Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 5

Last Lecture:
- FD on non-uniform meshes
- FD on general domains

Today's lecture:

- Basic notions of functional analysis
- Intro to weak formulations

Computer practical
- FD for more general two-points BVPs

also, e.g. Quarteroni-Valli, Larson-Thomée

Basic notions on function spaces (Q. Chap 2)

Motivation: recall model problem for Poisson:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

towards weak formulation, we test with some v and integrate:

$$\stackrel{?}{=} \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \qquad \forall v \in ?$$

which space?

$= (u,v)_1$

- $C^k(\bar\Omega) = \{ f : \bar\Omega \to \mathbb{R} : f \text{ is } k\text{-time cont. diff. in } \Omega \}$

  ($k=0$ $C(\bar\Omega)$ cont. functions)

- $C^k_0(\Omega) = \{ f \in C^k(\Omega) : f \text{ has compact support in } \Omega \}$

we have:

- $C^k(\bar\Omega)$ is complete (Banach) w.r.t.

$$\|v\|_{C^k(\bar\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha v\|_{C(\bar\Omega)}$$

where $D^\alpha v = \dfrac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$, α = multi index

- but, for instance, $C^1_0(\Omega)$ is <u>not</u> complete w.r.t.

$$\|v\| = \int_\Omega |\nabla u|^2$$

(norm induced by I.P. $(u,v)_1 = \int \nabla u \cdot \nabla v$)

<u>Def</u>

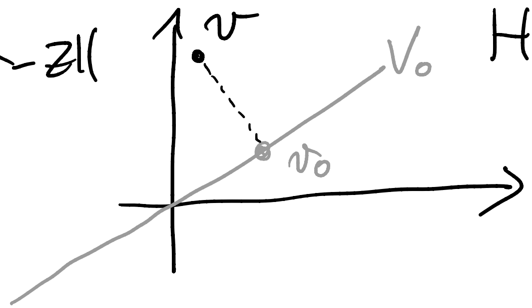
Hilbert space: $(H, (\cdot,\cdot))$

- H real linear space
- $(\cdot,\cdot)$ scalar product

- H complete w.r.t. $\|\cdot\|_H = (\cdot,\cdot)^{1/2}$

Properties, e.g.:

- $|(v,w)| \leq \|v\|\,\|w\|$ (Schwarz)
- Let H_0 closed linear subspace, then $\forall v \in H$, $v = v_0 + w$ $\exists!\, v_0 \in H_0$ and $w \in V$: $(w,z) = 0$ $\forall z \in H_0$, and $\underbrace{\|w\|}_{=\|v-v_0\|} = \min_{z \in V_0} \|v - z\|$



Crucial examples

L_p-spaces: for $1 \leq p \leq +\infty$

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ Lebesgue measurable} : \begin{cases} \int_\Omega |f|^p < \infty & p < +\infty \\ \sup_x |v(x)| < \infty \end{cases} \right.$$

normed with $\begin{cases} \|v\|_{L^p(\Omega)} = \left(\int_\Omega |v|^p\right)^{1/p} & p < +\infty \\ \|v\|_{L^\infty(\Omega)} = \sup_x |v(x)| & p = \infty \end{cases}$

Hölder ineq: given p, let $p' : \frac{1}{p} + \frac{1}{p'} = 1$ (with $p' = \infty$ if $p=1$), then

$$\left| \int_\Omega v w \, dx \right| \leq \|v\|_{L^p(\Omega)} \|w\|_{L^{p'}(\Omega)}$$

In particular, for $p = 2 = p'$, we have

$$\left| \int_\Omega u v \right| \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

(Schwarz $\quad \forall u, v \in L^2(\Omega)$)

L^2 is I.P. space (actually, Hilbert) w.r.t. $(u, v)_0 = \int_\Omega u v$

(To be made precise soon...) From $L^2(\Omega)$, we define $H^1(\Omega)$ and $H^1_0(\Omega)$

L^2 with 1st derivative in L^2

then $\forall f \in L^2(\Omega)$, can def. weak problem

find $u \in H^1_0(\Omega)$: $\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v$

$= (u, v)_1 \qquad (f, v)_0 =$

but, this indeed only valid for $f \in L^2$. We'd like to include also $\delta_x \notin L^2$ (Dirac)

Dual space: Let $(V, \|\cdot\|)$ normed space.

