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ORIGINAL RESEARCH

A new integer linear programming formulation for the problem of political districting

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Abstract
The problem of dividing political territories in electoral process is a very important factor which contributes to the development of democracy in modern political systems. The most significant criteria for fairness of electoral process are demographic, geographic and political. Demographic criterion in the first place refers to the population equality, while the geographic one is mostly represented by compactness, contiguity and integrity. In this paper we propose a new integer linear programming formulation for the problem of political districting. The model is based on the graph representation of political territory, where territorial units are vertices and direct links between them are edges. The correctness of integer linear programming formulation is mathematically proven. In contrast to the most of the previous formulations, all three major criteria, population equality, compactness and contiguity, are completely taken into consideration. There are two models, one which deals with afore mentioned criteria where compactness is taken as an objective function, and the other one which takes into account interests of the decision maker, i.e. the political ruling body which organizes elections. Several numerical examples for the presented models are given which illustrate general aspects of the problem. The experimental results are obtained using CPLEX solver.

Keywords Political districting · Integer linear programming · Graph partitioning · Combinatorial optimization

Mathematics Subject Classification 90C11 · 91F10 · 05C70

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1 Introduction

The process of governing large areas or a large number of people almost always require division and delegation of power to smaller districts and/or smaller groups of people to make the process effective. In modern democracies, where the political power is obtained through elections, this is even more obvious. The problem of dividing political territories is often called political districting (PD). In order to be fair or at least to have a semblance of fairness the process of dividing political territories must satisfy different criteria. The most common criteria are:

- *Demographic criterion* In the first place it means population equality. Political districts must be almost equal in population number to respect the principle "one man - one vote". Exceptions from the exact number depend from country to country and range from 1% in USA up to 50% in Canada due to various organizational and technical problems. The minorities representation must also be taken into consideration in this criterion. This means that minorities are getting the opportunity to elect representatives of their own volition;
- *Geographic criterion* The most important aspects of this criterion are compactness, contiguity, and integrity. The most common mathematical explanation of compactness is that the region should be round-shaped and undistorted. Even as this is intuitively clear, the strict definition does not exist. Various definitions have been proposed, but the two most common ones are defined using the center of region. Suppose that region has N units and that distances between them are given with d_{ij}. Harrary (1994) proposed a unit that minimizes $\max_{i \in N} d_{ij}$ or rather $\sum_{i \in N} d_{ij}$ as the center of a region. Contiguity simply means that territories should be connected. It is also important that territories are fully covered i.e. with no left out space. These two criterions, compactness and contiguity, are very closely linked to the idea of geographic compactness. Lack of any of them surely leads to poor results of compactness indicators. Integrity guarantees that populated or specific areas would not be divided by regional boundaries;
- *Political criterion* To arrange district boundaries the political data usage is much disputed amongst researchers. The reasons for using such data can be found in Bozkaya et al. (2003).

More about the PD criterions and their practical purposes and meanings can be found in Grilli di Cortona et al. (1999).

The interest for the political districting problem in the operation research community started in the 1960s. Since then, a rising number of the optimization models and algorithms has been proposed. For an overview of different approaches to the PD problem, the reader is referred to the more recent surveys in Ricca et al. (2013), Kalcsics et al. (2005) and Duque et al. (2007). Furthermore, Mallozzi and Puerto (2018) presented interesting results by observing partitions geometry of optimal districting of regions.

This paper is considering a similar approach to the PD problem with a new mathematical model defined as an integer linear programming formulation.

2 Previous mathematical formulations of PD problem

In this section, we will present various existing mathematical formulations of the PD problem.

The first paper depicting the PD problem as a mathematical formulation (obtained from a warehouse location problem) is the paper presented by Hess et al. (1965). The presented integer linear programming model does not consider contiguity of districts. Therefore, a posteriori changes in districts boundaries are required to obtain contiguity.

The improved version of the mathematical formulation from Hess et al. (1965) can be found in Hojati (1996). The idea is very similar to the ones stated in Hess et al. (1965) but a posteriori revision is significantly improved.

An alternative approach to the PD problem is proposed by Garfinkel and Nemhauser (1970) who presented it as an integer linear optimization problem. They proposed a two-phase PD algorithm based on a set partitioning approach.

Similar set partitioning approaches implementing a somewhat modified procedure and an objective function can be found in Nygreen (1988) and Mehrotra et al. (1998).

Some new exact approaches and corresponding mathematical formulations have been developed based on network flows.

Nemoto and Hotta (2003) proposed new integer linear programming formulation based on the idea of network flows. The point was to construct network $H = (\bar{N}, \bar{A})$ from contiguity graph $G = (N, E)$ where N is representing territorial units and E is connection among them. Because there should be k districts, k copies of this graph are considered. It should be noted that the model takes into account integrity, contiguity and population equality, but the compactness of the districts is not guaranteed.

Yet another mathematical formulation is proposed in Li et al. (2007). The territory in consideration is represented by a contiguity graph G, and the model has a quadratic formulation. The criteria taken into account are population equality and compactness. There are no explicit constraints for contiguity which implies that the optimal solution may be non-contiguous.

Shirabe (2009) introduced a new integer-programming-based approach to districting modeling, which enforced contiguity constraints independently of any other criteria that might be additionally imposed. The criteria taken into account are population equality, compactness and contiguity.

In Duque et al. (2011) the three MIP p-regions models (PRM) are presented, inspired by different areas of spatial optimization research and are accordingly named: $Tree$, a forest of trees, with one tree per region. Cycles are prevented in each tree based upon the properties of three sets of constraints; $Order$, areas are added to a given region in a specified order, where order prevents cycles and ensures contiguity and $Flow$, a model inspired by works of Shirabe (2005, 2009). This approach ensures contiguity by establishing a unit flow from each area (node) within a region to a selected sink of the region. The criteria taken into account are population equality, compactness and contiguity.

Yet another mathematical model, this time as Mixed Integer Program, for solving similar problem of map sectorization is given by Tang et al. (2014). Objective function can be presented as a piecewise linear function. Binary variables exist to indicate whether a territorial unit is within the given sector. There are also binary variables for determining which edges belong to the spanning arborescence associated with the given sector. Solution consists of vertex set partition into subsets. Each subset induces connected subgraph (sector). In order to restrict solution space, authors added some valid inequalities and also proposed preprocessing procedure to reduce the number of variables.

