Saddle Point Problems

$$(1)\begin{cases} Au + B^T p = F & \text{in } V' \\ Bu = G & \text{in } Q' \end{cases}$$

$A : V \longrightarrow V'$ cont. (linear)

$B : V \longrightarrow Q'$ " "

1) $G \in \operatorname{Im}(B) \Rightarrow \exists\, u_g$ s.t. $Bu_g = G$

Rewrite (1) As $u = u_0 + u_g$, $u_0 \in \operatorname{Ker}(B) = Z$

$$\begin{cases} Au_0 + B^T p = F - Au_g = \tilde{F} & \text{in } V' \\ Bu_0 = 0 \end{cases}$$

Restrict (1) to $Z = \ker(B) \subset V$

Find u_0 in Z s.t.

$$(2)\quad \langle Au_0, v_0 \rangle + \cancel{\langle B^T p, v_0 \rangle}^{\,0} = \langle F - Au_g, v_0 \rangle \quad \forall v_0 \in Z$$

$A\big|_Z : Z \longrightarrow Z'$ is isomorphism

$\exists \alpha > 0 \mid \quad \|Az\|_{V'} \geq \alpha \|z\|_V \quad \forall z \in Z = \ker(B)$

$\left(\exists A^{-1} \text{ and } \|A^{-1} F\| \leq \frac{1}{\alpha} \|F\|_{V'} \right)$

Given u_0 solution to (2), Find p s.t.

$$\langle B^T p, v\rangle = \langle F, v\rangle - \langle A u_g, v\rangle - \langle A u_0, v\rangle \quad \forall v \in V$$

$$\exists \beta > 0 \text{ s.t. } \|B^T p\|_{V'} \geq \beta \|p\|_Q \quad \forall p \in Q$$

Property ① $\forall v_0 \in Z$, $\quad (F - Au_g - Au_0 =: L)$

$$\langle L, v_0\rangle = \langle B^T p, v_0\rangle = 0 \qquad L \in V'$$

$$L \in Z^0 \quad \left(\text{polar space w.r.t. } Z: \ f \in V' \text{ s.t. } \langle f, v\rangle = 0 \ \forall v \in Z\right)$$

$$\mathrm{Im}(B) = \mathrm{Im}(\overline{B}) \iff \mathrm{Im}(B^T) = \overline{\mathrm{Im}(B^T)}$$

$$\iff \exists \beta \text{ s.t. } \|Bu\|_{Q'} \geq \beta \|u\|_V \quad \forall u \in Z^\perp$$

$$\iff \exists \beta \text{ s.t. } \|B^T p\|_{V'} \geq \beta \|p\|_Q \quad \forall p \in H^\perp \equiv \emptyset \quad \text{iif } \mathrm{Ker}(B^T) = \{0\}$$

$$\iff \exists \text{ right inverse for } B^T \text{ s.t. } \quad H = \mathrm{Ker}(B^T)$$

$$\|B^T (B^T)^{-1} F\|_{Q'} \leq \frac{1}{\beta} \|F\|_{V'}$$

$$B^T (B^T)^{-1} F = F \qquad \forall G \in \mathrm{Im}(B^T)$$

$$\iff \exists \text{ right inverse for } B \text{ s.t.}$$

$$\|B(B^{-1}) G\|_{V'} \leq \frac{1}{\beta} \|G\|_{Q'}$$

$$B(B^{-1}) G = G \quad \text{in } Q'$$

i) "ELL-KER"

$$\inf_{u \in Z} \sup_{v \in Z} \frac{\langle Au, v\rangle}{\|u\| \|v\|} \geq \alpha$$

$$\inf_{v \in Z} \sup_{u \in Z} \frac{\langle Au, v\rangle}{\|u\| \|v\|} \geq \alpha$$

ii) "INF-SUP"

LBB

$$\inf_{q \in Q \setminus H} \sup_{v \in V} \frac{\langle Bv, q\rangle}{\|v\|_V \|q\|_Q} \geq \beta$$

$$\left(\inf_{v \in V \setminus Z} \sup_{q \in Q} \frac{\langle Bv, q\rangle}{\|v\|_V \|q\|_Q} \geq \beta \right)$$

$$\mathcal{A} : \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} : V \times Q \longrightarrow V' \times Q'$$

i) + ii) $\Longleftrightarrow$ $\mathcal{A}$ satisfies BNB

$$\begin{cases} \eta(\underline{\underline{\nabla u}}, \underline{\underline{\nabla v}}) - (\operatorname{div} \underline{v}, p) = (f, v) & \forall v \in [H_0^1(\Omega)]^d \\ -(\operatorname{div} \underline{u}, q) = 0 & \forall q \in L_0^2(\Omega) \end{cases}$$

