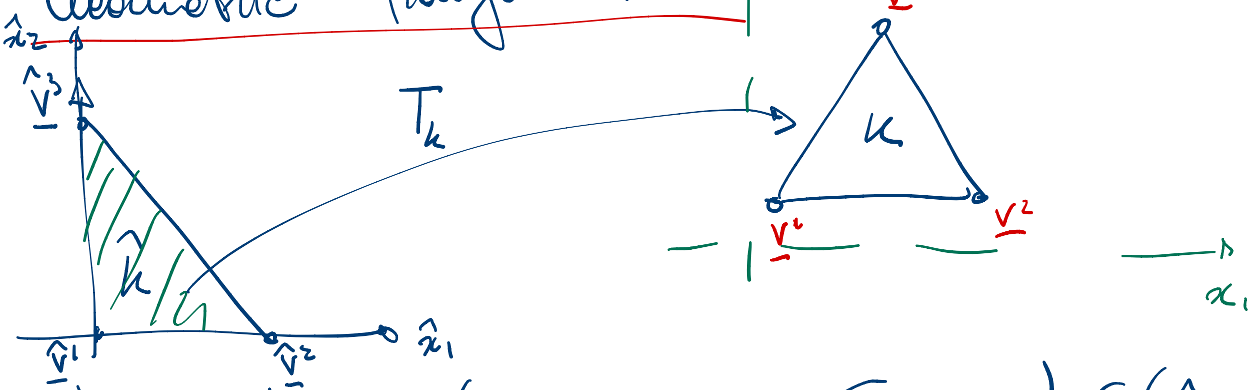
Geometric Transformation

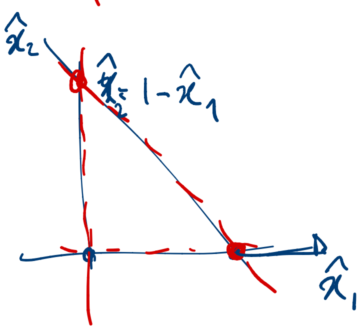


$$FE_{geo} \left\{ \hat{K},\ \mathbb{P}^1,\ \Sigma \right\} \qquad \underline{\Sigma} := \left\{ \delta(\hat{x} - \hat{v}^i) \right\}_{i=1}^{N}$$

$\mathbb{P}^1$: $\hat{v}^i$ are the vertices of reference simplex.

$$T_K := \underline{V}^i \, v_i(\hat{x}) \qquad v^j(v_i) = \delta^i_j$$

Two dimensions.



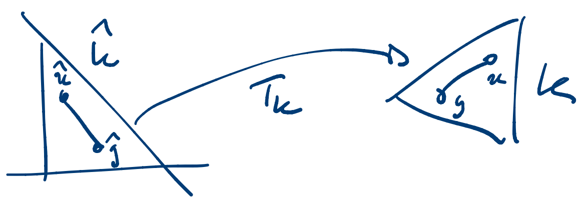
$$v_1 = 1 - \hat{x}_2 - \hat{x}_1$$
$$v_2 = \hat{x}_1$$
$$v_3 = \hat{x}_2$$

$$T_K = \underline{V}^i \, v_i = \underline{\underline{B}}\hat{x} + \hat{b}$$

Affine

$$T_K : \hat{K} \longrightarrow K$$
$$\hat{x} \longrightarrow v^i \, v_j(\hat{x}) = \underline{\underline{B_K \hat{x} + \hat{b}_K}}$$

$$T_K(\hat{v}^i) = v^i = v^i \, v_j(\hat{v}_j)$$



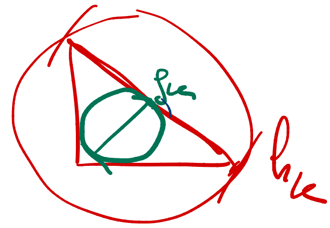
$$\xi = x - y$$
$$\hat{\xi} = \hat{x} - \hat{y}$$