All before mentioned papers have mathematical formulation of PD problem, but they do not have a proof that optimal solution of mathematical formulation generates optimal solution of PD problem. Lack of proof can lead to that, though some solutions of mathematical formulation are optimal, they are not correct solutions of PD problem.

3 New integer linear programming formulations of the PD problem

Political territory is given by m basic territorial units, with the information which of them are neighbors. The population number of ith, $i = 1, \ldots, m$ territorial unit is given by p_i. The problem of political districting (PD) is to partition the territory in n ($n \leq m$) districts according to democratic principles:

- Population of every district should be between beforehand given a and b;
- Every district should be connected (e.g. any two territorial units in a district are either neighbors or there exists a chain of neighboring territorial units from the same district between them);
- Every district should have a certain measure of geographical compactness.

The solution of PD problem which satisfies these democratic principles will be called a *democratic solution.*

The PD problem can be considered as the optimization problem on graphs.

Let $G = (V, E)$ be an undirected graph obtained from the political territory where vertices $v_i, i = 1, \ldots, m$ from V are territorial units with weights p_i and edges are links that exist among neighboring territorial units given by adjacency matrix $A = A_{ij}, i, j = 1, \ldots, m$. Furthermore, we assume that graph G is connected. The mathematical interpretation of PD problem is to partition the vertex set of the graph G into n non-empty subsets, each one of them inducing connected subgraph. These subsets will represent districts with given properties.

3.1 Partitioning of the vertex set

Let us consider the following system of binary variables $X_{ij}, S_{ij}, i = 1, \ldots, m, j = 1, \ldots, n, n < m$.

$$\sum_{j=1}^{n} X_{ij} = 1, \quad i = 1, \ldots, m \tag{1}$$

each territory assign to 1 district

$$\sum_{i=1}^{m} S_{ij} = 1, \quad j = 1, \ldots, n \tag{2}$$

$$S_{ij} \leq X_{ij}, \quad i = 1, \ldots m, j = 1, \ldots, n \tag{3}$$

This system always has a solution, for instance, $X_{11} = X_{22} = \ldots = X_{n-1,n-1} = 1$, $X_{kn} = 1$ for $k = n, \ldots, m$ and $X_{ij} = 0$ otherwise; $S_{11} = S_{22} = \ldots = S_{n-1,n-1} = 1$, $S_{nn} = 1$ and all others $S_{ij} = 0$.

Moreover, we use the following well-known Lemma (see Duque et al. 2011).

Lemma 1 *For every solution of the system (1)–(3), the non-empty sets $V_j = \{v_i | i : X_{ij} = 1\}$, $j = 1, \ldots, n$ constitute a partition of V and there exists exactly one vertex v_i such that $S_{ij} = 1$ which will be called the representative of V_j and referred as r_j, and vice versa, for every partition of V into non-empty sets V_j with chosen representative r_j, $j = 1, \ldots, n$, values X_{ij}, S_{ij} $i = 1, \ldots, m$, $j = 1, \ldots, n$, defined as*

$$X_{ij} = \begin{cases} 1, & v_i \in V_j \\ 0, & \text{otherwise} \end{cases}$$

and

$$S_{ij} = \begin{cases} 1, & v_i = r_j \in V_j \\ 0, & \text{otherwise} \end{cases}$$

make a solution of the system (1)–(3).

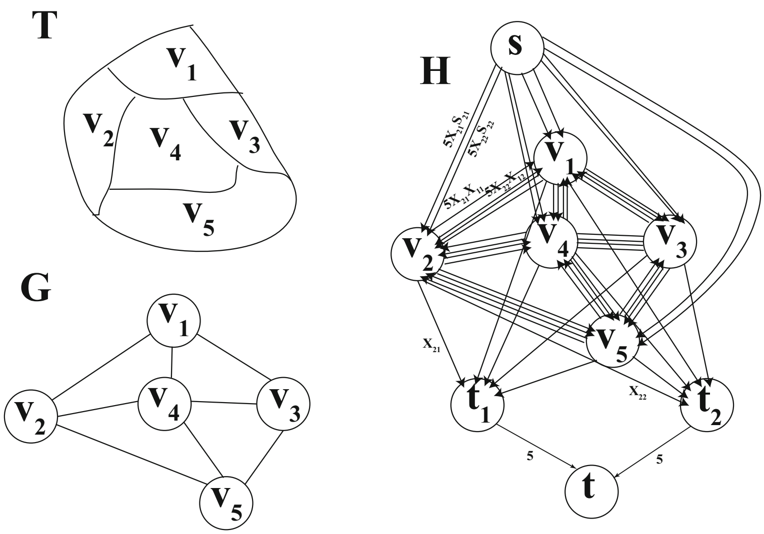

Fig. 1 Example of transportation network; capacities are determined according to the presented procedure

In order to partition vertices of undirected graph $G = (V, E)$, $|V| = m$ into n, $n \leq m$ non-empty sets V_j, $j = 1, \ldots, n$ such that induced subgraphs $G_j = (V_j, E_j)$ are connected, we construct an assigned transportation network $H = (W, F)$ with source-node s and sink-node t and functional capacities obtained from the solution of system (1)–(3) in the following way:

- Set of vertices $W = V \cup \{s, t_1, \ldots, t_n, t\}$;
- Source-node s is connected with every vertex $v_i, i = 1, \ldots, m$ from V with n parallel directed edges $(s, v_i)_j$, $j = 1, \ldots, n$, each with the capacity $mX_{ij}S_{ij}$;
- Every vertex v_i from V is connected with every vertex t_j, $j = 1, \ldots, n$ with a directed edge (v_i, t_j) with the capacity X_{ij};
- Every vertex t_j, $j = 1, \ldots, n$ is connected with t through a directed edge (t_j, t) with the capacity m;
- For every edge $< v_i, v_k >\in E$, n parallel directed edges $(v_i, v_k)_j$, $j = 1, \ldots, n$ and n parallel directed edges $(v_k, v_i)_j$, $j = 1, \ldots, n$ are constructed, each with the capacity $mX_{ij}X_{kj}$;

Since G is connected, it can be concluded that H is well defined transportation network with functional capacities. Note that every flow f from s to t through H has the quantity $|f| \leq m$ because (S, T), where $S = \{s, v_1, \ldots, v_m\}$ and $T = \{t_1, \ldots, t_n, t\}$, is a cut and it holds

$$|f| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} = m.$$

Figure 1 depicts an example of transforming political territory T with five territorial units into graph G with five vertices and assigned transportation network H with functional capacities for partitioning into two districts.

