Applied Stochastic Analysis
Homework assignment 7

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Exercise 1

Since every Y_t^i, $i = 1, \dots, d$, is a linear combination of continuous independent (centered) Gaussian processes then every Y_t^i is a continuous (centered) Gaussian process too. Hence, we only need to examine its covariance function. We have

$$\begin{aligned}\mathbb{E}[Y_s^i Y_t^j] &= \sum_{k=1}^d \sum_{n=1}^d Q_{ik} Q_{jn} \, \mathbb{E}[W_s^k W_t^n] = (s \wedge t) \sum_{k=1}^d \sum_{n=1}^d Q_{ik} Q_{jn} \delta_{kn} \\ &= (s \wedge t) \sum_{k=1}^d Q_{ik} Q_{jk} = (s \wedge t)(Q \cdot Q^t)_{ij} = (s \wedge t)\delta_{ij}.\end{aligned}$$

The above concludes the proof that $\mathbf{Y}_t = (Y_t^1, \dots, Y_t^d)$ is a d-dimensional Brownian motion.

Exercise 2

(a) First of all, we notice that, by definition of that T_a, $\tilde{B}_t$ is a.s. continuous. Hence, to show that $\tilde{B}_t$ is a B.m. (Brownian motion), it is sufficient to show that

$$P(\tilde{B}_s \le x, \tilde{B}_t \le y) = P(B_s \le x, B_t \le y)$$

for every $0 \le s \le t$ and $x, y \in \mathbb{R}$ (i.e. that $(\tilde{B}_t, \tilde{B}_s)$ and (B_t, B_s) have the same distribution). Let $\mathcal{F}_t = \sigma(B_s : 0 \le s \le t)$. By the strong Markov property, we know that $W_t \doteq B_{t+T_a} - B_{T_a}$ is a B.m. independent by $\mathcal{F}_{T_a}$. Notice that

$$\tilde{W}_t \doteq \tilde{B}_{t+T_a} - \tilde{B}_{T_a} = 2a - B_{t+T_a} - B_{T_a} = B_{T_a} - B_{t+T_a} = -W_t,$$

so $\tilde{W}_t$ is a B.m. independent by $\mathcal{F}_{T_a}$ too. We can write

$$B_t = B_t \mathbb{1}_{\{t \le T_a\}} + (W_{t-T_a} + B_{T_a})\mathbb{1}_{\{t > T_a\}} \qquad \text{and} \qquad \tilde{B}_t = B_t \mathbb{1}_{\{t \le T_a\}} + (\tilde{W}_{t-T_a} + B_{T_a})\mathbb{1}_{\{t > T_a\}}.$$

Now

$$\begin{aligned}P(\tilde{B}_s \le x, \tilde{B}_t \le y) = {}& P(B_s \le x, B_t \le y, t \le T_a) + P(B_s \le x, \tilde{W}_{t-T_a} + B_{T_a} \le y, s \le T_a < t) \\ & + P(\tilde{W}_{s-T_a} + B_{T_a} \le x, \tilde{W}_{t-T_a} + B_{T_a} \le y, T_a < s).\end{aligned} \tag{1}$$

We can evaluate the second term in the rhs as follows:

$$P(B_s \le x, \tilde{W}_{t-T_a} + B_{T_a} \le y, s \le T_a < t) = \mathbb{E}[P(B_s \le x, \tilde{W}_{t-T_a} + B_{T_a} \le y \,|\, \mathcal{F}_{T_a})\mathbb{1}_{\{s \le T_a < t\}}],$$

where

$$P(B_s \le x, \tilde{W}_{t-T_a} + B_{T_a} \le y \,|\, \mathcal{F}_{T_a}) = \tilde{f}(B_s, T_a)$$

with

$$\tilde{f}(z,r) \doteq \mathbb{1}_{\{z\le x\}} P(\tilde{W}_{t-r} + a \le y) = \mathbb{1}_{\{z\le x\}} P(W_{t-r} + a \le y) \doteq f(z,r).$$

Hence

$$\begin{aligned} P(B_s \le x, \tilde{W}_{t-T_a} + B_{T_a} \le y, s \le T_a < t) &= \mathbb{E}[f(B_s, T_a)\mathbb{1}_{\{s\le T_a < t\}}] \\ &= P(B_s \le x, W_{t-T_a} + B_{T_a} \le y, s \le T_a < t). \end{aligned} \tag{2}$$

Similarly one can prove that

$$P(\tilde{W}_{s-T_a} + B_{T_a} \le x, \tilde{W}_{t-T_a} + B_{T_a} \le y, T_a < s) = P(W_{s-T_a} + B_{T_a} \le x, W_{t-T_a} + B_{T_a} \le y, T_a < s). \tag{3}$$

From (1), (2) and (3) one gets that

$$\begin{aligned} P(\tilde{B}_s \le x, \tilde{B}_t \le y) = {} & P(B_s \le x, B_t \le y, t \le T_a) + P(B_s \le x, W_{t-T_a} + B_{T_a} \le y, s \le T_a < t) \\ & + P(W_{s-T_a} + B_{T_a} \le x, W_{t-T_a} + B_{T_a} \le y, T_a < s) \\ = {} & P(B_s \le x, B_t \le y). \end{aligned}$$

This completes the proof that $\tilde{B}_t$ is a B.m.

