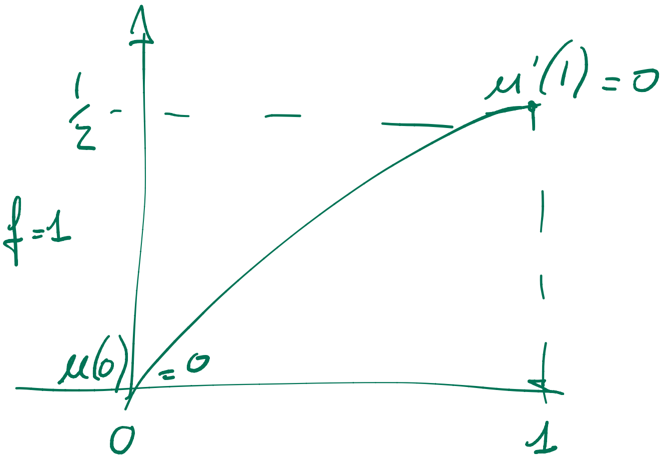
Advanced Finite Analysis

Motivational Example



$$1)\quad \begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = 0 & u'(1) = 0 \end{cases}$$

Variational Formulation

$$-\int_0^1 u'' v = \int_0^1 f v$$

for any "v regular enough" ($\Rightarrow$ any integral has to make sense) s.t. $v(0)=0$

$\Downarrow$ Integrate by parts

$$2)\quad \int_0^1 u'v' = \int_0^1 f v + u'v\Big|_0^1$$

disappears if $u'=0$ or if $v=0$

$$a(u,v) = (f,v) \qquad \forall v \in V := \{ v \in L^2(0,1) \text{ s.t. } a(v,v) < +\infty,\ v(0)=0 \}$$

Then (2) reads

Given $f \in L^2(0,1)$, find $u \in V$ s.t.

$$(3)\quad a(u,v) = (f,v) \qquad \forall v \in V$$

$$a(u,v) := \int_0^1 u'v'$$

$$(f,v) := \int_0^1 f v$$

i) what definition to use for u', v'

ii) when is a solution to (3) also a sol. to pb. (1)

i) weak derivatives, and Sobolev Spaces.

ii) when $f \in C^0([0,1]) \Rightarrow u$ sol to (3) also solves (1)

Ritz - Galerkin approx to (3)

Take a finite dimensional subspace of V

$$V_h = \text{span}\{v_i\}_{i=1}^{N} \qquad v_i \in V \quad i=1,\dots,N$$

Restrict Pb (3) to V_h:

Given $f \in L^2((0,1])$ find $u_h \in V_h$ s.t.

$$a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

Exploit $u_h = \underbrace{u^i}_{\text{real coefficients}} \underbrace{v_i}_{\text{basis functions}}$ Einstein summation convention

$$(4) \quad a(u^j v_j, v_i) = (f, v_i) \qquad i = 1,\dots,N$$

$$\Longleftrightarrow$$

$$A_{ij} u^j = F_i \qquad A_{ij} := a(v_j, v_i)$$

$$F_i := (f, v_i)$$

$A \in \mathbb{R}^{N\times N}$, $F \in \mathbb{R}^N$, $\{u\} \in \mathbb{R}^N$

Theorem $\exists!$ sol. to (4)

Proof If I have one $\tilde{u}$ for which

$$A\tilde{u} = 0 \quad \text{when } \tilde{u} \neq 0$$

$$\tilde{u}^i A_{ij} \tilde{u}^j = 0;$$

$$a(\tilde{u}, \tilde{u}) = \int_0^1 (\tilde{u}')^2 = 0 \Rightarrow \tilde{u}' = 0 \quad \forall x$$

$\tilde{u}' = 0 \Rightarrow \tilde{u}$ is constant: for B.C.

$$\tilde{u} = 0$$

contradiction: $\exists!$ sol. u to pb. 4

1.2: The variational pb. 3 is satisfied also on $V_h \subset V$

$$a(u, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

subtract 4

$$a(u_h, v_h) = (f, v_h)$$

5 $\quad a(u - u_h, v_h) = 0$ Orthogonality property

let's define $\|u\|_E^2 := a(u,u) \quad \forall u \in V$

$$\|u - u_h\|_E^2 = a(u - u_h, u - u_h) \overset{\text{by } 5}{=} a(u - u_h, u - v_h)$$

