Sobolev Norms under affine transformations

$$\mathbb{P}^1(\hat{K}) = \operatorname{span}\{\hat{\phi}_i\}_{i=1}^{d+1}$$

$$T : \hat{K} \longrightarrow K$$

$$\hat{x} \longrightarrow v^i \underline{\hat{\phi}_i}(\hat{x}) = B\hat{x} + \hat{b}$$

$v \in P_K$ can be seen as push forward of $\hat{v} \in P_{\hat{K}}$:

$$v(T(\hat{x})) = \hat{v}(\hat{x}) \quad \hat{x} \in \hat{K}$$

$$v(x) = \hat{v}(T^{-1}(x)) \quad x \in K$$

In general for Affine T, $DT = B$ constant, $J = \det B$ constant

$$D^\alpha(v \circ T) = [(D^\alpha v) \circ T](DT)^\alpha = D^\alpha \hat{v}$$

$$|\hat{v}|^2_{m,\hat{K}} := \sum_{|\alpha|=m} \int_{\hat{K}} (D^\alpha \hat{v})^2 \, d\hat{x}$$

$$\leq \sum_{|\alpha|=m} \left[\|B\|^{2m} J^{-1} \int_{\hat{K}} (D^\alpha \hat{v} \, B^{-\alpha})^2 J \, d\hat{x} \right]$$

$$= \sum_{|\alpha|=m} \left[\|B\|^{2m} J^{-1} \int_K (D^\alpha v)^2 \, dx \right]$$

That is:

$$|\hat{v}|_{m,\hat{K}} \leq \|B\|^m J^{-\frac{1}{2}} |v|_{m,K}$$

and moreover

$$|v|_{m,K} \leq \|B^{-1}\|^m J^{\frac{1}{2}} |\hat{v}|_{m,\hat{K}}$$

using $\|B\| \leq C h_K$ $\|B^{-1}\| \leq C \rho_K^{-1}$

$$\boxed{\begin{aligned} |\hat{v}|_{m,\hat{K}} &\leq C h_K^m J^{-\frac{1}{2}} |v|_{m,K} \\ |v|_{m,K} &\leq C \rho_K^{-m} J^{\frac{1}{2}} |\hat{v}|_{m,\hat{K}} \end{aligned}}$$

For $v \in \mathbb{P}^\ell(K)$, given $0 \leq m, s \leq \ell$

$$c|\hat{v}|_{m,\hat{K}} \leq |\hat{v}|_{s,\hat{K}} \leq C|\hat{v}|_{m,\hat{K}}$$

$$|v|_{m,K} \leq C \rho_K^{-m} J^{\frac{1}{2}} |\hat{v}|_{m,\hat{K}} \leq C \rho_K^{-m} J^{\frac{1}{2}} |\hat{v}|_{s,\hat{K}} \leq C \rho_K^{-m} h_K^s |v|_{s,K}$$

for $v \in H^s(K)$, with full norms I can still say

$$\|v\|_{m,K} \leq C \|v\|_{s,K} \quad 0 \leq m \leq s$$

Bramble Hilbert Lemma

$m \geq 0$, $s \geq 0$, $\ell \geq 0$ $\quad \tau : H^s(K) \longrightarrow H^m(K)$

such that
- $\ker(\tau) \supset \mathbb{P}^\ell(K)$ $\quad [\tau p = 0 \ \forall p \in \mathbb{P}^\ell(K)]$
- τ is linear

Then $\forall v \in H^s(K)$

$$\| \tau v \|_{m,K} \leq \| \tau \|_* \inf_{p \in \mathbb{P}^\ell(K)} \| v + p \|_{s,K}$$

Proof:

τ is linear, and $\tau p = 0 \quad \forall p \in \mathbb{P}^\ell(K)$

$$\| \tau v \|_{m,K} = \| \tau(v+p) \|_{m,K} \leq \| \tau \|_* \| v+p \|_{s,K} \quad \forall p \in \mathbb{P}^\ell(K)$$

By definition of $\boxed{\| \tau \|_* := \sup \frac{\| \tau v \|_{m,K}}{\| v \|_{s,K}}}$

Application to FEM: Take $s = \ell+1$, $0 \leq m \leq \ell$,
$\tau := I - \Pi_K$

$$\| u - \Pi_K u \|_{m,K} \leq \| I - \Pi_K \|_* \inf_{p \in \mathbb{P}^\ell(K)} \| u + p \|_{\ell+1,K} \quad (1)$$

Now let's show that:

$$\inf_{p \in \mathbb{P}^\ell(K)} \| u + p \|_{\ell+1,K} \leq C \, | u |_{\ell+1,K}$$

