Windowed Fourier Transform:

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Let's take a "window" function ($g \in L^2(\mathbb{R})$)

$$f_t(x) := g(x-t)\, f(x)$$

$$\boxed{\mathrm{WFT}} := \hat{f_t}(\omega) \equiv \tilde{f}(t,\omega) = \int_{\mathbb{R}} \bar{g}(x-t) f(x) e^{-i2\pi x\omega}\, dx$$

$$g_{\omega,t} := g(x-t)\, e^{i2\pi x\omega} \in L^2(\mathbb{R}) \quad \forall\, \omega, t$$

$$\boxed{\mathrm{WFT}}(u) := \langle g_{\omega,t}, u \rangle$$

$g_{\omega,t}$: "notes" function

Expected values and deviations of g and $\hat{g}$

center: $t_0(g) = \int_{\mathbb{R}} t\, |g|^2\, dt$

width: $T(g) := \left(\int (t-t_0)^2 |g|^2\, dt \right)^{\frac{1}{2}}$

center of $\hat{g}$ $\quad \hat{\omega}_0(g) := \int_{\mathbb{R}} \omega\, |\hat{g}|^2\, d\omega$

width of $\hat{g}$ $\quad \Omega(g) := \left(\int_{\mathbb{R}} (\omega-\omega_0)^2 |\hat{g}|^2\, d\omega \right)^{\frac{1}{2}}$

Heisenberg uncertainty principle

$$\boxed{T(g) \cdot \Omega(g) \geq \frac{1}{4\pi}}$$

The g that achieves $T(g) \cdot \Omega(g)$ is the Gaussian

$$g_{\omega_0, t_0} := \pi^{-\frac{1}{4}} \frac{1}{\sqrt{2\pi}} e^{-2i\pi\omega_0 t} e^{-\pi(t-t_0)^2}$$

WFT with ... is called "Gabor" Transform

Proof:

- hypothesis $\|g\|_{L^2} = 1 = \|\hat{g}\|_{L^2}$
- assume $g' \in L^2$
- assume $t_0 = 0$, $\omega_0 = 0$

$$\Omega(g)^2 = \int_{\mathbb{R}} \omega^2 |\hat{g}|^2 \, d\omega$$

$$T(g)^2 = \int_{\mathbb{R}} t^2 |g|^2 \, dt$$

$\|\widehat{(g')}\| = \|g'\|$

$$\int_{\mathbb{R}} |\omega \hat{g}|^2 = \int_{\mathbb{R}} \left| \frac{\widehat{(g')}}{2\pi i} \right|^2 = \frac{1}{4\pi^2} \|g'\|^2$$

$$T(g)^2 \Omega(\hat{g})^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}} |t g|^2 \int_{R} |g'|^2$$

$$|\langle u, v\rangle|^2 \leq \|u\|^2 \|v\|^2 \qquad u = |tg| \quad v = |g'|$$

$$T(g)^2 \Omega(\hat{g})^2 \geq \frac{1}{4\pi^2} \left(\int |tg \, g'| \right)^2$$

$$|ab| = |a\bar{b}| \geq \mathrm{Re}(a\bar{b}) = (a\bar{b} + \bar{a}b)\tfrac{1}{2}$$

$$T(g)^2\,\Omega(\bar{g})^2 \geq \frac{1}{4\cdot 4\pi^2}\left(\cancel{\tfrac{1}{2}}\int_{\mathbb{R}} \underbrace{t\bar{g}g' + tg\bar{g}'}\,dt\right)^2$$

$$\frac{d}{dt}|g|^2 = \frac{d}{dt}(g\bar{g}) = g'\bar{g} + g\bar{g}'$$

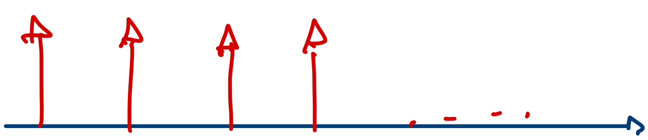
$$\geq \frac{1}{16\pi^2}\left(\int_{\mathbb{R}} t\,\frac{d}{dt}|g|^2\,dt\right)^2$$

$$\geq \frac{1}{16\pi^2}\left(-\int_{\mathbb{R}} |g|^2\,dt\right)^2$$

$$T(g)^2\,\Omega(\hat{g})^2 \geq \frac{1}{16\pi^2}\underbrace{\|g\|^4}_{=1} = \left(\frac{1}{4\pi}\right)^2$$

$\boxed{\text{DTFT}}$

Define $D(x) := \sum_{k\in\mathbb{Z}} \delta(x-k)$



$$\mathrm{DTFT}(u) := \mathcal{F}(uD) := \sum_{k\in\mathbb{Z}} u(k)\,e^{-2\pi i\omega k}$$

