Master in High Performance Computing

Continuum mechanics and fluid-structure interaction problems: mathematical modelling and numerical approximation

Kinematics

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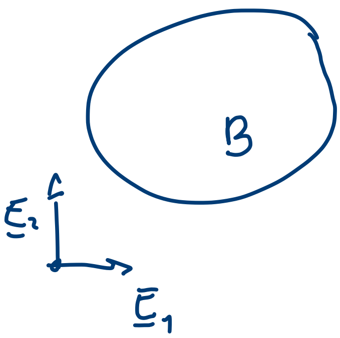
Kinematics := geometry of motions

→ how bodies deform (not why)

> we identify a "body" $\mathcal{B}$ with the region it occupies at a given reference time.



$\mathcal{B}$ is a (possibly unbounded) region of $\mathcal{E}^d$

$$\mathcal{E}^d \equiv \text{span}\{\underline{E}_\alpha\}_{\alpha=1}^{d\,(m)} \qquad m \le d$$

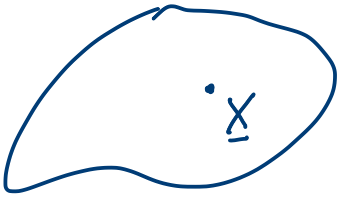
W is the supporting vector space of "$\mathcal{E}^d$"

$W \equiv T\mathcal{B}$

The ambient space is $\mathcal{E}^d$, V is its support vectorspace

$V = \text{span}\{\underline{e}_i\}_{i=1}^d$ we take $\underline{e}_i$ to be orthonormal

- Capital letters and greek indices: $\underline{X}, \underline{U}, \underline{F}$ reference
- small letters and latin indices: $\underline{x}, \underline{u}, \underline{f}$ space/ambient



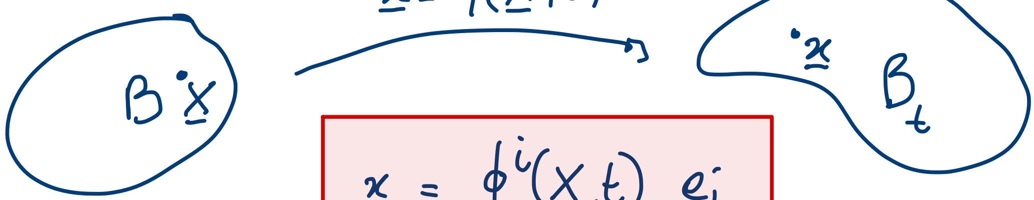
- $\underline{X}$ is a material point
  (identified with its position w.r.t. $\underline{E}_\alpha$)
  $$\underline{X} := X^\alpha \underline{E}_\alpha$$

- $\underline{x}$ is a spatial point
  (a point of observation)
  $$\underline{x} = x^i \underline{e}_i$$

A motion (a deformation) is a one parameter family of mappings: $\phi : B \times [0,T] \longrightarrow \mathcal{E}^d$ (ambient space)

At fixed t: $\phi(B,t) = B_t$

$$\underline{x} = \phi(\underline{X},t)$$



$$\boxed{\underline{x} = \phi^i(\underline{X},t)\ \underline{e}_i}$$

usually $\phi(\underline{X},0) \equiv \underline{X} \rightsquigarrow X^i \underline{e}_i \rightsquigarrow \phi \equiv$ "id"

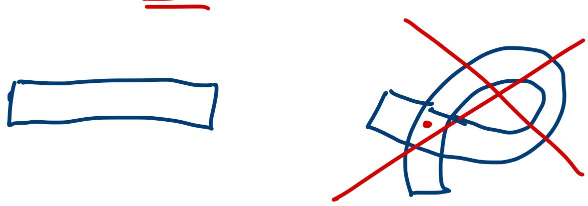
$$\boxed{X^i = \delta^i_\alpha X^\alpha}$$

ϕ must satisfy:

i) $\forall t$, ϕ is invertible
ii) ϕ must "preserve orientation"

$\Big\}$ $\det(\nabla_{\underline{X}} \phi) > 0 \quad \forall t$

iii) ϕ must not self intersect



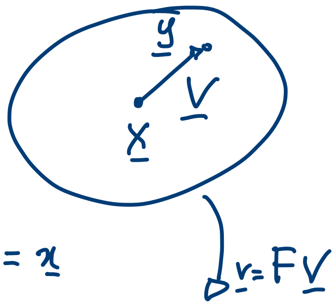
$$\det(\nabla_{\underline{x}} \phi^{-1}) > 0$$

