Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 13

Last Lecture:
- Intro to parabolic problems
- Finite Difference methods for parabolic problems

Today's lecture:
- Implicit FD methods → LT, online notes NSPDE.pdf
- Weak form of parabolic problems
- Semidiscrete in space FEM ← Quarteroni

Morton Mayers

Computer practical:
- Computing errors
- FEM implementation in 2D

General theory

Consider IVP

$$(\mathrm{IVP})\begin{cases} u_t = A\,u(t) \\ u(0)=u_0 \end{cases} \qquad t \in I = (0,T]$$

$A : B \longrightarrow B$ $\quad B$ Banach, A linear assume (IVP) is well posed.

Let $S_k : B \to B$ a FD formula (k = time step), so that $v^{n+1} = S_k v^n$

- S_k is of order p if $\forall t \in I$, given u smooth sol. of (IVP), we have

$$\| u(t+k) - S_k u(t) \| = O(k^{p+1}) \text{ as } k \to 0$$

w.r.t. some given norm $\| \cdot \|$.

S_k is consistent if of order $p > 0$.

($\Rightarrow$ Consistency about local errors: these must be small at the time they are introduced)

S_k is convergent if $\forall t \in I$

$$\lim_{\substack{k \to 0 \\ nk = t}} \| u(t) - S_k^n u_0 \| = 0$$

for all initial values u_0

S_k is stable iff $\exists\, C > 0$:

$$\| S_k^n \| \le C \qquad \forall n, k : nk < T$$

(• does not depend on IVP
• stability $\equiv$ errors are not amplified as the computation progresses)

Theorem (Lax Equivalence): Let S_k be a consistent scheme for the well-posed (IVP). Then S_k is convergent $\Longleftrightarrow$ it is stable

(all notions measured with same norm!)

Apply theory to θ-method . . .

• $$\begin{cases} u_t = u_{xx} & (0,1) \times (0,T] \\ u(x,0) = u_0(x) & \text{in } (0,1) \\ u(0,t) = u(1,t) = 0 \end{cases}$$

ϑ-method : $\vartheta \in [0,1]$

$$\frac{U_{l+1}^{n+1} - 2U_l^{n+1} + U_{l-1}^{n+1}}{h^2} \; /\!/$$

$$\frac{U_l^{n+1} - U_l^n}{k} = \vartheta \left(\delta_h^x\right)^2 U_l^{n+1} + (1-\vartheta)\left(\delta_h^x\right)^2 U_l^n$$

$\mu = k/h^2$ Courant, reduces to

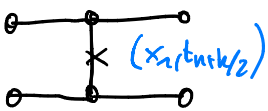
$$-\mu\vartheta U_{l+1}^{n+1} + (1 + 2\mu\vartheta) U_l^{n+1} - \mu\vartheta U_{l-1}^{n+1}$$

$$= \mu(1-\vartheta) U_{l+1}^n + (1 - 2\mu(1-\vartheta)) U_l^n + \mu(1-\vartheta) U_{l-1}^n$$

$$\rightsquigarrow \quad B U^{n+1} = A U^n \Rightarrow U^{n+1} = B^{-1} A U^n =: S_k U^n$$

tridiagonal

Consistency: Truncation error (max norm)



$$T_l^{n+1/2} = \delta_k^t u(t_{n+1/2}, x_l) - \vartheta \left(\delta_h^x\right)^2 u(t_{n+1}, x_l) - (1-\vartheta)\left(\delta_h^x\right)^2 u(t_n, x_l)$$

is of $\mathcal{O}(k, h^2)$ unless

$\vartheta = 1/2$ (CN) in which case

$$\mathcal{O}(k^2, h^2).$$

Stability: θ-method is stable in ℓ_∞ if

$$\mu(1-2\theta)\le 1/2$$

Proof:
(max principle)

$$(1+\mu\theta)\,U_i^{n+1} = \mu\theta\left(U_{i+1}^{n+1}+U_{i-1}^{n+1}\right) + (1-\theta)\mu\left(U_{i+1}^{n}+U_{i-1}^{n}\right) + \left(1-2\mu(1-\theta)\right)U_i^n$$

Coeffs are all positive if $\left(1-2\mu(1-\theta)\right)\ge 0$

$\therefore$ $\boxed{\mu(1-\theta)\le 1/2}$

$\Rightarrow$ take abs values $\Rightarrow$

$$(1+2\mu\theta)\,|U_i^{n+1}| \le 2\mu\theta\|U^{n+1}\|_{\infty,h} + \|U^n\|_{\infty,h}$$

$$\Downarrow$$

$$\|U^{n+1}\|_{\infty,h} \le \|U^n\|_{\infty,h} \le \dots \le \|U^0\|_{\infty,h}$$

Lax $\Rightarrow$ convergence !

