Politecnico di Milano

Financial Engineering

Assignment 2 - Group 1

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Contents

Interest rate derivatives are instruments whose payoffs are dependent in some way on the level of interest rates. Some examples are swaptions, caps and floors. The volume of IR rates derivatives in both the over-the-counter and exchange-traded-markets had a significant increase over the last decades. Pricing these products is challenging since the behaviour of a single IR is more complicated than that of a stock price. Moreover, it is necessary to develop a model describing the behaviour of the entire zero ratescurve. To do this one the most used techniques is the boostrap.

In order to develop the boostrap procedure with settlement date February 19, 2008, we have at our disposal the I.B. deposites rates, the STIR futures and the IR swaps.

1 Bootstrap for Euribor 3M Interbank curve

The objective is constructing the Euribor 3 months Interbank curve using the bootstrap technique. The settlement date (t_0) is February 15, 2008 at 10:45 C.E.T.

Provided I.B. deposits, STIR futures and IR swaps which overlaps in some intervals of time, we choose to consider the first seven swaps as reference. This is because these instruments are the most liquid ones in the market.

As a consequence, the bootstrap procedure divides the IR curve in three term buckets: the short end is derived using I.B. deposit rates, the middle end is constructed using the first seven futures, while the long end is derived from the swap market.

Up to the settlement date of the first future (March 19, 2008), we consider the depos: we can derive the discount factors using Formula 1

$$B(t_0, t_i) = \frac{1}{1 + \delta(t_0, t_i)L(t_0, t_i)} \tag{1}$$

where $\delta(t_0, t_i)$ is computed with day count $Act/360$ and $L(t_0, t_i)$ is the mid-market rate of the deposit with expiry t_i. In our case, we take into consideration the first three depos.

Let us now proceed with the middle term of the curve. For each future we compute the discount factor at expiry using Formula 2

$$B(t_0; t_0, t_{i+1}) = B(t_0; t_0, t_i)B(t_0; t_i, t_{i+1}) \tag{2}$$

where t_i is the settlement date of the future, t_{i+1} is its expiry,

$$B(t_0; t_i, t_{i+1}) = \frac{1}{1 + \delta(t_i, t_{i+1})L(t_0; t_i, t_{i+1})} \tag{3}$$

(considering that the day count for futures is $Act/360$) and $B(t_0; t_0, t_i)$ is the discount factor at settlement date. The latter is obtained by inspecting the curve up to that point:

- if at t_i the discount factor has already been computed, we use it directly
- if t_i is not among the dates we have already considered and it is before the last one of them, we use interpolation. In particular, we do linear interpolation of the zero rates, expressed as function of the discount factors in Formula 4.

$$y = -\frac{ln(B(t_0, t))}{\delta(t_0, t)} \tag{4}$$

  The day count used to compute the year fraction for zero rates is $Act/365$.

- if t_i is not among the dates we have already considered and it is after the last one of them, we use flat extrapolation of the zero rates. We usually use this technique only if t_i is at most 2 days after the last date.

Finally, we model the long end of the curve, that consists of the swaps with expiry after the seventh future's one. Hence, the aim is to compute the discount factor at swap expiries after the 2y swap one. Firstly, we need to find $B(t_0, t_1)$, where t_1 is the expiry of the first swap, not included in our bootstrap curve, but necessary in order to compute the following discount factors. We do this by interpolation, using the zero rates already computed.

Now, for the date t_i, having the set $B(t_0, t_n)_{n=1,\dots,i-1}$, we compute $B(t_0, t_i)$ with Formula 5

$$B(t_0, t_i) = \frac{1 - S_{IR}(t_0; t_i) \sum_{n=1}^{i-1} \delta(t_{n-1}, t_n) B(t_0, t_n)}{1 + \delta(t_{i-1}, t_i) S_{IR}(t_0; t_i)} \tag{5}$$

where the day convention for the fixed leg is 30/360 (European).

Once we finish this procedure, we obtain the curve of the interest rates, shown in Figure 1.

