25.05.2023

A is a linear operator, $A \in \mathcal{L}(V, W')$

Given $F \in W'$, find $u \in V$ s.t.

(1) $\qquad Au = F \quad$ in W'

- V is Banach
- W' is Banach and reflexive

For pb (1) to be well posed: $\exists \alpha$, s.t. $\forall F \in W'$

$\exists! u$ s.t. $\quad Au = F \quad$ and $\quad \|u\|_V \leq \frac{1}{\alpha} \|F\|_{W'}$

$$\Leftrightarrow$$

$$\|Au\| \geq \alpha \|u\| \quad *$$

$$\|Au\| \leq C \|u\| \quad \text{linearity}$$

1) A surjective
2) A injective
3) A bounding

$\Leftrightarrow$ closed range / open map theorem

$A : V \longrightarrow W' \qquad\qquad A \in \mathcal{L}(V, W')$

$A^T : W \longrightarrow V'$

$K := \ker(A) := \{ v \in V \text{ s.t. } Av = 0 \quad (\langle Av, w \rangle = 0 \ \forall w \in W) \}$

$H := \ker(A^T) := \{ w \in W \text{ s.t. } A^T w = 0 \quad (\langle v, A^T w \rangle = 0 \ \forall v \in V) \}$

Z is a subset of V, we call Z° (Annilator of Z)

$$Z^\circ := \{ f \in V' \text{ s.t. } {}_{V'}\langle f, z \rangle_V = 0 \quad \forall z \in Z \}$$

i) By linearity of A

$$\boxed{\ker(A) = \overline{\ker(A)}}$$

$v_n \in K$, Cauchy $\quad Av_n = 0 \ \forall n \quad v_n \to v \quad Av = 0$

ii) By linearity $\langle \cdot, \cdot \rangle$ (duality pairing

$$\boxed{Z^\circ = \overline{Z^\circ}}$$

iii) $$\boxed{\ker(A) = \operatorname{Im}(A^T)^\circ}$$

$$0 = \langle Av, w \rangle = \langle v, A^T w \rangle \qquad \forall v \in K, \forall w \in W$$

iv) $$\boxed{\ker(A^T) = \operatorname{Im}(A)^\circ}$$

$$0 = {}_{V'}\langle A^T w, v \rangle_V = {}_W\langle w, Av \rangle_{W'} \qquad \forall w \in H, \forall v \in V$$

v) $(Z^\circ)^\circ = Z \iff Z = \overline{Z}$

$$\boxed{\begin{aligned} \ker(A)^\circ &= \operatorname{Im}(A^T) \\ \ker(A^T)^\circ &= \operatorname{Im}(A) \end{aligned}} \qquad \begin{aligned} &\iff \operatorname{Im}(A^T) = \overline{\operatorname{Im}(A^T)} \\ &\iff \operatorname{Im}(A) = \overline{\operatorname{Im}(A)} \end{aligned}$$

Simple "illuminating" example

$$A : V \equiv H^1_0(\Omega) \longrightarrow L^2(\Omega)$$

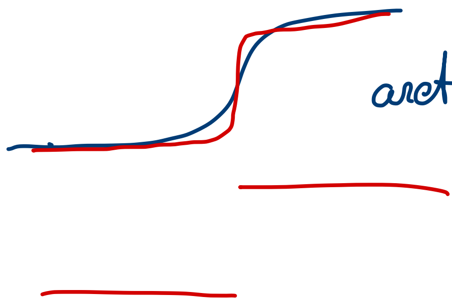
$Av = v$ canonical immersion

$H^1_0(\Omega)$ is dense in $L^2(\Omega)$ but <u>not</u> closed

$v_n \in H^1_0(\Omega)$, s.t. v_n is cauchy in $L^2(\Omega)$

$\Rightarrow \exists \ v = \lim_{n\to\infty} v_n \quad v \notin V, \quad v \in W$

$H^1_0(\Omega) \equiv \mathrm{Im}(A)$



arctan(nx)

$\forall v_n \ \exists \ w_n \in V$ s.t.

