$H := L^2([t_0, t_0+T])$ of periodic functions. $u(t+T) = u(t)$ f.a.e. $t \in \mathbb{R}$

$$u \in H, \quad \int_{t_0}^{t_0+T} \bar{u} u \, dt = \int_{t_0}^{t_0+T} |u|^2 dt =: \|u\|_{L^2}^2 < +\infty$$

$$\langle u, g \rangle := \int_{t_0}^{t_0+T} \bar{u} g \; \Big/ \; \tilde{u}(g) \; \Big/ \; u_g$$

$$e_n(x) := e^{2\pi i \omega_n x} \qquad \omega_n := \frac{n}{T}$$

$$\langle e_n, e_m \rangle := \int_{t_0}^{t_0+T} \overline{e_n} e_m = T \delta_{nm} =: g_{nm}$$

$$g^{nm} g_{nk} = \delta^n_k$$

$$e^n := g^{nm} e_m = \frac{1}{T} \delta^{nm} e_m = \frac{1}{T} e_n$$

$$\boxed{\text{Bessels Inequality}}$$

Given a sequence of linearly independent basis $\{e_n\}$ with reciprocal basis $\{e^n\}$, induced by $\langle \, , \, \rangle$ on H

$$\|\langle e^n, u \rangle e_n\|^2 \leq \|u\|^2$$

Proof.

$$0 \leq \|\langle e^n, u \rangle e_n - u\|^2 = \langle \langle e^n, u \rangle e_n - u, \langle e^m, u \rangle e_m - u \rangle$$

$$0 \leq \langle \underbrace{\langle e^n, u \rangle}_{u^n} e_n, \underbrace{\langle e^m, u \rangle}_{u^m} e_m \rangle - \langle u, \underbrace{\langle e^m, u \rangle}_{u^m} e_m \rangle - \langle \underbrace{\langle e^n, u \rangle}_{u^n} e_n, u \rangle + \|u\|^2$$

$$\langle e^n, u \rangle =: u^n \qquad \langle e_n, u \rangle = u_n$$

$$0 \leq \langle u^n e_n, u^m e_m \rangle - \langle u, u^m e_m \rangle - \langle u^n e_n, u \rangle + \|u\|^2$$

$$0 \leq \cancel{\bar{u}^n u_m} - \cancel{\bar{u}^n u_n} - \bar{u}^n u_n + \|u\|^2$$

$$\bar{u}^n u_n \leq \|u\|^2_{L^2}$$

$$\|\{u\}\|^2_{\ell^p} := \bar{u}^n g_{nm} u^m = \bar{u}^n u_n$$

$$\bar{u}^n u_n = \|\{u\}\|_{\ell^p} \leq \|u\|^2_{L^2}$$

Parseval's identity: if $\{e_n\}$ is complete (span $\{e_n\}$ is dense in H.) then

$$\boxed{\|\{u\}\|_{\ell^p} = \|u\|^2_{L^2}}$$

$$u_m = \langle e_n, u \rangle$$

$$u^m = \frac{1}{T} u_m$$

$$u = \langle e^m, u \rangle e_m = \frac{1}{T} \sum_{-\infty}^{+\infty} u_m e_m = \sum_{-\infty}^{+\infty} \hat{u}_T(\omega_n) \Delta\omega \, e_n$$

$$\Delta\omega = \omega_n - \omega_{n-1} = \frac{1}{T}$$

Fourier Analysis:

$$u^m := \underbrace{\langle e^m, u \rangle}_{\text{Fourier coefficients}} = \int_{t_0}^{t_0+T} \bar{e}^m u \, dt = \frac{1}{T} \int_{t_0}^{t_0+T} e^{-2\pi i \omega_n t} u \, dt$$

Fix t_0 to be $-\frac{T}{2}$ (it is arbitrary)

$$\hat{u}_T(\omega) := \int_{-T/2}^{T/2} e^{-2\pi i \omega t} u(t)\, dt$$

$\boxed{\text{CTFT}}$ Continuous time Fourier Transform

$$\hat{u}(\omega) := \int_{\mathbb{R}} e^{-2\pi i \omega t} u(t)\, dt$$

$$\check{u}(t) := \int_{\mathbb{R}} e^{2\pi i \omega t} \hat{u}(\omega)\, d\omega$$

Properties:

$(ug)^\wedge = \hat{u} * \hat{g}$ $\qquad$ $(u')^\wedge = (2\pi i \omega \hat{u})$

$(u * g)^\wedge = \hat{u}\hat{g}$

Play time!