Max norm analysis comments

- gives conditional convergence under
  $$\mu(1-\vartheta) \leq 1/2$$
- coincides with regime of monotonicity (sufficient for convergence !)
- apart from EE case (monotone $\Leftrightarrow$ convergent)
- $\vartheta = 1/2$ CN monotone $\Leftrightarrow \mu \leq 1$
  but in fact we can prove that CN (and $\forall \vartheta \geq 1/2$) always convergent $\forall \mu$!
- this analysis, however, generalises immediately to non-constant coeffs

Let's check von Neumann for θ-method plug $U_i^n = \lambda(\xi)^n e^{i\xi i h}$

$$\lambda(\xi) = \frac{1 - 4\mu(1-2\vartheta)\sin^2(\xi h/2)}{1 + 4\mu\vartheta\sin^2(\xi h/2)} \quad (ex)$$

stability condition is

$$|\lambda(\xi)| \leq 1$$

$$\Longleftrightarrow \mu(1-2\vartheta) \leq 1/2 \quad \text{if } \vartheta < 1/2$$

stability predicted under much less stringent conditions than monotonicity condition !

l_2 - norm for $V, W \in \mathbb{R}^{N_x+1}$

$$(V, W)_h = h \sum_{i=0}^{N_x} V_i W_i$$

$$\Rightarrow \|V\|_{2,h} = (V, V)_h^{1/2}$$

ℓ_2 space of $V \in \mathbb{R}^{M_x+1}$: $\|V\|_{2,h} < +\infty$

$$\ell_2^\circ = \{ V \in \ell^2 : V_0 = V_{M_x+1} = 0 \}$$

$$\overset{!}{=} \mathrm{span} \langle \varphi_j \rangle_{j=1}^{M_x-1}$$

$$\varphi_{j,i} = \sqrt{2} \sin(\pi j i h)$$

$\forall i = 0, \dots, M_x$

- $(\varphi_\ell, \varphi_m)_h = \delta_{\ell m}$ orthonormal basis

- φ_j's are eigenfunctions of $(\delta_h^x)^2$:

$$-(\delta_h^x)^2 \varphi_{j,i} = \frac{2}{h^2}\left(1-\cos(\pi j h)\right)\varphi_{j,i}$$

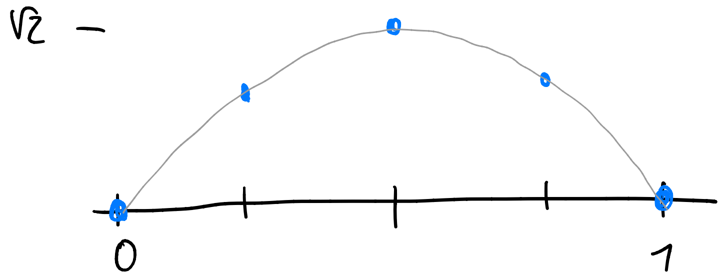
$i = 1, \dots, M_x - 1$

example : $N_x = 4$

$j=1 \qquad u_{1,i} = \sqrt{2}\,\sin(\pi i h)$

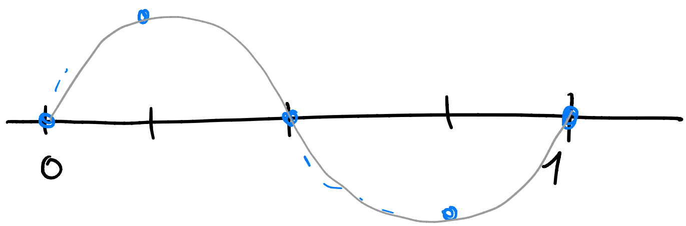
$i=4$

$4h=1$



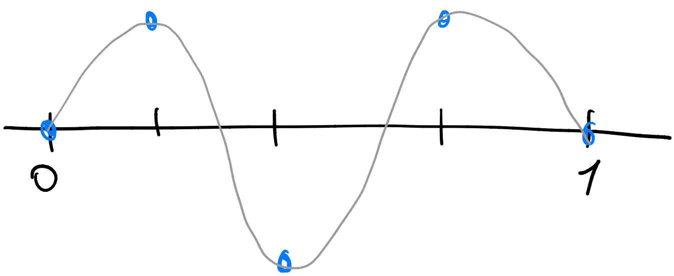
$j=2 \qquad u_{2,i} = \sqrt{2}\,\sin(\pi\, \underbrace{2ih})$

