Applied Stochastic Analysis
Homework assignment 11

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Exercise 1

(a) The corresponding Fokker-Plank equation is

$$\rho_t = -\partial_x(\lambda x \rho) + \partial^2_{xx}\left(\frac{1}{2}\sigma^2 x^2 \rho\right) = (\sigma^2 - \lambda)\rho + (2\sigma^2 - \lambda)x\rho_x + \frac{1}{2}\sigma^2 x^2 \rho_{xx}.$$

(b) If $\rho = \rho(x)$ is a stationary distribution, then it must satisfy

$$0 = \mathcal{L}^*\rho = (\sigma^2 - \lambda)\rho + (2\sigma^2 - \lambda)x\rho' + \frac{1}{2}\sigma^2 x^2 \rho''. \tag{1}$$

This is a second order ODE, so it admits a 2 dimensional family of solutions. In particular we look for solutions of the form $\rho(x) = x^\alpha$, $\alpha \in \mathbb{R}$. In this case (1) becomes

$$0 = x^\alpha \left[(\sigma^2 - \lambda) + \alpha(2\sigma^2 - \lambda) + \frac{1}{2}\sigma^2\alpha(\alpha - 1)\right]$$

which implies

$$\frac{1}{2}\sigma^2\alpha^2 + \left(\frac{3}{2}\sigma^2 - \lambda\right)\alpha + (\sigma^2 - \lambda) = 0,$$

i.e.

$$\begin{aligned}\alpha &= \frac{1}{\sigma^2}\left[\left(\lambda - \frac{3}{2}\sigma^2\right) \pm \sqrt{\left(\frac{3}{2}\sigma^2 - \lambda\right)^2 - 2\sigma^2(\sigma^2 - \lambda)}\right] \\ &= \frac{1}{\sigma^2}\left[\left(\lambda - \frac{3}{2}\sigma^2\right) \pm \sqrt{\left(\frac{1}{2}\sigma^2 - \lambda\right)^2}\right] = \begin{cases} -1 \\ \frac{2(\lambda - \sigma^2)}{\sigma^2} \end{cases}.\end{aligned}$$

Therefore the stationary distribution must be of the form

$$\rho_s(x) = \frac{c_1}{x} + c_2 x^{\frac{2(\lambda-\sigma^2)}{\sigma^2}}$$

for some constants $c_1, c_2 \in \mathbb{R}$ (a part from the case $\sigma^2 = 2\lambda$; in this case it must have the form $\rho_s(x) = \frac{c_1}{x} + c_2\frac{\log x}{x}$). In any case no one of these functions is non negative integrable; therefore they can not be distributions. This means no stationary distribution exists.

(c) The n-th moment M_n must satisfy the PDE given by the backward Kolmogorov equation:

$$\partial_t M_n(x,t) = \mathcal{L} M_n(x,t) = \lambda x \partial_x M_n(x,t) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} M_n(x,t),$$

along with the initial conditions $M_n(x,0) = x^n$.

Exercise 2

(a) The boundary conditions for the operator $\mathcal{L}$ acting on $f = f(x,t)$ are given by

$$(\mathbf{j}\cdot\mathbf{n})f + \rho(a\cdot\nabla f)\cdot\mathbf{n} = 0 \quad \text{at } x = 0 \quad \Leftrightarrow \quad j(0,t)f(0,t) + \frac{1}{2}\rho(0,t)f'(0,t) = 0.$$

Using the boundary condition for $\mathcal{L}$, the above equation becomes

$$\rho(0,t)\left(\kappa f(0,t) + \frac{1}{2}f'(0,t)\right) = 0.$$

(b) We have that

$$\begin{aligned}
\dot{P}_{tot}(t) &= \int_0^\infty \partial_t \rho(x,t)\,dx = \int_0^\infty \mathcal{L}^*\rho(x,t)\,dx = \alpha\int_0^\infty \partial_x(x\rho(x,t))\,dx + \frac{1}{2}\int_0^\infty \partial_{xx}\rho(x,t)\,dx \\
&= \alpha[x\rho(x,t)]_0^\infty + \frac{1}{2}[\partial_x\rho(x,t)]_0^\infty = -\frac{1}{2}\partial_x\rho(0,t),
\end{aligned}$$

where we assumed that $\lim_{x\to+\infty}(x\rho(x,t)) = 0$ and $\lim_{x\to+\infty}\partial_x\rho(x,t) = 0$. Using the boundary condition

$$\kappa\rho(0,t) = -j(0,t) = -\frac{1}{2}\partial_x\rho(0,t),$$

we get that $\dot{P}_{tot}(t) = \kappa\rho(0,t)$. Thus the total probability is only conserved if $\rho(0,t) \equiv 0$.

