First Strang Lemma

Consider Given $f \in V'$, Find $u \in V$ s.t.

1) $a(u,v) = \langle f, v \rangle \quad \forall v \in V$ $\qquad Au = f$ in V'

$V_h \subset V$, and consider a family of problems:

$a_h : V_h \times V_h \longrightarrow \mathbb{R}$ $\qquad$ Approximation for $\underline{a}$

$f_h : V_h'$

So that the discrete pb. reads:

2) find $u_h \in V_h$ s.t.

$a_h(u_h, v_h) = \langle f_h, v_h \rangle$ $\qquad A_h u_h = f_h$ in V_h'

Uniform V_h ellipticity of a_h.

Assume $\exists\ \alpha_0 > 0$ s.t. for all h

$$\rightarrow \underbrace{\underbrace{a_h(u_h, v_h) \geq \alpha_h \|u_h\|^2}_{\text{ellipticity for } a_h} \geq \alpha_0 \|u_h\|^2}_{\text{Uniformly } V_h \text{ elliptic}}$$

First Strang lemma

$$\|u-u_e\| \le C\left[\inf_{v_e\in V_e}\left[\|u-v_e\| + \sup_{w_e\in V_e}\frac{|a(v_e,w_e)-a_e(v_e,w_e)|}{\|w_e\|}\right] + \sup_{w_e\in V_e}\frac{|\langle f,w_e\rangle-\langle f_e,w_e\rangle|}{\|w_e\|}\right]$$

Define V_e^*-norm $:=$ $\|f_e\|_{*,e} := \sup_{w_e\in V_e}\frac{|\langle f_e, w_e\rangle|}{\|w_e\|}$

$$\|u-u_e\| \le C\left(\inf_{v_e\in V_e}\left[\|u-u_e\| + \underbrace{\|Av_e - A_e v_e\|_{*,e}}_{\text{consistency errors in } A_e}\right] + \|f-f_e\|_{*,e}\right)$$

consistency error in f_e

By construction:

$a_e(u_e,w_e) = \langle f_e, w_e\rangle \quad \forall w_e\in V_e \subset V$

$a(u,w_e) = \langle f, w_e\rangle \quad \forall w_e \in V_e \subset V$

$$\alpha_0\|u_e-v_e\|^2 \le a_e(u_e-v_e,\, u_e-v_e) \quad \pm a(u-v_e,\, u_e-v_e)$$

$$\le \langle f_e, u_e-v_e\rangle - a_e(v_e, u_e-v_e) + a(u, u_e-v_e)$$

$$-\langle f, u_e-v_e\rangle + a(v_e, u_e-v_e)$$

$$\le a(u-v_e, u_e-v_e) + \langle f-f_e, u_e-v_e\rangle + \langle Av_e - A_e v_e, u_e-v_e\rangle$$

$\Rightarrow$ Continuity

$$\cancel{\alpha_0}\|u_e-v_e\|^{\cancel{2}} \le \frac{C}{\alpha_0}\|u-v_e\|\cancel{\|u_e-v_e\|} + \frac{1}{\alpha_0}\|f-f_e\|_{*,e}\cancel{\|u_e-v_e\|} + \frac{1}{\alpha_0}\|Av_e-A_e v_e\|_{*,e}\cancel{\|u_e-v_e\|}$$

$$\|u-u_e\| \le \|u-v_e\| + \|v_e-u_e\|$$

$$\|u-u_\ell\| \le \cancel{\left(\frac{C}{\alpha_0}+1\right)}\, C \inf_{v_\ell\in V_h}\left(\|u-v_\ell\| + \|A v_\ell - A_\ell v_\ell\|_{*,\ell}\right) + \frac{1}{\alpha_0}\|f-f_\ell\|_{*,\ell}$$

Approx. error

Consistency errors.

