A priori Estimates

$\operatorname{dist}(V, V_h) := \sup_{u \in V} \inf_{v_h \in V_h} \|u - v_h\|$

1 Construct splitting of $\Omega = \mathring{\overline{\bigcup K}}$ into d-simplices
or tensor product structures

2 Construct local basis on K

3 Make sure they glue together correctly

4 Estimate what happens on one reference element $\hat{K}$

5 write all elements as $K = T_K(\hat{K})$

6 Estimate $|\cdot|_{m,\hat{K}}$ w.r.t. $|\cdot|_{m,K}$ and T_K

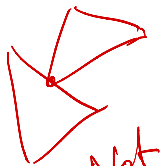
7 Exploit equivalence of $|\cdot|_{m,K}$ with $\|[\cdot]\|_{m,K}$ for polynomials (Bramble Hilbert)

8 Sum up, "glue" together get estimate on Ω

1: Triangulation

Ω is split into elements $K \subset \mathcal{T}_h$

- $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$ $\quad$ K open, $K \neq \emptyset$
- $K_1 \cap K_2 = \emptyset$
- ∂K Lip. Continuous
- $\overline{K_1} \cap \overline{K_2}$ vertex, edge, or face



Not possible (yet)

2: local (finite dimensional) spaces of dimension $\underline{\underline{N_K}}$

- $P_K := \{ v_h|_K \quad v_h \in V_h \}$
- P_K should contain polynomials (or be close to polynomials)
- A "finite element" is the triple $\{K, P_K, \Sigma_K\}$ where $\Sigma_K := \{\phi^i\}_{i=1}^{N_K}$ are linear functionals on P_K, such that Σ_K is $\underline{P_K\text{-unisolvant}}$

$\Leftrightarrow$ given N_K real numbers $\{p^i\}_{i=1}^{N_K}$, $\exists!\, p \in P_K$ such that $p^i = \langle \phi^i, p \rangle$

$\forall_K$, $\exists!$ set $\{\phi_i\}_{i=1}^{N_K}$ s.t. $\langle \phi^j, \phi_i \rangle = \delta^j_i$

- Σ_K is a basis for P_K^*
- $p^i = \langle \phi^i, p \rangle$ are the local degrees of freedom of p
- ϕ_i are the local basis functions on K
- $\tilde{\phi}^i$ are extensions of ϕ^i to V^* s.t. $\tilde{\phi}^i(p) = \phi^i(p) \quad \forall p \in P_K$

- $\Pi_K : V \longrightarrow P_K$

$$u \longrightarrow \langle \tilde{\phi}^i, u \rangle \phi_i$$

local P_K-interpolation operator

3 If $V_h \subset C^m(\bar{\Omega}) \Rightarrow V_h \subset H^{m+1}(\Omega)$

Proof: Start with $C^0(\bar{\Omega})$, $H^1(\Omega)$, proceed by induction

We need to show that ∇u_h is well defined (1) and in $L^2(\Omega)$

(1): $\forall \varphi \in (C_c^\infty(\Omega))^d$

$$\int_\Omega \nabla u_h \varphi = \sum_K \left[-\int_K u_h \nabla \cdot \varphi + \int_{\partial K} u_h \varphi \right]$$

$$\mathcal{E} := \bigcup_{K \in \mathcal{T}_h} \partial K$$

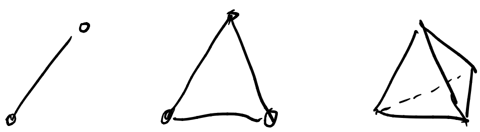
$$= -\int_{\Omega \setminus \mathcal{E}} u_h \nabla \varphi + \cancel{\int_{\mathcal{E}} [u_h] \varphi} \to 0$$

$u \in C^0(\Omega)$

$\Rightarrow$ ∇u is well defined in the weak sense

How do we enforce $\boxed{\text{global continuity?}}$

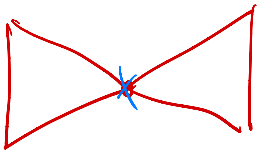
Most important case:

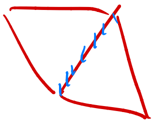
* K: d-simplices  ...

