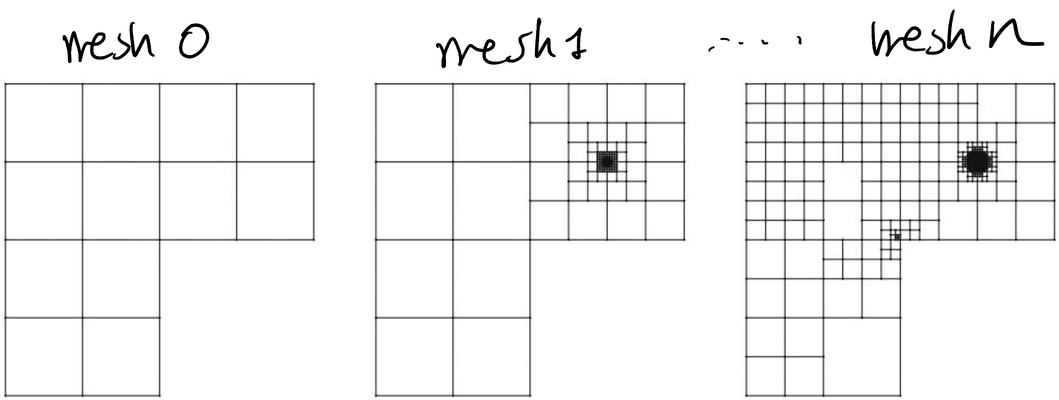
Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 12

Last Lecture:
- Convection-diffusion-reaction problems
- The Streamline diffusion method

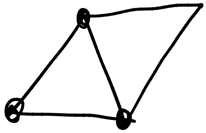
Today's lecture:
- A posteriori error analysis

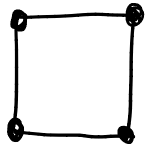


$u_0 \Rightarrow \mathcal{E}(u_0)$ refine where $\mathcal{E}_T(u_0)$ is relatively big

$$\mathcal{E}(u_0) = \sum_{T \in \mathcal{T}_h} \mathcal{E}_T(u_0)$$

$\boxed{\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}}$





automatically adaptive algorithm based on a posteriori error estimates

$\mathbb{P}_1 = \text{span}\{1, x, y\}$ $\qquad$ $\mathbb{Q}_1 = \text{span}\{1, x, xy\}$

<u>Justification</u>: sol. of lin. system $AU = b$

suppose U_0 some approx of U. Then

$$A(U - U_0) = \underbrace{b - AU_0}_{\substack{\text{residual} \\ \text{only dep. on} \\ \text{available } U_0\,!}}$$

$$\Rightarrow \boxed{\|U - U_0\| = \|A^{-1}(b - AU_0)\|}$$

A posteriori error bounds (Q. chap. 4)

Let • V Hilbert, $V_h < V$

• a, F satisfying Lax Milgram hypothesis

$$\Rightarrow \exists!\, u \in V: a(u, v) = F(v) \quad \forall v \in V \qquad \text{(WP)}$$

$$\exists!\, u_h \in V_h: a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h \qquad \text{(GM)}$$

Residual : $R \in V'$

$$\langle R, v \rangle = F(v) - A(u_h, v) \quad \forall v \in V$$

$$\Rightarrow \|R\|_{V'} = \sup_{\substack{v \in V \\ v \neq 0}} \frac{\langle R, v \rangle}{\|v\|_V}$$

Proposition: We have

$$\alpha \|u - u_h\|_V \leq \|R\|_{V'} \leq \gamma \|u - u_h\|_V$$

(α = coerc, γ = cont of A)

