Intent Matching and partially fillable orders

Theory:

Let us consider a set of users sending swap orders with limit price.

When a swap order for the i-th user is executed, we have that:

$$\begin{cases} S_i^e \\ B_i^e \\ \pi_i^e \end{cases}$$

Where S_i^e = executed sell amount (of token x)

B_i^e = executed buy amount (of token y)

π_i^e = executed price

Where $\pi_i^e = \frac{S_i^e}{B_i^e} \leq \frac{S_i^{max}}{B_i^{min}} = \pi_i^{lim}$

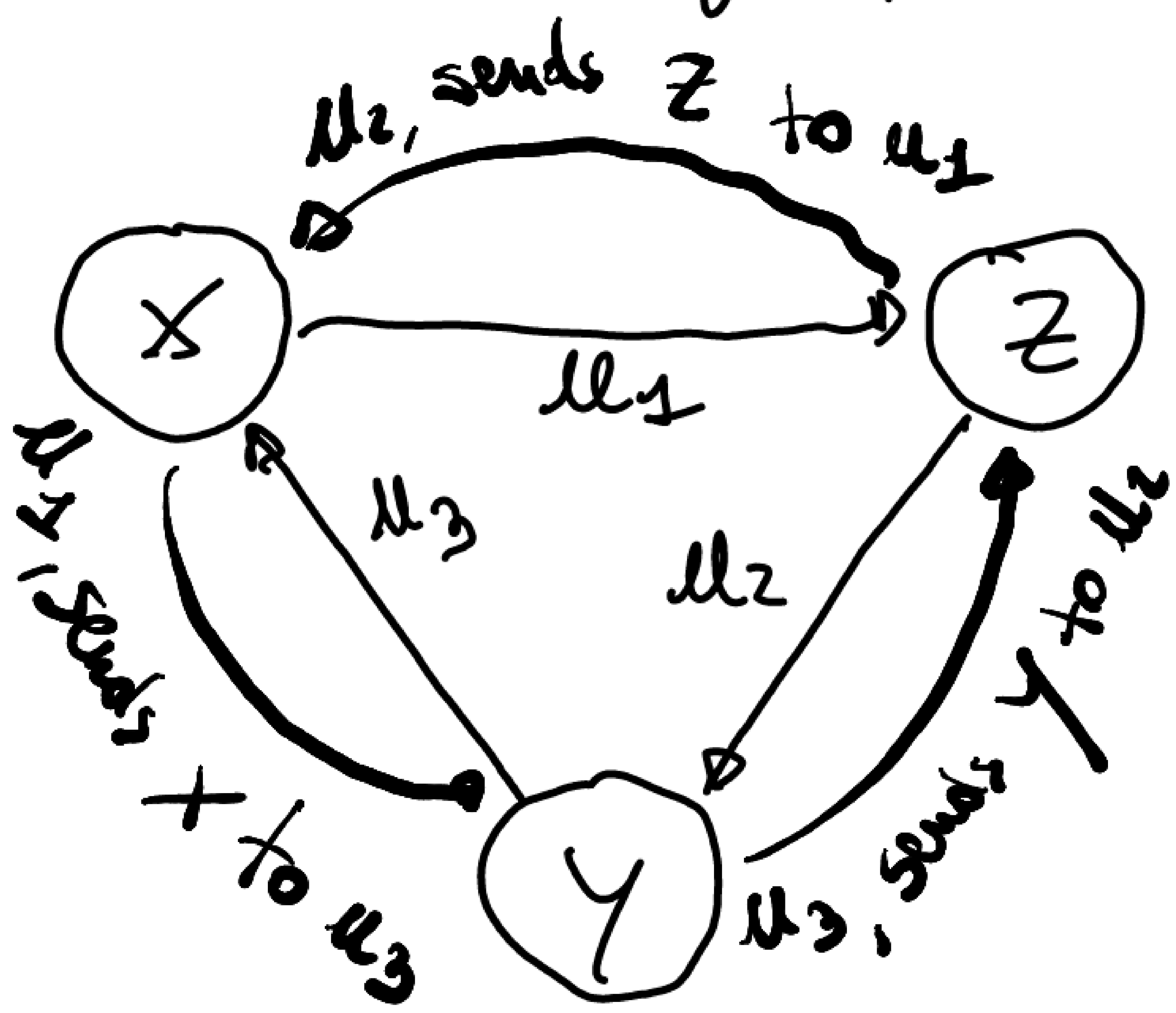
S_i^{max} → Max sell amount

B_i^{min} → Min buy amount

We can think each user's intent as a directed edge connecting two nodes, which are the x and y tokens

User 1: Sells x
Buys y

Thanks to this, finding matching intents, means to find closed loops in such intent-multigraph.



