Applied Stochastic Analysis
Homework assignment 3

Luca Venturi

February 17, 2017

Exercise 1

(a) Consider $P \in \mathbb{R}^{2\times 2}$ the transition matrix of this process. We know there exists a stationary distribution π. Now, if there exists another stationary distribution $\mu \neq \pi$, then π, μ are independent. Therefore they form a basis for $\mathbb{R}^2$ of 1-eigenvectors of P, which implies that P is similar to the identity matrix $I_2 \in \mathbb{R}^{2\times 2}$. But the only matrix similar to I_2 is I_2 itself. Therefore, it follows that if $P \neq I_2$, then the stationary distribution must be unique; $P = I_2$ is clearly the pathological case. It remains to show that P is reversible with respect to π. To prove this, we only need to show that $\pi_1 P_{12} = \pi_2 P_{21}$. This suddenly follows:

$$\pi_1 P_{12} = \pi_1 - \pi_1 P_{11} = \pi_1 - (\pi_1 - \pi_2 P_{21}) = \pi_2 P_{21},$$

where we used that $P_{12} = 1 - P_{11}$ and that $\pi_1 = \pi_1 P_{11} + \pi_2 P_{21}$, being π stationary.

(b) Consider the 3×3 transition matrix

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}.$$

Its stationary distribution is given by $\pi = (1/3, 1/3, 1/3)$ and P is not reversible with respect to π, indeed:

$$\pi_1 P_{12} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \neq \frac{1}{9} = \frac{1}{3} \cdot \frac{1}{3} = \pi_2 P_{21}.$$

Exercise 2

The transition probabilities are given by $P = (P_{ij})_{i,j=0,\dots,N}$ such that

$$P_{ij} = \begin{cases} i^2/N^2 & \text{if } j = i-1 \\ 2i(N-i)/N^2 & \text{if } j = i \\ (N-i)^2/N^2 & \text{if } j = i+1 \\ 0 & \text{o.w.} \end{cases}$$

One can verify that the stationary distribution is given by $\pi = (\pi_j)_{j=0,\dots,N}$ defined as $\pi_j = \binom{N}{j}^2\binom{2N}{N}^{-1}$. Indeed, if for example $j \in \{1, \dots, N-1\}$, we have

$$\begin{aligned}
\sum_{i=0}^{N} \pi_i P_{ij} &= \pi_{j-1}P_{j-1,j} + \pi_j P_{jj} + \pi_{j+1}P_{j+1,j} \\
&= \binom{2N}{N}^{-1}\left[\binom{N}{j-1}^2\frac{(N-j+1)^2}{N^2} + 2\binom{N}{j}^2\frac{j(N-j)}{N^2} + \binom{N}{j+1}^2\frac{(j+1)^2}{N^2}\right] \\
&= \frac{1}{N^2}\binom{2N}{N}^{-1}\left[j^2\binom{N}{j}^2 + 2j(N-j)\binom{N}{j}^2 + (N-j)^2\binom{N}{j}^2\right] \\
&= \frac{1}{N^2}\binom{2N}{N}^{-1}\binom{N}{j}^2[j+(N-j)]^2 = \binom{N}{j}^2\binom{2N}{N}^{-1} = \pi_j.
\end{aligned}$$

The equation above can be proved similarly for $j = 0, N$. The chain is also reversible in equilibrium. Indeed it suffices to show that $\pi_i P_{i,i+1} = \pi_{i+1}P_{i+1,i}$ for $i = 0, \dots, N-1$. This holds since:

$$\pi_i P_{i,i+1} = \binom{2N}{N}^{-1}\frac{1}{N^2}\binom{N}{i}^2(N-i)^2 = \binom{2N}{N}^{-1}\frac{1}{N^2}\binom{N}{i+1}^2(i+1)^2 = \pi_{i+1}P_{i+1,i}.$$

Exercise 3

It holds that:

$$\begin{aligned}
\langle Pu, v\rangle_\pi &= \sum_i \pi_i (Pu)_i v_i = \langle Pu, v\rangle_\pi = \sum_i \pi_i v_i \sum_j P_{ij}u_j = \sum_j u_j \sum_i \pi_i P_{ij} v_i = \sum_j u_j \sum_i \pi_j P_{ji} v_i \\
&= \sum_j \pi_j u_j \sum_i P_{ji} v_i = \sum_j \pi_j u_j (Pv)_j = \langle u, Pv\rangle_\pi.
\end{aligned}$$

Exercise 4

(a) First of all $\pi = (\pi_i)_i$, where $\pi_i = \frac{d_i}{\sum_k d_k}$ is the stationary distribution since

$$\sum_i \pi_i P_{ij} = \sum_i \frac{d_i}{\sum_k d_k}\frac{w_{ij}}{d_i} = \frac{1}{\sum_k d_k}\sum_i w_{ij} = \frac{d_j}{\sum_k d_k} = \pi_j.$$

Also, the transition matrix P satisfies the detailed balance since

$$\pi_i P_{ij} = \frac{d_i}{\sum_k d_k}\frac{w_{ij}}{d_i} = \frac{w_{ij}}{\sum_k d_k} = \frac{w_{ji}}{\sum_k d_k} = \frac{d_j}{\sum_k d_k}\frac{w_{ji}}{d_j} = \pi_j P_{ji}.$$

