Applied Stochastic Analysis
Homework assignment 4

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Exercise 1

(a) Calculating its eigenvalues/vectors, Q can be decomposed as

$$Q = \begin{pmatrix} 1 & \mu \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\lambda - \mu \end{pmatrix} \begin{pmatrix} \lambda/(\lambda+\mu) & \mu/(\lambda+\mu) \\ 1/(\lambda+\mu) & -1/(\lambda+\mu) \end{pmatrix}.$$

Therefore the transition probabilities matrix is given by

$$\begin{aligned} P(t) = e^{tQ} &= \begin{pmatrix} 1 & \mu \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-(\lambda+\mu)t} \end{pmatrix} \begin{pmatrix} \lambda/(\lambda+\mu) & \mu/(\lambda+\mu) \\ 1/(\lambda+\mu) & -1/(\lambda+\mu) \end{pmatrix} \\ &= \frac{1}{\lambda+\mu} \begin{pmatrix} \lambda + \mu e^{-(\lambda+\mu)t} & \mu - \mu e^{-(\lambda+\mu)t} \\ \lambda - \lambda e^{-(\lambda+\mu)t} & \mu + \lambda e^{-(\lambda+\mu)t} \end{pmatrix}. \end{aligned}$$

(b) The stationary distribution is given by

$$\pi = \begin{pmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{pmatrix}.$$

Moreover, since $\lambda + \mu > 0$, for $t \to +\infty$ we have $e^{-(\lambda+\mu)t} \to 0$ and thus

$$P(t) \to \frac{1}{\lambda+\mu} \begin{pmatrix} \lambda & \mu \\ \lambda & \mu \end{pmatrix} = \begin{bmatrix} \pi \\ \pi \end{bmatrix}.$$

(c) Thanks to previous part, as $t \to +\infty$ it holds that

$$\mathbb{E}[X(t)^2 \,|\, X(0) = 1] = 1 \cdot p_{11}(t) + 4 \cdot p_{12}(t) \to \frac{\lambda + 4\mu}{\lambda+\mu}$$

and

$$\mathbb{E}[X(t)^2 \,|\, X(0) = 2] = 1 \cdot p_{21}(t) + 4 \cdot p_{22}(t) \to \frac{\lambda + 4\mu}{\lambda+\mu}.$$

Clearly, the value above is exactly the 2-nd moment of the distribution π.

Exercise 2

Consider

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}$$

for $0 < \alpha < 1$. If there exists a ctMC with transition probabilities $P(t)$ such that $P(1) = P$, then $P(t)$ must have the form

$$P(t) = e^{tQ}.$$

Now the above equation implies that $\det(P(t)) > 0$ for every $t \geq 0$. Indeed it must be $\det(P(t)) \neq 0$ since $P(t)$ has inverse $P(t)^{-1} = e^{-tQ}$ and it must also be $\det(P(t)) \geq 0$ since

$$\det(P(t)) = \det(P(t/2)^2) = \det(P(t/2))^2 \geq 0.$$

So if the dtMC with transition matrix P can be embedded in a ctMC, it must hold

$$2\alpha - 1 = \det(P) = \det(P(1)) > 0,$$

i.e. $\alpha \in (\frac{1}{2}, 1)$. On the other hand, if $\alpha \in (\frac{1}{2}, 1)$ then one can take

$$\begin{aligned} P(t) &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (2\alpha - 1)^t \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 + 1/2 \cdot (2\alpha - 1)^t & 1/2 - 1/2 \cdot (2\alpha - 1)^t \\ 1/2 - 1/2 \cdot (2\alpha - 1)^t & 1/2 + 1/2 \cdot (2\alpha - 1)^t \end{pmatrix}, \end{aligned}$$

for $t \geq 0$, i.e. the dtMC with transition matrix P can be embedded in a ctMC with transition probabilities $P(t)$ such that $P(1) = P$.

