Master in High Performance Computing

Continuum mechanics and fluid-structure interaction problems: mathematical modelling and numerical approximation

Kinematics - part II

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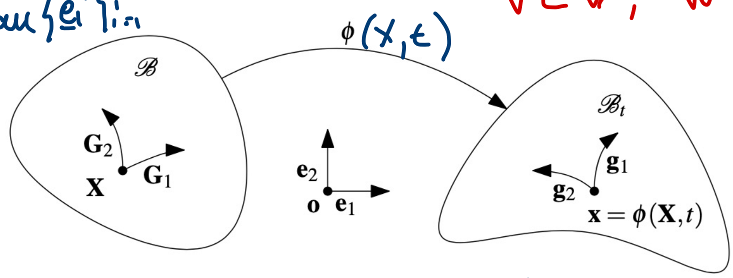
Recap

$G_\alpha, \; g_\alpha \in \mathbb{V} = \text{span}\{e_i\}_{i=1}^{d}$

$V \subset \mathbb{V}, \quad W \subseteq \mathbb{V}$



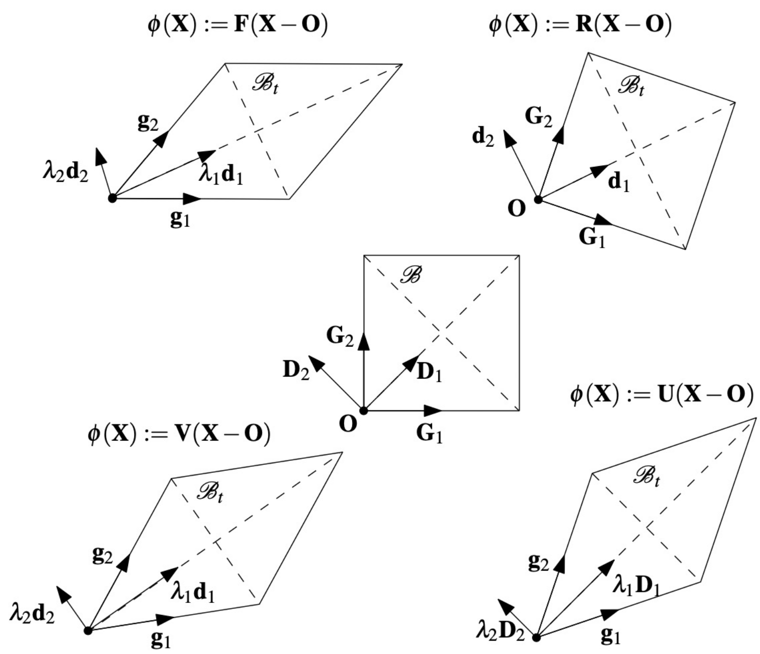
- $\underline{X} = X^\alpha \underline{G}_\alpha$
- $\underline{\phi} = \phi^i \underline{e}_i = X^\alpha \underline{g}_\alpha$

$W = TB, \quad V = TB_t = T\phi(B_t)$

$\underline{F} = \text{Grad}\,\underline{\phi} = \frac{\partial \phi}{\partial X^\alpha} \underline{G}^\alpha$

$= F^i_\alpha \underline{G}^\alpha \otimes \underline{e}_i$

$= \delta^\beta_\alpha \underline{G}^\alpha \otimes \underline{g}_\beta = \underline{G}^\alpha \otimes \underline{g}_\beta$



$C := F^T F = C_{\alpha\beta} \underline{G}^\alpha \otimes \underline{G}^\beta$

$B := F F^T = B^{\alpha\beta} \underline{g}_\alpha \otimes \underline{g}_\beta$

$= b^{ij} \underline{e}_i \otimes \underline{e}_j$

Polar dec. theorem: $\exists!\; R, U, V$ s.t. $F = RU = VR$

$U^2 = F^T F, \quad V^2 = F F^T \qquad U = \sum_\alpha \left(\lambda_\alpha \underline{D}_\alpha \otimes \underline{D}_\alpha\right) \qquad \mathcal{B} = \sum_\alpha \left(\lambda_\alpha \underline{d}_\alpha \otimes \underline{d}_\alpha\right)$

