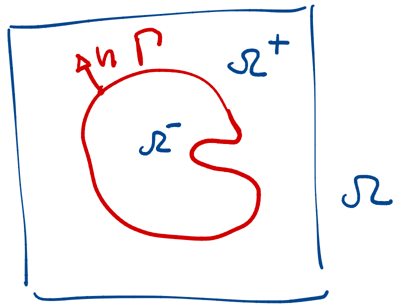
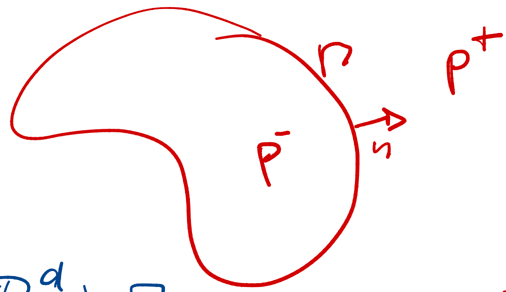
Γ: Lip. closed curve $\subset \Omega$
($\mathrm{dist}(\Gamma, \partial\Omega) > 0$)



Solve pb on Γ, one on Ω
and couple the two problems
efficiently —

- Non-matching
- Allows for changing Γ without changing Ω and $\mathcal{T}_h(\Omega)$

Prototypical Pb: $\Omega \equiv \mathbb{R}^d$



Γ Lip. closed co-dim one surface
with intrinsic dimension $= d-1$

Find $p \in \mathbb{R}^d$, Harmonic in $\mathbb{R}^d \setminus \Gamma$ s.t., Given $f \in H^{-\frac{1}{2}}(\Gamma)$

$H^{-\frac{1}{2}}(\Gamma) = (H^{\frac{1}{2}}(\Gamma))'$

1) $$\left\| \begin{aligned} -\Delta p &= 0 \\ [\![p]\!] &= 0 \quad \text{on } \Gamma \\ [\![\nabla p]\!] &= f \quad \text{on } \Gamma \end{aligned} \right.$$

$\leftarrow \; p^+ n^+ + p^- n^- = p^- n - p^+ n$

$\leftarrow \; \nabla p^+ \cdot n^+ + \nabla p \cdot n^- = \nabla p^- n - \nabla p \cdot n$

B.C. at infty

$\exists$ Explicit solution, given G: Green function of $-\Delta$

$$-\Delta_x G(x-y) = \delta(x-y) : \text{Dirac delta centered in } \underline{\underline{y}}$$
$$\delta_y(x)$$

$$\langle \delta_y, v \rangle = v(y) \qquad \forall v \in C^0(\mathbb{R}^d)$$

$$G(x) = \begin{cases} -\frac{1}{\pi} \ln |x| & d=2 \\ \frac{1}{4\pi} \frac{1}{|x|} & d=3 \end{cases} \qquad G(x) \notin H^1_{loc}(\mathbb{R}^d)$$

Take ①, multiply by G, and integrate by parts twice:

$$0 = \int_{\mathbb{R}^d \setminus \Gamma} -\Delta p \, G_y = \int_{\mathbb{R}^d \setminus \Gamma} \nabla_x p \, \nabla_x G_y - \int_\Gamma [\![\nabla p]\!] G_y = 0$$

$$= -\int_{\mathbb{R}^d \setminus \Gamma} p \underbrace{\Delta_x G_y}_{-\delta_y} - \int_\Gamma \underbrace{[\![\nabla p]\!]}_{\equiv f} G_y + \int_\Gamma \underbrace{[\![p]\!]}_{\equiv 0} \nabla G_y$$

$$0 = p(y) - \int_\Gamma f \, G_y + \cancel{\int_\Gamma [\![p]\!] \nabla G_y}^{\,0}$$

$$p(y) = \int_\Gamma [\![\nabla p]\!](x) G(x-y) \, d\Gamma_x + \int_\Gamma [\![p]\!](x) \nabla_x G(x-y) \, d\Gamma_x, \quad \forall y \in \mathbb{R}^d \setminus \Gamma$$

