Applied Stochastic Analysis
Homework assignment 9

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Exercise 1

Using the formula to transform an Itô SDE into a Stratonovich one (or viceversa) we get:

(a)

$$dX_t = \left(aX_t + \frac{1}{2}b^2 X_t\right) dt + bX_t \, dB_t.$$

(b)

$$dX_t = \frac{1}{2}\left(\sin X_t \cos X_t - t^2 \sin X_t\right) dt + (t^2 + \cos X_t) \, dB_t.$$

(c)

$$dX_t = \left(r - \frac{1}{2}\alpha^2\right) X_t \, dt + \alpha X_t \circ dB_t.$$

(d)

$$dX_t = \left(2e^{-X_t} - X_t^3\right) dt + X_t^2 \circ dB_t.$$

Exercise 2

(a) By Itô formula we have

$$d(X_t^n) = \left(\lambda n + \frac{1}{2}n(n-1)\sigma^2\right) X_t^n \, dt + X_t^n \, dW_t,$$

i.e.

$$X_t^n = x_0^n + \int_0^t \left(\lambda n + \frac{1}{2}n(n-1)\sigma^2\right) X_s^n \, ds + \sigma n \int_0^t X_s^n \, dW_s.$$

Taking the expectation we get

$$M_n(t) = \mathbb{E}\, X_t^n = \mathbb{E}\, x_0^n + \int_0^t \left(\lambda n + \frac{1}{2}n(n-1)\sigma^2\right) M_n(s) \, ds.$$

The above formula is equivalent to say that M_n satisfies the equation

$$\frac{dM_n}{dt} = \left(\lambda n + \frac{\sigma^2}{2}n(n-1)\right) M_n, \qquad M_n(0) = \mathbb{E}\, x_0^n. \tag{1}$$

(b) The solution to equation (1) is

$$M_n(t) = (\mathbb{E}\, x_0^n)\, e^{n(\lambda+\sigma^2(n-1)/2)t}.$$

(c) It must be

$$\lambda + \frac{\sigma^2}{2}(n-1) < 0, \qquad \text{i.e.} \qquad \lambda < -\frac{\sigma^2}{2}(n-1).$$

(d) One just need to take $N = \inf\{n \geq 1 \,:\, \lambda + \sigma^2(n-1)/2 > 0\}$.

Exercise 3

(a) Suppose that X_t satisfies the SDE

$$\begin{cases} dX_t = a(X_t)\, dY_t \\ dY_t = b(X_t)\, dW_t \end{cases} \tag{2}$$

The terms in the Riemann sums for (2) are

$$\begin{cases} \Delta_j X = a(X_j)\Delta_j Y \\ \Delta_j Y = b(X_j)\Delta_j W \end{cases} \qquad \Longrightarrow \qquad \Delta_j X = a(X_j)b(X_j)\Delta_j W.$$

So the above gives

$$dX_t = (a(X_t)b(X_t))\, dW_t.$$

(b) Suppose that X_t satisfies the SDE

$$\begin{cases} dX_t = a(X_t)\, dY_t \\ dY_t = b(X_t) \circ dW_t \end{cases} \tag{3}$$

and that this implies

$$dX_t = \alpha(X_t)\, dt + \beta(X_t)\, dW_t \tag{4}$$

for some functions α, β. The terms in the Riemann sums for (3) are

$$\begin{cases} \Delta_j X = a(X_j)\Delta_j Y \\ \Delta_j Y = \frac{1}{2}(b(X_j) + b(X_{j+1}))\Delta_j W \end{cases} \tag{5}$$

Now, using (4), we have

$$b(X_{j+1}) = b(X_j) + b'(X_j)\Delta_j X \quad \Longrightarrow \quad b(X_{j+1})\Delta_j W = b(X_j)\Delta_j W + b'(X_j)\beta(X_j)\Delta_j t,$$

and thus from (5) we get

$$\Delta_j X = \frac{1}{2} a(X_j)b'(X_j)\beta(X_j)\Delta_j t + a(X_j)b(X_j)\Delta_j W$$

Comparing the above with (4), we get that X must satisfy:

$$dX_t = \frac{1}{2} a^2(X_t)b(X_t)b'(X_t)\, dt + a(X_t)b(X_t)\, dW_t.$$

(c) Suppose that X_t satisfies the SDE

$$\begin{cases} dX_t = a(X_t) \circ dY_t \\ dY_t = b(X_t)\, dW_t \end{cases} \tag{6}$$

and that this implies

$$dX_t = \alpha(X_t)\, dt + \beta(X_t)\, dW_t \tag{7}$$

for some functions α, β. The terms in the Riemann sums for (6) are

$$\begin{cases} \Delta_j X = \frac{1}{2}(a(X_j) + a(X_{j+1}))\Delta_j Y \\ \Delta_j Y = b(X_j)\Delta_j W \end{cases} \tag{8}$$