$i=4 \Rightarrow 8h=2$



$j=3 \qquad u_{3,i} = \sqrt{2}\,\sin(\pi\, 3ih)$

$i=4 \Rightarrow 12h=3$



Let U^0 initial data, then

$$U^0 = \sum_{j=1}^{M_x-1} \hat{U}_j^0 \varphi_j \quad \text{with } \hat{U}_j^0 = (U_0, \varphi_j)_h$$

Then, for instance,

- EE method:

$$U_i^1 = U_i^0 + k (\delta_h^x)^2 U_i^0$$

$$= \sum_{j=1}^{M_x-1} \hat{U}_j^0 \underbrace{\left(1 - 2\mu\left(1-\cos(\pi j h)\right)\right)}_{S_k(jh)} \varphi_{j,i}$$

$$\Rightarrow U_i^n = \sum_{j=1}^{M_x-1} \hat{U}_j^0 \underline{S_k(jh)^n} \varphi_{j,i} \quad \forall i = 1, \dots, M_x - 1$$

growth factor

$$\|U^n\|_{2,h} = \left(\sum_{j=1}^{M_x-1} (\hat{U}_j^0)^2 (S_k(jh))^{2n}\right)^{1/2}$$

Parseval identity

$$\leq \max_{\gamma} |S_k(\gamma h)^n| \; \|U^0\|_{2,h}$$

stability in ℓ_2 - norm $\iff |S_k(\gamma h)| \leq 1$

$\iff \boxed{\mu \leq 1/2}$

same result as before !

Implicit Euler ($\vartheta = 1$) :

$$S_k(\gamma h) = 1 + 2\mu(1 - \cos(\pi\gamma h))$$

$|S_k| < 1 \Rightarrow$ uncond. stable !

different from before !

CN ($\vartheta = 1/2$)

$$S_k(\gamma h) = \frac{1 - 2(1-\vartheta)\mu(1 - \cos(\pi\gamma h))}{1 + 2\vartheta\mu(1 - \cos(\pi\gamma h))}$$

$\vartheta \geq 1/2 \qquad |S_k| < 1 \Rightarrow$ uncond. stab !

$\Rightarrow$ CN 2^{nd} order for all μ

in the $(\underbrace{\ell_\infty}_{\text{time}} - \underbrace{\ell_2}_{\text{space}})$ - norm

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Weak form of heat equation

$$\begin{cases} u_t - \Delta u = f & \Omega \times I \\ u(0,x) = u_0(x) & x \in \Omega \\ u = 0 & \partial\Omega \times I =: \Sigma \end{cases}$$

for any given t, test with $v \in H_0^1(\Omega)$:

$\Downarrow$

$$\frac{d}{dt}\int_\Omega u(t)\, v + \int_\Omega \nabla u \cdot \nabla u = \int_\Omega f\, v \quad \forall v \in V$$

$$\Leftrightarrow \frac{d}{dt}\underbrace{\left(u(t), v\right)}_{L^2\text{- product}} + A(u,v) = F(v) = (f, v)$$

Issue: $u(t)$??

$$u(t) : \Omega \longrightarrow \mathbb{R}$$

$$u : I \longrightarrow W \quad \in \text{ some banach space}$$

$$t \longmapsto u(t) = u(\cdot, t)$$

Hence, define

space w.r.t. origins is made!

$$u \in L^2(I; V) \cap C^0(\bar{I}; L^2(\Omega))$$

iff $u(t) \in V$ and $\int_I \|u(t)\|_V^2 \, dt < \infty$

iff $\forall \bar{t} \in \bar{I}$ $\lim_{t \to \bar{t}} \|u(t) - u(\bar{t})\|_{L^2(\Omega)} = 0$

weak form: $f \in L^2(I; L^2(\Omega))$

$u_0 \in L^2(\Omega)$

find $u \in L^2(I; V) \cap C^0(\bar{I}; L^2(\Omega))$:

$$\frac{d}{dt}(u(t), v) + a(u, v) = F(v) \quad \forall v \in V$$

$t > 0$

same for more general elliptic op ∇

well-posedness

we will require

- a cont and coercive

although, coercivity can

be relaxed, to:

<u>Def</u>: $\mathcal{A}$ is WEAKLY COERCIVE in V with $H_0^1(\Omega) \subseteq V \subseteq H^1(\Omega)$ if $\exists\, \alpha > 0,\ \lambda \geq 0 : \forall v \in V$

$$\mathcal{A}(v,v) + \lambda \|v\|_0^2 \geq \alpha \|v\|_1^2$$

($\lambda = 0$ $\therefore$ coercive)

- F continuous

Theorem (well-posedness):
Assume $\mathcal{A}$ cont. and coerc.,
$f \in L^2(I; L^2(\Omega))$, $u_0 \in L^2(\Omega)$

Then $\exists!\ u \in L^2(I;V) \cap C^0(\bar{I};L^2(\Omega))$ sol. of the weak problem.

