Applied Stochastic Analysis
Homework assignment 5

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Exercise 1

If $t_1, \ldots, t_n \in \mathbb{R}$, $B = (B(t_i, t_j))_{1 \le i,j \le n}$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, then

$$\begin{aligned}\mathbf{v}^t B \mathbf{v} &= \sum_{i,j=1}^n v_i B(t_i, t_j) v_j = \sum_{i,j=1}^n v_i \, \mathbb{E}[X_{t_i} X_{t_j}] v_j = \sum_{i,j=1}^n \mathbb{E}[v_i X_{t_i} v_j X_{t_j}] \\ &= \sum_{i=1}^n \mathbb{E}\left[v_i X_{t_i} \Big(\sum_{j=1}^n v_j X_{t_j} \Big) \right] = \mathbb{E}\left[\Big(\sum_{j=1}^n v_j X_{t_j} \Big)^2 \right] \ge 0\end{aligned}$$

This proves that $B(s, t)$ is a positive semi-definite function.

Exercise 2

Since we want to generate $(X_t)_t$ only for $t = t_1, \ldots, t_m$, then consider $B = (B(t_i, t_j))_{1 \le i,j \le m}$. Let K be the matrix in the Cholesky decomposition of B, i.e., $B = KK^t$. We first generate a sample of $\mathbf{Y} = (Y_1, \ldots, Y_m) \sim N(\mathbf{0}, I_m)$: this can be done simply generating independently $Y_i \sim N(0, 1)$, $i = 1, \ldots, m$. If we consider $\mathbf{X} = K\mathbf{Y}$, then $X_1, \ldots, X_m$ are samples from the desired distribution (i.e., that of $X_{t_!}, \ldots, X_{t_m}$). Indeed $\mathbf{X} \sim N(\mathbf{0}, B)$ since

$$\mathrm{Cov}(\mathbf{X}) = \mathbb{E}[\mathbf{X}\mathbf{X}^t] = K \, \mathbb{E}[\mathbf{Y}\mathbf{Y}^t] K^t = K I_m K^t = KK^t = B.$$

Exercise 3

If $N = (N_t)_t$ is a Poisson process, then, for every $0 < t_1 < \cdots < t_n$, $N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent and the distribution of $N_{t_2} - N_{t_1}$ only depends on $t_2 - t_1$.

Exercise 4

(a) Since $\{\xi_n\}_n$ are uncorrelated, we have

$$\mathbb{E}\, X_n = \mu \sum_{i=1}^m a_i \qquad \text{and} \qquad \mathrm{Var}(X_n) = \sigma^2 \sum_{i=1}^m a_i^2.$$

Moreover, since X_n depends only on $\xi_n, \dots, \xi_{n-m+1}$, if $|n-k| \geq m$ then X_n and X_k are uncorrelated. Otherwise suppose $n \geq k$ and $l = n-k < m$. Then

$$\begin{aligned}
\mathrm{Cov}(X_n, X_k) &= \mathrm{Cov}\,\Big(\sum_{i=1}^{m} a_i \xi_{n-i+1}, \sum_{j=1}^{m} a_j \xi_{k-j+1}\Big) \\
&= \mathrm{Cov}\,\Big(\sum_{i=1}^{l} a_i \xi_{n-i+1} + \sum_{i=l+1}^{m} a_i \xi_{n-i+1}, \sum_{j=1}^{m-l} a_j \xi_{k-j+1} + \sum_{j=m-l+1}^{m} a_j \xi_{k-j+1}\Big) \\
&= \mathrm{Cov}\,\Big(\sum_{i=l+1}^{m} a_i \xi_{n-i+1}, \sum_{j=1}^{m-l} a_j \xi_{k-j+1}\Big) = \mathrm{Cov}\,\Big(\sum_{i=1}^{m-l} a_{i+l} \xi_{k-i+1}, \sum_{j=1}^{m-l} a_j \xi_{k-j+1}\Big) \\
&= \sigma^2 \sum_{i=1}^{m-l} a_{i+l} a_i.
\end{aligned}$$

Therefore we can write, for every $n, k \in \mathbb{Z}$,

$$\mathrm{Cov}(X_n, X_k) = \sigma^2 \mathbb{1}_{\{|n-k|<m\}} \sum_{i=1}^{m-|n-k|} a_{i+|n-k|} a_i.$$

(b) In the case $a_k = 1/\sqrt{m}$ for $k = 1, \dots, m$ the covariance function is

$$\mathrm{Cov}(X_n, X_k) = \sigma^2 \mathbb{1}_{\{|n-k|<m\}} \frac{m - |n-k|}{m}.$$

If $m = 1$ then $(X_n)_n$ are uncorrelated and their variance is $\mathrm{Var}(X_n) = \sigma^2$. If $m \to \infty$ then

$$\mathrm{Cov}(X_n, X_k) \to \sigma^2 \qquad \text{for every } n, k \in \mathbb{Z}.$$

This implies that for every $n, k \in \mathbb{Z}$

$$\mathbb{E}[(X_n - X_k)^2] = 0,$$

i.e., $X_n = X_0$ a.s. for every $n \in \mathbb{Z}$.

Exercise 5

(a) The condition in the exercise is sufficient to mean-square differentiability because it's basically Cauchy condition for L^2 convergence. Now, it holds that

$$\begin{aligned}
\mathbb{E}\left|\frac{X_{t+h} - X_t}{h} - \frac{X_{t+h'} - X_t}{h'}\right|^2 &= \mathbb{E}\left|\frac{X_{t+h} - X_t}{h}\right|^2 + \mathbb{E}\left|\frac{X_{t+h'} - X_t}{h'}\right|^2 - 2\,\mathbb{E}\left|\frac{X_{t+h} - X_t}{h}\frac{X_{t+h'} - X_t}{h'}\right| \\
&= 2\frac{C(0) - C(h)}{h^2} + 2\frac{C(0) - C(h')}{(h')^2} - 2\frac{C(h'-h) - C(h') - C(h) + C(0)}{hh'}.
\end{aligned}$$

Now

$$2\frac{C(0) - C(h)}{h^2} = \frac{-C(-h) + 2C(0) - C(h)}{h^2} \to -C''(0)$$

as $h \to 0$, and similarly

$$2\frac{C(0) - C(h')}{(h')^2} \to -C''(0)$$

as $h' \to 0$, if C has continuous second derivative in a neighborhood of the origin. Moreover

$$\begin{aligned}\lim_{h\to 0}\lim_{h'\to 0}\frac{C(h'-h)-C(h')-C(h)+C(0)}{hh'} &= \lim_{h\to 0}\frac{1}{h}\left[\lim_{h'\to 0}\frac{C(h'-h)-C(h)}{h'}+\lim_{h'\to 0}\frac{C(0)-C(h')}{h'}\right]\\ &= \lim_{h\to 0}\frac{C'(-h)-C'(0)}{h} = -C''(0).\end{aligned}$$

Therefore

$$\lim_{h\to 0,\, h'\to 0}\mathbb{E}\left|\frac{X_{t+h}-X_t}{h}-\frac{X_{t+h'}-X_t}{h'}\right|^2 = -2C''(0)+2C''(0)=0.$$

(b) If C is C^2, it holds that

$$\mathbb{E}\left[\frac{X_{s+t+h}-X_{s+t}}{h}\frac{X_{s+h}-X_s}{h}\right] = -\frac{C(t+h)-2C(t)+C(t-h)}{h^2} \to -C''(t)$$

as $h \to 0$. (This implies that $-C''(t)$ is the covariance function of X_t' because of the mean-square differentiability of X_t).