$V' = \mathcal{L}(V; \mathbb{R})$ is the DUAL space of V:

- space of linear continuous functionals from $V \to \mathbb{R}$

  $\Updownarrow$ $L \in V' \Rightarrow L$ bounded: $|L(v)| \leq C \|v\|_V$

- V' is normed w.r.t. operator norm:

$$\|L\|_{V'} := \sup_{v \in V \setminus \{0\}} \frac{|L(v)|}{\|v\|}$$

- $(V', \|\cdot\|_{V'})$ is Banach

- $\langle \cdot, \cdot \rangle : V' \times V \to \mathbb{R}$ bilinear form

$$\langle L, v \rangle := L(v)$$

DUALITY PAIRING

Theorem (Riesz-Frèchet) Characterisation of the dual of an Hilbert space:

Let V Hilbert, $L \in V'$. Then

$$\exists!\ u \in V : \underset{= \langle L, v \rangle}{L(v)} = (u,v)_V \quad \forall v \in V.$$

Also $\|L\|_{V'} = \|u\|_V$.

example: $V = L^2(\Omega)$, $f \in (L^2(\Omega))'$ $\exists!\ u \in L^2(\Omega)$:

$$\underset{= \langle f, v \rangle}{f(v)} = (u, v)_0 = \int_\Omega u\, v$$

With this in mind, using concept of distributions we shall be able to give meaning in general to the Poisson problem

Distribution: Given $C_0^\infty(\Omega)$, named $\mathcal{D}(\Omega)$ when provided with the topology given by uniform convergence

We Define space $\mathcal{D}'(\Omega)$ = SPACE OF DISTRIBUTIONS or dual of $\mathcal{D}(\Omega)$ so made $T : \mathcal{D}(\Omega) \to \mathbb{R}$ linear and cont.

$$\varphi \to T(\varphi) = \langle T, \varphi \rangle$$

We have,

- $L^p(\Omega) \subsetneq \mathcal{D}'(\Omega) \quad \forall\, p \in [1, +\infty]$

Indeed ($p = 2$) : $f \in L^2$ defines a distribution by $f \to \underbrace{\int_\Omega f \varphi}_{= L(\varphi)} \quad \forall \varphi \in \mathcal{D}(\Omega)$

or $\int_\Omega f\varphi \leq \|f\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \Rightarrow \varphi$ bounded

- δ = Dirac's delta is a distribution but is not representable by any $f \in L^p(\Omega)$ $\forall p$

Weak derivatives : For $T \in \mathcal{D}'(\Omega)$

we define the weak derivatives $D^\alpha T \in \mathcal{D}'(\Omega)$ as

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$

$\forall \alpha$ multiindex.

Hence

- weak derivatives are always defined
- but, for instance Heaviside function

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \in L^2_{loc}$$

(any interval)

but $DH = \delta \notin L_{loc}$

$\Rightarrow$ starting from L^2, weak derivatives may not be in L^2

$\Downarrow$

Sobolev spaces

$$W^{k,p}(\Omega) = \{ v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \ \forall |\alpha| \leq k \}$$

In particular, for $p=2$

$W^{k,2}(\Omega) = H^k(\Omega)$ which is Hilbert

w.r.t. $$(v,w)_k := \sum_{|\alpha| \leq k} (D^\alpha v, D^\alpha w)$$

note from def. $$\int_\Omega \frac{\partial u}{\partial x_i} \varphi \overset{\text{by def. of weak deriv.}}{=} -\int_\Omega u \frac{\partial \varphi}{\partial x_i}$$

$\Rightarrow$ integration by parts! In particular, if classical derivative exists $\equiv$ to weak derivative

Again, for us, crucial example is

$$H^1(\Omega) = \{ v \in L^2(\Omega) : \nabla v \in (L^2(\Omega))^d)$$

Hilbert w.r.t. $(u,v)_{H^1} = (u,v)_0 + \underbrace{(\nabla u, \nabla v)_0}_{= (u,v)_1}$

and norm $\|v\|_{H^1} = \left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right)^{1/2}$

$$H_0^1(\Omega) = \{ v \in H^1(\Omega) : v_{|\partial\Omega} = 0 \}$$

Hilbert with $\|v\|_{H_0^1} = (\nabla u, \nabla v)_0$

Same is true for

$$H_\Gamma^1(\Omega) = \{ v \in H^1(\Omega) : v_{|\Gamma} = 0 \}$$

whenever $\Gamma \subset \partial\Omega$ with $|\Gamma| \neq 0$

Theorem (Poincaré ineq.): $\Omega \subset \mathbb{R}^d$ $\begin{cases}\text{open,}\\ \text{bounded}\end{cases}$

$\Gamma \subset \partial\Omega$, Lipshitz cont., $\exists$

$C_\Omega > 0$: $\forall v \in H^1_\Gamma(\Omega)$

$$\int_\Omega v^2 dx \le C_\Omega \underbrace{\int_\Omega |\nabla v|^2 dx}_{|v|_{H^1(\Omega)}}$$

$\Rightarrow$ it is a norm for $H^1_\Gamma(\Omega)$:

$$\|v\|_{H^1_\Gamma(\Omega)} := \|\nabla v\|_{L^2} = |v|_{H^1(\Omega)}$$

Weak formulations

Consider, once again poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

recall :

- $\mathcal{A}(u,v) := \int_\Omega \nabla u \cdot \nabla v$ — symmetric bilinear form

- $f \in \mathcal{D}'(\Omega)$, $\langle f, v\rangle$ — linear functional

then, we have WEAK FORMULATION

find $u \in H^1_0(\Omega)$: $\mathcal{A}(u,v) \overset{(WP)}{=} \langle f, v\rangle$

In particular, $f \in L^2(\Omega)$ $\langle f, v\rangle = (f, v)$

Theorem : $u \in H^1_0(\Omega)$ solves weak problem (WP)