Proof: $$\|R\|_{V'} = \sup_{\substack{v \in V \\ v \neq 0}} \frac{\langle R, v \rangle}{\|v\|_V}$$

$$= \sup_{\substack{v \in V \\ v \neq 0}} \frac{F(v) - A(u_h, v)}{\|v\|_V}$$

by (WP) $$= \sup_{\substack{v \in V \\ v \neq 0}} \frac{A(u - u_h, v)}{\|v\|_V}$$

by cont of $\mathcal{A}$ $\leq \gamma \|u-u_h\|_V$

By coercivity,

$$\alpha \|u-u_h\|_V^{\cancel{2}} \leq \mathcal{A}(u-u_h, u-u_h)$$

$$= \frac{\langle R, u-u_h \rangle}{\|u-u_h\|_V} \|u-u_h\|_V$$

$$\leq \|R\|_{V'} \cancel{\|u-u_h\|_V}$$

□

Issue: provide computable estimates of $\|R\|_{V'}$

Fix: $V = H_0^1(\Omega)$ $\quad \Omega \in \mathbb{R}^d$

$\mathcal{A}(u,v) = (\nabla u, \nabla v)_\Omega := \int_\Omega \nabla u \cdot \nabla v$

$F(v) = (f, v)_\Omega$

$\Rightarrow$ Poisson problem in weak form

$\langle R, v \rangle = \mathcal{A}(u-u_h, v)$ (by def of R, WP)

$= \mathcal{A}(u-u_h, v-v_h)$ (by Galerkin orthog $\forall v_h \in V_h$

$|$

where eq. V_h standard FE space.

$$= F(v-v_h) - a(u_h, v-v_h)$$

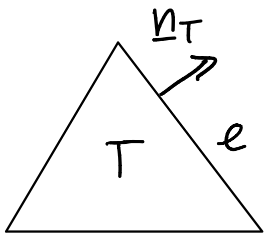
$$= \int_\Omega f(v-v_h) - \int_\Omega \nabla u_h \cdot \nabla(v-v_h)$$

$$= \sum_{T\in\mathcal{T}_h} \left[\int_T \underbrace{(f+\Delta u_h)}\,(v-v_h) - \int_{\partial T} \frac{\partial u_h}{\partial \underline{n}_T}(v-v_h) \right]$$

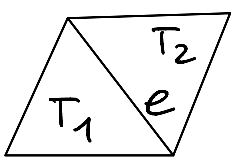
$\underline{n}_T$ = unit outward pointing normal on ∂T

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u_h}{\partial \underline{n}_T}(v-v_h) = \sum_{T\in\mathcal{T}_h} \sum_{e\in\partial T} \int_e \frac{\partial u_h}{\partial \underline{n}_T}(v-v_h)$$

↑ the edges of T



$$= \sum_{e\in\mathcal{E}^0} \int_e \underbrace{\left(\frac{\partial u_h}{\partial \underline{n}_{T_1}}\Big|_{T_1} + \frac{\partial u_h}{\partial \underline{n}_{T_2}}\Big|_{T_2} \right)}_{=:\, [\![\frac{\partial u_h}{\partial \underline{n}}]\!]} (v-v_h)$$



$\mathcal{E}^0$ = set of the mesh's internal edges

$\Rightarrow$

$$\frac{\langle R, v\rangle}{\|v\|_V} = \sum_{T\in\mathcal{T}_h} \underbrace{\int_T (f+\Delta u_h)(v-v_h)}_{\text{elemental interior residual}} - \underbrace{\frac{1}{2}\int_{\partial T} \left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right](v-v_h)}_{\text{flux jump residual}} \Big/ \|v\|_V$$

Recall $v \in V = H^1_0(\Omega)$

Choice of v_h: Lagrange interpolant used in a priori analysis <u>not</u> define in H^1_0!

Use instead <u>quasi interpolant</u> based on averages!

For instance, Clément or Scott-Zhang interpolants...

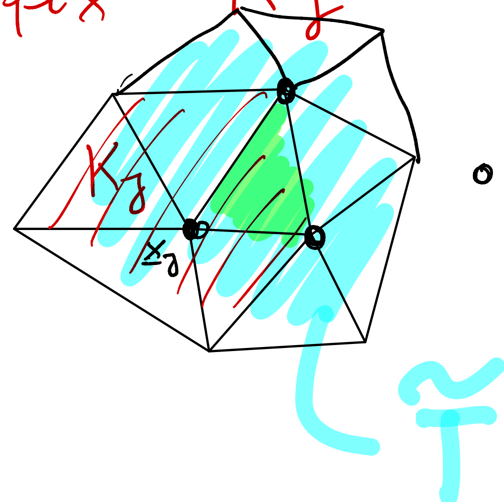
Clément: $R_h : H^1(\Omega) \longrightarrow V_h$

$$v \longrightarrow \sum_j (P_j v)(x_j)\varphi_j$$

$\underline{x}_j$ j-th node (deg.-of-freedom) of FE

where $P_j v$ local (patch-wise) L^2-projection of v onto $\mathbb{P}^1$:

- fix K_j = patch of elements sharing node $\underline{x}_j$



- fix $P_j v \in \mathbb{P}^1(K_j)$:

$$\int_{K_j} P_j v\, \varphi = \int_{K_j} v\, \varphi$$

$$\varphi = 1, x, y$$

we have

$$(*) \quad \| v - R_h v \|_{L^2(T)} \le C_1 h_T |v|_{H^1(\tilde{T})}$$

where $\tilde{T}$ = patch of elements sharing at least a node with T.

We also have

$$\|v - R_h v\|_{L^2(\partial T)} \le C_2 h_T^{1/2} \|v\|_{H^1(\tilde{T})}$$

comes from (*) after the

Trace estimate: $\forall v \in H^1(T)$

$$\|v\|_{L^2(\partial T)} \le C \left(h_T^{-1/2} \|v\|_{L^2(T)} + h_T^{1/2} \underbrace{\|\nabla v\|_{L^2(T)}}_{= |v|_{H^1(T)}} \right)$$

$$\langle R, v \rangle = \sum_{T \in \mathcal{T}} \left[\int_T (f + \Delta u_h)(v - v_h) - \frac{1}{2} \int_{\partial T} \left[\frac{\partial u_h}{\partial \underline{n}_T} \right] (v - v_h) \right]$$

$\forall v \in V, \forall v_h \in V_h$

Schwarz $\le \sum_{T \in \mathcal{T}_h} \left[\|f + \Delta u_h\|_{L^2(T)} \|v - v_h\|_{L^2(T)} + \frac{1}{2} \left\| \left[\frac{\partial u_h}{\partial \underline{n}_T} \right] \right\|_{L^2(\partial T)} \|v - v_h\|_{L^2(\partial T)} \right]$

fix v_h as Clément quasi interpolant

$$\leq C \sum_{T \in \mathcal{T}_h} \Big[\|f + \Delta u_h\|_{L^2(T)} h_T |v|_{H^1(\tilde{T})} + \frac{1}{2} \|[\![\frac{\partial u_h}{\partial n_T}]\!]\|_{L^2(\partial T)} h_T^{1/2} \|v\|_{H^1(\tilde{T})} \Big]$$

C depends on C_1, C_2

$$\leq C \Big[\sum_T \mathcal{E}_T(u_h)^2 \Big]^{1/2} \Big[\sum_T \|v\|^2_{H^1(\tilde{T})} \Big]^{1/2}$$

$$\mathcal{E}_T := h_T \|f + \Delta u_h\|_{L^2(T)} + \frac{1}{2} h_T^{1/2} \|[\![\frac{\partial u_h}{\partial n_T}]\!]\|_{L^2(\partial T)}$$

If $\mathcal{T}_h$ is shape-regular ($\exists C : h_T/\rho_T < C$ $\forall T \in \mathcal{T}_h$)

then # set of neighbours of T is bounded uniformly by some $n \in \mathbb{N}$.

$$\Rightarrow \langle R, v \rangle \leq C \sqrt{n} \Big[\sum_T \mathcal{E}_T(u_h)^2 \Big]^{1/2} \|v\|_{H^1(\Omega)}$$

$$\leq C \sqrt{n} \sqrt{1 + C_\Omega^2} \Big[\sum_T \mathcal{E}_T(u_h)^2 \Big]^{1/2} \|v\|_V$$

$$\left(|v|_{H^1(\Omega)} \right)$$

C_Ω = Poincaré constant

$$\Rightarrow \boxed{\|R\|_{V'} \leq C \left[\sum_T \mathcal{E}_T(u_h)^2\right]^{1/2}}$$

$$\underbrace{\qquad}_{\text{local error estimator}}$$

$\mathcal{E}$''

- $\mathcal{E}$ only dep. on u_h, h, n, C_Ω, C_1, C_2