Numerical solution of PDEs - NSPDE
Advanced Numerical Analysis - ANA

LECTURE 9

Last Lecture:
- FEM in more dimensions
- Conditioning of the FEM system

Today's lecture:
- FEM a priori error analysis

(See Quarteroni)

Computer practical
- more FEM for two-points bvps

FEM analysis in $\Omega \subset \mathbb{R}^d$, $d=2,3$

FEM: Given V Hilbert, $V_h = V_h^k \leq V$ subspace, defined starting from a triangulation $\mathcal{T}_h$ define Galerkin method as: find $u_h \in V_h$:

$$\mathcal{A}(u_h, v_h) = \mathcal{F}(v_h) \qquad \forall v_h \in V_h$$

- Lax-Milgram: $\exists !$ solution, $\|u_h\|_V \leq \frac{1}{\alpha} \|\mathcal{F}\|_{V'}$ (stability)
- Galerkin Orthogonality

$$\mathcal{A}(\underbrace{u - u_h}_{e_h}, v_h) = 0 \qquad \text{(consistency)}$$

• stab + cons. ⇒ Cèa lemma:

$$\|u-u_h\|_V \le \frac{\gamma}{\alpha} \inf_{v_h\in V_h} \|u-v_h\|_V$$

• to assess convergence w.r.t. h, use

$$\le \frac{\gamma}{\alpha} \|u - I_h^k u\|_V$$

for some $I_h^k u \in V_h$ well-chosen

Idea: use Lagrange Interpolant for $I_h^k u$.

Analysis of FE interpolation error

1. <u>Define interpolant.</u>

• fix $v \in H^s(\Omega)$ with $s \ge 2$ as this ensures (by Sobolev imbedding) that
$$H^s(\Omega) \hookrightarrow \mathcal{C}^0(\bar\Omega)$$

• **Def** (Lagrange Interpolant).
Let $V_h^k \subseteq \mathcal{C}^0(\Omega)$ the FE related to $\mathcal{T}_h$, so
$$V_h^k = \{v \in \mathcal{C}^0(\bar\Omega) : v|_T \in \mathbb{P}_k(T) \quad T\in\mathcal{T}_h\}$$

and let $\{\varphi_j\}$ basis of V_h^k s.t.

$$\varphi_j(x_i) = \delta_{ij}$$

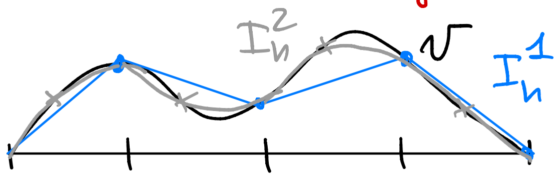
for each node x_i of the triangulation

Then the Lagrange Interpolant is

$$I_h^k : C^0(\bar{\Omega}) \longrightarrow V_h^k$$

$$v \longrightarrow I_h^k v = \sum_j v(x_i)\, \varphi_j(x)$$

- eg. in 1D



- note $I_h^k v|_T \in \mathbb{P}^k(T)$ and the issue is assess error $v|_T - I_h^k v|_T$ or then

$$\| v - I_h^k v \|_V = \left(\sum_T \| v - I_h^k v \|_T^2 \right)^{1/2}$$

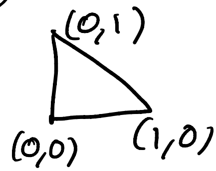
CRUCIAL IDEA: reduce to reference element

issue: scaling of norms. for instance take

T triangle with $\mathrm{diam}(T) = h_T$, then

$$\int_T \nabla \varphi_i \cdot \nabla \varphi_j \simeq \frac{|T|}{h_T^2} \qquad \int_T \varphi_i \, \varphi_j \simeq \text{area}(T)$$

Recall reference element $\hat{T}$



and

reference-to-physical affine map:

$$F_T : \hat{T} \longrightarrow T$$
$$\hat{x} \longrightarrow F_T(\hat{x}) = B_T \hat{x} + \underline{b}_T$$

B_T invertible matrix, $\underline{b}_T$ translation vector

Lemma 1 (scaling of seminorms) For each $m \geq 0$ and $v \in H^m(T)$, let $\hat{v} : \hat{T} \to \mathbb{R}$ def. by $\hat{v} = v \circ F_T$

