Applied Stochastic Analysis
Homework assignment 6

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Exercise 1

(a) First, note that if $N \sim \text{Poi}(\lambda)$, then

$$\mathbb{E}[N_t] = \lambda t, \qquad \mathbb{E}[N_t^2] = \lambda t(1+\lambda t) \qquad \text{and} \qquad \mathbb{E}[X_t] = \lambda(t+\alpha) - \lambda t = \lambda\alpha.$$

Moreover

$$\begin{aligned}\text{Cov}(N_t, N_s) &= \mathbb{E}[N_t N_s] - \mathbb{E}[N_t]\,\mathbb{E}[N_s] = \mathbb{E}[(N_{t\vee s} - N_{t\wedge s})N_{t\wedge s}] + \mathbb{E}[N_{t\wedge s}^2] - \mathbb{E}[N_t]\,\mathbb{E}[N_s] \\ &= \mathbb{E}[N_{t\vee s} - N_{t\wedge s}]\,\mathbb{E}[N_{t\wedge s}] + \mathbb{E}[N_{t\wedge s}^2] - \mathbb{E}[N_t]\,\mathbb{E}[N_s] = \mathbb{E}[N_{t\wedge s}^2] - \mathbb{E}[N_{t\wedge s}]^2 = \lambda(t\wedge s).\end{aligned}$$

Since X is strongly stationary, its covariance function is given by

$$C(t) = \text{Cov}(X_{|t|}, X_0) = \text{Cov}(N_{|t|+\alpha} - N_{|t|}, N_\alpha) = \lambda(\alpha - |t| \wedge \alpha) = \lambda(\alpha - |t|)\mathbb{1}_{\{|t|\le\alpha\}}.$$

Since $C \in L^1(\mathbb{R})$, the spectral density exists and its given by its Fourier transform:

$$f(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \lambda(\alpha - |t|)\mathbb{1}_{\{|t|\le\alpha\}} e^{-i\xi t} dt = \frac{\lambda}{\pi}\frac{1-\cos\alpha\xi}{\xi^2}.$$

(b) Since W is a Brownian motion, then

$$\mathbb{E}[W_t] = \mathbb{E}[X_t] = 0, \qquad \text{and} \qquad \mathbb{E}[W_t W_s] = s \wedge t.$$

Hence the covariance function of X is given by

$$\begin{aligned}C(t) &= \mathbb{E}[X_{|t|+s}X_s] = \mathbb{E}[(W_{|t|+\alpha} - W_{|t|})(W_{s+\alpha} - W_s)] \\ &= (\alpha + |t| \wedge s) - |t| \wedge (s+\alpha) - s \wedge (|t| + \alpha) + |t| \wedge s = (\alpha - |t|)\mathbb{1}_{\{|t|\le\alpha\}}\end{aligned}$$

As in the part (a), the spectral density exists and its given by its Fourier transform:

$$f(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} (\alpha - |t|)\mathbb{1}_{\{|t|\le\alpha\}} e^{-i\xi t} dt = \frac{1-\cos\alpha\xi}{\pi\xi^2}.$$

Exercise 2

(a) Using the notation on the notes, we have that the spectral representation of X is

$$X_t = \int_{\mathbb{R}} e^{i\xi t}\, dZ(\xi).$$

Derivating under the integral sign, we get the spectral representation of X':

$$X'_t = \int_{\mathbb{R}} e^{i\xi t}\, i\xi dZ(\xi).$$

Its covariance function is given by

$$\begin{aligned}
C_{X'}(t) &= \mathbb{E}[X'_{t+s}\overline{X'_s}] = \mathbb{E}\Big[\int_{\mathbb{R}^2} e^{i\xi(t+s)}e^{-i\xi' s}\, \xi\xi' dZ(\xi)\overline{dZ(\xi')}\Big] \\
&= \int_{\mathbb{R}^2} e^{i\xi(t+s)}e^{-i\xi' s}\, \xi\xi'\, \mathbb{E}\Big[dZ(\xi)\overline{dZ(\xi')}\Big] = \int_{\mathbb{R}^2} e^{i\xi(t+s)}e^{-i\xi' s}\, \xi\xi'\delta(\xi-\xi')dF(\xi)d\xi' \\
&= \int_{\mathbb{R}} e^{i\xi t}\, \xi^2 dF(\xi) = -\frac{d^2}{dt^2}\int_{\mathbb{R}} e^{i\xi t}\, dF(\xi) = -\frac{d^2}{dt^2}C(t).
\end{aligned}$$

(This confirms what found in exercise 5 of homework 5).

(b) It holds that

$$\begin{aligned}
C_{X,X'}(t) &= \mathbb{E}[X_{t+s}\overline{X'_s}] = -i\,\mathbb{E}\Big[\int_{\mathbb{R}^2} e^{i\xi(t+s)}e^{-i\xi' s}\, \xi' dZ(\xi)\overline{dZ(\xi')}\Big] \\
&= -i\int_{\mathbb{R}^2} e^{i\xi(t+s)}e^{-i\xi' s}\, \xi'\, \mathbb{E}\Big[dZ(\xi)\overline{dZ(\xi')}\Big] = -i\int_{\mathbb{R}^2} e^{i\xi(t+s)}e^{-i\xi' s}\, \xi'\delta(\xi-\xi')dF(\xi)d\xi' \\
&= -i\int_{\mathbb{R}} e^{i\xi t}\, \xi dF(\xi) = -\frac{d}{dt}\int_{\mathbb{R}} e^{i\xi t}\, dF(\xi) = -\frac{d}{dt}C(t).
\end{aligned}$$

(c) For every $t \in \mathbb{R}$ it holds

$$\mathbb{E}[X_t\overline{X'_t}] = C_{X,X'}(0) = -C'(0).$$

Since it is constant, its derivative is zero.

(d) A sufficient condition is that X is a C^{2n} process.