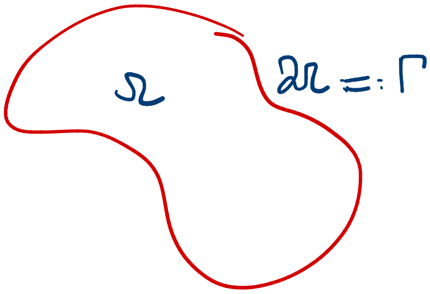
Hilbert Case: Lax Milgram Lemma

Context: V Hilbert Space (Sobolev Space on Ω Lip. subset of $\mathbb{R}^d$)

think of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega = \Gamma \end{cases}$$



Idea: Hilbert Space (separable) V.

1) $Au = F$ in V' $\qquad A: V \longrightarrow V'$

$\equiv$

$\langle Au, v\rangle = \langle F, v\rangle \quad \forall v \in V$

$a(u,v) := \langle Au, v\rangle \quad \exists\ C > 0,\ \alpha > 0$ s.t.

$a(u,v)$ is bounded i) $|a(u,v)| \le C \|u\|_V \|v\|_V \quad \forall u,v \in V$

" " coercive ii) $|a(v,v)| \ge \alpha \|v\|_V^2$

Then

2) $a(u,v) = \langle F, v\rangle \quad \forall v \in V$

Has a unique solution $u \in V$ and $\|u\|_V \le \frac{1}{\alpha} \|F\|_{V'}$

Rietz operator: $\tau: V' \longrightarrow V$

$$(\pi F, v)_V = \langle F, v \rangle \qquad \forall v \in V$$

$$-\rho\tau(Av - F) + v =: T_\rho(v)$$

$=$ if u is fixed point of T_ρ, then u solves. 2)

$$T_\rho(u) = u \quad \rightsquigarrow \quad \rho\tau(Au - F) = 0$$

Find ρ for which T_ρ is contraction

$$\underline{\| T_\rho(v_1) - T_\rho(v_2) \|_V^2} = \| v_1 - v_2 - \rho\tau A(v_1 - v_2) \|_V^2$$

$$\|v_1 - v_2\|^2 + \rho^2 \|\tau A(v_1 - v_2)\|_V^2 - \underline{2\left((v_1 - v_2), \rho\tau A(v_1 - v_2)\right)_V}$$

$$-2\rho\langle A v_1 - v_2, v_1 - v_2 \rangle$$

$$\leq \underline{\|v_1 - V_2\|^2} \underbrace{\left(1 + C\rho^2 - 2\alpha\rho\right)}$$

if $|1 + C\rho^2 - 2\alpha\rho| < 1$

Then $\exists!$ fixed point to T_ρ $\Rightarrow$ $\rho < \dfrac{2\alpha}{C}$

$$\|Au\|_{V'} \leq \|A\|_* \|u\| \leq C\|u\|$$

$$\|Au\|_{V'} \geq \|A^{-1}\|_* \|u\| \geq \alpha\|u\|$$

Take $V_h \subset V$, restrict A to V_h

Find $u_h \in V_h$ s.t.

$$\langle A u_h, v_h \rangle = \langle F, v_h \rangle \quad \forall v_h \in V_h$$

$$\underline{\langle A(u - u_h), v_h \rangle = 0} \quad \forall v_h \in V_h$$

orthogonality

Cea's Lemma:

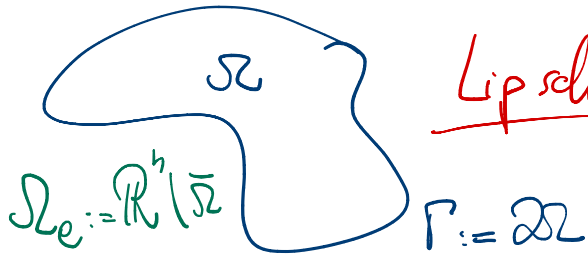
$$\alpha \| u - u_h \|_V^2 \leq \langle A(u-u_h), (u - \underline{\underline{u_h}}) \rangle \leq \langle A(u-u_h), u - v_h \rangle \quad \forall v_h \in V_h$$

$$\leq C \| u - u_h \|_V \| u - v_h \|_V \quad \forall v_h \in V_h$$

$$\boxed{\| u - u_h \|_V^2 \leq \inf_{v_h \in V_h} \frac{C}{\alpha} \| u - v_h \|}$$

1) construct V_h such that $\mathrm{dist}(V, V_h)$ is small $\rightarrow$ a-priori
2) Given $F \in V'$, try to construct best V_h for F $\rightarrow$ a-posteriori
3) Relax the constraint that $V_h \subset V$
$\rightarrow$ "Variational Crimes" ($V_h \not\subset V$)
4) Mixed formulations / Stabilized formulations

