V, Q Hilbert Spaces 30.5.2023

$B : V \to Q$

$K := \ker(B)$

$H := \ker(B^T)$

These are all equivalent

i) $\operatorname{Im}(B) = \overline{\operatorname{Im}(B)}$

ii) $\operatorname{Im}(B^T) = \overline{\operatorname{Im}(B^T)}$

iii) $K^0 = \operatorname{Im}(B^T)$

iv) $H^0 = \operatorname{Im}(B)$

v) $\exists L_B \in \mathcal{L}(\operatorname{Im}(B), K^\perp)$, $\exists \beta > 0$ s.t.

$$B(L_B g) = g \qquad \beta \|L_B g\|_V \leq \|g\|_{Q'} \quad \forall g \in \operatorname{Im}(B)$$

vi) $\exists L_{B^T} \in \mathcal{L}(\operatorname{Im}(B^T), H^\perp)$, $\exists \beta > 0$ s.t.

$$B^T(L_{B^T} f) = f \qquad \beta \|L_{B^T} f\|_Q \leq \|f\|_{V'} \quad \forall f \in \operatorname{Im}(B^T)$$

Important case: $\operatorname{Im}(B) = Q'$ closed! Then

- $H = \{0\}$
- B^T is injective
- $\exists \beta > 0 : \|L_B\|_{\mathcal{L}} \leq \frac{1}{\beta}$
- $\|B^T q\|_{V'} \geq \beta \|q\|_Q \quad \forall q \in Q \iff \inf_{q \in Q} \sup_{v \in V} \frac{\langle B^T q, v \rangle}{\|q\| \|v\|} \geq \beta$

Mixed Laplacian

$V := H_{div}(\Omega) \qquad Q := L^2(\Omega)$

Given $g \in L^2(\Omega)$ find $u, p \in V \times Q$ s.t.

$$\boxed{\begin{aligned} (u,v) - (\operatorname{div} v, q) &= 0 \quad \forall v \in V \\ (\operatorname{div} u, q) &= (g,q) \quad \forall q \in Q \end{aligned}}$$

Darcy flow

$B := -\text{"div"}$

$\ker B := \{ v \in V \text{ s.t. } (\operatorname{div} v, q) = 0 \ \forall q \in L^2(\Omega) \}$

$\operatorname{div} v = 0 \quad \text{in } L^2(\Omega)$

ELL-KER (INF-SUP conditions on A):

- $a(u,u) \geq \alpha \|u\|_V^2 \quad \forall u \in K$
- $\|Au\|_{V'} \geq \alpha \|u\|_V \quad \forall u \in K$

  $\|A^T u\|_{V'} \geq \alpha \|u\|_V \quad \forall u \in K$

$\|u\|_V^2 := \|u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2$

$$a(u,u) = (u,u) + \underbrace{(\operatorname{div} u, \operatorname{div} u)}_{=0 \text{ in } K} \geq \alpha \|u\|_V^2 \quad \forall u \in K$$

$\Rightarrow$ ELL ker is OK with $\boxed{\alpha = 1}$

Is INF SUP condition satisfied?

$\forall g \in L^2(\Omega)$, can we build $v_g \in V$ s.t. $B v_g = g$

surjectivity?

$-\operatorname{div}(v_g) = g \quad \text{in } L^2 \qquad v_g = \nabla \phi$

$\boxed{v_g = -\nabla \phi}$

$(\nabla \phi, \nabla v) = (g, v) \quad \forall v \in H_0^1(\Omega)$

$\Rightarrow \exists! \ \phi \in H_0^1(\Omega) \cap H^2(\Omega) \Rightarrow -\nabla \phi \in L^2(\Omega), \quad -\nabla\phi \in H_{div}$

$\Rightarrow$ B satisfies the inf sup

Stokes?

$V = (H_0^1(\Omega))^d \qquad Q = L_0^2(\Omega) := \{ q \in L^2(\Omega), (q,1) = 0 \}$

$(\nabla u, \nabla v) - (\operatorname{div} v, p) = (f, v) \qquad \forall v \in V$

$(\operatorname{div} u, q) = 0 \qquad \forall q \in L_0^2(\Omega)$

ELL KER : $\quad a(u,u) := \|\nabla u\|_0^2 \geq \alpha \|u\|_1^2 \quad \forall v \in V$

poincare inequality

INF SUP : $\quad B(V) \equiv L_0^2(\Omega) \quad$ (long proof : Temam)

Discrete version of Abstract saddle point problems.

