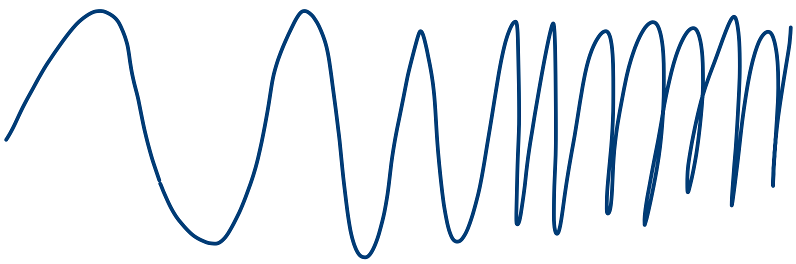
13. Feb. 2023

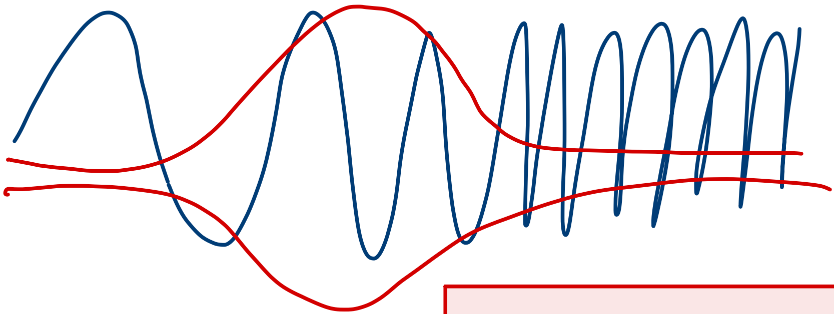
WFT: $f \longrightarrow g(u-t)\, f(u) \longrightarrow (g_t f)^\wedge$

$$L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}^2)$$



chirp.

$g(u-t)\, f(u)$



Idea of Wavelets:

$\boxed{\psi \text{ must satisfy the Wavelet condition}}$

Wavelet function: $\psi(u) \rightsquigarrow \psi^{s,t}(u) := |s|^{-\frac{1}{2}}\, \psi\left(\frac{u-t}{s}\right)$

$\boxed{\text{CTWT}}$: $\tilde{f}(s,t) := \langle \psi^{s,t}, f \rangle = \int_{\mathbb{R}} |s|^{-\frac{1}{2}}\, \overline{\psi\left(\frac{u-t}{s}\right)}\, f(u)\, du$

$$\boxed{C_\psi := \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega} = 1$$

H: Hilbert space induced by:

$$\langle \tilde{f}, \tilde{g} \rangle_H := \int_{\mathbb{R}^2} \overline{\tilde{f}}\, \tilde{g}\, \frac{ds\,dt}{s^2}$$

$$f(u) = \langle \psi^{s,t}, \tilde{f} \rangle_H = \int_{\mathbb{R}^2} \overline{\psi}^{s,t}(u)\, \tilde{f}(s,t)\, \frac{ds\,dt}{s^2}$$

$\tilde{f} \in H$ iff $\langle \tilde{f}, \tilde{f} \rangle_H < +\infty$

What is the wavelet condition?

ψ is such that $\displaystyle\int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < +\infty$ $\boxed{WC}$

Wavelet condition

$\forall f \in L^2, \forall \varepsilon \in \mathbb{R}^+ \; \exists \psi$ s.t. WC is satisfied and

$\|f - \psi\|_{L^2} \leq \varepsilon$ Wavelets are dense in L^2

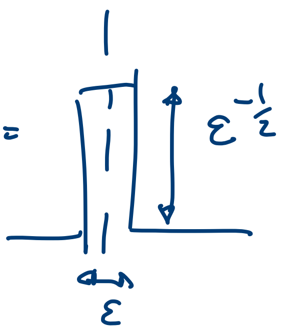
take $\psi_{f,\varepsilon}$ inverse f.t of $\hat{\psi}_{f,\varepsilon} := \begin{cases} \hat{f}(\omega) & |\omega| \geq \varepsilon c \\ 0 & \text{otherwise} \end{cases}$

$\psi_{f,\varepsilon}$ satisfies $\boxed{WC}$

$$\|f - \psi_{f,\varepsilon}\|^2_{L^2(\mathbb{R})} = \|\hat{f} - \hat{\psi}_{f,\varepsilon}\|^2_{L^2(\mathbb{R})} := \int_{-c\varepsilon}^{c\varepsilon} |\hat{f}(\omega)|^2 d\omega$$

I can choose c s.t. $\displaystyle\int_{-c\varepsilon}^{c\varepsilon} |\hat{f}(\omega)|^2 d\omega \leq \varepsilon$

morally: $\hat{f}(0) = 0 \Rightarrow \displaystyle\int_{\mathbb{R}} f \, d\mu = 0$ Zero average

$f_\varepsilon :=$ 

$\|f_\varepsilon\|_{L^2} := 1$ $\displaystyle\left(\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (\varepsilon^{-\frac{1}{2}})^2 dt \right)^{\frac{1}{2}}$

Multiresolution Analysis (MRA)

MRA of $L^2(\mathbb{R})$ is a set $\{V_J\}_{J\in\mathbb{Z}}$ subspaces of $L^2(\mathbb{R})$ s.t.

