A priori estimates

Find $u \in \underline{V}$ s.t.

$$\langle Au, v\rangle = \langle f, v\rangle \quad \forall v \in \underline{V} \qquad \text{\& also true in } V_h$$

Find $u_h \in \underline{V_h \subset V}$ s.t.

$$\langle Au_h, v_h\rangle = \langle f, v_h\rangle \quad \forall v_h \in V_h \subset V$$

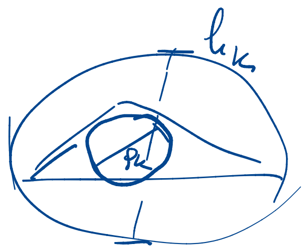
$$\langle A(u-u_h), v_h\rangle = 0$$

$$\alpha \|u-u_h\|_V^2 \le \langle A(u-u_h), u-\underset{v_h}{\underline{\underline{u_h}}}\rangle$$

$$\le \|A\|_* \|u-u_h\| \, \|u-v_h\| \qquad \forall v_h \in V_h$$

$$\|u-u_h\|_V \le \frac{\|A\|_*}{\alpha} \inf_{v_h \in V_h} \|u-v_h\| \quad \ldots$$

$$\|u-\Pi_K u\|_{m,K} \le C \rho_K^{-m} h_K^{l+1} |u|_{l+1,K}$$



Shape regular triangulation

$$\exists \sigma \in (0,1) \mid \rho_K > \sigma h_K \qquad \forall K \in \mathcal{T}_h(\Omega)$$

call $h = \max_{K \in \mathcal{T}_h(\Omega)} \{h_K\}$ $\qquad 0 \le m \le l$

$$\sum_K \|u-\Pi_K u\|_{m,K} \le C_\sigma h^{l+1-m} \sum_K |u|_{l+1,K}$$

For example:

$$A: \text{"}-\Delta\text{"} \quad \rightsquigarrow \quad \langle Au, v\rangle = \int_\Omega \nabla u \nabla v$$

$$V = H_0^1(\Omega) \equiv \overline{C_c^\infty}^{\|\cdot\|_{1,\Omega}}$$

$$\|u - \Pi_h u\|_{m,\Omega} \le C_6 \, h^{\ell+1-m} \, |u|_{\ell+1,\Omega} \qquad \text{for } 0 \le m \le 1$$

$$V_h = \{ v \in C^0(\overline{\Omega}) \mid v|_K \in \mathbb{P}^\ell(K) \}$$

$$\|u - u_h\|_{1,\Omega} \le \frac{\|A\|_*}{\alpha} C_6 h^\ell \, |u|_{\ell+1,\Omega}$$

Abstract Setting for Aubin-Nitsche trick

let H be Hilbert space s.t. $V \to H$ continuously and densely embedded in V

i) $\|u\|_H \le \|u\|_V \quad \forall u \in V$

ii) $H = \overline{V}^{\|\cdot\|_H}$

We identify H with H^* so that $\tilde{f}$ extension of $f \in H$ to V'

$$\langle \tilde{f}, v\rangle := (f, v)_H \qquad \forall f \in H$$

$$|\langle \tilde{f}, v\rangle| := |(f,v)_H| \le \|f\|_H \|v\|_H \le \|f\|_H \|v\|_V \qquad \tilde{f} \in V'$$

From $\overline{V}^{\|\cdot\|_H} = H \Rightarrow \tilde{f}$ is injective $\langle \tilde{f}, v\rangle = 0$

$$\Rightarrow \underline{f = 0}$$

Consider now $f \in H$, and find $u \in V$

$$\langle Au, v\rangle = \langle \tilde{f}, v\rangle \qquad \forall v \in V$$
$$= (f, v)_H$$

Theorem 1) The adjoint pb. has a unique solution

Find φ_g s.t.

$$\langle Av, \varphi_g\rangle = \langle g, v\rangle \qquad \forall v \in V \qquad \langle A^T\varphi_g, v\rangle = \langle g, v\rangle$$

2) if $g \in H$ then

$$\|u - u_h\|_H \leq C\|u - u_h\|_V \left(\sup_{g \in H} \frac{1}{\|g\|_H} \inf_{\varphi_H \in V_h} \|\varphi_g - \varphi_H\|_V\right)$$

