On the dimension of Koch-style fractals

Luca Vercelli

2011-2014

Notation. Let O, O' be geometrical objects (i.e. subsets of $\mathbf{R}^n$ for some $n \in \mathbf{N}$). Let $\lambda \in \mathbf{R}^+$. We shall write $O + O'$ for their union, and λO for the object obtained from O with a dilatation of λ.

Dimension. Informally speaking, the dimension d of a geometrical object O and the d-measure μ_d are respectively a number and a function such that:

- $\mu_d(O) \in \mathbf{R}^+ \cup \{0, \infty\}$.
- $\forall e < d \, (\mu_e(O) = \infty)$ and $\forall e > d \, (\mu_e(O) = 0)$.
- $\forall \lambda \in \mathbf{R}^+ \left(\mu_d(\lambda O) = \lambda^d \mu_d(O)\right)$.
- $\forall O, O' \, (\mu_d(O + O') = \mu_d(O) + \mu_d(O'))$.

Observation 1 *The two latter properties hold, in fact, for every dimension d, non necessarily the same dimension of the objects O, O'.*

Observation 2 *Let $O = O_1 + \ldots + O_n$. Let moreover $O_i = \lambda_i O$. Then*

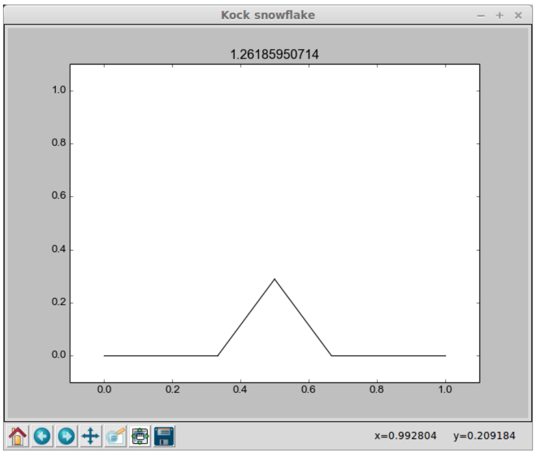
$$\mu_d(O) = \sum_{i=1}^{n} \lambda_i^d \mu_d(O). \tag{1}$$

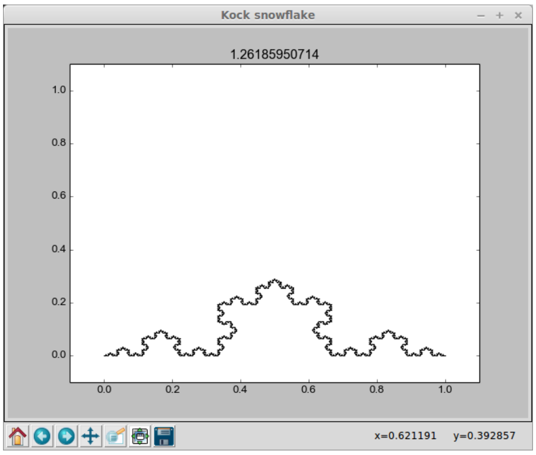
and

$$1 = \sum_{i=1}^{n} \lambda_i^d. \tag{2}$$

Such equation allows to calculate the fractal dimension of the object O, at least numerically.

The well-known **Koch snowflake** satisfies equation (2) with $n = 4$, and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{3}$.





$$
\begin{aligned}
\mu(O) &= 4\left(\frac{1}{3}\right)^d \mu(O) \\
3^d &= 4 \\
d &= \log_3 4
\end{aligned}
$$

More generally speaking,

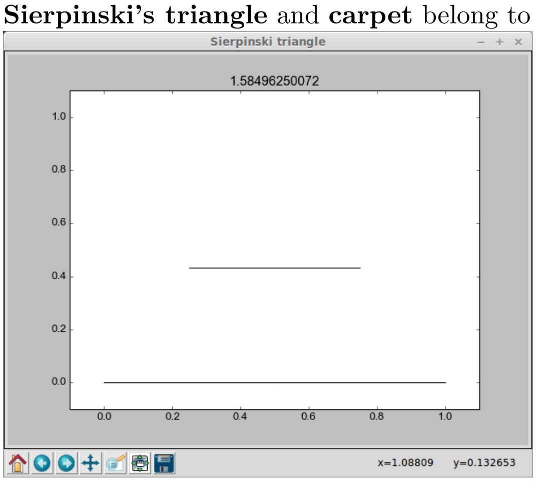
Observation 3 *If $O = O_1 + \ldots + O_n$ and $\forall i\,(O_i = \lambda O)$ (i.e. the large figure is made up of n equal smaller pieces), then equation (2) becomes*

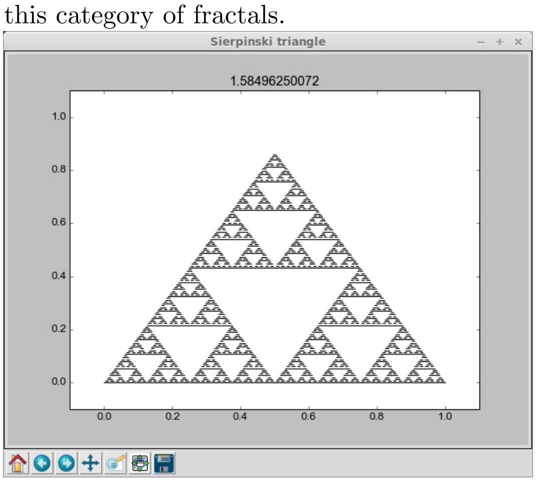
$$1 = n\lambda^d \tag{3}$$

which can be algebrically solved:

$$d = -\log_\lambda n. \tag{4}$$

Sierpinski's triangle and **carpet** belong to this category of fractals.





Now, let's assume that O is an object made up of pieces with different shapes. For example let

$$O = \sum_{j=1}^{n} O^{(j)} \tag{5}$$

and each piece be linear composition of the other ones (we can allow some $\lambda_i^{(j)}$ to be 0):

$$O^{(j)} = \sum_{i=1}^{n} \lambda_i^{(j)} O^{(i)} \qquad 1 \leq j \leq n. \tag{6}$$

So, if d is the fractal dimension of O,

$$\mu_d\left(O^{(j)}\right) = \sum_{i=1}^{n} \left(\lambda_i^{(j)}\right)^d \mu_d\left(O^{(i)}\right) \qquad 1 \leq j \leq n. \tag{7}$$

This is a system of n equations whose unknown are the n measures $\mu_d\left(O^{(j)}\right)$ and d. The system is linear in the first n unknowns and not linear in d. Considering d as a a parameter, the linear system is not determined because there are no constants. The determinant of the resolving $n \times n$-matrix is 0, and this give us a trascendental equation in d. We have just shown that:

Observation 4 *If a fractal O is a defined by equations (5) and (6), then its fractal dimension d can be determined by*

$$\det(A - I) = 0 \tag{8}$$

where

$$A = \left(\sum_i \left(\lambda_i^{(j)}\right)^d\right)_{ij}$$

is the matrix of coefficients of the linear system (7).