(b) The transition matrix P has a full set of eigenvalues (i.e. it is similar to a diagonal real matrix) since it is similar to a symmetric matrix (and so spectral theorem applies). Indeed if $V = \Lambda P \Lambda^{-1}$, where $\Lambda = \mathrm{diag}(\sqrt{\pi_1}, \dots, \sqrt{\pi_n})$, then V is symmetric. Indeed, thank to detailed balance, it holds

$$V_{ij} = \frac{\sqrt{\pi_i}}{\sqrt{\pi_j}}P_{ij} = \frac{\pi_i}{\sqrt{\pi_i\pi_j}}P_{ij} = \frac{\pi_j}{\sqrt{\pi_i\pi_j}}P_{ji} = \frac{\sqrt{\pi_j}}{\sqrt{\pi_i}}P_{ji} = V_{ji}.$$

(c) Since $V = \Lambda P \Lambda^{-1}$ is symmetric we can write it as $V = O \Sigma O^T$, where $O = (w_1 | \cdots | w_n)$ is an orthogonal matrix and $\Sigma = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ (i.e. λ_i are the eigenvalues of V and w_i the respective eigenvectors). This is also equivalent to write $V = \sum_{k=1}^n \lambda_k w_k w_k^T$. From this we have that

$$P^t = \Lambda^{-1} V^t \Lambda = \sum_{k=1}^n \lambda_k^t \, \Lambda^{-1} w_k w_k^T \Lambda = \sum_{k=1}^n \lambda_k^t \, \Lambda^{-1} w_k w_k^T \Lambda^{-1} \Lambda^2 = \sum_{k=1}^n \lambda_k^t \, \phi_k \phi_k^T \Lambda^2,$$

where $\phi_k = \Lambda^{-1} w_k$ are the right eigenvectors of P. Now we can write

$$D^t(i,j) = (e_i^T P^t - e_j^T P^t) \, \Lambda^{-2} \, (e_i^T P^t - e_j^T P^t)^T = (e_i - e_j)^T \, P^t \Lambda^{-2} (P^t)^T \, (e_i - e_j).$$

Since

$$\begin{aligned} P^t \Lambda^{-2} (P^t)^T &= \left(\sum_{k=1}^n \lambda_k^t \, \phi_k \phi_k^T \Lambda^2 \right) \Lambda^{-2} \left(\sum_{j=1}^n \lambda_j^t \, \phi_j \phi_j^T \Lambda^2 \right)^T \\ &= \sum_{j,k=1}^n \lambda_k^t \lambda_j^t \phi_k \phi_k^T \Lambda^2 \phi_j \phi_j^T = \sum_{j,k=1}^n \lambda_k^t \lambda_j^t \phi_k w_k^T w_j \phi_j^T = \sum_{j,k=1}^n \lambda_k^t \lambda_j^t \phi_k \delta_{ij} \phi_j^T = \sum_{k=1}^n \lambda_k^{2t} \phi_k \phi_k^T, \end{aligned}$$

we get that

$$D_t^2(i,j) = (e_i - e_j)^T \left(\sum_{k=1}^n \lambda_k^{2t} \phi_k \phi_k^T \right) (e_i - e_j) = \sum_{k=1}^n \lambda_k^{2t} ((\phi_k)_i - (\phi_k)_j)^2.$$

Exercise 5

Let's consider a distribution $\pi = (\pi_i)_i$ and a proposal probability matrix $H = (H_{ij})_{ij}$. We want to define an acceptance probability matrix $A = (A_{ij})_{ij}$ such that the transition probability matrix $P = (P_{ij})_{ij}$ defined as

$$P_{ij} = \begin{cases} A_{ij} H_{ij} & \text{if } i \neq j \\ 1 - \sum_k A_{ik} H_{ik} & \text{if } i = j \end{cases}$$

is in detailed balance with π. Let $F : [0, +\infty] \to [0,1]$ be a function such that $F(z) = zF(1/z)$ for every $z \in [0,1]$ (note that every function $f : [0,1] \to [0,1]$ can be extended to such an F) and define A as

$$A_{ij} = F\left(\frac{\pi_j H_{ji}}{\pi_i H_{ij}} \right).$$

Note that, since

$$\sum_{j \neq i} P_{ij} = \sum_{j \neq i} H_{ij} A_{ij} \leq \sum_{j \neq i} H_{ij} \leq 1,$$

then P defined above is a stochastic matrix. Moreover it is in detailed balance with π, indeed:

$$\begin{aligned} \pi_i P_{ij} = \pi_i H_{ij} A_{ij} = \pi_i H_{ij} F\left(\frac{\pi_j H_{ji}}{\pi_i H_{ij}} \right) &= \pi_j H_{ji} \frac{\pi_i H_{ij}}{\pi_j H_{ji}} F\left(\frac{\pi_j H_{ji}}{\pi_i H_{ij}} \right) \\ = \pi_j H_{ji} \left(\frac{\pi_i H_{ij}}{\pi_j H_{ji}} \right) &= \pi_j H_{ji} A_{ji} = \pi_j P_{ji}. \end{aligned}$$

With this method we can find a whole family of acceptance probability matrices. In particular the standard Metropolis-Hastings algorithm is obtained taking $F(z) = \min\{1, z\}$.