ϕ is one to one and onto

$$F := \operatorname{Grad}\phi := \frac{\partial \phi}{\partial X} : \underset{\substack{\equiv \\ W}}{T_X B} \longrightarrow \underset{\substack{\subset \\ V}}{T_{\phi(X,t)}(\phi(B,t))} \quad \forall t$$

the deformation gradient : $\boxed{F}$

$$F = F^i{}_\alpha \, \underline{e}_i \otimes \underline{E}^\alpha := \frac{\partial \phi^i}{\partial X^\alpha} \underline{e}_i \otimes \underline{E}^\alpha$$



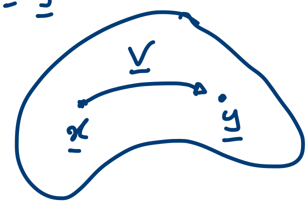
$$\underline{v} = F(\underline{V}) = F\underline{V}$$

$$v^i = F^i{}_\alpha V^\alpha$$

$$\phi(\underline{X},t) = \underline{x}$$

$$\phi(\underline{Y},t) = \underline{y}$$

$$\underline{v} = v^i \underline{e}_i = F^i{}_\alpha V^\alpha \underline{e}_i = F\underline{V}$$



$$\phi(\underline{Y}) = \phi(\underline{X}) + \underset{\sim}{F}(\underline{Y} - \underline{X}) + o(|\underline{Y} - \underline{X}|^2),$$

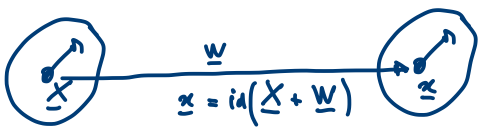
$$\underline{y} - \underline{x} = F(\underline{Y} - \underline{X}) + o(|Y - X|^2)$$

If $\operatorname{Grad}\phi$ is constant : homogeneous deformation

i) translations: $\quad \phi(\underline{X},t) = \mathrm{id}(\text{"}\underline{X}\text{"} + \underline{W}(t)) \quad \forall t$

$$\operatorname{Grad}\phi = \nabla_{\underline{X}}\phi = F = \text{"}\mathrm{Id}\text{"}$$

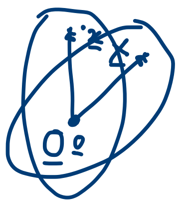
$$F^i{}_\alpha = \delta^i{}_\alpha$$



$$\boxed{\phi(\underline{X},t) = \mathrm{id}(\underline{X}) + \underline{w}(t)}$$

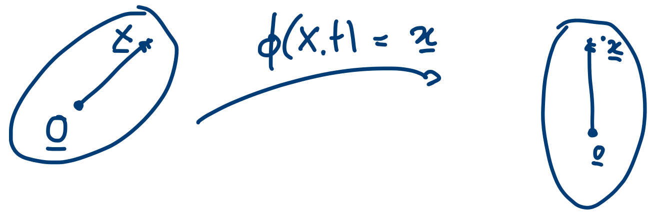
ii) rotations: $\quad \phi(X,t) := R(t)(\underline{X} - \underline{O})$

$$R^T = R^{-1}$$



iii) Rigid motion i) + ii)

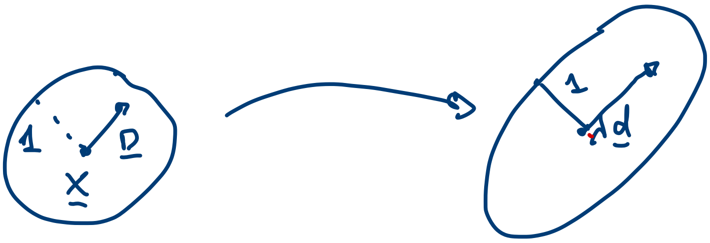
$$\phi = R(\underline{X} - \underline{X}_0)$$
$$= R\underline{X} - R\underline{X}_0$$
$$= R\underline{X} - \underline{w}$$



