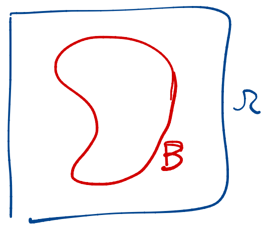
Regularized Elliptic Problems.

Prototypical problems. :

$B \subset \mathbb{R}^d$

$$F(x) := \int_B \delta(x-y)\, f(y)\, dy$$



- $f : L^2(B)$

For variational formulation :

$$\langle F, v \rangle_{V' \times V} := \int_B f\, v\, dB$$

- B has co-dimension 0

$$\rightarrow \quad F = \chi_B f \quad \Rightarrow \quad F \in L^2(\Omega)$$

- B has co-dimension 1

$$\rightarrow \quad \langle F, v \rangle_{V \times V'} = \int_B f, v\, dB \quad \Rightarrow \quad F \in H^{-\frac{1}{2}-}(\Omega) \qquad -\tfrac{1}{2}- \equiv -\tfrac{1}{2} - \varepsilon \ \forall \varepsilon > 0$$

- B has co-dimension 2 in $d=2$

$$\rightarrow \quad \langle F, v \rangle_{V \times V'} = f\, v(B) \quad \Rightarrow \quad F \in H^{-1-}(\Omega)$$

- B has co-dimension 3 (in dim. 3)

$$\rightarrow \quad \langle F, v \rangle_{V \times V'} = f\, v(B) \quad \Rightarrow \quad F \in H^{-\frac{3}{2}-}(\Omega)$$

Idea behind regularization comes from Mollification in $\mathbb{R}^d$: We define $*$ convolution as:

$$(v * g)(x) := \int_{\mathbb{R}^d} v(y)\, g(x-y)\, dy$$

$$v * \delta = v \qquad \forall v \in C_c^\infty(\mathbb{R}^d)$$

$$v * \delta^\varepsilon = v^\varepsilon \quad \Rightarrow \quad v^\varepsilon \text{ has at least same reg. of } \delta^\varepsilon$$

$$v * \delta^\varepsilon = v^\varepsilon = \int_{\mathbb{R}^d} v(y)\, \delta^\varepsilon(x-y)\, dy$$

We go from $\int_B \delta(x-y) f(y) dy$ to $\int_B \delta^\varepsilon(x-y) f(y) dy$

$$F \to F^\varepsilon(z) = \int_{\mathbb{R}^d} F(y)\, \delta^\varepsilon(z-y)\, dy = \int_{\mathbb{R}^d} \underbrace{\int_B f(x)\, \delta(y-x)\, dx}_{F(y)}\, \delta^\varepsilon(z-y)\, dy$$

$$= \int_B \underbrace{\int_{\mathbb{R}^d} \delta(y-x)\, \delta^\varepsilon(z-y)\, dy}_{\delta^\varepsilon(y-x)}\, f(x)\, dx$$

$$\int_B \delta^\varepsilon(y-x)\, f(x)\, dx = F^\varepsilon$$

Two working hypothesis. (A)

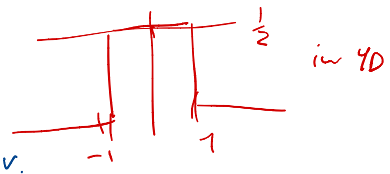
- $\exists\, k \geq 0 \qquad k \in \mathbb{N}$
- $\psi \in L^\infty(\mathbb{R}^d)$ s.t. $\operatorname{supp} \psi \subset B_1(0)$

$$\int_{\mathbb{R}^d} y_i^\alpha\, \psi(x-y)\, dy = x_i^\alpha \quad i = 1,..,d, \quad 0 \leq \alpha \leq k, \quad \forall x \in \mathbb{R}^d$$

We define a family δ^ε of functions based on (A) s.t. $\delta^\varepsilon := \frac{1}{\varepsilon^d}\,\psi\left(\frac{x}{\varepsilon}\right)$

$$\lim_{\varepsilon\to 0}\delta^\varepsilon(x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^d}\left(\psi\left(\frac{x}{\varepsilon}\right)\right) = \delta(x)$$

