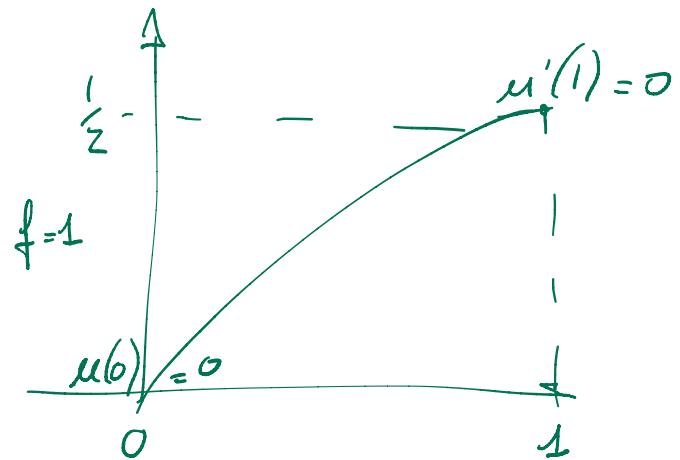


# Advanced Finite Analysis

## Motivational Example



$$1) \begin{cases} -u'' = f & \text{in } (0, 1) \\ u(0) = 0 & \\ u'(1) = 0 & \end{cases}$$

## Variational Formulation

$$\int_0^1 u'' v = \int_0^1 f v \quad \text{for any "v regular enough"} \\ (\Rightarrow \text{any integral has to make sense}) \\ \text{s.t. } v(0) = 0$$

Integrate by parts  $\Downarrow$

$$2) \int_0^1 u' v' = \int_0^1 f v + u' v \Big|_0^1 \quad \begin{array}{l} \text{disappears if } u' = 0 \\ \text{or if } v = 0 \end{array}$$

$$a(u, v) = (f, v) \quad \forall v \in V := \left\{ v \in L^2(0, 1) \text{ s.t. } a(v, v) < +\infty, v(0) = 0 \right\}$$

Then ② yields

Given  $f \in L^2(0, 1)$ , find  $u \in V$  s.t.

$$3) a(u, v) = (f, v) \quad \forall v \in V$$

$$a(u, v) := \int_0^1 u' v'$$

$$(f, v) := \int_0^1 f v$$

i) What definition to use for  $u', v'$

ii) When is a solution to ③ also a sol. to pb. ①

- i) weak derivatives, and Sobolev Spaces.  
ii) when  $f \in C^0([0,1]) \Rightarrow u$  sol to ③ also solves ①

Ritz-Galerkin approx to ③

Take a finite dimensional subspace of  $V$

$$V_h = \text{span} \left\{ v_i \right\}_{i=1}^N \quad v_i \in V \quad i=1, \dots, N$$

Restrict Pb ③ to  $V_h$ .

Given  $f \in L^2([0,1])$  find  $u_h \in V_h$  s.t.

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Exploit  $u_h = \sum_i u_i^i v_i$  Einstein summation convention  
↓ basis functions.  
real coefficients

$$\textcircled{4} \quad a(u^T v_j, v_i) = (f, v_i) \quad i=1, \dots, N$$

$\Leftrightarrow$

$$A_{ij} u^j = F_i$$

$$A_{ij} := -a(v_j, v_i)$$

$$F_i := (f, v_i)$$

$$A \in \mathbb{R}^{N \times N}, \quad F \in \mathbb{R}^N, \quad \{u\} \in \mathbb{R}^N$$

Theorem  $\exists!$  sol. to ④

Proof If I have one  $\tilde{u}$  for which

$$\tilde{A}\tilde{u} = 0 \quad \text{when } \tilde{u} \neq 0$$

$$\underbrace{\tilde{u}^i A_{ij} \tilde{u}^j}_{= 0}$$

$$a(\tilde{u}, \tilde{u}) = \int_0^1 (\tilde{u}')^2 = 0 \Rightarrow \tilde{u}' = 0 \text{ f.s.}$$

