

LECTURE 13

Last Lecture:

- A posteriori error analysis

Today's lecture:

- Intro to parabolic problems
- Finite Difference methods for parabolic problems

Computer practical:

- Intro to FEniCSx
- FEM implementation in 1D

see
Larsson -
Thomeé
my notes
(Morton-Myers)

Parabolic problems

A parabolic PDE is of the form

$$u_t + Lu = f$$

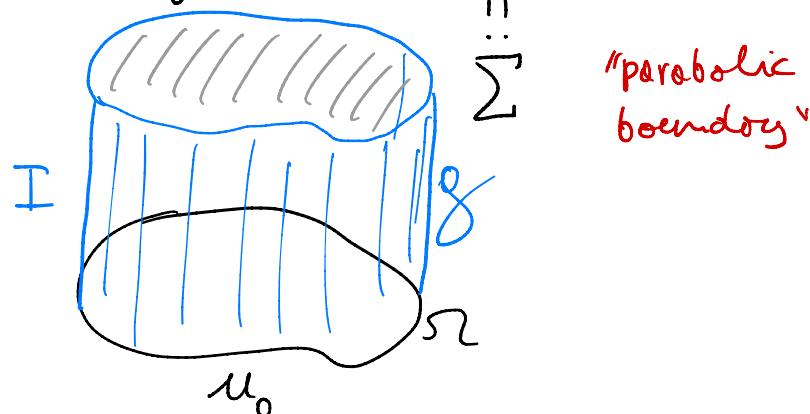
with L a linear elliptic operator of 2nd order in \mathbb{R}^d .

Prototypes : $\begin{cases} \text{Heat eq: } Lu = -\Delta u \\ \text{diffusion-convection-reaction in divergence form} \end{cases}$

Prototype problem : $\begin{cases} u_t + \mathcal{L} u = f & \text{on } \Sigma \times I \\ u(x, 0) = u_0(x) & \text{on } \Sigma \\ u(t, x) = g(t, x) & \text{on } \partial\Sigma \times I \end{cases}$

Cauchy-
Dirichlet
Pbm

$\Sigma \subset \mathbb{R}^n$ of
 $I = (0, T]$



with $g(0, x) = u_0(x)$

A priori bounds / max principle

- fix $\mathcal{L} = -\Delta u$ (heat eq.)
- $g = 0$
- call $E(t)u_0$ = solution operator of homog. problem ($f \equiv 0$)

Then, $u(\cdot, t) = u(t)$

$$= E(t)u_0 + \int_0^t E(t-s)f(s) ds$$

superposition of homog. sol.

Duhamel principle

$$\Rightarrow \|u(t)\|_{0,\Omega} \leq \|u_0\|_{0,\Omega} + \int_0^t \|f(s)\|_{0,\Omega} ds$$

a priori bound (stability)

Maximum Principle:

Let $u: u_t - \Delta u \leq 0$ in $\Omega \times I$. Then
 u attains ~~not~~ on parabolic boundary Σ
or at $t=0$.

If $\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times I \\ u(\cdot, 0) = u_0 & \text{in } \Omega \times \{0\} \\ u = g & \text{on } \Sigma \end{cases}$

then $\max |u| \leq \max \left\{ \max_{\Sigma} |g|, \max_{\Omega} |u_0| \right\} + C \max_{\Omega \times I} |f|$

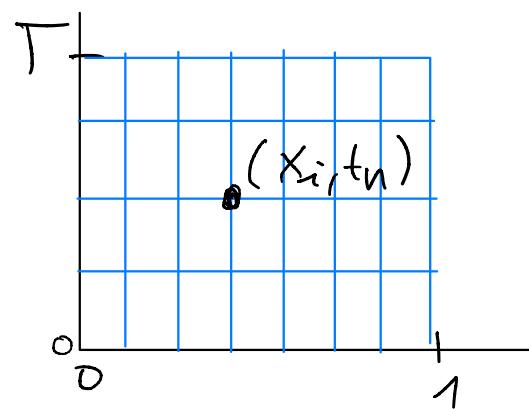
FINITE DIFFERENCE METHODS

$d = 1$, Heat problem in 1D:

