Advanced Finite Element Methods

Model problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \Omega \subset \mathbb{R}^d, \text{ Lipschitz}$$

Weak formulation:

Given $f \in H^{-1}(\Omega)$, find $u \in H_0^1(\Omega)$ s.t.

$$(\nabla u, \nabla v) = \langle f, v \rangle \qquad \forall v \in H_0^1(\Omega)$$

Particular case of: V, Hilbert

Given $f \in V^*$, find $u \in V$ s.t.

$$① \quad a(u,v) = f(v) \qquad \forall v \in V$$

Lax Milgram:

If a is bilinear, coercive, and continuous, $\exists!\, u$, s.t. ① is satisfied, and

- $a(u,v) \leq C \|u\| \, \|v\| \qquad \forall u, v \in V$
- $a(u,u) \geq \alpha \|u\|^2 \qquad \forall u \in V$

$$\Rightarrow \quad \|u\| \leq \frac{C}{\alpha} \|f\|_{V^*} \qquad \|f\|_{V^*} := \sup_{0 \neq v \in V} \frac{|f(v)|}{\|v\|}$$

Galerkin Method (Apply Lax Milgram to Finite dimensional space $V_h \subset V$)

Given $V_h \subset V$, $f \in V^*$, find $u_h \in V_h$ s.t.

$$a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h$$

Orthogonality:

main motivation for FEM

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h \Rightarrow \boxed{\|u - u_h\|_V \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V}$$

- Construct $V_h \subset V$ s.t. $\text{dist}(V_h, V)$ is small

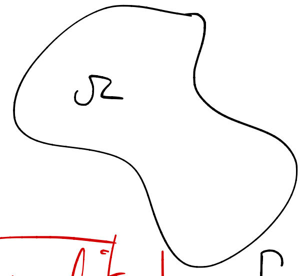
$$\boxed{\text{dist}(V_h, V) := \sup_{u \in V} \inf_{v_h \in V_h} \|u - v_h\|_V}$$

- Estimate $\text{dist}(V_h, V)$ (a priori estimates)
- Given particular data $f \in V^*$, find "optimal" V_h for the solution u_h (a posteriori estimates)
- For difficult problems find "better" formulations
  - Stabilization mechanisms
  - Variational crimes
  - Mixed formulations
  - Non-matching discretizations.

Prerequisites:
- Familiarity with Sobolev Spaces

Sobolev Spaces 101

- Ω : bounded open set in $\mathbb{R}^d$ (domain)
- $\overline{\Omega}$: closure of Ω
- $\Gamma := \partial\Omega$: boundary of Ω
- $\Omega_e := \mathbb{R}^d \setminus \overline{\Omega}$ exterior domain



$\Gamma := \partial\Omega$

$\boxed{\Omega \text{ is Lipschitz for us}}$

For functions $u : \Omega \longrightarrow \mathbb{R}$, $D^\alpha u$ is its partial derivative of order $|\alpha|$ –

$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ multi index, $|\alpha| = \sum \alpha_i$, its modulus

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$C^m(\Omega)$: linear space of functions with continuous derivatives up to order m, i.e.

$$C^m(\Omega) := \{ u : \Omega \longrightarrow \mathbb{R} \text{ s.t. } D^\alpha u, |\alpha| \leq m \text{ is continuous} \}$$

With norm

$$\|u\|_{C^m} := \max_{0 \leq \alpha \leq m} \sup_{x \in \Omega} |D^\alpha u(x)|$$

$C^m(\Omega)$ is a Banach space

We call $L^p(\Omega)$, $p \in [1, \infty)$ the space of pth order integrable functions (Banach)

$$L^p(\Omega) := \{ u : \Omega \longrightarrow \mathbb{R} \text{ s.t. } \left(\int_\Omega |u|^p \right)^{\frac{1}{p}} < +\infty \}$$

$$L^\infty(\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|$$

For $p=2$, $L^2(\Omega)$ is Hilbert with scalar product

$$(u,v) := \int_\Omega u\,v$$

Let $u \in L^1(\Omega)$, $\alpha \in \mathbb{N}_0^d$, u has weak α derivative D_w^α if there exists $v \in L^1(\Omega)$ s.t.

$$\langle v, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle \quad \forall \varphi \in C_c^\infty(\Omega)$$

C^∞ with compact support on Ω.