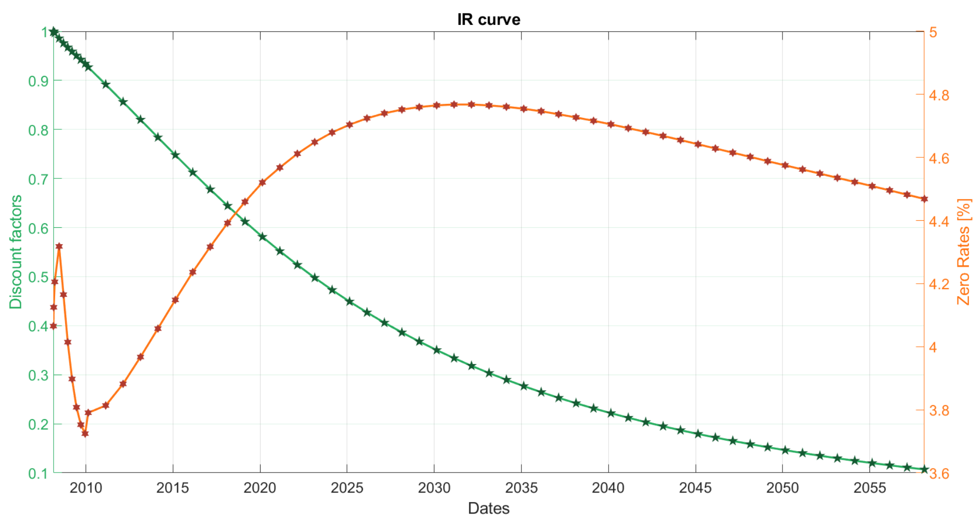

Figure 1: Zero rates curve and discount factors, obtained with bootstrap

As we can observe, the curve of the zero rates in the very short term has a rapid increase because of the need of liquidity, instead, later on, the rates are lower because - considering the historical period - a recession is expected. After that, the curve normalizes as it should.

2 The importance of bootstrap

Bootstrap is a simple method to get Discount Factors from quoted rates. Obtaining Discount Factors is useful in finance both for mark to market fixed-income securities - that is valuing a financial product that reflects current market conditions - and for hedging. In particular, marking to market is done by discounting the future cash-flows of a fixed-income financial product using Discount Factors derived from the bootstrap. Moreover, it is possible, starting from Discount Factors, to easily compute zero-rates, using Formula 4, and so to obtain the term structure of zero-rates. This curve is important since it reflects the future expectations on the market.

3 Sensitivities

Interest Rate Sensitivities are a set of measures that quantify the NPV variation of a portfolio composed by fixed income products, given a variation in the IR curve.

The portfolio in exam is composed by a 6y plain vanilla IR swap vs Euribor 3m with a fixed rate 2.8173% and a Notional of 10 Mln.

DV01 is a sensitivity measure that is computed as

$$DV01 = NPV^{shifted} - NPV \tag{6}$$

where $NPV^{shifted}$ is related to the portfolio when all rates used for the bootstrap are increased by 1bp, while NPV is related to the portfolio with the original curve.

DV01$^{(z)}$ is still computed with Formula 6, but in this case $NPV^{shifted}$ regards the portfolio when the zero rates of the original bootstrap curve are increased by 1bp.

BPV, instead, considers a payer swap. In Formula 6 $NPV^{shifted}$ is related to the swap where the fixed rate S is increased by 1bp, while $NPV^{shifted}$ is related to the swap with the original S. This, in practice, corresponds to: $BPV = 1bp \sum_{i=1}^{N} \delta(t_{i-1}, t_i) B(t_0, t_i)$.

Lastly, the Macaulay Duration is the variation in the value of an I.B. coupon bond for the parallel shift of the corresponding zero rate curve of 1bp, normalized by the bond price. It represents the weighted average number of years that the investor must maintain the position in the bond until the present value of the bond's cash flows equals the amount paid for the bond. We are considering a I.B. coupon bond with same expiry, fixed rate and reset dates of the IRS, and face value equal to IRS Notional.

The results, in absolute value, are in Table 1.

DV01	**DV01$^{(z)}$**	**BPV**	**Duration**
5014.1720	5204.9717	5239.7880	5.5852

Table 1: Sensitivieties mesasures. DV01, DV01$^{(z)}$ and BPV are expressed in €, instead the duration is expressed in years

In general, if a swap is traded at par then its DV01 coincides with its BPV. This gives a ”rule of thumb” to compute the DV01 in a more quick way, avoiding doing an extra bootstrap. In our case, we cannot use this shortcut because our swap is not at par ($|NPV| = 0.0686$), but we can still notice that the two values are quite similar. A similar reasoning holds between the Macaulay duration of an I.B. Coupon Bond at par and the DV01$^{(z)}$ of the corresponding par swap.