$$\begin{cases} u_t = \alpha u_{xx} & \text{in } (0, 1) \times \mathbb{R} \\ u(x, 0) = u_0(x) & \text{in } (0, 1) \\ u(0, t) = u(1, t) = 0 \end{cases}$$

$$\alpha = \alpha(x, t)$$

- Fix grid: $N_x, N_t \in \mathbb{N}, h = \frac{1}{N_x}, k = \frac{1}{N_t}$
(uniform)



$$U_i^n \approx u(x_i, t_n)$$

Explicit (forward) Euler (EE) method

$$u_t(x, t) \approx \sum_{k=+}^t u(x, t) = \frac{u(x, t+k) - u(x, t)}{k}$$

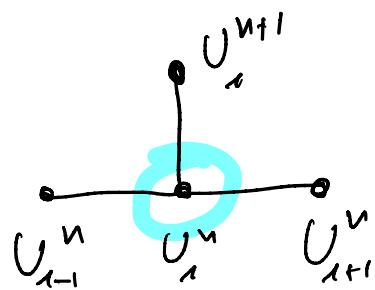
$$u_{xx}(x, t) \approx \left(\sum_h\right)^2 u(x, t) = \frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2}$$

$$\left\{ \begin{array}{l} U_i^0 = u(x_i) \\ \frac{U_i^{n+1} - U_i^n}{k} - \alpha_i^n \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0 \\ U_0^{n+1} = 0 = U_{N_x}^{n+1} \end{array} \right. \quad \forall n = 0, \dots, N_t - 1$$

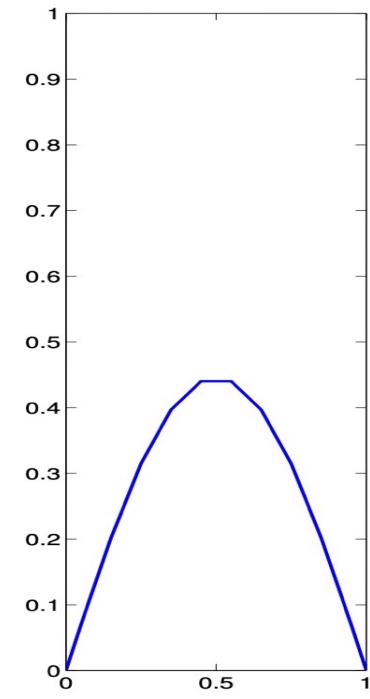
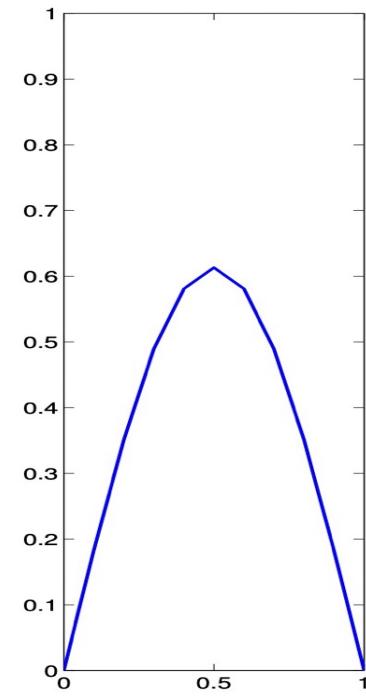
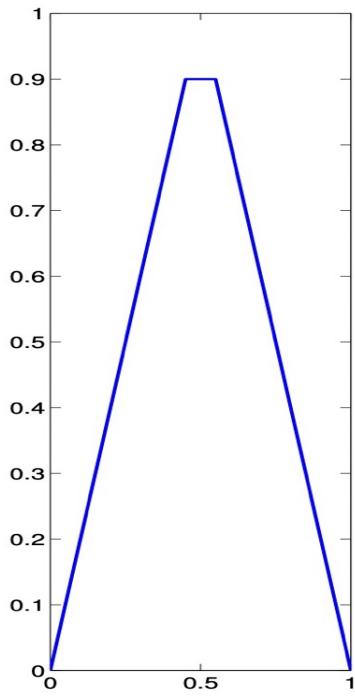
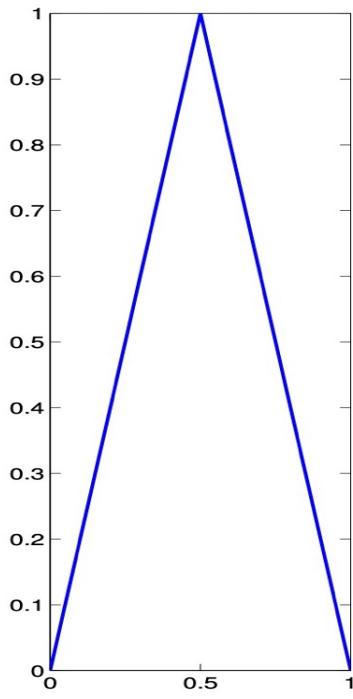
$$\Leftrightarrow \boxed{U_i^{n+1} = \alpha_i^n \mu U_{i+1}^n + \alpha_i^n (1-2\mu) U_i^n + \alpha_i^n \mu U_{i-1}^n}$$