We set $D_w^\alpha u = v$

Sobolev Space $W^{m,p}(\Omega)$

$$W^{m,p}(\Omega) := \{ u \mid D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \le m \}$$

Banach with norm:

$$\|u\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha|\le m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

$$\|u\|_{W^{m,\infty}(\Omega)} := \max_{|\alpha|\le m} \|D^\alpha u\|_{L^\infty(\Omega)}$$

$W^{m,2}(\Omega)$ is Hilbert: $(u,v)_{W^{m,2}} := \sum_{|\alpha|\le m} \int D^\alpha u \cdot D^\alpha v$

$$C^{m,*} := \{ u \in C^m(\Omega) \mid \|u\|_{W^{m,p}(\Omega)} < +\infty \}$$

Define $\quad H^{m,p}(\Omega) = \overline{C^{m,*}}^{\|\cdot\|_{W^{m,p}(\Omega)}}$

Meyers/Serrin: Ω Lipschitz $\Rightarrow H^{m,p}(\Omega) = W^{m,p}(\Omega)$

We'll restrict to $p = 2$ (Hilbert). and use

$$H^m(\Omega) \quad \text{to denote} \quad H^{m,2}(\Omega) \underbrace{= W^{m,2}(\Omega)}_{\text{se } \Omega \text{ Lip}}$$

with norm

$$\boxed{\|u\|_{m,\Omega} := \|u\|_{W^{m,2}(\Omega)}}$$

and seminorm

$$\boxed{|u|_{m,\Omega} := \|D^m u\|_{0,\Omega}}$$

We define $\quad H^m_0(\Omega) = \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{m,\Omega}}$

(The closure of $C^\infty_c(\Omega)$ w.r.t. the norm $\|.\|_{m,\Omega}$)

LAX MILGRAM

$\Omega \subset \mathbb{R}^d$

$V, \|\cdot\|$ Hilbert space on Ω Lip

V^* Algebraic and Topological dual of V

Abstract Problem

$a : V \times V \longrightarrow \mathbb{R}$ bilinear

$f \in V^*$

Find $u \in V$ s.t.

$$a(u,v) = f(v) = \langle f, v \rangle \quad \forall v \in V \qquad (1)$$

Theo (LAX MILGRAM)

If a is bounded and V-elliptic, $\exists!\, u$ solution to (1)

$$|a(u,v)| \leq C \|u\| \|v\| \quad \forall u,v \in V \quad \text{Boundedness}$$

$$|a(u,u)| \geq \alpha \|u\|^2 \quad \forall u \in V \quad \text{V-ellipticity}$$

Proof

We use the Rietz operator.

Denote with $A : V \longrightarrow V^*$ the linear operator associated with a:

$$Au(v) = \langle Au, v \rangle := a(u,v) \qquad u,v \in V$$

A is a bounded linear operator:

$$\|Au\|_{V^*} := \sup_{v\in V} \frac{|Au(v)|}{\|v\|} \leq C\|u\|$$

$$\|A\| := \sup_{u\in V} \frac{\|Au\|_{V^*}}{\|u\|} \leq C$$

Problem ① can be rewritten in operator form:

$$Au = f \quad \text{in } V^* \qquad ②$$

Consider now the lifting from V^* to V (Rietz operator)

$$\tau : V^* \longrightarrow V$$
$$f \longrightarrow \tau f$$

such that

$$f(v) = \langle f, v\rangle = (\tau f, v)_V \quad , \; v \in V$$

We rewrite ② in V:

$$\tau(Au - f) = 0 \quad \text{in } V$$

And show that, under the hypothesis of LM,

$$T_\rho : V \longrightarrow V$$

$$v \longrightarrow v - \rho\tau(Av - f)$$

is a contraction for some choices of ρ,

hence, $\exists!$ $\tau(Au - f) = 0 \Rightarrow Au = f$
(Banach fixed point theorem)

$$\|T_\rho v_1 - T_\rho v_2\| \leq \gamma \|v_1 - v_2\| \qquad \gamma < 1$$

$$\|T_\rho v_1 - T_\rho v_2\|^2 = \|v_1 - v_2 - \rho\tau(Av_1 - Av_2)\|^2$$

$$= \|v_1 - v_2\|^2 - 2(\rho\tau A(v_1 - v_2), v_1 - v_2)_V + \|\rho\tau A(v_1 - v_2)\|^2$$

$$= \|v_1 - v_2\|^2 - 2\rho\, a(v_1 - v_2, v_1 - v_2) + \rho^2 \|A(v_1 - v_2)\|^2_{V^*}$$

continuity + coercivity

$$\leq (1 - 2\rho\alpha + \rho^2 C^2)\|v_1 - v_2\|^2$$

$$1 - 2\rho\alpha + \rho^2 C^2 < 1 \Leftrightarrow \rho < \frac{2\alpha}{C^2}$$

For $\rho < \frac{2\alpha}{C^2}$ T_ρ is a contraction

$$\Rightarrow \exists! \; u \text{ s.t. } T_\rho u = u \Rightarrow \exists! u \text{ s.t. } \boxed{Au = f}$$