Theorem 1 *Let sets $V_j = \{v_i \in V | i : X_{ij} = 1\}$, $j = 1, \dots, n$ constitute a partition of V such that induced subgraphs $G_j = (V_j, E_j)$ are connected and let r_j are respective representatives of V_j. Then the maximal flow from s to t through the transportation network H is equal to $m = |V|$.*

Proof Remark that capacities of edges (s, r_j), $\quad j = 1, \dots, n$ are equal to m and capacities of all other edges from s are zero.

Since every G_j is connected, there exists a spanning tree in G_j, which induce arborescence $T_j = (V_j, F_j)$ in H rooted in r_j. We define the flow through every edge of H as follows:

- $f(s, r_j) = |V_j|$;
- for every $v \in V_j \backslash \{r_j\}$, a unique directed edge $(u, v) \in F_j$ exists. Let U be the subarborescence of T_j containing the vertex v after cutting the edge (u, v) from F_j and let $f(u, v) = |U|$;.
- $f(v_i, t_j) = 1$ if $v_i \in V_j$ and 0 otherwise;
- $f(t_j, t) = |V_j|$;
- Flow through all other edges in H is 0.

Let us prove that flow f is correctly defined i.e. for every vertex v, $v \notin \{s, t\}$ input flow is equal to output flow.

Let $v = r_j$. Input flow is $f(s, r_j) = |V_j|$ and output flow is again $|V_j|$.

For $v = t_j$, $j = 1, \dots, n$ it is true since input flow is $\sum_{i=1}^{m} f(v_i, t_j) = \sum_{i=1}^{m} X_{ij} = \sum_{i:v_i \in V_j} 1 = |V_j|$ and output flow is $f(t_j, t) = |V_j|$.

Let $v \in V_j \backslash \{r_j\}$ and $(u, v) \in F_j$. If v has a maximal depth in T_j, then input flow is equal to 1. Output flow is equal to $\sum_{k=1}^{n} f(v, t_k) = f(v, t_j) = 1$. Let us assume that the flow is correct for all vertices $v \in V_j \backslash \{r_j\}$ with depth greater then k, $k > 0$. Let $v \in V_j$ be a vertex with depth equal to k and let (v, u_i), $i = 1, \dots, p$ are all outgoing edges in F_j from v. By induction hypothesis $f(v, u_i) = |U_i|$, $i = 1, \dots, p$ where U_i is subarborescence rooted in u_i. Since U_i, $i = 1, \dots, p$ are disjoint, by definition of flow, input flow in v is equal to $1 + \sum_{l=1}^{p} |U_l|$ and output flow is $\sum_{j=1}^{n} f(v, t_j) + \sum_{l=1}^{p} f(v, u_l) = 1 + \sum_{l=1}^{p} |U_l|$. Hence, flow f is correctly defined for vertices of depth k. By induction, f is correctly defined for H.

The quantity of flow f is equal to $\sum_{j=1}^{n} f(t_j, t) = \sum_{j=1}^{n} |V_j| = m$, hence it is maximal. □

Theorem 2 *Let H be a transportation network defined as above and let the quantity of the flow from s to t through H be equal to m. Then sets $V_j = \{v_i \in V | i : X_{ij} = 1\}$, $j = 1, \dots, n$ constitute a partition of V which in G induce n connected subgraphs $G_j = (V_j, E_j)$, $j = 1, \dots, n$.*

Proof According to Lemma 1, sets $V_j = \{v_i | i : X_{ij} = 1|\}$, $j = 1, \dots, n$ constitute a partition of V into n non-empty sets. The quantity of flow f that reaches each t_j cannot be greater than $\sum_{i=1}^{m} X_{ij} = |V_j|$. That implies $f(t_j, t) \le |V_j|$, $j = 1, \dots, n$. Since $\sum_{j=1}^{n} |V_j| = m = \sum_{j=1}^{n} f(t_j, t)$, the quantity of flow f that reaches each t_j must be equal to $|V_j|$. Therefore, $f(v_i, t_j) = 1$ for each $v_i \in V_j$. Hence, the cut (S, T), $S = \{s, v_1, \dots, v_m\}$ and $T = \{t, t_1, \dots, t_n\}$ has capacity equal to m. Therefore, this cut has the minimal capacity. Now, Ford-Fulkerson theorem yields that there exists unsaturated path from s through r_j to every $v \in V_j$. From this fact, we can conclude that there is a path from the representative r_j to each vertex in V_j through edges of subgraph $G_j = (V_j, E_j)$. This proves that G_j is connected for every j. □

3.2 Mathematical formulation for PD problem

Our goal is to make a partition which will include connectivity, compactness and equality of population of districts. As a measure of compactness we propose average of moments of inertia of districts. Moment of inertia will be calculated as the sum of squared geographical distances of each unit in a district to the district representative. If p_k is the population of the unit v_k, $v_k \in V_j$ and d_{ij} is geographical distance from unit v_i to unit v_j, $i, j = 1, \ldots, m$, moment of inertia of the district V_j is calculated as $\sum_{i=1}^{m} \sum_{k=1}^{m} d_{ik}^2 p_k S_{ij} X_{kj}$.