If $A_\ell : V\times V \longrightarrow \mathbb{R}$ then this is the same as

$$\|u-u_\ell\| \le C \inf_{v_\ell\in V_h} \|u-v_\ell\| + \|A_\ell u - f_\ell\|_{*,\ell}$$

$$\langle A_\ell u_\ell, v\rangle = \langle f_\ell, v\rangle$$

Diffusion Transport Reaction

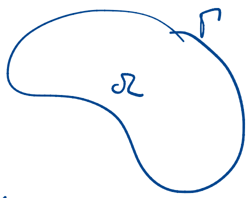
$\mu, \sigma \in L^\infty(\Omega)$

$b \in L^2(\Omega)^d$

1) $-\operatorname{div}(\mu\nabla u - b\, u) + \sigma u = f$ in Ω

$u = 0$ on Γ

$\mu \geq \mu_0 > 0$ a.e.



Bilinear form associated to 1)

$$a(u,v) = \underbrace{\int_\Omega \mu\nabla u\nabla v}_{\text{diffusion term}} - \underbrace{\int_\Omega b\,u\cdot\nabla v}_{\text{Transport term}} + \underbrace{\int_\Omega \sigma u v}_{\text{reaction term}}$$

ellipticity: 1 $\left|\int_\Omega \mu \nabla u \nabla u\right| \geq \mu_0 \|\nabla u\|^2_{0,\Omega} \geq \frac{\mu_0}{1+|\Omega|}\|u\|^2_{1,\Omega}$

② $-\int_\Omega \underline{b}\,u\nabla u = -\frac{1}{2}\int_\Omega \underline{b}\,\nabla(u^2) = \int_\Omega \frac{1}{2}\nabla\cdot\underline{b}\,\underline{\underline{u^2}} - \cancel{\int_{\partial\Omega} \underline{b}\cdot n\, u^{0}}$

③ $\int_\Omega \sigma u^2$

②+③ $\int_\Omega \left(\sigma + \frac{1}{2}\nabla\cdot\underline{b}\right) u^2 \geq 0$ if $\underline{\underline{\sigma + \frac{1}{2}\nabla\cdot b \geq 0 \text{ a.e.}}}$

$\underline{|a(u,u)| \geq C\mu_0 \|u\|^2}$ if $\sigma + \frac{1}{2}\nabla\cdot b \geq 0$ a.e.

Continuity $\left|\int_\Omega \mu \nabla u \nabla v\right| \leq \|\mu\|_{\infty,\Omega} \|\nabla u\|_{0,\Omega} \|\nabla v\|_{0,\Omega}$

$\left|\int_\Omega u\,\underline{b}\cdot\nabla v\right| \leq \|b\|_{\infty,\Omega} \|u\|_{0,\Omega} \|\nabla v\|_{0,\Omega}$

$\left|\int_\Omega \sigma u v\right| \leq \|\sigma\|_{L^\infty,\Omega} \|u\|_{0,\Omega} \|v\|_{0,\Omega}$

$M : \|\mu\|_{L^\infty,\Omega} + \|b\|_{L^\infty,\Omega} + \|\sigma\|_{L^\infty,\Omega}$

$a(u,v) \leq M \|u\|_{1,\Omega} \|v\|_{1,\Omega}$

$\frac{M}{\alpha} \quad \frac{(1+|\Omega|)}{\mu_0}\cdot\left(\|\mu\|_{L^\infty,\Omega} + \|b\|_{L^\infty,\Omega} + \|\sigma\|_{L^\infty}\right)$

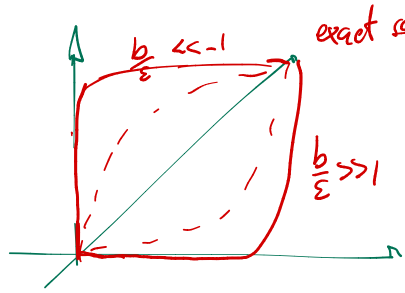
Problems if $\frac{\|b\|_{L^\infty}}{\mu_0} \gg 1$ or $\frac{\|\sigma\|_{L^\infty}}{\mu_0} \gg 1$

transport dominated — reaction dominates

1D example

1) $$\begin{cases} -\varepsilon u'' + (bu)' = 0 & 0 < x < 1 \\ u(0) = 0 \quad u(1) = 1 \end{cases}$$

exact solution $\dfrac{\left(e^{\frac{b}{\varepsilon}x} - 1\right)}{\left(e^{\frac{b}{\varepsilon}} - 1\right)}$