Exercise 3

We have that

$$\begin{aligned} \sum_i \pi_i P\{Y_1 = j \,|\, Y_0 = i\} &= \sum_i \pi_i P\{X_{T_1} = j \,|\, X_0 = i\} \\ &= \sum_i \pi_i \int_0^{+\infty} P\{X_{T_1} = j \,|\, X_0 = i, T_1 = t\} \, dP_{T_1}(t) \\ &= \sum_i \pi_i \int_0^{+\infty} P\{X_t = j \,|\, X_0 = i, T_1 = t\} \, dP_{T_1}(t) \\ &= \sum_i \pi_i \int_0^{+\infty} P\{X_t = j \,|\, X_0 = i\} \, dP_{T_1}(t) \\ &= \int_0^{+\infty} \sum_i \pi_i P\{X_t = j \,|\, X_0 = i\} \, dP_{T_1}(t) = \int_0^{+\infty} \pi_j \, dP_{T_1}(t) = \pi_j, \end{aligned}$$

i.e., π is a stationary distribution for Y too.

Exercise 4

If T_n is the time of the n-th arrival of X, where $X = \{X_t\}_{t \geq 0} \sim \mathrm{Poi}(\lambda)$, then what we want to show is that

$$P\{T_1 \leq s \,|\, T_2 > t,\, T_1 \leq t\} = \frac{s}{t} \tag{1}$$

for $s \in [0, t]$. If we consider $S_n = T_n - T_{n-1}$ for $n > 1$, $S_1 = T_1$, it holds that the $\{S_n\}_{n \geq 1}$ are independent and have distribution $\exp(\lambda)$. Note that we can rewrite the lhs of (1) as

$$P\{T_1 \leq s \,|\, T_2 > t,\, T_1 \leq t\} = P\{S_1 \leq s \,|\, S_1 + S_2 > t,\, S_1 \leq t\} = \frac{P\{S_1 \leq s,\, S_1 + S_2 > t\}}{P\{S_1 \leq t,\, S_1 + S_2 > t\}}. \tag{2}$$

Now if $s, t \geq 0$ we have that

$$P\{S_1 \leq s,\, S_1 + S_2 > t\} = \int_A \lambda^2 e^{-\lambda(x+y)}\, dxdy,$$

where $A = \{(x, y) \in [0, +\infty) \,:\, x \leq s,\, x+y > t\}$. Applying the change of variables $u = x, v = x+y$, the above becomes

$$P\{S_1 \leq s,\, S_1 + S_2 > t\} = \int_0^s \int_t^{+\infty} \lambda^2 e^{-\lambda v}\, dudv = s\lambda e^{-\lambda t}.$$

Therefore (2) gives

$$P\{T_1 \leq s \,|\, T_2 > t,\, T_1 \leq t\} = \frac{s\lambda e^{-\lambda t}}{t\lambda e^{-\lambda t}} = \frac{s}{t},$$

which is exactly (1), i.e., what we wanted to prove.

Exercise 5

(a) If $t > 0$, $i, j \in S$, we have that

$$\begin{aligned} |p_{ij}(t+h) - p_{ij}(t)| = \left| \sum_k p_{ik}(h) p_{kj}(t) - p_{ij}(t) \right| &\leq p_{ij}(t)(1 - p_{ii}(h)) + \sum_{k \neq i} p_{ik}(h) p_{kj}(t) \\ &\leq (1 - p_{ii}(h)) + \sum_{k \neq i} p_{ik}(h) = 2(1 - p_{ii}(h)) \to 0, \end{aligned}$$

as $h \to 0$ (to take into account $h < 0$ one can simply define $p_{ij}(t)$ for $t \in \mathbb{R}$ as an even function).

(b) Since $p_{ij}(t) \in (0, 1)$ for every $i, j \in S$ and $t \in (0, 1) \geq 0$ and $t \mapsto -\log t$ is a decreasing function for $t \in (0, 1)$, we have

$$\begin{aligned} g(t+s) = -\log p_{ii}(t+s) &= -\log \sum_k p_{ik}(t) p_{ki}(s) \leq -\log p_{ii}(t) p_{ii}(s) \\ &= -\log p_{ii}(t) - \log p_{ii}(s) = g(t) + g(s). \end{aligned}$$

