Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 3

Last Lecture:
- Poisson problem in strong form: Well-posedness
- Numerical methods and the Lax-Richtmyer equivalence theorem
- Finite Difference discretisation: consistency, stability, convergence

Today's lecture:
- Divided differences
- Elliptic PDEs
- General DMP result
- Applications: more general two-points BVPs, multi-D problems

Computer practical
- Intro to Python for Scientific Computing
- Divided differences and FD for two-points BVPs

Divided differences

$f : \mathbb{R} \to \mathbb{R}$ bounded, $h > 0$

$$\delta_h f(x) = \frac{f(x+h/2) - f(x-h/2)}{h}$$ central diff.

$$\delta_{h,-} f(x) = \frac{f(x) - f(x-h)}{h}$$ backward diff.

$$\delta_{h,+} f(x) = \frac{f(x+h) - f(x)}{h}$$ forward diff

$$\delta_{h,+2} f(x) = \left(\delta_{h,+} - \frac{h}{2}\,\delta_{h,+}^2\right) f(x)$$

$$= \frac{1}{2h}\left(-f(x+2h) + 4f(x+h) - 3f(x)\right)$$

$$\delta_{h,-,2}\, f(x) = \left(\delta_{h,-} + \frac{h}{2}\,\delta_{h,-}^2\right) f(x)$$

one-sided higher-order formulas

<u>Lemma:</u> $f \in C^2([a,b])$, $a < b \in \mathbb{R}$, $h > 0$

$$|f'(x) - \delta_{h,+} f(x)| \leq \frac{h}{2} \|f''\|_{C([a,b])} \quad \forall x \in [a, b-h]$$

$$|f'(x) - \delta_{h,-} f(x)| \leq \frac{h}{2} \|f''\|_{C([a,b])} \quad \forall x \in [a+h, b]$$

$$|f'(x) - \delta_h f(x)| \leq \frac{h^2}{24} \|f'''\|_{C([a,b])} \quad \forall x \in [a+\tfrac{h}{2}, b-\tfrac{h}{2}]$$

if $f \in C^3([a,b])$

$$|f'(x) - \delta_{h,+,2} f(x)| \le h^2 \, \|f'''\|_{C([a,b])}$$
$$x \in [a, b-2h]$$

$$\underbrace{|f'(x) - \delta_{h,+,2} f(x)|}_{e_h} \le h^2 \, \|f'''\|_{C([a,b])}$$
$$x \in [a+2h, b]$$

plotting rates in loglog plots

$$|e_h| \le C \, h^p$$

$$\log |e_h| \le p \log h + \log C$$

estimation of p EOC

$$\frac{|e_h|}{|e_{h/2}|} \approx \frac{\cancel{C} h^p}{\cancel{C} (h/2)^p} \sim 2^p \Rightarrow \log \frac{|e_h|}{|e_{h/2}|}$$

$$\approx p \log 2$$

where we say that g_h converges to g

if $\lim_{h \to 0^+} (g - g_h) = 0$

and it is of order p if

$$\lim_{h \to 0^+} \frac{|g - g_h|}{h^p} \in \mathbb{R}$$

we say $|g - g_h| = O(h^p)$.

Elliptic PDEs

2D 2^{nd} order PDE

$$a\, u_{xx} + 2b\, u_{xy} + c\, u_{yy} + L.O.T. = 0$$

elliptic: if $-D = ac - b^2 > 0$

Def: general 2^{nd} order operator in $\mathbb{R}^d$ in DIVERGENCE FORM or

$$\mathcal{L}u = -\sum_{i,j=1}^{d} D_i\left(a_{ij}(x) D_j u\right) + \sum_{i=1}^{d} D_i(b_i u) + c\,u$$

by testing and integrating, $\mathcal{L}$ defines the bilinear form:

$$a(u,v) = \int_\Omega \sum_{ij} a_{ij} D_j u\, D_i v + \int_\Omega \sum_i b_i u\, D_i v + \int_\Omega c\, u\, v$$

note: valid for $a_{ij}, b_i, c \in L^\infty$

$u, v \in H^1_0$

<u>Def</u>: The general 2^{nd} order op. $\mathcal{L}$ is <u>ELLIPTIC</u> if the matrix