Now, using (7), we have

$$a(X_{j+1}) = a(X_j) + a'(X_j)\Delta_j X \quad \Longrightarrow \quad a(X_{j+1})\Delta_j W = a(X_j)\Delta_j W + a'(X_j)\beta(X_j)\Delta_j t,$$

and thus from (8) we get

$$\Delta_j X = \frac{1}{2}a'(X_j)b(X_j)\beta(X_j)\Delta_j t + a(X_j)b(X_j)\Delta_j W$$

Comparing the above with (7), we get that X must satisfy:

$$dX_t = \frac{1}{2}a(X_t)a'(X_t)b^2(X_t)\, dt + a(X_t)b(X_t)\, dW_t.$$

(d) Suppose that X_t satisfies the SDE

$$\begin{cases} dX_t = a(X_t) \circ dY_t \\ dY_t = b(X_t) \circ dW_t \end{cases} \tag{9}$$

and that this implies

$$dX_t = \alpha(X_t)\, dt + \beta(X_t)\, dW_t \tag{10}$$

for some functions α, β. The terms in the Riemann sums for (6) are

$$\begin{cases} \Delta_j X = \frac{1}{2}(a(X_j) + a(X_{j+1}))\Delta_j Y \\ \Delta_j Y = \frac{1}{2}(b(X_j) + b(X_{j+1}))\Delta_j W \end{cases} \tag{11}$$

Now, using (10), we have

$$a(X_{j+1}) = a(X_j) + a'(X_j)\Delta_j X, \qquad b(X_{j+1}) = b(X_j) + b'(X_j)\Delta_j X,$$

and thus from (11) we get

$$\begin{aligned} \Delta_j X &= \frac{1}{4}(a(X_j) + a(X_{j+1}))(b(X_j) + b(X_{j+1}))\Delta_j W \\ &= \frac{1}{4}(2a(X_j) + a'(X_j)\beta(X_j)\Delta_j W)(2b(X_j) + b'(X_j)\beta(X_j)\Delta_j W)\Delta_j W \\ &= \frac{1}{2}(a \cdot b)'(X_j)\beta(X_j)\Delta_j t + a(X_j)b(X_j)\Delta_j W \end{aligned}$$

Comparing the above with (10), we get that X must satisfy:

$$dX_t = \frac{1}{2}(a \cdot b)(X_t)(a \cdot b)'(X_t)\, dt + a(X_t)b(X_t)\, dW_t.$$

Exercise 4

Be $Y_t = f(X_t)$ and suppose that $dX_t = \alpha(X_t)\,dt + \beta(X_t) \circ dW_t$. Then

$$dX_t = (\alpha(X_t) + \frac{1}{2}\beta(X_t)\beta'(X_t))dt + \beta(X_t)\,dW_t$$

and so, by Itô formula

$$\begin{aligned} dY_t &= f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\ &= \left[\alpha(X_t)f'(X_t) + \frac{1}{2}\beta(X_t)\beta'(X_t)f'(X_t) + \frac{1}{2}\beta(X_t)^2 f''(X_t)\right]dt + \beta(X_t)f'(X_t)\,dW_t \\ &= \left[\alpha(X_t)f'(X_t) + \frac{1}{2}\beta(X_t)(\beta \cdot f')'(X_t)\right]dt + \beta(X_t)f'(X_t)\,dW_t. \end{aligned} \tag{12}$$

If $g = f^{-1}$, $h = f'$, $\hat{\alpha} = \alpha \circ g$, $\hat{\beta} = \beta \circ g$ and $\hat{h} = h \circ g$, then using the change of variable $y = f(x)$ (which we can do locally) (12) becomes

$$\begin{aligned} dY_t &= \left[\hat{\alpha}(Y_t)\hat{h}(Y_t) + \frac{1}{2}\hat{\beta}(Y_t)\hat{h}(Y_t)\frac{d}{dy}(\hat{\beta} \cdot \hat{h})(Y_t)\right]dt + \hat{\beta}(Y_t)\hat{h}(Y_t)\,dW_t \\ &= \hat{\alpha}(Y_t)\hat{h}(Y_t)\,dt + \hat{\beta}(Y_t)\hat{h}(Y_t) \circ dW_t \\ &= \hat{h}(Y_t)\left[\hat{\alpha}(Y_t)\,dt + \hat{\beta}(Y_t) \circ dW_t\right] \\ &= h(X_t)[\alpha(X_t)\,dt + \beta(X_t) \circ dW_t] = h(X_t) \circ dX_t, \end{aligned}$$

i.e. $d(f(X_t)) = f'(X_t) \circ dX_t$.

