Applied Stochastic Analysis
Homework assignment 10

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Exercise 1

(a) By applying Itô formula, we have

$$
\begin{aligned}
I_t^{(3)} &\doteq \int_0^t \int_0^{s_1} \int_0^{s_2} dW_{s_3}\, dW_{s_2}\, dW_{s_1} = \int_0^t \int_0^{s_1} W_{s_2}\, dW_{s_2}\, dW_{s_1} \\
&= \int_0^t \left[\int_0^{s_1} d\Big(\frac{1}{2} W_{s_2}^2\Big) - \frac{1}{2}\int_0^{s_1} ds_2 \right] dW_{s_1} = \frac{1}{2}\int_0^t W_{s_1}^2\, dW_{s1} - \frac{1}{2}\int_0^t s_1\, dW_{s1} \\
&= \frac{1}{6}\int_0^t d\big(W_{s_1}^3\big) - \frac{1}{2}\int_0^t W_{s_1}\, ds_1 - \frac{1}{2}\int_0^t d(s_1 W_{s_1}) + \frac{1}{2}\int_0^t W_{s_1}\, ds_1 \\
&= \frac{1}{6} W_t^3 - \frac{1}{2} t W_t .
\end{aligned}
$$

(b) We proceed by induction. For $k = 3$ the equality holds due to part (a) and to

$$
I_t^{(2)} = \frac{1}{2} W_t^2 - \frac{1}{2} t, \qquad I_t^{(1)} = W_t .
$$

First of all, we notice that

$$
I_t^{(k+1)} = \int_0^t I_{s_1}^{(k)}\, dW_{s_1} \qquad \Longleftrightarrow \qquad dI_t^{(k+1)} = I_t^{(k)}\, dW_t .
$$

Then, if we suppose that the recursion formula holds for k, for $k+1$ we get

$$
dI_t^{(k+1)} = I_t^{(k)}\, dW_t = \frac{1}{k}\Big(W_t I_t^{(k-1)} - t I_t^{(k-2)}\Big)\, dW_t .
$$

On the other hand

$$
\begin{aligned}
d\left[\frac{1}{k+1}\Big(W_t I_t^{(k)} - t I_t^{(k-2)}\Big)\right] &= \frac{1}{k+1}\Big[d\Big(W_t I_t^{(k)}\Big) - d\Big(t I_t^{(k-1)}\Big)\Big] \\
&= \frac{1}{k+1}\Big[I_t^{(k)} dW_t + W_t\, dI_t^{(k)} + dW_t\, dI_t^{(k)} - t dI_t^{(k-1)} - I_t^{(k-1)}\, dt - dt dI_t^{(k-1)} \Big] \\
&=
\end{aligned}
$$