Applied Stochastic Analysis
Homework assignment 7

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Exercise 1

Since every Y_t^i, $i = 1, \ldots, d$, is a linear combination of continuous independent (centered) Gaussian processes then every Y_t^i is a (centered) Gaussian process too. Hence, we only need to examine its covariance function. We have

$$\mathbb{E}[Y_s^i Y_t^j] = \sum_{k=1}^{d} \sum_{n=1}^{d} Q_{ik} Q_{jn} \mathbb{E}[W_s^k W_t^n] = (s \wedge t) \sum_{k=1}^{d} \sum_{n=1}^{d} Q_{ik} Q_{jn} \delta_{kn}$$

$$= (s \wedge t) \sum_{k=1}^{d} Q_{ik} Q_{jk} = (s \wedge t)(Q \cdot Q^t)_{ij} = (s \wedge t)\delta_{ij}.$$

The above concludes the proof that $\mathbf{Y}_t = (Y_t^1, \ldots, Y_t^d)$ is a d-dimensional Brownian motion.

Exercise 2

(a) First of all, we notice that, by definition of that T_a, $\tilde{B}_t$ is a.s. continuous. Hence, to show that $\tilde{B}_t$ is a B.m. (Brownian motion), it is sufficient to show that

$$P(\tilde{B}_s \leq x, \tilde{B}_t \leq y) = P(B_s \leq x, B_t \leq y)$$

for every $0 \leq s \leq t$ and $x, y \in \mathbb{R}$ (i.e. that $(\tilde{B}_t, \tilde{B}_s)$ and (B_t, B_s) have the same distribution). Let $\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)$. By the strong Markov property, we know that $W_t \doteq B_{t+T_a} - B_{T_a}$ is a B.m. independent by $\mathcal{F}_{T_a}$. Notice that

$$\tilde{W}_t \doteq \tilde{B}_{t+T_a} - \tilde{B}_{T_a} = 2a - B_{t+T_a} - B_{T_a} = B_{T_a} - B_{t+T_a} = -W_t,$$

so $\tilde{W}_t$ is a B.m. independent by $\mathcal{F}_{T_a}$ too. We can write

$$B_t = B_t \mathbb{1}_{\{t \leq T_a\}} + (W_{t-T_a} + B_{T_a}) \mathbb{1}_{\{t > T_a\}} \qquad \text{and} \qquad \tilde{B}_t = B_t \mathbb{1}_{\{t \leq T_a\}} + (\tilde{W}_{t-T_a} + B_{T_a}) \mathbb{1}_{\{t > T_a\}}.$$

Now

$$P(\tilde{B}_s \leq x, \tilde{B}_t \leq y) = P(B_s \leq x, B_t \leq y, t \leq T_a) + P(B_s \leq x, \tilde{W}_{t-T_a} + B_{T_a} \leq y, s \leq T_a < t)$$
$$+ P(\tilde{W}_{t-T_a} + B_{T_a} \leq x, \tilde{W}_{t-T_a} + B_{T_a} \leq y, T_a < s). \tag{1}$$

We can evaluate the second term in the rhs as follows:

$$P(B_s \leq x, \tilde{W}_{t-T_a} + B_{T_a} \leq y, s \leq T_a < t) = \mathbb{E}[P(B_s \leq x, \tilde{W}_{t-T_a} + B_{T_a} \leq y \,|\, \mathcal{F}_{T_a}) \mathbb{1}_{\{s \leq T_a < t\}}],$$

where
$$P(B_s \leq x, \tilde{W}_{t-T_a} + B_{T_a} \leq y \,|\, \mathcal{F}_{T_a}) = \tilde{f}(B_s, T_a)$$

with
$$\tilde{f}(z, r) \doteq \mathbb{1}_{\{z \leq x\}} P(\tilde{W}_{t-r} + a \leq y) = \mathbb{1}_{\{z \leq x\}} P(W_{t-r} + a \leq y) \doteq f(z, r).$$

