Politecnico di Milano

Financial Engineering

Assignment 5 - Group 1

Authors:
Luca Amadori
Nicola Baldoni
Greta Belgeri

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1 Certificate Pricing

We are asked to compute the participation coefficient α of a certificate, in such way that its Net Present Value is zero. In particular, since α is involved only in the swap part, we analyze this one. We neglect the counterparty risk between the two parties.

In this instrument, the certificate's issuer pays to the investment bank the 3 months Euribor and the spread over Libor every three months, plus a fraction of the notional, given by one minus the protection coefficient at maturity.
On the other hand, the certificate's issuer receives the upfront at start date and the coupon at maturity.

In this case study we consider a portfolio composed by Enel and AXA stocks, with the same weights, and a coupon given by the product between α and the positive part of the difference between the mean of the portfolio returns in 4 year and the protection coefficient:

$$\alpha\left(\frac{1}{4}\sum_{s=1}^{4}\sum_{n=1}^{2} W_n \frac{E_s^n}{E_{s-1}^n} - P\right)^+ \tag{1}$$

We compute the coupon exploiting the Monte Carlo method: since the values of the stocks are distributed as correlated Geometric Brownian motions, we generate a set of standard Gaussian random variables and we apply an affine transformation to choose their variance and covariance, so that they would represent the Brownian motion term in the exponential of the Geometric Brownian motion. Then we compute the drift term and we perform the computation of the exponential term of the GBM. Actually, it is enough to compute the value of the coupon, since:

$$\frac{E_s^n}{E_{s-1}^n} = \frac{E_{s-1}^n \cdot e^{\mu+B}}{E_{s-1}^n} = e^{\mu+B}$$

where B is the Brownian motion, $\mu = (r - d - \frac{\sigma^2}{2})\delta(s-1, s)$, r is the forward zero rate we obtain by the forward discounts, d is the dividend yield, σ^2 is the variance and $\delta(s-1, s)$ is the year fraction between time $s-1$ and time s in Act/365.
In all the computations of the coupon we use a 3D matrix with dimensions 2x4xN_{sim}, where N_{sim} is the number of Monte Carlo simulations. We decide to use this data structure in order to have, for every Monte Carlo simulation, a 2x4 matrix whose elements are the values of the two stocks at any monitoring date.

Discounting all payments we get the Net Present Value and, imposing it equal to zero, we obtain the participation coefficient.
Fixing the random seed at 42, we obtain the results of Table 1. The confidence interval at 95% is computed as:

$$[\bar{\alpha} + t_q SE; \bar{\alpha} - t_q SE]$$

where t_q is the t percent point value with N - 1 degrees of freedom (N is the number of simulation) and SE is the standard error $std(\alpha)/\sqrt{N}$.

N simulations	Method	α	IC	L_{IC}
10^5	MC	3.3496	[3.3492; 3.3500]	$8.1287 \cdot 10^{-4}$
	MC AV	3.3538	[3.3536; 3.3539]	$3.3879 \cdot 10^{-4}$
10^6	MC	3.3512	[3.3511; 3.3513]	$2.5725 \cdot 10^{-4}$
	MC AV	3.3516	[3.3516; 3.3517]	$1.0764 \cdot 10^{-4}$
10^7	MC	3.3510	[3.3510; 3.3511]	$8.1377 \cdot 10^{-5}$
	MC AV	3.3506	[3.3506; 3.3506]	$3.4113 \cdot 10^{-5}$

Table 1: Values of α

In order to have more precise results we also implement the antithetic variables technique, that gives us almost the same values of α, but with a tighter confidence interval, as we can see in the last column of Table 1.
To have a balance between precision and elapsed time we choose $N_{sim} = 10^6$ and AV technique, which takes about 10 times less than the $N_{sim} = 10^7$ case.

2 Pricing Digital option

The importance of the Black '76 formula is mainly due to the possibility of synthesizing option prices through a unique parameter, the implied volatility. Indeed, the implied volatility is the main tool to price plain vanillas: σ can be easily converted into the price at which the related option can be exchanged. However, this technique cannot be used directly to price exotic options.

In this section we focus on digital options. The objective is to verify the difference between the price of a digital option computed according to the Black model and in the case where one takes into account the smile in the curve of the implied volatility.
We are given the S&P500 volatility surface (varying the strikes), which is shown in Figure 1. The parameters of the ATM-spot contract are the following: 1 year expiry $(t - t_0)$, initial price $S_0 = 2973.8740$ (from which we can derive $F_0 = 2971.8273$ through the Garman & Kohlhagen) and consequently strike $K = 2973.8740$.

Firstly, we compute the price of the digital using the Black formula:

$$dc_B(K) = B(t_0, t)N(d_2) = -\frac{\partial c_B(K)}{\partial K} \tag{2}$$

where

$$\begin{aligned} d_1 &= \frac{\log(F_0/K)}{\sigma\sqrt{(t-t_0)}} + \frac{1}{2}\sigma\sqrt{(t-t_0)}, \\ d_2 &= d_1 - \sigma\sqrt{t-t_0} \end{aligned} \tag{3}$$