$V_h \subset V \qquad Q_h \subset Q$

1) A_h is invertible on K_h $(\equiv \ker B_h)$

2) B_h is surjective

$\langle A_h v_h, w_h \rangle := a(v_h, w_h) \qquad \forall v_h, w_h \in V_h$

$\langle B_h v_h, q_h \rangle := b(v_h, q_h) \qquad \forall v_h, q_h \in V_h \times Q_h$

$K_h := \{ \ker(B_h) : u_h \in V_h, \text{ s.t. } b(u_h, q_h) = 0 \quad \forall q_h \in Q_h \}$

$K_h \subset K$

ELL-KER (2 INF SUP) on A

$\Longleftrightarrow$

A_h is invertible on K_h

INF-SUP on B_h

B_h is full rank (B_h surjective ...)

Must hold uniformly w.r.t. h

ELL-KER

$$\boxed{\begin{aligned} &\circ \ \|A_h u_h\| \geq \alpha_h \|u_h\| \quad \forall u_h \in K_h \\ &\circ \ \|A_h^T u_h\| \geq \alpha_h \|u_h\| \quad \forall u_h \in K_h \end{aligned} \qquad \alpha_h \geq \alpha_0 > 0 \quad \forall h}$$

INF SUP

$$\boxed{\cdot \ \|B_h^T p_h\| \geq \beta_h \|p_h\| \quad \forall p_h \in Q_h \qquad \beta_h \geq \beta_0 > 0 \ \forall h}$$

Error analysis

$$a(u,v) + b(v,p) = F(v) \qquad \forall v \in V \qquad \text{pick } v \equiv v_h \in V_h$$

$$a(u_h,v_h) + b(v_h,p_h) = F(v_h) \qquad \forall v_h \in V_h$$

$$\boxed{a(u-u_h, v_h) + b(v_h, p-p_h) = 0}$$

$$b(u,q) = G(q) \qquad \forall q \in Q \qquad \text{pick } q \equiv q_h \in Q_h$$

$$b(u_h,q_h) = G(q_h) \qquad \forall q_h \in Q_h$$

$$\boxed{b(u-u_h, q_h) = 0 \qquad \forall q_h \in Q_h}$$

Goal: estimate $\|u - u_h\|$ and $\|p - p_h\|$

Idea: prove an estimate for $\|u - u^I\|$ and $\|p - p^I\|$ and then use triangle inequality:

$$\boxed{a(u_h - u^I, v_h) + b(v_h, p_h - p^I) = a(u - u^I, v_h) + b(v_h, p - p^I)}$$

$$\boxed{b(u_h - u^I, q_h) = b(u - u^I, q_h)}$$

$$\mathcal{F}(v_h) := a(u - u^I, v_h) + b(v_h, p - p^I)$$

$$\mathcal{G}(q_h) := b(u - u^I, q_h)$$

depends on α, β

$$\|u_h - u^I\|_V + \|p_h - p^I\| \leq c\left(\|\mathcal{F}\| + \|\mathcal{G}\|\right)$$

$$\lesssim c\left(\|u - u^I\| + \|p - p^I\|\right)$$

$$\|u - u_h\| \leq \|u - u^I\| + \|u^I - u_h\|$$

$$\leq \|u - u^I\| + c_2\left(\|u - u^I\| + \|p - p^I\|\right)$$

$$\|p - p_h\| \leq \|p - p^I\| + \|p^I - p_h\|$$

$$\leq \|p - p^I\| + c_3\left(\|u - u^I\| + \|p - p^I\|\right)$$

$$\Downarrow$$

$$\|u - u_h\| + \|p - p_h\| \leq c\left(\|u - u^I\| + \|p - p^I\|\right) \quad \forall u^I, p^I \in V_h \times Q_h$$

very similar to: $\|u - u_h\| \leq c \inf\limits_{v_h \in V_h} \|u - v_h\|$

If $\quad K_h \subset K \quad$ then

$$\boxed{\begin{aligned} \|u-u_h\| &\leq c\,\|u-u^I\| \qquad \forall u^I \in K_h \subset K \\ \|p-p_h\| &\leq c\left(\|u-u^I\| + \|p-p^I\|\right) \quad \forall u^I, p^I \in K_h \times Q_h \end{aligned}}$$