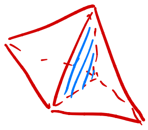
* $P_K = \mathbb{P}^n(K)$ complete polynomial spaces

Let K_i, K_j be such that $\overline{K_i} \cap \overline{K_j} = X$

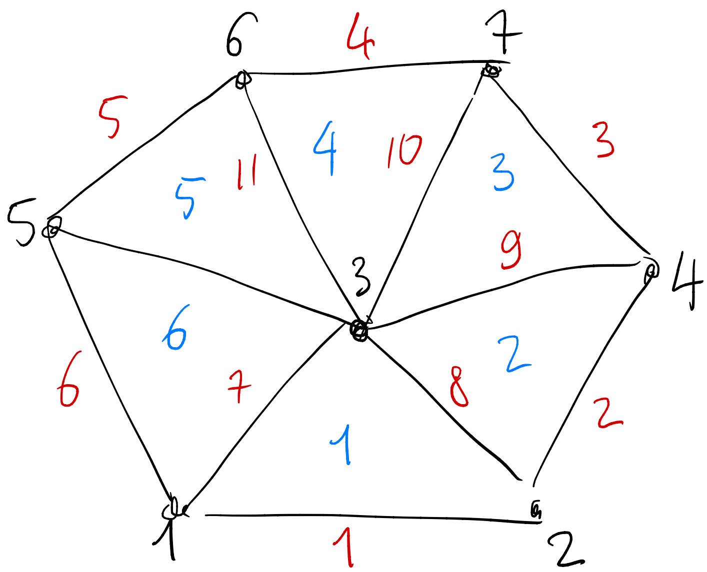
for example:

$X = v$: vertex 

$X = e$: edge 

$X = T$: triangle 

★ Associate a global number to every (shared) object in $\mathcal{T}_h$

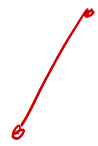


★ Count the dimension of $\mathbb{P}^n(K)|_X$, starting from vertices

Convention: $\dim\left(\mathbb{P}^n(K)|_v\right) \equiv \begin{cases} 1 & n>0 \\ 0 & n=0 \end{cases}$

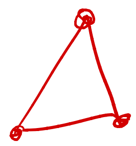
* Start with vertices: 1-dof associated to each vertex (if $n>0$) $\boxed{N_v = 1}$ for scalar problems.

* Edges: each edge has two vertices.
associate to each edge N_e dofs



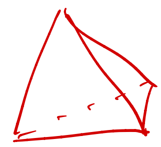
$$N_e := \left[\dim\left(P^n(K)\big|_e\right) - 2N_v\right]$$

* Triangles. three vertices, three edges



$$N_T := \left[\dim\left(P^n(K)\big|_T\right) - 3N_e - 3N_v\right]$$

* Tetrahedron: 4 Triangles, 4 vertices, 6 edges



$$N_{Tet} := \left[\dim\left(P^n(K)\right) - 4N_T - 6N_e - 4N_v\right]$$

Example: $\mathbb{P}^1$ on Triangles.

$N_v = 1$

$\dim(\mathbb{P}^1)\big|_e = 2 \Rightarrow N_E = 0$

$\dim(\mathbb{P}^1)\big|_T = 3 \Rightarrow N_T = 0$

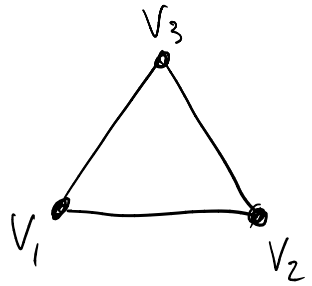
All dofs are associated to vertices

★ Notation: "•" represents a Dirac Delta

- "↗" " directional derivative of the Dirac Delta

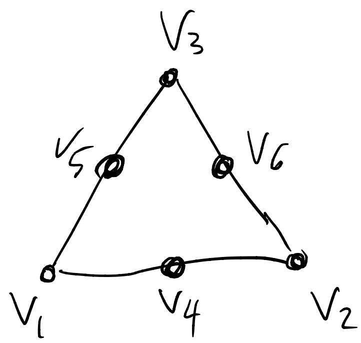
- "O" gradient of Dirac Delta

For $\mathbb{P}^1$:



$$\Sigma_K := \{ \delta(x - V_i) \}_{i=1}^{3}$$

$\mathbb{P}^2$:



$$\Sigma_K := \{ \delta(x - V_i) \}_{i=1}^{6}$$

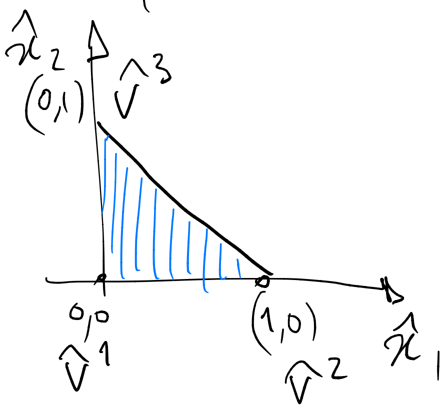
Corresponding basis functions;