$A w_n = v_n \qquad (v_n \equiv w_n)$

$\ker(A) = \{0\} \qquad A^T : \overset{W}{L^2(\Omega)} \longrightarrow V' = H^{-1}(\Omega) \equiv (H^1_0(\Omega))'$

$$\langle A^T w, v \rangle := \int_\Omega w v = (w, v) = {}_{V'}\langle w, v \rangle_V$$

$\langle w, v \rangle = 0 \quad \forall w \in L^2 \Rightarrow v = 0$

$\ker(A^T) = \{0\}$

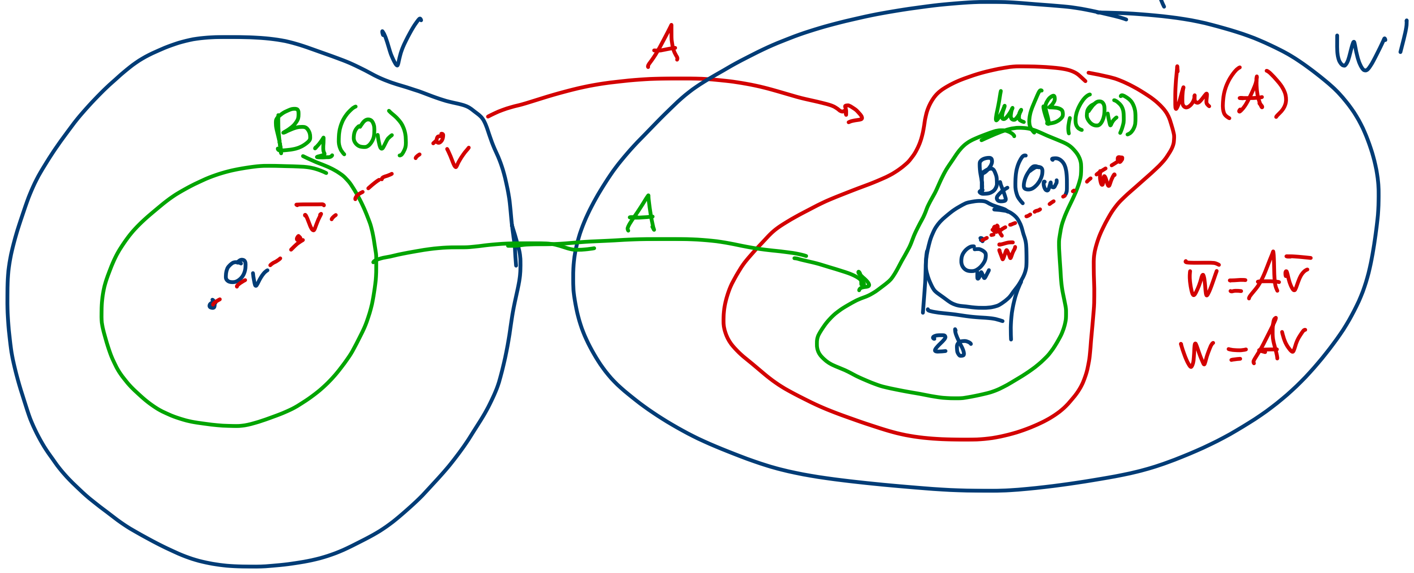
Theo Open mapping, closed range

$$\boxed{\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}} \Leftrightarrow \boxed{\begin{aligned} &\exists \alpha \mid \forall w \in \mathrm{Im}(A)\ \exists u \text{ s.t. } Au = w \\ &\|w\|_{W'} = \|Au\|_{W'} \geq \alpha \|u\|_V \end{aligned}}$$

Open Mapping: if A is surjective, it maps open sets to open sets.