Exercise 5

It is sufficient to show that $dY_t = 0$, where $Y_t \doteq X_{1,t}^2 + X_{2,t}^2$. Indeed, using the result from Exercise 4, it holds:

$$\begin{aligned} dY_t &= d(X_{1,t}^2) + d(X_{2,t}^2) = 2X_{1,t} \circ dX_{1,t} + 2X_{2,t} \circ dX_{2,t} \\ &= \frac{1}{|X_t|^2}\left[2X_{1,t}(X_{2,t}^2 \circ dW_{1,t} - X_{1,t}X_{2,t} \circ dW_{2,t}) + 2X_{2,t}(-X_{1,t}X_{2,t} \circ dW_{1,t} + X_{1,t}^2 \circ dW_{2,t})\right] \\ &= \frac{1}{|X_t|^2}\left[(2X_{1,t}X_{2,t}^2 - 2X_{1,t}X_{2,t}^2) \circ dW_{1,t} + (2X_{1,t}^2X_{2,t} - 2X_{1,t}^2X_{2,t}) \circ dW_{2,t}\right] = 0. \end{aligned}$$

Exercise 6

(a) The spectral form of the SDE is

$$i\xi\hat{X}_\xi = -\alpha\hat{X}_\xi + \sigma\hat{\eta}_\xi \quad \Longrightarrow \quad \hat{X}_\xi = \frac{\sigma}{\alpha + i\xi} \cdot \hat{\eta}_\xi$$

where $\hat{X}_\xi = dZ_X(\xi)$, $\hat{\eta}_\xi = dZ_\eta(\xi)$ (η is a Gaussian process with mean 0 and covariance function $C_\eta(t,s) = \delta(t-s)$). Since the covariance function of η has the spectral representation

$$C_\eta(t) = \delta(t) = \int e^{i\xi t}\frac{d\xi}{2\pi}$$

then the covariance function of X has the spectral representation

$$C_X(t) = \mathbb{E}[X_t \overline{X}_0] = \int e^{i\xi t}\, \mathbb{E}\Big[\hat{X}_\xi \overline{\hat{X}_\lambda}\Big] = \int e^{i\xi t} \frac{\sigma}{(\alpha + i\xi)} \frac{\sigma}{(\alpha - i\lambda)} \mathbb{E}[\hat{\eta}_\xi \overline{\hat{\eta}}_\lambda] = \int e^{i\xi t} \frac{\sigma^2}{\alpha^2 + \xi^2} \frac{d\xi}{2\pi},$$

i.e. the spectral density of X is

$$F_X(\xi) = \frac{\sigma^2}{2\pi(\alpha^2 + \xi^2)}.$$

(b) If X, Y are independent then

$$\begin{aligned}
C_W(t,s) &= \cos kt\, \cos ks\, C_X(t-s) + \sin kt\, \sin ks\, C_Y(t-s) \\
&= \int e^{i\xi(t-s)} \cos k(t-s) \frac{\sigma^2}{\alpha^2 + \xi^2} \frac{d\xi}{2\pi} \\
&= \int e^{i\xi(t-s)} (e^{ik(t-s)} + e^{-ik(t-s)}) \frac{\sigma^2}{\alpha^2 + \xi^2} \frac{d\xi}{4\pi} \\
&= \int e^{i\xi(t-s)} \frac{\sigma^2}{4\pi} \left[\frac{1}{\alpha^2 + (\xi - k)^2} + \frac{1}{\alpha^2 + (\xi + k)^2} \right] d\xi,
\end{aligned}$$

i.e. the spectral density of W is

$$F_W(\xi) = \frac{\sigma^2}{4\pi} \left[\frac{1}{\alpha^2 + (\xi - k)^2} + \frac{1}{\alpha^2 + (\xi + k)^2} \right].$$

Exercise 7

(a) Equation (a) means that

$$X_t = \xi + \int_0^t s\, ds + 2 \int_0^t dB_s = \xi + \frac{t^2}{2} + 2B_t.$$

This implies $\mathbb{E}\, X_t = t^2/2$.