Consider mathematical programming problem (MP):

$$\min \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} d_{ik}^2 p_k S_{ij} X_{kj} \tag{4}$$

subject to

$$\sum_{j=1}^{n} X_{ij} = 1, \quad i = 1, \ldots, m \tag{5}$$

$$\sum_{i=1}^{m} S_{ij} = 1, \quad j = 1, \ldots, n \tag{6}$$

$$S_{ij} \leq X_{ij}, \quad i = 1, \ldots m, j = 1, \ldots, n \tag{7}$$

$$0 \leq \alpha_{ij} \leq m X_{ij} S_{ij}, \quad i = 1, \ldots m, j = 1, \ldots, n \tag{8}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} = m \tag{9}$$

$$0 \leq \beta_{ikj} \leq m X_{ij} X_{kj} A_{ik}, \quad i, k = 1, \ldots m, j = 1, \ldots, n \tag{10}$$

$$0 \leq \gamma_{ij} \leq X_{ij}, \quad i = 1, \ldots m, j = 1, \ldots, n \tag{11}$$

$$\sum_{i=1}^{m} \gamma_{ij} = \delta_j, \quad j = 1, \ldots, n \tag{12}$$

$$\alpha_{lj} + \sum_{i=1}^{m} \beta_{ilj} A_{il} = \gamma_{lj} + \sum_{k=1}^{m} \beta_{lkj} A_{lk}, \quad l = 1, \ldots, m, \quad j = 1, \ldots, n \tag{13}$$

$$a \leq \sum_{i=1}^{m} p_i X_{ij} \leq b, \quad j = 1, \ldots, n \tag{14}$$

$$X_{ij}, S_{ij} \in \{0, 1\}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \tag{15}$$

$$\alpha_{ij}, \beta_{ikj}, \gamma_{ij}, \delta_j \in Z, \quad i, k = 1, \ldots, m, \quad j = 1, \ldots, n \tag{16}$$

In the formulation there are $2mn$ binary and $2mn + n + m^2 n$ integer variables, while there are $m + 3n + 4mn + m^2 n + 1$ constraints with appropriate flow interpretation which considerably make easier the proof that feasible solution of mathematical formulation generates feasible solution of PD problem and vice versa. We prove that (MP) is mathematical formulation for the PD problem which considers compactness, connectedness and population equality.

Theorem 3 1. *For every democratic solution of PD problem, there is a feasible solution of (MP) with the value of objective function equal to the measure of compactness of the solution for the PD problem;*

2. *For every feasible solution of (MP), there is a democratic solution of PD with the measure of compactness equal to the objective function's value of (MP).*

Proof 1. From the feasible democratic solution of the PD problem with chosen centers, the values of variables X_{ij} and S_{ij} can be directly calculated. Since every territorial unit belongs to exactly one district, constraints (5) are satisfied and also the population numbers are in given boundaries which means that constraints (14) are satisfied. Due to the fact that every district has exactly one district representative, constraints (6)–(7) are satisfied. Capacities of the assigned transportation network H can be evaluated. According to the Theorem 1, a feasible solution of the MaxFlow Problem through assigned transportation network H exists with maximum flow value $m = |V|$. If we interpret $\alpha_{ij}, \beta_{ijk}, \gamma_{ij}, \delta_j$ as

- α_{ij}, flow through the edge $(s, v_i)_j$, $\quad i = 1, \ldots, m,\ j = 1, \ldots, n$;
- β_{ikj}, flow through the edge $(v_i, v_k)_j$, $\quad i, k = 1, \ldots, m,\ j = 1, \ldots, n$;
- γ_{ij}, flow through the edge (v_i, t_j), $\quad i = 1, \ldots, m,\ j = 1, \ldots, n$;
- δ_j, flow through the edge (t_j, t), $\quad j = 1, \ldots, n$,

then constraints (8)–(13) are satisfied. Value of the objective function of (MP) is equal to the measure of compactness for the given PD democratic solution.

2. From Theorem 2, sets $V_j = \{v_i \in V | i : X_{ij} = 1\},\ j = 1, \ldots, n$ form a partition of V which in G induce connected subgraphs $G_j,\ j = 1, \ldots, n$. This gives the solution of the PD problem with measure of compactness which is equal to objective function's value. □

Clearly, optimal solutions of PD and (MP) are corresponding, too. That shows that (MP) is a mathematical formulation for PD problem.

The presented mathematical formulation is in the class of quadratic programming and it can be linearized. To do so, every product $x \cdot y$ of two binary variables x and y can be replaced with an additional binary variable z satisfying the following constraints

$$x + y - 1 \leq 2z \leq x + y, \quad z \in \{0, 1\}. \tag{17}$$

since $z = xy$, $x, y \in \{0, 1\}$ is equivalent to (17). In accordance to this, the objective function can be replaced with

$$\min \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} d_{ik}^2 p_k R_{ikj} \tag{18}$$

and we add additional constraints

$$S_{ij} + X_{kj} - 1 \leq 2R_{ikj} \leq S_{ij} + X_{kj}, \quad i, k = 1, \ldots, m,\ j = 1, \ldots, n \tag{19}$$

where R_{ikj} are the new binary variables. Also, constraints (8) can be replaced with

$$0 \leq \alpha_{ij} \leq m R_{iij}, \quad i = 1, \ldots m, j = 1, \ldots, n \tag{20}$$

$$X_{ij} + S_{ij} - 1 \leq 2R_{iij} \leq X_{ij} + S_{ij}, \tag{21}$$

Similarly, constraint (10) can be replaced with

$$0 \leq \beta_{ikj} \leq m P_{ikj} A_{ik}, \quad i, k = 1, \ldots m, j = 1, \ldots, n \tag{22}$$

$$X_{ij} + X_{kj} - 1 \leq 2P_{ikj} \leq X_{ij} + X_{kj}, \quad i, k = 1, \ldots, m,\ j = 1, \ldots, n \tag{23}$$

where P_{ikj} are the new binary variables.

Finally, constraint (15) should be replaced with

$$X_{ij}, S_{ij}, P_{ikj}, R_{ikj} \in \{0, 1\} \quad i, k = 1, \ldots, m, \quad j = 1, \ldots, n \tag{24}$$

Now, the mathematical formulation described by constraints (18)–(19), (5)–(7), (9), (11)–(14), (16) and (20)–(24) is the integer linear programming (ILP) formulation for the PD problem. Process of linearization requests $2m^2n$ new binary variables and additional $6m^2n$ new constraints.

3.3 Mathematical formulation for PD problem including standpoint of decision maker

The procedure of political districting is implicitly, if not explicitly, conducted by the government representatives. If there are several democratic solutions of the PD problem, those who have a choice will always choose the solution which best suits their interests. From the standpoint of the decision maker, mathematical formulation will consider different objective function than in the previous section. Since every district delegates one representative in parliament, the goal of those who conduct this partitioning is to have the maximal number of representatives in the parliament.

Let parameters c_i, $i = 1, \ldots, m$ represents the expected difference in the votes pro or contra the party in power for ith territorial unit.