$$p(y) = \int_\Gamma f(x) \, G(x-y) \, d\Gamma_x = \int_\Gamma f(x) \, G(y-x) \, d\Gamma_x \qquad \forall y \in \mathbb{R}^d \setminus \Gamma$$

$$\lim_{y \to \Gamma} p(y) = \gamma p \qquad \gamma : H^1_{loc}(\mathbb{R}^d) \to H^{\frac{1}{2}}(\Gamma)$$

$$V : H^{-\frac{1}{2}}(\Gamma) \longrightarrow H^1_{loc}(\mathbb{R}^d \setminus \Gamma)$$

$$f \longrightarrow \int_\Gamma f(y) \, G(x-y) \, d\Gamma_y$$

Γ-convolution of f with G

$$S : H^{-\frac{1}{2}}(\Gamma) \longrightarrow \left(H^1_{loc}(\mathbb{R}^d \setminus \Gamma)\right)'$$

$$f \longrightarrow -\Delta V f$$

$$\int_\Gamma f(y)\, \delta(x-y)\, d\Gamma_y$$

Γ-convolution with δ

$$-\Delta p = \int_\Gamma f\, \delta\, d\Gamma$$

+ B.C. for p at $|x| \to +\infty$

$$p(x) = \int_\Gamma f(y)\, G(x-y)\, d\Gamma_y$$

$\Longleftrightarrow$

$$(-\Delta p)(x) = \int_\Gamma f(y)\, \delta(x-y)\, d\Gamma_y = Sf$$

$$\neq f(x) \;!!$$

$\Longleftrightarrow$

$$-\Delta p = 0 \quad \text{in } (H^1_{loc}(\Omega))'$$

$$[\![p]\!] = 0 \quad \text{in } H^{\frac{1}{2}}(\Gamma)$$

$$[\![\nabla p]\!] = f \quad \text{in } H^{-\frac{1}{2}}(\Gamma)$$

What properties do Sf have?

$$\langle v, Sf \rangle = \int_{\mathbb{R}^d} v(x) \int_\Gamma \delta(x-y)\, f(y)\, d\Gamma_y\, dx \quad \forall v \in C_c^\infty(\mathbb{R}^d)$$

defining property of δ !!

$$\int_\Gamma f(y) \underbrace{\int_{\mathbb{R}^d} v(x)\, \delta(x-y)\, dx}_{\equiv v(y)}\, d\Gamma_y$$

$$= \int_\Gamma f(y)\, v(y)\, d\Gamma_y \qquad \forall v \in C_c^\infty(\mathbb{R}^d)$$

Sf distribution that given f, computes duality between v and f on Γ

$V := H^1_{loc}(\mathbb{R}^d)$

$$\langle Sf, v \rangle_{V' \times V} = \langle f, \gamma v \rangle_{H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)}$$

$$S = \gamma^T$$

<u>Theo 1</u>:
$$S : H^{s-\frac{1}{2}}(\Gamma) \longrightarrow H^{-1+s}(\Omega) \quad \text{cont.} \quad s \in [0, \tfrac{1}{2})$$
$$f \longrightarrow \int_\Gamma \delta f$$

$$\left| \langle Sf, v \rangle_{V' \times V} \right| = \left| \langle f, v \rangle_{H^{s-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}-s}(\Gamma)} \right| \leq \|f\|_{s-\frac{1}{2},\Gamma} \|v\|_{\frac{1}{2}-s,\Gamma}$$

$$\leq \underline{\underline{C_T}} \|f\|_{s-\frac{1}{2},\Gamma} \|v\|_{1-s,\Omega}$$

$$\Rightarrow \quad \|Sf\|_{-s+1,\Omega} := \sup_{v \in H^{1-s}(\Omega)} \frac{|\langle Sf, v \rangle|}{\|v\|_{1-s,\Omega}} \leq C_T \|f\|_{s-\frac{1}{2},\Gamma}$$