$$\mu = \frac{k}{h^2} \quad \text{"current number" of scheme}$$

\Downarrow
Stencil:



$h=0.05, k=0.00125$ giving $\mu=0.5$

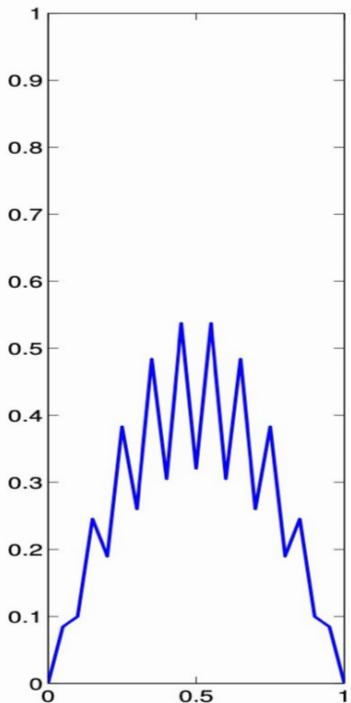
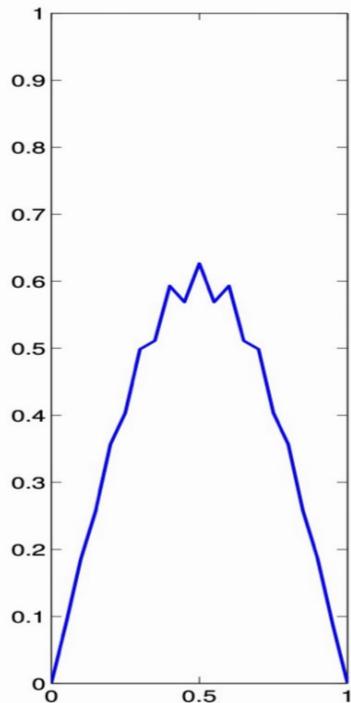
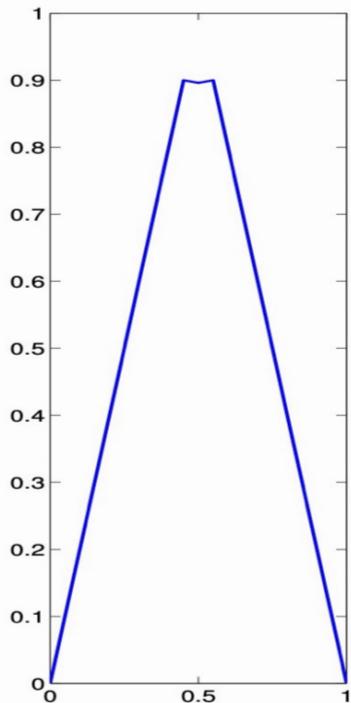
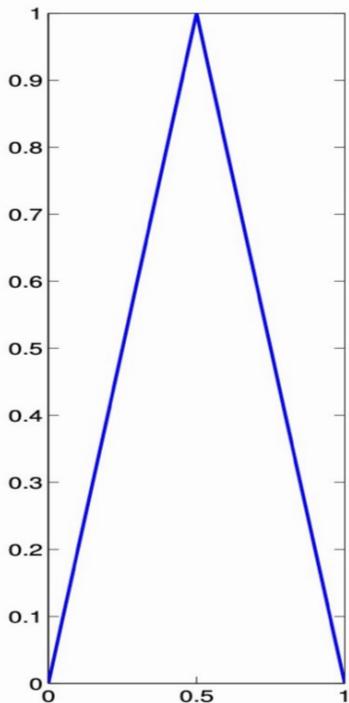


