Applied Stochastic Analysis
Homework assignment 7

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Exercise 1

Since every Y_t^i, $i = 1, \dots, d$, is a linear combination of continuous independent (centered) Gaussian processes then every Y_t^i is a (centered) Gaussian process too. Hence, we only need to examine its covariance function. We have

$$\begin{aligned}\mathbb{E}[Y_s^i Y_t^j] &= \sum_{k=1}^{d}\sum_{n=1}^{d} Q_{ik}Q_{jn}\,\mathbb{E}[W_s^k W_t^n] = (s \wedge t)\sum_{k=1}^{d}\sum_{n=1}^{d} Q_{ik}Q_{jn}\delta_{kn} \\ &= (s \wedge t)\sum_{k=1}^{d} Q_{ik}Q_{jk} = (s \wedge t)(Q \cdot Q^t)_{ij} = (s \wedge t)\delta_{ij}.\end{aligned}$$

The above concludes the proof that $\mathbf{Y}_t = (Y_t^1, \dots, Y_t^d)$ is a d-dimensional Brownian motion.

Exercise 2

(a)

(b) It holds that

$$P(M_t > a) = P(M_t > a, B_t > a) + P(M_t > a, B_t \leq a) = P(B_t > a) + P(M_t > a, B_t \leq a),$$

since $B_t > a$ implies $M_t = \sup_{s \leq t} B_s \geq B_t > a$.

(c) Since $\{M_t > a\} = \{T_a \leq t\}$ and if $T_a \leq t$ then $\tilde{B}_t = 2a - B_t$, then

$$P(M_t > a, B_t \leq a) = P(T_a \leq t, 2a - B_t \geq a) = P(T_a \leq t, \tilde{B}_t \geq a) = P(\tilde{B}_t \geq a),$$

since $\{T_a > t, \tilde{B}_t \geq a\} = \{T_a > t, B_t \geq a\} = \emptyset$.

(d) Thanks to part (a), (b) and (c) we have that

$$P(M_t > a) = P(B_t > a) + P(\tilde{B}_t \geq a) = 2\int_a^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\,dx = \int_a^{\infty} \sqrt{\frac{2}{\pi t}} e^{-x^2/2t}\,dx.$$

Exercise 3

Exercise 4

(a) We showed in class that

$$Q_t^\sigma(W_t) \xrightarrow{L^2} t \qquad \text{as } |\sigma| \to 0. \tag{1}$$

Hence it is sufficient to notice that

$$\begin{aligned} Q_t^\sigma(aW_t + bt) &= \sum_{i=0}^{n-1} |aW_{t_{i+1}} + bt_{i+1} - aW_{t_i} - bt_i|^2 = \sum_{i=0}^{n-1} |a(W_{t_{i+1}} - W_{t_i}) + b(t_{i+1} - t_i)|^2 \\ &= a^2 Q_t^\sigma(W_t) + ab \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) + b^2 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2. \end{aligned}$$

Now

$$\mathbb{E}\left[\Big(\sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i)\Big)^2\right] = \sum_{i=0}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2](t_{i+1} - t_i)^2 = \sum_{i=0}^{n-1} (t_{i+1} - t_i)^4 \leq |\sigma|^3 t,$$

where the first equality holds since $\mathbb{E}[(W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})] = 0$ for every $i \neq j$, and so

$$ab \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0. \tag{2}$$

Combining (1) and (2) and the fact that $\sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 \leq |\sigma| t \xrightarrow{L^2} 0$ as $|\sigma| \to 0$, we get that

$$Q_t^\sigma(aW_t + bt) \xrightarrow{L^2} t \qquad \text{as } |\sigma| \to 0.$$

(b) Let's consider

$$Q_{t,4}^\sigma(W_t) = \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^4.$$

We have

$$\mathbb{E}[(Q_{t,4}^\sigma(W_t))^2] = \sum_{i=0}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^8] + \sum_{\substack{i,j=0 \\ i \neq j}}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^4 (W_{t_{j+1}} - W_{t_j})^4]$$

