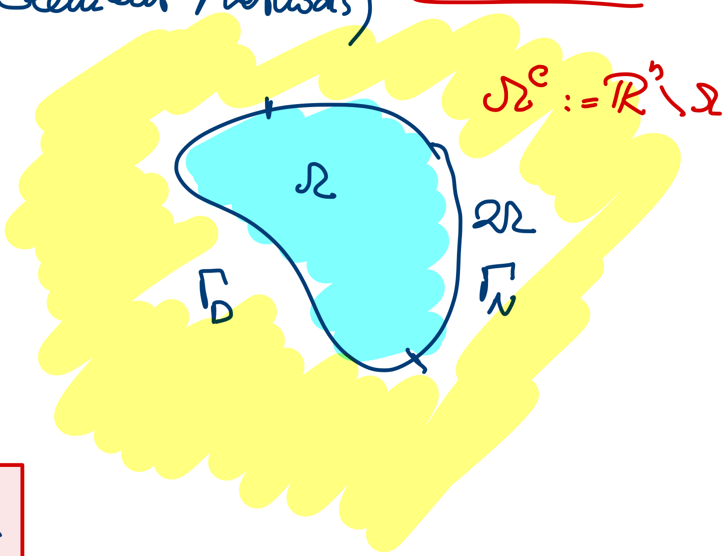
Principles of BEM (Boundary Element Methods) 20.4.23



$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g_D & \text{on } \Gamma_D \\ \dfrac{\partial u}{\partial n} = g_N & \text{on } \Gamma_N \end{cases}$$

$$\boxed{\text{look for a } \underline{\text{fundamental solution}}}$$

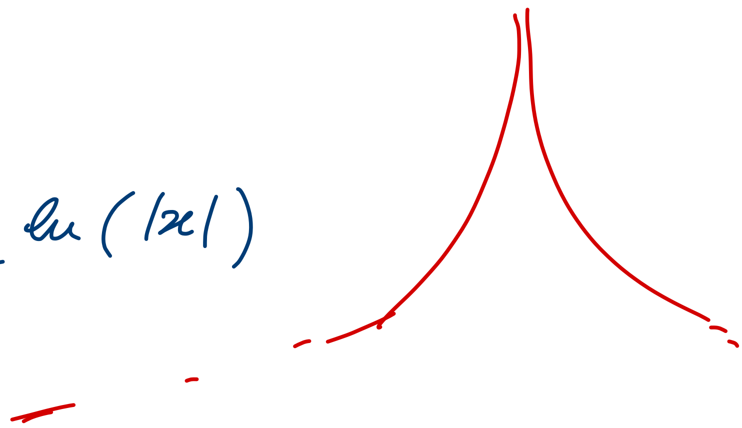
$$-\Delta G = \delta \qquad \text{in } \mathbb{R}^n$$

Dirac delta distribution

Fundamental solution

$$\left(\lim_{|x|\to\infty} G = 0 \qquad \text{in } \mathbb{R}^3 \right)$$

$$n = 3 \quad \rightsquigarrow \quad G(x) = \frac{1}{4\pi|x|}$$



$$n = 2 \quad \rightsquigarrow \quad G(x) = -\frac{1}{2\pi} \ln(|x|)$$

Trick:

$$\int_\Omega -\Delta_x u(x)\, G(y-x)\, dx = 0$$

$$\int_\Omega \nabla_x u \, \nabla_x G(y-x) dx - \int_{\partial\Omega} (\nabla_x u)(x)\, n(x)\, G(y-x)\, dS_x = 0$$

$$\underbrace{\int_\Omega u(x)\underbrace{\left(-\Delta_x G(y-x)\right)}_{\delta(y-x)}dx}\ + \int_{\partial\Omega} u(x)\ \nabla_x G(y-x)\, n(x)\, dS_x$$

