Applied Stochastic Analysis
Homework assignment 8

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Exercise 1

(a) We use the same notation as in the text of the assignment. Here $\sum_j$ means $\sum_{j=0}^{n-1}$. It holds that

$$\sum_j \frac{1}{2}(W_{j+1}+W_j)\Delta W_j = \frac{1}{2}\sum_j (W_{j+1}^2 - W_j^2) = \frac{1}{2}W(t)^2.$$

Therefore it is straightforward that

$$\sum_j \frac{1}{2}(W_{j+1}+W_j)\Delta W_j \quad \xrightarrow{L^2} \quad \frac{1}{2}W(t)^2$$

as $|\sigma| \to 0$.

(b) We consider

$$\begin{aligned}\sum_j \frac{1}{2}(W_{j+1}+W_j)\Delta W_j - \sum_j W_{j+1/2}\Delta W_j &= \frac{1}{2}\sum_j (W_{j+1} - 2W_{j+1/2} + W_j)\Delta W_j \\ &= \frac{1}{2}\sum_j (W_{j+1} - W_{j+1/2})^2 - \frac{1}{2}\sum_j (W_{j+1/2} - W_j)^2.\end{aligned}$$

Now

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_j [(W_{j+1} - W_{j+1/2})^2 - (t_{j+1} - t_{j+1/2})]\Big)^2\Big] = \\ &= \sum_{i,j} \mathbb{E}[((W_{j+1} - W_{j+1/2})^2 - (t_{j+1} - t_{j+1/2}))((W_{i+1} - W_{i+1/2})^2 - (t_{i+1} - t_{i+1/2}))] \\ &= \sum_i \mathbb{E}[((W_{i+1} - W_{i+1/2})^2 - (t_{i+1} - t_{i+1/2}))^2] \leq \sum_i \mathbb{E}[(W_{i+1} - W_{i+1/2})^4] + \sum_i (t_{i+1} - t_{i+1/2})^2 \\ &= 4\sum_i (t_{i+1} - t_{i+1/2})^2 \leq 2|\sigma|\sum_i (t_{i+1} - t_{i+1/2}) = |\sigma|,\end{aligned}$$

which implies that

$$\sum_j (W_{j+1} - W_{j+1/2})^2 \quad \xrightarrow{L^2} \quad \frac{t}{2}$$

as $|\sigma| \to 0$. Similarly one can prove that

$$\sum_j (W_{j+1/2} - W_j)^2 \quad \xrightarrow{L^2} \quad \frac{t}{2}$$

as $|\sigma| \to 0$. Thus, it follows that

$$\sum_j \frac{1}{2}(W_{j+1} + W_j)\Delta W_j - \sum_j W_{j+1/2}\Delta W_j \quad \xrightarrow{L^2} \quad \frac{t}{4} - \frac{t}{4} = 0$$

as $|\sigma| \to 0$, which, thanks to part (a) of the exercise, implies that

$$\sum_j W_{j+1/2}\Delta W_j \quad \xrightarrow{L^2} \quad \frac{1}{2}W(t)^2$$

as $|\sigma| \to 0$.

(c) It holds that

$$\begin{aligned}
\sum_j W_j^2 \Delta W_j &= \sum_j W_j(W_j - W_{j+1} + W_{j+1})\Delta W_j = -\sum_j W_j(\Delta W_j)^2 + \sum_j W_j W_{j+1}\Delta W_j \\
&= -\sum_j W_j(\Delta W_j)^2 + \frac{1}{3}\sum_j (W_{j+1}^3 - W_j^3) - \frac{1}{3}\sum_j (W_{j+1} - W_j)^3 \\
&= -\sum_j W_j(\Delta W_j)^2 + \frac{1}{3}W(t)^3 - \frac{1}{3}\sum_j (W_{j+1} - W_j)^3. \qquad (1)
\end{aligned}$$

Now

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_j W_j((W_{j+1} - W_j)^2 - (t_{j+1} - t_j))\Big)^2\Big] = \\
&= \sum_{i,j} \mathbb{E}[W_i W_j((W_{i+1} - W_i)^2 - (t_{i+1} - t_i))((W_{j+1} - W_j)^2 - (t_{j+1} - t_j))] \\
&= \sum_i \mathbb{E}[W_i^2((W_{i+1} - W_i)^2 - (t_{i+1} - t_i))^2] = 2\sum_i t_i(t_{i+1} - t_i)^2 \le 2|\sigma| t^2
\end{aligned}$$

which implies that

$$-\sum_j W_j(\Delta W_j)^2 \quad \xrightarrow{L^2} \quad -\int_0^t W(s)\, ds \qquad (2)$$

as $|\sigma| \to 0$. Also

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_j (W_{j+1} - W_j)^3\Big)^2\Big] &= \sum_{i,j} \mathbb{E}[(W_{i+1} - W_i)^3(W_{j+1} - W_j)^3] = \sum_j \mathbb{E}[(W_{j+1} - W_j)^6] \\
&= 15\sum_j (t_{j+1} - t_j)^3 \le 15|\sigma|^2 t \quad \to \quad 0
\end{aligned}$$

as $|\sigma| \to 0$, which implies that

$$-\frac{1}{3}\sum_j (W_{j+1} - W_j)^3 \quad \xrightarrow{L^2} \quad 0 \qquad (3)$$

as $|\sigma| \to 0$. Therefore (1), (2) and (3) imply that

$$\sum_j W_j^2 \Delta W_j \quad \xrightarrow{L^2} \quad \frac{1}{3}W(t)^3 - \int_0^t W(s)\, ds$$

as $|\sigma| \to 0$.