$\underline{w} = R\underline{X}_0$ $R^T\underline{w} = \underline{X}_0$

iv) Stretching along $\underline{d}$

$\underline{d} = id(\underline{D})$



$$\phi = id(X) + (\lambda - 1)\, \underline{d} \otimes \underline{D}$$

example: say $\underline{E}_\alpha = \delta^i{}_\alpha\, \underline{e}_i$ $\underline{d} = \underline{D} = \underline{e}_1$ $F^i_\alpha := \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$

Polar decomposition of F

Premise : $\forall$ SPSD matrix $\in \mathbb{R}^{d \times d}$: $\exists! \{\lambda_i, \underline{\sigma}_i\}_{i=1}^{d}$ $\lambda_i \geq 0$

s.t. $\boxed{A \underline{\sigma}_i = \lambda_i \underline{\sigma}_i \quad \text{NO SUM}}$ $\lambda_i \in \mathbb{R}^+ \cup \{0\}$

$\underline{\sigma}_i \cdot \underline{\sigma}_j = \delta_{ij}$ $A := \sum_i \lambda_i\, \underline{\sigma}_i \otimes \underline{\sigma}_i$ (No sum)

$\forall\ F,\quad \exists!\quad R\in\mathcal{L}(W,V),\ U\in\mathcal{L}(W,W),\ V\in\mathcal{L}(V,V),$

i) $R^T = R^{-1}$, $\det(R) = 1$

ii) $U^T = U$, $U > 0$ (U SPD)

iii) $V^T = V$, $V \geq 0$ (V SPSD) if $d=m$ V SPD

iv) $\boxed{F = RU}$ $\quad U$ is the Right Stretch tensor

v) $\boxed{F = VR}$ $\quad V$ is the left Stretch tensor

Note: $F^TF = U^TR^TRU = U^2$ $\quad U := \text{"}\sqrt{F^TF}\text{"}$

$FF^T = VRR^TV^T = V^2$ $\quad V := \text{"}\sqrt{FF^T}\text{"}$

$\forall\ A$ S.P.SD s.t. $A = \sum_i \left(\boxed{\lambda_i}\, 6_i \otimes 6_i\right)$

$\sqrt{A}$ is defined as $\sqrt{A} = \sum_i \left(\boxed{\sqrt{\lambda_i}}\ 6_i \otimes 6_i\right)$

$$F^TF = \sum_{\alpha=1}^{d(m)} \lambda_\alpha^2\ 6_\alpha \otimes 6_\alpha$$

$$FF^T = \sum_{\alpha=1}^{d(m)} \lambda_\alpha^2\ R6_\alpha \otimes R6_\alpha$$

$F = RU$

$F = VR$

$FR^T = \boxed{RUR^T = V}$

$FR^T = VRR^T = V$

$6_\alpha R^T v = R6_\alpha\, v$

$C = F^TF :=$ Right Cauchy Green tensor (U^2)

$B = FF^T :=$ left Cauchy Green tensor (V^2)

$$(F^TF)_{\alpha\beta} := F^i{}_\alpha \, \overset{\delta_{ij}}{g_{ij}} \, F^j{}_\beta$$

$$F^TF = (F^TF)_{\alpha\beta} \, \underline{E}^\alpha \otimes \underline{E}^\beta$$

$$(FF^T)^{ij} = F^i{}_\alpha \, \underline{\underline{g}}^{\alpha\beta} \, F^j{}_\beta \qquad (FF^T) = (FF^T)^{ij} \, \underline{e}_i \otimes \underline{e}_j$$

convected coord: $\underline{g}_\alpha : F\underline{E}_\alpha$

$$g_{\alpha\beta} = \underline{g}_\alpha \cdot \underline{g}_\beta = F\underline{E}_\alpha \cdot F\underline{E}_\beta = \underline{E}_\alpha \cdot F^TF\,\underline{E}_\beta$$

$$g_{\alpha\beta} = (F^TF)_{\alpha\beta} = \underline{g}_\alpha \cdot \underline{g}_\beta$$

$$F = F^\alpha{}_\beta \, \underline{g}_\alpha \otimes \underline{E}^\beta = \delta^\alpha{}_\beta \, \underline{g}_\alpha \otimes \underline{E}^\beta = \underline{g}_\alpha \otimes \underline{E}^\alpha$$

$$\underline{X} = X^\alpha \underline{E}_\alpha \qquad \underline{x} = X^\alpha \underline{g}_\alpha$$