$n=0$

1

25

50



$h=0.05, k=0.0013$ giving $\mu=0.52$

analysis of EE

- $U^n = (U_0^n, \dots, U_{N_x}^n)$

$$\|U^n\|_{\infty,h} = \max_i |U_i^n|$$

- Discrete sol. op. E_k or $E_k U^n = U^{n+1}$
 $(\Rightarrow U^n = E_k^n U_0)$

- $e_i^n = u_i^n - U_i^n \quad u_i^n := u(x_i, t_n)$

- Truncation (local disc.) error

$$T(x, t) := \int_{k,+}^t u(x, \tau) - \alpha (\delta_h^x)^2 u(x, \tau)$$

$$T_i^n := T(x_i, t_n), \quad \|T_i^n\| = \max |T_i^n|$$

Consistency: $|T_i^n| \leq \frac{k}{2} M_{tt} + \frac{h^2}{12} \bar{\alpha} M_{xxxx}$

$$= \frac{k}{2} (M_{tt} + \frac{\mu}{6} \bar{\alpha} M_{xxxx})$$

where $\bar{\alpha} = \max_{\Omega \times I} |\alpha(x, t)|$

$$M_{tt} = \max |u_{tt}|, \quad M_{xxxx} = \max |u_{xxxx}|$$

$$\text{Proof: } \frac{u_t^{n+1} - u_t^n}{\Delta t} = (u_t)_x^n + \frac{\epsilon}{2} u_{ttt}(x_i, t_n) \\ \sum_{k=1}^t u =$$

$t_n \in (t_n, t_{n+1})$

$$(\delta_h^2) u_i^n = (u_{xx})_x^n + \frac{h^2}{24} (u_{xxxx}(z_i, t_n) + u_{xxxx}(z_{i+1}, t_n))$$

$z_i \in [x_{i-1}, x_i]$
 $z_{i+1} \in [x_i, x_{i+1}]$

take difference and apply PDE :

$$T_i^n = \underbrace{(u_t)_x^n - \alpha_i^n (u_{xx})_x^n}_{=: 0}$$

+ remainder term S

Stability : The BE method is stable \Leftrightarrow

$$\alpha \mu \leq 1/2 \quad \text{and} \quad \| U^n \|_{\infty, h} \leq \| U^0 \|_{\infty, h}$$

Proof : ① suff.