$\forall F,\ \exists!\ R, U, V$ s.t. $F = RU = VR$

$R : W \longrightarrow V \qquad R^{i}{}_{\alpha}$

$U : W \longrightarrow W \qquad U_{\alpha\beta}$

$V : V \longrightarrow V \qquad V^{ij}$

Generally $\exists\ \tilde{R} : \mathcal{V} \longrightarrow \mathcal{V}$, $\det(\tilde{R}) = +1 \qquad \tilde{R}^{-1} = \tilde{R}^T$

$\boxed{\tilde{R} = -\tilde{R}^T}$

R is restriction of $\tilde{R}$ to $W \longrightarrow V$, i.e.

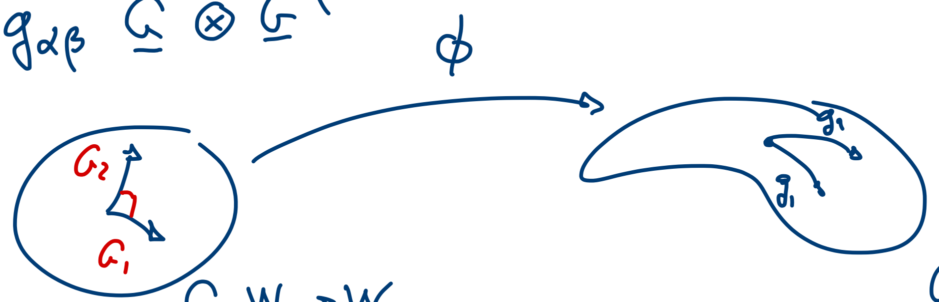
$\forall\ \underline{v} \in \mathcal{V} \quad \mathcal{V} \ni \tilde{R}\underline{v} = \underline{v} = Rv \in V$

$C = F^T F = U^2 \qquad C_{\alpha\beta} = (F^T F)_{\alpha\beta}$

$\underline{g}_\alpha = F \underline{G}_\alpha \rightarrow \underline{g}_\alpha \cdot \underline{g}_\beta = g_{\alpha\beta} = (F\underline{G}_\alpha)\cdot(F G_\beta)$

$= G_\alpha \cdot (F^T F) G_\beta$

$C = \overset{\underline{g}_\alpha \cdot \underline{g}_\beta}{g_{\alpha\beta}}\ \underline{G}^\alpha \otimes \underline{G}^\beta$



$C : W \rightarrow W$

$\underline{G}_\alpha \cdot \underline{G}_\beta$

$B = FF^T \rightarrow (FF^T)^{\alpha\beta}\ \underline{g}_\alpha\ \underline{g}_\beta = G^{\alpha\beta}\ \underline{g}_\alpha \otimes \underline{g}_\beta$

Green-Lagrangian Strain:

$\underline{\underline{E}} := \frac{1}{2}(C - I_W) = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{G}}) = \frac{1}{2}(g_{\alpha\beta} - G_{\alpha\beta})\ \underline{G}^\alpha \otimes \underline{G}^\beta$

$\underline{\underline{E}} : W \longrightarrow W$

Euler - Almansi Strain:

$$\underline{\underline{e}} := \frac{1}{2}\left(\underline{\underline{I}}_V - \underline{\underline{B}}\right) = \frac{1}{2}\left(\underline{\underline{g}} - \underline{\underline{B}}\right) = \frac{1}{2}\left(g_{\alpha\beta} - G_{\alpha\beta}\right) \underline{g}^{\alpha} \otimes \underline{g}^{\beta}$$