$$L_0^2 := \{ v \in L^2 \mid (v, 1) = 0 \}$$

$$\Psi := \frac{\eta}{2}(\underline{\nabla u}, \underline{\nabla u}) - (p, \operatorname{div} u) - (f, u)$$

$$u, p : \arg \inf_{v \in V} \arg \sup_{p \in Q} \Psi$$

$$\langle D_u \Psi, v\rangle = 0 \quad \forall v \in V$$

$$\langle D_p \Psi, q\rangle = 0 \quad \forall q \in V$$

Discrete version

$\Pi_h : V \longrightarrow V_h$

$P_h : Q \longrightarrow Q_h$

$$\begin{array}{ccc} V & \xrightarrow{B} & Q' \\ \Pi_h \downarrow & & \downarrow P_h^* \\ V_h & \xrightarrow{B_h} & Q_h' \end{array}$$

$\langle B_h u_h, v_h \rangle := b(u_h, q_h)$

$\ker(B_h^T) \subset \ker(B^T)$

$B_h v_h = P_h^* B v_h \neq B v_h$

$\langle Au, v\rangle =: a(u,v)$

$\langle Bu, q\rangle =: b(u,q)$

$V_h \subset V \qquad Q_h \subset Q$

$$(1_h)\begin{cases} a(u_h, v_h) + b(v_h, p_h) = (f, v_h) & \forall v_h \in V_h \\ b(u_h, q_h) = 0 & \forall q_h \in Q_h \end{cases}$$

$a(u, v_h) + b(v_h, p) = (f, v_h) \quad \forall v_h \in V_h$

$b(u, q_h) = 0 \quad \forall q \in Q_h$

1) $Z_h \subset Z \quad \rightarrow$ Restriction to Z_h:

$$\|u - u_h\|_V \le \frac{\|A\|}{\alpha} \inf_{z \in Z_h} \|z - u\|_V$$

2) $Z_h \not\subset Z \qquad$ 2nd Strang lemma with $|||\cdot||| \equiv \|\cdot\|$

$$\|u - u_h\|_V \le \frac{\|A\|}{\alpha_h} \inf_{z \in Z_h} \|z - u\|_V + \frac{1}{\alpha_h} \sup_{w_h \in Z_h} \frac{|a(u - u_h, w_h)|}{\|w_h\|_V}$$

$$a(u-u_h, w_h) = \left(a(u-u_h, w_h) - \langle f, w_h\rangle + b(w_h, p)\right) - b(w_h, p)$$

$$= -b(w_h, p) \quad \text{non zero} \qquad \forall w_h \in Z_h$$

$$= -b(w_h, p - q_h) \qquad \forall q_h \in Q_h$$

$$Z_h := \{ v_h \in V_h \ \text{s.t.} \ b(v_h, q_h) = 0 \ \forall q_h \in Q_h \}$$

$$a(u-u_h, w_h) \leq \|B\| \, \|w_h\| \, \|p - q_h\| \qquad \forall q_h \in Q_h$$

$$\|u - u_h\| \leq \frac{\|A\|}{\alpha_h} \inf_{v_h \in Z_h} \|u - v_h\| + \frac{\|B\|}{\alpha_h} \inf_{q_h \in Q_h} \|p - q_h\|$$

For the second equation:

$$\beta_h \|q_h - p_h\| \, \|v_h\| \leq b(v_h, q_h - p_h) \qquad \forall v_h \in V_h$$

$$\leq b(v_h, p - p_h) + b(v_h, q_h - p)$$

$$\leq -a(u-u_h, v_h) + b(v_h, q_h - p)$$

$$\leq \|A\| \, \|u - u_h\| \, \|v_h\| + \|B\| \, \|v_h\| \, \|q_h - p\|$$

$$\|p - p_h\| \leq \|p - q_h\| + \|q_h - p_h\|$$

$$\leq \left(1 + \frac{\|B\|}{\beta_h}\right) \|p - q_h\| + \frac{\|A\|}{\beta_h} \|u - u_h\|$$

$$\|p - p_h\| \leq \left(1 + \frac{\|B\|}{\beta_h}\right)\left(\frac{\|A\|}{\alpha_h}\right) \inf_{q_h \in Q_h} \|p - p_h\| + \frac{\|A\|^2}{\beta_h \alpha_h} \inf_{v_h \in Z_h} \|u - v_h\|$$