A priori Estimate in L^2 norm

Assume shape regularity!

We saw from Bramble Hilbert that

$$\|u - \Pi_h u\|_{m,\Omega} \leq C h^{\ell+1-m} |u|_{\ell+1,\Omega} \qquad \text{for } 0 \leq m \leq \ell$$

However, we cannot say that

$$\|u - u_h\|_{0,\Omega} \leq C h^{\ell+1} |u|_{\ell+1,\Omega}$$ FALSE IN GENERAL

Abstract Setting

Let H Hilbert Space s.t. $V \to H$ continuously and densely

$(\|u\|_H \leq \|u\|_V \quad \forall u \in V) \qquad (H = \overline{V}^{\|\cdot\|_H})$

If we identify H with its dual H^*, we may interpret H as a subspace of V.

Then, if $f \in H$

$$|\langle \tilde{f}, v\rangle| := |(f,v)_H| \leq \|f\|_H \|v\|_H \leq \|f\|_H \|v\|_V \qquad \text{i.e. } \tilde{f} \in V^*$$

From $\overline{V}^{\|\cdot\|_H} \equiv H$ it follows that $\tilde{f}$ is injective $\langle \tilde{f}, v\rangle = 0 \Rightarrow f = 0$

Consider

- a elliptic
- a bounded

$$a(u,v) = \langle f, v\rangle \qquad v \in V$$

and

$$a(u_h, v_h) = \langle f, v_h\rangle \qquad v_h \in V_h \subset V$$

Then (1) the Adjoint variational problem

$$a(v, \varphi_g) = (g, v)_H \qquad v \in V$$

Admits a unique solution and

(2) $$\|u - u_h\|_H \leq C \|u - u_h\|_V \left(\sup_{g \in H} \frac{1}{\|g\|_H} \inf_{\varphi_h \in V_h} \|\varphi_g - \varphi_h\|_V \right)$$

1 Since $g \in H$, $\tilde{g} \in V^*$, Lax Milgram: $\exists!\ \varphi_g$

choose $v = u - u_h$ in 1:

$$a(u-u_h, \varphi_g) = (g, u-u_h)_H$$

and $a(u-u_h, \varphi_h) = 0$

$$a(u-u_h, \varphi_g - \varphi_h) = (g, u-u_h)_H$$

$$\Rightarrow \left|(g, u-u_h)_H\right| \leq C\|u-u_h\|_V \inf_{\varphi_h \in V_h} \|\varphi_g - \varphi_h\|_V$$

Definition of H^* norm of $\|u-u_h\|_H$

$$\|u-u_H\|_H := \sup_{g \in H} \frac{|(u-u_H, g)_H|}{\|g\|_H}$$

$$\Rightarrow \|u-u_H\|_H \leq C\|u-u_H\|_V \sup_{g\in H} \inf_{\varphi_h \in V_h} \frac{\|\varphi_g - \varphi_h\|_V}{\|g\|_H}$$

A priori Estimate in L^2 $\quad (V \equiv H_0^1(\Omega),\ H := L^2(\Omega))$

If 1 is 2-regular,

$$\inf_{\varphi_h \in V_h} \|\varphi_g - \varphi_h\|_1 \leq \underbrace{\|\varphi_g - \Pi\varphi_g\|_{1,\Omega} \leq C h\, |\varphi_g|_{2,\Omega}}_{BH} \underbrace{\leq C h\, \|g\|_{0,\Omega}}_{2\text{-regularity}}$$

$$\sup_{g\in L^2}\left[\inf_{\varphi_h \in V_h} \frac{\|\varphi_g - \varphi_h\|_1}{\|g\|_0}\right] \leq Ch\|g\|_{L^2}$$

$$\Rightarrow \|u-u_h\|_{0,\Omega} \leq C h^{\ell} |u|_{\ell+1,\Omega}$$

$$\leq \tilde{C} h^{\ell+1} |u|_{\ell+1,\Omega}$$

Inverse Estimates (General Case)

In the continuous case, we can say that $H^s(K) \to H^m(K)$ when $m \geq s$, that is:

$$\|u\|_{s,K} \leq \|u\|_{m,K}$$

Moreover, for seminorms and $u \in \mathbb{P}^e(K)$, when $0 \leq s,m \leq e$, we have

$$|u|_{s,K} \leq C h^{m-s} |u|_{m,K} \qquad u \in \mathbb{P}^e(K)$$

Assume that $\underline{H^{e+1}(\hat{K}) \to C^s(\hat{K})}$, if $P_K \neq \mathbb{P}^e(K)$, but we have that $\mathbb{P}^e(K) \subset P_K(K)$, we can still show that

$$\|u\|_{s,K} \leq C h^{e+1-s} \|u\|_{e+1,K} \qquad 0 \leq e+1 \leq s$$

We exploit the Denis-Lions norm $\left(N = \dim(P_K) \geq \dim(\mathbb{P}^e(K))\right)$

$$|||u|||_{e+1,K} := |u|_{e+1,K} + \sum_{i=0}^{N} |\langle \tilde{\phi}^i, u \rangle| \qquad \phi^i : \text{basis for } P_K^*$$

And

$$\|u\|_{e+1,K} \leq C\, |||u|||_{e+1,K}$$

Start with seminorm on $\hat{K}$

$$|\hat{v}|_{s,\hat{K}} \leq |\hat{v} - \Pi_{\hat{K}}\hat{v}|_{s,\hat{K}} + \cancel{|\Pi_{\hat{K}}\hat{v}|_{s,\hat{K}}} \to 0, \quad s \geq e+1$$

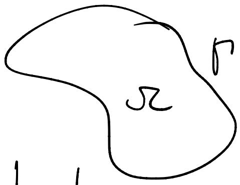
$$\leq \|\hat{v} - \Pi_{\hat{K}}\hat{v}\|_{s,\hat{K}} \leq \|\hat{v} - \Pi_{\hat{K}}\hat{v}\|_{e+1,\hat{K}}$$

finite dimension, all norms are equivalent.

$$\leq C|||\hat{v} - \Pi_{\hat{K}}\hat{v}|||_{e+1,\hat{K}} \leq C|\hat{v}|_{e+1,\hat{K}}$$

$$\leadsto |\langle \tilde{\phi}^i, \hat{v} - \Pi_{\hat{K}}\hat{v} \rangle| = 0$$

Trace Spaces



For Ω Lip., with boundary Γ,
if $u \in C^0(\bar{\Omega})$ it makes sense to talk about
$u\big|_\Gamma$, simply by the pointwise restriction —
However, if $u \in H^m(\Omega)$, it does not make sense pointwise, unless
$H^m(\Omega) \to C^0(\Omega)$ (Imbedding Theorems).

Trace Theorem

Ω Lipschitz, $s \in (\frac{1}{2}, 1]$

1) $\exists!$ linear bounded mapping

$$\boxed{\begin{aligned} \gamma_0 : H^s(\Omega) &\longrightarrow H^{s-\frac{1}{2}}(\Gamma) \\ v &\longrightarrow \gamma_0(v) = v\big|_\Gamma \end{aligned}}$$

such that

$$\|\gamma_0 v\|_{s-\frac{1}{2},\Gamma} \leq C \|v\|_{s,\Omega}$$

2) γ_0 has a bounded right inverse E_0

$$E_0 : H^{s-\frac{1}{2}}(\Gamma) \longrightarrow H^s(\Omega)$$

$$\gamma_0 E_0 v = v \qquad \forall v \in H^{s-\frac{1}{2}}(\Gamma)$$

and

$$\|E_0 v\|_{s,\Omega} \leq C \|v\|_{s-\frac{1}{2},\Gamma}$$

What are the spaces $H^m(\Omega)$ when $0 < m < 1$?

Fractional Spaces

Three possible definitions:

1) $\Omega \equiv \mathbb{R}^d$ Fourier Transform

$$\mathcal{F}v := \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left(-i\xi\cdot x\right) v(x)\,dx$$

Define $\quad \|v\|_{s,\mathbb{R}^d} := \left\|\left(1+|\cdot|^2\right)^{\frac{s}{2}} \hat{v}(\cdot)\right\|_{0,\mathbb{R}^d}$

and set $\quad H^s(\mathbb{R}^d) := \overline{C_c^\infty}^{\|\cdot\|_{s,\mathbb{R}^d}}$

2) Ω Lip. $\quad s = m + \lambda \quad , m \in \mathbb{N}_0 \ , \lambda \in (0,1)$

$$\|u\|_{s,\Omega} := \left(\|u\|^2_{m,\Omega} + |u|^2_{\lambda,\Omega}\right)^{\frac{1}{2}}$$

$$|u|_{\lambda,\Omega} := \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{\left|D^\alpha u(x) - D^\alpha u(y)\right|^2}{|x-y|^{d+2\lambda}}\,dx\,dy$$

and again set $\quad H^s(\Omega) := \overline{C_c^\infty}^{\|\cdot\|_{s,\Omega}}$

3) with the usual definition, setting

$$D^\lambda u = \frac{1}{\Gamma(1-\lambda)} D^\lambda \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_d} \frac{f(t)}{(x-t)^\lambda}\,dt \qquad \lambda \in (0,1)$$

($\Gamma(1-\lambda)$ here is the "Gamma function": $\Gamma(n+1) = n! \quad \forall n \in \mathbb{N}_0$)

Traces of finite element functions

Let $u \in \mathbb{P}^{\ell}(K) \subset H^{m}(K)$, $m > \frac{1}{2}$

$$\|u\|_{m-\frac{1}{2},\partial K} \leq C_T \|u\|_{m,K} \leq C_T h^{-m+s} \|u\|_{s,K}$$

$$\|\nabla u \cdot n\|_{0,\partial K} \leq \|\nabla u\|_{0,\partial\Omega} \leq \|\nabla u\|_{\frac{1}{2},K} \leq h^{-\frac{1}{2}} \|\nabla u\|_{0,K} = h^{-\frac{1}{2}} |u|_{1,K}$$

Combination of trace theorem + inverse estimates.

A posteriori Error Estimates (Residual type)

Once a solution has been computed (u_h), if we could estimate the local error function $e_h := u - u_h$ in terms of Ω, u_h, and f only, we could refine the grid in a way that ensures that the error is well distributed.

Given
$$a(u,v) = \langle f, v\rangle \qquad a(u,v) := \int_\Omega \nabla u \cdot \nabla v$$
$$a(u_h, v_h) = \langle f, v_h\rangle$$

$$\boxed{a(e_h, v) = \langle r, v\rangle \qquad r := \langle f, v\rangle - a(u_h, v)} \quad (1)$$

We would like to estimate e_h from u_h, Ω, f by constructing on each cell an estimator

$$\eta^2 := \sum_{K \in \mathcal{T}_h} \eta_K^2$$

such that

$$\underbrace{c\,\eta \leq |e_h|_{1,\Omega}}_{\text{efficiency}} \overbrace{\leq C\eta}^{\text{reliability}}$$

The mesh is then refined according to some criterion: e.g.:

mark K to be refined if $\eta_K \geqslant \sigma \bar{\eta}_K$ (average)

Residual based estimator:

set $v = e$ in (1)

$$\alpha \|e\|_1^2 \leqslant a(e,e) = \langle r, e \rangle$$

$\tilde{K}$:= set of elements that share at least a vertex with K

$$\tilde{K} := \{ K_J \mid K_J \cap K \neq \emptyset \}$$

$$\langle r, v \rangle := \int_\Omega f v - a(u_h, v) = \int_\Omega f v - \int_\Omega \nabla u_h \nabla v$$

$$= \sum_K \left[\int_K f v + \int_K \Delta u_h v - \int_{\partial K} n \cdot \nabla u_h v \right]$$

$$= \sum_K \left[\int_K (f + \Delta u_h) v - \frac{1}{2} \int_{\partial K} [\![\nabla u_h]\!] v \right]$$

$$[\![\nabla u_h]\!]\Big|_e := \begin{cases} n^+ \nabla u_h^+ + n^- \nabla u_h^- & \text{if } e \in \mathring{\Omega} \\ 2 n^+ \nabla u_h^+ & \text{if } e \in \Gamma \end{cases}$$

$$a(e, v) = a(e, v - \Pi v) = \langle r, v - \Pi v \rangle =$$

$$= \sum_K \left[\int_K (f + \Delta u_h)(v - \Pi v) - \frac{1}{2} \int_{\partial K} [\![\nabla u_h]\!](v - \Pi v) \right]$$

$$\leqslant \sum_K \left[\|f + \Delta u_h\|_{0,K} \|v - \Pi v\|_{0,K} + \frac{1}{2} \|[\![\nabla u_h]\!]\|_{0,\partial K} \|v - \Pi v\|_{0,\partial K} \right]$$

$$\|v - \Pi v\|_{0,K} \leqslant C h_K |v|_{1,\tilde{K}}$$

$$\|v - \Pi v\|_{0,\partial\Omega} \leqslant C_T \|v - \Pi v\|_{\frac{1}{2},K} \leqslant C_T h_K^{\frac{1}{2}} |v|_{1,\tilde{K}}$$

$$a(e, v - \Pi v) \leq \sum_K \left[C_1 h_K \|r_K\|_{0,K} + C_2 h_K^{\frac{1}{2}} \| [\![\nabla u_h]\!] \|_{0,\partial K} \right] |v|_{1,\tilde{K}}$$

$$\left| \sum a_i b_i \right| \leq c \left(\sum a_i^2 \right)^{\frac{1}{2}} \left(\sum b_i^2 \right)^{\frac{1}{2}}$$

$$a(e, v) \leq c \left[\sum_K \left[h_K^2 \|r_u\|_{0,K}^2 + h_K \| [\![\nabla u]\!] \|_{0,\partial K}^2 \right] \right]^{\frac{1}{2}} |v|_{1,\Omega}$$

$$\Rightarrow \quad |e|_1 \leq C \, \eta$$

$$\eta^2 = \sum_K \eta_K^2 \qquad \boxed{\eta_K^2 := h_K^2 \|r_u\|_{0,K}^2 + h_K \| [\![\nabla u]\!] \|_{0,\partial K}^2}$$

Efficiency is more complicated.