

Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 2

Quar teroni (Q)
Lorsson - Thorée (L.T.)

Last Lecture:

- Three different points of view: FD, FV, and FEM

Today's lecture:

- Poisson problem in strong form: Well-posedness
- Numerical methods and the Lax-Richtmyer equivalence theorem
- Finite Difference discretisation: consistency, stability, convergence

(1)

(2)

(3)

+ See also notes online

① Poisson problem strong form:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subseteq \mathbb{R}^d \\ u = g & \text{on } \partial\Omega \end{cases}$$

Boundary
of Ω

Def: An harmonic function in Ω is
a function $u: \Omega \rightarrow \mathbb{R}$: $\begin{cases} u \in C^2(\Omega) \\ \Delta u = 0 \end{cases}$

Theorem (mean-value formula):

[see Evans, p. 25]

$u \in \text{harmonic}$ iff

$$u(x) = \frac{1}{\omega(B_r(x))} \int_{B_r(x)} u \, ds \quad \forall x \in \Omega$$

and we also have

$$u(x) = \frac{1}{\omega(B_r(x))} \int_{B_r(x)} u \, dv$$

for each ball $B_r(x) \subset \Omega$.

Theorem (Strong Maximum Principle)

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ harmonic
in Ω ,

Then

(i) $\max_{\bar{\Omega}} u = \max_{\Omega} u$

(ii) If Ω connected and
 $\exists x_0 \in \Omega$ s.t.

$$u(x_0) = \max_{\bar{\Omega}} u$$

then $u \equiv \text{const}$ in Ω

Proof : (ii) suppose $x_0 \in \Omega$:

$$u(x_0) = M := \max_{\bar{\Omega}} u$$

then

$$M = u(x_0) = \int_{B_{x_0}(r)} u \, dV$$

$$\leq M$$

$$\Rightarrow u \geq M \quad \text{in } B_{x_0}(r)$$

Hence $\{x \in \Omega : u(x) = M\} = S$

- open
- relatively closed
in Ω

$\Rightarrow S = \Omega$ when

Ω is connected.

(i) immediate from (ii)

Corollary: (positiveness)

If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

With Ω connected, then

• If $g \geq 0 \Rightarrow u \geq 0$ in Ω

• If $\begin{cases} g \geq 0 \\ g(x) > 0 \exists x \end{cases} \Rightarrow u > 0$ in Ω

Corollary (uniqueness)

Let $g \in C(\partial\Omega)$, $f \in C(\Omega)$.

Then there exist at most 1 solution of the Poisson prob:

$$(P) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Proof: ex

Theorem: Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$
sol of (P). Then,

$$\|u\|_{C(\bar{\Omega})} \leq \|g\|_{C(\partial\Omega)} + C \|f\|_{C(\Omega)}$$

$$\left(\|w\|_{C(\omega)} := \sup_{x \in \omega} |w(x)| \right)$$

Corollary: If u, \tilde{u} are sol. of (P) with data f, g and \tilde{f}, \tilde{g} , respectively, then

$$\|u - \tilde{u}\| \leq \|g - \tilde{g}\|_{C(\omega)} + C \|f - \tilde{f}\|_{C(\omega)}$$

Numerical discretisations (Q. 1.2)

Let $P(u, g) = 0$ the PDE problem $\begin{cases} u = u(g) \\ g \end{cases}$ the solution data

$P_H(u_H, g_H) = 0$ the

approximate "numerical" PDE problem , w.r.t. H = disc. parameter, e.g. dimension of discrete problem

We say that the numerical method is

- convergent if $\|u - u_H\| \rightarrow 0$
or $H \rightarrow +\infty$
↑
some
appropriate
norm

- consistent if

$$P_H(u, g) \rightarrow 0 \Leftrightarrow H \rightarrow \infty$$

(equivalently, $P_H(u, \delta) - P(u, \delta) \geq 0$

$\Leftrightarrow H \rightarrow \infty$

strongly consistent if

$$P_H(u, \delta) = 0$$

• stable if

$\forall \epsilon > 0 \exists S_\epsilon > 0 : \text{if } \| \delta g_H \| < S$

$$\Rightarrow \| \delta u_N \| \leq \epsilon \quad \forall H$$

where $u_H + \delta u_H$ is the
numerical solution of the
perturbed problem

$$P_H(u_H + \delta u_H, \delta g_H) = 0$$

Theorem: (Lox-Richtmyer equivalence).

A discretization $P_\pi(u_\pi, g_\pi) = 0$ to a PDE problem

$P(u, g) = 0$ is convergent \Leftrightarrow it is consistent + stable

Note: analogue to Dahlquist equivalence

Consider Poisson in 1D:

$$\begin{cases} -u'' = f & \text{in } (0, 1) \\ u(0) = 0 = u(1) \end{cases}$$

and its FD discretization on

$$x_i = ih \quad i=0, 1, \dots, N \text{ with } h = 1/N$$

$$\begin{cases} -\delta_h^2 u_i = f_i & i=1, \dots, N-1 \\ u_0 = 0 \\ u_N = 0 \end{cases}$$

$$\sum_h u(x) = \sum_h \left(\sum_h u(x) \right) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