recall the scheme :

$$U_i^{n+1} = \alpha_i^n \mu U_{i+1}^n + (1 - \alpha_i^n \mu) U_i^n + \alpha_i^n \mu U_{i-1}^n$$

$$\Rightarrow |U_i^{n+1}| \leq \alpha_i^n \mu |U_{i+1}^n| + |(1 - \alpha_i^n \mu)| |U_i^n| + \alpha_i^n \mu |U_{i-1}^n|$$

if $\alpha_i^n \mu \leq 1/2$ \Rightarrow all coeffs are positive
and add up to 1:

$$= \alpha_i^n \mu |U_{i+1}^n| + (1 - \alpha_i^n \mu) |U_i^n| + \alpha_i^n \mu |U_{i-1}^n|$$

$$\leq \|U^n\|_{\infty, h} \leq \dots \leq \|U^0\|_{\infty, h}$$

② nec.

$$\text{let } U_i^0 = (-1)^i \varepsilon, \quad \varepsilon > 0$$

$$\Rightarrow \|U^0\|_{\infty, h} = \varepsilon.$$

and apply scheme:

$$U_i^1 = \alpha_i^1 \mu (-1)^{i+1} + (1 - 3\mu \alpha_i^1) (-1)^i + \alpha_i^1 \mu (-1)^{i-1}$$

$$= (-1)^i (1 - 4 \alpha_i^1 \mu) \varepsilon$$

$$\Rightarrow U_i^n = (1 - 4 \alpha_i^1 \mu) \dots (1 - 4 \alpha_i^n \mu) (-1)^i \varepsilon$$

if $\alpha = \alpha(x)$ then

$$= (1 - 4\alpha_1)^n (-1)^i \epsilon$$

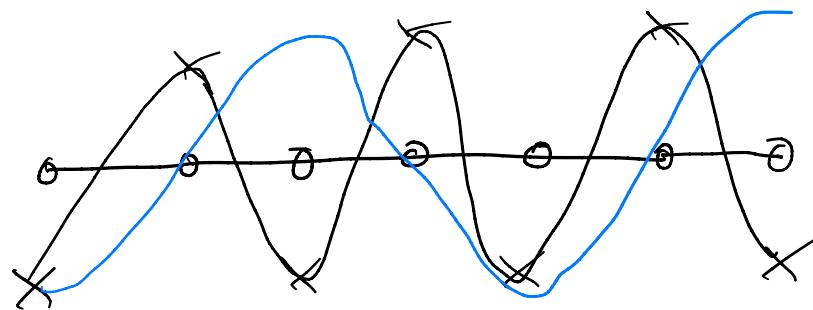
if $\alpha_i^n \mu > 1/2$ $\|U^n\|_{\infty, h} \xrightarrow[h \rightarrow 0]{} \infty$

for any ϵ fixed \square

(LT)

Remarks

- sawtooth type initial cond is the ✓ scheme can carry



and is the one most likely amplified.

- due to roundoff errors, instability expected of log 5.
- a closer look to stability

(\Rightarrow Von Neumann stability analysis)

Motivation:

$$u(t, \gamma) = \sum_{j=1}^{\infty} \hat{u}_j e^{-\gamma_j t} \varphi_j(x)$$

\uparrow

countable
modes carried
by exact sol

similarity /

$$U_t^n = \sum_{-\infty}^{\infty} a_j \gamma_j^n e^{i \gamma_j (ih)}$$

(in fact this reduces to finite sum)

\Rightarrow makes sense to study

$$U_t^0 = e^{i \gamma_0 (ih)} \quad (\alpha = 0)$$

$$\begin{aligned} \Rightarrow U_t^1 &= \mu e^{i \gamma_1 (1+h)} + (1-\mu) e^{i \gamma_0 (1+h)} + \mu e^{i \gamma_1 (1+h)} \\ &= e^{i \gamma_0 h} (\mu e^{i \gamma_1 h} + 1 - \mu + \mu e^{i \gamma_1 h}) \\ &= \mu \left(e^{\frac{i \gamma_1 h}{2}} - e^{-\frac{i \gamma_1 h}{2}} \right)^2 \end{aligned}$$

$$U_1^1 = e^{i\gamma_0 h} \underbrace{\left(1 - 4\mu \sin^2 \left(\frac{\gamma h}{2} \right) \right)}_{\lambda(\gamma)}$$