User	intent	result
u_1:	buy z sell x	$z \xleftarrow{\text{from}} u_2$ $x \xrightarrow{\text{to}} u_3$
u_2:	buy y sell z	$y \xleftarrow{\text{from}} u_3$ $z \xrightarrow{\text{to}} u_1$
u_3:	buy x sell y	$x \xleftarrow{\text{from}} u_1$ $y \xrightarrow{\text{to}} u_2$

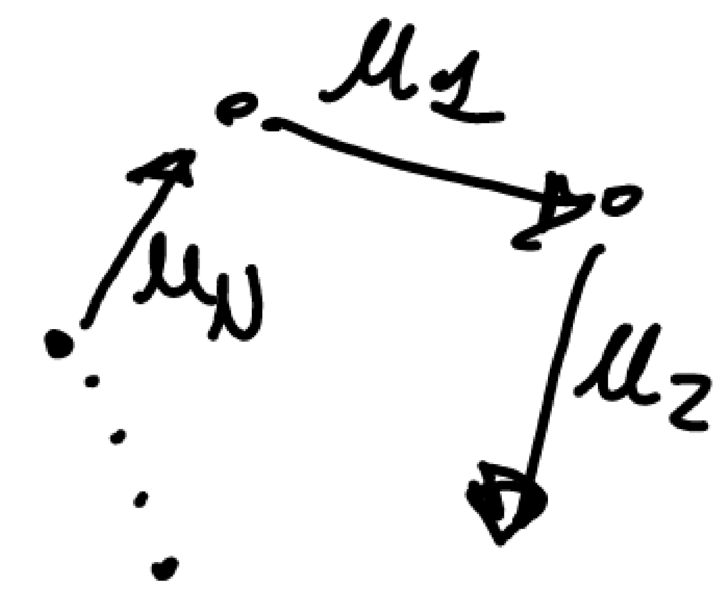
This will be valid only if:

$$\frac{S_i^e}{B_i^e} = \pi_i^e \leq \pi_i^{lim}, \quad \forall i \in \text{loop}.$$

Which, in an N-edged loop reads also:

$$\frac{S_i^e}{S_{i-1}^e} = \pi_i^e \leq \pi_i^{lim},$$

where, if $i = 1$, $i-1 = N$



Now let's search for the solutions.

No Partial fill:

If no intent is partially fillable, then, for each user $S_i^e \equiv S_i^{max}$.

Thus, the solution to the problem exists only if:

$$\frac{S_i^{max}}{S_{i-1}^{max}} \leq \Pi_i^{lim}, \quad \forall i \in loop.$$

This condition can be checked in parallel for each i. Thus it's efficiently verifiable.

If the solution exists, then $S_i^e = S_i^{max}$, $\forall i \in loop$.

All Partially Fillable

If the intents will all admit partial fill, the situation is trickier.

Let's consider L, a N-dimensional loop.

A solution to the intent-matching problem exists if $\frac{S_i^e}{S_{i-1}^e} \leq \pi_i^{lim}$, $\forall i \in L$.

This problem is self-consistent, i.e. S_i^e will depend on S_{i-1}^e.

The minimal condition acceptable by each user is such that:

$$\frac{S_i^e}{S_{i-1}^e} = \pi_i^{lim}, \forall i \in L.$$

Let's now apply the log function:

$$\log\left(\frac{S_i^e}{S_{i-1}^e}\right) = \log\left(\pi_i^{lim}\right).$$

Let's call $\log(a) = \tilde{a}$.

Thus we have that:

$\tilde{S}_i^e - \tilde{S}_{i-1}^e = \tilde{\pi}_i^{lim}$. Which can be written in matrix form as:

$$\underline{\underline{M}}\,\underline{\tilde{S}} = \underline{\tilde{\pi}}^{lim}. \quad \text{Where } \underline{\tilde{a}} = \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \vdots \\ \tilde{a}_N \end{bmatrix},$$

and

$$M = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & -1 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & \\ & & \ddots & \ddots & & \\ & & & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{N \times N}$$

M is a $N \times N$ matrix, with ones on the diagonal and -1 on the lower sub-diagonal.

However $\det(M) = 0$, $\forall N$.

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break:

indeed $N=2$, $M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. $\det = 0$.

$N=3$, $M = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{\text{Laplace}} \det = 0$

$\det(M^{N \times N}) \equiv 0$, since $\text{row}_1 = (-1) \cdot \sum_{i>1} \text{row}_i$

Thus our problem either has $\emptyset$ or ∞ solutions.

