2 feb. 2024

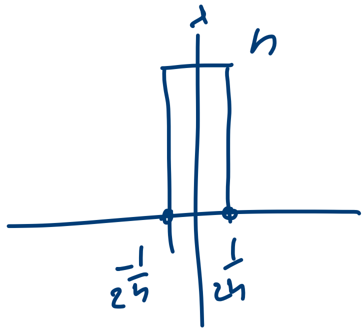
$$\hat{\mu}(\omega) := \int_{\mathbb{R}} \mu(x)\, e^{-i2\pi\omega x}\, dx$$

$$\mu(x) := \delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$ not the right way

$$\int_{\mathbb{R}} \delta f := f(0) \qquad \forall f \in C^0(I(0))$$

Box

$$\delta := \underbrace{\lim_{n\to\infty} \delta^n := n\, \underline{\chi}\left(-\frac{1}{2n}, \frac{1}{2n}\right)}$$



$$\int_{\mathbb{R}} \delta^n = 1$$

$$\hat{\delta}(\omega) = \int_{R} \delta\, \underbrace{e^{-i2\pi\omega x}}_{1_{x=0}}\, dx = 1$$

$\mathcal{F}$: action of taking the Fourier transform

$$\mathcal{F} : L^1 \longrightarrow \; ??$$

$$\mu \longrightarrow \hat{\mu} \in \, ?$$

$$\mathcal{F}(\alpha u + \beta v) = \alpha \mathcal{F}(u) + \beta \mathcal{F}(v)$$

$$\mathcal{F}(u(t-t_0)) := \int_{\mathbb{R}} u(t-t_0)\, e^{-i2\pi\omega(t-t_0 \underset{\sim}{+t_0})}\, dt$$

$$t \to t - t_0$$

$$= e^{-i2\pi\omega t_0}\, \mathcal{F}(u)$$

$$(u * v)(t) := \int_{\mathbb{R}} v(t-z)\, u(z)\, dz$$

$$\mathcal{F}(u * v) = \mathcal{F}(u)\, \mathcal{F}(v)$$

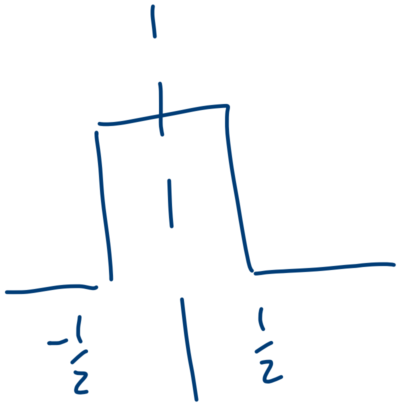
$$u * \delta = u$$

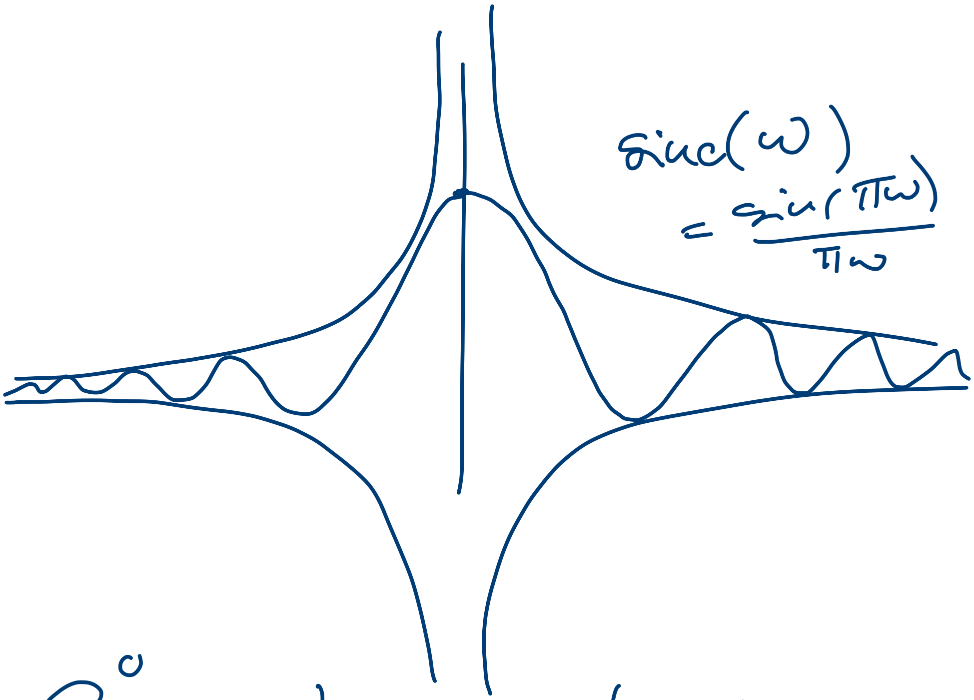
$$\mathcal{F}\left(\chi\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right)(\omega) \overset{?}{=} \int_{\mathbb{R}} \chi\left[-\tfrac{1}{2}, \tfrac{1}{2}\right] e^{-i2\pi\omega t}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i2\pi\omega t}\, dt$$

$$= \frac{1}{-i2\pi\omega} e^{-i2\pi\omega t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = + \frac{\sin(\pi\omega)}{\pi\omega}$$