Deformation $\underline{W}$: $\quad \underline{\phi}(\underline{X},t) := \underline{X} + \underline{W}(\underline{X},t)$

$$\varepsilon(W) := \frac{1}{2}\left(\mathrm{Grad}\, W^T + \mathrm{Grad}\, W\right)$$

$$F = I_W + \mathrm{Grad}\, \underline{W}$$

$$C = F^T F = I_W + \mathrm{Grad}\, \underline{W}^T + (\mathrm{Grad}\, \underline{W}) + (\mathrm{Grad}\, \underline{W})^T (\mathrm{Grad}\, \underline{W})$$

$$B = F F^T = I_W + \mathrm{Grad}\, \underline{W} + (\mathrm{Grad}\, \underline{W})^T + (\mathrm{Grad}\, \underline{W})(\mathrm{Grad}\, \underline{W})^T$$

$$\underline{\underline{E}} := \varepsilon(W) + \frac{1}{2}\mathrm{Grad}(W)^T(\mathrm{Grad}\, W)$$

$$\underline{\underline{e}} := \varepsilon(W) + \frac{1}{2}\mathrm{Grad}(W)(\mathrm{Grad}\, W)^T$$

Two perspectives:

1. Lagrangian: fixed $\underline{X} \in B$, varying t

(capital letters)

2. Eulerian Representation: fixed $\underline{x}$ in $\mathbb{R}^d$

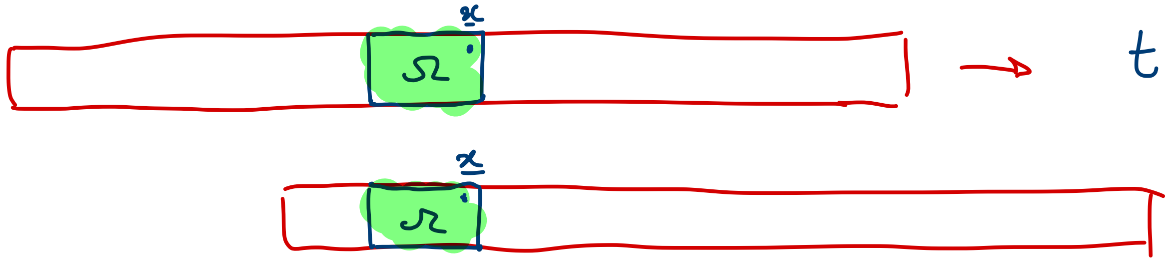
at some t, $\forall \underline{x} \in B_t \subset \mathbb{R}^d \quad \exists! \; \underline{X} \in B$ s.t.

$$\underline{x} = \phi(\underline{X},t) \rightarrow \boxed{\underline{X} = \phi^{-1}(\underline{x},t)}$$

Let's assume that $\Omega \subset B_t \quad \forall t \in [0,T]$

Ω is a spatial control Volume. $\Omega \subset \mathbb{R}^d$

FIXED



$\to \quad t = 0$

$t = T$

$\underline{x}$ in Ω is a spatial point (Eulerian)
fixed in time

$\Omega \subset B_t \quad \forall \quad t \in [0,T] \implies \forall\, t \in [0,T] \ \exists! \underline{X}$ s.t.

$$\underline{x} = \phi(\underline{X}, t)$$

Field φ on $\underline{x} \in \Omega$ is an Eulerian field, or spatial field
if it describes physical properties of whatever particle appears to be
in $\underline{x}$ at time t.

- Temperature field. $\theta : \Omega \longrightarrow \mathbb{R}$
  $\forall t$, it is the temperature of the material point $\underline{X}$
  that happens to be in $\underline{x} = \phi(\underline{X}, t)$
  $$\implies \Theta(\underline{X}, t) = \theta(\phi(X, t), t)$$

- Time derivatives at fixed Material particles are
  called Material time derivatives, and are indicated
  with "$\cdot$" dot notation

- Time derivatives at fixed spatial point are
  called time derivatives, and are indicated
  with "$'$" prime notation or "∂_t"

"Material" velocity of particle $\underline{X}$

- $\dot{\phi}(X,t) := \dfrac{\partial \phi}{\partial t}(X,t) = U(X,t)$

"Material" acceleration of particle $\underline{X}$

- $\ddot{\phi}(X,t) = \dfrac{\partial}{\partial t} U(X,t) = \dfrac{\partial^2 \phi}{\partial t^2}(X,t)$

$U : B \times [0,T] \longrightarrow \mathbb{R}^d$

$u : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$

Eulerian velocity field.