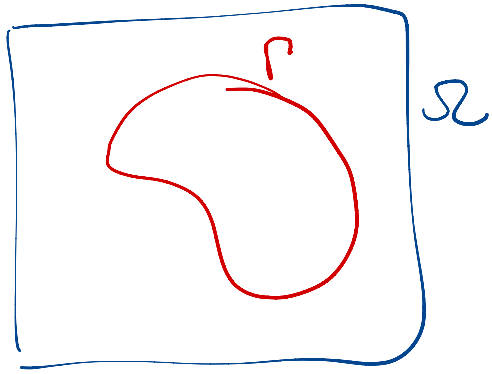
Taking f more regular than $H^{-\varepsilon}(\Gamma)$ $\varepsilon > 0$, won't help because even when $f \in L^2(\Gamma)$, I need $v \in H^\varepsilon(\Gamma)$ to apply trace inequality — $\left(\frac{1}{2} - s > 0\right)$

For $f \in H^r(\Gamma)$, , $S : H^r(\Gamma) \to H^{-\frac{1}{2}-\varepsilon}(\Omega)$

$\forall \varepsilon \in (0, -r]$ if $r < 0$

$\forall \varepsilon > 0$ if $r > 0$

$$(2)\quad \begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \Gamma \\ [u] = 0 & \text{on } \Gamma \\ [\nabla u] = f & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



$\exists!\, u \in H^{\frac{3}{2}-\varepsilon}(\Omega)$. Pb (2) is equivalent to

$$\begin{cases} -\Delta u = Sf = \int f \delta & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Use sol. to pb 1: $u = p + z$

$$(3)\quad \begin{cases} -\Delta z = 0 & \text{in } \Omega \\ z = -p & \text{on } \partial\Omega \end{cases} \qquad (1)\quad \begin{cases} -\Delta p = Sf & \text{in } \mathbb{R}^d \\ + \text{B.C. } |x| \to \infty \end{cases}$$

$u = z + p$ is the unique sol. to (2)

- $-\Delta$ is 2-regular when Ω is convex and Lip.

$$\Rightarrow \quad |u|_{k+2} \leq \|Sf\|_k \quad \forall -1 \leq k \leq 0$$

• Apply th. 1:

$$\|u\|_{\frac{3}{2}-\varepsilon,\Omega} \leq C \|Sf\|_{-\frac{1}{2}-\varepsilon,\Omega} \leq C_T C \|f\|_{-\varepsilon,\Gamma}$$

$$\Rightarrow \quad \|u-u_h\|_{m,\Omega} \leq C\, h^{\frac{3}{2}-\varepsilon-m} \|f\|_{-\varepsilon,\Gamma} \qquad m=0,1$$

Take $f \in L^2 \Rightarrow$

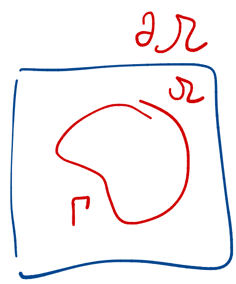
$$\|u-u_h\|_{m,\Omega} \leq C\, h^{\left(\frac{3}{2}-m\right)^{-}} \|f\|_{0,\Gamma} \qquad {}^{-} = \left(\frac{3}{2}-m-\varepsilon\right) \quad \forall \varepsilon>0$$

$$\|u-u_h\|_{1,\Omega} \simeq h^{\frac{1}{2}}$$

$$\|u-u_h\|_{0,\Omega} \simeq h^{3/2}$$

Fictitious Domain Method

$$\begin{cases} -\Delta u + \int_\Gamma p\,\delta = 0 & \text{on } \Omega \\ \gamma u = g & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