District V_j, $j = 1, \ldots, n$ is won by the decision maker if and only if $1 \leq \sum_{i:v_i \in V_j} c_i = \sum_{i=1}^{m} c_i X_{ij}$,. We introduce the following binary variables

$$Y_j = \begin{cases} 1, & \sum\limits_{i:v_i \in V_j} c_i \geq 1 \\ 0, & \text{otherwise} \end{cases}, \quad j = 1, \ldots, n$$

which will indicate that in district V_j the party of decision maker wins. Let $M = 1 + \sum_{i=1}^{m} |c_i|$. Now, the following equivalency holds:

$$\left(1 \leq \sum_{i=1}^{m} c_i X_{ij} \wedge Y_j = 1\right) \vee \left(\sum_{i=1}^{m} c_i X_{ij} \leq 0 \wedge Y_j = 0\right) \Leftrightarrow -M(1 - Y_j) + 1 \leq \sum_{i=1}^{m} c_i X_{ij} \leq M Y_j,$$

which can be proved directly by analyzing the cases. Hence, the goal of decision maker is to maximize $\sum_{j=1}^{n} Y_j, \quad j = 1, \ldots, n$.

The mathematical formulation (MPD) of the PD problem can be represented as the following, using the notation from 3.2.1:

$$\max \quad \sum_{j=1}^{n} Y_j \tag{25}$$

subject to

$$\sum_{j=1}^{n} X_{ij} = 1, \quad i = 1, \ldots, m \tag{26}$$

$$-M(1 - Y_j) + 1 \leq \sum_{i=1}^{m} c_i X_{ij} \leq M Y_j, \quad j = 1, \ldots, n \tag{27}$$

$$\sum_{i=1}^{m} S_{ij} = 1, \quad j = 1, \ldots, n \tag{28}$$

$$S_{ij} \leq X_{ij}, \quad i = 1, \ldots m, j = 1, \ldots, n \tag{29}$$

$$0 \leq \alpha_{ij} \leq m X_{ij} S_{ij}, \quad i = 1, \ldots m, j = 1, \ldots, n \tag{30}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} = m \tag{31}$$

$$0 \leq \beta_{ikj} \leq m X_{ij} X_{kj} A_{ik}, \quad i, k = 1, \ldots m, j = 1, \ldots, n \tag{32}$$

$$0 \leq \gamma_{ij} \leq X_{ij}, \quad i = 1, \ldots m, j = 1, \ldots, n \tag{33}$$

$$\sum_{i=1}^{m} \gamma_{ij} = \delta_j, \quad j = 1, \ldots, n \tag{34}$$

$$\alpha_{lj} + \sum_{i=1}^{m} \beta_{ilj} A_{il} = \gamma_{lj} + \sum_{k=1}^{m} \beta_{lkj} A_{lk}, \quad l = 1, \ldots, m, \quad j = 1, \ldots, n \tag{35}$$

$$a \leq \sum_{i=1}^{m} p_i X_{ij} \leq b, \quad j = 1, \ldots, n \tag{36}$$

$$R_1 \leq \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} d_{ik}^2 p_k S_{ij} X_{kj} \leq R_2 \tag{37}$$

$$X_{ij}, Y_j, S_{ij} \in \{0, 1\}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \tag{38}$$

$$\alpha_{ij}, \beta_{ikj}, \gamma_{ij}, \delta_j \in Z, \quad i, k = 1, \ldots, m, \quad j = 1, \ldots, n \tag{39}$$

Parameters R_1 and R_2 are lower and upper bounds on the measure of compactness, which was set as objective function in the ILP formulation in the previous subsection (3.2). These parameters are obtained from the solution of the ILP formulation (4)–(16) as an acceptable deviation from the optimal compactness measure. Parameters a and b are deviations from the average population in every district. The presented formulation could be linearized similarly as in Sect. 3.2.

Theorem 4 1. *For every democratic solution of the PD problem which considers the interest of decision maker, there is a feasible solution of (MPD) with the value of objective function equal to the number of districts in favor of the decision maker's party;*
2. *For every feasible solution of (MPD) there is a democratic solution of PD with the number of districts in favor of decision maker's party equal to the value of the objective function.*

Proof 1. From the solution of PD we can construct solution of (MPD) analogous to Theorem 3, thereby values of Y_j are calculated from the values of X_{ij}. Constraints (27) are satisfied as explained above. Part of the proof which corresponds to connectedness and population equality is the same. Solution of (MPD) has a measure of compactness in an acceptable range of deviation from measure of compactness of PD problem solution. Objective function in (25) measures the number of districts in favour of the decision maker.
2. As in the proof of Theorem 2, sets $V_j = \{v_i \in V | i : X_{ij} = 1\}$, $j = 1, \ldots, n$ form a partition of V which in G induce connected subgraphs G_j, $j = 1, \ldots, n$. This gives the democratic solution of PD problem with the number of districts in favour of decision maker equal to the value of the objective function.

□

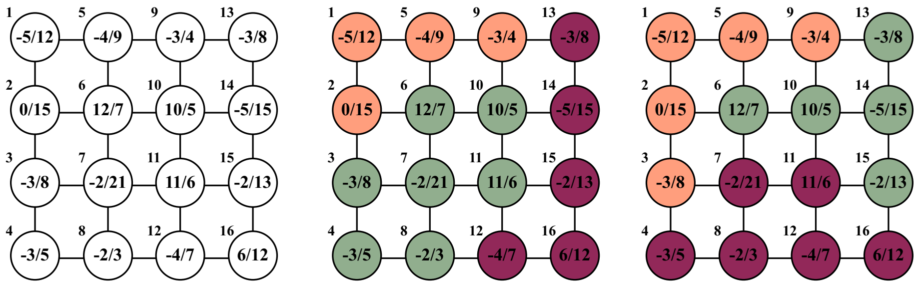

Fig. 2 Graph in Example 1 with optimal solution of two ILP formulations

Clearly, optimal solutions of PD and (MPD) are corresponding. This shows that (MPD) is mathematical formulation for PD problem from the standpoint of decision maker.

4 Numerical examples

In this section some examples which were solved by using the proposed ILP formulations and commercial solver will be presented.