Exercise 3

(a) The process $X_t \in [0,L]$ should satisfy the two SDEs:

$$\begin{aligned}
dX_t &= -v\,dt + \sigma\,dW_t \qquad && \text{if } X_t \in [0,d), \\
dX_t &= \sigma\,dW_t && \text{if } X_t \in [d,L].
\end{aligned}$$

(b) The backward equation for $\rho = \rho(x,t)$, $t \geq 0, x \in [0,L]$, is

$$\rho_t = \mathcal{L}^*\rho = -v\mathbb{1}_{\{x\in[0,d]\}}\rho_x + \frac{1}{2}\sigma^2\rho_{xx},$$

along with the boundary conditions

$$0 = \partial_x\rho(L,t) = v\rho(0,t) - \frac{1}{2}\sigma^2\partial_x\rho(0,t)$$

and initial condition $\rho(0,t) = \rho_0(t)$. The forward equation for $u = u(x,t) = \mathbb{E}^x f(X_t)$, $t \geq 0, x \in [0,L]$, is

$$u_t = \mathcal{L}u = v\mathbb{1}_{\{x\in[0,d]\}}u_x + \frac{1}{2}\sigma^2 u_{xx},$$

along with the boundary conditions $0 = \partial_x u(L,t) = \partial_x u(0,t)$ and initial condition $u(x,0) = f(x)$.

(c) The stationary distribution $\rho_s(x) = \rho^{(1)}(x) + \rho^{(2)}(x)$, where we denote $\rho^{(1)} = \rho_s|_{x\in[0,d]}$ and $\rho^{(2)} = \rho_s|_{x\in[d,L]}$. Then $\rho^{(1)}$ satisfies

$$0 = v\rho^{(1)}_x + \frac{1}{2}\sigma^2\rho^{(1)}_{xx} \quad \Rightarrow \quad \rho^{(1)}(x) = A_1 e^{2vx/\sigma^2} - \frac{1}{v}B_1$$

and $\rho^{(2)}$ satisfies

$$0 = \frac{1}{2}\sigma^2\rho^{(2)}_{xx} \quad \Rightarrow \quad \rho^{(2)}(x) = A_2 x + B_2.$$

If we impose the boundary (and continuity) conditions we get

$$\begin{aligned} v\rho^{(1)}(0) - \frac{1}{2}\sigma^2\partial_x\rho^{(1)}(0) = 0 \quad &\Rightarrow \quad vA_1 - B_1 - \frac{1}{2}\sigma^2 A_1 \frac{2v}{\sigma^2} = 0 \quad \Rightarrow \quad B_1 = 0, \\ \partial_x\rho(L) = 0 \quad &\Rightarrow \quad A_2 = 0, \\ \rho^{(1)}(d) = \rho^{(2)}(d) \quad &\Rightarrow \quad B_2 = A_1 e^{2vd/\sigma^2}. \end{aligned}$$

Also, we can find A_1 by imposing $\int_0^L \rho_s(x)\,dx = 1$, i.e.

$$A_1 = \left[\frac{\sigma^2}{2v}\left(e^{2vd/\sigma^2} - 1\right) + (L-d)e^{2vd/\sigma^2}\right]^{-1}.$$

Therefore the stationary distribution is given by

$$\rho_s(x) = A_1 \exp\left\{\frac{2v}{\sigma^2}\left(x\mathbb{1}_{\{x\in[0,d]\}} + d\mathbb{1}_{\{x\in(d,L]\}}\right)\right\}.$$

Exercise 4

(a) Be $V_t = \dot{X}_t$. Then we can write the Langevin equation as a first order system:

$$\begin{cases} dX_t = V_t\,dt \\ dV_t = -\frac{1}{m}(\gamma V_t + \nabla U(X_t))\,dt + \frac{1}{m}\sqrt{2}\sigma\,dW_t \end{cases}$$

i.e.

