

Numerical solution of PDEs - NSPDE

Advanced Numerical Analysis - ANA

LECTURE 4

Last Lecture:

- Divided differences
- Elliptic PDEs
- General DMP result
- Applications: more general two-points BVPs, multi-D problems

Today's lecture:

- FD on non-uniform meshes
- FD on general domains
- Basic notions of functional analysis
- Intro to weak formulations

Computer practical

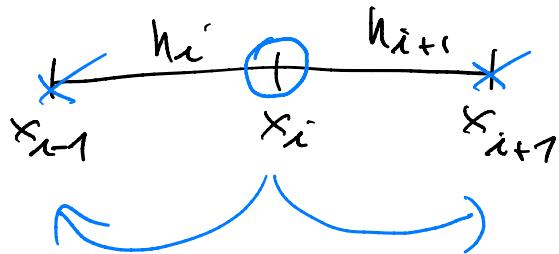
- Divided differences and FD for two-points BVPs

$$\begin{cases} -u'' = f & \text{on } (0, 6) \\ u(0) = 0 = u(6) \end{cases}$$

(see online notes)
Morton-Rogers

fix h_i , $i=1, \dots, N$ such that $\sum_i h_i = b-a$

$$x_i := x_{i-1} + h_i$$



Method of indetermined coefficients:

determine coeffs $\alpha, \beta, \gamma \in \mathbb{R}$ in

$$L := \alpha u_{i+1} + \beta u_i + \gamma u_{i-1}$$

so as to obtain as high as possible

approx of $u''(x_i) =: u''_i$ ($u_i = u(x_i)$)

expand by x_i

$$L' = \alpha(u_i + h_{i+1} u'_i + \frac{h_{i+1}^2}{2} u''_i + \frac{h_{i+1}^3}{6} u'''(\xi_{i+1}))$$

if $h_{i+1} = h_i \Rightarrow //$
 and gain 1 order

$\xi_{i+1} \in [x_i, x_{i+1}]$

$$+ \beta u_i$$

$$+ \gamma(u_i - h_i u'_i + \frac{h_i^2}{2} u''_i - \frac{h_i^3}{6} u'''(\xi_i))$$

$\xi_i \in [x_{i-1}, x_i]$

$$\left\{ \begin{array}{l} \alpha + \beta + \gamma = 0 \\ \alpha h_{i+1} = \gamma h_i \Rightarrow \\ \alpha \frac{h_{i+1}^2}{2} + \gamma \frac{h_i^2}{2} = 1 \end{array} \right\} \quad \begin{array}{l} \beta = -\alpha - \gamma \\ \alpha = \frac{\gamma h_i}{h_{i+1}} \gamma \\ \frac{h_i h_{i+1}}{2} \gamma + \frac{h_i^2}{2} \gamma = 1 \end{array}$$

$$\left\{ \begin{array}{l} \beta = -\frac{2(h_i + h_{i+1})}{n_i h_{i+1} (h_i + h_{i+1})} \\ \lambda = \frac{2}{h_{i+1} (h_i + h_{i+1})} \\ \gamma = \frac{2}{n_i (h_i + h_{i+1})} \end{array} \right.$$

FD scheme: Find $U = \{U_i\}_{i=0}^H$:

$$\left\{ \begin{array}{l} U_0 = 0 \\ -\left(\frac{2}{h_{i+1} (h_i + h_{i+1})} U_{i+1} - \frac{2}{h_i h_{i+1}} U_i + \frac{2}{h_i (h_i + h_{i+1})} \right) \\ \qquad \qquad \qquad = f_i \\ U_H = 0 \end{array} \right. \quad i = 1, \dots, H-1$$

Truncation error

$$|T_i| \leq \frac{2}{3} h_{\max} \|u''\| C + \frac{1}{12} \frac{h_{\max}^3}{h_{\min}} \|u'''\| \frac{\|u^{IV}\|}{C}$$

→ error $\in \mathcal{O}(h_{\max})$

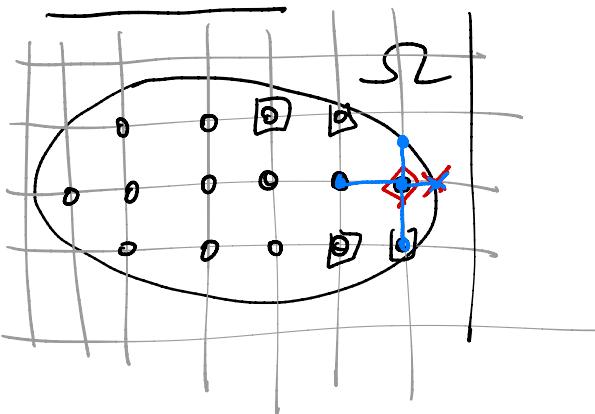
but need $\frac{h_{\max}^3}{h_{\min}} \rightarrow 0$ for convergence

FD for general domains

$$\Omega \subset \mathbb{R}^2, \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

open, bounded domain

convex



S_h = internal nodes with only internal neighboring nodes

ω_h = set of internal nodes with external weights

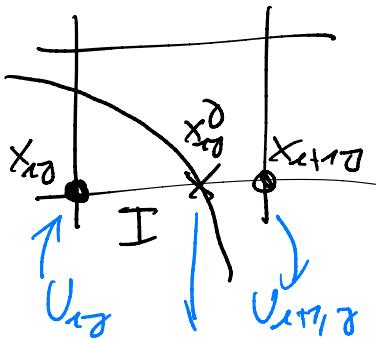
Shortley-Weller scheme: use the nonuniform grid scheme at non-boundary points on

- Truncation error is
 - quadratic at nodes in S_h
 - linear $\approx \dots \approx w_h$
 - 5 done
- however by the generalized DMP only now it is possible to use these truncation error functions (comparison functions) to show that

$$(u_{ij} - U_{ij}) \leq \frac{d^2}{36} h^2 \|u''\|$$

Hence, local lower-order truncation at boundary does not necessarily reduce the overall order?

- From LT: different approach based on interpolating data on interval I cut by boundary coincide to similar scheme



from state at x_{i0}^d to $x_{i+1,j}$ \downarrow
 $y(x_{i0}^d)$

$$V_{i,j} = \lambda V_{i+1,j} + (1-\lambda) y(x_{i0}^d)$$

where λ related to distance between the points