$$-\int_{\partial\Omega} \nabla_x u(x)\cdot n(x)\ G(y-x)\, dS_x = 0$$

$$\boxed{u(y) = \int_{\partial\Omega}\frac{\partial u}{\partial n}\, G\, dS - \int_{\partial\Omega} u\ \frac{\partial G}{\partial n}\, dS} \qquad \forall y \in \mathring{\Omega}$$

$$\Downarrow$$

$$u(y) = \underline{\underline{\int_{\Gamma_D}\frac{\partial u}{\partial n}\, G\, dS}} + \underbrace{\int_{\Gamma_N} g_N\, G}_{\text{known}} - \underbrace{\int_{\Gamma_D} g_D \frac{\partial G}{\partial n}}_{\text{known}} - \underline{\underline{\int_{\Gamma_D} u\,\frac{\partial G}{\partial n}\, dS}}$$

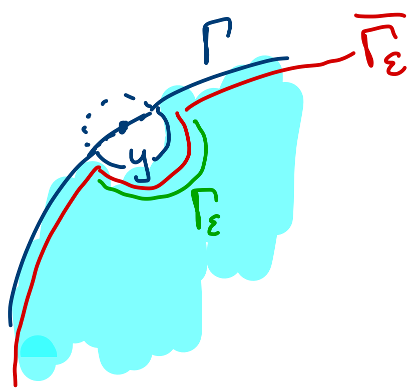
$$\lim_{y\to y_0\in\Gamma}\qquad \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\ \vdash\!\!-\!\!\dashv \qquad \Gamma_\varepsilon :=$$

$$\left(PV\int_{\Gamma_N}\right)(y) := \left(\lim_{\varepsilon\to 0}\int_{\Gamma_{\varepsilon(y)}}\right)(y)$$

$$\frac{\partial G}{\partial n}\ ,\ G \quad \text{on } \Gamma \setminus B_\varepsilon \qquad \underline{\text{is ok}}\ \ \forall \varepsilon$$

y on Γ we have

$\Gamma_\varepsilon := \partial B_\varepsilon \cap \Omega$



$\Gamma \backslash B_\varepsilon \cup \Gamma_\varepsilon = \overline{\Gamma_\varepsilon}$

$$\int_{\Gamma_\varepsilon} G \frac{\partial u}{\partial n} \qquad \int_{\Gamma_\varepsilon} \frac{\partial G}{\partial n} \mu \qquad \nabla G : -\frac{x-y}{|x-y|^{n-1}}$$

$\downarrow$ $\quad c_1 = 4$ in 3d, 2 in 2d $\quad n = \frac{x-y}{\underbrace{|x-y|}_{\varepsilon}}$ $\quad x \in \Gamma_\varepsilon$

$$\frac{1}{4\pi}\int_{\Gamma_\varepsilon} \frac{1}{\varepsilon} \frac{\partial u}{\partial n} \qquad \frac{1}{c_1 \pi} \int_{\Gamma_\varepsilon} \frac{\varepsilon^2}{\varepsilon^n} \mu \, dS \qquad \frac{\partial G}{\partial n}\Big|_{\Gamma_\varepsilon} := \frac{\varepsilon^2}{\varepsilon^n}$$

$\frac{1}{2\pi} - \ln(\varepsilon)$

$$\int_{\Gamma_\varepsilon} \underset{\uparrow \, \mu(y)}{\mu} \, dS$$

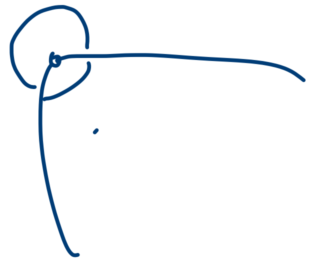
$$\boxed{\alpha := \frac{\int_{\Gamma_\varepsilon} dS}{\int_{\partial B_\varepsilon} dS}} \quad (y)$$

$$\int_{\partial B_\varepsilon} dS = \begin{matrix} 2\pi\varepsilon & \text{in 2D} \\ 4\pi\varepsilon^2 & \text{in 3D} \end{matrix}$$

