$\boxed{\text{Preliminaries}}$ Vector spaces, span, basis, dual basis

23 Jan 2023

$V, \mathbb{K}, +, \circ$ $\quad\quad \mathbb{K} = \mathbb{R} / \mathbb{C}$ scalars

main properties of vector spaces on field $\mathbb{K}$

$$+ : \quad V \times V \longrightarrow V$$
$$\quad\quad u, v \longrightarrow u + v$$
$$\circ : \quad V \times \mathbb{K} \longrightarrow V$$
$$\quad\quad u, \alpha \longrightarrow \alpha u$$

- $\exists \; 0_V : \quad u + 0 = u \quad \forall u$
- $\exists \; 1_{\mathbb{K}} : \quad 1_{\mathbb{K}} \cdot u = u \quad \forall u$
- $\alpha(u + v) = \alpha u + \alpha v$
- $\alpha(\beta u) = (\alpha\beta) u$

Typical examples:

$$V = \underbrace{\mathbb{C}^n}_{\text{finite dimensional}} \quad \text{or} \quad V = \underbrace{L^2([a,b])}_{\text{infinite dimensional}}$$

B is a linear subspace of V of dimension n generated by a set of linearly independent elements of V

$$\{b_i\}_{i=1}^{n} \quad b_i \in V \quad i \in \mathbb{N} \cap [1, n]$$

$$B = \operatorname{span}\{b_i\}_{i=1}^{n} \iff$$

$$B = \{u \in B \iff \exists! \{u^i\} \in \mathbb{C} \text{ s.t. } u = \sum_{i=1}^{n} u^i b_i\}$$

$F \in \underbrace{\mathcal{L}(V,W)}_{\text{linear functions from } V \text{ to } W}$ $\quad V, W$ are complex vector spaces

$$W \ni w = F(\alpha u + \beta v) = \alpha F(u) + \beta F(v) \quad \forall\, u, v \in V \quad \alpha, \beta \in \mathbb{C}$$

We call $\mathcal{L}(V, \mathbb{C})$ "linear functionals"

$\mathcal{L}(V, \mathbb{C}) \equiv V^*$ $\quad$ Dual space of V

$$B = \text{span}\{b_i\}_{i=1}^{n}$$

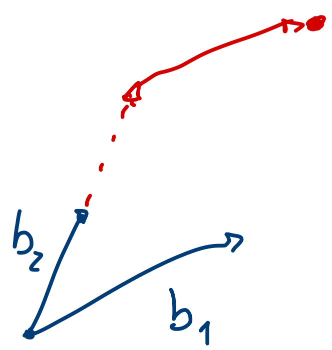
choose $b^i \in B^*$ s.t.

$$b^i(b_j) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example: $\quad$ if $u = \alpha b_i \Rightarrow b^i(u) = \alpha$

$\{b^i\}$ is the $\boxed{\text{dual basis}}$ of $\{b_i\}$

$b^i \in B^*$ $\quad$ is a basis for B^*, i.e., $B^* = \text{span}\{b^i\}$



$$u = u^i b_i = b^i(u)\, b_i$$

$$u = b_1 + 2 b_2$$

$$u^1 = b^1(b_1 + 2 b_2) = \overbrace{b^1(b_1)}^{1} + 2\, \overbrace{b^1(b_2)}^{0}$$

$$u^2 = b^2(b_1 + 2b_2) = \underbrace{b^2(b_1)}_{0} + \underbrace{2\,b^2(b_2)}_{2}$$

$$\overline{B} : B \longrightarrow \mathbb{C}^n$$
$$u \longrightarrow \{b^i(u)\}$$

Given a vector in B find its coefficients $\{u^i\} \in \mathbb{C}^n$ s.t. $u = u^i b_i$

$$\underline{B} : \mathbb{C}^n \longrightarrow B$$
$$\{u^i\} \longrightarrow u^i b_i$$

Given a set of n coefficients $\{u^i\} \in \mathbb{C}^n$, build $u \in B$ $u = u^i b_i$

$$\underline{B} \circ \overline{B}(u) = \underline{B}(\overline{B}(u))$$

in B, this is the identity!

$$\underline{B} \circ \overline{B} : B \longrightarrow B$$
$$u \longrightarrow b^i(u)\, b_i$$

$P : \underline{B} \circ \overline{B}$ prove that $P^2 = P$

$$P(u) = b^i(u)\, b_i \qquad P^2(u) = b^J(\underbrace{b^i(u)\, b_i}_{P(u)})\, b_J$$

$$= b^i(u)\, b^J(b_i)\, b_J$$

$$= b^i(u)\, \delta^J{}_i\, b_J$$

$$= b^i(u)\, b_i$$

More interestingly, we can extend b^i to operate on the full space V (by Hahn-Banach extension theorem)

i.e., given $b^i \in B^*$, choose a $\tilde{b}^i$ in V^* s.t.

$$\tilde{b}^i \in V^*, \quad \tilde{b}^i(u) = b^i(u) \quad \forall u \in B$$

and now define an operator $\tilde{B} : V \longrightarrow C^n$:

$$\tilde{B} : V \longrightarrow C^n \qquad \tilde{b}^i \in V^*$$
$$u \longrightarrow \tilde{b}^i(u)$$

The resulting projection operator $\tilde{P} := \underline{B} \circ \tilde{B}$ can be used to project elements of V to B

(Remember! B is a finite dimensional subspace of V)

According to

- choice of $\{b^i\}$ (and therefore of $\{b_i\}$)
- dimension of V
- dimension of B
- family of elements of V we want to approximate

we are interested in estimating the error we make if we replace u with $\tilde{P}u$ in a subspace $B \subset V$ of finite dimension

$$e := u - \tilde{P}u = (I - \tilde{P})u$$