1) $g \in H \Rightarrow \tilde{g} \in V'$, $LM \Rightarrow \exists!\ \varphi_g$

$v = u - u_h$ in the dual pb.

$$a(u - u_h, \varphi_g) = (g, u - u_H)_H$$

$$a(u - u_h, \varphi_h) = 0 \qquad \forall \varphi_h \in V_h$$

$$a(u - u_h, \varphi_g - \varphi_h) = (g, u - u_H)_H$$

$$|(g, u - u_h)_H| \leq C\|u - u_h\|_V \inf_{\varphi_H \in V_h} \|\varphi_g - \varphi_h\|$$

Definition of norm in H':

$$\|u-u_e\|_H := \sup_{g\in H} \frac{(u-u_e, g)_H}{\|g\|_H}$$

$$\|u-u_e\|_H \leq C\|u-u_e\|_V \sup_{g\in H}\left(\frac{1}{\|g\|_H}\cdot \inf_{\varphi_H\in V_e}\|\varphi_g-\varphi_e\|\right)$$

Apply to $L^2(\Omega)$ estimates in $H^1_0(\Omega)$ $\quad (V\equiv H^1_0(\Omega),\ H=L^2(\Omega))$

If the problem is 2-regular (laplacian on convex Lip. is 2-regular)

$$\|\varphi_g\|_{2,\Omega} \leq C\|g\|_{0,\Omega}$$

$$\inf_{\varphi_e\in V_e}\|\varphi_g-\varphi_e\|_{1,\Omega} \leq \|\varphi_g - \Pi\varphi_g\|_{1,\Omega} \leq Ch\,|\varphi_g|_{2,\Omega} \leq Ch\|g\|_{0,\Omega}$$

$$\|u-u_e\|_H \leq \underbrace{C\|u-u_e\|_V}_{\leq C h^e |u|_{e+1,\Omega}} \underbrace{\sup_{g\in H}\left(\frac{1}{\|g\|_H}\inf_{\varphi_H\in V_e}\|\varphi_g-\varphi_e\|\right)}_{Ch}$$

$$\|u-u_e\|_{0,\Omega} \leq C h^{e+1}\,|u|_{e+1,\Omega}$$

Inverse Estimates (assume that $\mathcal{T}_h$ is shape regular)

In the continuous case $H^m(K) \longrightarrow H^s(K)$

when $m \geq s$: $\|u\|_{s,K} \leq \|u\|_{m,K}$

with seminorms and $u \in \mathbb{P}^\ell(K)$, when $0 \leq s, m \leq \ell$

$$|u|_{s,K} \leq c\, h^{m-s} |u|_{m,K} \qquad \forall u \in \mathbb{P}^\ell(K)$$

Assume the $H^{\ell+1}(\hat{K}) \longrightarrow C^s(\hat{K})$, $\mathbb{P}_K \neq \mathbb{P}^\ell(K)$

but we have $\mathbb{P}^\ell(K) \subset \mathbb{P}_K(K)$, we can still show that

$$\|u\|_{s,K} \leq c\, h^{\ell+1-s} \|u\|_{\ell+1,K} \qquad 0 \leq \ell+1 \leq s$$

$$\underline{\underline{|||u|||_{\ell+1,K} := |u|_{\ell+1,K} + \sum_{i=0}^{N} |\langle \hat{v}^i, u\rangle|}}$$

$\hat{v}^i$ basis for $(\mathbb{P}_K)'$

$$\|u\|_{\ell+1,K} \leq C\, |||u|||_{\ell+1,K}$$

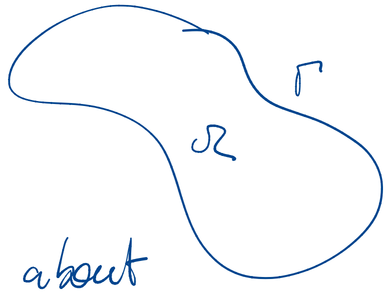
$s \geq \ell+1$

$$|\hat{v}|_{s,\hat{K}} \leq |\hat{v} - \Pi_{\hat{K}}\hat{v}|_{s,K} + \cancel{|\Pi_{\hat{K}}\hat{v}|_{s,K}} \xrightarrow{} 0 \quad s \geq \ell+1$$

$$\leq \|\hat{v} - \Pi_{\hat{K}}\hat{v}\|_{s,K} \leq \|\hat{v} - \Pi_{\hat{K}}\hat{v}\|_{\ell+1,\hat{K}} \quad \text{equivalent}$$

$$\leq c\, |||\hat{v} - \Pi_{\hat{K}}\hat{v}|||_{\ell+1,\hat{K}} \leq c\, |\hat{v}|_{\ell+1,\hat{K}}$$