To evaluate if at least 1 solution, and thus ∞, exists, we use the Rouché-Capelli theorem, i.e.:

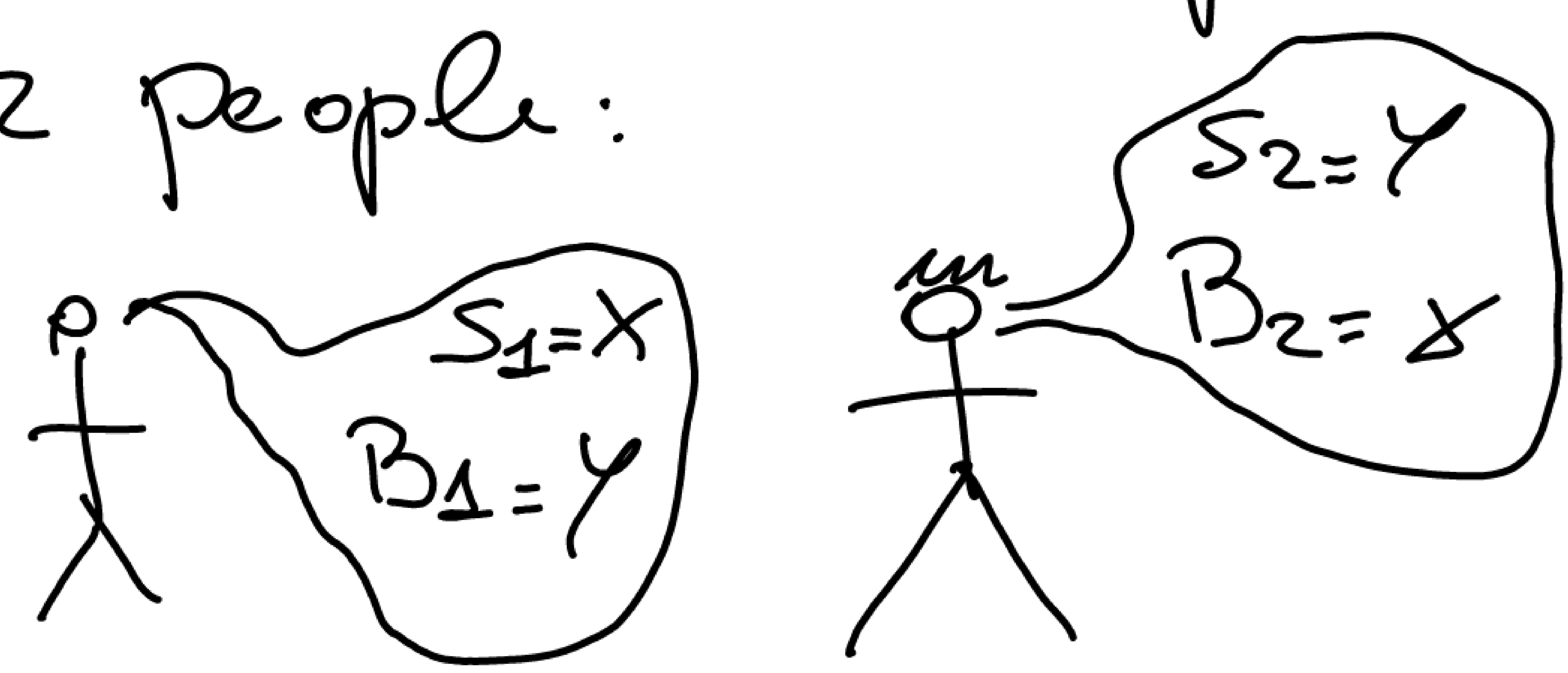
Given $\underline{\underline{M}}\ \underline{\hat{S}} = \underline{\tilde{\pi}}$

if $\text{Rank}(M) = \text{Rank}([M|\tilde{\pi}])$,

then $\exists\ \underline{\hat{S}_0}$ s.t. $\underline{\underline{M}}\ \underline{\hat{S}_0} = \underline{\tilde{\pi}}$.

Example:

2 people:



Let us assume that they agree on the relative price of x and y.

Then $\pi_2 = \frac{1}{u_1}$.

Our system will be:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{S_1} \\ \widetilde{S_2} \end{bmatrix} = \begin{bmatrix} \widetilde{u_1} \\ -\widetilde{u_1} \end{bmatrix}$$

$Rk(M) = 1$, but

$$Rk\left(\begin{bmatrix} 1 & -1 & \widetilde{u_1} \\ -1 & 1 & -\widetilde{u_1} \end{bmatrix}\right) = 1.$$

Thus, there are ∞ solutions.

This makes sense, since if u_1 and u_2 agree on the price π_1, then u_1 and u_2 can exchange whatever amount of x and y

under the only condition that π is ok.

Now Rouché-Capelli can be used rather efficiently (e.g. with Python).

However, once we checked that at least a solution exists, how can we search for the optimal one? (i.e. the solution that leaves the fewest "leftovers"?)

Stay tuned.