lemma: if $u \in C^4([0, 1])$, $h > 0$, $x \in [h, 1-h]$

$$u''(x) - \sum_h u(x) = \frac{h^2}{24} (u^{(IV)}(\xi_1) + u^{(IV)}(\xi_2))$$

Taylor $\xi_1 \in [x-h, x]$, $\xi_2 \in [x, x+h]$

$\sum_h u(x)$ is $O(h^2)$

error term

$$|u''(x) - \sum_h u(x)| \leq \frac{h^2}{12} \|u^{(IV)}\|_{C[0,1]}$$

A priori only if

Def: For \mathcal{S}_h a finite difference operator,

$$T(x) = \mathcal{S}u(x) - \mathcal{S}_h u(x)$$

\uparrow diff. operator

$$(e.g. \mathcal{S}u = -u''; \mathcal{S}_h = \mathcal{S}_h^2)$$

is the TRUNCATION ERROR

$$\begin{aligned} & (\cancel{f} - \mathcal{S}_h u) - (\cancel{f} - \mathcal{S}u) = \mathcal{S}u - \mathcal{S}_h u \\ & P_H(u, f) - P(u, f) \\ & = f(x) - \mathcal{S}_h u(x) \end{aligned}$$

$$\begin{aligned}
 T(x) &= f(x) - S_h u(x) \\
 &\leq f(x) - S_h^2 u(x) \\
 &= \cancel{f(x)} - \left[u''(x) + \frac{h^2}{24} (u^{(4)}(\xi_1) + u^{(4)}(\xi_2)) \right] \\
 \Rightarrow |T(x)| &\leq \frac{h^2}{12} \|u^{(4)}\|_{C(0,1)}
 \end{aligned}$$

thus $T(x) \xrightarrow{h \rightarrow 0} 0$ and it is $O(h^2)$.

Note: consistency does not imply that the discrete solution converges. Stability is also required.

See Richtmyer and Morton 1967 for examples of disc. solutions which get farther and farther from the

exact solution or $h \rightarrow 0$

Stability comes from

Theorem (discrete max principle)

- DMP

let $V = \{V_i\}_{i=0}^H$.

If $L_h V_i \leq 0 \quad \forall i=1, \dots, H-1$,

then $\max_i V_i = \max \{V_0, V_H\}$

Proof: by contradiction, let

$$V_n = \max_{1 \leq i \leq H-1} V_i, \quad V_n > V_0 \\ \text{or/and} \\ V_n > V_H$$

then

$$0 \leq -h^2 \sum_n V_n = V_{n+1} - 2V_n + V_{n-1}$$

{
hypothesis

$$\Rightarrow V_n \leq \frac{V_{n+1} + V_{n-1}}{2} \leq V_n$$

V_n max

$$\Rightarrow V_n = \frac{V_{n+1} + V_{n-1}}{2} \Rightarrow V_n = V_{n-1} = V_{n+1}$$

repetition of argument $\Rightarrow V_n = V_i \forall i$
contradicting $V_n > V_0$ or $V_n \downarrow$

Lemma (stability) : Let $S_n = S_n^2$,
then for any $V = \{V_i\}_{i=0}^H$,

$$\|V\|_\infty \leq \max\{|V_0|, |V_H|\} + C \|S_n V\|_\infty$$

$= \max_i |V_i|$

Proof: by "comparison function"

$$w(x) = x - x^2 = \frac{1}{4} - \frac{(x-\frac{1}{2})^2}{4} \leq \frac{1}{4}$$
$$w(x) = x - x^2, \quad w_i := w(x_i)$$

$$\sum_n w_i = \sum_n w(x_i^{=ih})$$

$$= \frac{\cancel{(i+1)}h - \cancel{i}h + \cancel{(i-1)}h}{h^2} + \frac{(i+1)^2 h^2 - 2i^2 h^2 + (i-1)^2 h^2}{h^2}$$
$$= 2$$

So, defining

$$\tilde{V}_i^\pm := \pm V_i - \frac{1}{2} |\sum_n V|_\infty w_i$$

we have

$$\sum_n \tilde{V}_i^\pm = \pm \sum_n V_i - \frac{1}{2} |\sum_n V|_\infty \sum_n w_i$$

||
2

$$\|\mathcal{L}_h V\|_\infty$$

$$\leq 0$$

$$(DMP) \quad \tilde{V}_i^\pm \leq \max \left\{ \pm V_0, \pm V_H \right\}$$

$$= \pm V_i - \frac{1}{2} \|\mathcal{L}_h V\|_\infty w_i$$

$$\Rightarrow \pm V_i \leq \max \left\{ |V_0|, |V_H| \right\} + \frac{1}{2} \|\mathcal{L}_h V\|_\infty$$

□

$$w_i \leq \frac{1}{4}$$

$$\Rightarrow \text{result with } c = \frac{1}{8}$$

Theorem : (error estimate)

$\forall h > 0, \exists!$ discrete

solution U of the
FD scheme and

$$\max_i |u(x_i) - v_i| \leq C h^2 \|u^{(4)}\|_{C(0,1)}$$

Proof :- uniqueness by contradiction
- existence by problem being on
finite dim
- convergence : $e_i = u(x_i) - v_i$

$$\begin{aligned} \textcircled{1} \text{ stability } |e_i| &\leq \max\{|e_0|, |e_H|\} \\ &+ C \|f_n e\|_\infty \\ &= C \|f_n e\|_\infty \end{aligned}$$

because $e_0 = e_H = 0$

$$\textcircled{2} f_n e_i = f_n u(x_i) - f_n v_i$$

$$\text{by FD} = \sum u(x_i) - f_i$$

$$\begin{aligned}\text{by PDE} &\stackrel{!}{=} \sum u(x_i) - \sum u(x_i) \\ &\stackrel{!}{=} T(x_i)\end{aligned}$$

consistency

$$\Rightarrow |\sum \epsilon_i| \leq c h^2 \|u\|_{C^4(0,1)}$$