Applied Stochastic Analysis
Homework assignment 10

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Exercise 1

(a) By applying Itô formula, we have

$$
\begin{aligned}
I_t^{(3)} &\doteq \int_0^t \int_0^{s_1} \int_0^{s_2} dW_{s_3}\, dW_{s_2}\, dW_{s_1} = \int_0^t \int_0^{s_1} W_{s_2}\, dW_{s_2}\, dW_{s_1} \\
&= \int_0^t \left[\int_0^{s_1} d\left(\frac{1}{2}W_{s_2}^2\right) - \frac{1}{2}\int_0^{s_1} ds_2 \right] dW_{s_1} = \frac{1}{2}\int_0^t W_{s_1}^2\, dW_{s1} - \frac{1}{2}\int_0^t s_1\, dW_{s1} \\
&= \frac{1}{6}\int_0^t d(W_{s_1}^3) - \frac{1}{2}\int_0^t W_{s_1}\, ds_1 - \frac{1}{2}\int_0^t d(s_1 W_{s_1}) + \frac{1}{2}\int_0^t W_{s_1}\, ds_1 \\
&= \frac{1}{6}W_t^3 - \frac{1}{2}tW_t.
\end{aligned}
$$

(b) We proceed by induction. For $k = 3$ the equality holds due to part (a) and to

$$
I_t^{(2)} = \frac{1}{2}W_t^2 - \frac{1}{2}t, \qquad I_t^{(1)} = W_t.
$$

First of all, we notice that

$$
I_t^{(k+1)} = \int_0^t I_{s_1}^{(k)}\, dW_{s_1} \qquad \Longleftrightarrow \qquad dI_t^{(k+1)} = I_t^{(k)}\, dW_t.
$$

Then, if we suppose that the recursion formula holds for k, for $k+1$ we get

$$
dI_t^{(k+1)} = I_t^{(k)}\, dW_t = \frac{1}{k}\Big(W_t I_t^{(k-1)} - tI_t^{(k-2)}\Big) dW_t.
$$

On the other hand

$$
\begin{aligned}
d\left[\frac{1}{k+1}\Big(W_t I_t^{(k)} - tI_t^{(k-2)}\Big)\right] &= \frac{1}{k+1}\Big[d\Big(W_t I_t^{(k)}\Big) - d\Big(tI_t^{(k-1)}\Big)\Big] = \\
&= \frac{1}{k+1}\Big[I_t^{(k)} dW_t + W_t\, dI_t^{(k)} + dW_t\, dI_t^{(k)} - t dI_t^{(k-1)} - I_t^{(k-1)}\, dt - dt dI_t^{(k-1)}\Big] \\
&= \frac{1}{k+1}\Big[I_t^{(k)} dW_t + W_t I_t^{(k-1)} dW_t + I_t^{(k-1)} (dW_t)^2 - tI_t^{(k-2)}\, dW_t - I_t^{(k-1)}\, dt\Big] \\
&= \frac{1}{k+1}\Big[I_t^{(k)} + W_t I_t^{(k-1)} - tI_t^{(k-2)}\Big] dW_t \\
&= \frac{1}{k+1}\left[\frac{1}{k}\Big(W_t I_t^{(k-1)} - tI_t^{(k-2)}\Big) + W_t I_t^{(k-1)} - tI_t^{(k-2)}\right] dW_t \\
&= \frac{1}{k+1}\left(1 + \frac{1}{k}\right)\Big(W_t I_t^{(k-1)} - tI_t^{(k-2)}\Big) dW_t = \frac{1}{k}\Big(W_t I_t^{(k-1)} - tI_t^{(k-2)}\Big) dW_t.
\end{aligned}
$$

This shows that

$$dI_t^{(k+1)} = d\left[\frac{1}{k+1}\Big(W_t I_t^{(k)} - t I_t^{(k-1)}\Big)\right],$$

which implies

$$I_t^{(k+1)} = \frac{1}{k+1}\Big(W_t I_t^{(k)} - t I_t^{(k-1)}\Big),$$

i.e. what we wanted to prove.

Exercise 3

If we consider the SDE

$$dX_t = \lambda X_t\, dt + \mu X_t\, dW_t,$$

the Milstein scheme gives

$$X_{n+1} = X_n + \lambda X_n \Delta t + \mu X_n \Delta W + \frac{1}{2}\mu^2 X_n((\Delta W)^2 - \Delta t),$$

where ΔW is a r.v. $\sim N(0, \Delta t)$. Then

$$\begin{aligned}
\mathbb{E}|X_{n+1}|^2 &= \mathbb{E}|X_n|^2\, \mathbb{E}\left[\left(1 + \lambda\Delta t + \mu\Delta W + \frac{1}{2}\mu^2((\Delta W)^2 - \Delta t)\right)^2\right] \\
&= \mathbb{E}|X_n|^2 \left[(1+\lambda\Delta t)^2 + \mu^2\Delta t + \frac{1}{4}\mu^4\, \mathbb{E}\big[((\Delta W)^2 - \Delta t))^2\big]\right] \\
&= \mathbb{E}|X_n|^2 \left[(1+\lambda\Delta t)^2 + \mu^2\Delta t + \frac{1}{2}\mu^4(\Delta t)^2\right].
\end{aligned}$$

Therefore the stability region for the Milstein scheme is

$$R_1 = \{\ (x,y) = (\lambda\Delta t, \mu^2\Delta t)\ :\ (1+x)^2 + y + \frac{1}{2}y^2 < 1\ \}.$$

We know that the stability region for the SDE is $R_2 = \{\ (x,y)\ :\ y < -2x\ \}$ and the one for the EM scheme is $R_3 = \{\ (x,y)\ :\ (1+x)^2 + y < 1\ \}$.