The results of all numerical examples are obtained by using the commercial solver CPLEX 12.6 on Intel Core i7-4710MQ CPU 2.50 Ghz 2.50 GHz with six core processor with 16Gb memory and Intel Core i7-4712MQ CPU 2.30 Ghz 2.30 GHz with 8 core processors and 6 Gb memory. Working time significantly increases with the increase of the dimension of problems, which is expected as the PD problem is NP-hard. The problems of greater dimensions should be addressed by some kind of heuristics where optimal values obtained by our method could provide good reference points.

Example 1 Let us present optimal solutions for grid graph with 4 rows and 4 columns ($m = 16$). It should be partitioned in 3 districts ($n = 3$). Populations in territorial units (p_i, $i = 1, \ldots, 16$) are 12, 15, 8, 5, 9, 7, 21, 3, 4, 5, 6, 7, 8, 15, 13, 12 and expected differences in the votes (c_i, $i = 1, \ldots, 16$) are $-5, 0, -3, -3, -4, 12, -2, -2, -3, 10, 11, -4, -3, -5, -2$. The first array is generated as random numbers in range $[0, 25]$ while the second array is generated as random numbers in range $[-12, 12]$. Distance in the graph is calculated geographically, i.e as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, where (x_i, y_i) are grid coordinates of the vertex i.

Figure 2 shows grid graphs with appropriate information, partition in districts according with democratic principles, and partition which takes into the account interest of the decision maker.

Optimal value for the first model is 157 (democratic principles) while for the second is 2 (interest of decision maker). The optimal solution for the first model represents minimal total moment of inertia from the centers of districts while for the second model it is a number of districts that will vote for decision maker (green and purple districts). Both solutions were calculated with population tolerance of 25%. In the second formulation tolerance for the difference from optimal value of moment of inertia is also set on 25%, i.e parameter R_1 is equal to 157 and R_2 is set on 210.

Variables with non-zero values for the first model are:

$$X_{11} = X_{121} = X_{15} = X_{19} = X_{2,12} = X_{2,13} = X_{2,14} = X_{2,15} = X_{2,16} = X_{33}$$
$$= X_{34} = X_{36} = X_{37} = X_{385} = X_{3,10} = X_{3,12} = 1;$$

$S_{1,1} = S_{15,2} = S_{7,3} = 1$ (centers of districts);

$$\begin{aligned}
P_{1,2,1} &= P_{=}P_{1,5,1} = P_{1,9,1} = P_{2,1,1} = P_{2,5,1}\\
&= P_{2,9,1} = P_{3,4,3} = P_{3,6,3} = P_{3,7,3} = P_{3,8,3}\\
&= P_{3,10,3} = P_{3,11,3} = P_{4,3,3} = P_{4,6,3} = P_{4,7,3}\\
&= P_{4,8,3} = P_{4,10,3} = P_{4,11,3} = P_{5,1,1} = P_{5,2,1}\\
&= P_{5,9,1} = P_{6,3,3} = P_{6,4,3} = P_{6,7,3} = P_{6,8,3}\\
&= P_{6,10,3} = P_{6,11,3} = P_{7,3,3} = P_{7,4,3} = P_{7,6,3}\\
&= P_{7,8,3} = P_{7,10,3} = P_{7,11,3} = P_{8,3,3} = P_{8,4,3}\\
&= P_{8,6,3} = P_{8,7,3} = P_{8,10,3} = P_{8,11,3} = P_{9,1,1}\\
&= P_{9,2,1} = P_{9,5,1} = P_{10,3,3} = P_{10,4,3} = P_{10,6,3}\\
&= P_{10,7,3} = P_{10,8,3} = P_{10,11,3} = P_{11,3,3}\\
&= P_{11,4,3} = P_{11,6,3} = P_{11,7,3} = P_{11,8,3}\\
&= P_{11,10,3} = P_{12,13,2} = P_{12,14,2} = P_{12,15,2}\\
&= P_{12,16,2} = P_{13,12,2} = P_{13,14,2} = P_{13,15,2}\\
&= P_{13,16,2} = P_{14,12,2} = P_{14,13,2} = P_{14,15,2}\\
&= P_{14,16,2} = P_{15,12,2} = P_{15,13,2} = P_{15,14,2}\\
&= P_{15,16,2} = P_{16,12,2} = P_{16,13,2} = P_{16,14,2} = P_{16,15,2} = 1;\\
R_{1,1,1} &= R_{1,2,1} = R_{1,5,1} = R_{1,9,1} = R_{15,12,2}\\
&= R_{15,13,2} = R_{15,14,2} = R_{15,15,2} = R_{15,16,2} = R_{7,3,3}\\
&= R_{7,4,3} = R_{7,6,3} = R_{7,7,3} = R_{7,8,3} = R_{7,10,3} = R_{7,11,3} = 1;\\
\alpha_{11} &= 4, \alpha_{37} = 7, \alpha_{2,15} = 5;\\
\beta_{121} &= 1, \beta_{151} = 2, \beta_{343} = 1, \beta_{591} = 1, \beta_{733}\\
&= 2, \beta_{763} = 1, \beta_{783} = 1, \beta_{7,11,3} = 2, \beta_{11,10,3} = 1, \beta_{14,13,2} = 1, \beta_{15,14,2}\\
&= 2, \beta_{15,16,2} = 2, \beta_{16,12,2} = 1;\\
\gamma_{11} &= \gamma_{21} = \gamma_{33} = \gamma_{43} = \gamma_{51} = \gamma_{63} = \gamma_{73} = \gamma_{83}\\
&= \gamma_{91} = \gamma_{10,3} = \gamma_{11,3} = \gamma_{12,2}\\
&= \gamma_{13,2} = \gamma_{14,2} = \gamma_{15,2} = \gamma_{16,2} = 1;\\
\delta_1 &= 4, \delta_2 = 5, \delta_3 = 7.
\end{aligned}$$

□

Example 2 (Giannuli 1993) Let $m = 30$, $n = 10$, $p_i = 10$ for each $i = 1, \ldots, m$. The upper and the lower bound on population are set on 31 and 29, respectively. Since population in all districts are equal, it means that all district will have exactly 3 units, so moment of inertia for any district is the same. The diagram representing political territory is presented on Fig. 3a.