Weak formulation:

$$\begin{cases} (\nabla u, \nabla v)_\Omega + \langle p, v\rangle_\Gamma = 0 & \forall v \in H^1_0(\Omega) \\ \langle u, q\rangle_\Gamma = \langle g, q\rangle & \forall q \in H^{-\frac{1}{2}}(\Gamma) \end{cases}$$

$$\begin{cases} Au + B^T p = 0 & \text{in } V' \\ Bu = G & \text{in } Q' \end{cases}$$

LBB + BNB

$\ker(B) := \{ v \in H^1_0(\Omega) \mid \gamma v|_\Gamma = 0 \}$

cond 1) ELL-KER ✓ trivial because A is elliptic on V

$\exists \alpha > 0$ s.t. $\|Au\| \geq \alpha \|u\| \;\forall u \in V$

$\|A^T u\| \geq \alpha \|u\| \;\forall v \in V$

$|\langle Au, u \rangle| \geq \alpha \|u\|^2 \;\forall u \in V$

2) INF SUP

$\exists \beta$ s.t. $\|B^T p\|_{V'} \geq \beta \|p\|_Q$

$$\langle B^T p, v \rangle_{V' \times V} = \langle p, Bv \rangle_{Q \times Q'}$$

$\exists$ right inverse (B^{-1}) of γ (B): $\quad B(B^{-1} g) = g$

inf sup is satisfied with $\|B^{-1} g\| \leq \frac{1}{\beta} \|g\|$

$\Rightarrow \beta = C_T$

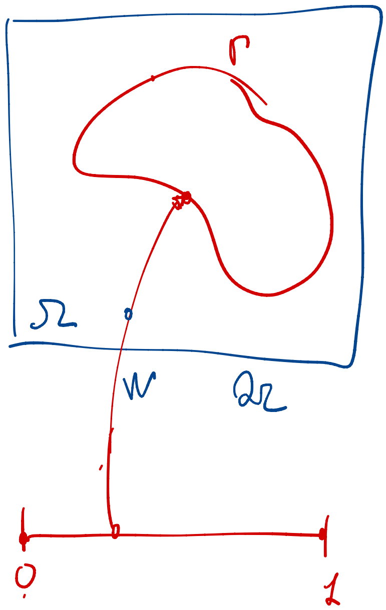
Finite Element Immersed Boundary Method

IBM (72 Peskin)

$$
\begin{aligned}
-\Delta u + \nabla p + S\lambda &= 0 && \text{on } \Omega \\
\nabla \cdot u &= 0 && \text{on } \Omega \\
-W'' - \lambda &= 0 && \text{on } \Gamma_0 \\
u(\xi(s)) - \dot{W}(s) &= 0 && \text{on } \Gamma_0
\end{aligned}
$$

Non-slip condition

$$
\begin{aligned}
& && \text{on } \Gamma \\
W(0) &= W_0 && \\
u &= 0 && \text{on } \partial\Omega
\end{aligned}
$$



$$\xi(0) = \xi(1)$$

$$\Gamma = I + W(\Gamma_0) = \xi(\Gamma_0)$$

$$\xi := I + W$$

$$\Gamma_0 = [0,1]$$

$$\xi : [0,1] \longrightarrow \mathbb{R}^2 \subset \Omega$$

$W(s,t)$

$u(x,t)$

$$
\begin{array}{cc|cc|c}
\boxed{\begin{matrix} Au & \\ & KW \end{matrix}} & + & \boxed{\begin{matrix} B^T p + C^T\lambda \\ -D\lambda \end{matrix}} & = & \begin{matrix} 0 \\ 0 \end{matrix} \\
\boxed{\begin{matrix} Bu \\ Cu - D\dot{w} \end{matrix}} & & \begin{matrix} 0 \quad 0 \\ 0 \quad 0 \end{matrix} & = & \begin{matrix} 0 \\ 0 \end{matrix}
\end{array}
$$

$$
\begin{aligned}
Au + B^T p + C^T \underline{\underline{D^{-1}}} K W &= 0 \\
Bu &= 0 \\
Cu - D\dot{w} &= 0
\end{aligned}
$$