Sobolev Notations:



Lipschitz

$u : \Omega \longrightarrow \mathbb{R}$

α multi index

$D^{\alpha} u$: weak derivative of u of order $m = |\alpha|$

$C^m(\Omega)$: linear space with continuous derivatives

$$L^p(\Omega) := \left\{ v : \Omega \to \mathbb{R} \ \text{ s.t. } \left(\int_\Omega v^p \right)^{\frac{1}{p}} := \|v\|_{L^p(\Omega)} < +\infty \right\}$$

$$L^\infty(\Omega) := \operatorname*{ess\,sup}_{x \in \Omega} |v(x)|$$

$$W^{m,p}(\Omega) := \left\{ v : \Omega \to \mathbb{R} \ \middle| \ D^\alpha v \in L^p(\Omega) \ \text{ for } |\alpha| \le m \right\}$$

$$\|v\|_{m,p,\Omega} := \left(\sum_{|\alpha| \le m} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

three subscripts : Sobolev norm

When $p = 2$ $\quad \|v\|_{m,\Omega} := \|v\|_{m,2,\Omega}$

Hilbert Space, $\quad$ two subscripts : $p = 2$

with inner product $\quad (v, v)_{m,\Omega} := \sum_{|\alpha| \le m} \int_\Omega D^\alpha v \, D^\alpha v \, d\Omega$

Seminorm $\quad |u|_{m,p,\Omega} := \|D^m u\|_{L^p(\Omega)}$

$$C^{m,p,*} := \{ u \in C^m(\Omega) \mid \|u\|_{m,p,\Omega} < +\infty \}$$

$$H^{m,p}(\Omega) := \overline{C^{m,p,*}}^{\|\cdot\|_{m,p,\Omega}}$$

Meyers / Serrin : Ω Lip. $\Rightarrow$ $\underline{\underline{H^{m,p}(\Omega) \equiv W^{m,p}(\Omega)}}$

Formal Definition of Finite Element (Ciarlet)

Triple K, P, Σ

i) K is a compact, connected, Lip. subset of $\mathbb{R}^d$
s.t. $\mathring{K} \neq \{\phi\}$

ii) P is a vector space of dimension N from K to $\mathbb{R}^m$
$m=1$ or d

iii) $\Sigma := \{v^i\}_{i=1}^N$ a basis for $\mathcal{L}(P, \mathbb{R})$

iii) $\Leftrightarrow$ $P \ni p : \longrightarrow \{v^i(p)\}_{i=1}^N \in \mathbb{R}^N$
is bijective

Σ is the set of "degrees of freedom"

$\exists!$ Canonical basis $\{v_i\}_{i=1}^N$ for P s.t.

$$v^j(v_i) = \delta^j_i = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Local basis functions

Ex 1 Lagrange Finite Element Spaces.

A $\{K, P, \Sigma\}$ is a Lagrange FE space if $\exists \{a_i\}_{i=1}^N$
(support points) $a_i \in \mathbb{R}^d$ s.t.

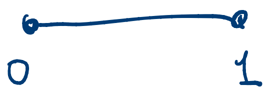
$$v^i(p) = p(a_i) \quad \forall p \in P, \quad i = 1, \dots, N$$

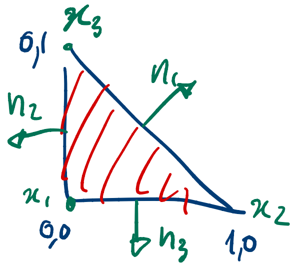
or Nodal Finite Element (and a_i are the "nodes")

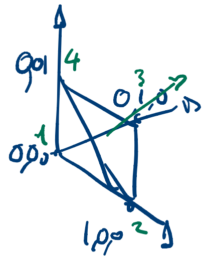
Simplicial Lagrangian FE (in $\mathbb{R}^d$)

$$K := \left\{ x \text{ s.t. } x_i \geq 0 , \sum_{i=1}^{d} x_i \leq 1 \right\}$$

$d=1$





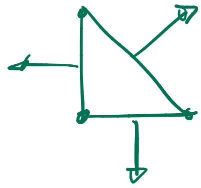


P = Polynomial space of order $\underline{\underline{\leq k}}$: $N = \binom{d+k}{k}$

$d=1 \quad N = K+1$

$d=2 \quad N = \frac{1}{2}(k+1)(k+2)$

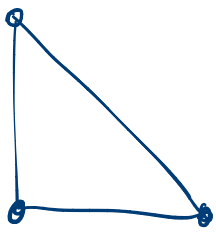
$$N = \frac{1}{6}(k+1)(k+2)(k+3)$$



Example $k=1$ set a_i = vertices of K

$$v_i(x) = 1 - \frac{(x - a_i)\cdot n_i}{(a_j - a_i)\cdot n_i} \quad \text{for any } j \neq i$$

