26. Jan 2023

$$\|\cdot\| : V \longrightarrow \mathbb{R}^+ \cup \{0\}$$

1) $\|u\| \geq 0 \quad \forall u \in V$

2) $\|\alpha u\| = |\alpha| \, \|u\| \quad \forall u \in V, \forall \alpha \in \mathbb{C}$

3) $\|u\| = 0 \iff u = 0_V$ (if false, $|u|$ is used)

4) $\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in V$

Vector space + a norm is a Banach space (if the vector space is complete w.r.t. the norm)

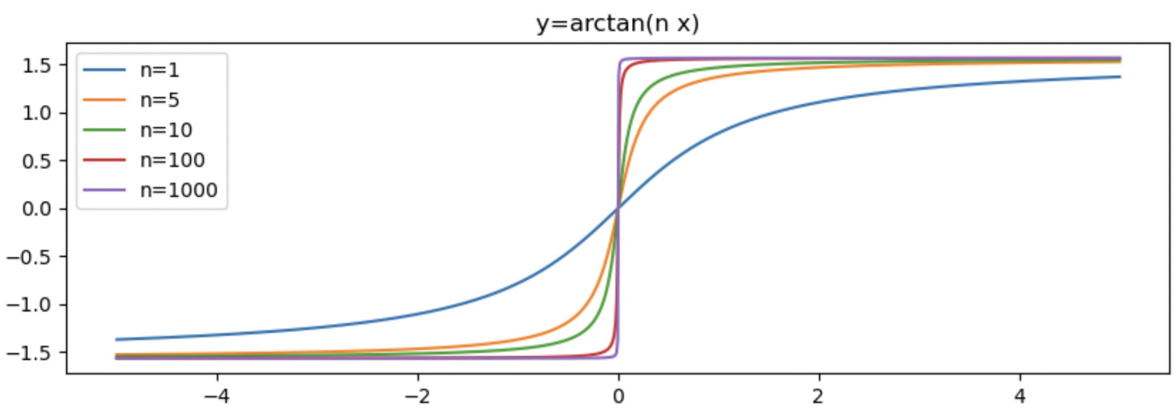
negative example: $C^0([a,b])$ $\quad u : [a,b] \longrightarrow \mathbb{C}$
s.t. u is continuous

Complete: the limit of every Cauchy sequence belongs to V

Cauchy: $\{u_n\}_{n=1}^{\infty} \in V$ s.t. $\forall \varepsilon \in \mathbb{R}$ $\exists \bar{n}$ s.t. $\forall n, m \geq \bar{n}$ $\|u_n - u_m\| \leq \varepsilon$

$C^0([a,b])$ with norm $\|u\|_\infty := \max_{x \in [a,b]} |u(x)|$ is $\boxed{\text{NOT}}$ complete

Example:
- $\operatorname{atan}(nx) \in C^0([-1,1]) \quad \forall n$
- it is Cauchy $\quad \lim_{n\to\infty} \operatorname{atan}(nx) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}$
- the limit is Not C^0



$$f : \mathbb{Z} \longrightarrow \mathbb{C} \qquad \{f_n\}_{n=-\infty}^{+\infty}$$

$$\|f\|_{\ell^p} := \left(\sum_{n=-\infty}^{+\infty} |f|^p \right)^{\frac{1}{p}} \qquad \ell_p \text{ norm}$$

The space of $\{f_n\}$ s.t. $\|f_n\|_{\ell^p} < +\infty$ is a Banach space

$$(f+g)(i) := f(i) + g(i) \quad \forall i \in \mathbb{Z}$$

$$(\alpha f)(i) := \alpha f(i) \qquad \forall i \in \mathbb{Z}, \ \forall \alpha \in \mathbb{C}$$

$$f, g \in \ell^p \implies \alpha f + \beta g \in \ell^p \quad \forall \alpha, \beta \in \mathbb{C}$$

$$\|\alpha f + \beta g\|_{\ell^p} \leq \|\alpha f\|_{\ell^p} + \|\beta g\|_{\ell^p} \qquad \text{by } 4)$$

$$\leq |\alpha| \, \|f\|_{\ell^p} + |\beta| \, \|g\|_{\ell^p} \qquad \text{by } 2)$$

$$L^p(\mathbb{R}) : \quad f : \mathbb{R} \longrightarrow \mathbb{C}$$

$$L^p(\mathbb{R}) := \left\{ f \text{ s.t. } \int_{\mathbb{R}} |f|^p \, dx < +\infty \right\}$$

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}} |f|^p \, dx \right)^{\frac{1}{p}}$$

$$f(x) = \begin{cases} 1 & \text{when } x \in \mathbb{Q} \\ 0 & \text{when } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$$\int_{\mathbb{R}} f(x)\, dx = 0$$

we identify $f(x)$ with $\underline{\underline{0}}$

<u>Inner product</u>

$$\boxed{\begin{array}{l} \langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{C} \qquad \text{" $\cdot$ "} \\ P) \quad \langle u, u \rangle > 0 \quad \forall u \neq 0 \in V \\ H) \quad \langle u, v \rangle = \overline{\langle v, u \rangle} \\ L) \quad \langle u, \underline{\underline{c}}v + w \rangle = \underline{\underline{c}} \langle u, v \rangle + \langle u, w \rangle \end{array}}$$

axioms for inner product.