By using second proposed ILP formulation, we have proved that the solution proposed in Giannuli (1993), is optimal. If we let party A to be a decision maker, the optimal solution value is 6, which is presented on Fig. 3b. Districts which voted for party A are: II, V, VI, VII, IX and X. On the other hand, if party B is a decision maker, the optimal solution value

7-3	**4-6**	**9-1**	**9-1**	**9-1**
3-7	**4-6**	**4-6**	**4-6**	**6-4**
4-6	**6-4**	**4-6**	**6-4**	**4-6**
4-6	**4-6**	**4-6**	**4-6**	**3-7**
3-7	**4-6**	**4-6**	**7-3**	**3-7**
7-3	**5-5**	**6-4**	**5-5**	**7-3**

(a)

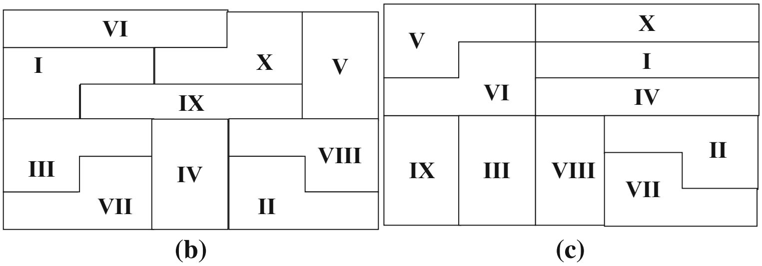

Fig. 3 **a** Political data. **b** Optimal solution for party A. **c** Optimal solution for party B

Table 1 Computational results

	n	Moment of inertia	t	Number of districts	t	Number of districts	t
$graph03xx03$	6	10	5.625	6	0.343	6	0.156
$graph04x03$	3	117	2.625	2	0.375	2	0.968
$graph04x04$	3	154	15.515	2	19.203	2	8.046
$graph04x07$	5	312	527661.297	4	2739.657	4	39, 724.127
$graph05x04$	4	196	1265.48	4	102.269	4	42.519
$graph05x05$	4	263	92206.457	3	210.085	3	1125.057
$graph05x07$	5	389*	1000, 001.112	5	14094.43	5	124, 800.643
$graph06x04$	4	207	5663.25	3	457.34	3	80.625
$graph06x05$	5	286*	1000, 004.186	5	7111.72	5	22, 692.02

is equal to 8, which is presented on Fig. 3c. In this case, districts voted for party B are I, II, III, IV, V, VI, VII and X.

As it can be seen, the optimal solution for party A is to obtain the maximum of 6 districts and for party B to obtain 8 districts (though the total number of votes for party A is 153 and for party B 147). □

The additional experiments were conducted on graphs similar in kind to those that were tested in the benchmark test for heuristic approaches. Tests were run on 9 grid graphs, with 9 to 35 vertices. Due to the CPLEX solver limitation and the NP-hard difficulty of the PD problem, we have tested our formulation on the graphs with smaller number of vertices in compare to, for example Ricca et al. (2008). Population numbers are randomly chosen from the interval [1,25] while the results of the previous elections are randomly selected from the interval [−12, 12]. The results of this additional test are presented in Table 1. Tolerance in

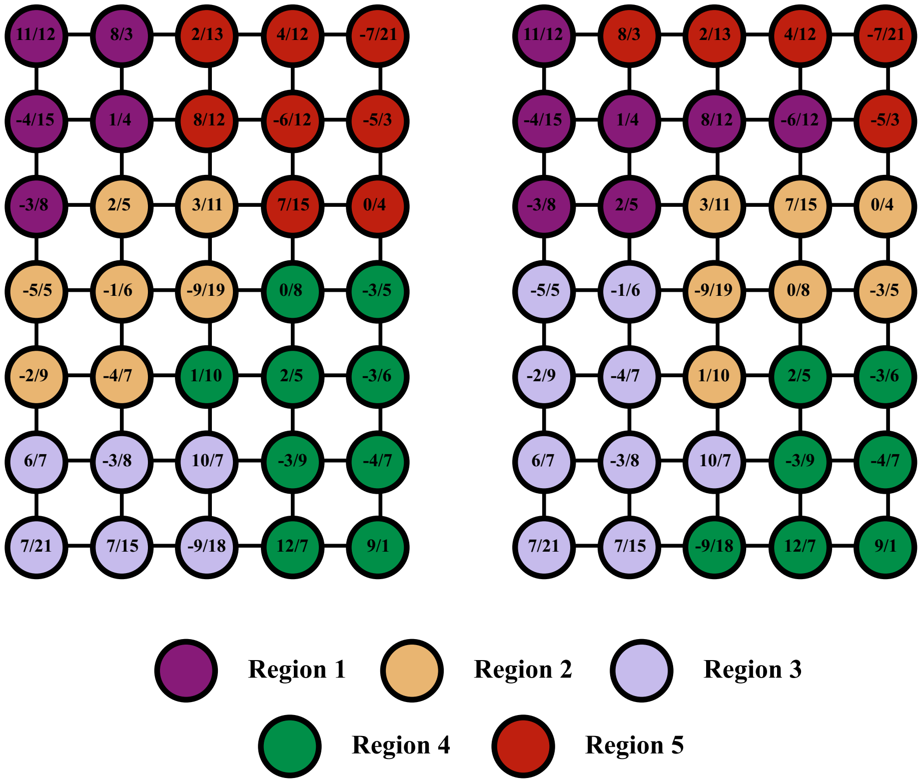

Fig. 4 Grid graph 05 $\times$ 07: best solution for the first ILP formulation and optimal solution for the second ILP formulation

population of the districts for all graphs was set to 40%. It means that a and b are respectively:

$$a = \frac{(1-0.4)}{m}\sum_{i=1}^{m} p_i, \quad b = \frac{(1+0.4)}{m}\sum_{i=1}^{m} p_i$$

The first column contains the instance name. For example graph $graph04x07$ is a grid graph with 4 columns and 7 rows. The number of districts is given in the second column. Next two columns represent the optimal solution and running time for the first formulation. The optimal soluton gives minimal total moment of inertia. The last four columns represent solutions and running time of the second ILP formulations. In the columns 5 and 6 are results of cases where a total moment of inertia is bounded by solution of the first formulation as a lower bound, while upper bound is set approximately as double value of the lower bound. In the last pair of columns are given results where upper bound is set on 25% larger than lower bound. In cases where running time exceeded 10^6 seconds the best found result is presented, which is indicated by asterisk (*). If the running time for the first formulation was greater than 10^6 s, lower bounds for the second formulation where set to be 10% smaller than the best solution of the first formulation.

