quadratic of nodes in } \Omega_h \\ \text{linear} \quad " \quad " \quad " \ \omega_h \end{cases}$ for second central scheme
- however by the generalised DMP analysis it is possible to use two truncation error functions (comparison functions) to show that

$$|u_{ij} - U_{ij}| \leq \underset{\text{sum}}{\overset{}{\frac{d^2}{96}}} h^2 \|u^{IV}\|_C$$

Hence, local lower-order truncation at boundary does not necessarily reduce the overall order!

- From LT: different approach based on interpolating data on interval I cut by boundary arrive to similar scheme



from data at x^{∂}_{i0} to $x_{i+1,j}$ $\Downarrow$ $g(x^{\partial}_{i0})$

$$U_{ij} = \alpha U_{i+1,j} + (1-\alpha)\, g(\underline{x}^{\partial}_{i0})$$

where α related to distance between the points