$d=2 \rightsquigarrow 1 - x - y \,,\; x \,,\; y$



VS
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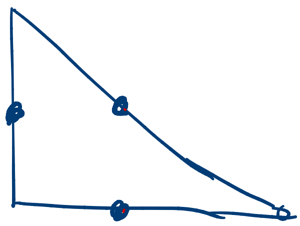


1) Any simplex can be constructed as an affine transformation from the reference simplex

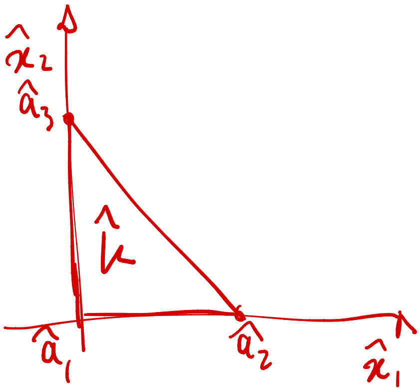
Start by defining a ref. FE $\hat{K}, \hat{P}, \hat{\Sigma}$

assume it is Lagrangian of order k.

Then write

$$T_K \hat{x} = \sum_{i=1}^{N} a_i \hat{v}_i$$

where $a_i \in \mathbb{R}^d$ and $\hat{v}_i$ is the canonical basis for $\hat{P}$



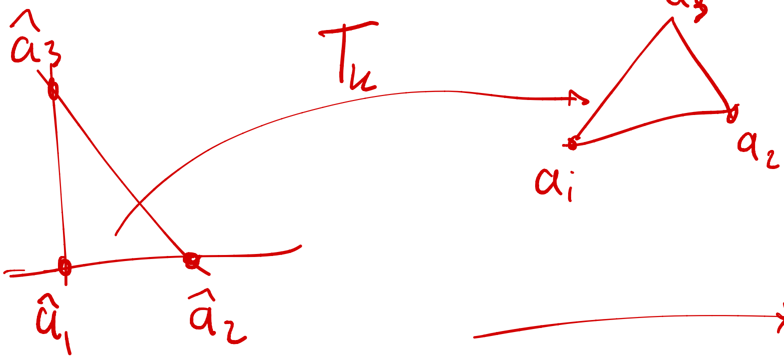
$\hat{a}_i :=$ support points of $\hat{v}^i$ (or $\hat{v}_i$)

$\hat{v}^i := \delta(\hat{x} - \hat{a}_i)$

Define K as image under T_K of $\hat{K}$

$$K = T_K(\hat{K})$$ known when you know $\{a_i\}_{i=1}^N$

order of $P^{(k=1)} = 1 \rightarrow d = 2, N = 3$



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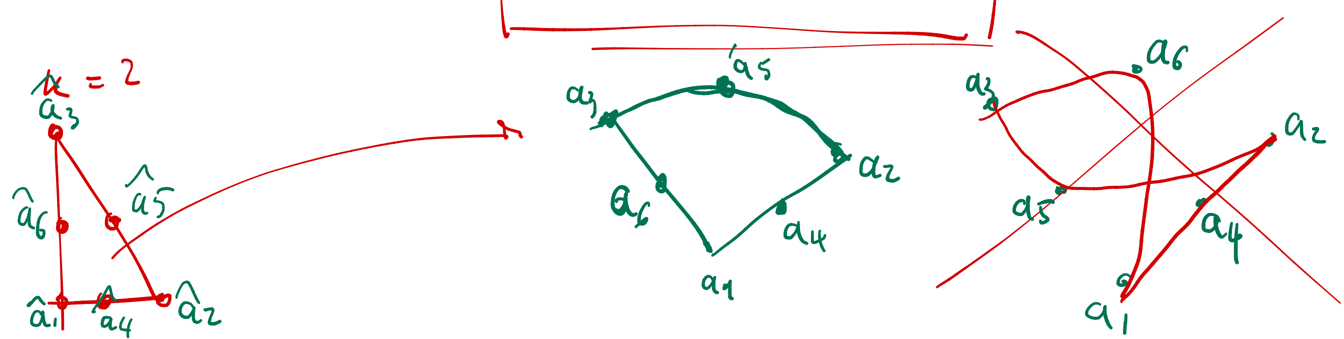
We ask that $\boxed{\det JT_K > 0}$ $\forall \hat{x}$