$\tilde{u}' = 0 \Rightarrow \tilde{u}$  is constant: f.s. B.C.

$$\tilde{u} = 0$$

contradiction:  $\exists$  sol.  $u$  to pb. ④

2. The variational pb. ③ is satisfied also on  $V_h \subset V$

$$a(u, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

subtract ④

$$a(u_h, v_h) = (f, v_h)$$

⑤  $\underbrace{a(u - u_h, v_h)}_{= 0} \quad \text{Orthogonality property}$

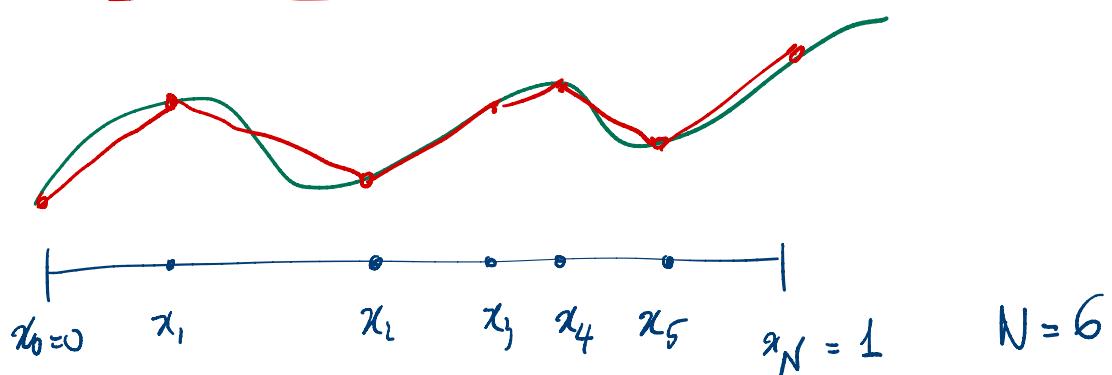
Let's define  $\|u\|_E^2 := a(u, u) \quad \forall u \in V$

$$\begin{aligned} \|u - u_h\|_E^2 &= a(u - u_h, u - u_h) \stackrel{\text{by ⑤}}{=} a(u - u_h, u - v_h) \\ &\leq \|u - u_h\|_E \|u - v_h\|_E \end{aligned}$$

$$\|u - u_h\|_E \leq \inf_{v_h \in V_h} \|u - v_h\|_E$$

1D let's fix  $\{x_i\}_{i=0}^N$  with  $x_0 = 0, x_N = 1$

$x_i < x_j$  when  $i < j$



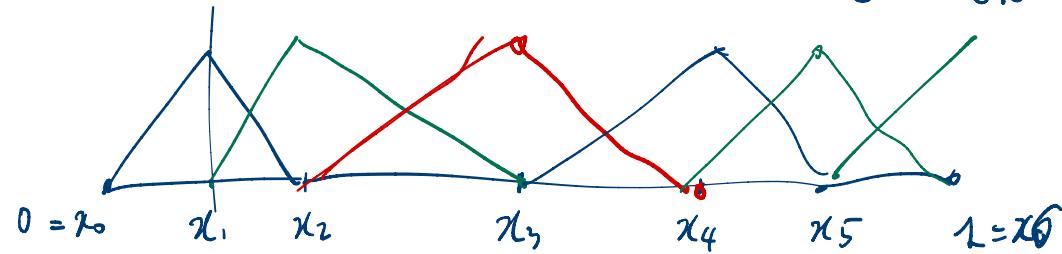
Define  $P_1(\{x\}) := \{v \in C^0([0, 1]), v|_{(x_i, x_{i+1})} \in P^1, v(0) = 0\}$

Canonical basis function for  $P_1$  is usually constructed from  $\{\delta^i\}_{i=1}^N$  where  $\delta \in \text{Lo}(P_1)$