$\Rightarrow$

$A \in \mathcal{L}(V, \mathrm{Im}(A))$ $\mathrm{Im}(A)$ is closed then it's a lin. subspace



$B_1(O_V) := v \in V \text{ s.t. } \|v\|_V < 1$

$\mathrm{Im}(B_1(O_V))$ is open, and $\mathrm{Im}(B_1(O_V)) \supset O_W$

$\exists \gamma > 0 \text{ s.t. } B_\gamma(O_W) \subset \mathrm{Im}(B_1(O_V))$

$\forall w \in \mathrm{Im}(A)$, $\bar{w} := \frac{\gamma}{2} \frac{w}{\|w\|_{W'}}$ $\quad \bar{w} \in B_\gamma(O_W)$

$B_\gamma(O_W) \subset \mathrm{Im}(B_1(O_V)) \Rightarrow \exists\ \bar{v} \in B_1(O_V) \text{ s.t.}$

$A\bar{v} = \bar{w} \Rightarrow Av = w$ where $v = \frac{2\|w\|_{W'}}{\gamma} \bar{v}$

$\|\bar{v}\| < 1 \qquad \|Av\| = \|w\|$

$$\|v\| = \frac{2}{\gamma}\|w\|_{W'}\|\bar{v}\| \leq \frac{2}{\gamma}\|w\|_{W'} = \|Av\|\frac{2}{\gamma}$$

$$\Rightarrow \|Av\|_{W'} \geq \underbrace{\frac{\gamma}{2}}_{\alpha}\|v\|_V$$

$\boxed{\Leftarrow}$ $\exists \alpha$ s.t. $\|Av\| \geq \alpha\|v\| \Rightarrow \mathrm{Im}(A) = \overline{\mathrm{Im}(A)}$

1) $w_n \in \mathrm{Im}(A)$ cauchy $\Rightarrow \exists w \in W'$ s.t. $\lim_{n\to\infty} w_n = w$

2) $\forall w_n \; \exists v_n$ s.t. $Av_n = w_n$

$\|w_n\| = \|Av_n\| \geq \alpha\|v_n\| \Rightarrow v_n$ is also cauchy

$\|w_n - w_m\| = \|A(w_n - w_m)\| \geq \alpha\|v_n - v_m\| \Rightarrow \exists v$ s.t. $Av = \underline{\underline{w}}$

$\Rightarrow \mathrm{Im}(A) = \overline{\mathrm{Im}(A)}$

Equivalent statements. (same for A^T)

i) A^T is surjective

ii) A is injective and $\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}$

iii) A is bounding $\quad \exists \alpha$ s.t. $\|Av\|_{W'} \geq \alpha\|v\|_V$

iv) the inf sup condition is satisfied

$$\exists \alpha \text{ s.t. } \inf_{v\in V}\sup_{w\in W}\frac{\langle Av, w\rangle}{\|v\|_V\|w\|_W} \geq \alpha$$

Hilbert case   BNB

$A : V \longrightarrow W'$ V, W Hilbert Spaces.

$\exists \alpha$ s.t $\forall f \in W'$ $\exists! u \in V$ s.t. $Au = F$

$$\|u\| \leq \frac{1}{\alpha}\|f\| = \frac{1}{\alpha}\|Au\|$$

$\Longleftrightarrow$

$\exists \alpha > 0$ s.t.

i) $$\inf_{v \in V} \sup_{w \in W} \frac{\langle Au, w\rangle}{\|v\| \, \|w\|_W} \geq \alpha$$

ii) $$\inf_{w \in W} \sup_{v \in V} \frac{\langle Au, w\rangle}{\|v\| \, \|w\|_W} \geq \alpha$$

INF SUP conditions

Banach : ii) becomes $\ker(A^T) = \{0\}$

Lax Milgram . $V \equiv W \Longrightarrow$ inf sup

$\Longleftarrow$ is false

Mixed problems

$V \quad Q$ two operators.

$A : V \longrightarrow V'$ $\quad A \in \mathcal{L}(V, V')$

$B : V \longrightarrow Q'$ $\quad B \in \mathcal{L}(V, Q')$

$(B^T : Q \longrightarrow V')$

$\langle Au, v \rangle \leq \|A\| \, \|u\|_V \, \|v\|_V$ continuity

$\langle Bu, q \rangle \leq \|B\| \, \|u\|_V \, \|q\|_Q$ continuity

Given $f \in V'$, $g \in Q'$ find $(u, p) \in V \times Q$ s.t.