Then $\hat{v} \in H^m(\hat{T})$, and $\exists\, C = C(m) > 0$:

$$\boxed{|\hat{v}|_{m,\hat{T}} \leq C \, \|B_T\|^m \, |\det B_T|^{-1/2} \, |v|_{m,T}},$$

$$\boxed{|v|_{m,T} \leq C \, \|B_T^{-1}\|^m \, |\det B_T|^{1/2} \, |\hat{v}|_{m,\hat{T}}}.$$

where $|\cdot|_{m,\omega} = |\cdot|_{H^m(\omega)}$ ω domain, and $\|\cdot\|$ is matrix norm assoc. to Euclidean norm of vectors $|\cdot|$, so $\|B_T\| := \sup_{\substack{\xi \in \mathbb{R}^d \\ \xi \neq 0}} \frac{|B_T \xi|}{|\xi|}$.

Proof: See Quarteroni or Quarteroni–Valli or Ciarlet

based on chain rule, eg.

$$\hat{v} = v \circ F_T \Rightarrow D_j \hat{v} = \sum_i (D_i v) \circ F_T \, D_j (F_T)_i$$

$$= \sum_i (D_i v) \circ F_T (B_T)_{ij}$$

Problem 1: estimation of $\|B_T\|$ and $\|B_T^{-1}\|$

Def (diameter, sphericity, regularity): Let $\mathcal{T}_h = \{T\}$ triangulation of Ω. We define

- h_T = diam T "diameter"
- $\rho_T = \sup\{\text{diam}(S) : S \subset T \text{ sphere}\}$ "sphericity"

($\hat{h}, \hat{\rho}$ diam, sphericity of $\hat{T}$)

We say that a family of meshes $\{\mathcal{T}_h\}_h$ is REGULAR if $\exists \sigma > 1 : \forall h, \forall T \in \mathcal{T}_h$

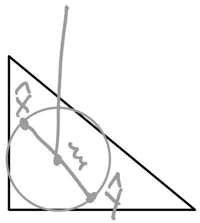
$$h_T / \rho_T \leq \sigma .$$

Lemma 2 (estimates of $\|B_T\|$, $\|B_T^{-1}\|$): we have

- $\|B_T\| \le \dfrac{h_T}{\hat\rho}$
- $\|B_T^{-1}\| \le \dfrac{\hat h}{\rho_T}$

Proof: $(d=2)$

$$\|B_T\| = \sup_{\substack{\xi\in\mathbb{R}^2\\ \xi\neq 0}} \frac{|B_T\xi|}{|\xi|} = \sup_{|\xi|=\hat\rho} \frac{|B_T\xi|}{|\xi|} = \frac{1}{\hat\rho}\sup|B_T\xi|$$



(given B_T linear, ratio is const along rays from center of maximal circ.

$$\forall \xi \ \ \exists \hat x, \hat y \in \hat T : \ \xi = \hat x - \hat y$$

Then

$$B_T\xi = B_T(\hat x - \hat y) = B_T\hat x + \underline{b}_T - B_T\hat y - \underline{b}_T$$

$$= F_T\hat x - F_T\hat y$$

$$\Rightarrow \|B_T\| = \frac{1}{\hat\rho}\sup_{|\xi|=\hat\rho}|B_T\xi| = \frac{1}{\hat\rho}\sup_{(\hat x,\hat y)_\rho}|F_T\hat x - F_T\hat y|$$

$$\le \frac{1}{\hat\rho}\,\mathrm{diam}\,T = \frac{h_T}{\hat\rho}$$

similarly for bound of $\|B_T^{-1}\|$.