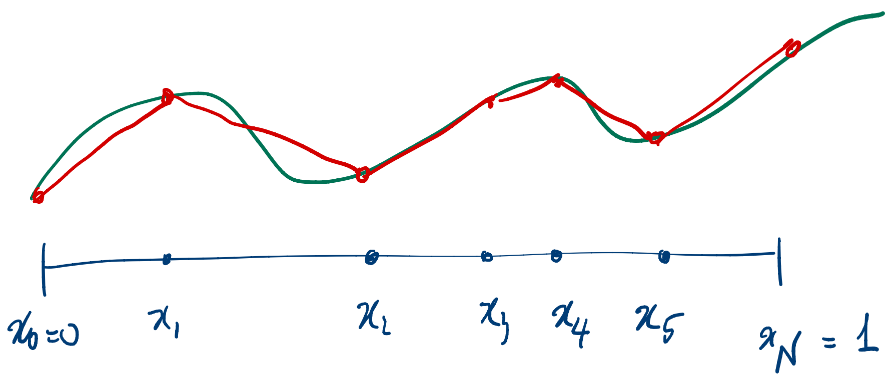
Schwartz

$$\leq \|u - u_h\|_E \, \|u - v_h\|_E$$

$$\boxed{\|u-u_h\|_E \le \inf_{v_h\in V_h}\|u-v_h\|_E}$$

1D Let's fix $\{x_i\}_{i=0}^{N}$ with $x_0=0$, $x_N=1$

$x_i < x_j$ when $i<j$



$N=6$

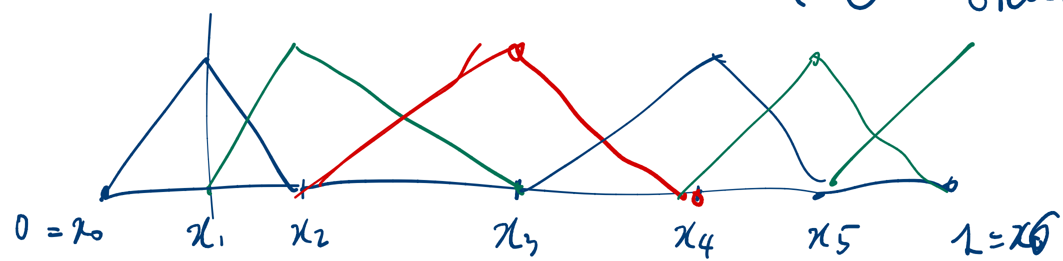
Define $P_1(\{x\}) := \{ v\in C^0([0,1]),\ v|_{(x_i,x_{i-1})} \in P^1,\ v(0)=0 \}$

Canonical basis function for P_1 is usually constructed from $\{\sigma\}_{i=1}^{N}$ where $\sigma\in \mathcal{L}(P_1)$

(6) $\sigma^i(v) := v(x_i)$ $\quad i=1,\dots,N \quad \forall v\in C^0([0,1])$

The canonical basis is the set $\{v_i\}_{i=1}^{N} \in P^1([0,1])$

$$\sigma^i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$



Define an Interpolation Operator

$$I_h v := \underline{\sigma^i(v)}\ \underline{v_i} \quad \in V_h = \operatorname{span}\{v_i\}_{i=1}^N$$

Nodal linear operator defined in 6 | basis function

$$I_h : C^0([0,1]) \longrightarrow V_h$$

$$I_h^2 = I_h \qquad I_h v_j := \sigma^i(v_j)\, v_i = \delta^i_j v_i = v_j$$

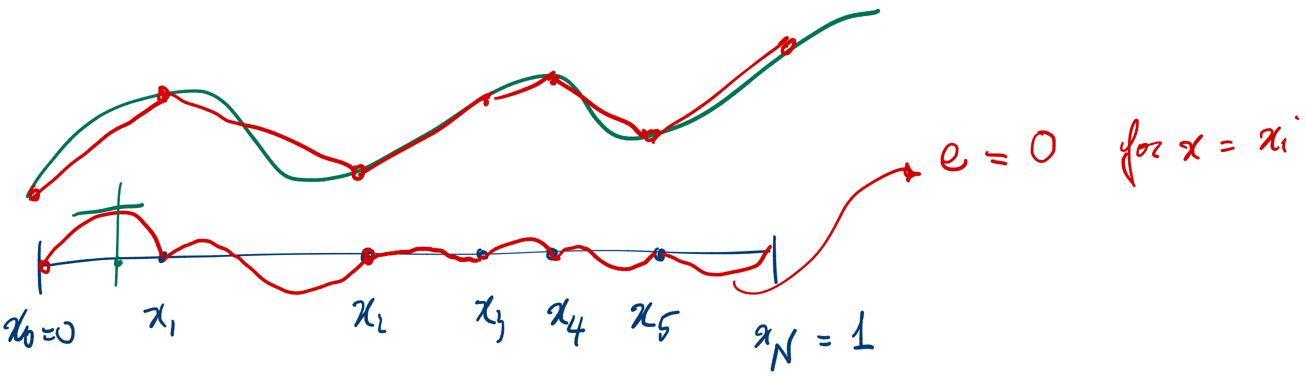
Main Question: What is the error in V?

$$\| I_h v - v \|_E \quad ??$$

<u>Theorem</u>: let $h = \max_{i\in[1,N]} |x_i - x_{i-1}|$

$$\| I_h v - v \|_E \leq c\, h\, \| v'' \|_{L^2([0,1])} \quad \forall v \in C^2([0,1])$$

