Applied Stochastic Analysis
Homework assignment 11

Luca Venturi

May 1, 2017

Exercise 1

(a) The corresponding Fokker-Plank equations are

$$\begin{aligned}\rho_t &= -\partial_x(\lambda x \rho) + \partial^2_{xx}\left(\frac{1}{2}\sigma^2 x^2 \rho\right) \\ &= (\sigma^2 - \lambda)\rho + (2\sigma^2 - \lambda)x\rho_x + \frac{1}{2}\sigma^2 x^2 \rho_{xx}.\end{aligned}$$

Exercise 3

(a) X_t should satisfy the SDE

$$dX_t = -v\mathbb{1}_{\{X_t \in [0,d]\}}\, dt + \sigma\, dW_t, \qquad X_t \in [0, L].$$

(b) X_t should satisfy the SDE

$$dX_t = -v\mathbb{1}_{\{X_t \in [0,d]\}}\, dt + \sigma\, dW_t, \qquad X_t \in [0, L].$$

Exercise 4

(a) Be $V_t = \dot{X}_t$. Then we can write the Langevin equation as a first order system:

$$\begin{cases} dX_t = V_t\, dt \\ dV_t = -\frac{1}{m}(\gamma V_t + \nabla U(X_t))\, dt + \frac{1}{m}\sqrt{2}\sigma\, dW_t \end{cases}$$

i.e.

$$d\mathbf{X}_t = \mathbf{b}(\mathbf{X}_t)\, dt + \Sigma\, d\mathbf{W}_t,$$

where

$$\mathbf{X}_t = \begin{pmatrix} X_t \\ V_t \end{pmatrix}, \quad \mathbf{b}(\mathbf{X}_t) = \begin{pmatrix} V_t \\ -\frac{1}{m}(\gamma V_t + \nabla U(X_t)) \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 0 & 0 \\ \frac{1}{m}\sqrt{2}\sigma & 0 \end{pmatrix}.$$

(b) The corresponding Fokker-Plank equation is

$$\rho_t = \mathcal{L}^*_{(x,v)}\rho = -\nabla_{(x,v)} \cdot (\mathbf{b}\rho) + \nabla^2_{(x,v)} : (A\rho) \tag{1}$$

where

$$A = \frac{1}{2}\Sigma\Sigma^T = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma^2}{m^2} \end{pmatrix}.$$

Equation (1) can be written more explicitly as

$$\rho_t = -v\rho_x + \frac{1}{m}[\rho_v(\gamma v + \nabla U(x)) + \gamma\rho] + \frac{\sigma^2}{m^2}\rho_{vv} \tag{2}$$

(c) If $\rho = \rho_s(x,v) = Z^{-1}e^{-\beta H(x,v)}$, we have

$$\begin{aligned} -v\rho_x &= -v(-\beta H_x)\rho = \beta v \nabla U(x)\rho, \\ \frac{1}{m}\rho_v(\gamma v + \nabla U(x)) &= -\frac{\beta}{m}H_v(\gamma v + \nabla U(x))\rho = -\beta v\nabla U(x)\rho - \beta\gamma v^2\rho, \\ \frac{\sigma^2}{m^2}\rho_{vv} &= \frac{\sigma^2}{m^2}(\rho H_v)_v = -\beta\frac{\sigma^2}{m^2}H_{vv} + \frac{\sigma^2}{m^2}\beta^2(H_v)^2 = \frac{\gamma}{m}\rho + \beta\gamma v^2\rho. \end{aligned}$$

Then, using the above equations, (2) gives

$$\rho_t = \mathcal{L}^*_{(x,v)}\rho = \beta v\nabla U(x)\rho - \beta v\nabla U(x)\rho - \beta\gamma v^2\rho + \frac{\gamma}{m}\rho + \frac{\gamma}{m}\rho + \beta\gamma v^2\rho = 0.$$

(d) The steady-state flux is

$$\mathbf{j}_s = \mathbf{b}\rho - \nabla_{(x,v)} \cdot (A\rho) = \rho \begin{pmatrix} v \\ -\frac{1}{m}(\gamma v + \nabla U(x)) \end{pmatrix} - \rho_v \begin{pmatrix} 0 \\ \frac{\sigma^2}{m^2} \end{pmatrix}.$$

From this expression we see that $\mathbf{j}_s$ is not zero in general and neither if $\rho = \rho_s(x,v) = Z^{-1}e^{-\beta H(x,v)}$.

(e) fare