Master in High Performance Computing

# Continuum mechanics and fluid-structure interaction problems: mathematical modelling and numerical approximation

## Reference configuration, deformation gradient

**Luca Heltai** <luca.heltai@sissa.it>

International School for Advanced Studies (www.sissa.it)
Mathematical Analysis, Modeling, and Applications (math.sissa.it)
Master in High Performance Computing (www.mhpc.it)
SISSA mathLab (mathlab.sissa.it)
King Abdullah University of Science and Technology (kaust.edu.sa)

جامعة الملك عبدالله للعلوم والتقنية
King Abdullah University of Science and Technology

The Abdus Salam International Centre for Theoretical Physics

SISSA

Lec. 4

$$g_{ij} := \underline{e}_i \cdot \underline{e}_j$$

$$g_{ij}\, \underline{e}^i \otimes \underline{e}^j = \bar{\bar{g}} \qquad \bar{\bar{g}} \in V^* \otimes V^*$$

use Riesz liberally

$$g^{ij}\, \underline{e}_i \otimes \underline{e}_j = \underline{\underline{g}} \qquad \bar{\bar{g}} \in \underline{V \otimes V}$$

Only for Euclidean

$$\mathcal{L}(V, V) \equiv \underline{\mathcal{L}}(V \times V, \mathbb{R})$$

$$\bar{\bar{g}}(\underline{v}, \underline{u}) := (g_{ij}\, \underline{e}^i \otimes \underline{e}^j)(v^k \underline{e}_k, u^m \underline{e}_m)$$

$$= g_{ij}\, \underline{e}^i \cdot \underline{e}_k\ \underline{e}^j \cdot \underline{e}_m\ v^k u^m = g_{ij} v^i u^j = \underline{v} \cdot \underline{u}$$

$$V \ni g(\underline{v}) = (g_{ij}\, \underline{e}^i \otimes \underline{e}^j)(v^k \underline{e}_k) = g_{ij}\, \underline{e}^i v^j = v_i \underline{e}^i = \underline{v}$$

$$T \in \mathcal{L}(V, W) \qquad \underline{w} = T(\underline{v}) \in W$$

$V = \mathrm{span}\{\underline{e}_i\}$
$W = \mathrm{span}\{E_\alpha\}$

$$E^\alpha(T(v^i \underline{e}_i)) = w^\alpha$$

$$E^\alpha \in W^* \qquad W \otimes V^* \qquad T^\alpha{}_i\, \underline{E}_\alpha \otimes \underline{e}^i$$