Simplest example: $\psi := \frac{1}{|B_1|}\chi_{B_1}$



We define a Regularization in $\mathbb{R}^d$ of v.

$$v^\varepsilon := v * \delta^\varepsilon$$

Lemma 1 Growth control. δ^ε (constructed from (A)) Satisfies

$$\| |x|^m \delta^\varepsilon \|_{L^1(\mathbb{R}^d)} \le c\, \varepsilon^m$$

$$\to \int_{B_\varepsilon} \left| |x|^m \frac{1}{\varepsilon^d}\psi\left(\frac{x}{\varepsilon}\right)\right| dx \qquad \to x = \varepsilon\xi$$

$$\int_{B_1} \left| |\varepsilon\xi|^m \psi(\xi)\right| d\xi \le \underbrace{\|\psi\|_{L^\infty} \||\xi|^m\|_{L^1(B_1)}}_{c}\, \varepsilon^m$$

Goal: estimate $\|v - v_\varepsilon\|_{m,\mathbb{R}^d}$?

We will use Young's inequality for convolutions:

Given $f, g \in L^2(\mathbb{R}^d)$, $h \in L^1(\mathbb{R}^d)$ then

$$|(f, g*h)| \le \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)} \|h\|_{L^1(\mathbb{R}^d)}$$

$$\left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\, g(y)\, h(x-y)\, dx\, dy \right|$$

Theorem ①

$$\|v - v^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq c\, \varepsilon^s |v|_{H^s(\mathbb{R}^d)} \qquad 0 \leq s \leq k+1$$

Given any point x, we can expand $v \in C_c^\infty(\mathbb{R}^d)$ as $\in \mathbb{P}^{s-1}$

$$v(y) = P_x(y) + R_x(y)$$

$$= \sum_{|\alpha| \leq s-1} \frac{D^\alpha v(x)}{\alpha!} (y-x)^\alpha + \sum_{|\beta|=s} R_\beta(y)(y-x)^\beta$$

$$R_\beta(y) := \frac{|\beta|}{\beta!} \int_0^1 (1-t)^{|\beta|-1} D^\beta v(x + t(y-x))\, dt$$

$$v^\varepsilon(y) = p_x^\varepsilon(y) + R_x^\varepsilon(y) = P_x(y) + R_x^\varepsilon(y) \qquad 0 \leq s \leq k+1$$

$$(v - v^\varepsilon, \theta) = \int_{\mathbb{R}^d} \Big(\cancel{P_x(x)} + R_x(x) - \cancel{p_x^\varepsilon} - R_x^\varepsilon(x) \Big)\theta(x)\, dx \qquad R_x(x) = 0$$

$p_x^\varepsilon = P_x$

$$= \int_{\mathbb{R}^d} - R_x^\varepsilon(x)\, \theta(x)\, dx = \int_{\mathbb{R}^d} - \underbrace{\int_{\mathbb{R}^d} \delta^\varepsilon(x-y)\, R_x(y)\, dy}_{R_x^\varepsilon(x)}\, \theta(x)\, dx$$

$$-\int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{|\beta|=s} \Big(\overbrace{\frac{|\beta|}{s!} (1-t)^{|\beta|-1} D^\beta v(x + t(y-x))}^{R_\beta(y)} \Big) (y-x)^\beta\, \delta^\varepsilon(x-y)\, \theta(x)\, dx\, dy\, dt$$

$\rightarrow \quad \xi = x + t(y-x) \qquad$ for fixed $x \in \mathbb{R}^d$

$$y - x = \frac{\xi - x}{t} \quad \rightarrow \quad -\int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{|\beta|=s} \Big((1-t)^{|\beta|-1} \frac{|\beta|}{\beta!}$$

$$D^\beta v(\xi) \left(\frac{\xi - x}{t}\right)^\beta \delta^\varepsilon\!\left(-\frac{\xi - x}{t}\right) \theta(x)\, d\xi\, dx\, dt$$

$$(v-v^\varepsilon,\theta)_{\mathbb{R}^d} \leq |v|_{H^s(\mathbb{R}^d)} \|\theta\|_{L^2(\mathbb{R}^d)} \int_0^1 \frac{(1-t)^{s-1}}{t^d (s-1)!} \underline{\left\| \left|\frac{x}{t}\right|^s \delta^\varepsilon\left(-\frac{x}{t}\right) \right\|_{L^1(\mathbb{R}^d)}} dt$$

