Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 7

Last Lecture:
- Weak formulations:
  - different boundary conditions
  - general elliptic differential operators
  - well-posedness by Lax-Milgram Lemma
- The method of Galerkin

Today's lecture:
- Cea's lemma
- The Finite Element Method
- Error analysis for two-points BVPs
- FEM in higher-dimensions
- Conditioning of the FEM system

} Chap 4 in quarteroni

Computer practical
- FD in two dimensions

Method Galerkin analysis

Given
- $(V, (\cdot,\cdot))$ Hilbert
- $\mathcal{A} : V \times V \to \mathbb{R}$ $\begin{cases} \text{bilinear} \\ \text{continuous } (\gamma) \\ \text{coercive } (\alpha) \end{cases}$
- $F \in V'$

Then, Lax-Milgram, there exists a unique

$u \in V: \quad \mathcal{A}(u, v) = F(v) \quad \forall v \in V \quad$ (WP)

Galerkin method:

- Consider sequence $\{V_n\}_n$ of subspaces of V
- Restrict (WP) in V_n:

find $u_n \in V_n$: $A(u_n, v_n) = F(v_n) \quad \forall v_n \in V_n$

(GM)

analysis:

by Lax-Milgram, $\exists!\, u_n \in V_n$ sol. of (GM) and

a priori bound holds:

$$\|u_n\|_V \leq \frac{1}{\alpha} \|F\|_{V'} \qquad \text{(STABILITY)}$$

Also, we have Galerkin orthogonality:

$$\mathcal{A}(u - u_n, v) = 0 \quad \forall v \in V_n$$

(FULL CONSISTENCY)

Convergence:

Lemma of Céa: Under assumptions of LM

$$\|u - u_n\|_V \leq \frac{\gamma}{\alpha} \inf_{v_n \in V_n} \|u - v_n\|$$

(quasi-optimality property)

<u>Proof</u>: apply Galerkin orthog with $v = v_n - u_n$ for any $\underline{v_n \in V_n}$

$$= a(u-u_n, v_n - u_n) = 0$$

$$= a(u-u_n, v_n - u) + a(u-u_n, u - u_n)$$

$$\Rightarrow \quad \geq a(u-u_n, u-v_n) = a(u-u_n, u-u_n) \geq$$

continuity of a — coercivity of a

$$\leq \gamma \cancel{\|u-u_n\|} \|u-v_n\| \qquad \alpha \|u-u_n\|^{\cancel{2}} \leq$$

$$\Rightarrow \quad \|u-u_n\| \leq \gamma/\alpha \ \|u-v_n\| \qquad \forall v_n \in V_n$$

□

Convergence now follows, for instance, if

- Fix $\{\varphi_j\}_{j=1}^{\infty}$ complete orthonormal system for V
- Define $V_n = \text{span}\{\varphi_1, .., \varphi_n\}$
- Then, as $\lim_{n\to\infty} V_n$ is dense in V

we have $\forall u \in V, \forall \varepsilon > 0 \; \exists n:$

$$\exists w_n \in V_n : \|u - w_n\| \leq \varepsilon$$

- This + Céa $\Rightarrow$ convergence

or $\inf_{v_n \in V_n} \|u - v_n\| \to 0$

FEM, spectral, etc. are instances of practical choices of subspaces !

Finite Element Method (FEM)

FEM for two-points BVP

model problem: reaction-diffusion

$$\begin{cases} Lu = -(a u')' + cu = f & \Omega = (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

$a(x) \geq a_0 > 0$, $c \geq 0$, $f \in L^2(0,1)$

Weak-formulation: $u \in H_0^1(\Omega) =: V$

$$\underbrace{A(u,v) := (a u', v') + (cu, v)}_{= \int_0^1 a u' v' \ldots} = \underbrace{(f, v)}_{=: F(v)} \quad \forall v \in V$$

since prob. is symmetric have more refine analysis (general !)

Proposition (Céa symmetric case):

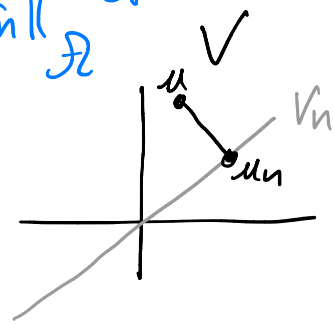
Under ass. of Céa, if a is symmetric, then

$$\|u-u_n\|_V \le \sqrt{\gamma/\alpha} \min_{u_n\in V_n} \|u-u_n\|_V$$

better constant !

$$\|u-u_n\|_a = \min_{v_n\in V_n} \|u-v_n\|_a$$

optimality !



