Applied Stochastic Analysis
Homework assignment 10

Luca Venturi

April 30, 2017

Exercise 1

(a) By applying Itô formula, we have

$$\begin{aligned}
I_t^{(3)} &\doteq \int_0^t \int_0^{s_1} \int_0^{s_2} dW_{s_3}\, dW_{s_2}\, dW_{s_1} = \int_0^t \int_0^{s_1} W_{s_2}\, dW_{s_2}\, dW_{s_1} \\
&= \int_0^t \left[\int_0^{s_1} d\left(\frac{1}{2}W_{s_2}^2\right) - \frac{1}{2}\int_0^{s_1} ds_2 \right] dW_{s_1} = \frac{1}{2}\int_0^t W_{s_1}^2\, dW_{s1} - \frac{1}{2}\int_0^t s_1\, dW_{s1} \\
&= \frac{1}{6}\int_0^t d(W_{s_1}^3) - \frac{1}{2}\int_0^t W_{s_1}\, ds_1 - \frac{1}{2}\int_0^t d(s_1 W_{s_1}) + \frac{1}{2}\int_0^t W_{s_1}\, ds_1 \\
&= \frac{1}{6}W_t^3 - \frac{1}{2}tW_t.
\end{aligned}$$

(b) We proceed by induction. For $k = 3$ the equality holds due to part (a) and to

$$I_t^{(2)} = \frac{1}{2}W_t^2 - \frac{1}{2}t, \qquad I_t^{(1)} = W_t.$$

First of all, we notice that

$$I_t^{(k+1)} = \int_0^t I_{s_1}^{(k)}\, dW_{s_1} \qquad \Longleftrightarrow \qquad dI_t^{(k+1)} = I_t^{(k)}\, dW_t.$$

Then, if we suppose that the recursion formula holds for k, for $k+1$ we get

$$dI_t^{(k+1)} = I_t^{(k)}\, dW_t = \frac{1}{k}\Big(W_t I_t^{(k-1)} - tI_t^{(k-2)}\Big)dW_t.$$

On the other hand

$$\begin{aligned}
d\left[\frac{1}{k+1}\Big(W_t I_t^{(k)} - tI_t^{(k-2)}\Big)\right] &= \frac{1}{k+1}\left[d\Big(W_t I_t^{(k)}\Big) - d\Big(tI_t^{(k-1)}\Big)\right] = \\
&= \frac{1}{k+1}\left[I_t^{(k)}dW_t + W_t\, dI_t^{(k)} + dW_t\, dI_t^{(k)} - t dI_t^{(k-1)} - I_t^{(k-1)}\, dt - dt dI_t^{(k-1)}\right] \\
&= \frac{1}{k+1}\left[I_t^{(k)}dW_t + W_t I_t^{(k-1)}dW_t + I_t^{(k-1)}(dW_t)^2 - tI_t^{(k-2)}\, dW_t - I_t^{(k-1)}\, dt\right] \\
&= \frac{1}{k+1}\left[I_t^{(k)} + W_t I_t^{(k-1)} - tI_t^{(k-2)}\right]dW_t \\
&= \frac{1}{k+1}\left[\frac{1}{k}\Big(W_t I_t^{(k-1)} - tI_t^{(k-2)}\Big) + W_t I_t^{(k-1)} - tI_t^{(k-2)}\right]dW_t \\
&= \frac{1}{k+1}\left(1 + \frac{1}{k}\right)\Big(W_t I_t^{(k-1)} - tI_t^{(k-2)}\Big)dW_t = \frac{1}{k}\Big(W_t I_t^{(k-1)} - tI_t^{(k-2)}\Big)dW_t.
\end{aligned}$$

This shows that

$$dI_t^{(k+1)} = d\left[\frac{1}{k+1}\Big(W_t I_t^{(k)} - t I_t^{(k-1)}\Big)\right],$$

which implies

$$I_t^{(k+1)} = \frac{1}{k+1}\Big(W_t I_t^{(k)} - t I_t^{(k-1)}\Big),$$

i.e. what we wanted to prove.

Exercise 2

Let's consider the SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t.$$

We know that the Milstein scheme has order 1 and it's based on the expansion:

$$X_t = X_0 + b(X_0)\int_0^t ds + \sigma(X_0)\int_0^t dW_s + (\sigma\sigma')(X_0)\int_0^t \frac{1}{2}\big[d(W_s^2) - ds\big] + R_1,$$

where

$$\begin{aligned} R_1 = &\int_0^t\int_0^s \mathcal{L}_0 b(X_z)\,dzds + \int_0^t\int_0^s \mathcal{L}_0\sigma(X_z)\,dzdW_s + \int_0^t\int_0^s \mathcal{L}_1 b(X_z)\,dW_z ds \\ &+ \int_0^t\int_0^s\int_0^z \mathcal{L}_1\mathcal{L}_1\sigma(X_u)\,dW_u dW_z dW_s + \int_0^t\int_0^s\int_0^z \mathcal{L}_0\mathcal{L}_1\sigma(X_u)\,du dW_z dW_s \end{aligned}$$