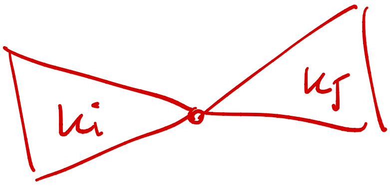


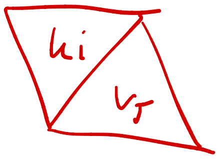
Construction of V_h : partitioning Ω into Ω_h

$$\Omega_h = \left(\bigcup_{i=1}^{M} \overline{K_i} \right)$$

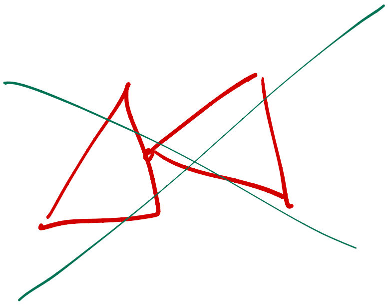
$\overline{K_i} \cap \overline{K_j}$ is common vertex or common line in 2d

or common triangle in 3d





NOT OK



Once the partition is known

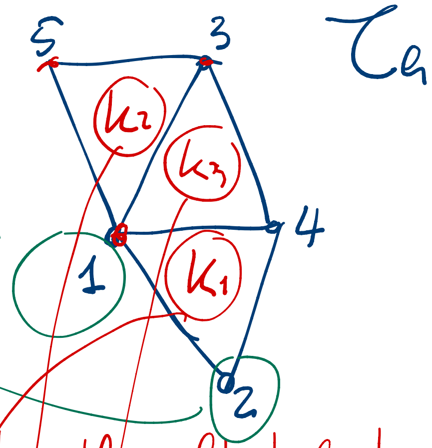
$\forall k \quad \exists \{a_i(k_m)\}_{i=1}^{N}$ local support points for all $m = 1, \dots, M$

m cells = a list of length m of N points in $\mathbb{R}^d$

Structure:

list of points with id $\quad N_p$ points

1	p_{11}	p_{12}	p_{13}
2	p_{21}	p_{22}	p_{23}
⋮			
N_p			



list of cells with id. → indices into the first list

1	1	2	4
2	1	3	5
3	1	4	3

How to construct V_h on $\mathcal{T}_h$, s.t. $V_h \subset H^1(\Omega)$

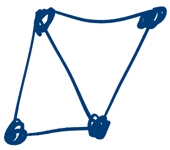
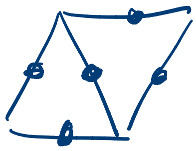
$$V_h^0 = \{ v \in C^0(\overline{\mathcal{T}_h}) \mid v|_K \in P^1(K) \quad \forall K \in \mathcal{T}_h \}$$

Simpler case:

$V_h^{-1} \subset L^2(\Omega)$, $V_h = \{ v \in L^2(\mathcal{T}_h) \mid v|_K \in P^1(K) \; \forall K \in \mathcal{T}_h \}$

$\dim_{d=2}(V_h^{-1}) = 3M$ = 3 times number of cells.

<u>How about V_h^0</u> : <u>number vertices</u>

<u>Idea</u> : $N_0 = \dim(P^k)$ for $d=0$ is $\begin{cases} 1 & k \geq 1 \\ 0 & k=0 \end{cases}$

$N_1 = \dim(P^k)$ for $d=1$ is $(k+1)$

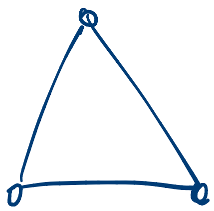
$N_2 = \dim(P^k)$ for $d=2$ is $\frac{1}{2}(k+1)(k+2)$

$N_3 = \dim(P^k)$ for $d=3$ is $\frac{1}{6}(k+1)(k+2)(k+3)$

To ensure continuity for $\{K, P, \Sigma\}$

$\#$ support points on subelements of K (i.e. edges, faces, vertices) coincides with $\dim(P^k|_X)$ where X is edge, face or vertex.

Start with $d=0$ (Two dimensional example)



3 vertices $d=1$

3 edges $d=1$

1 triangle $d=2$

$N_i = \dim(P^k)$ in $d=1$

1 degree of freedom ($\equiv$ 1 support point) per vertex: N_v

$\boxed{3}$ For each edge, I assign $N_E = N_1 - 2N_0$ to the edge.

For each triangle I assign $N_T = N_2 - 3N_E - 3N_v$

For each tetrahedron I assign $N_{TET} = N_3 - 4N_E - 4N_v$

"Degrees of freedom per object" dpo

$dpo \in \mathbb{R}^{d+1}$ s.t. $dpo_0 = N_v$

$dpo_1 = N_E$

$dpo_2 = N_T$

$dpo_3 = N_{TET}$

P^3