$$e^{i\alpha} - e^{-i\alpha} = +2i\sin(\alpha)$$





$\mathrm{sinc}(\omega) = \dfrac{\sin(\pi\omega)}{\pi\omega}$

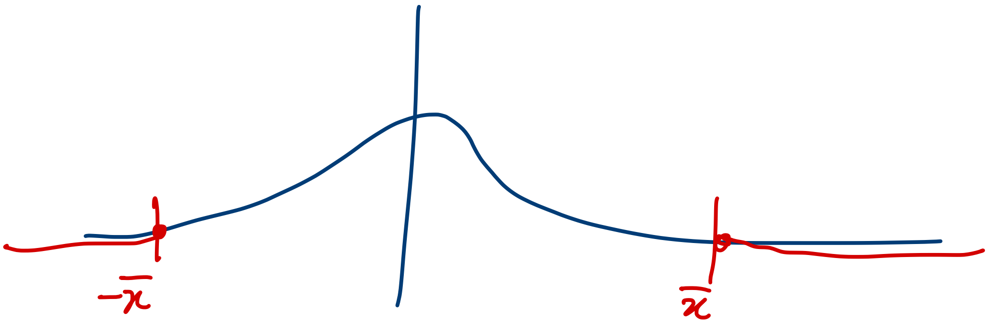
$$\mathcal{F} : L^1 \longrightarrow C_0^0 := \{u \in C^0 \text{ s.t. } u(x) \to 0,\ |x| \to \infty\}$$

$u \in L^1$ then $\exists\ u_n \in C_c^\infty(\mathbb{R})$ s.t.

$\lim_{n\to\infty} \|u_n - u\| \to 0$ $\qquad C_c^\infty$ is dense in L^p, $1 \le p < \infty$

$\Rightarrow$ $1 \le p < \infty$, $\forall u \in L^p$, $\forall \varepsilon > 0$ $\exists\ \bar{x}_{u,\varepsilon}$ s.t.

$$\|u - \chi[-\bar{x}, \bar{x}]u\|_{L^p} \le \varepsilon$$



$u_n \in C_c^\infty$

$\forall \mu_n$ s.t $\lim_{n\to\infty} \|\mu_n - \mu\|_{L^1} = 0$

$$\hat{\mu}_n := \int_{\mathbb{R}} \mu_n e^{-i2\pi\omega t} dt = + \int_{\mathbb{R}} \mu_n' \frac{e^{-i2\pi\omega t}}{+i2\pi\omega} dt$$

$$|\hat{\mu}^n| \leqslant \left|\frac{1}{2\pi\omega}\right| \|\mu_n'\|_{L^1} \Rightarrow |\hat{\mu}^n| \to 0$$

$$|\omega| \to \infty$$

$$\hat{\mu} = \int_{\mathbb{R}} (\mu - \mu^n + \mu^n) e^{-i2\pi\omega t}$$

$$= \underbrace{\int_{\mathbb{R}} (\mu - \mu^n) e^{-i2\pi\omega t}}_{\leqslant \varepsilon} + \underbrace{\int_{\mathbb{R}} \mu^n e^{-i2\pi\omega t}}_{\leqslant \varepsilon}$$

$\exists \bar{n}$ s.t $\forall n \geqslant \bar{n}$ $\|\mu - \mu^n\|_{L^1} < \varepsilon$

$$\leqslant \varepsilon + \varepsilon$$

Along the way we also proved:

$$\mathcal{F}(\mu') = i2\pi\omega \mathcal{F}(\mu)$$

$$\mathcal{F}\left(\int \mu\right) = \frac{1}{i2\pi\omega} \mathcal{F}(\mu) \ldots$$

$$\ddot{u} - c\dot{u} + ku = \delta$$

$$\dot{u}(0) = v_0$$

$$u(0) = u_0$$

$$\mathcal{F}(\ddot{u} - c\dot{u} + ku) = \left(\underline{(i2\pi\omega)^2 - c(i2\pi\omega) + k}\right)\hat{u}$$

$$\mathcal{F}(L^1 \cap L^2) \longrightarrow L^2$$

Hahn Banach $\exists!\ \tilde{\mathcal{F}} : L^2 \longrightarrow L^2$

$\hat{u}(\omega)$ does not make sense pointwise

but

$$\|\hat{u}\|_{L^2} = \|u\|_{L^2}$$ Plancharel equality

$$\int_{\mathbb{R}} \bar{u} f = \int_{\mathbb{R}} \bar{\hat{u}}\, \bar{\hat{f}} \qquad \forall u, f \in L^2$$

$e_n \in L^2(t_0, t_0+T)$

Fourier Series: $\quad \langle e_n, f \rangle e^n = \langle e^n, f \rangle e_n$

$$\langle e_n, e_m \rangle = T\,\delta_{nm} \qquad e_\omega(t) := e^{i2\pi\omega t}$$

$$\langle e_\omega, f \rangle := \int_R \bar{e}_\omega f = \hat{f}(\omega) \in L^2$$

$$(\hat{f})^\vee := \int_R \hat{f}(\omega)\, e^{i2\pi\omega t}\, d\omega$$

$$= \int_R \int_R f(t)\, e^{-i2\pi\omega t}\, dt\, e^{i2\pi\omega t}\, d\omega$$

$$= \int_R \underbrace{\int_R e^{-i2\pi\omega t}\, e^{i2\pi\omega t}\, d\omega}_{\text{careful !!}}\, f(t)\, dt$$

$$e^{-i2\pi\omega t} \notin L^2(R)$$

$$f(t) : \underset{\subset \mathbb{R}}{[0,T]} \longrightarrow \mathbb{R}, \mathbb{C}\mathbb{C}$$

Audio signal of a song (pressure value on ear drum)

$$\hat{f}(\omega) : \int_0^T f(t)\, e^{-i2\pi\omega t}\, dt$$