i) $V_J \subset V_{J+1} \qquad J\in\mathbb{Z}$

ii) $\bigcap_{J\in\mathbb{Z}} V_J = \{0\} \qquad \overline{\bigcup_{J\in\mathbb{Z}} V_J} = L^2(\mathbb{R})$

iii) $\exists\ \varphi$ (a scaling function) $\varphi\in L^2(\mathbb{R})$

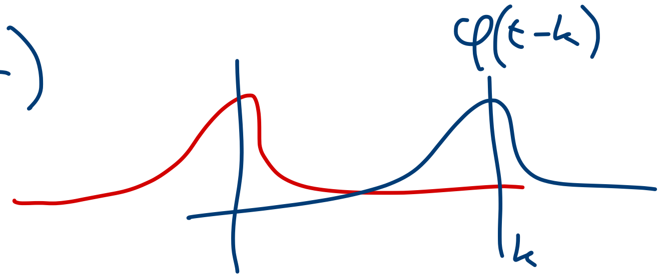
$\{\varphi(x-k)\}_{k=-\infty}^{+\infty}$ is a Rietz basis for V_0

$$\{e_k\}_{k=-\infty}^{+\infty} \iff \exists\ A, B \text{ s.t.}$$

$$A\sum|c_k|^2 \leq \left\| \sum_{k=-\infty}^{+\infty} c_k e_k \right\|_{L^2} \leq B\sum|c_k|^2$$

Rescaling of φ:

$$\varphi_{J,k}(t) := 2^{J/2}\varphi(2^J t - k)$$



$$u\in V_J \iff u(2\,\cdot)\in V_{J+1}$$

$\varphi_{J,k}$ is a basis for V_J then

$\varphi_{J,k}(2\,\cdot)$ is a basis for V_{J+1}

$\varphi_{J+1,k}$ " "

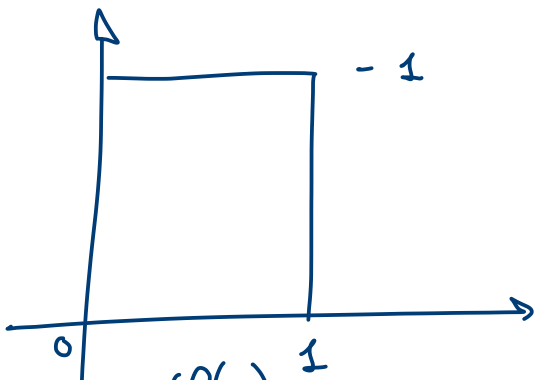
$\tilde{\varphi}_{J,k}$ is a reciprocal basis for $\varphi_{J,k}$

$\langle \tilde{\varphi}_{J,k}, \varphi_{J,m} \rangle = \delta_{km}$ $\qquad$ $g_{Jmn} := \langle \varphi_{J,m}, \varphi_{J,n} \rangle$

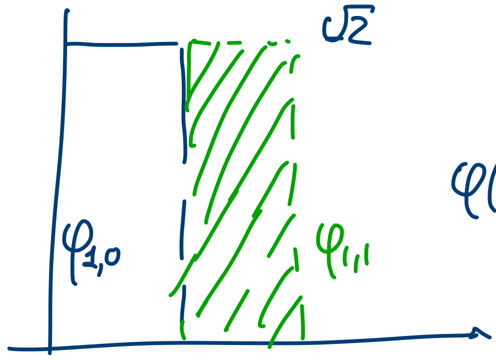
$\tilde{\varphi}_{J,k} := \sum_n g_J^{kn} \varphi_{J,n}$ $\qquad$ $\sum_n g_J^{kn} g_{Jne} = \delta^k_e$

$P_J \mu := \sum_k \langle \tilde{\varphi}_{J,k}, \mu \rangle \varphi_{J,k}$ $\quad$ L^2 projection on V_J

$V_0 \subset V_1 \Rightarrow \exists\, h_k$ s.t. $\varphi(x) = \sum_k h_k \varphi_{1,k}(x)$



$\varphi(x)$



$\varphi(x) = (\varphi_{1,0} + \varphi_{1,1}) \frac{1}{\sqrt{2}}$

$\rightarrow h_0 = \frac{1}{\sqrt{2}} \quad h_1 = \frac{1}{\sqrt{2}} \quad , h_k = 0 \quad \forall k \neq 0,1$

φ is refineable, and h_k are the refinement coefficients

$Q_J := P_{J+1} - P_J$ $\qquad$ $W_J = Q_J(L^2)$

$W_J := \ker(P_J)$ in V_{J+1}

$g_k := (-1)^k \tilde{h}_{1-k}$ $\qquad$ $\tilde{\varphi} := \sum_k \tilde{h}_k \tilde{\varphi}_{1,k}$

$\tilde{g}_k := (-1)^k h_{1-k}$

$$\boxed{\psi(x) = \sum_k g_k \varphi(2x-k)}$$

$$\boxed{\tilde{\psi}(x) = \sum_k \tilde{g}_k \tilde{\varphi}(2x-k)}$$

ψ, $\tilde{\psi}$ form basis for W_J

$$\boxed{V_{J+1} := V_J \oplus W_J}$$

$$V_{J+1} := V_0 \bigoplus_{k=0}^{J} W_k$$