$$T_K \hat{\xi} = \xi = B_K \hat{\xi}$$

$$\|B_K\| := \sup \frac{|B\hat{\xi}|}{|\hat{\xi}|}$$

$$h_K := \max\{ |x-y| \mid x,y \in K \}$$

$$\rho_K := \sup\{ 2\rho \mid B_\rho \subseteq K \}$$



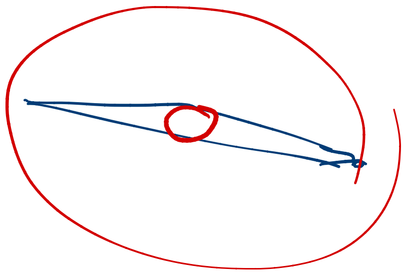
$$\| B_K \| := \sup_{|\hat{\xi}_K| = \rho_{\hat{K}}} \frac{|B_K \hat{\xi}_K|}{|\hat{\xi}_K|} \leq \frac{h_K}{\rho_{\hat{K}}} \leq C_1 h_K$$

$\to \ell_2$ norm

$$\| B_K^{-1} \| := \sup_{|\xi| = \rho_K} \frac{|B_K^{-1} \xi_K|}{|\xi_K|} \leq \frac{h_{\hat{K}}}{\rho_K} \leq C_2 \rho_K^{-1}$$

$$h_{\hat{K}} = \sqrt{2} \qquad \rho_{\hat{K}} \leq \frac{\sqrt{2}}{2}$$

Shape Regular Triangulations



$\exists\, \sigma \in (0,1)$ s.t. $\rho_K \geq \sigma h_K$

$$\Rightarrow \quad \|B_K\| \leq C h_K \qquad \|B_K^{-1}\| \leq C_\sigma h_K^{-1}$$

Estimate Sobolev Norms under affine transform.

T_K defines naturally a push forward:

$$v \circ T_K \Big|_K = \hat{v} \Big|_{\hat{K}} \qquad v(T_K(\hat{x})) = \hat{v}(\hat{x})$$

$K = T_K(\hat{K})$, $\mathbb{P}^K = \# T_K$, $\Sigma_K = \# T_K$ is a FE in particular $\quad v_i = \hat{v}_i \circ T_k^{-1}$

$$v_i(T_k(\hat{x})) = \hat{v}_i(\hat{x})$$

$$T : k \longrightarrow \hat{k} \qquad \mathbb{P}'(\hat{k}) \longrightarrow \mathbb{P}'(k)$$

derivative is taken w.r.t. $\hat{x}$

$\hat{D}^\alpha \hat{v}$ $\qquad\qquad$ $D^\alpha v$

Derivative is taken w.r.t. x

$$\hat{D}^\alpha \hat{v} = \hat{D}^\alpha (v \circ T_k) = \left[(D^\alpha v)\circ T\right] \underbrace{(DT_k)}_{B_k}{}^{|\alpha|}$$

$$|\hat{v}|_{m,p,\hat{k}} = \left(\sum_{|\alpha|=m} \int_{\hat{k}} \left(\hat{D}^\alpha \hat{v}\right)^p d\hat{x} \right)^{\frac{1}{p}} \qquad J := \det(B)$$

$$|\hat{v}|_{m,p,\hat{k}} = \left(\sum_{|\alpha|=m} \int_{\hat{k}} \left((D^\alpha v)\circ T_k \, B^m \right)^p J^{-1} J \, d\hat{x} \right)^{\frac{1}{p}} =$$

$$= \left(\sum_{|\alpha|=m} \int_{k} \left(D^\alpha v \, B^m \right)^p J^{-1} dx \right)^{\frac{1}{p}}$$

$$|\hat{v}|_{m,p,\hat{k}} \leq \|B\|^m J^{-\frac{1}{p}} |v|_{m,p,k}$$

$$|\hat{v}|_{m,p,\hat{k}} \leq C\, h_k^m \, J^{-\frac{1}{p}} |v|_{m,p,k}$$

Applying the same to $\hat{v} \circ T_k^{-1} = v$

$$|v|_{m,p,k} \leq C \rho_k^{-m} J^{p} |\hat{v}|_{m,p,\hat{k}}$$

For $v \in \mathbb{P}^{e}(k)$ for $0 \leq m, s \leq e$

$\exists\, c_1, c_2$ s.t. $\quad c_1 |\hat{v}|_{m,p,\hat{k}} \leq |\hat{v}|_{s,p,\hat{k}} \leq c_2 |\hat{v}|_{m,p,\hat{k}}$