Exercise 2

(a) Let $Y_t = \frac{1}{3}W_t^3$. By Itô formula we have that

$$dY_t = W_t^2\, dW_t + W_t\,(dW_t)^2 = W_t^2\, dW_t + W_t\, dt.$$

Therefore it holds that

$$\int_0^t W_s^2\, dW_s = \int_0^t d(\frac{1}{3}W_s^3) - \int_0^t W_s\, ds = \frac{1}{3}W_t^3 - \int_0^t W_s\, ds.$$

Note that this result is in accordance with what found in Exercise 1.(c).

(b) By Itô isometry and Fubini theorem, it holds that

$$\mathbb{E}\Big[\Big(\int_0^t W_s^2\, dW_s\Big)^2\Big] = \mathbb{E}\Big[\int_0^t W_s^4\, ds\Big] = \int_0^t \mathbb{E}[W_s^4]\, ds = \int_0^t 3s^2\, ds = t^3.$$

Exercise 3

By applying Itô formula we find:

(a) If $Y_t = W_t/(1+t)$ then

$$dY_t = -\frac{1}{(1+t)^2} W_t\, dt + \frac{1}{1+t}\, dW_t.$$

(b) If $Y_t = \sin W_t$ then

$$dY_t = \cos W_t\, dW_t - \frac{1}{2}\sin W_t (dW_t)^2 = -\frac{1}{2}\sin W_t\, dt + \cos W_t\, dW_t.$$

(c) If $X_t = a\cos W_t$, $Y_t = b\sin W_t$ $(ab \neq 0)$ then

$$\begin{aligned} dX_t &= -a\sin W_t\, dW_t - \frac{a}{2}\cos W_t\,(dW_t)^2 = -\frac{1}{2}X_t\, dt - \frac{a}{b}Y_t\, dW_t,\\ dY_t &= b\cos W_t\, dW_t - \frac{b}{2}\sin W_t\,(dW_t)^2 = -\frac{1}{2}Y_t\, dt + \frac{b}{a}X_t\, dW_t. \end{aligned}$$

We can write this as

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \begin{pmatrix} 0 & -a/b \\ b/a & 0 \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} dW_t.$$

Exercise 4

Let's consider $Z_t = f(X_t, Y_t) = X_t Y_t$. Since

$$\nabla f(x,y) = (y,x) \qquad \text{and} \qquad \nabla^2 f(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$\nabla f(X_t,Y_t)\cdot d(X_t,Y_t) = Y_t dX_t + X_t dY_t \qquad \text{and} \qquad d(X_t,Y_t)^T\,\nabla^2 f(X_t,Y_t)\, d(X_t,Y_t) = 2dX_t dY_t.$$

It follows from the multi-dimensional Itô formula that

$$dZ_t = \nabla f(X_t,Y_t)\cdot d(X_t,Y_t) + \frac{1}{2}d(X_t,Y_t)^T\,\nabla^2 f(X_t,Y_t)\, d(X_t,Y_t) = Y_t dX_t + X_t dY_t + dX_t dY_t.$$

Exercise 5

We have that

$$(dX_t)^T\,\nabla^2 f(X_t)\,dX_t = \sum_{i,j} dX_t^i\,\partial_{ij} f(X_t)\,dX_t^j.$$

Now

$$dX_t^i\,dX_t^j = \Big(b^i\,dt + \sum_k \sigma^{ik}\,dW_t^k\Big)\Big(b^j\,dt + \sum_k \sigma^{jl}\,dW_t^l\Big) = \Big(\sum_k \sigma^{ik}\sigma^{jk}\Big)\,dt = (\sigma\,\sigma^T)_{ij}\,dt,$$

thanks to the formal rules $dt\,dt = dt\,dW_t^i = dW_t^i\,dW_t^j = 0$ if $i \neq j$ and $(dW_t^i)^2 = dt$. Therefore

$$(dX_t)^T\,\nabla^2 f(X_t)\,dX_t = \sum_{i,j} \partial_{ij} f(X_t)\,(\sigma\,\sigma^T)_{ij}\,dt = (\sigma\,\sigma^T \;:\; \nabla^2 f)\,dt.$$

Exercise 6

Let $R_t = f(B_t)$ where $f(\mathbf{x}) = |\mathbf{x}|$, $\mathbf{x} = (x_1, \ldots, x_n)$. Since $\nabla f(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$ it follows that

$$\nabla f(B_t)\cdot dB_t = \frac{1}{|B_t|} B_t \cdot dB_t = \frac{1}{R_t}\sum_i B_i\,dB_i. \tag{4}$$

Moreover $\partial_{ii} f(\mathbf{x}) = (|\mathbf{x}|^2 - x_i^2)/|\mathbf{x}|^3$, so $\sum_i \partial_{ii} f(\mathbf{x}) = (n-1)/|\mathbf{x}|$. Hence

$$(dB_t)^T\,\nabla^2 f(B_t)\,dB_t = (\sigma\,\sigma^T \;:\; \nabla^2 f)\,dt = \mathrm{tr}(\nabla^2 f)\,dt = \frac{n-1}{|B_t|}\,dt = \frac{n-1}{R_t}\,dt, \tag{5}$$

thanks to what we proved in Exercise 5 and to the fact that $\sigma = I$ in this case. Therefore, by (4), (5) and the multi-dimensional Itô formula we get:

$$dR_t = \nabla f(B_t)\cdot dB_t + \frac{1}{2}(dB_t)^T\,\nabla^2 f(B_t)\,dB_t = \frac{1}{R_t}\sum_i B_i\,dB_i + \frac{1}{2}\frac{n-1}{R_t}\,dt.$$