(b) If we multiply both sides of equation (b) by $e^{\cos t}$, we get

$$d(e^{\cos t} X_t) = e^{\cos t}\, dX_t - (\sin t) e^{\cos t} X_t\, dt = e^{\cos t}\, dB_t,$$

which gives

$$\begin{aligned}
X_t &= \xi\, e^{(1-\cos t)} + e^{-\cos t} \int_0^t e^{\cos s}\, dB_s \\
&= \xi\, e^{(1-\cos t)} + e^{-\cos t} \int_0^t d(e^{\cos s} B_s) - e^{-\cos t} \int_0^t (\sin s) B_s\, e^{\cos s}\, ds \\
&= \xi\, e^{(1-\cos t)} + B_t - e^{-\cos t} \int_0^t (\sin s) B_s\, e^{\cos s}\, ds.
\end{aligned}$$

In particular $\mathbb{E}\, X_t = 0$.

(c) If we multiply both sides of equation (c) by e^t, we get

$$d(e^t X_t) = e^t\, dX_t + e^t X_t\, dt = e^t dt + e^t dB_t,$$

which gives

$$\begin{aligned} X_t &= \xi e^{-t} + e^{-t}\int_0^s e^s ds + e^{-t}\int_0^t e^s dB_s = \xi e^{-t} + 1 - e^{-t} + e^{-t}\int_0^t d(e^s B_s) - e^{-t}\int_0^t e^s B_s\, ds \\ &= e^{-t}(\xi - 1) + 1 + B_t - e^{-t}\int_0^t e^s B_s\, ds. \end{aligned}$$

In particular $\mathbb{E}\, X_t = 1$.

Exercise 8

(a) By Itô formula, it holds that

$$dG_t = \frac{\alpha^2}{2} G_t\, dt - \alpha G_t\, dB_t + \frac{\alpha^2}{2} G_t\, dt = \alpha^2 G_t\, dt - \alpha G_t\, dB_t,$$

and that

$$\begin{aligned} d(X_t G_t) &= X_t\, dG_t + G_t\, dX_t + dX_t\, dG_t \\ &= \alpha^2 X_t G_t\, dt - \alpha X_t G_t\, dB_t + \frac{G_t}{X_t}\, dt + \alpha X_t G_t\, dB_t - \alpha^2 X_t G_t\, dt = \frac{G_t}{X_t}\, dt. \end{aligned}$$

(b) By part (a), we have

$$dY_t = \frac{G_t}{X_t}\, dt \quad \Longrightarrow \quad Y_t = Y_0 + \int_0^t \frac{G_s}{X_s}\, ds = Y_0 + \int_0^t \frac{G_s^2}{Y_s}\, ds \quad \Longrightarrow \quad \frac{dY_t}{dt} = \frac{G_t^2}{Y_t}. \tag{13}$$

(c) Equation (13) can be written as

$$\frac{d}{dt}\left(\frac{1}{2} Y_t^2\right) = G_t^2 \quad \Longrightarrow \quad Y_t^2 = Y_0^2 + 2\int_0^t G_s^2\, ds \quad \Longrightarrow \quad X_t^2 = \frac{X_0^2 + 2\int_0^t G_s^2\, ds}{G_t^2},$$

which implies

$$X_t = e^{\alpha B_t - \alpha^2 t/2}\left[X_0^2 + 2\int_0^t e^{\alpha^2 s - 2\alpha B_s}\, ds\right]^{1/2}.$$

Exercise 9

Suppose that X_t satisfies the SDE

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t) * dW_t \tag{14}$$

and that this is equivalent to

$$dX_t = \alpha(t, X_t)\, dt + \beta(t, X_t)\, dW_t \tag{15}$$

for some functions α, β. The terms in the Riemann sums for (14) is

$$\begin{aligned} \Delta_j X &= b(t_j, X_j)\Delta_j t + \sigma(t_{j+1}, X_{j+1})\Delta_j W \\ &= b(t_j, X_j)\Delta_j t + \sigma(t_j, X_j)\Delta_j W + \partial_x \sigma(t_j, X_j)(\Delta_j X \Delta_j W) \\ &= b(t_j, X_j)\Delta_j t + \sigma(t_j, X_j)\Delta_j W + \partial_x \sigma(t_j, X_j) \cdot \beta(t_j, X_j)\Delta_j t \end{aligned}$$

where we used (15) in the last equality. This means

$$dX_t = [b(t, X_t) + \partial_x \sigma(t, X_t) \cdot \beta(t, X_t)]\, dt + \sigma(t, X_t)\, dW_t. \tag{16}$$

Comparing (16) with (15) we conclude that it must be

$$\beta(t, X_t) = \sigma(t, X_t) \qquad \text{and} \qquad \alpha(t, X_t) = b(t, X_t) + \partial_x \sigma(t, X_t) \cdot \sigma(t, X_t),$$

and therefore X must satisfy

$$dX_t = [b(t, X_t) + \partial_x \sigma(t, X_t) \cdot \sigma(t, X_t)]\, dt + \sigma(t, X_t)\, dW_t.$$