

# Applied Stochastic Analysis

## Homework assignment 6

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March 16, 2017

### Exercise 1

(a) First, note that if  $N \sim \text{Poi}(\lambda)$ , then

$$\mathbb{E}[N_t] = \lambda t, \quad \mathbb{E}[N_t^2] = \lambda t(1 + \lambda t) \quad \text{and} \quad \mathbb{E}[X_t] = \lambda(t + \alpha) - \lambda t = \lambda \alpha.$$

Moreover

$$\begin{aligned} \text{Cov}(N_t, N_s) &= \mathbb{E}[N_t N_s] - \mathbb{E}[N_t] \mathbb{E}[N_s] = \mathbb{E}[(N_{t \vee s} - N_{t \wedge s}) N_{t \wedge s}] + \mathbb{E}[N_{t \wedge s}^2] - \mathbb{E}[N_t] \mathbb{E}[N_s] \\ &= \mathbb{E}[N_{t \vee s} - N_{t \wedge s}] \mathbb{E}[N_{t \wedge s}] + \mathbb{E}[N_{t \wedge s}^2] - \mathbb{E}[N_t] \mathbb{E}[N_s] = \mathbb{E}[N_{t \wedge s}^2] - \mathbb{E}[N_{t \wedge s}]^2 = \lambda(t \wedge s). \end{aligned}$$

Since  $X$  is strongly stationary, its covariance function is given by

$$C(t) = \text{Cov}(X_{|t|}, X_0) = \text{Cov}(N_{|t|+\alpha} - N_{|t|}, N_\alpha) = \lambda(\alpha - |t| \wedge \alpha) = \lambda(\alpha - |t|) \mathbb{1}_{\{|t| \leq \alpha\}}.$$

Since  $C \in L^1(\mathbb{R})$ , the spectral density exists and its given by its Fourier transform:

$$f(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \lambda(\alpha - |t|) \mathbb{1}_{\{|t| \leq \alpha\}} e^{-i\xi t} dt = \frac{\lambda}{\pi} \frac{1 - \cos \alpha \xi}{\xi^2}.$$

(b) Since  $W$  is a Brownian motion, then

$$\mathbb{E}[W_t] = \mathbb{E}[X_t] = 0, \quad \text{and} \quad \mathbb{E}[W_t W_s] = s \wedge t.$$

Hence the covariance function of  $X$  is given by

$$\begin{aligned} C(t) &= \mathbb{E}[X_{|t|+s} X_s] = \mathbb{E}[(W_{|t|+\alpha} - W_{|t|})(W_{s+\alpha} - W_s)] \\ &= (\alpha + |t| \wedge s) - |t| \wedge (s + \alpha) - s \wedge (|t| + \alpha) + |t| \wedge s = (\alpha - |t|) \mathbb{1}_{\{|t| \leq \alpha\}} \end{aligned}$$

As in the part (a), the spectral density exists and its given by its Fourier transform:

$$f(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} (\alpha - |t|) \mathbb{1}_{\{|t| \leq \alpha\}} e^{-i\xi t} dt = \frac{1 - \cos \alpha \xi}{\pi \xi^2}.$$

### Exercise 2

(a) Using the notation on the notes, we have that the spectral representation of  $X$  is

$$X_t = \int_{\mathbb{R}} e^{i\xi t} dZ(\xi).$$

Derivating under the integral sign, we get the spectral representation of  $X'$ :

$$X'_t = \int_{\mathbb{R}} e^{i\xi t} i\xi dZ(\xi).$$

Its covariance function is given by

$$\begin{aligned} C_{X'}(t) &= \mathbb{E}[X'_{t+s} \overline{X'_s}] = \mathbb{E} \left[ \int_{\mathbb{R}^2} e^{i\xi(t+s)} e^{-i\xi' s} \xi \xi' dZ(\xi) \overline{dZ(\xi')} \right] \\ &= \int_{\mathbb{R}^2} e^{i\xi(t+s)} e^{-i\xi' s} \xi \xi' \mathbb{E} \left[ dZ(\xi) \overline{dZ(\xi')} \right] = \int_{\mathbb{R}^2} e^{i\xi(t+s)} e^{-i\xi' s} \xi \xi' \delta(\xi - \xi') dF(\xi) d\xi' \\ &= \int_{\mathbb{R}} e^{i\xi t} \xi^2 dF(\xi) = -\frac{d^2}{dt^2} \int_{\mathbb{R}} e^{i\xi t} dF(\xi) = -\frac{d^2}{dt^2} C(t). \end{aligned}$$

(This confirms what found in exercise 5 of homework 5).

(b) It holds that

$$\begin{aligned} C_{X,X'}(t) &= \mathbb{E}[X_{t+s} \overline{X'_s}] = -i \mathbb{E} \left[ \int_{\mathbb{R}^2} e^{i\xi(t+s)} e^{-i\xi' s} \xi' dZ(\xi) \overline{dZ(\xi')} \right] \\ &= -i \int_{\mathbb{R}^2} e^{i\xi(t+s)} e^{-i\xi' s} \xi' \mathbb{E} \left[ dZ(\xi) \overline{dZ(\xi')} \right] = -i \int_{\mathbb{R}^2} e^{i\xi(t+s)} e^{-i\xi' s} \xi' \delta(\xi - \xi') dF(\xi) d\xi' \\ &= -i \int_{\mathbb{R}} e^{i\xi t} \xi dF(\xi) = -\frac{d}{dt} \int_{\mathbb{R}} e^{i\xi t} dF(\xi) = -\frac{d}{dt} C(t). \end{aligned}$$

(c) For every  $t \in \mathbb{R}$  it holds

$$\mathbb{E}[X_t \overline{X'_t}] = C_{X,X'}(0) = -C'(0).$$

Since it is constant, its derivative is zero.

(d) A sufficient condition is that  $C$  is a  $C^{2n}$  function. Indeed we have shown in homework 5 (exercise 5) that if  $C^2$  then  $X$  is  $C^1$ . Applying this reasoning on the derivatives of  $X$ , one finds that if  $C$  is  $C^{2n}$  then  $X$  is a  $C^n$  process.

### Exercise 3