$$|v|_{m,p,k} \leq C \rho_k^{-m} J^{p} |\hat{v}|_{m,p,\hat{k}} \leq C \rho_k^{-m} J^{p} |\hat{v}|_{s,p,\hat{k}} \leq C \rho_k^{-m} h_k^{s} |v|_{s,p,k}$$

$0 \leq m \leq s \leq e$

$$\|v\|_{m,p,k} \leq C \rho_k^{-m} h_k^{s} \|v\|_{s,p,k}$$

Bramble Hilbert Lemma

$m \geq 0$, $s \geq 0$, $e \geq 0$ $\quad \tau : H^{s}(k) \to H^{m}(k)$

such $\cdot$ $\ker(\tau) \supset \mathbb{P}^{e}(k)$ $[\tau p = 0 \ \forall p \in \mathbb{P}^{e}(k)]$

$\cdot$ τ is linear

$\forall v \in H^{s}(k)$

$$\|\tau v\|_{m,k} \leq \|\tau\|_{*} \inf_{p \in \mathbb{P}^{e}(k)} \|v - p\|_{s,k}$$

$$\|\tau(v)\|_{m,k} = \|\tau(v-p)\|_{m,k} \leq \|\tau\|_* \, \|v-p\|_{s,k} \quad \forall p$$

$$\Rightarrow \quad \|\tau(v)\|_{m,k} \leq \|\tau\|_* \inf_{p \in \mathbb{P}^\ell(k)} \|v-p\|_{s,k}$$

Apply to FEM : $\tau := I - \Pi$

$$V_h := \operatorname{span}\{v_i\}_{i=1}^N \qquad \Sigma := \{v^i\}_{i=1}^N$$

$$v^i \in \mathcal{L}(V_h)$$

Given $V \supset V_h \to \exists\, \tilde{\Sigma} := \{\tilde{v}^i\}_{i=1}^N$

$\tilde{v}^i$ are extensions of v^i to V' Hahn-Banach

$\to$ "Interpolation" operator:

$$\Pi : V \longrightarrow V_h$$

$$u \longrightarrow \tilde{v}^i(u)\, v_i$$

Π is a projection $(\Pi^2 = \Pi)$

because $\forall\, v \in V_h \quad v = v^i(v)\, v_i$

and $\quad v^i(v_j) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\Pi v = \Pi\left(v^j(v)\, v_j\right) = \tilde{v}^i(v_j)\, v^j(v)\, v_i = v^i(v_j)\, v^j(v)\, v_i$$

$$= \delta^i_J v^J(v) v_i = \underline{\underline{v^i(v)\, v_i}} \qquad \forall v \in V_\ell$$

$$s = \ell+1, \qquad 0 \le m \le \ell$$

$$I - \Pi \Rightarrow \|u - \Pi u\|_{m,\ell} \le \|I-\Pi\|_* \inf_{p \in \mathbb{P}^\ell} \underline{\underline{\|u+p\|_{\ell+1,k}}}$$

<u>Theorem</u>

$$\underline{\inf_{p\in\mathbb{P}^\ell} \|u-p\|_{\ell+1,k} \le C\,|u|_{\ell+1,k} \qquad \forall \ell \ge 0 \quad \forall u \in H^{\ell+1}(k)}$$

$\downarrow$

Denis Lion lemma

Show that $\exists\, C$ s.t. $\forall v \in H^{\ell+1}(k)$

1) $$\|v\|^2_{\ell+1,k} \le C\left[\,|v|^2_{\ell+1,k} + \sum_{i=1}^N |\tilde{v}^i(v)|^2\right]$$