<u>Proof</u>: $e := I_h v - v$



$\exists\ \xi_i \in (x_{i-1}, x_i) \quad \text{s.t.} \quad e'(\xi_i) = 0$ Rolle's Theorem

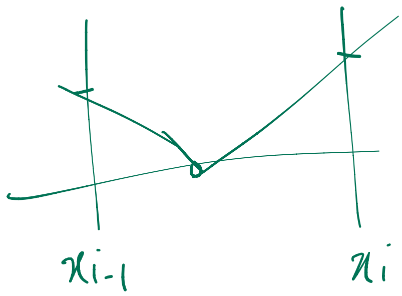
$$e'(y) = \int_{\xi}^{y} e''(s)\, ds \qquad \forall y \in [x_{i-1}, x_i]$$

$$e'(y) = \int_{\xi}^{y} 1 \cdot e''(s)\, ds \leq \left| \int_{\xi}^{y} 1 \right|^{\frac{1}{2}} \left| \int_{\xi}^{y} \underline{\underline{(e''(s))^2}}\, ds \right|^{\frac{1}{2}}$$

$= v''(s)$

$$e'(s) \leq |y - \xi_i|^{\frac{1}{2}} \, \| v'' \|_{L^2([x_{i-1}, x_i])}$$

$$\int_{x_{i-1}}^{x_i} |e'(s)|^2 \leq \int_{x_{i-1}}^{x_i} |y - \xi_i| \, \| v'' \|^2_{L^2([x_{i-1}, x_i])}$$



$$\int_{x_{i-1}}^{x_i} |e'(s)|^2 \leq \frac{1}{2}(x_i - x_{i-1})\, h \, \| v'' \|^2_{L^2}$$

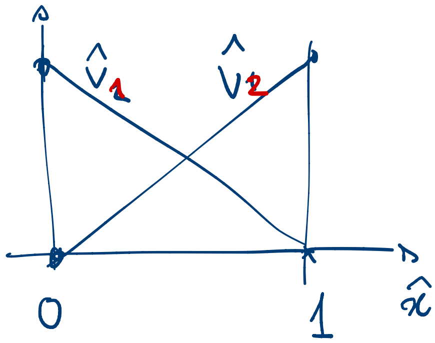
$$\int_{x_{i-1}}^{x_i} |e'(s)|^2 \leq \frac{1}{2} h^2 \, \| v'' \|^2_{L^2([x_{i-1}, x_i])}$$

$$\| e \|_E \leq \frac{1}{2} h \, \| v'' \|_{L^2([0,1])}$$

Implementation details:

In inner intervals:

Define $\hat{I} : [0,1]$



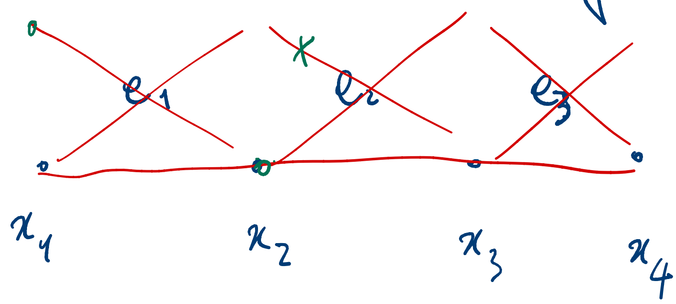
$\hat{v}_2 := \hat{x}$

$\hat{v}_1 := 1 - \hat{x}$

Any interval x_{i-1}, x_i, can be seen as an affine transformation of $\hat{I}$: $T_e : \hat{I} \longrightarrow [x_{e-1}, x_e]$

$$T_e : \hat{v}_1 x_{e-1} + \hat{v}_2 x_e$$

There is a local-to-global numbering, i.e.,



$i(e,\ell) := e + \ell - 1$

input: . element index "e"
. local basis index "ℓ"

output: global basis function index

Global basis v_i can be written by push forward of the local basis $\hat{v}_\ell$.

In particular we have that on the element e the global index $i(e,e)$ identifies:

$$v_{i(e,l)} \circ T_e = \hat{v}_e$$

$$v_{i(e,e)} = \hat{v}_e \circ T_e^{-1}$$

With this it is "easy" to assemble A_{ij} locally:

$$A_{ij} = a(v_i, v_j) = \int_0^1 v_i' \, v_j' = \sum_e \int_{I_e} v_i' \, v_j'$$

for e in all intervals:

compute

$$a_{lm} := \int_{I_e} v_{i(e,l)}' \, v_{i(e,m)}' = \int_0^1 \frac{1}{h_e} \hat{v}_e'(\hat{x}) \cdot \frac{1}{h_e} \hat{v}_m'(\hat{x}) \, h_e \, d\hat{x}$$

$$= \sum_{q=1}^{N_q} \frac{1}{h_e} \hat{v}_e'(\hat{x}_q) \frac{1}{h_e} \hat{v}_m'(\hat{x}_q) \, h_e \cdot w_q$$

Known

Depend on T_e

sum into $A_{i(e,l)\,i(e,m)} \mathrel{+}= a_{lm}$