Now

$$\sum_{i=0}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^8] = 105 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^4 \leq 105 |\sigma|^3 t \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0 \tag{3}$$

and

$$\begin{aligned}
\sum_{\substack{i,j=0\\ i\neq j}}^{n-1} \mathbb{E}[(W_{t_{i+1}}-W_{t_i})^4(W_{t_{j+1}}-W_{t_j})^4] &= \sum_{\substack{i,j=0\\ i\neq j}}^{n-1} \mathbb{E}[(W_{t_{i+1}}-W_{t_i})^4]\,\mathbb{E}[(W_{t_{j+1}}-W_{t_j})^4] \\
&= 225\sum_{i=0}^{n-1}(t_{i+1}-t_i)^2\sum_{\substack{j=0\\ j\neq i}}^{n-1}(t_{j+1}-t_j)^2 \\
&\leq 225\sum_{i=0}^{n-1}(t_{i+1}-t_i)^2|\sigma|t \\
&\leq 225|\sigma|^2t^2 \xrightarrow{L^2} 0 \qquad \text{as } |\sigma|\to 0. \qquad (4)
\end{aligned}$$

Hence (3) and (4) imply that

$$Q^{\sigma}_{t,4}(W_t) \xrightarrow{L^2} 0 \qquad \text{as } |\sigma|\to 0.$$

Exercise 5

Let us denote by X_t^x the solution to $\dot{X}_t = a(X_t)$, $X_0 = x$. If $f \in C_b^2(\mathbb{R})$, then

$$(\mathcal{L}f)(x) = \lim_{t\to 0^+}\frac{f(X_t^x)-f(x)}{t} = \frac{d}{dt}f(X_t^x)\Big|_{t=0} = f'(X_0^x)\frac{d}{dt}X_t^x\Big|_{t=0} = f'(x)a(X_0^x) = a(x)f'(x),$$

i.e. $\mathcal{L} = a\,\partial_x$.

Exercise 6

Given $f \in C_b^2$, we have, for $x \in K$, where $K \subset \mathbb{R}$ is a compact set such that $\operatorname{supp}\{p(\cdot,t|\cdot,0)\} \subseteq K^2$,

$$(\mathcal{L}f)(x) = \lim_{t\to 0^+}\frac{\mathbb{E}_x[f(X_t)]-f(x)}{t} = \lim_{t\to 0^+}\frac{1}{t}\int_K (f(y)-f(x))\,p(y,t|x,0)\,dy. \qquad (5)$$

If $\varepsilon > 0$ we can split the integral in (5) as

$$\begin{aligned}
t^{-1}\int_K (f(y)-f(x))\,p(y,t|x,0)\,dy &= t^{-1}\int_{y\in K,|x-y|\geq\varepsilon}(f(y)-f(x))\,p(y,t|x,0)\,dy \\
&\quad + t^{-1}f'(x)\int_{y\in K,|x-y|<\varepsilon}(y-x)\,p(y,t|x,0)\,dy \\
&\quad + t^{-1}/2\int_{y\in K,|x-y|<\varepsilon} f''(\xi_y)(y-x)^2\,p(y,t|x,0)\,dy,
\end{aligned}$$

where ξ_y is a point between x and y. Now,

$$t^{-1}\int_{y\in K,|x-y|\geq\varepsilon}(f(y)-f(x))\,p(y,t|x,0)\,dy \quad \xrightarrow[t\to 0^+]{} \int_{y\in K,|x-y|\geq\varepsilon}(f(y)-f(x))\,W(y|x)\,dy + O(\varepsilon),$$

$$t^{-1}f'(x)\int_{y\in K,|x-y|<\varepsilon}(y-x)\,p(y,t|x,0)\,dy \quad \xrightarrow[t\to 0^+]{} f'(x)O(\varepsilon),$$

$$t^{-1}/2\int_{y\in K,|x-y|<\varepsilon} f''(\xi_y)(y-x)^2\, p(y,t|x,0)\, dy \leq$$

$$\leq \|f''\|_\infty t^{-1}/2\int_{y\in K,|x-y|<\varepsilon} (y-x)^2\, p(y,t|x,0)\, dy \quad \xrightarrow[t\to 0^+]{} \|f''\|_\infty O(\varepsilon)/2.$$

hence, if we send $\varepsilon \to 0^+$, we get

$$(\mathcal{L}f)(x) = \int_K (f(y)-f(x))\, W(y|x)\, dy = \int_{\mathbb{R}} f(y)\, W(y|x)\, dy - f(x)\lambda(x).$$