If 1) is false $\exists$ sequence

$u_J \in H^{\ell+1}(k)$ s.t. $\|u_J\| = 1 \ \forall J$ and

$$* \qquad |u_J|^2_{\ell+1,k} + \sum_{i=1}^N |\tilde{v}^i(u)|^2 \le \frac{1}{J}$$

$$H^{\ell+1} \hookrightarrow H^\ell$$

$\exists$ subsequence (which I still call u_J) s.t.

strongly covergent in H^ℓ. from $*$ $|u_J|_{\ell+1,k} \to 0$

$\Rightarrow$ it strongly converges also in $H^{\ell+1}$ (to $\underline{\underline{w}}$)

- $\|w\|_{\ell+1,k} = 1$
- $|w|_{\ell+1,k} = 0 \Rightarrow w \in \mathbb{P}^{\ell}(k)$, $\|w\|_{\ell} = 1$
- by $*$ $\sum |\tilde{v}^i(w)|^2 = \sum |v^i(w)|^2 = 0$

$\Rightarrow w = v^i(w)\, v_i = 0$

contradiction $\|w\|_{\ell} = 1$

$$\inf_{p \in \mathbb{P}^{\ell}} \|u - p\|_{\ell+1,k} \le C\, |u|_{\ell+1,k}$$

$$\Downarrow$$

$$\|u - \Pi u\|_{\ell+1,k} \le \underline{\underline{\|I - \Pi\|_{*}}}\; |u|_{\ell+1,k}$$

$H^{\ell+1} \hookrightarrow H^{m}$ $m \le \ell$

$$\|u - \Pi u\|_{m,k} \le C \rho_k^{-m} J^{\frac{1}{2}} \|\hat{u} - \hat{\Pi}\hat{u}\|_{m,\hat{k}} \le C \rho_k^{-m} J^{\frac{1}{2}} \|\hat{u} - \hat{\Pi}\hat{u}\|_{\ell+1,\hat{k}}$$

$$\le C \rho_k^{-m} h_k^{\ell+1} |u - \Pi u|_{\ell+1,k}$$

$$\|u - \Pi u\|_{m,k} \le C \rho_k^{-m} h_k^{\ell+1} |u|_{\ell+1,k}$$

Apply this to $Au = f$ in V'

$V_h \subset V$

$\Downarrow$

Find $u_h \in V_h$ s.t.

$\langle Au_h, v_h \rangle = \langle f, v_h \rangle \quad \forall v_h \in V_h$

$V_h\big|_K = \mathbb{P}^k$

$h = \max_K h_K$

$\rho = \min_K \rho_K$

Assume $V = H^\ell$ and $V_h\big|_K = \mathbb{P}^\ell$

$$\Rightarrow \quad \|u - u_h\|_{\ell,\Omega} \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_{\ell,\Omega}$$

$$\leq \frac{C}{\alpha} \sum_K \|u - \Pi u\|_{\ell,K}$$

$$\leq \frac{C}{\alpha} \rho^{-\ell} h^{\ell+1} \sum_K |u|_{\ell+1,K}$$

If $u \in H^{\ell+1}(\Omega)$ then

$$\|u - u_h\|_{\ell,\Omega} \leq \frac{C}{\alpha} \rho^{-\ell} h^{\ell+1} |u|_{\ell+1,\Omega}$$

For shape regular $\mathcal{T}_h$:

$$\|u - u_h\|_{\ell,\Omega} \leq \frac{C}{\alpha} h^{1} |u|_{\ell+1,\Omega}$$

In general : $V \subset H^m(\Omega)$ pol. of order $\ell \geq m$

$$\|u - u_h\|_{m,\Omega} \leq \frac{C}{\alpha} h^{\ell+1-m} |u|_{\ell+1,\Omega} \quad \forall u \in H^{\ell+1}(\Omega)$$

If pb is k-regular, that is : the solution $u \in V \cap H^{\ell+1}$

$\exists\ C$ s.t.

$$\|u\|_{k+p} \leq C \|f\|_{p,\Omega} \quad \text{for some } p$$

Then if $p \geq \ell+1-k \geq 0$ then

$$\|u - u_h\|_{m} \leq \frac{C}{\alpha} h^{\ell+1-m} \|f\|_{\ell+1-k,\Omega}$$