Recap DG Methods

$$\begin{cases} -\Delta u - f = 0 & \text{in } \mathcal{T} \\ [\![u]\!]|_e = 0 & \text{on } \mathcal{E} \\ [\![\nabla u]\!]_e = 0 & \text{on } \mathcal{E}^0 \end{cases}$$

$$(-\Delta u - f, B_0 v) + \langle [\![u]\!], \underline{B_1 v}\rangle_{\underline{\mathcal{E}}} + \langle [\![\nabla u]\!], B_2 v\rangle_{\underline{\mathcal{E}^0}} = 0$$

$$\sum_K \int_{\underline{\partial K}} a(\underline{b}\cdot n) = \sum_{\underline{e \in \mathcal{E}}} \langle [\![a]\!], \{\underline{b}\}\rangle + \sum_{\underline{e\in\mathcal{E}^0}} \langle \{a\}, [\![b]\!]\rangle$$

$B_0 v \longrightarrow v$

$$\underline{(\nabla u, \nabla v)_{\mathcal{T}} - (f, v)_{\mathcal{T}} - \langle [\![v]\!], \{\nabla u\}\rangle_{\mathcal{E}} - \langle \{v\}, [\![\nabla u]\!]\rangle_{\mathcal{E}^0}}$$

$$+ \langle [\![u]\!], \underline{B_1} v\rangle_{\mathcal{E}} + \langle [\![\nabla u]\!], B_2 v\rangle_{\mathcal{E}^0} = 0$$

SIPG:
$$B_2 v = \{v\}$$
$$B_1 v = s_1 [\![v]\!] - \{\nabla v\}$$

$$\Rightarrow (\nabla u, \nabla v)_{\mathcal{T}} - (f, v)_{\mathcal{T}} - \langle [\![v]\!], \{\nabla u\}\rangle_{\mathcal{E}} - \langle [\![u]\!], \{\nabla v\}\rangle_{\mathcal{E}}$$
$$+ \langle s_1 [\![u]\!], [\![v]\!]\rangle_{\mathcal{E}} = 0$$

Banach, Necas (62), Babuska (72) (BNB)

Given $F \in V'$, find $u \in W$ s.t.

(1) $\langle Bu, v\rangle = \langle F, v\rangle \quad \forall v \in V$

i) W Banach

ii) V reflexive Banach

iii) B is bounded $\quad (B : W \longrightarrow V')$

$w \longrightarrow b(w,\cdot)$

$b(w,v) = \langle Bw, v\rangle$

(1) is well posed if $\exists c$ s.t.

$\forall F \in V', \ \exists!\, u \in W$ s.t. $\|u\|_W \le c \|F\|_{V'}$

BNB Theorem:

(1) is well posed iif:

BNB1) inf-sup condition $\quad \exists \alpha > 0$ s.t. $\displaystyle\inf_{w \in W} \sup_{v \in V} \frac{\langle Bw, v\rangle}{\|w\|_W \|v\|_V} \ge \alpha$

BNB2) $\forall v \in V \quad (\forall w \ \langle Bw, v\rangle = 0) \Rightarrow (v = 0)$

BNB1) $\Longleftrightarrow$ $\ker(B) = \{0\}$ and $\overline{\mathrm{Im}(B)} = \mathrm{Im}(B)$ $\Longleftrightarrow$ B^T surjective

BNB2) $\Longleftrightarrow$ $\ker(B^T) = \{0\}$ $\Longleftrightarrow$ B^T injective

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BNB1): $\|Bw\|_{V'} \ge \alpha \|w\| \quad \forall w \in W$

$\alpha \|u\|_W \le \|Bu\|_{V'} = \|F\|_{V'} \Rightarrow \|u\|_W \le \frac{1}{\alpha} \|F\|_{V'}$

$$\frac{\langle Au,u\rangle}{\|u\|} \geq \frac{\alpha\|u\|^2}{\cancel{\|u\|}}$$

$$\sup_{v\in V} \frac{\langle Au,v\rangle}{\|v\|} \geq \frac{\langle Au,u\rangle}{\|u\|} \geq \alpha\|u\|$$

$$\|Au\|_{V'} \geq \alpha\|u\|$$

BNB does not translate automatically in the discrete setting

Petrov Galerkin method (Generalized Galerkin)

- two discrete spaces W_h, V_h
- two "extended" spaces $\tilde{W} = W_h + W$, $\tilde{V} = V_h + V$
- two possibly h-dependent norms $|||\cdot|||_W$ $|||\cdot|||_V$
- $|||u|||_W \leq c\|u\|_W \quad \forall u \in W$
- $|||v|||_V \leq c\|v\|_V \quad \forall v \in V$

W, V are cont. embedded in $\tilde{W}, \tilde{V}$

- Conformity $V_h \subset V$, $W_h \subset W$
- Approximability 1) $\exists \Pi_w$ s.t. $\|u - \Pi_w u\|_W \leq c \inf_{w_h \in W_h} \|u - w_h\|$