$A = \{a_{ij}(x)\}_{ij}$ is <u>positive definite</u> almost everywhere in Ω, that is,

$$(A\xi, \xi) = \sum_{i,j=1}^{d} a_{ij}(x)\xi_i \xi_j > 0$$

DMP (Morton + Mayers book, p. 172)

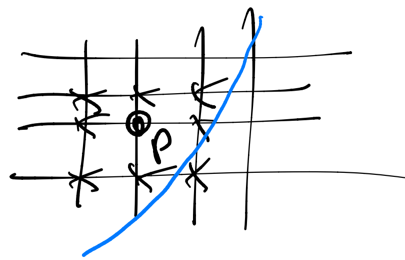
Given any elliptic problem on some domain $\Omega \subseteq \mathbb{R}^d$

- $\overline{\Omega}_h$ set of grid points
  ($\partial\Omega_h$ = set of boundary points
  Ω_h = " internal points)
- $\forall P \in \Omega_h$, $\mathcal{L}_h U_P + \underbrace{f_P}_{\text{forcing}} + \underbrace{g_P}_{\text{boundary}} = 0$

such that

$$\mathcal{L}_h U_P = c_P U_P - \sum_K c_K U_K$$

K ranges on neighbouring points



with • all coeffs are positive

• $c_P \geq \sum_K c_K$

then <u>DMP</u>: If a grid function U satisfies $\mathcal{L}_h U_P \leq 0 \quad \forall P \in \Omega_h$

then $\max\limits_{P\in\Omega_h} U_P \leq \max\left\{\max\limits_{Q\in\partial\Omega_h} U_Q, 0\right\}$

≥ 0

$\|$ min

Proof: exercise

More general two-points BVP.

Consider

$$\begin{cases} \mathcal{L}u = -a u'' + b u' + c u = f & \text{in } \Omega = (0,1) \\ u(0) = u_0 \ , \quad \underline{u'(1) = 0} & \end{cases}$$

homogeneous Neumann b.c.

$a = a(x), \ \dots$ smooth enough

(LT chapter 2, Gilbarg-Trudinger "Elliptic PDEs of 2nd order")

FD discretisation:

again, let $x_i = h i \quad i = 0, \dots, N$ and look for $U_i \approx u(x_i)$:

$$\begin{cases} U_0 = u_0 \\ -a_i \dfrac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + b_i \dfrac{U_{i+1} - U_{i-1}}{2h} + c_i U_i = f_i \\ ?? \quad u_N \end{cases}$$

$a_i = a(x_i), \dots,$

??

- Introduce ficticious node $x_{M+1} = (M+1)h$
- approx $u'(x_M) = 0$ $\qquad \dfrac{U_{M+1} - U_{M-1}}{2h} = 0$

$$\Rightarrow U_{M+1} = U_{M-1}$$

- apply our FD scheme at U_M :

$$-a_M \frac{\overset{= U_{M-1}}{U_{M+1}} - 2U_M + U_{M-1}}{h^2} + b_M \cancel{\frac{U_{M+1} - U_{M-1}}{2h}} + c_M U_M = f_M$$

$$\Rightarrow \boxed{2a_M \frac{U_M - U_{M-1}}{h^2} + c_M U_M = f_M}$$

<u>Matrix Form</u> : $L U = F$

$$L = A + B + C \qquad M \times M$$

$$A = \frac{1}{h^2} \begin{array}{c} \left(\begin{array}{ccccc} 2a_1 & -a_1 & & & 0 \\ -a_i & \ddots & \ddots & & \\ & -a_i & 2a_i & -a_i & \\ & & \ddots & \ddots & \ddots \\ 0 & & & -2a_M & 2a_M \end{array} \right) \end{array} \begin{array}{c} 1 \\ \\ \\ \\ M \end{array}$$

Analysis (LT, online notes)

Consistency

$$|T(x)| \leq \left(\frac{h^2}{12}\|a\|_C + \frac{h^2}{6}\|b\|_C\right)\|u^{(4)}\|_C$$

$$= \mathcal{O}(h^2)$$

DMP: Let $V = \{V_i\}$. If $\mathcal{L}_h V_i \leq 0$

$\forall\, i = 1, \dots, M-1$, then

(i) if $c = 0$, $\overbrace{a_j}^{>0} \pm \frac{1}{2} h\, b_j \geq 0 \Rightarrow \max_i V_i =$

$= \max\{V_0, V_M\}$

(ii) if $c \geq 0$ $\quad$ " $\quad \Rightarrow \max_i V_i =$

$\max\{V_0, V_M, 0\}$

coeffs of U_{i+1}, U_{i-1} need to be positive !

