Recap 1st Strang lemma

Replace A by A_h, f by f_h, ask A_h is uniformly V_h-elliptic

$\exists\, \tilde\alpha$ s.t. $\langle A_h u, u\rangle \geq \alpha_h \|u\|_V^2 \geq \tilde\alpha \|u\|_V^2$

$$\|u - u_h\| \leq \frac{C}{\tilde\alpha}\left[\inf_{v_h\in V_h}\left(\|v_h - u\| + \|A_h v_h - A v_h\|_{*,h}\right) + \|f - f_h\|_{*,h}\right]$$

if $A_h : V \longrightarrow V'$ $\inf_{v_h\in V_h} \|v_h - u\| + \|A_h u - f_h\|_{*,h}$

II Strang lemma

Apply I Strang lemma for A_h defined on $V_h \not\subset V$

Define $\underline{\underline{\tilde V}} = \underset{\sim}{V_h} + \underline{V}$ think of V_h as rich DG space

i) Define a norm on $\tilde V$: $|||\cdot|||$ such that

ii) $\exists\, c_1, c_2$ s.t. $c_1\|u\|_V \leq |||u||| \leq c_2\|u\|_V$ $\forall u \in V$

keep in mind that $|||\cdot|||$ maybe mesh dependent.

Now we define $A_h : \tilde V \longrightarrow \tilde V'$, assume the hypot. of 1st. Strang lemma are satisfied, and apply it:

$\exists\, C$ s.t. $\langle A_h u, v\rangle \leq C\, |||u|||\ |||v|||$ $\forall u, v \in \tilde V$

$\exists\, \alpha$ s.t. $\langle A_h u, u\rangle \geq \alpha\, |||u|||^2$ $\forall u \in \tilde V$

$$\Rightarrow |||u - u_h||| \leq \frac{C}{\alpha}\left[\inf_{v_h\in V_h} \underbrace{|||v_h - u|||}_{\text{approx error}} + \underbrace{|||A_h u - f_h|||_{*,h}}_{\text{consistency error.}}\right]$$

Discontinuous Galerkin Methods

$V_h := \{ v \in L^2(\Omega) \mid v|_K \in \mathbb{P}^\ell(K) \}$ $\quad V_h \notin H^1(\Omega)$

$\ast \quad \mathcal{T} := \bigcup_K \mathring{K} \quad \left(\mathcal{T}_h(\Omega) \setminus \bigcup_K \partial K \right)$ interior of all elements.

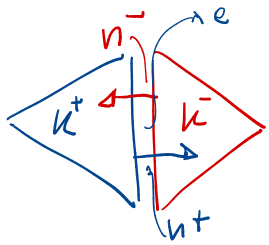
$\ast \quad \mathcal{E}^\circ := \bigcup_{\substack{K_i \in \mathcal{T}_h \\ K_j \in \mathcal{T}_h}} \overline{K_i} \cap \overline{K_j} =: \underline{\underline{e_{ij}}}$ Skeleton of all interior faces

$\ast \quad \mathcal{E}^{\partial\Omega} := \bigcup_{K_j} \partial K_j \cap \Gamma = \partial\Omega$ Boundary faces

$\ast \quad \mathcal{E} := \mathcal{E}^\circ \cup \mathcal{E}^{\partial\Omega}$ All faces

$\ast \quad (u,v)_{\mathcal{T}} := \sum_K (u,v)_K$ "broken inner product"

$\ast \quad \langle u,v \rangle_{\mathcal{E}} := \int_{\mathcal{E}} u\,v = \sum_{e \in \mathcal{E}} \int_e u\,v$ "broken inner product on faces"



n^+ : outer normal to K^+

n^- : outer normal to K^-

$\ast \quad [\![u]\!] := \begin{cases} u^+ n^+ + u^- n^- & x \in \mathcal{E}^\circ \\ u \cdot n & x \in \mathcal{E}^{\partial\Omega}_\circ \end{cases}$

scalar when u is vector, vector when u is a scalar

$\ast \quad \{u\} := \begin{cases} \frac{1}{2}u^+ + \frac{1}{2}u^- & x \in \mathcal{E} \\ u & x \in \mathcal{E}^{\partial\Omega} \end{cases}$

$$a^+ n^+ \underline{b^+} + a^- n^- b^- = [\![a]\!]\{b\} + \{a\}[\![b]\!]$$

Model Problem

| $f \in L^2(\Omega)$, Ω is convex and polygonal

$$1)\ \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega = \Gamma \end{cases} \qquad \Rightarrow u \in H^2(\Omega) \cap H^1_0(\Omega)$$