2) $\lim_{h\to 0} \left(\inf_{w_h\in W_h} \|u - w_h\| \right) = 0$

- Consistency if $B_h : \tilde{W} \to \tilde{V}'$

$$B_h u - f_h = 0$$

• Asimptotically consistent: $B_h : W_h \longrightarrow V_h'$

$$\lim_{h\to 0} \underbrace{|||B_h \Pi u - f_h|||_{*,h}}_{R_h(u)} = 0 \quad \text{consistency error}$$

<u>Consistency</u> $\langle Bu - Bu_h, v_h\rangle = 0$

• Cea's lemma
• Galerkin orthogonality

consistent and conformal approx: BNB $\rightarrow$ BNB_h

For general case

i) BNB_h are satisfied uniformly in h for $|||\cdot|||_W$, $|||\cdot|||_V$

ii) B_h is uniformly bounded w.r.t. $|||\cdot|||_W$ $|||\cdot|||_V$

iii) V_h and W_h have approx. property with $\Pi_W : W \longrightarrow W_h$, $\Pi_V : V \longrightarrow V_h$

iv) B_h is at least asimptotically consistent.

Then: $$|||u-u_h|||_W \leq \frac{1}{\alpha_h} R_h(u) + c \inf_{w_h \in W_h} |||w_h - u|||_W$$

$$|||f_h - B_h \Pi_h u_h|||_{*,h} =: R_h(u)$$

(1) $\langle B_h u_h - B_h \Pi u_h, v_h\rangle = \langle f_h, v_h\rangle - \langle B_h \Pi u_h, v_h\rangle$

take sup of $\frac{(1)}{\|v_h\|}$ and use BNB_h

(1) $\alpha_h |||u_h - \Pi u_h|||_W \leq |||B_h u_h - B_h \Pi u_h|||_{*,h} = R_h(u)$

$$|||u-u_h|||_W \leq |||u - \Pi u|||_W + |||\Pi u - u_h|||_W \overset{(1)}{\leq} \frac{1}{\alpha_h} R_h(u) + |||u - \Pi u|||_W$$

$$|||u - u_h|||_W \leq \frac{1}{\alpha_h} R_h(u) + c \inf_{w_h \in W_h} |||u - w_h|||_W$$

$$\Rightarrow \lim_{h \to \infty} \left(|||u - u_h|||_W \right) = 0$$ by asymptotical consistency and approximability

Non consistent / conformal $\Rightarrow$ 1st Strang Lemma

Non " / non conformal $\Rightarrow$ 2nd " "

Examples

Given $f \in V'$, find $u \in W \equiv H^1(\Omega)$ s.t.

$$(\nabla u, \nabla v) = \langle Bu, v \rangle = \langle f, v \rangle \qquad \forall v \in V \equiv H^1_0(\Omega)$$

$$\gamma_0 u = g \qquad \text{in } Z \equiv H^{\frac{1}{2}}(\Gamma) \qquad \leftarrow \text{(A)}$$

1) Assume that γ_0 is ~~surjective~~ bounded surjective (Trace theorem)

We call $W_0 = \ker(\gamma_0) \equiv H^1_0(\Omega) \equiv V$

by 1): $\exists u_g \in W$ s.t. $\gamma_0 u_g = g$ $\qquad \phi = \underset{\in W_0}{u} + u_g$

(A) is equivalent to solving find $\phi \in W_0$ s.t.

$$\langle B\phi, v \rangle = \langle f, v \rangle - \underline{\underline{\langle Bu_g, v \rangle}} \qquad \forall v \in V$$

$$|||u|||_W \leq c \left(\|f\|_{V'} + \|g\|_Z \right)$$

$$|\langle f, v \rangle - \langle Bu_g, v \rangle| \leq \|f\|_{V'} \|v\|_V + \alpha \|u_g\|_W \|v\|_V$$

$\alpha C_T \|g\|_Z \|v\|_V$

Second Example: Mixed problems (LBB condition)

$$\begin{cases} -\Delta \underline{u} + \nabla p = f \\ \nabla \cdot \underline{u} = 0 \end{cases}$$

$u = 0$ on $\partial\Omega$

$$\begin{cases} \underline{u} + \nabla p = 0 \\ \nabla \cdot \underline{u} = g \end{cases} \quad : \quad -\Delta p = g$$

General case

$A : V \longrightarrow V'$ $\quad V = \left(H_0^1(\Omega) \right)^d$ think of

$B : V \longrightarrow Q'$ $\quad Q = L_0^2(\Omega) := \{ v \in L^2 \text{ s.t. } (v,1) = 0 \}$

$$\begin{cases} \langle Au, v \rangle + \langle B^T p, v \rangle = \langle f, v \rangle & \forall v \in V \\ \langle Bu, q \rangle = \langle g, q \rangle & \forall q \in Q \end{cases}$$