$$T^\alpha{}_i := E^\alpha\left(T(e_i)\right) \equiv E^\alpha \cdot T(e_i)$$

$T^\alpha{}_i \, E_\alpha \otimes e^i$ ←

$e^i \in V$

$\boxed{\text{What are basis vectors?}}$

$E^n$ := Euclidean space of dim $n$

we perform identification $E^n$ with $(E^n)^*$ via Rietz

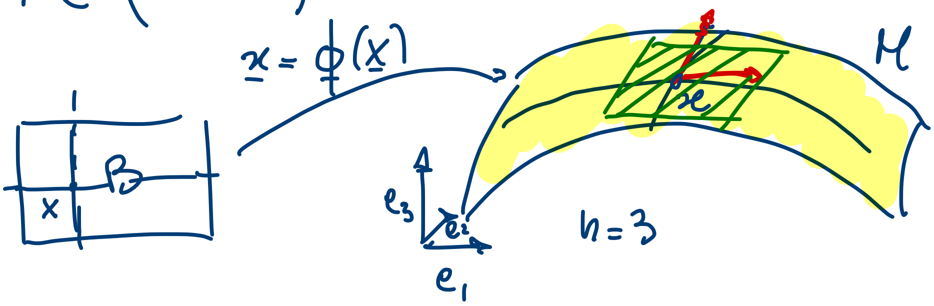
$$E^n := \text{span}\{e_i\}_{i=1}^{n}$$

We also introduce $E^m = \text{span}\{E_\alpha\}_{i=1}^{m}$ $\quad m \le n$

we perform identification $E^m$ with $(E^m)^*$ via Rietz

$B \subset E^m$ open subset

$M \ (\equiv \Omega)$ is a $m$-dimensional manifold in $E^n$



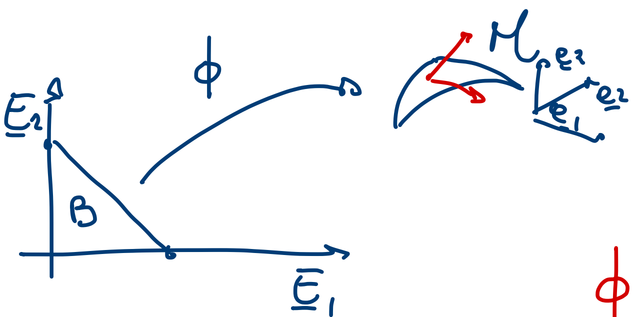

$$\phi : B \longrightarrow M$$

$\phi$ is invertible

$\phi$ is as smooth as necessary

* Tangent vectors
* normal vectors
* how they transform

$\boxed{\text{connected coordinate basis}}$




$$\begin{array}{ccc} \phi : B & \longrightarrow & M \\ \downarrow E_\alpha & & \downarrow e_e \\ E^m & \longrightarrow & E^n \end{array}$$

$$\phi \in E^n \to \phi^i e_i = \phi^i(X)\, e_i$$

$\mathbb{E}^n$ $\quad \underline{u}, \underline{v}, \underline{x}, \underline{e}_i$ small letters $\qquad i, j, k, l : 1, \dots, n$

$\mathbb{E}^m$ $\quad \underline{U}, \underline{V}, \underline{X}, \underline{E}_\alpha$ capital letters $\qquad \alpha, \beta, \gamma : 1, \dots, m$

$X \in \mathcal{B} \qquad x \in \mathcal{M} \qquad x = \phi(X) \qquad \phi^i \equiv x^i$

$$\boxed{\underline{g}_\alpha = \frac{\partial \phi^i}{\partial X^\alpha} \underline{e}_i}$$

$\underline{g}_\alpha \in \mathbb{E}^n$ convected coordinate system

$\underline{g}_\alpha$ is different at every $\underline{X}$

$T_x M := \mathrm{span}\{\underline{g}_\alpha\}$

IF $M$ IS SMOOTH ENOUGH

$\phi^i(X) \; : \; \mathbb{R}^m \equiv \mathbb{E}^m \longrightarrow$

Riesz!

$f : \mathcal{B} \to \mathbb{R} : f(X) \qquad \frac{\partial}{X^\alpha} f = f_\alpha$

$$\boxed{\begin{array}{l} f_\alpha \underline{E}^\alpha \in (\mathbb{E}^m)^* \\ f_\alpha \underline{e}^\alpha \in \mathbb{E}^m \end{array}}$$

$$\frac{\partial \phi^i}{\partial X^\alpha} = \phi^i{}_{,\alpha} =: F^i{}_\alpha$$

$$F^i{}_\alpha \, \underline{e}_i \otimes \underline{E}^\alpha = F$$

$F$ is known as the $\boxed{\text{Deformation Gradient}}$

$$F = F^i{}_\alpha \boxed{\underline{e}_i \otimes \underline{E}^\alpha}$$

$$\mathrm{Grad}\, f := \frac{\partial f}{\partial X^\alpha} g^{\alpha\beta} \underline{g}_\beta \qquad \forall f : \mathcal{B} \longrightarrow \mathbb{R}$$

$\underline{\underline{\text{Grad}}}$ taking derivatives w.r.t. $\underline{X}$

grad taking derivatives w.r.t. $\underline{x}$

$$w : \mathbb{E}^n \longrightarrow \mathbb{R} \qquad w(x) = w(x^i \underline{e}_i)$$

$$\text{grad}(w) := \frac{\partial w}{\partial x^i} g^{ij} \underline{e}_j \equiv \frac{\partial w}{\partial x^i} \underline{e}^i$$

$\mathbb{E}^n$ $g_{ij} \equiv \delta_{ij}$ if orthonormal

$$w_{,i} := \frac{\partial w}{\partial x^i}$$

$$d[w] := \frac{\partial w}{\partial x^i} \bar{e}^i \qquad \langle d[w], v \rangle = \frac{\partial w}{\partial x^i} v^i$$

$$\underline{g}_\alpha := \frac{\partial \phi^i}{\partial x^\alpha} \underline{e}_i = F^i{}_\alpha \underline{e}_i \qquad (\text{grad}\, w) \cdot v = \frac{\partial w}{\partial x^i} v^i$$

$$g_{\alpha\beta} = \underline{g}_\alpha \cdot \underline{g}_\beta = F^i{}_\alpha g_{ij} F^j{}_\beta$$

$\mathbb{E}^n \Rightarrow \delta_{ij}$

$$g_{\alpha\beta} := (F^T F)_{\alpha\beta} := F^i{}_\alpha g_{ij} F^T{}_\beta \qquad C = F^T F$$