$$\Rightarrow \quad \langle cu, v \rangle = \bar{c} \langle u, v \rangle$$

Banach space with norm $\sqrt{\langle u, u \rangle}$ is called Hilbert space H

ℓ^p is Hilbert for $p = 2$

L^p " $p = 2$

ℓ^2 : inner product $\langle u, v \rangle := \sum_{n=-\infty}^{+\infty} u_n \bar{v}_n$

L^2 : " $\langle u, v \rangle := \int_{\mathbb{R}} u \bar{v}\, dx$

with $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ (Hilbert Spaces)

we use the notation $\tilde{u}$ to indicate linear functionals

$\tilde{u} : V \longrightarrow C$

we have called $\{b_i\}_{i=1}^{n}$ basis for $B \subset V$

$\dim(B) = n$

$\forall \; u \in B \quad \exists! \; \{u^i\}_{i=1}^{n} \quad$ s.t. $\quad u = \underline{u^i b_i}$

We now call $\tilde{b}^i \in V^*$ s.t. $\tilde{b}^i(b_j) = \delta^i_j$
canonical dual basis of $\{b_i\}$

$\{\tilde{b}^i\}$ is a basis for B^*

$\forall u \in B \qquad u = \tilde{b}^i(u) \, b_i$

that is : $\quad b_i \tilde{b}^i \equiv I_B$

$b_i \tilde{b}^i : B \longrightarrow B$

$b_i \tilde{b}^i(u) \;=\; u^i b_i \;=\; u$

$B^{**} \equiv B$

Rietz representation theorem.

$\forall\ \tilde{u} \in H^*$ with H Hilbert, $(\langle\cdot,\cdot\rangle\ ,\ \|\cdot\|)$

$\exists!\ u \in H$ s.t. $\tilde{u}(v) = \langle u, v\rangle \quad \forall\ v \in V$

Opposite is simple: $\forall u \in H \quad \tilde{u}(v) := \langle u, v\rangle \in H^*$

Define Reciprocal basis of $\{b_i\}_{i=1}^n$ as

$b^J \in B$ s.t. $\tilde{b}^J(v) = \langle b^J, v\rangle \quad \forall v \in H$

Metric tensor matrix $g_{iJ} := \langle b_i, b_J\rangle$

Defines how to compute $\langle\cdot,\cdot\rangle$ in B:

$\forall u, v \in B \qquad u = u^i b_i \qquad v = v^J b_J$

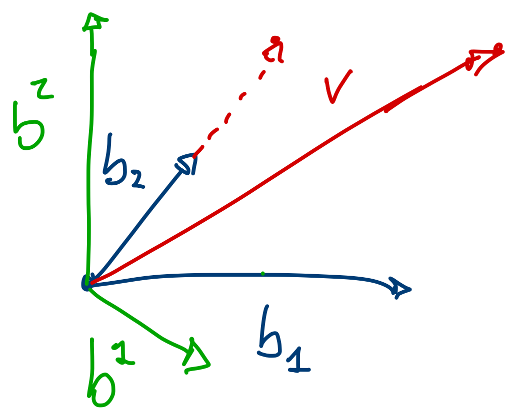
$$\langle u, v\rangle := \bar{u}^i \langle b_i, b_J\rangle v^J = \bar{u}^i g_{iJ} v^J$$

g is Hermitian positive definite
(invertible in general cases)

$$g_{iJ} = \overline{g_{Ji}}$$

$$b^j \in B \Rightarrow \tilde{b}^i(b^j)\, b_i = b^j$$

$$\tilde{b}^i(b^j)\, b_i := \underbrace{\langle b^i, b^j \rangle}_{g^{ij}} b_i = g^{ij} b_i$$



$$\langle b_1, b^2 \rangle = \langle b_2, b^1 \rangle = 0$$

$$\langle b_1, b^1 \rangle = 1$$

$$\langle b_2, b^2 \rangle = 1$$

$v = 2b_2 + b_1$

$$\langle b^i, b^j \rangle = g^{ij} \neq \delta^{ij}$$

$$\boxed{\langle b^i, b_j \rangle = \delta^i_j}$$

$$\langle b^i, b_j \rangle = \delta^i_j = \langle \langle b^k, b^i \rangle b_k, b_j \rangle$$

$$= \overline{\langle b^k, b^i \rangle} \langle b_k, b_j \rangle$$

reciprocal metric matrix is inverse of metric matrix

$$= \overline{g^{ki}}\, g_{kj} = \boxed{g^{ik} g_{kj} = \delta^i_j}$$

$$b_i = g_{ij} b^j \qquad u = u_i b^i = u^i b_i$$

$$b^i = g^{ij} b_j \qquad u_i = g_{ij} u^j \qquad u^i = g^{ij} u_j$$