$$d\mathbf{X}_t = \mathbf{b}(\mathbf{X}_t)\,dt + \Sigma\,d\mathbf{W}_t,$$

where

$$\mathbf{X}_t = \begin{pmatrix} X_t \\ V_t \end{pmatrix}, \quad \mathbf{b}(\mathbf{X}_t) = \begin{pmatrix} V_t \\ -\frac{1}{m}(\gamma V_t + \nabla U(X_t)) \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 0 & 0 \\ \frac{1}{m}\sqrt{2}\sigma & 0 \end{pmatrix}.$$

(b) The corresponding Fokker-Plank equation is

$$\rho_t = \mathcal{L}^*_{(x,v)}\rho = -\nabla_{(x,v)}\cdot(\mathbf{b}\rho) + \nabla^2_{(x,v)} : (A\rho) \tag{2}$$

where

$$A = \frac{1}{2}\Sigma\Sigma^T = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma^2}{m^2} \end{pmatrix}.$$

Equation (2) can be written more explicitly as

$$\rho_t = -v\rho_x + \frac{1}{m}[\rho_v(\gamma v + \nabla U(x)) + \gamma\rho] + \frac{\sigma^2}{m^2}\rho_{vv} \tag{3}$$

(c) If $\rho = \rho_s(x,v) = Z^{-1}e^{-\beta H(x,v)}$, we have

$$\begin{aligned}
-v\rho_x &= -v(-\beta H_x)\rho = \beta v \nabla U(x)\rho, \\
\frac{1}{m}\rho_v(\gamma v + \nabla U(x)) &= -\frac{\beta}{m}H_v(\gamma v + \nabla U(x))\rho = -\beta v \nabla U(x)\rho - \beta\gamma v^2\rho, \\
\frac{\sigma^2}{m^2}\rho_{vv} &= \frac{\sigma^2}{m^2}(\rho H_v)_v = -\beta\frac{\sigma^2}{m^2}H_{vv} + \frac{\sigma^2}{m^2}\beta^2(H_v)^2 = \frac{\gamma}{m}\rho + \beta\gamma v^2\rho.
\end{aligned}$$

Then, using the above equations, (3) gives

$$\rho_t = \mathcal{L}^*_{(x,v)}\rho = \beta v\nabla U(x)\rho - \beta v \nabla U(x)\rho - \beta\gamma v^2\rho + \frac{\gamma}{m}\rho + \frac{\gamma}{m}\rho + \beta\gamma v^2\rho = 0.$$

(d) The steady-state flux is

$$\mathbf{j}_s = \mathbf{b}\rho - \nabla_{(x,v)} \cdot (A\rho) = \rho\begin{pmatrix} v \\ -\frac{1}{m}(\gamma v + \nabla U(x)) \end{pmatrix} - \rho_v \begin{pmatrix} 0 \\ \frac{\sigma^2}{m^2} \end{pmatrix}.$$

From this expression we see that $\mathbf{j}_s$ is not zero in general and neither if $\rho = \rho_s(x,v) = Z^{-1}e^{-\beta H(x,v)}$.

(e) Be $\mathbf{X}_t \in \mathbb{R}^n$, $\mathbf{V}_t = \dot{\mathbf{X}}_t$ and $\mathbf{Y}_t = (\mathbf{X}_t, \mathbf{V}_t)$. Then $\mathbf{Y}_t$ satisfies the SDEs system

$$d\mathbf{Y}_t = \mathbf{b}(\mathbf{Y}_t)\,dt + \Sigma(\mathbf{Y}_t)\,d\mathbf{W}_t,$$

where

$$\mathbf{b}(\mathbf{Y}_t) = \begin{pmatrix} \mathbf{V}_t \\ -\frac{1}{m}(\gamma(\mathbf{X}_t)\mathbf{V}_t + \nabla U(\mathbf{X}_t)) \end{pmatrix} \quad \text{and} \quad \Sigma(\mathbf{Y}_t) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \frac{1}{m}\sqrt{2}\sigma(\mathbf{X}_t) & \mathbf{0} \end{pmatrix}.$$

The corresponding Fokker-Plank equation is

$$\rho_t = \mathcal{L}^*_{(x,v)}\rho = -\nabla_{(x,v)} \cdot (\mathbf{b}\rho) + \nabla^2_{(x,v)} : (A\rho) \tag{4}$$

where

$$A = A(\mathbf{x}) = \frac{1}{2}\Sigma(\mathbf{x})\Sigma(\mathbf{x})^T = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\beta}{m^2}\gamma(\mathbf{x}) \end{pmatrix}.$$