where $\|v\|_a = a(v,v)^{1/2}$

Proof: exercise

... back to model problem and FEM

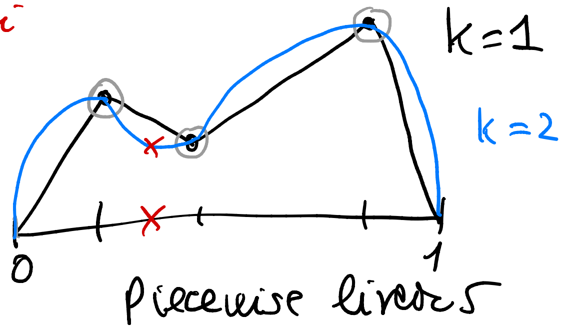
- partition $\Omega=(0,1)$ by $\begin{cases} x_0=0 \\ x_i = x_{i-1}+h_i \end{cases}$ $i=1,\dots,M$

and let $h = \max_i h_i$

Define the space of Finite Elements as

$$V_h := V_h^k = \{ v \in C_0(\bar{\Omega}) : v|_{I_i} \in \mathbb{P}^k(I_i),\ \forall i = 1, \dots, M \}$$

$$I_i = (x_{i-1}, x_i)$$



FEM(k): Find $u_h \in V_h$: $A(u_h, v_h) = F(v_h)$ $\forall v_h \in V_h$

Here $\mathbb{P}^k(\omega)$, $\omega \in \mathbb{R}^d$

Space of polynomials of degree $\leq d$

<u>algebric formulation of FEM:</u>

- fix a basis $\{\varphi_m\}_{m=1}^{\dim V_h}$
- write $u_h = \sum_m U_m \varphi_m$

- test with basis:

$$\sum_m U_m \Big[\underbrace{(a\varphi_m', \varphi_\ell')}_{S_{\ell m}} + \underbrace{(c\varphi_m, \varphi_\ell)}_{M_{\ell m}} \Big] = (f, \varphi_\ell) \quad \forall \ell = 1, \dots, \dim V_h$$

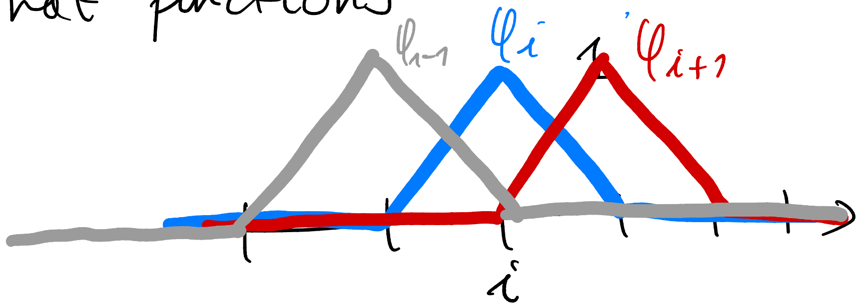
$$U = \{U_m\}_{m=1}^{\dim V_h}, \quad F = \{F_\ell\}, \; F_\ell = (f, \varphi_\ell)$$

$$\Rightarrow \begin{cases} AU = F \leftarrow \text{load vector} \\ A = S + M \end{cases}$$

S ← stiffness, M ← mass matrix

Crucial issue: <u>practical</u> choice of the FE basis !!

$k=1$: hat functions



- fix nodal values as degrees of freedom (DOF) and then the corresponding Lagrange basis $\{\varphi_i\}$:

$$\varphi_i(x_j) = \delta_{ij}$$

the hat functions

- note $\operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_j = 0$

if $|i-j| > 1$

$\Rightarrow$ A is sparse!

Practical implementation

- record implementations of of the basis functions
- loop over "elements" (that is, the intervals composing mesh) to evaluate local contributions to the integrals defining FEM

For instance

$$S_{\ell m} = \int_0^1 a(x)\,\varphi_m(x)\,\varphi_\ell(x) = \sum_i \int_{I_i} a(x)\,\varphi_m(x)\,\varphi_\ell(x)$$

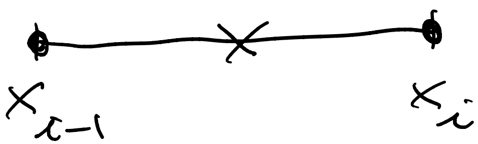
Each such integral is in general evaluated by quadrature!

$$\rightarrow \qquad \approx \sum_i \sum_j w_j\, a(x_j)\,\varphi_m(x_j)\,\varphi_\ell(x_j)$$