$$\leq c\, \varepsilon^s |v|_{H^s(\mathbb{R}^d)} \|\theta\|_{L^2(\mathbb{R}^d)}$$

$$\sup_{\theta\in L^2} \frac{|(v-v^\varepsilon,\theta)|}{\|\theta\|_{L^2}} = \|v-v^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq c\, \varepsilon^s |v|_{H^s(\mathbb{R}^d)}$$

We can extend this for $F \in H^{-s}$ by defining.

$$\underline{\langle F^\varepsilon, v\rangle := \langle F, v^\varepsilon\rangle} \qquad \forall v \in H^s(\Omega)$$

$$\|F-F^\varepsilon\|_{H^{-k-1}(\mathbb{R}^d)} \leq \sup_{w\in H^{k+1}(\mathbb{R}^d)} \frac{|\langle F-F^\varepsilon, w\rangle|}{\|w\|_{H^{k+1}(\mathbb{R}^d)}} =$$

$$\sup_{w\in H^{k+1}(\mathbb{R}^d)} \frac{|\langle F, w-w^\varepsilon\rangle|}{\|w\|_{H^{k+1}(\mathbb{R}^d)}} \leq \frac{\|F\|_{H^m(\mathbb{R}^d)} \|w-w^\varepsilon\|_{H^{-m}(\mathbb{R}^d)}}{\|w\|_{H^{k+1}(\mathbb{R}^d)}}$$

$$* \quad \|w-w^\varepsilon\|_{H^{-m}(\mathbb{R}^d)} \leq \varepsilon^{k+1+m} \|w\|_{H^{k+1}(\mathbb{R}^d)}$$

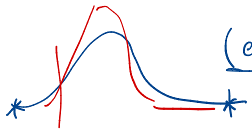
$$\Longrightarrow \|F-F^\varepsilon\|_{-k-1,\mathbb{R}^d} \leq \varepsilon^{k+1+m} \|F\|_{H^m(\mathbb{R}^d)} \qquad \underline{-k-1 \leq m \leq 0}$$

* Theorem 2

$$\|v-v^\varepsilon\|_{H^s(\mathbb{R}^d)} \leq c\, \varepsilon^{m-s} \|v\|_{H^m(\mathbb{R}^d)}$$

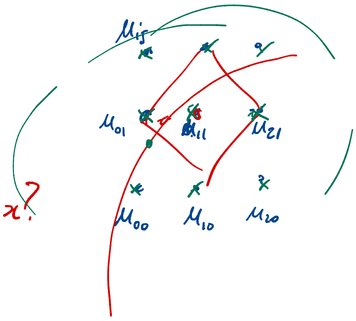
Observing that $\underline{(D^\alpha v)^\varepsilon = D^\alpha(v^\varepsilon)}$ trivial in $\mathbb{R}^d$

Usually we have φ either C^∞ with F.D.



$$\frac{(\cos(\pi x)+1)}{2}\cdot\chi_{[-1,1]}$$

$$F(x) = \int_\Gamma \delta(x-y)\, f(y)\, d\Gamma$$



$$-\Delta = \left(4u_{ii} - u_{i(j-1)} - u_{(i-1)j} - u\right)\frac{1}{h^2}$$

$-u'' = F$ where do you put x?

if you "replace" δ with δ^ε

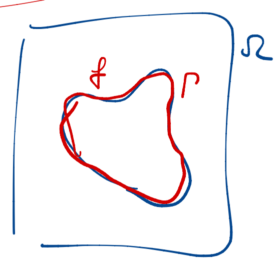
$$\rightarrow F^\varepsilon = \delta^\varepsilon * F$$

$$= \int_\Gamma \delta^\varepsilon(x-y) f(y)$$

$$F^\varepsilon(x_{ij}) := \sum \delta^\varepsilon(x_{ij}-y_q)\, f(y_q)\, w_q$$

Take as an example $F(x) = \int_\Gamma f(y)\,\delta(x-y)\,d\Gamma_y$

$$\langle \nabla u^\varepsilon, \nabla v\rangle = \langle F^\varepsilon, v\rangle \quad \forall v \in H_0^1(\Omega)$$