Solutions for both ILP formulation (results from columns 3 and 7) for instance grid05x07 are presented on Fig. 4.

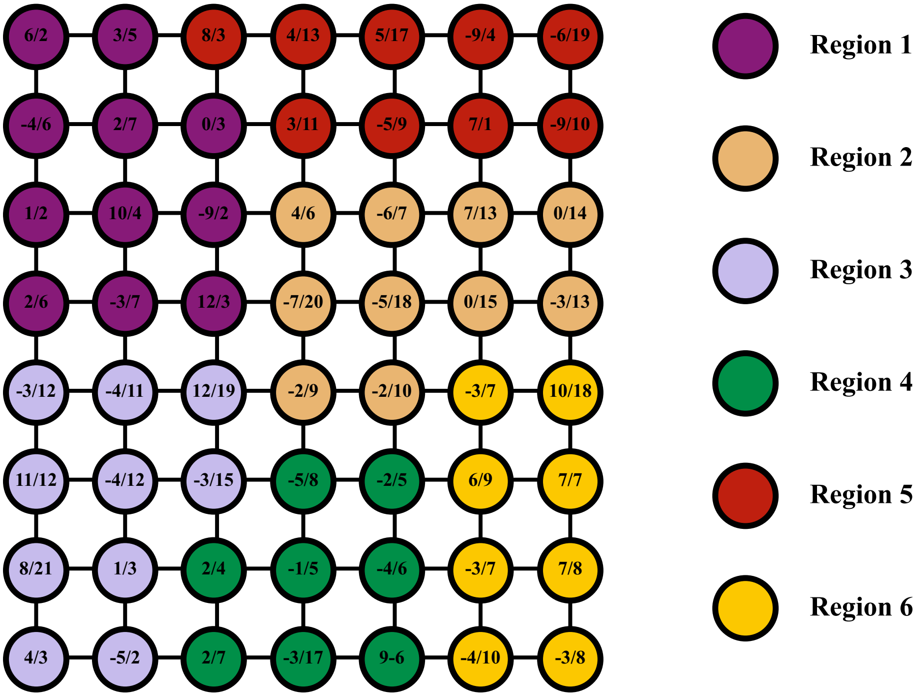

Fig. 5 Best solution for the first ILP formulation on instance grid graph with 8 rows and 7 columns which was partitioned in 6 districts

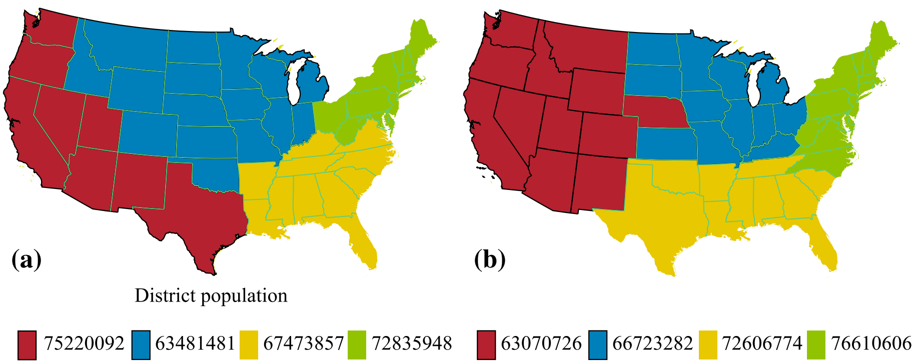

Fig. 6 Best solution for the first ILP formulation on 4 region partition of conterminous forty-eight US states

On Fig. 5, is represented the best solution of the first ILP formulation for the instance $graph07x08$ obtained after 10^6 s, since the largest instances presented in Duque et al. (2011) are up to 49 vertices. The value of objective function which measures total moment of inertia is 814. The best solution is 814 . Solution for the second ILP formulation with 25% tolerance of the moment of inertia could not be obtained due to insufficient memory (running time was 274,231.206 s). Political data and populations of territorial units are same as in previous example.

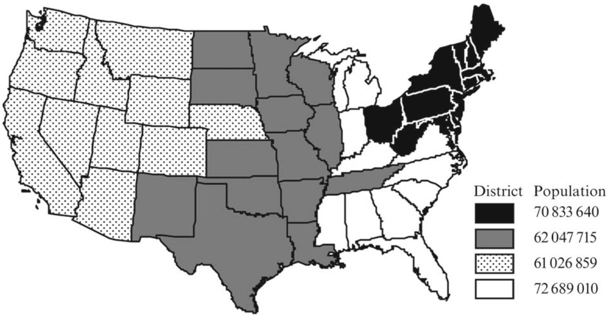

Fig. 7 Best solution for the ILP formulation from Shirabe (2009) on 4 region partition of conterminous forty-eight US states

Finally, in order to compare our results with other solutions obtained by solving other mathematical formulations of PD problem we present on Fig. 6 our results obtained by solving the first ILP formulation on example presented in Shirabe (2009), where study area is conterminous forty-eight US states partitioned in 4 regions. Population numbers are for US in year 2005 while geographical centers can be found in https://en.wikipedia.org/wiki/List_of_geographic_centers_of_the_United_States. On Fig. 6a is presented solution where distances between states are the shortest paths in corresponding graph while on Fig. 6b distances are euclidean distances between geographical centers of given states. The tolerance on population numbers is set on 10%. Both solutions were obtained after 10^6 s by CPLEX solver. Solution obtained in paper Shirabe (2009) is presented in Fig. 7. By comparing the pictures it can be easily concluded that both our solutions, though still not optimal, achieved much better level of compactness.

5 Conclusions

This paper considers the political districting problem. A new ILP formulations are presented with the proof of their correctness. The population equality, contiguity and compactness are completely modeled. The experimental results illustrate general aspects of all major criteria. The future work can evolve in several directions, including construction of some new methods based on this presented formulations, by solving some similar optimization problems, and testing the proposed model on large scale instances using some powerful multiprocessor systems.

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