Trace Spaces



For Ω Lip., with boundary Γ.

if u $C^0(\overline{\Omega})$, it make sense to talk about

$u|_\Gamma$ (point wise restriction)

However if $u \in H^m(\Omega)$, then "pointwise" doesn't make sense – (unless $H^m(\Omega) \to C^0(\overline{\Omega})$)

Trace theorem

Ω Lip. $\quad s \in (\frac{1}{2}, 1]$

1) $\exists !$ linear bounded mapping

$$\gamma_0 : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma)$$

such that

$$\|\gamma_0 v\|_{s-\frac{1}{2},\Gamma} \leq C \|v\|_{s,\Omega}$$

2) γ_0 has a bounded right inverse E_0

$$E_0 : H^{s-\frac{1}{2}}(\Gamma) \longrightarrow H^s(\Omega)$$

$$\gamma_0 E_0 v = v \qquad \forall v \in H^{s-\frac{1}{2}}(\Gamma)$$

$$\| E_0 v\|_{s,\Omega} \leq C \|v\|_{s-\frac{1}{2},\Gamma}$$

What are $H^m(\Omega)$ spaces for $0 < m < 1$?

Fractional Spaces

1) $\Omega \equiv \mathbb{R}^d$ Fourier Transform

$$F v := \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp(-i\xi \cdot x)\, v(x)\, dx$$

Define

$$\| v \|_{s,\mathbb{R}^d} := \left\| \left(1 + |\cdot|^2\right)^{\frac{s}{2}} \hat{v}(\cdot) \right\|_{0,\mathbb{R}^d}$$

$$H^s(\mathbb{R}^d) := \overline{C_c^\infty}^{\|\cdot\|_{s,\mathbb{R}^d}}$$

2) Ω Lip. $\quad s = m + \lambda \quad m \in \mathbb{N}_0, \ \lambda \in (0,1)$

$$\| u \|^2_{s,\Omega} := \| u \|^2_{m,\Omega} + | u |^2_{\lambda,\Omega}$$

$$| u |^2_{\lambda,\Omega} := \sum_{|\alpha| = m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{d-2\lambda}}\, dx\, dy$$

$$H^s_0(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{s,\Omega}}$$

3) We use def. of fract. derivative

$$D^\lambda u = \frac{1}{\Gamma(1-\lambda)} D^\lambda \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_d} \frac{f(t)}{(x-t)^\lambda}\, dt \qquad \lambda \in (0,1)$$

Γ : is Gamma Function $\left(\Gamma(n+1) = n! \quad \forall n \in \mathbb{N}_0\right)$

Traces of FE spaces

Let $u \in \mathbb{P}^e(K) \subset H^m(K)$ $\quad m > \frac{1}{2}$

$$\|u\|_{m-\frac{1}{2},\partial K} \le C_T \|u\|_{m,K} \le C_T h^{-m+s} \|u\|_{s,K}$$

For example:

$$\|\nabla u \cdot n\|_{0,\partial K} \le \|\nabla u\|_{0,\partial K} \le C \|\nabla u\|_{\frac{1}{2},K} \le C h^{-\frac{1}{2}} \|\nabla u\|_{0,K} = C h^{-\frac{1}{2}} |u|_{1,K}$$

A posteriori Error estimates — Residual Based

- Once u_h is computed
- estimate the local error computed on $u_h|_K$ only using known quantities: $u_h, f, \Omega, \mathcal{T}_h(\Omega)$
- Find $\eta_K(u_h, f, \Omega, \mathcal{T}_h)$ s.t.