Split on each K, and study the following: ($u \in H^2(\Omega) \cap H^1_0(\Omega)$ *)

$$\text{i)}\quad -\Delta u - f = 0 \quad \text{on every } K$$

$$\text{ii)}\quad [\![u]\!]\big|_e = 0 \quad \text{on } \mathcal{E} \qquad (2)$$

$$\text{iii)}*\quad [\![\nabla u]\!]\big|_e = 0 \quad \text{on } \mathcal{E}^0$$

A sol. to (1) is also a sol to i–iii)

Space $H^2_{\mathcal{E}_h} := \{ v \in L^2(\Omega) \mid v|_K \in H^2(K) \}$

Variational form of (2), as general as possible
(Brezzi, Cockburn, Marini, Süli, 2004)

B_0, $\underline{B}_1$, B_2 s.t.

$$\left(-\Delta u - f, B_0 v\right)_{\mathcal{Z}} + \langle [\![u]\!], \underline{B}_1 v \rangle + \langle [\![\nabla u]\!], B_2 v \rangle = 0$$

$$B_0 v = v \qquad \sum_K -\int_{\partial K} \nabla u \cdot n\, v = -\langle [\nabla u], \{v\} \rangle_{\mathcal{E}}$$

$$\underbrace{(\nabla u, \nabla v)_{\mathcal{Z}} - (f, v)_{\mathcal{Z}} - \langle [\![\nabla u]\!], \{v\} \rangle}_{(-\Delta u - f, B_0 v)} + \langle [\![u]\!], \underline{B}_1 v \rangle + \langle [\![\nabla u]\!], B_2 v \rangle = 0$$

Set $B_2 v = \{v\}$

$$\Rightarrow (\nabla u, \nabla v)_{\mathcal{Z}} - (f, v)_{\mathcal{Z}} + \langle [\![u]\!], B_1 v \rangle = 0$$

$\alpha = 1$

Method	$B_0 v$	$\mathbf{B}_1 v$	$B_2 v$
classical C^1-conforming	v	$([\![u]\!] \equiv 0)$	$([\![\boldsymbol{\sigma}(u)]\!] \equiv 0)$
classical C^0-conforming	v	$([\![u]\!] \equiv 0)$	v
IP [20]	v	$([\![u]\!] \equiv 0)$	$v + s_2 [\![\alpha \nabla v]\!]$
B.O. [7]	v	$\{\alpha \nabla v\}$	$\{v\}$
NIPG [26]	v	$\{\alpha \nabla v\} + s_1 [\![v]\!]$	$\{v\}$
IP [4, 28, 1]	v	$-\{\alpha \nabla v\} + s_1 [\![v]\!]$	$\{v\}$
D.S.W. [19]	v	$s_1 [\![v]\!]$	$\{v\}$

SIPG

(3) $(\nabla u, \nabla v)_{\mathcal{T}} - \langle \{\nabla u\}, [\![v]\!] \rangle_{\mathcal{E}} - \langle \{\nabla v\}, [\![u]\!] \rangle_{\mathcal{E}} + \langle s_1 [\![u]\!], [\![v]\!] \rangle_{\mathcal{E}} = (f, v)_{\mathcal{T}}$

Assume a norm: $|||u|||^2 := |u|^2_{1,\Omega} + \sum_e \frac{1}{|e|} \| [\![u]\!] \|^2_{0,e}$ on $\tilde{V} = V_h + V$