(here $\mathcal{L}_0 = b\partial_x + (1/2)\sigma^2\partial^2_{xx}$ and $\mathcal{L}_1 = \sigma\partial_x$). Applying the rule

$$f(X_t) = f(X_0) + \int_0^t \mathcal{L}_0 f(X_s)\,ds + \int_0^t \mathcal{L}_1 f(X_s)\,dW_s$$

to the first four components of R_1, we get the new expansion:

$$\begin{aligned} X_t = X_0 &+ b(X_0)\int_0^t ds + \sigma(X_0)\int_0^t dW_s + (\sigma\sigma')(X_0)\int_0^t \frac{1}{2}\big[d(W_s^2) - ds\big] \\ &+ \frac{1}{2}\left[(bb')(X_0) + \frac{1}{2}(\sigma^2 b'')(X_0)\right]\int_0^t d(s^2) + (\sigma b')(X_0)\int_0^t\int_0^s dW_z ds \\ &+ \left[(b\sigma')(X_0) + \frac{1}{2}(\sigma^2\sigma'')(X_0)\right]\int_0^t [d(sW_s) - W_s ds] \\ &+ \frac{1}{2}\sigma(X_0)\big[(\sigma')^2(X_0) + (\sigma\sigma'')(X_0)\big]\int_0^t \left[\frac{1}{3}d(W_s^3) - d(sW_s)\right] + R_2 \end{aligned}$$

for some remainder term R_2 (we used the result from exercise 1.a for the last term). From this expansion we get the following discrete scheme:

$$\begin{aligned} X_{n+1} = X_n &+ b(X_n)\Delta t + \sigma(X_n)\Delta W + \frac{1}{2}(\sigma\sigma')(X_n)((\Delta W)^2 - \Delta t) \\ &+ \frac{1}{2}\left[(bb')(X_n) + \frac{1}{2}(\sigma^2 b'')(X_n)\right](\Delta t)^2 \\ &+ \left[(b\sigma')(X_n) + \frac{1}{2}(\sigma^2\sigma'')(X_n)\right](\Delta t\Delta W - \Delta Z) + (\sigma b')(X_n)\Delta Z \\ &+ \frac{1}{2}\sigma(X_n)\big[(\sigma')^2(X_n) + (\sigma\sigma'')(X_n)\big]\left(\frac{1}{3}(\Delta W)^3 - \Delta W\Delta t\right), \end{aligned}$$

where ΔW is a r.v. $\sim N(0, \Delta t)$ and ΔZ is a r.v. $\sim \int_0^{\Delta t} \int_0^s dW_z ds \sim N(0, (\Delta t)^3/3)$. In particular $\mathbb{E}[\Delta W \Delta Z] = (\Delta t)^2/2$, so, at every step we must compute an instance of the 2d r.v. $(\Delta W, \Delta Z) \sim N(\mathbf{0}, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \Delta t & (\Delta t)^2/2 \\ (\Delta t)^2/2 & (\Delta t)^3/3 \end{pmatrix}$$

(we can do this using the method based on Choleski decomposition of Σ for example). We expect this method to have strong order of convergence 1.5 cause we derived every term in the remainder of the Milstein scheme, which has order 1, such that now in the remainder term R_2 we only have triple (or quadruple) (stochastic) integrals, but none of the type $dW_t dW_s dW_z$.

Exercise 3

If we consider the SDE

$$dX_t = \lambda X_t \, dt + \mu X_t \, dW_t,$$

the Milstein scheme gives

$$X_{n+1} = X_n + \lambda X_n \Delta t + \mu X_n \Delta W + \frac{1}{2}\mu^2 X_n ((\Delta W)^2 - \Delta t),$$

where ΔW is a r.v. $\sim N(0, \Delta t)$. Then

$$\begin{aligned} \mathbb{E}|X_{n+1}|^2 &= \mathbb{E}|X_n|^2 \, \mathbb{E}\left[\left(1 + \lambda \Delta t + \mu \Delta W + \frac{1}{2}\mu^2((\Delta W)^2 - \Delta t) \right)^2 \right] \\ &= \mathbb{E}|X_n|^2 \left[(1 + \lambda \Delta t)^2 + \mu^2 \Delta t + \frac{1}{4}\mu^4 \, \mathbb{E}\big[((\Delta W)^2 - \Delta t))^2\big] \right] \\ &= \mathbb{E}|X_n|^2 \left[(1 + \lambda \Delta t)^2 + \mu^2 \Delta t + \frac{1}{2}\mu^4 (\Delta t)^2 \right]. \end{aligned}$$

Therefore the stability region for the Milstein scheme is

$$R_1 = \{ \ (x, y) = (\lambda \Delta t, \mu^2 \Delta t) \ : \ (1 + x)^2 + y + \frac{1}{2} y^2 < 1 \ \}.$$

We know that the stability region for the SDE is $R_2 = \{ \ (x, y) \ : \ y < -2x \ \}$ and the one for the EM scheme is $R_3 = \{ \ (x, y) \ : \ (1 + x)^2 + y < 1 \ \}$.