$\phi_i \in \mathbb{P}^{1/2}(K)$ | $\phi_i(V_j) = \delta_{ij}$

If Σ contains only Dirac $\Rightarrow$ Lagrange Basis

If Σ " also derivatives of Dirac $\Rightarrow$ Hermite Basis.

Any d-simplex K can be represented as an affine transformation of the reference d-simplex: $\hat{K}$

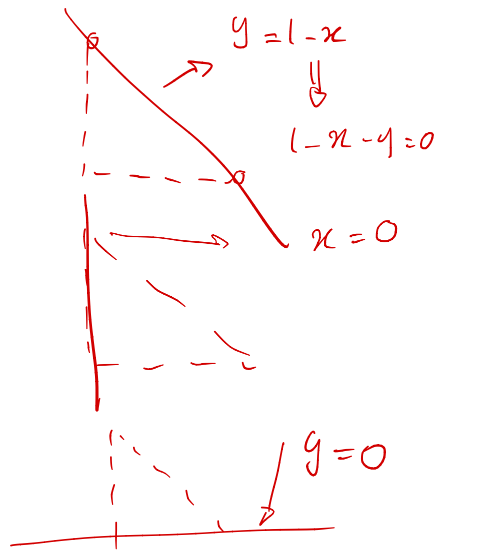


$$\Sigma_{\hat{K}} := \{ \delta(\hat{x} - \hat{v}^i) \}_{i=1}^{3}$$

$$\hat{\phi}_1 := 1 - x - y$$

$$\hat{\phi}_2 := x$$

$$\hat{\phi}_3 := y$$



Affine transformation from $\hat{K}$ to K:

$$T_K : \hat{K} \longrightarrow K$$
$$\hat{x} \longrightarrow v^i \hat{\phi}_i(\hat{x})$$

By definition: $T_K(\hat{v}^i) = v^i$, F:R

$$T_K(\hat{x}) \in \left(\mathbb{P}^1(K)\right)^2$$

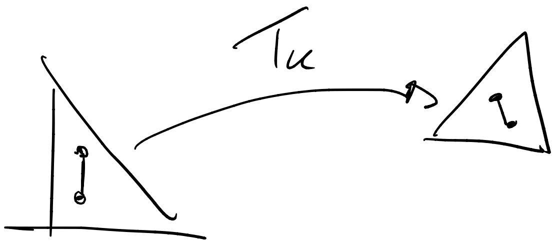
$$\boxed{T_K := B_K \hat{x} + \hat{b}_K}$$

$$h_K := \max \{ |x-y| \quad x,y \in K \}$$

$$\rho_K := \sup \{ 2\rho \mid B_\rho \subseteq K \}$$

$$\hat{\xi} := \hat{x} - \hat{y} \qquad \text{with} \quad \hat{x}, \hat{y} \in \hat{K}$$

$$\xi = B_K \hat{\xi} = T_K \hat{x} - T_K \hat{y} = x - y \qquad \text{with} \quad x, y \in K$$



$$\| B_K \| := \sup_{|\hat{\xi}_K| = \rho_{\hat{K}}} \frac{|B_K \hat{\xi}|}{|\hat{\xi}|} \leq \frac{h_K}{\rho_{\hat{K}}} \leq c\, h_K$$

$$\| B_K^{-1} \| := \sup_{|\xi| = \rho} \frac{|B_K^{-1} \xi|}{|\xi|} \leq \frac{h_{\hat{K}}}{\rho_K} \leq c\, \rho_K^{-1}$$

If triangulation is <u>shape regular</u>

$$\rho_K \geq \sigma h_K \quad \forall K$$

$$\| B_K \| \leq c\, h_K \qquad \| B_K^{-1} \| \leq c\, h_K^{-1}$$