⑥  $\delta^i(v) := v(x_i) \quad i = 1, \dots, N \quad \forall v \in C^0([0, 1])$

The canonical basis is the set  $\{\delta^i\}_{i=1}^N \in P^1([0, 1])$

$$\delta^i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$



Define an interpolation Operator

$$I_h v := \sum_{i=1}^N G^i(v) v_i \quad \in V_h = \text{span } \{v_i\}_{i=1}^N$$

Nodal linear operator | basis function  
defined in ⑥

$$I_h: C^0([0,1]) \longrightarrow V_h$$

$$I_h^2 = I_h \quad I_h v_j := G^j(v_j) v_i = \delta_{ij} v_i$$

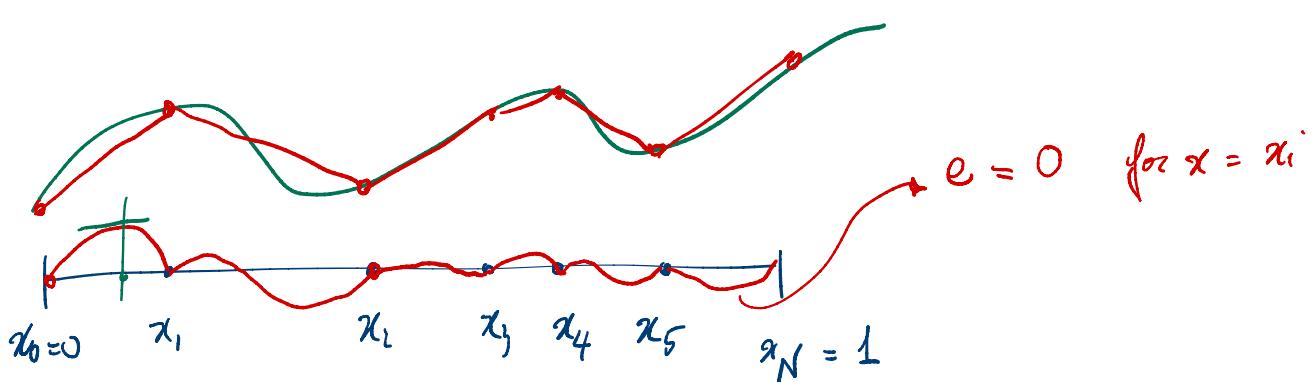
Main Question: What is the error in  $V_h$ ?  $= v_j$

$$\|I_h v - v\|_E \quad ??$$

Theorem: let  $h = \max_{i \in \{1, N\}} |x_i - x_{i-1}|$

$$\|I_h v - v\|_E \leq Ch \|v''\|_{L^2([0,1])} \quad \forall v \in C^2([0,1])$$

Proof:  $e := I_h v - v$



$\exists \xi_i \in (x_{i-1}, x_i)$  s.t.  $e'(x_i) = 0$  Rolle's Theorem

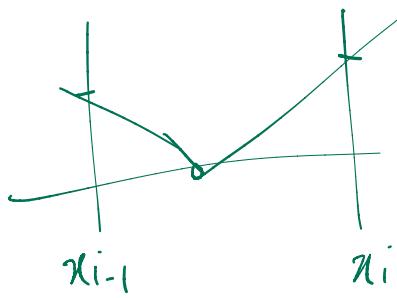
$$e'(y) = \int_{\xi}^y e''(s) ds \quad \forall y \in [x_{i-1}, x_i]$$

$$e'(y) = \int_{\xi}^y 1 \cdot e''(s) ds \leq \left| \int_{\xi}^y 1 \right|^{\frac{1}{2}} \left| \int_{\xi}^y (e''(s))^2 ds \right|^{\frac{1}{2}}$$

$$= \|v''\|$$

$$|e'(s)| \leq |y - \xi|^{\frac{1}{2}} \|v''\|_{L^2([x_{i-1}, x_i])}$$

$$\int_{x_{i-1}}^{x_i} |e'(s)|^2 \leq \int_{x_{i-1}}^{x_i} |y - \xi|^2 \|v''\|_{L^2([x_{i-1}, x_i])}^2$$