By construction $\mathrm{DTFT}(u)(\omega)$ is periodic in $[0,1]$

$$\mathrm{DTFT}: \underbrace{\ell^2(\mathbb{Z})}_{\text{discrete time}} \longrightarrow L^2(\underbrace{[0,1]}_{\omega})$$

IDTFT : is the Fourier series :

$$u_n := \langle e_n , \hat{u} \rangle \qquad e_n := e^{2\pi i n x}$$

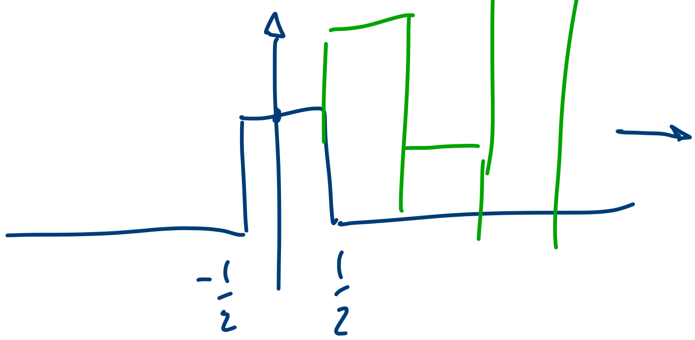
$$\Uparrow$$

$u(\omega)$

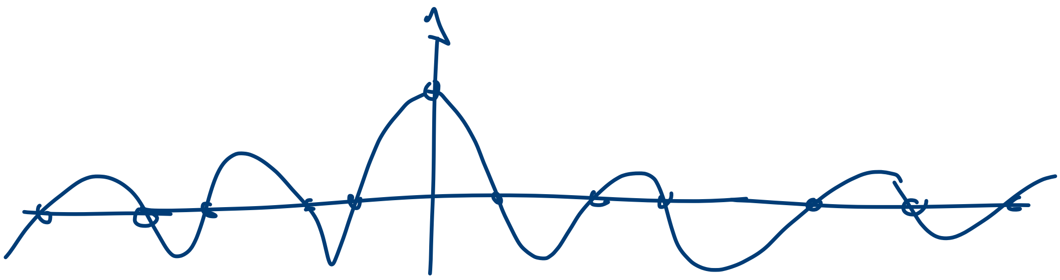
$$\langle e_n , e_m \rangle = T \delta_{nm} = 1 \; \delta_{nm} \quad \Rightarrow \quad e^n \equiv e_n$$

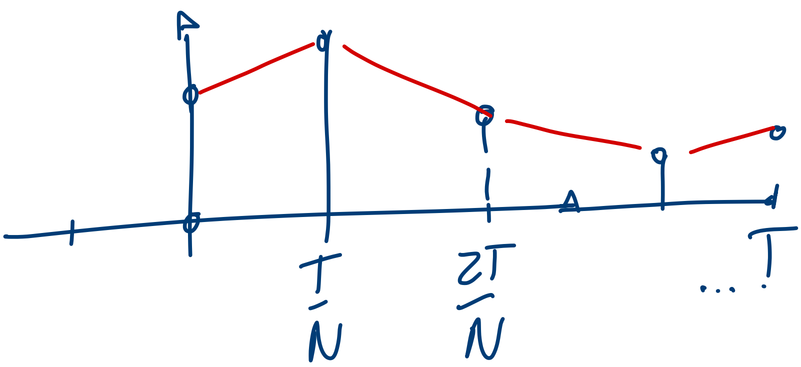
$$\langle e_n , \hat{u} \rangle_{L^2([0,1])} e^n(\omega) = \hat{u}(\omega) \in L^2([0,1]_1)$$

$$\langle e_n , \hat{u} \rangle \in \ell^2(\mathbb{Z}) \qquad \| u(\mathbb{Z}) \|_{\ell^2} = \| \hat{u} \|_{L^2([0,1])}$$



$$\frac{\sin(\pi\omega)}{\pi\omega}$$





$$\Delta t = t_n - t_{n-1} = \frac{T}{N}$$

freq: $\frac{1}{\Delta t}$

Remember : $\mathcal{F}\left(u\left(\frac{t}{a}\right)\right) = |a| \, \mathcal{F}(u)(a\omega)$

$$\boxed{\mathrm{IDTFT}}(\hat{u}) := \int_0^1 \hat{u}(\omega)\, e^{2\pi i \omega k}\, d\omega \qquad k \in \mathbb{Z}$$

$$\mathrm{DFT}(u) := \sum_{k=-N}^{N} u(k)\, e^{-2\pi i k \omega_n}$$

with $\omega_n \in \left[0, \frac{1}{2N+1}, \frac{2}{2N+1}, \dots, 1\right]$ $\quad \cancel{\#}\ \omega_n \equiv t_n$