Applied Stochastic Analysis
Homework assignment 9

Luca Venturi

April 17, 2017

Exercise 1

(a)

$$dX_t = \left(aX_t + \frac{1}{2}b^2 X_t\right) dt + bX_t\, dB_t.$$

(b)

$$dX_t = \frac{1}{2}\left(\sin X_t \cos X_t - t^2 \sin X_t\right) dt + (t^2 + \cos X_t)\, dB_t.$$

(c)

$$dX_t = \left(r - \frac{1}{2}\alpha^2\right) X_t\, dt + \alpha X_t \circ dB_t.$$

(d)

$$dX_t = \left(2e^{-X_t} - X_t^3\right) dt + X_t^2 \circ dB_t.$$

Exercise 2

(a) By Itô formula we have

$$d(X_t^n) = \left(\lambda n + \frac{1}{2}n(n-1)\sigma^2\right) X_t^n\, dt + X_t^n\, dW_t,$$

i.e.

$$X_t^n = x_0^n + \int_0^t \left(\lambda n + \frac{1}{2}n(n-1)\sigma^2\right) X_s^n\, ds + \sigma n \int_0^t X_s^n\, dW_s.$$

Taking the expectation we get

$$M_n(t) = \mathbb{E}\, X_t^n = \mathbb{E}\, x_0^n + \int_0^t \left(\lambda n + \frac{1}{2}n(n-1)\sigma^2\right) M_n(s)\, ds.$$

The above formula is equivalent to say that M_n satisfies the equation

$$\frac{dM_n}{dt} = \left(\lambda n + \frac{\sigma^2}{2}n(n-1)\right) M_n, \qquad M_n(0) = \mathbb{E}\, x_0^n.$$

(b)

$$M_n(t) = (\mathbb{E}\, x_0^n)\, e^{n(\lambda + \sigma^2(n-1)/2)t}$$

(c) It must be

$$\lambda + \frac{\sigma^2}{2}(n-1) < 0, \qquad \text{i.e.} \qquad \lambda < -\frac{\sigma^2}{2}(n-1).$$

(d)

Exercise 5

It is sufficient to show that $dY_t = 0$, where $Y_t \doteq X_{1,t}^2 + X_{2,t}^2$. Indeed, according to multidimensional Ito formula, it holds:

$$dY_t = 2X_{1,t}\,dX_{1,t} + 2X_{2,t}\,dX_{2,t} + (dX_{1,t})^2 + (dX_{1,t})^2 = 2X_{1,t}(X_{2,t}^2 \circ dW_{1,t})$$

Exercise 7

(a) The equation means

$$X_t = \xi + \int_0^t s\,ds + 2\int_0^t dB_s = \xi + \frac{t^2}{2} + 2B_t.$$

This implies $\mathbb{E}\,X_t = t^2/2$.

(b) If we multiply both sides by $e^{\cos t}$, we get

$$d(e^{\cos t}X_t) = e^{\cos t}\,dX_t - (\sin t)e^{\cos t}X_t\,dt = e^{\cos t}\,dB_t,$$

which gives

$$\begin{aligned} X_t &= \xi\, e^{(1-\cos t)} + e^{-\cos t}\int_0^t e^{\cos s}\,dB_s \\ &= \xi\, e^{(1-\cos t)} + e^{-\cos t}\int_0^t d(e^{\cos s}B_s) - e^{-\cos t}\int_0^t (\sin s)B_s\, e^{\cos s}\,ds \\ &= \xi\, e^{(1-\cos t)} + B_t - e^{-\cos t}\int_0^t (\sin s)B_s\, e^{\cos s}\,ds. \end{aligned}$$

In particular $\mathbb{E}\,X_t = 0$.

(c) If we multiply both sides by e^t, we get

$$d(e^tX_t) = e^t\,dX_t + e^tX_t\,dt = e^tdt + e^tdB_t,$$

which gives

$$\begin{aligned} X_t &= \xi e^{-t} + e^{-t}\int_0^s e^sds + e^{-t}\int_0^t e^sdB_s = \xi e^{-t} + 1 - e^{-t} + e^{-t}\int_0^t d(e^sB_s) - e^{-t}\int_0^t e^sB_s\,ds \\ &= e^{-t}(\xi - 1) + 1 + B_t - e^{-t}\int_0^t e^sB_s\,ds. \end{aligned}$$

In particular $\mathbb{E}\,X_t = 1$.