$g \in \mathrm{Im}(B)$

$$\underbrace{\langle Au, v \rangle}_{a(u,v)} \leq \|A\|_* \|u\|_V \|v\|_V \quad \forall u, v \in V$$

$$\underbrace{\langle Bu, q \rangle}_{b(u,q)} \leq \|B\|_* \|u\|_V \|q\|_Q \quad \forall u \in V, \ \forall q \in Q$$

Stokes:

$a(u,v) = (\nabla \underline{u}, \nabla \underline{v})$ $\qquad b(u,q) = -(\nabla \cdot \underline{u}, q)$

$g \in \mathrm{Im}(B) \Rightarrow \exists \, u_g$ s.t. $Bu_g = g$

$V_0 := \ker(B)$ $\qquad v \in V$ s.t. $(\nabla \cdot v, q) = 0 \ \forall q \in Q$

Rewrite the rhs. with $\tilde{f} = f - Au_g$ $\qquad u = u_0 + u_g$

Find $u_0 \in V$, $p \in Q$ s.t. $\qquad u_0 \in V_0$

$u_g \in V$ s.t. $Bu_g = g$

$$\begin{cases} Au_0 + B^T p = \tilde{f} & \text{in } V' \\ Bu_0 = 0 & \text{in } Q' \end{cases}$$

V_0 is $\ker(B)$: restrict $A|_{V_0}$ and $B|_{V_0}$:

Find $u_0 \in V_0$, $p \in Q$ s.t.

$\langle A u_0, v_0\rangle + \cancel{\langle B^T p, v_0\rangle}^{\to 0} = \langle \tilde{f}, v_0\rangle$ in V_0' $\qquad B u_0 = 0 \ \forall u_0 \in V_0$

$\cancel{\langle B u_0, q\rangle}^{\to 0} = 0$ in Q'

Find $u_0 \in V_0$ s.t.

$\langle A u_0, v_0\rangle = \langle \tilde{f}, v_0\rangle \quad \forall v_0 \in V_0$

First condition : BNB on $A|_{V_0}$ $\qquad$ ELL-KER

$\exists \alpha$ s.t. i) $\langle A u_0, u_0\rangle \geq \alpha \|u_0\|^2 \quad \forall u_0 \in V_0$

ii) $\|A u_0\|_* \geq \alpha \|u_0\|_V$

iii) $\inf_{v_0 \in V} \sup_{u_0 \in V} \dfrac{\langle A u_0, v_0\rangle}{\|u_0\|_V \|v_0\|_V} \geq \alpha$ $\qquad \|u_0\| \leq \frac{1}{\alpha}\|\tilde{f}\|$

$\leq \frac{1}{\alpha}\left(\|f\|_{V'} + \|A\|\|g\|_{Q'}\right)$

Once u_0 has been found then look at : Find $p \in Q$ s.t.

$\langle B^T p, v\rangle = \langle \tilde{f}, v\rangle - \langle A u_0, v\rangle \quad \forall v \in V$ (for uniqueness $\cancel{V_0}^{\perp}$)

Second condition : LBB condition, inf-sup condition

$\exists \beta > 0$ s.t. $\qquad \|B^T p\|_{V'} \geq \beta \|p\|_Q \quad \forall p \in Q$

To ensure uniqueness we restrict the second pb to $V_0^{\perp}$

Discrete level

$V_h \subset V \quad , \quad Q_h \in Q$

$$\langle A u_h, v_h \rangle + \langle B^T p_h, v_h \rangle = \langle \tilde{f}, v_h \rangle \quad \forall v_h \in V_h$$
$$\langle B u_h, q_h \rangle = 0 \quad \forall q_h \in V_h$$

$$V_{0,h} := \ker(B_h) := \{ v_h \in V_h \ \text{s.t.} \ \langle B v_h, q_h \rangle = 0 \ \forall q_h \in Q_h \}$$

$$V_{0,h} \not\subset V_0$$

Restrict to $V_{0,h}$, and find $u_{0,h}$. ELL-KER is not obviously satisfied because we only ask that A is elliptic on $V_0 \subset V$ but we know $V_{0,h} \subset V_0$

we need ELL-KER$_h$

Once we have $u_{0,h}$, find p_h s.t.

$$\langle B^T p_h, v_h \rangle = \langle \tilde{f}, v_h \rangle - \langle A u_{0,h}, v_h \rangle \quad \forall v_h \in V_h$$

we need INF-SUP$_h$

$$\inf_{q_h \in Q \setminus \ker B^T} \ \sup_{v_h \in V_h \setminus \ker B} \frac{\langle B v_h, q_h \rangle}{\|v_h\|_V \|q_h\|_Q} > \beta_h > \beta$$