$$\langle \nabla u, \nabla v\rangle = \langle F, v\rangle \quad \forall v \in H_0^1(\Omega)$$

$$\langle \nabla u^\varepsilon - \nabla u, \nabla v\rangle = \langle F^\varepsilon - F, v\rangle \qquad F \in H^{-\frac{1}{2}-}$$

$$\|u^\varepsilon - u\|_{1,\Omega} \leq \|F^\varepsilon - F\|_{-1,\Omega} \leq \varepsilon^{\frac{1}{2}-}\|F\|_{-\frac{1}{2}-,\Omega} \leq \underline{\underline{\varepsilon^{\frac{1}{2}-}}}\,\|f\|_{0,\Gamma}$$

$$1 \leq s+2 \leq 2$$

$$\langle u_q, v_q\rangle = \langle F, v_q\rangle \quad \forall v_q \in V_q \subset V$$

$$2 \text{ reg} \rightarrow |u|_{s+2,\Omega} \leq c\,|f|_{s,\Omega}$$

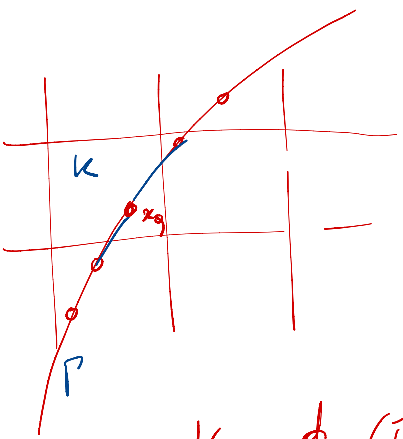
$$\Rightarrow \|u - u_h\|_{1,\Omega} \leq c\, h^{\frac{1}{2}-}\,|u|_{\frac{3}{2}-,\Omega} \leq c\,\underline{\underline{h^{\frac{1}{2}-}}}\,\|f\|_{0,\Gamma}$$

IF we fix $\varepsilon = c\,h \Rightarrow$

$$\| u - u_h^\varepsilon \|_{1,\Omega} \leq c\, h^{\frac{1}{2}-} \| f \|_{0,\Gamma} \leq c\, \varepsilon^{\frac{1}{2}-} \| f \|_{0,\Gamma}$$

$$\langle \nabla u_h^\varepsilon, \nabla v_h \rangle = \langle F^\varepsilon, v_h \rangle \quad \forall v_h \in V_h$$

$$\| u - u^\varepsilon \| + \| u^\varepsilon - u_h^\varepsilon \| \leq c\, \varepsilon^{\frac{1}{2}-} \| f \|_{0,\Gamma} + h^{\frac{1}{2}-} \| f \|_{0,\Gamma}$$



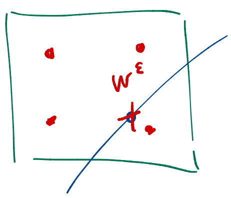
$$\langle F, v \rangle_{V' \times V} = \langle f, v \rangle_\Gamma$$

$$= \sum_q f(x_q) \cdot v(x_q)\, w_q$$

$$= \sum_q f(x_q)\, \hat{v}(\phi_K^{-1}(x_q))\, w_q$$

$$K = \phi_K(\hat{K}) \qquad \text{when} \quad x_q \in K$$

$$\langle F^\varepsilon, v \rangle = \int_\Omega v(y) \int_\Gamma \delta^\varepsilon(y - x) f(x)\, d\Gamma_x \, dy$$



$$= \sum_\beta v(x_\beta) w_\beta \sum_q \delta^\varepsilon(y_\beta - x_q) f(x_q)\, w_q$$

$$v^\varepsilon(x_q) = \sum_\beta v(y_\beta)\, w_\beta\, \delta^\varepsilon(y_\beta - x_q) = \sum_\beta \hat{v}(\hat{y}_\beta)\, \delta^\varepsilon(y_\beta - x_q)$$