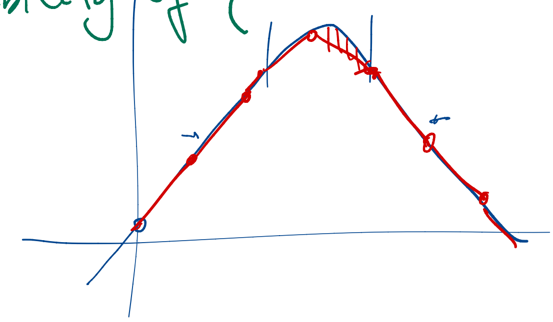
$$\sum_K \eta_K^2 = \eta^2 \quad \text{is equivalent to} \quad \|u-u_h\|_V$$

that is: $\exists\, c_1, c_2$ s.t.

efficiency

$$c_1 \eta^2 \le \|u-u_h\|_V^2 \le c_2 \eta^2$$

Reliability of η



$$a(u,v) = \langle f, v\rangle$$

$$a(u_h, v_h) = \langle f, v_h\rangle$$

Residual

$$R = \langle f, v\rangle - a(u_h, v)$$

(1) $$a(e_h, v) = \langle R, v\rangle$$

$$\langle R, v\rangle := \int_\Omega f\, v - \int_\Omega \nabla u_h \nabla v$$

$$= \sum_K \left[\int_K f v + \int_K \Delta u_h\, v - \int_{\partial K} n\cdot \nabla u_h \cdot v \right]$$

$$= \sum_K \left[\int_K (f + \Delta u_h)\, v - \frac{1}{2} \int_{\partial K} [\![\nabla u_h]\!] v \right]$$

$$[\![\nabla u_h]\!]\Big|_e := \begin{cases} n^+ \nabla u_h^+ + n^- \nabla u_h^- & \text{if } e \in \mathring{\Omega} \\ 2 n^+ \nabla u_h^+ & \text{if } e \in \Gamma \end{cases}$$

$$a(e, v) = a(e, v - \Pi v) = \langle R, v - \Pi v\rangle$$

$$= \sum_K \left[\int_K (f + \Delta u_h)(v - \Pi v) - \frac{1}{2} \int_{\partial K} [\![\nabla u_h]\!] (v - \Pi v) \right]$$

$$\leq \sum_K \left[\| f - \Delta u_h \|_{0,K} \, \| v - \Pi v \|_{0,K} + \frac{1}{2} \| [\![\nabla u_h]\!] \|_{0,\partial K} \| v - \Pi v \|_{0,\partial K} \right]$$

$$\| v - \Pi v \|_{0,K} \leq C\, h_K\, |v|_{1,\tilde{K}} \qquad \tilde{K} = \{ K_J \mid K_J \cap K \neq \phi \}$$

$0 \to \frac{1}{2}, \partial K \longrightarrow 1, K$

$$\| v - \Pi v \|_{0,\partial K} \leq C_T \| v - \Pi v \|_{\frac{1}{2}, K} \leq C_T\, h_K^{\frac{1}{2}} |v|_{1,\tilde{K}}$$

$\longrightarrow$ $$\| v \|_{0,\Gamma} \leq \| v \|_{1,\Omega} \| v \|_{0,\Omega} \qquad v \in H^1(\Omega)$$

$$a(e, v - \Pi v) \leq \sum_K \left[C_1 h_K \|r_K\|_{0,K} + C_2 h_K^{\frac{1}{2}} \| [\![\nabla u]\!] \|_{0,\partial K} \right] |v|_{1,K}$$

$$a(e, v) \leq C \left[\sum_K \left(h_K \|r_K\|_{0,K} + h_K^{\frac{1}{2}} \| [\![\nabla u_h]\!] \|_{0,\partial K} \right) |v|_{1,\Omega}$$

$$\alpha \|e\|_{1,\Omega}^{\cancel{2}} \leq a(e, e) \leq C \left(\sum_K \dots \right) \|e\|_{1,\Omega}$$

$$\|e\|_{1,\Omega} \leq C \eta$$

$$\eta^2 = \sum_K \eta_K^2$$

$$\eta_K^2 := h_K^2 \|r_K\|_{0,K}^2 + h_K \| [\![\nabla u_h]\!] \|_{0,\partial K}^2$$

$$\left| \sum a_i b_i \right| \leq c \left(\sum a_i^2 \right)^{\frac{1}{2}} \left(\sum b_i^2 \right)^{\frac{1}{2}}$$

Efficiency = next time :)