

Numerical solution of PDEs - NSPDE

Advanced Numerical Analysis - ANA

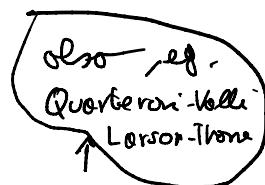
LECTURE 5

Last Lecture:

- FD on non-uniform meshes
- FD on general domains

Today's lecture:

- Basic notions of functional analysis
- Intro to weak formulations



Basic Notions on function spaces (Q. chap 2)

Motivation: recall model problem for Poisson's

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

towards weak formulation, we test with some
 v and integrate:

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} fv$$

HVE?
which space?

$$\stackrel{1}{=} (u, v)_1$$

- $\mathcal{C}^k(\bar{\Omega}) = \left\{ f: \bar{\Omega} \rightarrow \mathbb{R}: f \text{ is } k\text{-time cont. diff.} \right\}$
 $(k=0 \quad \mathcal{C}(\bar{\Omega}) \text{ cont. functions})$
- $\mathcal{C}_0^k(\Omega) = \left\{ f \in \mathcal{C}^k(\Omega): f \text{ has compact support} \right\}$

We have :

- $\mathcal{C}^k(\bar{\Omega})$ is complete (Banach) w.r.t.

$$\|v\|_{\mathcal{C}^k(\bar{\Omega})} = \max_{|D| \leq k} \|D^{|D|} v\|_{\mathcal{C}(\bar{\Omega})}$$

where $D^d v = \frac{\partial^d v}{\partial x_1^{d_1} \cdots \partial x_d^{d_d}}$, $d = \text{multi-index}$

- but, for instance, $\mathcal{C}_0^1(\Omega)$ is not complete w.r.t.

$$\|v\| = \int_{\Omega} |\nabla v|^2 \quad \begin{array}{l} (\text{norm induced by}) \\ \text{I.P. } (u, v) = \int_{\Omega} u v \end{array}$$

Def

Hilbert space : $(H, (\cdot, \cdot))$

- H real linear space
- (\cdot, \cdot) scalar product

• H complete w.r.t. $\|\cdot\|_H = (\cdot, \cdot)^{1/2}$

Properties, e.g.:

- $|(\nu, w)| \leq \|\nu\| \|w\|$ (Schwarz)
- Let H_0 closed linear subspace, then
if $v \in H$, $v = v_0 + w$ $\exists! v_0 \in H_0$
and $w \in V$: $(w, z) = 0 \quad \forall z \in H_0$,

$$\text{and } \begin{cases} \|w\| \\ \geq \|v - v_0\| \end{cases} = \min_{z \in V_0} \|v - z\|$$

Crucial example

L_p -spaces : for $1 \leq p \leq \infty$

$L^p(\Omega) = \{f: \Omega \rightarrow \mathbb{R} \text{ Lebesgue measurable}\} : \begin{cases} \int_{\Omega} |f|^p < \infty \\ p < \infty \\ \sup_x |f(x)| < \infty \end{cases}$

normed with $\begin{cases} \|\nu\|_{L^p(\Omega)} = \left(\int_{\Omega} |\nu|^p \right)^{1/p} & p < \infty \\ \|\nu\|_{L^\infty(\Omega)} = \sup_x |\nu(x)| & p = \infty \end{cases}$

Hölder inequality: given p , let $p' = \frac{1}{p} + \frac{1}{p'} = 1$
 (with $p' = \infty$ if $p=1$) , then

$$\left| \int_{\Omega} uv \, dx \right| \leq \|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}$$

In particular, for $p=2 = p'$, we have

$$\left| \int_{\Omega} uv \right| \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

(since $u, v \in L^2(\Omega)$)

L^2 is I.P. space (actually, Hilbert)

$$\text{w.r.t. } (u, v)_0 = \int_{\Omega} uv$$

(To be made precise soon ...) From $L^2(\Omega)$,
 we define $H^1(\Omega)$ and $H_0^1(\Omega)$

L^2 with 1st derivative in L^2

then $f \in L^2(\Omega)$, can def. weak problem

$$\text{find } u \in H_0^1(\Omega) : \begin{aligned} & \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} fv \\ & = (u, v)_1 \quad (f, v)_0 = \end{aligned}$$

but, this is valid only for L^2 . We'd like to include also $S_x \notin L^2$ (Dirac)

Dual space: Let $(V, \|\cdot\|)$ normed space.