- $u(\phi(X,t),t) = \dot{\phi}(X,t) = \dfrac{\partial}{\partial t}\phi(X,t) = U(X,t)$
- $a(\phi(X,t),t) = \ddot{\phi}(X,t) = \dfrac{\partial^2}{\partial t}\phi(X,t) = A(X,t)$

$$a \circ \phi = \frac{d}{dt} u(\phi(X,t),t) = \frac{\partial}{\partial t} u(\phi(X,t),t) + \operatorname{grad}(u) \cdot u \Big|_{x=\phi(X,t)}$$

$$u = \frac{\partial \phi}{\partial t} \circ \phi^{-1} \qquad a = \frac{\partial}{\partial t}\left(u \circ \phi\right) \circ \phi^{-1}$$

$$\equiv$$

$$\dot{u} := \frac{\partial}{\partial t} u + (\operatorname{grad} u)\, u$$

In general: for spatial field $\varphi : \Omega \times [0,T] \longrightarrow \mathbb{R}$

$$\dot{\varphi} := \frac{\partial}{\partial t}\varphi + (\operatorname{grad} \varphi) \cdot u := \left[\frac{\partial}{\partial t}(\varphi \circ \phi)\right] \circ \phi^{-1}$$

$\cancel{u \cdot \nabla u}$

- $u := \dot{\phi} \circ \phi^{-1} = U \circ \phi^{-1}$
- $a := \dot{u} := \ddot{\phi} \circ \phi^{-1} = \dfrac{\partial}{\partial t} u + (\operatorname{grad} u)\, u$
- $a := A \circ \phi^{-1}$

$\dot{\varphi}$: material time derivative

Grad: $\frac{\partial}{X}$ $\in TB$ Grad or ∇_X

grad: $\frac{\partial}{\partial x}$ $\in V$ grad or ∇

$$\dot{u} = \partial_t u + (\nabla u)\, u \qquad u : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$$

Theorem

$$\boxed{\dot{J} = J\, \mathrm{div}(u)} \qquad \dot{\left(J \circ \phi^{-1}\right)} = \left(J \circ \phi^{-1}\right) \mathrm{div}(u)$$

$$\boxed{J \circ \phi = \mathcal{J}}$$

Reynolds Transport Theorem

Let $\tilde{\Omega}(t)$ be a time dependent domain, with $v(x,t)$ a spatial field describing its velocity on the boundary: $\partial\tilde{\Omega}(t)$ moves with velocity $v(x,t)$

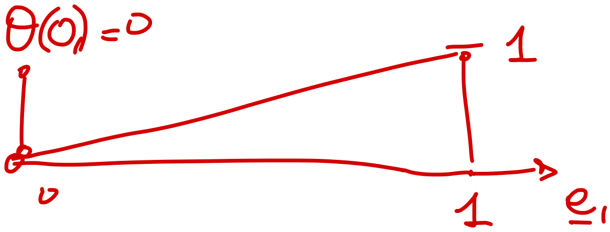
Let $\varphi : \Omega \times [0,T] \longrightarrow \mathbb{R}$ be a spatial field

Let $n(x,t)$: normal to $\partial\tilde{\Omega}(t)$

$$\boxed{\frac{d}{dt} \int_{\tilde{\Omega}(t)} \varphi(x,t)\, dx = \int_{\tilde{\Omega}(t)} \frac{\partial}{\partial t} \varphi(x,t)\, dx + \int_{\partial\tilde{\Omega}(t)} \varphi(x,t)\, v(x,t) \cdot n(x,t)\, da}$$

Q.A.

$\Theta(X,t)$ $\qquad \frac{\partial}{\partial t}\Theta(X,t) = 0 = \dot{\Theta}$



$\dot{\phi}(X,t) = 1\,\underline{e}_1$

$u(x,t) = 1\,\underline{e}_1$

$u(\phi(X,t),t) = U(X,t) = \dot{\phi}(X,t) = 1\,\underline{e}_1$

$\theta(x,t) = \Theta \circ \phi^{-1}$ $\qquad \theta(\phi(X,t),t) = \Theta(X,t) = \Theta(\underline{X})$

$\dot{\theta} = \partial_t \theta + \operatorname{grad}\theta \cdot u$