Denis – Lions Lemma

$\forall \ell \geq 0$, $\forall v \in H^{\ell+1}(K)$

$$\inf_{p \in \mathbb{P}^{\ell}(K)} \|v+p\|_{\ell+1,K} \leq |v|_{\ell+1,K} \qquad (2)$$

proof:

Show that

$$(3) \qquad \|v\|_{\ell+1,K} \leq C\left[\,|v|_{\ell+1,K} + \sum_{i=1}^{N} \left| \langle \tilde{\underline{\phi}}^i, v \rangle \right|^2 \right]$$

where $(\mathbb{P}^{\ell})^{*} = \operatorname{span}\{\underline{\phi}^i\}_{i=1}^{N}$, $N = \dim(\mathbb{P}^{\ell}(K))$

$\tilde{\phi}^i$ Extension of ϕ^i to $(H^{\ell+1})^{*}$

By contradiction, if (3) is false, then $\exists$ a sequence $v_J \in H^{\ell+1}(K)$ s.t. $\|v_J\| = 1$ $\forall J$ and

$$(4) \qquad |v_J|_{\ell+1,K} + \sum_{i=1}^{N} \left| \langle \tilde{\phi}^i, v \rangle \right|^2 \leq \frac{1}{J}.$$

- Observe that $H^{\ell+1} \hookrightarrow H^{\ell}$.

  $\Rightarrow \exists$ subsequence v_J strongly convergent in H^{ℓ}

- from (4), we have that $|v_J|_{\ell+1,K} \to 0$, therefore, v_J converge strongly in $H^{\ell+1}$ to $\underline{w}$,

- $\| w \|_{l+1,K} = 1$
- $| w |_{l+1,K} = 0 \Rightarrow w \in \mathbb{P}^l(K)$, $\|w\|_l = 1$
- $\sum |\langle \tilde{\phi}^i, w \rangle|^2 = 0$ by (4), then

$$\Rightarrow \underline{w = \langle \tilde{\phi}^i, w \rangle \phi_i} = \langle \phi^i, w \rangle \phi_i \underline{\underline{= 0}}$$

Contradiction, since $\| w \|_{l+1} = \| w \|_l = 1$

$$\star \quad \inf_{p \in \mathbb{P}^l(K)} \| u + p \|_{l+1,K} \leq \| u - \Pi_K u \|_{l+1,K}$$

$$\leq C \left[| u |_{l+1,K} + \sum_{i=1}^{N} | \langle \tilde{\phi}^i, (u - \Pi_K u) \rangle |^2 \right]$$

$$\langle \tilde{\phi}^i, u - \Pi_K u \rangle = \langle \tilde{\phi}^i, u \rangle - \langle \tilde{\phi}^i, \langle \tilde{\phi}^J, u \rangle \phi_J \rangle$$

$$= \langle \tilde{\phi}^i, u \rangle - \delta^i_J \langle \tilde{\phi}^J, u \rangle$$

$$= \langle \tilde{\phi}^i, u \rangle - \langle \tilde{\phi}^i, u \rangle = 0$$

cvd

Application to FEM

$H^{l+1} \hookrightarrow H^m$ for $m \leq l$

$$\| u - \Pi_K u \|_{m,K} \leq C \rho_K^{-m} J^{\frac{1}{2}} \| \hat{u} - \Pi_{\hat{K}} \hat{u} \|_{m,\hat{K}} \leq C \rho_K^{-m} J^{\frac{1}{2}} \| \hat{u} - \Pi_{\hat{K}} \hat{u} \|_{l+1,\hat{K}}$$

$$\leq C \rho_K^{-m} h_K^{l+1} \| u - \Pi_K u \|_{l+1,K}$$

$$\boxed{\| u - \Pi_K u \|_{m,K} \leq C \rho_K^{-m} h_K^{l+1} | u |_{l+1,K}}$$

Now take u, u_h sol. to $a(u,v) = \langle f, v\rangle \quad \forall v \in H^1_0$

$a(u_h, v_h) = \langle f, v_h\rangle \quad \forall v \in V_h \subset H^1_0$

with $V_h\big|_K \equiv \mathbb{P}^\ell(K)$

Then $$\boxed{\|u-u_h\|_{1,\Omega}} \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_1$$

$$\leq \frac{C}{\alpha} \|u - \Pi u\|_{1,\Omega}$$

$$\boxed{\leq \frac{C}{\alpha} C_{BH} \rho^{-1} h^{\ell+1} |u|_{\ell+1,\Omega}}$$

If the problem is k-regular $(k \geq 2)$, that is, the solution $u \in V \cap H^k$, and there exist C_{reg} s.t.

$\|u\|_{k,\Omega} \leq C \|f\|_{0,\Omega}$ then

$$\|u - u_h\|_{1,\Omega} \leq C h^2 \rho^{-1} \|f\|_{0,\Omega}$$

For L^2 estimates, we'll need some more work