$V' = \mathcal{L}(V; \mathbb{R})$ is the DUAL space of V :

- Space of linear continuous functions from $V \rightarrow \mathbb{R}$

$$\begin{aligned} &\text{if } L \in V' \Rightarrow L \text{ bounded:} \\ &|L(v)| \leq C \|v\|, \end{aligned}$$

- V' is normed w.r.t. operator norm:

$$\|L\|_{V'} := \sup_{v \in V \setminus \{0\}} \frac{|L(v)|}{\|v\|}$$

- $(V', \|\cdot\|_{V'})$ is Banach

- $\langle \cdot, \cdot \rangle : V' \times V \rightarrow \mathbb{R}$ bilinear form

$$\langle L, v \rangle := L(v)$$

DUALITY PAIRING

Theorem (Riesz-Fréchet) Characterisation of the dual of an Hilbert space :

Let V Hilbert, $L \in V'$. Then

$$\exists! u \in V : \begin{cases} L(v) = (u, v)_V \text{ for all } \\ = \langle L, v \rangle \end{cases}$$

Also $\|L\|_{V'} = \|u\|_V$.

example : $V = L^2(\Omega)$, $f \in (L^2(\Omega))'$ $\exists! u \in L^2(\Omega)$ such that

$$\begin{cases} f(v) = (u, v)_0 = \int_{\Omega} u v \\ = \langle f, v \rangle \end{cases}$$

With this in mind, using concept of distributions we shall be able to give meaning in general to the Poisson problem

Distribution : Given $\mathcal{D}'_0(\Omega)$, named $\mathcal{D}(\Omega)$ when provided with the topology given by uniform convergence

We Define space $\mathcal{D}'(\Omega)$ = SPACE OF DISTRIBUTIONS or dual of $\mathcal{D}(\Omega)$ so
 note $T : \mathcal{D}(\Omega) \rightarrow \mathbb{R}$ linear and cont.
 $\varphi \rightarrow T(\varphi) = \langle T, \varphi \rangle$

We have,

- $L^p(\Omega) \subsetneq \mathcal{D}'(\Omega) \quad \forall p \in [1, +\infty]$

Indeed ($p=2$) : $f \in L^2$ defines a distribution

by $f \rightarrow \int_{\Omega} f \varphi =_{L^2(\Omega)} \forall \varphi \in \mathcal{D}(\Omega)$

or $\left| \int_{\Omega} f \varphi \right| \leq \|f\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \Rightarrow f$ bounded

- $\delta =$ Dirac's delta is a distribution but is not representable by any $f \in L^p(\Omega) \forall p$

Weak derivatives : For $T \in \mathcal{D}'(\Omega)$

We define the weak derivatives $D^\alpha T \in \mathcal{D}'(\Omega)$

or $\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$

+ α multiindex.

Hence

- weak derivatives are always defined
- but, for instance Heaviside function

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \in L^2_{loc}$$

by
interval

but $DH = \delta \notin L_{loc}$

\Rightarrow starting from L^2 , weak derivatives may not be in L^2



Sobolev spaces

$$W^{k,p}(\Omega) = \left\{ v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \quad \forall |\alpha| \leq k \right\}$$

In particular, for $p=2$

$$W^{k,2}(\Omega) = H^k(\Omega) \quad \text{which is Hilbert}$$

$$\text{w.r.t. } (v, w)_k := \sum_{|\alpha| \leq k} (D^\alpha v, D^\alpha w)$$

by def. of weak deriv.

Note from def. $\int_{\Omega} \frac{\partial u}{\partial x_i} \varphi = \int_{\Omega} u \frac{\partial \varphi}{\partial x_i}$

\Rightarrow integration by parts! In particular, if classical derivative exists \equiv to weak derivative

Again, for us, crucial example is

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) : \nabla v \in (L^2(\Omega))^d \right\}$$

Hilbert a.r.t. $(u, v)_{H^1} = (u, v)_0 + (\nabla u, \nabla v)_0$

and norm $\|v\|_{H^1} = \left(\|v\|_0^2 + \|\nabla v\|_0^2 \right)^{1/2}$

$$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \right\}$$

Hilbert with $\|v\|_{H_0^1} = (\nabla u, \nabla v)_0$

Some is true for

$$H_\Gamma^1(\Omega) = \left\{ v \in H^1(\Omega) : v|_\Gamma = 0 \right\}$$

whenever $\Gamma \subset \partial\Omega$ with $|\Gamma| \neq 0$

Theorem (Poincaré inequality): $\Omega \subset \mathbb{R}^d$ open, bounded

$\Gamma \subset \partial\Omega$, Lipschitz cont., \exists

$C_\Omega > 0$: $\forall v \in H_\Gamma^1(\Omega)$

$$\int_{\Omega} v^2 dx \leq C_\Omega \int_{\Omega} |\nabla v|^2 dx$$

$\underbrace{\quad}_{|v|_{H_\Gamma^1(\Omega)}}$

$|v|_{H_\Gamma^1(\Omega)}$

\Rightarrow It is a norm for $H_\Gamma^1(\Omega)$:

$$\begin{aligned} \|v\|_{H_\Gamma^1(\Omega)} &= \|\nabla v\|_2 \\ &= |v|_{H^1(\Omega)} \end{aligned}$$