(b) It holds that

$$P(M_t > a) = P(M_t > a, B_t > a) + P(M_t > a, B_t \le a) = P(B_t > a) + P(M_t > a, B_t \le a),$$

since $B_t > a$ implies $M_t = \sup_{s\le t} B_s \ge B_t > a$.

(c) Since $\{M_t > a\} = \{T_a \le t\}$ and if $T_a \le t$ then $\tilde{B}_t = 2a - B_t$, then

$$P(M_t > a, B_t \le a) = P(T_a \le t, 2a - B_t \ge a) = P(T_a \le t, \tilde{B}_t \ge a) = P(\tilde{B}_t \ge a),$$

since $\{T_a > t, \tilde{B}_t \ge a\} = \{T_a > t, B_t \ge a\} = \emptyset$.

(d) Thanks to part (a), (b) and (c) we have that

$$P(M_t > a) = P(B_t > a) + P(\tilde{B}_t \ge a) = 2\int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, dx = \int_a^\infty \sqrt{\frac{2}{\pi t}} e^{-x^2/2t}\, dx.$$

Exercise 3

Let

$$Y_t^n = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} + \frac{1}{\sqrt{n}}(nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}.$$

By Donkser's theorem it holds that $Y^n \Rightarrow B$ as $n \to \infty$, where $Y^n = (Y_t^n)_{t\ge 0}$ and $B = (B_t)_{t\ge 0}$ is a B.m. In particular this implies that

$$\sup_{t\in[0,1]} Y_t^n \quad \Rightarrow \quad M_1 = \sup_{t\in[0,1]} B_t \tag{4}$$

as $n \to \infty$. We notice that

$$\sup_{t\in[0,1]} Y_t^n = \frac{1}{\sqrt{n}} \sup_{t\in[0,1]} (S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1})$$
$$= \frac{1}{\sqrt{n}} \sup\{S_i + tX_{i+1}, i = 0, \dots, n-1, t \in [0,1], S_n\} = \frac{1}{\sqrt{n}} \sup_{i=0,\dots,n} S_n = \frac{1}{\sqrt{n}} G_n.$$

Therefore, (4) tells us that (heuristically), for n large, G_n is distributed as $\sqrt{n}M_1$. Noticing that exercise 2 tells us that $\sqrt{n}M_1 \sim \sqrt{n}|B_1|$, we can (heuristically) say that

$$\mathbb{E}[G_n] \simeq \mathbb{E}[\sqrt{n}|B_1|] = \sqrt{\frac{2}{\pi n}} \int_0^\infty x e^{-x^2/2n}\, dx = \sqrt{\frac{2n}{\pi}}$$

and

$$\mathrm{Var}(G_n) = \mathbb{E}[G_n^2] - \mathbb{E}[G_n]^2 \simeq n\,\mathbb{E}[B_1^2] - \frac{2n}{\pi} = n\left(1 - \frac{2}{\pi}\right)$$

for n large.

Exercise 4

(a) We showed in class that

$$Q_t^\sigma(W_t) \xrightarrow{L^2} t \qquad \text{as } |\sigma| \to 0. \tag{5}$$

Hence it is sufficient to notice that

$$Q_t^\sigma(aW_t + bt) = \sum_{i=0}^{n-1} |aW_{t_{i+1}} + bt_{i+1} - aW_{t_i} - bt_i|^2 = \sum_{i=0}^{n-1} |a(W_{t_{i+1}} - W_{t_i}) + b(t_{i+1} - t_i)|^2$$
$$= a^2 Q_t^\sigma(W_t) + ab \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) + b^2 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$

Now

$$\mathbb{E}\left[\left(\sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i)\right)^2\right] = \sum_{i=0}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2](t_{i+1} - t_i)^2 = \sum_{i=0}^{n-1} (t_{i+1} - t_i)^4 \le |\sigma|^3 t,$$

where the first equality holds since $\mathbb{E}[(W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})] = 0$ for every $i \neq j$, and so

$$ab \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0. \tag{6}$$

Combining (5) and (6) and the fact that $\sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 \le |\sigma| t \xrightarrow{L^2} 0$ as $|\sigma| \to 0$, we get that

$$Q_t^\sigma(aW_t + bt) \xrightarrow{L^2} t \qquad \text{as } |\sigma| \to 0.$$

(b) Let's consider

$$Q_{t,4}^\sigma(W_t) = \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^4.$$