4 Price of a I.B. coupon bond

The goal is to price a 6y an I.B. coupon bond issued on February 15, 2008 with coupon rate c equal to 4.212%. We assume a 30/360 European day count. The settlement date t_0 is thus February 19,2008. Its price is in Formula 7.

$$P = \sum_{i=1}^{6} c\delta(t_{i-1}, t_i)B(t0, t_i) + B(t_0, t_6) \tag{7}$$

Since c is constant, Formula 7 becomes:

$$P = c\sum_{i=1}^{6} \delta(t_{i-1}, t_i)B(t0, t_i) + B(t_0, t_6) \tag{8}$$

Now we can substitute $\sum_{i=1}^{6} \delta(t_{i-1}, t_i)B(t0, t_i)$ with the already computed BPV of a 6y IR swap, with settlement date t_0 (Table 1).

$$P = c \cdot BPV + B(t_0, t_6) \tag{9}$$

By substituting this and by taking $B(t_0, t_6)$ from the bootstrap curve, we obtain that $P = 1.0045$.

5 Garman–Kohlhagen formula for an European Call option

We consider an European Call option written on an underlying with interest rate $r(t)$, dividend yield $d(t)$ and volatility $\sigma(t)$ which are deterministic functions of time and time to maturity $t - t_0$. The dynamics of the underlying is a Geometric Brownian Motion:

$$\begin{cases} dS_t = S_t[(r(t) - d(t))dt + \sigma(t)dW_t] \\ S_{t_0} = S_0 \end{cases}$$

In order to solve this stochastic differential equation with initial condition, we compute

$$d(ln(S_t)) = \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2} < dS_t^2 >= \Big[(r(t) - d(t))dt + \sigma(t)dW_t\Big] - \frac{1}{2}\sigma^2(t)dt =$$

$$= \Big[r(t) - d(t) - \frac{1}{2}\sigma^2(t)\Big]dt + \sigma(t)dW_t$$

Now we integrate between t_0 and t:

$$\int_{t_0}^{t} d\ln(S_u) = \int_{t_0}^{t}\Big[r(u) - d(u) - \frac{1}{2}\sigma^2(u)\Big]du + \int_{t_0}^{t}\sigma(u)dW_u \tag{10}$$

Defining the average values over the time-to-maturity of $d(t), r(t), \sigma^2(t)$ as

$$\overline{r(t)} = \frac{1}{t - t_0}\int_{t_0}^{t} r(u)du, \qquad \overline{d(t)} = \frac{1}{t - t_0}\int_{t_0}^{t} d(u)du, \qquad \overline{\sigma^2(t)} = \frac{1}{t - t_0}\int_{t_0}^{t}\sigma^2(u)du$$

we obtain from (10):

$$\ln(S_t) - \ln(S_0) = \Big[\overline{r(t)} - \overline{d(t)} - \frac{1}{2}\overline{\sigma^2(t)}\Big](t - t_0) + \int_{t_0}^{t}\sigma(u)dW_u$$

From this equation we can compute the expression of S_t:

$$S_t = S_0\exp\Big[(\overline{r(t)} - \overline{d(t)} - \frac{1}{2}\overline{\sigma^2(t)})(t - t_0) + \int_{t_0}^{t}\sigma(u)dW_u\Big] \tag{11}$$

Since the stochastic integral has a deterministic integrand function, it is a centred Gaussian process. To compute its variance we can use Itö's isometry and the fact that the mean of the stochastic integral is equal to zero, since it is centred:

$$\text{Var}\Big(\int_{t_0}^{t}\sigma(u)dW_u\Big) = \mathbb{E}\Big[(\int_{t_0}^{t}\sigma(u)dW_u)^2\Big] = \mathbb{E}\Big[\int_{t_0}^{t}\sigma^2(u)du\Big] = (t - t_0)\overline{\sigma^2(t)}$$

$$\Longrightarrow \int_{t_0}^{t}\sigma(u)dW_u \sim N(0, (t - t_0)\overline{\sigma^2(t)})$$

If we combine this result with (11) we get:

$$S_t \overset{\text{"law"}}{\sim} S_0 \exp\Big[(\overline{r(t)} - \overline{d(t)} - \frac{1}{2}\overline{\sigma^2(t)})(t-t_0) \pm g\sqrt{\overline{\sigma^2(t)}(t-t_0)}\ \Big]$$

with $g \sim N(0,1)$.
Thanks to this result we can price the European Call option:

$$C(t_0,t) = B(t_0,t)\mathbb{E}_0[(S_t - K)^+] =$$

$$= B(t_0,t)(\underbrace{\mathbb{E}_0[S_t 1_{S_t \geqslant k}]}_{(A)} - K\underbrace{\mathbb{E}_0[1_{S_t \geqslant k}]}_{(B)})$$