A is bounded and bounding:

$$\|Au\|_{V^*} \leq \|A\|_* \|u\| \leq C\|u\|$$

$$\|Au\|_{V^*} \geq \|A^{-1}\|_* \|u\| \geq \alpha\|u\|$$

Galerkin Method

Construct a series of spaces $V_h \subset V$, $\dim(V_h) = n_h$
Such that $\lim_{h\to 0} V_h = V$, $V_h := \mathrm{span}\{v_i\}_{i=1}^{n_h}$

V_h is Hilbert with norm $\|\cdot\|$ (∃! solution for LM)

Find $u_h \in V_h$ $\left(u_h = \sum_{i=1}^{n_h} u^i v_i = \underbrace{u^i v_i}_{\text{Einstein summation convention}} \right)$
such that

$$A u_h = f \quad \text{in } V_h^*$$

$$\left(\langle A u_h, v_h \rangle = \langle f, v_h \rangle \quad \forall v_h \in V_h \right)$$

Equivalent to Algebraic problem

$$A_{ij} u^j = F_i \qquad A_{ij} := \langle A v_j, v_i \rangle$$
$$F_i := \langle f, v_i \rangle$$

A is S.P.D. $(\exists A^{-1})$

$$u^T A u = u^i A_{ij} u^j = u^i \langle A v_i, A v_j \rangle u^j$$
$$= \langle A u_h, u_h \rangle \geq \alpha \|u_h\|^2$$

CEA's LEMMA

$$a(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h$$

$$\|u - u_h\| \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|$$

Proof

$$\alpha \|u - u_h\|^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h)$$
$$\leq C \|u - u_h\| \|u - v_h\| \qquad \forall v_h \in V_h$$

Notation

Given a basis for $V_h = \text{span}\{v_i\}_{i=1}^{n_h}$, the canonical dual basis for V_h^* is given by $\{v^i\}_{i=1}^{n_h}$ s.t.

$$\langle v^i , v_j \rangle = \delta^i_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

Any function $u \in V_h$ can be written as $u^i v_i$ where

$u^i := \langle v^i , u \rangle$ are the contravariant coefficients

By Hahn-Banach, $\exists$ extensions $\tilde{v}^i \in V^*$ of v^i such that $\tilde{v}^i(u_h) = \langle \tilde{v}^i , u_h \rangle = \langle v^i , u_h \rangle \ \forall u_h \in V_h$

Interpolation on V_h: $\Pi : V \longrightarrow V_h$

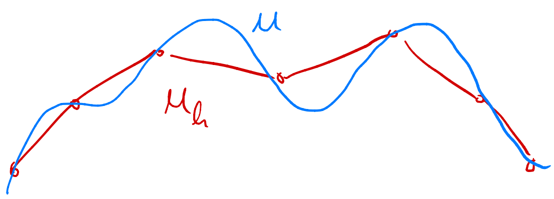
$$\Pi u := \langle \tilde{v}^i , u \rangle v_i = u^i v_i$$

For example: given $V_h := \{ u \in C^0([0,1]) \mid u|_{(x_i, x_{i+1})} \in P^1 \}$

and $v^i := \delta(x - x_i)$ $\Pi u = u(x_i)\, v_i(x)$ $\boxed{v_i(x_j) = \delta_{ij}}$

$(\Pi u)(x_i) = u(x_i)$ $u_h(x) = u(x_i) + [u(x_{i+1}) - u(x_i)]\, x$

for $x \in [x_i, x_{i+1}]$



A priori Estimates

$\operatorname{dist}(V, V_h) := \sup_{u \in V} \inf_{v_h \in V_h} \|u - v_h\|$

- Construct splitting of $\Omega = \overline{\overset{\circ}{\bigcup} K}$ into d-simplices or tensor product structures
- Make sure $u_h\big|_K \in \mathbb{P}^k$
- Estimate what happens on one reference element $\hat{K}$
- write all elements as $K = T_K(\hat{K})$
- Estimate $|\cdot|_{m,\hat{K}}$ w.r.t. $|\cdot|_{m,K}$ and T_K
- Exploit equivalence of $|\cdot|_{m,K}$ with $\|[\cdot]\|_{m,K}$ for polynomials (Bramble Hilbert)
- Sum up, "glue" together, get estimate on Ω.