We have

$$\mathbb{E}[(Q^{\sigma}_{t,4}(W_t))^2] = \sum_{i=0}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^8] + \sum_{\substack{i,j=0 \\ i\neq j}}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^4 (W_{t_{j+1}} - W_{t_j})^4]$$

Now

$$\sum_{i=0}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^8] = 105 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^4 \leq 105|\sigma|^3 t \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0 \tag{7}$$

and

$$\begin{aligned}
\sum_{\substack{i,j=0 \\ i\neq j}}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^4 (W_{t_{j+1}} - W_{t_j})^4] &= \sum_{\substack{i,j=0 \\ i\neq j}}^{n-1} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^4]\, \mathbb{E}[(W_{t_{j+1}} - W_{t_j})^4] \\
&= 225 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 \sum_{\substack{j=0 \\ j\neq i}}^{n-1} (t_{j+1} - t_j)^2 \\
&\leq 225 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 |\sigma| t \\
&\leq 225 |\sigma|^2 t^2 \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0.
\end{aligned} \tag{8}$$

Hence (7) and (8) imply that

$$Q^{\sigma}_{t,4}(W_t) \xrightarrow{L^2} 0 \qquad \text{as } |\sigma| \to 0.$$

Exercise 5

Let us denote by X^x_t the solution to $\dot{X}_t = a(X_t)$, $X_0 = x$. If $f \in C^2_b(\mathbb{R})$, then

$$(\mathcal{L}f)(x) = \lim_{t\to 0^+} \frac{f(X^x_t) - f(x)}{t} = \frac{d}{dt} f(X^x_t)\Big|_{t=0} = f'(X^x_0) \frac{d}{dt} X^x_t\Big|_{t=0} = f'(x) a(X^x_0) = a(x) f'(x),$$

i.e. $\mathcal{L} = a\, \partial_x$.

Exercise 6

Given $f \in C^2_b$, we have, for $x \in K$, where $K \subset \mathbb{R}$ is a compact set such that $\operatorname{supp}\{p(\cdot, t|\cdot, 0)\} \subseteq K^2$,

$$(\mathcal{L}f)(x) = \lim_{t\to 0^+} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t} = \lim_{t\to 0^+} \frac{1}{t} \int_K (f(y) - f(x))\, p(y, t|x, 0)\, dy. \tag{9}$$

If $\varepsilon > 0$ we can split the integral in (9) as

$$\begin{aligned}
t^{-1} \int_K (f(y) - f(x))\, p(y, t|x, 0)\, dy = t^{-1} &\int_{y\in K, |x-y|\geq\varepsilon} (f(y) - f(x))\, p(y, t|x, 0)\, dy \\
&+ t^{-1} f'(x) \int_{y\in K, |x-y|<\varepsilon} (y - x)\, p(y, t|x, 0)\, dy \\
&+ t^{-1}/2 \int_{y\in K, |x-y|<\varepsilon} f''(\xi_y)(y - x)^2\, p(y, t|x, 0)\, dy,
\end{aligned}$$

where ξ_y is a point between x and y. Now,

$$t^{-1}\int_{y\in K,|x-y|\geq\varepsilon}(f(y)-f(x))\,p(y,t|x,0)\,dy \quad \xrightarrow[t\to 0^+]{} \int_{y\in K,|x-y|\geq\varepsilon}(f(y)-f(x))\,W(y|x)\,dy+O(\varepsilon),$$

$$t^{-1}f'(x)\int_{y\in K,|x-y|<\varepsilon}(y-x)\,p(y,t|x,0)\,dy \quad \xrightarrow[t\to 0^+]{} f'(x)O(\varepsilon),$$

$$t^{-1}/2\int_{y\in K,|x-y|<\varepsilon}f''(\xi_y)(y-x)^2\,p(y,t|x,0)\,dy\leq$$

$$\leq \|f''\|_\infty t^{-1}/2\int_{y\in K,|x-y|<\varepsilon}(y-x)^2\,p(y,t|x,0)\,dy \quad \xrightarrow[t\to 0^+]{} \|f''\|_\infty O(\varepsilon)/2.$$

hence, if we send $\varepsilon\to 0^+$, we get

$$(\mathcal{L}f)(x)=\int_K(f(y)-f(x))\,W(y|x)\,dy=\int_{\mathbb{R}}f(y)\,W(y|x)\,dy-f(x)\lambda(x).$$