$$① \begin{cases} Au + B^T p = f \\ Bu = g \end{cases}$$

i) $g \in \mathrm{Im}(B) \Rightarrow \exists \, u_g$ s.t. $Bu_g = g$

ii) $Z := \ker(B) \Rightarrow u = u_0 + u_g \quad u_0 \in Z$

$$② \begin{cases} Au_0 + B^T p = f - Au_g = \left(\tilde{f} = f - Au_g\right) \\ Bu_0 + \cancel{Bu_g} = \cancel{g} \; 0 \end{cases}$$

$$\langle A u_0, v_0 \rangle + \cancel{\langle B^T p, v_0 \rangle}^{\,0} = \langle \tilde{f}, v_0 \rangle \qquad \forall v_0 \in Z$$

u-problem

$$\langle A u_0, v_0 \rangle = \langle \tilde{f}, v_0 \rangle \iff \text{BNB conditions.}$$

ELL-KER : $\boxed{\exists \alpha > 0 \text{ s.t. } \inf_{v \in Z} \sup_{u \in Z} \frac{\langle A u, v \rangle}{\|u\| \|v\|} \geq \alpha \qquad \inf_{u \in Z} \sup_{v \in Z} \frac{\langle A u, v \rangle}{\|u\| \|v\|} \geq \alpha}$ INF SUP on A

$$\Rightarrow \exists ! u_0 \in Z \text{ s.t. } A u_0 = \tilde{f}$$

$$\boxed{\|u_0\| \leq \frac{1}{\alpha} \|\tilde{f}\| \leq \frac{1}{\alpha} \|f\|_{V'} + \frac{1}{\alpha} \|A\| \|u_g\|_V}$$

Given u_0 we want to solve the p-problem:

$$\langle B^T p, v \rangle = \langle -A u, v \rangle + \langle f, v \rangle$$
$$= \langle -A u_0, v \rangle + \langle \tilde{f}, v \rangle \qquad \forall v \in V$$

$$\ker(B^T) = 0 \qquad \operatorname{Im}(B^T) = \overline{\operatorname{Im}(B^T)}$$

(B is surjective)

$$\exists \beta > 0 \quad \|B^T p\|_{V'} \geq \beta \|p\|_Q \qquad \forall p \in Q$$

$$\iff \boxed{\exists \beta > 0 \quad \text{s.t.} \quad \inf_{p \in Q} \sup_{v \in V} \frac{\langle B v, p \rangle}{\|v\| \|p\|}} \qquad \text{INF-SUP on B}$$

$$\|u\| \leqslant \frac{1}{\alpha}\|f\|_{V'} + \frac{1}{\alpha}\|A\|\,\underline{\underline{\|u_g\|}}$$

inf sup on B $\Leftrightarrow$ $\exists L$ left inverse of B (cont. lin. op.)

$$\|B^T p\| \geqslant \beta \|p\|$$

$\forall g \in \mathrm{Im}(B)$, $\exists\, u_g \in B$ s.t. $Bu_g = g$

$u_g = Lg$ where L is a linear op. s.t.

$BL g = g \quad \forall g \in \mathrm{Im}(B)$

$$\|g\| = \|BLg\| \geqslant \beta \|Lg\| \qquad \|Lg\| \leqslant \frac{1}{\beta}\|g\|$$

$$\boxed{\|u\|} \leqslant \frac{1}{\alpha}\|f\|_{V'} + \frac{1}{\alpha}\|A\|\,\underline{\underline{\|u_g\|}} \leqslant \boxed{\frac{1}{\alpha}\|f\|_{V'} + \frac{\|A\|\,\|g\|}{\alpha\beta}}$$

$$\|p\| \leqslant \frac{1}{\beta}\left(\|A\|\,\|u\| + \|f\|_{V'}\right)$$

$$\boxed{\|p\| \leqslant \frac{1}{\beta}\|f\|_{V'} + \frac{1}{\beta}\frac{\|A\|}{\alpha}\|f\|_{V'} + \frac{\|A\|^2\|g\|}{\alpha\beta^2}}$$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \in \mathcal{L}(V \times Q,\ V' \times Q')$$

$\mathbb{A} \iff$ INFSUP ou $\mathbb{A}$ ou $\mathbb{V} := V \times Q$