Mixed Laplacian in 1D $\qquad P^k_{c/d}$

$$\int_a^b u v - \int_a^b v' p = 0 \qquad \forall v \in V \equiv H^1([a,b])$$

$$\int_a^b u' q = \int_a^b g q \qquad \forall q \in Q \equiv L^2([a,b])$$

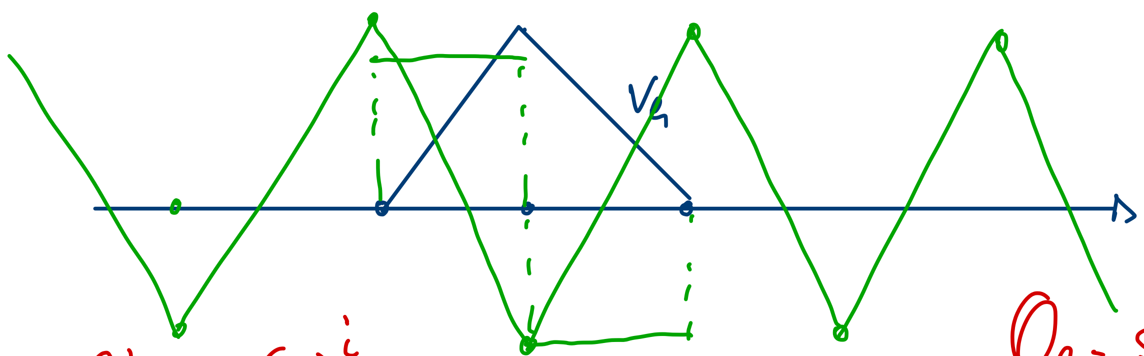
$\boxed{P^1_c - P^1_c}$?? is it ok?

ELL-KER : $\qquad \int_a^b u^2 \geq \alpha\left(\|u\|_0^2 + \|u'\|_0^2\right) \quad \forall u \in K_h$

$$K := \left\{ v \in H^1([a,b]) \;\; \int_a^b v' q = 0 \;\; \forall q \in L^2 \Rightarrow v' = 0 \right\}$$

global constants.

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{\int_a^b v_h' q_h}{\|v_h\|_1 \|q_h\|_0} \; \overset{?}{\geq} \; \beta_h \geq \beta_0 > 0$$



$Q_h = \mathrm{span}\{v_i\}$

take $\quad \tilde{q} := \sum (-1)^i v_i$

$$\int_a^b \tilde{q}\, v_h' = 0 \quad \forall\, v_h \in V_h \Rightarrow \boxed{\beta_h = 0}$$

INF SUP is violated

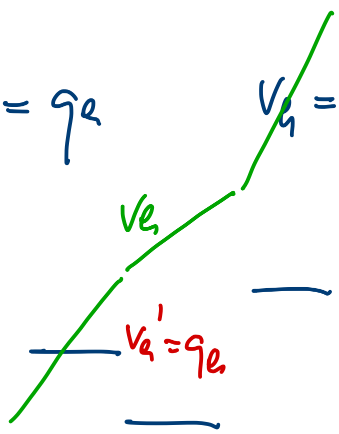
$$\boxed{P_c^1 - P_d^0}$$??

Given $q_h \in \boxed{P_d^0}$, look for $v_h \in \boxed{P_c^1}$ s.t.

- $b(v_h, q_h) = \|q_h\|^2$
- $\|v_h\|_1 \leq c\|v_h\|_0$

$$\Rightarrow \beta = \frac{1}{c}$$

$$\int_a^b v_h' q_h \Rightarrow v_h' = q_h \qquad v_h = \int_a^t q_h(x)\,dx$$



$\|v_h\|_1 \leq c\|v_h\|_0$ poincaré ineq. $\qquad \|v\|_0 \leq c\|v'\|_0$

$$\frac{b(v_h, q_h)}{\|q_h\|_0 \|v_h\|_1} = \frac{\|q_h\|_0^2}{\cancel{\|q_h\|_0} \|v_h\|_1} = \frac{\|q_h\|_0}{\|v_h\|_0 + \|v_h'\|_0} \leq \frac{\cancel{\|q_h\|_0}}{c\cancel{\|v_h'\|_0} + \cancel{\|v_h'\|_0}} = \frac{1}{c+1}$$

$$\boxed{\text{ELL-KER}} \qquad K_h \subset K$$

$$\boxed{P_c^2 - P_d^0}$$??