(c) By the previous part we know that

$$\lim_{t \to 0^+} \frac{\log p_{ii}(t)}{t} = \inf_{t > 0} \frac{\log p_{ii}(t)}{t} = \lambda \in [-\infty, 0].$$

Thanks to this and to the fact that $p_{ii}(t) \to 1$ as $t \to 0^+$, it holds that

$$q_{ii} = \lim_{t \to 0^+} \frac{p_{ii}(t) - 1}{t} = \lim_{t \to 0^+} \frac{p_{ii}(t) - 1}{\log p_{ii}(t)} \cdot \frac{\log p_{ii}(t)}{t} = \lim_{x \to 1} \frac{x - 1}{\log x} \cdot \lim_{t \to 0^+} \frac{\log p_{ii}(t)}{t} = 1 \cdot \lambda = \lambda \in [-\infty, 0].$$

Exercise 6

In the attached paper there is the `python` script. The matrix Q is obtained by considering the adjacency matrix E of the graph induced by the chessboard and replacing the diagonal with minus the sum of the entries of the corresponding rows. The matrix P is obtained by E by normalizing its rows so that they sum to 1 and λ is the vector of minus the diagonal entries of Q. The value of the mean first passage time obtained using backward Kolmogorov equations is $\text{mfpt}_a \simeq 14.857$. We can

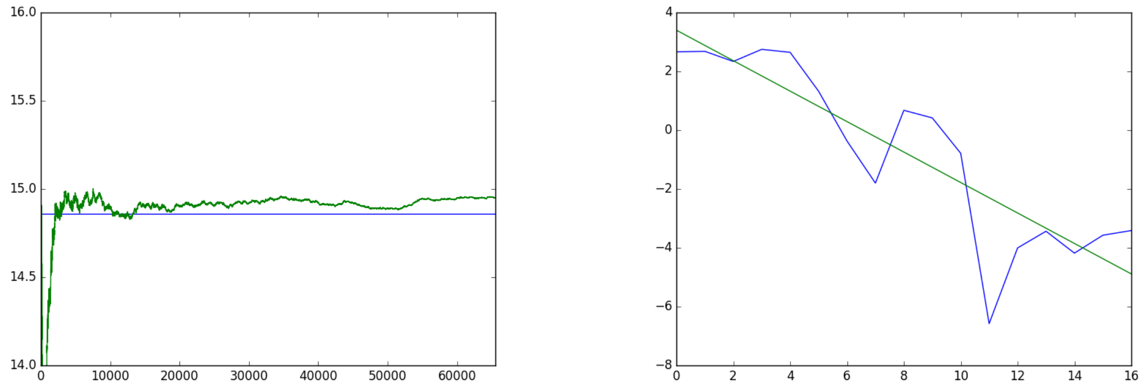

Figure 1

find almost the same value using Kinetic Monte Carlo, but this method requires an high number of iterations (I used $N = 2^{16}$) to converge and even for such a number of iteration (i.e. $N = 2^{16}$) it's not too much reliable. In the left figure in Figure 1 we plot the value mfpt_b as a function of N against the 'real' value mfpt_a. In the right figure in Figure 1 we plot the value of error $\log_2|\text{mfpt}_b - \text{mfpt}_a|$ for $N = 2^k, k = 0, \ldots, 16$. According to part (c), this should go approximately as a line, so we plot it against its linear regression. The linear regression gives an estimate of $\alpha \simeq 0.51$ (this is the value of α I obtained when I run the script who gave the figures in Figure 1 this value slightly changes every time we run the script). I think this value could be justified in the following way. If $\mathbb{E}[T]$ is the value of the mean first passage time and $Y_N = N^{-1}\sum_{k=1}^{N} T_k$ is our average approximation obtained generating N trajectories, then by Chebyshev inequality we have

$$P\{|Y_N - \mathbb{E}[T]| > \varepsilon\} \leq \frac{\text{Var}(T)}{N\varepsilon^2},$$

for any $\varepsilon \in (0, 1)$, which implies that with probability at least $1 - \varepsilon$, the Monte Carlo average Y_N lies within $(\text{Var}(T)/(N\varepsilon))^{1/2}$ of the true expected value (i.e. $\alpha = 0.5$).