PV of the Neumann part contains:

$$PV \int_{\Gamma_\varepsilon} \frac{\partial G}{\partial n} \mu = \lim_{\varepsilon \to 0} \int_{\Gamma \backslash B_\varepsilon} \frac{\partial G}{\partial n} \mu + \alpha \mu(y)$$

$c = \frac{1}{2}$ a.e. for Lip. Boundaries.



$$\alpha(y) = \begin{cases} 1 & \text{in } \Omega \\ 0 & \text{in } \Omega^c \\ \frac{1}{2} & \text{on } \partial\Omega \text{ smooth} \end{cases}$$

$$u(y)\left(1-\alpha(y)\right) = \mathrm{Pv}\int_\Gamma \frac{\partial u}{\partial n} G - \int_\Gamma \frac{\partial G}{\partial n} u$$

Say the same for Ω^c with $n^c = -n$

$$\alpha^c = 1 - \alpha$$

By summing:

$[u] : u^+ - u^-$

$$\boxed{\{u\}_\alpha = \int_\Gamma \left[\frac{\partial u}{\partial n}\right] G - \int_\Gamma [u] \frac{\partial G}{\partial n}}$$

$$\{u\}_\alpha := \begin{cases} \alpha u + (1-\alpha) u^c & \text{if } y \in \Gamma \\ u & \text{if } y \in \Omega \\ u^c & \text{if } y \in \Omega^c \end{cases}$$

$$\underline{u} = \begin{bmatrix} g_0 \\ u \end{bmatrix} \qquad \frac{\partial \underline{u}}{\partial n} = \begin{bmatrix} \frac{\partial u}{\partial n} \\ g_N \end{bmatrix}$$

$$\varphi := \begin{bmatrix} \frac{\partial u}{\partial n} \\ u \end{bmatrix} \begin{matrix} \text{on } \Gamma_D \\ \text{on } \Gamma_N \end{matrix} \qquad \psi := \begin{bmatrix} g_D \\ g_N \end{bmatrix}$$

$$W := \begin{bmatrix} G \\ \frac{\partial G}{\partial n} \end{bmatrix} \begin{matrix} \text{on } \Gamma_D \\ \text{on } \Gamma_N \end{matrix}$$

$$\begin{pmatrix} \frac{1}{2} & \\ & \end{pmatrix} \begin{pmatrix} \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \int_{\Gamma_D} \frac{\partial G}{\partial n} \mu + \int_{N_N} G \frac{\partial u}{\partial n} \\ \int_{\Gamma_N} \frac{\partial G}{\partial n} \mu \qquad \int_{\Gamma_D} G \frac{\partial u}{\partial n} \end{pmatrix} =$$

$$\begin{pmatrix} \int_{\Gamma_D} \frac{\partial G}{\partial n} g_D + \int_{N_N} G \, g_N \\ \int_{\Gamma_N} \frac{\partial G}{\partial n} g_D \qquad \int_{\Gamma_D} G \, g_N \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} g_D \end{pmatrix}$$

Neumann problem: g_N is known

$$\boxed{(1-\alpha)\mu + \int_\Gamma \frac{\partial G}{\partial n} \mu = \int_\Gamma G \, g_N} \leftarrow \quad \mu = \sum_{i \in [\![N]\!]} \varphi_i \mu^i$$

$*$

$$y_j \text{ s.t. } \varphi_i(y_j) = \delta_{ij}$$

evaluate $\ast$ on y_J and find u^i:

$$(1-\alpha)\, u^i \varphi_i(y_J) + \int_\Gamma \frac{\partial G}{\partial n}(x-y_J)\, u^i \varphi_i(x)\, dS_x = \int_\Gamma G\, g_N\, dS$$

$$A_{Ji}\, u^i \qquad A_{Ji} := \boxed{\Big(1-\alpha(y_J)\Big)\delta_{iJ} + \int_\Gamma \frac{\partial G}{\partial n}(x-y_J)\, \varphi_i(x)\, dS_x}$$

φ_i: