Master in High Performance Computing

Continuum mechanics and fluid-structure interaction problems: mathematical modelling and numerical approximation

Hilbert spaces, notations, short recap on FEM

Luca Heltai <luca.heltai@sissa.it>

International School for Advanced Studies (www.sissa.it)
Mathematical Analysis, Modeling, and Applications (math.sissa.it)
Master in High Performance Computing (www.mhpc.it)
SISSA mathLab (mathlab.sissa.it)
King Abdullah University of Science and Technology (kaust.edu.sa)

جامعة الملك عبدالله للعلوم والتقنية
King Abdullah University of Science and Technology

ICTP The Abdus Salam International Centre for Theoretical Physics

SISSA

Summary of previous lecture

V vector space of dimension n $\Leftrightarrow$ $V = \text{span}\{\underline{e}_i\}_{i=1}^{n}$ basis vectors

$V^* \equiv \mathcal{L}(V, \mathbb{R}) = \text{span}\{\bar{e}^i\}_{i=1}^{n}$ co-vectors

if $\boxed{\bar{e}^i(\underline{e}_J) = \delta^i{}_J = \begin{cases} 1 & i = J \\ 0 & i \neq J \end{cases}}$ then $\bar{e}^i$ is the canonical dual-basis

$\forall\, \underline{v} \in V \quad \exists! \quad \{v^i\}_{i=1}^{n} \quad \text{s.t.} \quad \underline{v} = v^i \underline{e}_i$

$\forall\, \bar{w} \in V^* \quad \exists! \quad \{w_i\}_{i=0}^{n} \quad \text{s.t.} \quad \bar{w} = w_i\, \bar{e}^i$ (sum convention)

$$\boxed{\begin{array}{ll} \underline{v} = \bar{v}^i(\underline{v})\, \underline{v}_i = v^i \underline{v}_i & v^i = \bar{v}^i(\underline{v}) \\ \bar{w} = \underline{v}_i(\bar{w})\, \bar{v}^i = w_i \bar{v}^i & w_i = \underline{v}_i(w) \end{array}}$$

$V^{**} \equiv V$

Tensor product. $\otimes$

Given $\underline{u} \in V$, $\overline{w} \in W^*$

$$\underline{u} \otimes \overline{w} \in \mathcal{L}(W, V) \equiv V \otimes W^*$$

$$V \ni v = \left(\underline{u} \otimes \overline{w}\right) \underline{\omega} = \underline{u}\ \overline{w}(\underline{\omega}) \equiv \underline{u} < \overline{w}, \underline{\omega} > \quad \forall\, \underline{\omega} \in W$$

$$\underline{e}_i \otimes \overline{e}^i\ \underline{v} = \overline{e}^i(\underline{v})\ \underline{e}_J = \underline{v} \quad \Rightarrow \quad \boxed{\mathbb{I} := \underline{e}_i \otimes \overline{e}^i}$$

$$\forall\ T \in \mathcal{L}(W = \text{span}\{\underline{e}_i\}, V = \text{span}\{E_\alpha\}), \quad \exists! \{T^\alpha{}_i\}_{\alpha=1,\, J=1}^{n_V \quad n_W}$$

s.t. $\quad T = T^\alpha{}_i\, E_\alpha \otimes \overline{e}^i$

$$T = \overline{E}^\alpha(T(\underline{e}_i))\, E_\alpha \otimes \overline{e}^i \qquad \boxed{T^\alpha{}_i = \overline{E}^\alpha(T(\underline{e}_i))} \in \mathbb{R}^{n_V \times n_W}$$

Adjoint

Transpose

Later

$$T : W \rightarrow V$$

$$T^* : V^* \rightarrow W^* \qquad T^* = T^\alpha{}_i\ \overline{e}^i \otimes E_\alpha = (T^*)_i{}^\alpha\ \overline{e}^i \otimes E_\alpha$$

Hilbert Case

If V is Hilbert, $(u, v) = u \cdot v$ denotes a ~~scalar~~ inner product

$$\|u\|_V^2 = (u, u) = u \cdot u$$

Riesz Representation Theorem:

$$\forall\ \overline{v} \in V^* \quad \exists!\ \underline{v} \in V \quad \text{s.t.} \quad \overline{v} = \tau \underline{v}$$

τ is invertible

$$\langle \overline{v}, \underline{\omega} \rangle = \overline{v}(\underline{\omega}) = (\underline{v}, \underline{\omega}) = \underline{v} \cdot \underline{\omega} \qquad \forall\, \underline{\omega} \in V$$

$\Rightarrow$ defines automatically inner product on V^*: $\langle\langle \overline{u}, \overline{\omega} \rangle\rangle$

$$\langle\langle \overline{v}, \overline{\omega} \rangle\rangle := (\tau^{-1}\overline{v}, \tau^{-1}\overline{\omega}) = (\underline{v}, \underline{\omega}) \qquad \forall\ \overline{v}, \overline{w} \in V^*$$

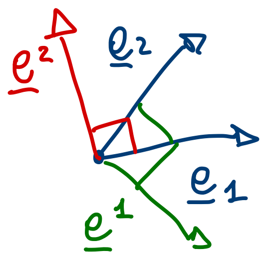
Two new sets of basis

$$\underline{e}_i \in V, \quad \overline{e}^i \in V^* \quad \Rightarrow \quad \overline{e}_i \in V^*, \quad \underline{e}^i \in V$$

primal $\qquad$ dual $\qquad$ reciprocal dual $\qquad$ reciprocal

$$\bar{e}^i(\underline{e}_j) = {}_{V^*}\langle \bar{e}^i, \underline{e}_j \rangle_V = (\underline{e}^i, \underline{e}_j) = \underline{e}^i \cdot \underline{e}_j = \delta^i{}_j$$

$$\underline{e}_i(\bar{e}^j) = {}_{V^{**}}\langle \underline{e}_i, \bar{e}^j \rangle_{V^*} = \langle\langle \bar{e}_i, \bar{e}^j \rangle\rangle = \delta_i{}^j \qquad V^{**} \equiv V$$



$$\underline{e}^2 \cdot \underline{e}_1 = 0 \qquad \underline{e}^2 \cdot \underline{e}^1 \neq 0$$

$$\underline{e}^1 \cdot \underline{e}_2 = 0$$

$$\underline{V} = \underline{\underline{v^i}}\, \underline{e}_i = (\underline{V} \cdot \underline{e}^i)\, \underline{e}_i$$

Let's assume that there exist matrices g^{ij} and g_{ij} s.t.

$$\underline{e}^i = g^{ij} \underline{e}_j \qquad \text{and} \qquad \underline{e}_i = g_{ij} \underline{e}^j$$

$$\underline{e}^j \cdot \underline{e}^i = g^{jk} \overbrace{\underline{e}_k \cdot \underline{e}^i}^{\delta_k{}^i} = g^{ji} =: g^{ij}$$

$$\underline{e}_j \cdot \underline{e}_i = g_{jk} \underbrace{\underline{e}^k \cdot \underline{e}_i}_{\delta^k{}_i} = g_{ji} =: g_{ij}$$

g^{ij} symmetric matrix, invertible

$$g_{ij} := \underline{e}_i \cdot \underline{e}_j \qquad \underline{e}^i \cdot \underline{e}_j = \delta^i{}_j$$

$$g^{ij} := \underline{e}^i \cdot \underline{e}^j \qquad \underline{e}^i \cdot \underline{e}_j = \delta^i{}_k\, g_{kj} \underline{e}^j$$

$$\boxed{\begin{aligned} g^{ij} g_{jk} &= \delta^i{}_k \\ g_{ij} g^{jk} &= \delta_i{}^k \end{aligned}}$$

Allows us to "raise an index" or "lower an index"

$$\underline{V} = v^i \underline{e}_i = v_i \underline{e}^i \qquad v^i = g^{ij} v_j$$

$$v_i = g_{ij} v^j$$

$$v^i = g^{ij} v_j \qquad \underline{e}_i \cdot \underline{e}_j =: g_{ij}$$

$$v_i = g_{ij} v^j \qquad \underline{e}^i \cdot \underline{e}^j =: g^{ij}$$

$$\underline{e}_i = g_{ij} \underline{e}^j$$

$$\underline{v} \cdot \underline{w} = (v^i \underline{e}_i) \cdot (w^j \underline{e}_j)$$

$$\underline{e}^i = g^{ij} \underline{e}_j$$

$$= v^i w^j g_{ij}$$

g_{ij} is called <u>metric matrix</u>

g is the metric tensor : $g = g_{ij} \underline{e}^i \otimes \underline{e}^j$

Slight change of notation $\boxed{\text{Hilbert } \underline{\text{case}} \quad \otimes}$

given $\underline{u} \in V$ and $\underline{w} \in W$ <u>both</u> Hilbert.

$\boxed{\text{usually}}$ $\quad \underline{u} \otimes \underline{w}$ is interpreted as

$$(\underline{u} \otimes \underline{w})\, \underline{v} = \underline{u}\, (\underline{w} \cdot \underline{v})$$

$$\underline{u} \otimes \underline{w} \in \mathcal{L}(W, V) \quad \underline{\text{not}} \quad \underline{\underline{\mathcal{L}(W^*, V)}}$$

$\boxed{\text{Transpose}}$

$$T : V \longrightarrow W$$

$$T^T : W \longrightarrow V \qquad \tau \circ T^T = T^*$$

$$T^* : W^* \longrightarrow V^* \qquad T^T = \tau^{-1} \circ T^*$$

If $V \equiv W$

$$T = T^{ij} \underline{e}_i \otimes \underline{e}_j = T^i{}_j \underline{e}_i \otimes \underline{e}^j = T_i{}^j \underline{e}^i \otimes \underline{e}_j = T_{ij} \underline{e}^i \otimes \underline{e}^j$$

contravariant indices   mixed   covariant.

$$T : V \longrightarrow V$$

$$T = T^T \quad \Rightarrow \quad T \text{ is symmetric}$$

$$T = T^{ij}\, \underline{e}_i \otimes \underline{e}_j \qquad T^T = T^{ij}\, \underline{e}_j \otimes \underline{e}_i$$

$$= (T^T)^{ji}\, e_j \otimes e_i$$

$$T^{ij} = (T^T)^{ji}$$

$$T = T^i{}_j\, \underline{e}_i \otimes \underline{e}^j \qquad T^T = T^i{}_j\, \underline{e}^j \otimes \underline{e}_i$$

$$T^T = (T^T)_j{}^i\, \underline{e}^j \otimes \underline{e}_i$$

$$(T^T)_j{}^i = T^i{}_j$$

Multilinear maps — Tensors of type (r,s)

r copies of V

s copies of V^* (or $\tau^{-1} V^*$)

$(0,0)$ scalars

$(1,0)$ vectors — order : $r+s$

$(0,1)$ co-vectors.

$(s,0)$ covariant tensors

$(0,s)$ contravariant tensor.

$T^{ij}{}_{kl}{}^{mn}$ 6th order mixed tensor of type $(4,2)$

$$T^{ij}{}_{kl}{}^{mn} \; \underline{e}_i \otimes \underline{e}_j \otimes \underline{\underline{e}}^k \otimes \underline{\underline{e}}^l \otimes \underline{e}_m \otimes \underline{e}_n$$

Tensor of order 2 $\quad \mathcal{L}(V, V) \;,\; \mathcal{L}(V \times V, \mathbb{R})$

" $\quad$ 3 $\quad \mathcal{L}(V \times V, V) \;,\; \mathcal{L}(V, V \times V) \;,$

$$\mathcal{L}(V \times V \times V, \mathbb{R})$$

2nd order : $\quad F : V \longrightarrow V$

and 4th order $\quad \mathbb{C} : V \times V \longrightarrow V \times V$

$$\mathbb{C} : \underline{\mathcal{L}(V, V) \longrightarrow \mathcal{L}(V, V)}$$

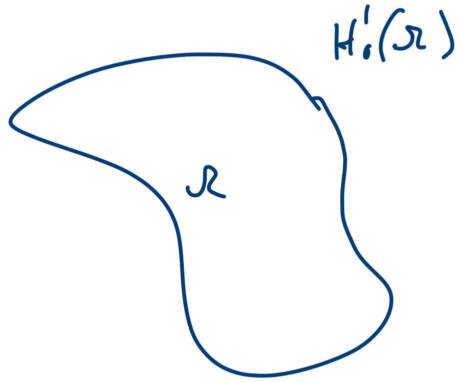
$\mathcal{L}(V \times V) \qquad \mathcal{L}(V, V)$

$$S : T_1 \longrightarrow T_2$$

$T_1 : V \longrightarrow V \qquad T_2 : V \longrightarrow V$

$$S_{ij}{}^{kl} = T_{1\,ij} \, T_2{}^{kl}$$

V: infinite dimensional $\qquad V := H_0^1(\Omega)$



$$A : V \longrightarrow V^* \equiv H^{-1}$$

$$\langle Au, v \rangle := (\nabla u, \nabla v)$$

$$F \in V^* \qquad \langle F, v \rangle \in \mathbb{R}$$

$$V_h = \operatorname{span}\{\underline{\varphi_i}\}_{i=1}^{n}$$

$$u_h = u^i \underline{\varphi_i}$$

$$\langle A\, u^j \underline{\varphi_j}, \varphi_i \rangle = \langle f, \underline{\varphi_i} \rangle$$

$$A_{ij} u^j = f_i \qquad\qquad u^i f_i = \langle f, u_h \rangle$$

Scalar product L^2: $(u, v) := \int_\Omega uv$

$$g_{ij} := (\underline{\varphi_i}, \underline{\varphi_j}) \quad \text{mass matrix}$$

$$f^i \underline{\varphi_i} = g^{ij} f_j \underline{\varphi_i} \equiv M^{-1} \text{rhs}$$

$$u^i u^i \qquad\qquad |\{u^i\}|_{\ell^2}^2$$

$$u^i g_{ij} u^j \equiv \int_\Omega u\, u$$

$$\underline{S^{ijklm}} = T^{ijk} G^{lm} = G^{lm} T^{ijk}$$

$$S : S^{ijklm} \, e_i \otimes e_j \otimes e_k \otimes e_l \otimes e_m$$

$$T^{ijk} G^{lm} \, e_i \otimes e_j \otimes e_k \otimes e_l \otimes e_m \equiv G^{lm} T^{ijk} \, e_i \otimes e_j \otimes e_k \otimes e_l \otimes e_m$$