$\mathbb{P}^e(\mathcal{T})$ $H^1_0(\Omega)$

Take $u \in H^1_0(\Omega)$, $\langle A_h u, u \rangle = (\nabla u, \nabla u) = |u|^2_{1,\Omega}$

for $u \in V \equiv H^1_0(\Omega)$ A_h is cont and coercive (in V)

for $u \in V_h$ $|||\cdot|||$ is a norm that is equiv. to $|u|_{1,\Omega}$ in V.

and it is indeed a norm on V_h $\Rightarrow$ $|||u||| = 0 \Leftrightarrow u = 0$ in $\mathcal{T}$

$$\left(\sqrt{\varepsilon}\, a \pm \frac{b}{\sqrt{\varepsilon}}\right)^2 = \varepsilon a^2 + \frac{b^2}{\varepsilon} \pm 2ab \geq 0$$

$$\pm ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon} \qquad \forall\, \varepsilon > 0$$

$u, v \in H^2(\overline{K})$, $u, v \in \mathbb{P}^e(K)$

$$\left| \int_e \nabla u \cdot n\, v \right| \leq \|\nabla u\|_{0,e} \|v\|_{0,e} \leq |u|_{1,e} \|v\|_{0,e} \leq C_I h_e^{-\frac{1}{2}} |u|_{\frac{1}{2},e} \|v\|$$

$$\leq C_T C_I h_e^{-\frac{1}{2}} |u|_{1,K} \|v\|_{0,e}$$

$$\langle A_h u, v \rangle \leq |u|_{1,\mathcal{T}} |v|_{1,\mathcal{T}} + C |u|_{1,\mathcal{T}} \left(\sum_{e \in \mathcal{E}} \frac{1}{h_e} \| [\![v]\!] \|_{0,e} \right) + C |v|_{1,\mathcal{T}} \left(\sum_{e \in \mathcal{E}} \frac{1}{h_e} \| [\![u]\!] \|_{0,e} \right)$$

$$+\sum_e s_1 \|[u]\|_{0,e} \|[v]\|_{0,e} \qquad \frac{c}{h_e} \dots$$

$\exists M$ s.t. $\langle A_h u, v\rangle \le M |||u||| \, |||v|||$ if $s_1 = \gamma \frac{1}{h_e^{1/2}}$

Ellipticity: $\exists \alpha$ s.t.

$$\langle A_h u, u\rangle \ge \alpha |||u|||^2$$

$$-ab \ge -\frac{1}{2\varepsilon} a^2 - \frac{\varepsilon}{2} b^2$$

$$\langle A_h u, u\rangle = |u|_{1,\Omega}^2 - \sum_e \left[2\int_e \{\nabla u\}[u] + s_1 \int_e [u][u] \right]$$

$$\ge |u|_{1,\Omega}^2 - \varepsilon C_T C_I h^{-\frac{1}{2}} |u|_{1,2}^2 - \sum_e \frac{1}{\varepsilon} \|[u]\|_{0,e}^2 + s_1 [u]_{0,e}^2$$

$$\alpha = \frac{1}{2} \Rightarrow -\varepsilon C_T C_I h^{-\frac{1}{2}} = -\frac{1}{2} \Rightarrow \varepsilon = \frac{h^{\frac{1}{2}}}{2 C_T C_I}$$

$$\frac{s_1}{2} = s_1 - \frac{2 C_T C_I}{h^{\frac{1}{2}}} \Rightarrow s_1 = \frac{4 C_T C_I}{h^{\frac{1}{2}}}$$

$$|||u - u_h||| \le \inf_{v_h \in V_h} \left[|||v - v_h||| + \cancel{|||A_h u - f_h|||_{*,h}} \right]$$

Theorem:

$$|||v - I_h v||| \le C h^{\ell} |v|_{\ell+1,\Omega} \qquad v \in H^{\ell+1} \qquad \ell \ge 1$$

$$|||v - I_h v|||_h^2 = |v - I_h v|_{1,h}^2 + \sum_{e \in \partial h} \frac{1}{h_e} \|[\cancel{v} - I_h v]\|_{0,e}^2$$