Lemma 3 (Bramble-Hilbert):

Let $\hat{L} : H^{r+1}(\hat{T}) \longrightarrow H^m(\hat{T})$, with $r, m \geq 0$.

linear and continuous transformation

(i.e. $\hat{L} \in \mathcal{L}(H^{r+1}(\hat{T}), H^m(\hat{T})) =: \mathcal{L}^m_{r+1}(\hat{T})$)

such that

$$\hat{L}(\hat{p}) = 0 \qquad \forall\, \hat{p} \in \mathbb{P}_r(\hat{T})$$

Then, for each $\hat{v} \in H^{r+1}(\hat{T})$, we have

$$|\hat{L}(\hat{v})|_{m,\hat{T}} \leq \|\hat{L}\|_{\mathcal{L}^m_{r+1}(\hat{T})} \inf_{\hat{p} \in \mathbb{P}_r(\hat{T})} \|\hat{v}+\hat{p}\|_{r+1,\hat{T}}$$

$$\left(\|\hat{L}\|_{\mathcal{L}^m_{r+1}(\hat{T})} := \sup_{\substack{\hat{v} \in H^{r+1}(\hat{T}) \\ \hat{v} \neq 0}} \frac{\|\hat{L}(\hat{v})\|_{m,\hat{T}}}{\|\hat{v}\|_{r+1,\hat{T}}} \right)$$

<u>Proof</u>: let $\hat{v} \in H^{r+1}(\hat{T})$, then

$$\begin{aligned} |\hat{L}(\hat{v})|_{m,\hat{T}} &= |\hat{L}(\hat{v}) + \hat{L}(\hat{p})|_{m,\hat{T}} \qquad \forall \hat{p} \in \mathbb{P}_r(\hat{T}) \\ &= |\hat{L}(\hat{v}+\hat{p})|_{m,\hat{T}} \\ &\leq \|\hat{L}\|_{\mathcal{L}^m_{r+1}(\hat{T})} \|\hat{v}+\hat{p}\|_{r+1,\hat{T}} \end{aligned}$$

result follows as $\hat{p}$ was arbitrary. $\square$

Lemma 4 (Deny–Lions): For each $r \geq 0$, $\exists\, C = C(r, \hat{T})$:

$$\inf_{\hat{p} \in \mathbb{P}_r(\hat{T})} \|\hat{v} + \hat{p}\|_{r+1,\hat{T}} \leq C\, |\hat{v}|_{r+1,\hat{T}} \qquad \forall \hat{v} \in H^{r+1}(\hat{T})$$

<u>Proof</u>: (full proof in Quarteroni-Valli, Ciarlet)

use without proof that $\exists\, C = C(r, d, \hat{T})$:

$$\|\hat{v}\|_{r+1,\hat{T}} \underset{\circledast}{\leq} C \left[|\hat{v}|_{r+1,\hat{T}} + \sum_{|\alpha| \leq r} \left(\int_{\hat{T}} D^\alpha \hat{v} \right)^2 \right]^{1/2}$$

Let $\hat{v} \in H^{r+1}(\hat{T})$ be given. I can construct $q \in \mathbb{P}_r(\hat{T})$:

$$\int_{\hat{T}} D^\alpha q = -\int_{\hat{T}} D^\alpha \hat{v} \qquad \forall\, |\alpha| \leq r$$

because
- conditions are linear constraints
- their number $\equiv$ dimension of $\mathbb{P}_r$

$$\left(\dim \mathbb{P}_r(\mathbb{R}^d) = \binom{d+r}{r} \right)$$

Then,

$$\inf_{\hat{p} \in \mathbb{P}_r(\hat{T})} \|\hat{v} + \hat{p}\|_{r+1,\hat{T}} \leq \|\hat{v} + q\|_{r+1,\hat{T}}$$

by ⊛ $\leq c\Big[\,|\hat v+\hat q|_{r+1,\hat T} + \sum_{|\alpha|\le r}\Big(\underbrace{\int_{\hat T} D^\alpha(\hat v+\hat q)\big)^2}_{=0}\Big]^{1/2}$

$= c\,|\hat v|_{r+1,\hat T}$ for $\hat q$ polynomial of degree r.

$\square$

Lemma 3 + Lemma 4 $\Rightarrow$

Corollary: if $\hat L \in \mathcal{L}^m_{r+1}(\hat T) : \hat L(\hat p)=0 \quad \forall \hat p\in \mathbb{P}_r(\hat T)$,
then $\exists\, C = C(r,\hat T)$:

$$\boxed{|\hat L(\hat v)|_{m,\hat T} \leq C\,\|\hat L\|_{\mathcal{L}^m_{r+1}(\hat T)}\,|\hat v|_{r+1,\hat T}}$$

we use the corollary on

$$\hat L = Id - \hat I_r$$

where $\hat I_r$ is the interpolant over the reference element,

so $\forall\, \hat v\in H^{r+1}(\hat T)$, $r\geq 1$, $\hat I_r\hat v\in \mathbb{P}_r(\hat T)$

$$\hat L\hat v = \hat v - \hat I_r\hat v$$

and for $\hat p \in \mathbb{P}_r(\hat T)$ $\hat L\hat p = \hat p - \hat I_r\hat p = 0$

$$\Rightarrow \quad \forall \hat{v} \in H^{r+1}(\hat{T}), \quad \boxed{\begin{array}{ll} |\hat{v} - \hat{I}_r \hat{v}|_{m,\hat{T}} \leq C |\hat{v}|_{r+1,\hat{T}} & \\ C = C(r,m,\hat{T}) & r \geq 1 \end{array}}$$

$$\hat{I}_r \hat{v} \in \mathbb{P}, \quad \hat{v} \in \mathbb{P}$$

Theorem (Interpolation error local estimate)

Let $r \geq 1$, $0 \leq m \leq r+1$. Then $\exists\, C = C(r, m, \hat{T}) > 0$:

$$|v - I_h^r v|_{m,T} \leq C \frac{h_T^{r+1}}{\rho_T^m} |v|_{r+1,T} \qquad \forall v \in H^{r+1}(T)$$

Proof: • $v \in H^{r+1}(T)$, $r \geq 1 \Rightarrow$ interpolant is welldefined

$$|v - I_h^r v|_{m,T} \leq C \|B_T^{-1}\|^m |\det B_T|^{1/2} |\hat{v} - \hat{I}_r \hat{v}|_{m,\hat{T}} \quad \text{(lemma 1)}$$

$$\leq C \frac{\hat{h}^m}{\rho_T^m} |\det B_T|^{1/2} |\hat{v} - \hat{I}_r \hat{v}|_{m,\hat{T}} \quad \text{(lemma 2)}$$

$$\hat{h}^m \simeq 1$$

$$\leq C \frac{1}{\rho_T^m} |\det B_T|^{1/2} |\hat{v}|_{r+1,\hat{T}} \quad \text{(Corollary of lemma 3+4)}$$

$$\leq C \frac{1}{\rho_T^m} \cancel{|\det B_T|^{1/2}} \|B_T^{-1}\|^{r+1} \cancel{|\det B_T|^{-1/2}} |v|_{r+1,T} \quad \text{(lemma 1)}$$

$$\leq C \frac{1}{\rho_T^m} \frac{h_T^{r+1}}{\hat{\rho}^{r+1}} |v|_{r+1,T} \qquad \text{(lemma 2)}$$

$$\leq C(r,m,\hat{T}) \frac{h_T^{r+1}}{\rho_T^m} |v|_{r+1,T} \qquad \square$$

Corollary: if moreover $\{\mathcal{T}_h\}$ is regular family, then $\exists$ $C = C(r, m, \hat{T}, \sigma)$ where σ is the regularity parameter of $\{\mathcal{T}_h\}$, such that

$$|v - I_h^r v|_{m,T} \leq C h_T^{r+1-m} |v|_{r+1,T}$$

$$\forall v \in H^{r+1}(T).$$

Proof: we had bound with $\frac{h_T^{r+1}}{\rho_T^m} = h_T^{r+1-m} \frac{h_T^m}{\rho_T^m} \leq \sigma^m h^{r+1-m}$

constant depends on σ^m

$\square$

In particular, $m = 1$

$$|v - I_h^r v|_{1,T} \leq C h_T^r |v|_{r+1,T}$$

Theorem (Global interp error estimation):

Let $r \geq 1$. There exists $C = C(r, m, \hat{T}, \sigma)$:

$$|v - I_h^r v|_{1,\Omega} \leq C_I \Big(\sum_{T \in \mathcal{T}_h} h_T^{2r} |v|_{r+1,T}^2 \Big)^{1/2} \quad \forall v \in H^{r+1}(\Omega)$$

and

$$\leq C_I h^r |v|_{H^{r+1}(\Omega)}$$

back to FEM of order k, we have

$$|u - u_h|_{1,\Omega} \leq \frac{\gamma}{2} C_I h^{\ell-1} |u|_{\ell,\Omega}$$

where $\ell = \min(k+1, s)$ if $u \in H^s(\Omega)$.