Weak formulations

Consider, once again Poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

recall :

$$\bullet \quad A(u, v) := \int_{\Omega} \nabla u \cdot \nabla v$$

symmetric
bilinear
form

$$\bullet \quad f \in \mathcal{D}'(\Omega), \quad \langle f, v \rangle \quad \text{linear functional}$$

then, we have WEAK FORMULATION

$$\text{find } u \in H_0^1(\Omega) : \quad A(u, v) = \langle f, v \rangle \quad (\text{WP})$$

$$\text{In particular, } f \in L^2(\Omega) \quad \langle f, v \rangle = (f, v)$$

Theorem: $u \in H_0^1(\Omega)$ solves weak problem (WP)

$$\Leftrightarrow -\Delta u = f \quad \text{in } \mathcal{D}'(\Omega)$$

Proof: " \Leftarrow " multiply $-\Delta u = f$ by $v \in H_0^1(\Omega)$ and integrate

$$\begin{aligned} \langle f, v \rangle &= \langle -\Delta u, v \rangle = (\nabla u, \nabla v) - \int_{\Omega} (u \cdot \nabla u) v \\ &= (\nabla u, \nabla v) \end{aligned}$$

" \Rightarrow " since $\mathcal{Z}_0^\infty(\Omega) \subset H_0^1(\Omega)$ can
test the WP with any $v \in \mathcal{Z}_0^\infty(\Omega)$:

$$= \int_{\Omega} \nabla u \cdot \nabla v \stackrel{\text{def}}{\rightarrow} \int_{\Omega} f v = \langle f, v \rangle$$

$$= - \int_{\Omega} \Delta u v = \underbrace{\langle -\Delta u, v \rangle}_{\text{as a distribution}}$$

$$\forall v \in \mathcal{C}_0^\infty$$

$$\Rightarrow -\Delta u = f \quad \text{in } \mathcal{D}'(\Omega)$$

□

- If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\bar{\Omega})$ is classical solution then u is weak solution
- Vice versa not true (weak problem is more general) :

take again Poisson, with $f \in L^2(\Omega)$

and Ω is at least \mathcal{C}^2 or

convex polygon

then it can be shown that

the solution to corresponding
b.v.p. for Poisson $u \in H_0^2(\Omega)$

but still not classical ?

For that, we need

in
 $d=2$!

$f \in H^2(\Omega) \Rightarrow u \in H^4(\Omega) \hookrightarrow C^2$
if $\Omega \in C^4$

$\Rightarrow u \in C^2(\Omega) \Rightarrow$ classical

from Sobolev Inequality:

for $\Omega \subset \mathbb{R}^d$ bounded smooth or polygonal

Then $H^k(\Omega) \subset C(\bar{\Omega})$ if $k > \frac{d}{2}$

and $\exists C(\Omega) : \|u\|_C \leq C \|u\|_K$

In conclusion

- 1) Weak problem can deal with more general data

2) much easier to prove well-posedness
(particularly for nonlinear problems!)

(Back to) wellposedness of W/P:

Theorem: Let $(V, (\cdot, \cdot)_V)$ Hilbert,

$A: V \times V \rightarrow \mathbb{R}$ symmetric bilinear form

which is continuous and coercive

$$(A(u,v) \leq c \|u\|_V \|v\|_V) \quad (A(v,v) \geq \alpha \|v\|_V^2)$$

$F: V \rightarrow \mathbb{R}$ continuous linear form

Then $\exists! u \in V: A(u, v) = F(v) \quad \forall v \in V$

$$\text{and } \|u\|_V \leq \frac{1}{\alpha} \|F\|_{V'}$$

Proof: • For A we have

(coercivity \Rightarrow) positive + symmetry

$\Rightarrow A(\cdot, \cdot)$ is I.P. in V

• $F \in V'$

By Riesz $\exists! u \in V: A(u, v) = F(v)$

Moreover,

$$0 \leq \|u\|_V^2 \leq F(u, u) = F(u) \leq \|F\|_V \|u\|_V$$

$$\Rightarrow \|u\|_V \leq \frac{1}{2} \|F\|_V$$

□

Note: limited to symmetric bilinear forms. We shall cover non-symmetric problems using Lax-Milgram lemma.

Regularity of Poisson's weak solutions

Let $\Omega \subset \mathbb{R}^2$, $\partial\Omega$ smooth.

If $f \in H^s(\Omega)$, $(s=0 \text{ means } L^2)$

then $u \in H^{s+2}(\Omega)$ shift

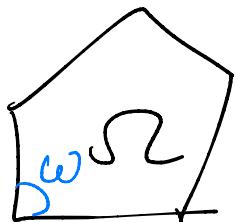
and $\|u\|_{H^{s+2}(\Omega)} \leq C \|f\|_{H^s(\Omega)}$ (R)

$$-\Delta u \quad \|L^2(\Omega)\|$$

For instance, $s=0$, then $\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^1(\Omega)}$
 that is all derivatives controlled by \max_u, u_{xy}

- If $\partial\Omega$ non-smooth, then (R) does not necessarily hold, even for $s=0$.

For instance take $\partial\Omega$ polygonal



near corners,

$$u(r,\vartheta) = r^{\frac{\pi}{\omega}} \alpha(\vartheta) + \beta(r,\vartheta)$$

where $\omega = \text{angle at corner}$

α, β smooth

If $\omega > \pi$ (nonconvex domain)

then $u \notin H^2(\Omega)$

While (R) holds if
all angles are $\leq \pi$, that
is for Ω convex.