Moreover

$$u_t \in L^2(I;V')$$

$$\max_{t\in\bar{I}} \|u(t)\|_0^2 + \alpha \int_I \|u(t)\|_1^2 \le \|u_0\|_0^2 + \frac{1}{\alpha}\int_I \|f(t)\|_0^2 \quad \text{(ee)}$$

$L^\infty - L^2$ $\quad\quad$ $L^2 - H^1$

"energy estimate"

<u>Proof</u>:

- existence via FAEDO-GALERKIN method i.e showing that a sequence of Galerkin discrete solutions exist on the spaces generated by an orthonormal basis and proving

$u_n \longrightarrow u \quad \exists u$

- uniqueness follows from (ee)

- proof of (ee) by energy argument

test weak problem with $v = u(t)$

$\therefore \quad (u_t, u) + a(u, u) = (f, u)$

- $(u_t, u) = \int_\Omega u_t u = \frac{1}{2} \int_\Omega (u^2)_t = \frac{1}{2} \frac{d}{dt} \|u\|_0^2$

- $a(u, u) \overset{\text{coerc}}{\geq} \alpha \|u\|_1^2$

- $(f, u) \overset{\text{cont}}{\leq} \|f\|_0 \|u\|_0 \leq \|f\|_0 \|u\|_1$

use Young's inequality

$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \qquad \forall \varepsilon > 0$
$\forall a, b \in \mathbb{R}$

with $\varepsilon = \frac{1}{2\alpha}$

$$\le \frac{1}{2\alpha}\|f\|_0^2 + \frac{\alpha}{2}\|u\|_1^2$$

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2 + \frac{\alpha}{2}\|u\|_1^2 \le \frac{1}{2\alpha}\|f\|_0^2 + \cancel{\frac{\alpha}{2}\|u\|_1^2}$$

$$\Rightarrow \quad \frac{d}{dt}\|u\|_0^2 + \alpha\|u\|_1^2 \le \frac{1}{\alpha}\|f\|_0^2$$

int. over $(0,t)$:

$$\|u(t)\|_0^2 + \alpha\int_0^t \|u\|_1^2\,d\tau \le \|u_0\|_0^2 + \frac{1}{\alpha}\int_0^t \|f\|_0^2\,d\tau$$

$\square$

Semidiscrete FEM

Let $\mathcal{T}_h$ be trianqulation (mesh) of Ω and $V_h^k \subset V$ corresponding finite element space of piecewise $\mathbb{P}^k$ continuous functions.

semidiscrete FEM: $\forall\ t \in [0, T]$ find $u_h(t) \in V_h^k$:

$$(SD) \begin{cases} \frac{d}{dt}(u_h(t), v) + a(u_h(t), v) = F(v) \quad \forall v \in V_h^k \\ u_h(0) = u_{0,h} \end{cases}$$

with $u_{0,h} \in V_h^k$ some approx. of u_0

Well posedness and energy estimate follows as before yielding $\exists!\ u_h$ sol of (SD) and

$$\|u_h(t)\|_0^2 + \alpha \int_0^t \|u(t)\|_1^2 \, d\tau \le \|u_{0,h}\|_0^2 + \frac{1}{\alpha}\int_0^t \|f(\tau)\|_0 \, d\tau$$

so if $\|u_{0,h}\|_0 \to \|u_0\|_0$ as $h \to 0$

we have stability estimate $L_\infty - L_2$ and $L_2 - H^1$.

Matrix Form

- fix $\{\varphi_j\}$ basis of V_h^k
- write $u(t) = \sum_j U_j(t)\, \varphi_j$
- " $u_{0/h} = \sum_j U_j^0\, \varphi_j$
- test (SD) with φ_i :

$$\begin{cases} M\, U'(t) + A\, U(t) = F(t) \\ U(0) = U^0 \end{cases}$$

where

$$M = (\varphi_i, \varphi_j) \qquad \text{MASS MATRIX}$$

$$A = \mathcal{A}(\varphi_i, \varphi_j)$$

$$(F(t))_i = (f(t), \varphi_i)$$