Shannon sampling theorem

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$f \in L^2(\mathbb{R})$ and $\operatorname{supp}(\hat{f}) \subset [-B, B]$

then

$$f(t) = \sum_{n \in \mathbb{Z}} \operatorname{sinc}(2B(t - t_n)) f(t_n) \qquad t_n := \frac{n}{2B} \quad n \in \mathbb{Z}$$

$$\operatorname{sinc}(2B(t_m - t_n)) = \delta_{mn}$$

$$\|\hat{f}\|^2_{L^2([-B,B])} = \int_{-B}^{B} \hat{f}^2 \, d\omega = \int_{-\infty}^{+\infty} \hat{f}^2 \, d\omega =$$

$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$$

$$g_n = e^{-i2\pi n\omega}$$

$$g_n \in L^2([-B,B])$$

frequency

$$\langle g_n, \hat{f} \rangle g^n = \hat{f}$$

$$\langle g_n, \hat{f} \rangle g^n$$

$$\langle g_n, g_m \rangle = \delta_{nm} 2B \qquad \langle g^n, g^m \rangle = \frac{1}{2B} \delta^{nm}$$

$$g^n = \frac{1}{2B} g_n$$

$$\langle g_n, \hat{f} \rangle g^n = \underbrace{\int_{-B}^{B} e^{2\pi i n\omega} \hat{f}(\omega) \, d\omega}_{\mathrm{IDTF}(\hat{f}) = f(t_n)} \frac{1}{2B} e^{-2\pi i n\omega}$$

$$\mathrm{IDTF}(\hat{f}) = f(t_n)$$

$$\text{DTFT}: \quad \ell^2(\mathbb{Z}) \longrightarrow L^2(-B,B)$$

$$\{x_n\}_{n\in\mathbb{Z}}$$

or

$$f \in L^2 \cap C^0 \qquad f(t_n) \quad n\in\mathbb{Z}, \quad t_n = \frac{n}{2B}$$

$$\boxed{\hat{f}(\omega) = \frac{1}{2B} \sum_{n\in\mathbb{Z}} f(t_n)\, e^{-2\pi i \omega t_n}}$$

$$(\hat{f})^{\vee}(t) = f(t) = \frac{1}{2B} \sum_{n\in\mathbb{Z}} \int_{-B \;\; -\infty}^{B \;\; +\infty} e^{-2\pi i \omega (t_n - t)}\, d\omega \; f(t_n)$$

$$= \sum_{n\in\mathbb{Z}} \operatorname{sinc}\big(2B\,(t-t_n)\big)\, f(t_n)$$

$2B = R$ Nyquist rate (minimal sampling rate)

Trick for finite signals: make them periodic

($\Rightarrow f(t+T) = f(t)$ $\Rightarrow$ frequency is finite set of frequencies.)

$$g_n = e^{\frac{-2\pi i n k}{N}}$$
$$e_n = e^{\frac{2\pi i n k}{N}}$$

$$\text{DFT}: \quad \mathbb{R}^N \longrightarrow \mathbb{R}_N$$

$$x \longrightarrow \left\{ \langle e_n, x\rangle := \sum_{k=0}^{N-1} x_k\, e^{\frac{-2\pi i n k}{N}} \right\}_{n=0}^{N-1}$$

$$\left\{ (e_n)^k := e^{\frac{-2\pi i n k}{N}} \right\}_{k=0}^{N-1} \in \mathbb{R}^N$$

$$\langle e_n, e_m \rangle = \delta_{nm}\, N$$

$$\mathrm{iDFT}: \quad \mathbb{R}_N \longrightarrow \mathbb{R}^N$$

$$x \longmapsto \langle g^n, x \rangle := \left\{ \frac{1}{N} \sum_k x_k e^{\frac{2\pi i n k}{N}} \right\}_{n=0}^{N-1}$$

$$\mathrm{DTFT}: \quad \ell^2(\mathbb{Z}) \longrightarrow L^2([0,1])$$

$$\mathrm{DTFT}(\mu) = FT(\mu \cdot D) \qquad D := \sum_{z \in \mathbb{Z}} \delta(t-z)$$

$$\mu * D \qquad \mu * \delta \overset{?}{=}$$

$$(\delta_0 * \mu)(t) := \int_{\mathbb{R}} \delta(t-s)\, \mu(s)\, ds = \mu(t)$$

$$(\delta_{t_0} * \mu)(t) := \int_{\mathbb{R}} \delta(t - t_0 - s)\, \mu(s)\, ds = \mu(t - t_0)$$

take μ s.t. $\mathrm{supp}(\mu) \subset [0,T]$

$$D_T := \sum_{n \in \mathbb{Z}} \delta(t - nT)$$

$$D_T * \mu = \sum_{n \in \mathbb{Z}} \mu(t - nT) \qquad \text{periodic of period } T$$

$$\mathcal{F}(D_T * \mu) = \hat{D}_T\, \hat{\mu} \overset{\text{Theorem}}{=} D_{\frac{1}{T}}\, \hat{\mu}$$

DFT : $\mathbb{R}^n \longrightarrow \mathbb{R}_n$

Fast Fourier Transform: Algorithm to compute DFT in $N\log(N)$ time

$\{x_n\} \longrightarrow \{X_n\}$

$\{x_n\} = \{x_{2n}\} + \{x_{2n+1}\}$

$x_{2n} := x_0, 0, x_2, 0, x_4, 0, \ldots$

$x_{2n+1} := 0, x_1, 0, x_3, 0, x_5, \ldots$

Cooley-Tuckey algorith

$$X_n = \sum_{k=0}^{N-1} x_n e^{\frac{-2\pi i n k}{N}}$$

$$= \underbrace{\sum_{k=0}^{\frac{N}{2}-1} x_{2n} e^{\frac{-2\pi i \, nk}{\frac{N}{2}}}}_{FFT(E_k)} + \underbrace{e^{\frac{-2\pi i n}{N}}}_{\text{twiddle factor}} \underbrace{\sum_{k=0}^{\frac{N}{2}-1} x_{2n+1} e^{\frac{-2\pi i n k}{\frac{N}{2}}}}_{FFT(O_k)}$$

for $n \in [0, \frac{N}{2})$

$X_{\frac{N}{2}+n} =$ ___ + corrections on exponents ___

$$X_n = FFT(E_k) + e^{\frac{-2\pi i n}{N}} FFT(O_k)$$

$$X_{\frac{N}{2}+n} = \quad '' \quad - e^{\frac{-2\pi i n}{N}} \quad ''$$