First we compute (B):

$$S_t \geqslant k \iff S_0 \exp\Big[(t-t_0)\Big(\overline{r(t)} - \overline{d(t)} - \frac{\overline{\sigma^2(t)}}{2}\Big) - \ g\sqrt{\overline{\sigma^2(t)}(t-t_0)}\ \Big] \geqslant K$$

$$\iff (t-t_0)\Big(\overline{r(t)} - \overline{d(t)} - \frac{\overline{\sigma^2(t)}}{2}\Big) - g\sqrt{\overline{\sigma^2(t)}(t-t_0)}\ \geqslant \ln\Big(\frac{K}{S_0}\Big)$$

$$\iff g \leqslant \frac{\ln\Big(\frac{S_0}{K}\Big) + \Big(\overline{r(t)} - \overline{d(t)} - \frac{\overline{\sigma^2(t)}}{2}\Big)(t-t_0)}{\sqrt{\overline{\sigma^2(t)}(t-t_0)}} := d_2$$

so

$$(B) = \int_{-\infty}^{+\infty} \frac{e^{-g^2/2}}{\sqrt{2\pi}} 1_{g \leqslant d_2} dg = \mathcal{N}(d_2)$$

where $\mathcal{N}$ is the cumulative density function of a standard gaussian random variable.
Now we compute (A):

$$(A) = S_0\mathbb{E}_0\Big[\exp\Big[(t-t_0)\Big(\overline{r(t)} - \overline{d(t)} - \frac{\overline{\sigma^2(t)}}{2}\Big) - g\sqrt{\overline{\sigma^2(t)}(t-t_0)}\ \Big]1_{g \leqslant d_2}\Big] =$$

$$S_0\mathbb{E}_0\Big[\exp\Big[\frac{-g^2}{2} + (t-t_0)\Big(\overline{r(t)} - \overline{d(t)} - \frac{\overline{\sigma^2(t)}}{2}\Big) - g\sqrt{\overline{\sigma^2(t)}(t-t_0)}\ \Big]$$

$$= S_0 e^{(\overline{r(t)} - \overline{d(t)})(t-t_0)} \int_{-\infty}^{d_2} \frac{\exp\Big[\frac{-g^2}{2} - \frac{\overline{\sigma^2(t)}}{2}(t-t_0) - g\sqrt{\overline{\sigma^2(t)}(t-t_0)}\ \Big]}{\sqrt{2\pi}} dg =$$

$$= S_0 e^{(\overline{r(t)} - \overline{d(t)})(t-t_0)} \int_{-\infty}^{d_2} \frac{\exp\Big[-\frac{1}{2}(g^2 + \overline{\sigma^2(t)}(t-t_0) + 2g\sqrt{\overline{\sigma^2(t)}(t-t_0)}\ )\Big]}{\sqrt{2\pi}} dg =$$

If we consider $g' = g + \sqrt{\overline{\sigma^2(t)}(t-t_0)}$, we can write (A) as:

$$S_0 e^{(\overline{r(t)} - \overline{d(t)})(t-t_0)} \int_{-\infty}^{d_2 + \sqrt{\overline{\sigma^2(t)}(t-t_0)}} \frac{e^{-g'/2}}{\sqrt{2\pi}} dg' = S_0 e^{(\overline{r(t)} - \overline{d(t)})(t-t_0)} \mathcal{N}(d_1)$$

where $d_1 = d_2 + \sqrt{\overline{\sigma^2(t)}(t-t_0)}$
We can finally compute the price of the European Call option as

$$C(t_0,t) = B(t_0,t)\{S_0 e^{(\overline{r(t)}-\overline{d(t)})(t-t_0)}\mathcal{N}(d_1) - K\mathcal{N}(d_2)\}$$

$$d_{1,2} = \frac{1}{\sqrt{\overline{\sigma^2(t)}(t-t_0)}}[\ln(\frac{S_0}{K}) + (\overline{r(t)}-\overline{d(t)})(t-t_0) \pm \frac{1}{2}\sqrt{\overline{\sigma^2(t)}(t-t_0)}]$$

In this way, we prove that the Garman-Kohlhagen formula for a European Call Option holds for an underlying with interest rates, continuous dividends and volatilities deterministic functions of time. Furthermore, we show that the constant values in the standard formula can be substituted with their average value over the time-to-maturity.