Hence
$$P(B_s \leq x, \tilde{W}_{t-T_a} + B_{T_a} \leq y, s \leq T_a < t) = \mathbb{E}[f(B_s, T_a)) \mathbb{1}_{\{s \leq T_a < t\}}]$$
$$= P(B_s \leq x, W_{t-T_a} + B_{T_a} \leq y, s \leq T_a < t). \quad (2)$$

Similarly one can prove that
$$P(\tilde{W}_{t-T_a} + B_{T_a} \leq x, \tilde{W}_{t-T_a} + B_{T_a} \leq y, T_a < s) = P(W_{t-T_a} + B_{T_a} \leq x, W_{t-T_a} + B_{T_a} \leq y, T_a < s). \quad (3)$$

From (1), (2) and (3) one gets that
$$P(\tilde{B}_s \leq x, \tilde{B}_t \leq y) = P(B_s \leq x, B_t \leq y, t \leq T_a) + P(B_s \leq x, W_{t-T_a} + B_{T_a} \leq y, s \leq T_a < t)$$
$$+ P(W_{t-T_a} + B_{T_a} \leq x, W_{t-T_a} + B_{T_a} \leq y, T_a < s)$$
$$= P(B_s \leq x, B_t \leq y).$$

This completes the proof that $\tilde{B}_t$ is a B.m.

(b) It holds that
$$P(M_t > a) = P(M_t > a, B_t > a) + P(M_t > a, B_t \leq a) = P(B_t > a) + P(M_t > a, B_t \leq a),$$

since $B_t > a$ implies $M_t = \sup_{s \leq t} B_s \geq B_t > a$.

(c) Since $\{M_t > a\} = \{T_a \leq t\}$ and if $T_a \leq t$ then $\tilde{B}_t = 2a - B_t$, then
$$P(M_t > a, B_t \leq a) = P(T_a \leq t, 2a - B_t \geq a) = P(T_a \leq t, \tilde{B}_t \geq a) = P(\tilde{B}_t \geq a),$$

since $\{T_a > t, \tilde{B}_t \geq a\} = \{T_a > t, B_t \geq a\} = \emptyset$.

(d) Thanks to part (a), (b) and (c) we have that
$$P(M_t > a) = P(B_t > a) + P(\tilde{B}_t \geq a) = 2 \int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \, dx = \int_a^\infty \sqrt{\frac{2}{\pi t}} e^{-x^2/2t} \, dx.$$

Exercise 3

Exercise 4

(a) We showed in class that
$$Q_t^\sigma(W_t) \xrightarrow{L^2} t \qquad \text{as } |\sigma| \to 0. \quad (4)$$

Hence it is sufficient to notice that
$$Q_t^\sigma(aW_t + bt) = \sum_{i=0}^{n-1} |aW_{t_{i+1}} + bt_{i+1} - aW_{t_i} - bt_i|^2 = \sum_{i=0}^{n-1} |a(W_{t_{i+1}} - W_{t_i}) + b(t_{i+1} - t_i)|^2$$
$$= a^2 Q_t^\sigma(W_t) + ab \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) + b^2 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$

Now

$$\mathbb{E}\left[\left(\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i)\right)^2\right] = \sum_{i=0}^{n-1}\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2](t_{i+1} - t_i)^2 = \sum_{i=0}^{n-1}(t_{i+1} - t_i)^4 \le |\sigma|^3 t,$$

where the first equality holds since $\mathbb{E}[(W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})] = 0$ for every $i \ne j$, and so

$$ab\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0. \tag{5}$$

Combining (4) and (5) and the fact that $\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2 \le |\sigma|t \xrightarrow{L^2} 0$ as $|\sigma| \to 0$, we get that

$$Q_t^\sigma(aW_t + bt) \xrightarrow{L^2} t \qquad \text{as } |\sigma| \to 0.$$

(b) Let's consider

$$Q_{t,4}^\sigma(W_t) = \sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^4.$$

We have

$$\mathbb{E}[(Q_{t,4}^\sigma(W_t))^2] = \sum_{i=0}^{n-1}\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^8] + \sum_{\substack{i,j=0\\i\ne j}}^{n-1}\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^4(W_{t_{j+1}} - W_{t_j})^4]$$