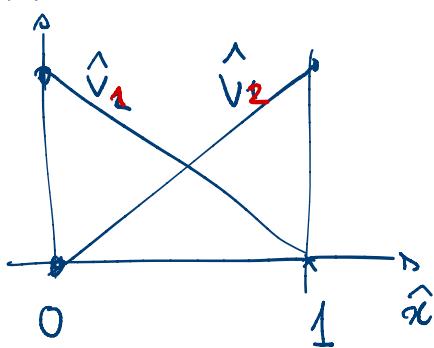
$$\int_{x_{i-1}}^{x_i} |e'(s)|^2 \leq \frac{1}{2}(x_i - x_{i-1}) h \|v''\|_{L^2}^2$$

$$\int_{x_{i-1}}^{x_i} |e'(s)|^2 \leq \frac{1}{2} h^2 \|v''\|_{L^2([x_{i-1}, x_i])}^2$$

$$\|e\|_E \leq \frac{1}{2} h \|v''\|_{L^2([0,1])}$$

## Implementation details:

In inner intervals:



Define  $\hat{I} : [0, 1]$

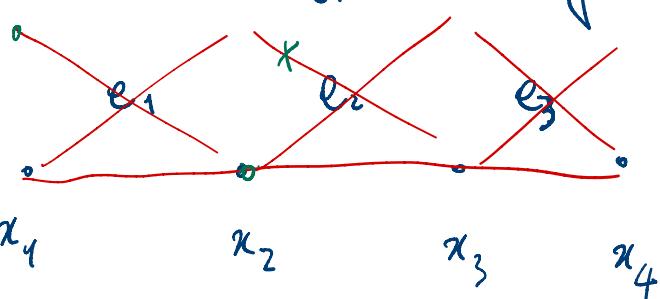
$$\hat{v}_2 := \hat{x}$$

$$\hat{v}_1 := 1 - \hat{x}$$

Any interval  $x_{e-1}, x_e$ , can be seen as an affine transformation of  $\hat{I}$ :  $\underline{T}_e : \hat{I} \rightarrow [x_{e-1}, x_e]$

$$T_e : \hat{v}_1 x_{e-1} + \hat{v}_2 x_e$$

There is a local-to-global numbering, i.e.,



$$i(e, l) := e + l - 1$$

input: . element index "e"  
. local basis index "l"

output: global basis function index

Global basis  $v_i$  can be written by push forward  
of the local basis  $\hat{v}_e$ .

In particular we have that on the element  $e$  the global index  $i(e, e)$  identifies:

$$v_{i(e, l)} \circ T_e = \hat{v}_e$$


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$$v_{i(e, e)} = \hat{v}_e \circ T_e^{-1}$$

With this it is "easy" to assemble  $A_{ij}$  locally:

$$A_{ij} = \alpha(v_i, v_j) = \int_0^1 v_i^\top v_j^\top = \sum_e \int_{I_e} v_i^\top v_j^\top$$


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for  $e$  in all intervals:

compute

$$\alpha_{lm} := \int_{I_e} v_{i(e, l)}^\top v_{i(e, m)}^\top = \int_0^1 \frac{1}{h_e} \hat{v}_e^\top(\hat{x}) \cdot \frac{1}{h_e} \hat{v}_m^\top(\hat{x}) h_e d\hat{x}$$

$$= \sum_{q=1}^{N_q} \left( \frac{1}{h_e} \hat{v}_e^\top(\hat{x}_q) \right) \cdot \left( \frac{1}{h_e} \hat{v}_m^\top(\hat{x}_q) \right) h_e \circ W_q$$

Known

Depend on  $T_e$

Sum into  $A_{i(e, l) i(e, m)} += \alpha_{lm}$