$\rightarrow$ INF SUP is ok for $P_c^1 - P_d^0$

same reason

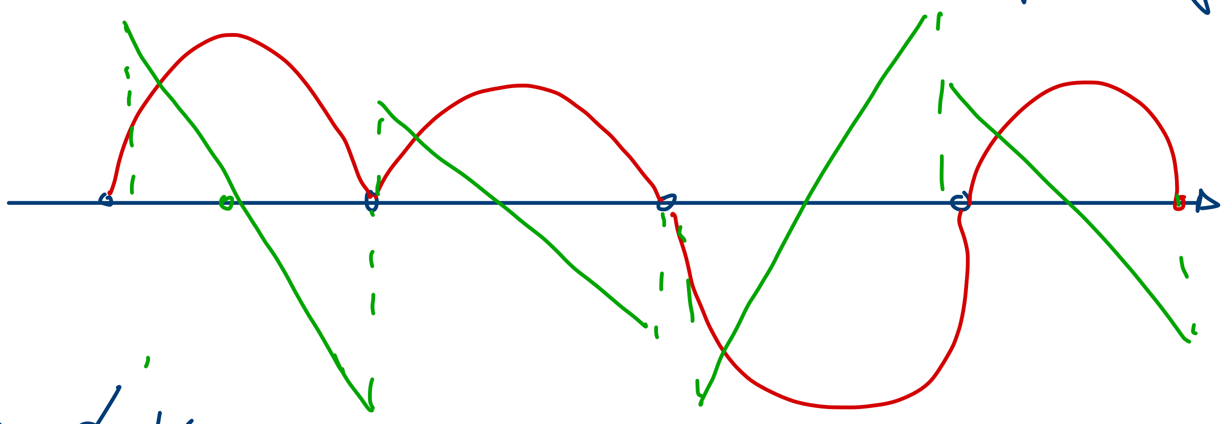
$\boxed{\text{ELL-KER}}$

$K_h := \{ v_h \in P_c^2 \text{ s.t. } \int_a^b v_h' q_h = 0 \ \forall q_h \in P_d^0 \}$

$\Leftrightarrow \quad v_h \in K_h \Rightarrow \int_{x_i}^{x_{i+1}} v_h' = 0 \Rightarrow v_h(x_{i+1}) - v_h(x_i) = 0$

$\forall i \in [0, N)$

N points of the grid.

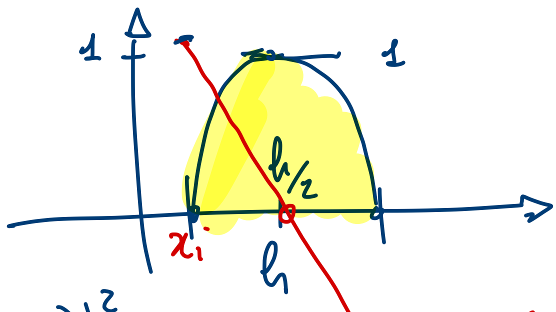


$K_h \not\subset K$

can we find $\alpha_h \geq \alpha_0 > 0$ s.t.

$a(v_h, v_h) \overset{?}{\geq} \alpha_h \|v_h\|_1^2 \overset{?}{\geq} \alpha_0 \|v_h\|_1^2 \qquad \forall v_h \in K_h$?

$\alpha_h \geq \alpha_0 > 0$



$a(v_h, v_h) = \|v_h\|_0^2$

$\|v_h\|_0^2 = \int_{x_i}^{x_i+h} \left(c_2(x-x_i)(x-x_i-h)\right)^2 \sim c_4 h$ $\qquad v' \sim c\frac{1}{h}\left(x - \frac{h}{2} - x_i\right)$

$\|v_h'\|_0^2 \simeq c_1 \frac{1}{h} \Rightarrow \boxed{\alpha_h \sim h}$

$\lim_{h \to} \alpha_h = 0$ $\qquad$ NO ELL-KER

$\boxed{P_c^k - P_d^{k-1}}$ $\qquad$ 1D it works