(w_q, x_q) a given quadrature formula

Second example: $k=2$

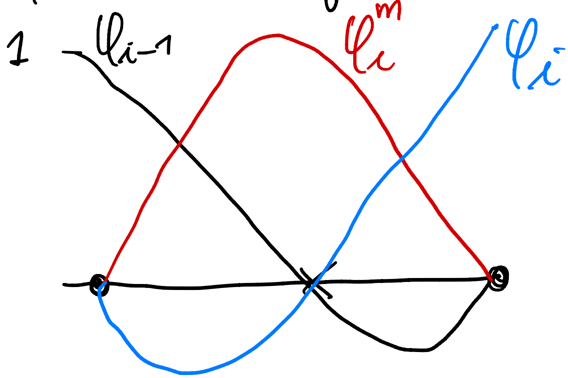
- on each interval, use $\mathbb{P}_2(I_i)$



three DOF: end points x_{i-1}, x_i

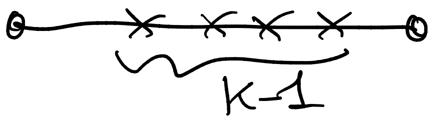
and $x_i^m := \frac{x_{i-1}+x_i}{2}$

- define Lagrange basis



$\dim V_h = V_h^2 = (N-1) + N$

In general $\dim V_h^k = (N-1) + N(k-1)$



- dim and stencil grows but higher order polynomials give higher order methods ?

A priori analysis of FE

Recall general result for Galerkin:

by Céa, we have

$$\|u-u_h\|_V \le \sqrt{\gamma/\alpha} \inf_{v_h \in V_h = V_h^k} \|u-v_h\|_V$$

with $V = H_0^1(0,1)$

$$\|v\|_V = \left(\int_0^1 (v')^2 \, dx \right)^{1/2} \qquad H_0^1\text{-norm}$$

or $$= \left(\int_0^1 (v')^2 dx + \int_0^1 v^2 dx \right)$$

H^1 norm

and also

$$\| u - u_h \|_{\Omega} = \min_{v_h \in V_h} \| u - v_h \|_{\Omega}$$

$$\| w \|_{\Omega} = \Omega(w, w)^{1/2} = \left(\int_0^1 a (w')^2 + c w^2 \right)^{1/2}$$

Issue: characterise $\inf_{v_h \in V_h} \| u - v_h \|_V$

$\forall\, u \in V$, in terms of h, k.

Clearly, $\inf_{v_h \in V_h} \| u - v_h \|_V \leq \| u - I_h u \|_V$

some "special" element from V_h

$\forall u \in V$.

note, can write

$$\|u - I_h u\|_V^2 = \sum_{i=1}^{N} \|u - I_h u\|_{I_i}^2$$

- recall, $\forall I_i \quad V_h|_{I_i} \in \mathbb{P}_k(I_i)$

$\Rightarrow$ local approx by polynomials question

Def (FEM interpolant): Define on each I_i the FE interpolant by setting $I_h u|_{I_i} \in \mathbb{P}^k(I_i)$ such that $I_h u|_{I_i}(x_j) = u(x_j)$ $\forall x_j$ "node" within I_i

indep. of h

Theorem (Q, LT)

$$\|v - I_h v\|_{L^2(I_i)} \le C_I h_i^{k+1} |v|_{H^{k+1}(I_i)} \quad \forall v \in H^{k+1}(I_i)$$

local poly interp

$$\|v' - I_h v'\|_{L^2(I_i)} \le C_I h_i^k |v|_{H^{k+1}(I_i)}$$

⇒ global bound follows:

$$\|v' - I_h v'\|^2_{L^2(\Omega)} = \sum_{i=1}^{M} \|v' - I_h v'\|^2_{L^2(I_i)}$$

$$\leq C_I \sum_{i=1}^{M} h_i^{2k} |v|^2_{H^{k+1}(I_i)}$$

if $v \in H^{k+1}(I_i)$ $\forall i$

also,

$$\leq C_I \, h^{2k} \, |v|_{H^{k+1}(\Omega)}$$

if $v \in H^{k+1}(\Omega)$

Theorem (a priori error convergence and rates):

$$\|u' - u_h'\|_{L^2(\Omega)} \leq C \, h^k \, |u|_{H^{k+1}(\Omega)}$$

$\exists$ $C > 0$ indep. of h.

if $u \in H^{k+1}(\Omega)$.

(also, $\leq C \left(\sum_i h_i^k \, |u|^2_{H^{k+1}(I_i)} \right)^{1/2}$

$$\|u - u_h\|_{L^2(\Omega)} \leq C h^{k+1} |u|_{H^{k+1}(\Omega)}$$

(also, $\leq C h \left(\sum_i h_i^k \, |u|^2_{H^{k+1}(I_i)} \right)^{1/2}$

global h !

If $u \in H^s(\Omega)$ $\quad s \leq k+1$

then $\|u' - u_h'\|_{L^2(\Omega)} \leq C h^s |u|_{H^s(\Omega)}$

again, with C independent of h.

(bounds explicit in k are also available)