Now

$$\sum_{i=0}^{n-1}\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^8] = 105\sum_{i=0}^{n-1}(t_{i+1} - t_i)^4 \le 105|\sigma|^3 t \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0 \tag{6}$$

and

$$\sum_{\substack{i,j=0\\i\ne j}}^{n-1}\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^4(W_{t_{j+1}} - W_{t_j})^4] = \sum_{\substack{i,j=0\\i\ne j}}^{n-1}\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^4]\,\mathbb{E}[(W_{t_{j+1}} - W_{t_j})^4]$$

$$= 225\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2 \sum_{\substack{j=0\\j\ne i}}^{n-1}(t_{j+1} - t_j)^2$$

$$\le 225\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2|\sigma|t$$

$$\le 225|\sigma|^2 t^2 \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0. \tag{7}$$

Hence (6) and (7) imply that

$$Q_{t,4}^\sigma(W_t) \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0.$$

Exercise 5

Let us denote by X_t^x the solution to $\dot{X}_t = a(X_t)$, $X_0 = x$. If $f \in C_b^2(\mathbb{R})$, then

$$(\mathcal{L}f)(x) = \lim_{t\to 0^+}\frac{f(X_t^x) - f(x)}{t} = \frac{d}{dt}f(X_t^x)\Big|_{t=0} = f'(X_0^x)\frac{d}{dt}X_t^x\Big|_{t=0} = f'(x)a(X_0^x) = a(x)f'(x),$$

i.e. $\mathcal{L} = a\,\partial_x$.

Exercise 6

Given $f \in C_b^2$, we have, for $x \in K$, where $K \subset \mathbb{R}$ is a compact set such that $\mathrm{supp}\{p(\cdot, t|\cdot, 0)\} \subseteq K^2$,

$$(\mathcal{L}f)(x) = \lim_{t\to 0^+} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t} = \lim_{t\to 0^+} \frac{1}{t} \int_K (f(y) - f(x))\, p(y, t|x, 0)\, dy. \tag{8}$$

If $\varepsilon > 0$ we can split the integral in (8) as

$$
\begin{aligned}
t^{-1} \int_K (f(y) - f(x))\, p(y, t|x, 0)\, dy = {} & t^{-1} \int_{y \in K, |x-y| \geq \varepsilon} (f(y) - f(x))\, p(y, t|x, 0)\, dy \\
& + t^{-1} f'(x) \int_{y \in K, |x-y| < \varepsilon} (y - x)\, p(y, t|x, 0)\, dy \\
& + t^{-1}/2 \int_{y \in K, |x-y| < \varepsilon} f''(\xi_y)(y - x)^2\, p(y, t|x, 0)\, dy,
\end{aligned}
$$

where ξ_y is a point between x and y. Now,

$$t^{-1} \int_{y \in K, |x-y| \geq \varepsilon} (f(y) - f(x))\, p(y, t|x, 0)\, dy \xrightarrow[t\to 0^+]{} \int_{y \in K, |x-y| \geq \varepsilon} (f(y) - f(x))\, W(y|x)\, dy + O(\varepsilon),$$

$$t^{-1} f'(x) \int_{y \in K, |x-y| < \varepsilon} (y - x)\, p(y, t|x, 0)\, dy \xrightarrow[t\to 0^+]{} f'(x) O(\varepsilon),$$

$$
\begin{aligned}
t^{-1}/2 \int_{y \in K, |x-y| < \varepsilon} f''(\xi_y)(y - x)^2\, p(y, t|x, 0)\, dy \leq {} & \\
\leq \|f''\|_\infty t^{-1}/2 \int_{y \in K, |x-y| < \varepsilon} (y - x)^2\, & p(y, t|x, 0)\, dy \xrightarrow[t\to 0^+]{} \|f''\|_\infty O(\varepsilon)/2.
\end{aligned}
$$

hence, if we send $\varepsilon \to 0^+$, we get

$$(\mathcal{L}f)(x) = \int_K (f(y) - f(x))\, W(y|x)\, dy = \int_{\mathbb{R}} f(y)\, W(y|x)\, dy - f(x)\lambda(x).$$