NOTE: if diffusion is "small" then require h small enough! (for stability)

DMP $\Rightarrow$ stability:

for any $V = \{V_i\}$

$$|V|_\infty \leq \max\{|V_0|, |V_M|\} + C\,|\delta_h V|_\infty$$

$$\left(|\cdot|_\infty = \max_i |V_i|\right)$$

Consistency + stability $\Rightarrow$ convergence $O(h^2)$

FD for elliptic PBMs in higher dimensions

Poisson 2D

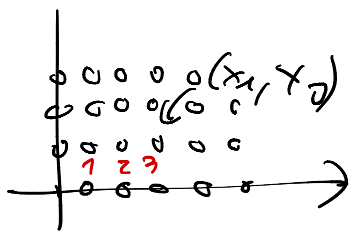
$$\begin{cases} -\Delta u = -u_{xx} - u_{yy} = f & \Omega = (0,1)^2 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- Fix $N_x \times N_y$ grid (x_i, y_j):

$$\begin{cases} x_i = i h_x & h_x = 1/N_x \\ y_j = j h_y & h_y = 1/N_y \end{cases}$$

- Fix disc. function $U = \{U_{ij}\}$

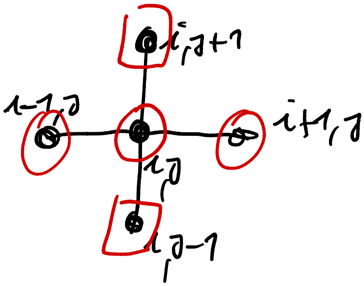
$$U_{ij} \approx u(x_i, y_j)$$



- FD:

$$\begin{cases} U_{ij} = 0 & (x_i, y_j) \text{ on border } \partial\Omega_h \\ (\delta_h^x)^2 U_{ij} + (\delta_h^y)^2 U_{ij} = f_{ij} := f(x_i, y_j) \end{cases}$$

STENCIL



$$\forall (x_i, y_j) \in \Omega_h$$

$$\left(\frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h_y^2} \right.$$

- write scheme in matrix form

$$A U = F$$

$$U = \{ U_{1,1}, U_{2,1}, \ldots, U_{M-1,1}, U_{1,2}, U \ldots$$

$$F = \{ f_{1,1}, f_{2,1} \ldots$$

$$A = - \begin{pmatrix} B & I & & \\ I & B & \ddots & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$ where

$I = (M-1) \times (M-1)$ identity matrix

$$B = \begin{pmatrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

DMP, stability, consistency, convergence follow as before (LT) giving

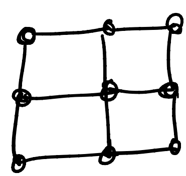
if $u \in C^4(\Omega) \cap C^0(\bar{\Omega})$ then

$$\underset{\substack{\| \\ u(x_i, y_j)}}{|u_{i,j} - U_{i,j}|} \le \frac{h^2}{96}\left(\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{C(\Omega)} + \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{C(\Omega)} \right)$$

$$= O(h^2)$$

- general 2nd order elliptic $\rightarrow$ trivial

$$u_{xy} \rightarrow \delta^{y}_{2h_y} \delta^{x}_{2h_x} u$$



u_{xx}, u_{xy}, u_{yy}

- In general, to devise higher-order formulas, requires larger stencils. However, there exist COMPACT higher order formulas

For example, a 9-points formula

for $-\Delta u = f$ of 4^{th} order can

be obtained by:

- consider instead $-\left[\Delta + \frac{1}{12} h^2 \Delta^2\right] u$

  $(h_x = h_y = h)$ $= f + \frac{1}{12} h^2 \Delta f$

- use standard central formulas

(details: ISERLES "A first course in the numerical analysis of differential equations", 2009, Cambridge)