$$\Rightarrow U_i^n = \lambda(\gamma)^n e^{i\gamma nh}$$

stability: $| \lambda(\gamma) | \leq 1$

$$\Leftrightarrow -1 \leq 1 - 4\mu \sin^2 \left(\frac{\gamma h}{2} \right) \leq 1$$

$$\Leftrightarrow 0 \leq 4\mu \sin^2 \left(\frac{\gamma h}{2} \right) \leq 2$$

$$\Leftrightarrow 0 \leq \mu \sin^2 \left(\frac{\gamma h}{2} \right) \leq \frac{1}{2}$$

$$\Leftrightarrow \mu \leq \frac{1}{2}$$

some condition on before γ_0

Convergence : If $\alpha\mu \leq \frac{1}{2}$, then

$$\|e^n\|_{\infty,h} \leq T \left(\frac{h}{2} \Pi_{tf} + \frac{h^2}{12} \Pi_{xxxx} \right)$$

Proof: apply scheme + def. of trunc. error:

$$e_i^{n+1} - \alpha_i^n \underbrace{\left(e_{i+1}^n + (1-2\mu) e_i^n + \mu e_{i-1}^n \right)}_G = k T_i^n$$

$n = 0, \dots, N_f - 1$
 $i = 1, \dots, N_x - 1$

using $\alpha\mu < \frac{1}{2}$,

$$|e_i^{n+1}| \leq \alpha_i^n \mu |e_{i+1}^n| + (1-2\mu\alpha_i^n) |e_i^n| + \alpha_i^n \mu |e_{i-1}^n|$$

$$\leq \|e^n\|_{\infty,h} + k |T_i^n|$$

$$\Rightarrow \|e^{n+1}\|_{\infty,h} \leq \|e^n\|_{\infty,h} + k T_M$$

$$\leq \dots \leq \|e^0\|_{\infty,h} + (n+1) k T_M$$

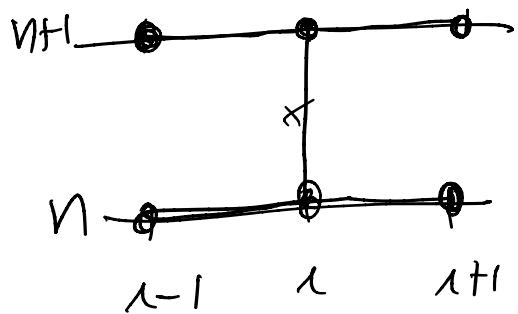
In particular,

$$\|e^n\|_{\infty,h} \leq T T_M$$

result follows from consistency bound (truncation error bound).

- if μ kept fixed, EE is $O(k)$

Implicit methods - ϑ method



$$\int_k^t U_l^{n+1/2} = \frac{U_l^{n+1} - U_l^n}{k}$$

$$= \vartheta (\delta_n^*)^2 U_l^{n+1} + (1-\vartheta) (\delta_n^*)^2 U_l^n$$

$$\vartheta \in [0, 1]$$

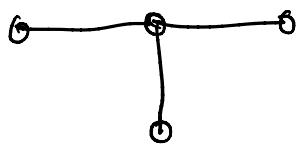
$$\vartheta = 0$$

EE



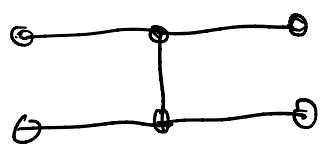
$$\vartheta = 1$$

IE



$$\vartheta = 1/2$$

CH



Truncation error

$$\vartheta\text{-method } O(k, h^2)$$

$$CH \quad (\vartheta = 1/2) \quad O(k^2, h^2)$$

Stability (von Neumann)

$$\mu(1 - z\vartheta) \leq 1/2$$

$$(z \equiv 1)$$

so stability is conditioned

$$\text{if } \vartheta < 1/2$$

} discrete
 ∞ -norm