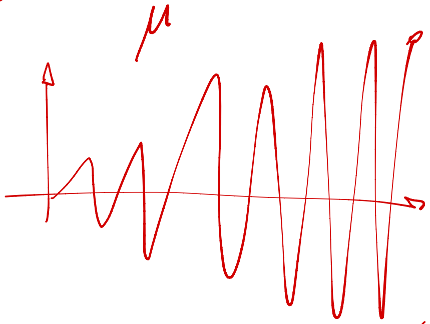
1D FEM

$$A = \text{tridiag}\left(-\frac{\varepsilon}{h} - \frac{b}{2}, \ \frac{2\varepsilon}{h}, \ -\frac{\varepsilon}{h} + \frac{b}{2}\right)$$

$$F = \left(0, \ldots, \frac{\varepsilon}{h} - \frac{b}{2}\right)$$

$$u_J = \left[\left(\frac{1+Pe}{1-Pe}\right)^J - 1\right] \Big/ \left[\left(\frac{1+Pe}{1-Pe}\right)^M - 1\right] \qquad J = 1, \ldots, M-1$$

$$Pe := \frac{h|b|}{\mu} > 1 \quad \text{if} \quad h > \frac{\mu}{|b|}$$



FD(BD) or BD(FD) CFD

$$-\frac{\varepsilon\left(u_{J-1} - 2u_J + u_{J+1}\right)}{h^2} + \frac{b\left(u_{J+1} - u_{J-1}\right)}{2h}$$

Simple remedy: UPWIND $(b>0)$

$$-\frac{\varepsilon\left(u_{J-1} - 2u_J + u_{J+1}\right)}{h^2} + \frac{b\left(u_J - u_{J-1}\right)}{h}$$

Exact Solution is: $\dfrac{\left[(1+2Pe)^J - 1\right]}{\left[(1+2Pe)^M - 1\right]}$ No more oscillation.

BUT error is now $O(h)$ instead of $O(h^2)$

$$b\frac{u_J-u_{J-1}}{h} = b\frac{u_{J+1}-u_{J-1}}{2h} - \underline{\underline{bh}}\left[\frac{u_{J+1}-2u_J+u_{J-1}}{\underline{\underline{h^2}}}\right]$$

Same as adding artificial diffusion term of order $\underline{\underline{bh}}$

For FEM : replace a by a_h :

$$a_h(u,v) := a(u,v) + \underline{\frac{h}{2}\|b\|_{L^\infty(\Omega)}}(\nabla u, \nabla v)$$

Error constant $\frac{M}{\alpha_h}$ for Strang's Lemma :

$$\|u-u_h\| \leq \frac{M}{\alpha_h} \inf_{v_h\in V_h}\left[\|u-u_h\|_{1,\Omega} + \underline{\underline{\|A_h v_h}} - f\|_{*,h}\right]$$

$$\alpha_h := \alpha + \frac{h\|b\|_{L^\infty(\Omega)}}{(1+|\Omega|)} = \frac{(1+|\Omega|)\alpha + h\|b\|_{L^\infty}}{(1+|\Omega|)}$$

$$\frac{M}{\alpha_h} \sim \frac{\cancel{\|\mu\|_{L^\infty}} + \|b\|_{L^\infty}}{(1+|\Omega|)\mu_0 + h\|b\|_{L^\infty}} \cdot (1+|\Omega|) \sim \frac{1}{h}(1+|\Omega|)$$

$$\frac{h\|b\|}{\mu_0} \gg 1$$