$\Longleftrightarrow$ $-\Delta u = f$ in $\mathcal{D}'(\Omega)$

<u>Proof</u>: "$\Leftarrow$" multiply $-\Delta u = f$ by $v \in H^1_0(\Omega)$ and integrate

$$\langle f, v\rangle = \langle -\Delta u, v\rangle = (\nabla u, \nabla v)_0 - \int_{\partial\Omega} (n \cdot \nabla u) \overset{0}{\overset{\shortparallel}{v}}$$
$$= (\nabla u, \nabla v)_0$$

"$\Rightarrow$" since $C^\infty_0(\Omega) \subset H^1_0(\Omega)$ can test the WP with any $v \in C^\infty_0(\Omega)$:

$$= \int_\Omega \nabla u \nabla v \overset{WP}{=} \int_\Omega f v = \langle f, v \rangle$$

$$= -\int_\Omega \Delta u \, v = \langle \underbrace{-\Delta u}_{\text{as a distribution}}, v \rangle$$

$$\forall v \in C_0^\infty$$

$$\Rightarrow -\Delta u = f \quad \text{in } \mathcal{D}'(\Omega)$$

$\square$

- If $u \in C^2(\Omega) \cap C_0(\bar{\Omega})$ is classical solution then u is weak solution
- Viceversa not true (weak problem is more general):

take again Poisson, with $f \in L^2(\Omega)$ and $\partial\Omega$ is at least C^2 or convex polygon

then it can be shown that

the solution to corresponding b.v.p. for Poisson $u \in H^2_0(\Omega)$

but still not classical !

For that, we need

$$f \in H^2(\Omega) \Rightarrow u \in H^4(\Omega) \hookrightarrow C^2$$

(in $d=2$!)

if $\partial\Omega \in C^4$

$$\Rightarrow u \in C^2(\Omega) \Rightarrow \text{classical}$$

from Sobolev Inequality :

for $\Omega \subset \mathbb{R}^d$ bounded smooth or polygonal

Then $H^k(\Omega) \subset C(\bar{\Omega})$ if $k > \frac{d}{2}$

and $\exists\, C(\Omega) : \|v\|_C \le C \|v\|_k$

In conclusion

1) Weak problem can deal with more general data

2) much easier to prove well-posedness (particularly for nonlinear problems!)

(Back to) wellposedness of WP:

Theorem: Let $(V, (\cdot,\cdot)_V)$ Hilbert,

$\mathcal{A} : V \times V \to \mathbb{R}$ symmetric bilinear form

which is continuous and coercive

$(\mathcal{A}(u,v) \leq \gamma \|u\|_V \|v\|_V)$ $(\mathcal{A}(v,v) \geq \alpha_0 \|v\|_V^2)$

$F : V \to \mathbb{R}$ continuous linear form

Then $\exists!\, u \in V : \mathcal{A}(u,v) = F(v) \quad \forall v \in V$

and $\|u\|_V \leq \frac{1}{\alpha_0} \|F\|_{V'}$

Proof: • For $\mathcal{A}$ we have

(coercivity $\Rightarrow$) positive + symmetry

$\Rightarrow \mathcal{A}(\cdot,\cdot)$ is I.P. in V

• $F \in V'$

By Riesz $\exists!\, u \in V : \mathcal{A}(u,v) = F(v)$

Moreover,

$$\alpha_0 \|u\|_V^2 \leq \mathcal{A}(u,u) = F(u) \leq \|F\|_{V'} \|u\|_V$$

$$\Rightarrow \|u\|_V \leq \frac{1}{\alpha_0} \|F\|_{V'} \qquad \square$$

Note: limited to symmetric bilinear forms. We shall cover non-symmetric problems using Lax-Milgram lemma.

Regularity of Poisson's weak solutions

Let $\Omega \subset \mathbb{R}^2$, $\partial\Omega$ smooth.

If $f \in H^s(\Omega)$, ($s=0$ means L^2)

then $u \in H^{s+2}(\Omega)$

and $\|u\|_{H^{s+2}(\Omega)} \leq C \|f\|_{H^s(\Omega)}$ (R)

shift

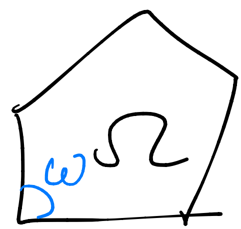
$$-\Delta u \qquad \|-\Delta u\|_{L^2}$$

For instance, $s=0$, then $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$

that is all derivatives controlled by u_{xx}, u_{xy}

- If $\partial\Omega$ non-smooth, then (R) does not necessarily hold, even for $s=0$.

For instance take $\partial\Omega$ polygonal



near corners,

$$u(r,\vartheta) = r^{\pi/\omega}\alpha(\vartheta) + \beta(r,\vartheta)$$

where ω = angle at corner

α, β smooth

If $\omega > \pi$ (nonconvex domain)

then $u \notin H^2(\Omega)$

while (R) holds if all angles are $\leq \pi$, that is for Ω convex.