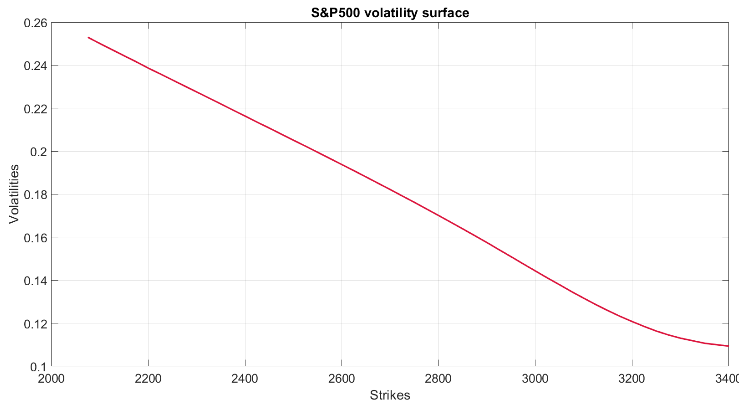

Figure 1: Volatility surface as a function of the strikes

and $c_B(K)$ is the price of a call option using the Black model. The volatility σ is taken from the surface by interpolation. We use spline interpolation since the graph has a positive concavity for higher values of the strike: although linear interpolation may work fine for the first part of the curve (since we can see from Figure 1 that the decreasing trend is linear), it could causes some discrepancies towards the end.
By applying Formula 2 and adding the notional term, we obtain a Black price of €$2.2527 \cdot 10^5$.

Secondly, we consider the digital option in an implied volatility approach. By approximating the digital as a bull spread (call) and taking the limit, we have that the price can be obtained as

$$dc_B(K) = -\frac{\partial c_B(K, \sigma(K))}{\partial K} - \frac{\partial \sigma(K)}{\partial K}\frac{\partial c_B(K, \sigma)}{\partial \sigma} \tag{4}$$

We can notice that the first addend of Formula 4 corresponds to the price of the digital using the Black formula, thanks to the second equality of Formula 2. For this reason it is called the "Black term". Thus, the second addend of Formula 4 represents the difference between the prices with the two approaches.

In particular, its first term, the slope impact of the surface, can be computed by approximating the derivative in this way:

$$\frac{\partial \sigma(K)}{\partial K} = \frac{\sigma_2 - \sigma_1}{K_2 - K_1}$$

where K_1, K_2 are closest points to K in the set of the strikes and σ_1, σ_2 are the corresponding volatilities.
The second term, instead, is the Vega obtained through the Black formula:

$$\upsilon = B(t_0, t) F_0 \sqrt{t - t_0} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}$$

Finally, we obtain a difference of $\frac{\partial \sigma(K)}{\partial K}\upsilon = 7.5883 \cdot 10^4$ and the price with the implied volatility approach is €$3.0116 \cdot 10^5$.

In conclusion, we can deduce that the Black model is not sufficient because it cannot detect the additional term due to the so called digital risk. For this reason, it is necessary to introduce a new model that tries to overcome this problem. The objective is, given the liquid market of plain vanillas, being able of pricing exotics, where there is a relevant impact of the smile.

3 Pricing with Normal Mean Variance Mixture model

One possible model that can be chosen to solve the issue mentioned above is the Normal Mean Variance Mixture. Having $F_t = F_0 e^{f_t}$, the dynamics of the forward returns is:

$$f_t = \sqrt{t - t_0}\sigma\sqrt{G}g - \left(\frac{1}{2} + \eta\right)(t - t_0)\sigma^2 G - \ln \mathcal{L}[\eta] \tag{5}$$

where g is a standard normal r.v. and G is a positive r.v. with unitary mean and $\kappa/(t-t_0)$ variance.

Once we have the model, Lewis formula (2001) permits to find the price of a European call option as a function of the logmoneyness x:

$$\frac{c(x)}{B(t_0,t)F_0} = 1 - e^{-x/2}\int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-i\xi x}\phi\left(-\xi - \frac{i}{2}\right)\frac{1}{\xi^2 + \frac{1}{4}}d\xi \tag{6}$$

where ϕ is the characteristic function, that in the NMVM case is:

$$\phi(\xi) = E[e^{i\xi f_t}] = e^{-i\xi ln\mathcal{L}[\eta]}\mathcal{L}\left[\frac{\xi^2 + i(1+2\eta)\xi}{2}\right] \tag{7}$$

We pick G as a tempered α-stable positive random variable. In this way, the Laplace transform $\mathcal{L}$, which depends on α, is:

$$ln\mathcal{L}[\omega] = \frac{\Delta t}{\kappa}\frac{1-\alpha}{\alpha}\left[1 - \left(1 + \frac{\omega\kappa\sigma^2}{1-\alpha}\right)^2\right] \tag{8}$$

where $\alpha \in (0, 1]$. It can be verified that the assumption of the Lewis formula ($\phi(\xi)$ analytical in the horizontal strip $\{\xi \in \mathbb{C} : -1 \leq Im(\xi) \leq 0\}$) is verified with the chosen process.

In the end, we have 3 parameters: σ (that represents the average volatility), η (the skew) and κ (the volatility of the volatility). Hence, with this model, we reach the parsimony goal: we want few parameters which have a financial interpretation.

In our case: $\alpha = 1/2$ (NIG process), $\sigma = 20\%$, $\kappa = 1$ and $\eta = 3$.

Firstly, in order to apply the Lewis formula (6), the Laplace transform must be well defined. To verify this, we check that

$$\eta \geq -\frac{1-\alpha}{\kappa\sigma^2} \tag{9}$$

In order to compute the call price we have 3 methods: applying the Lewis formula computing the integral with FFT (3.1), computing the same integral with a quadrature method (3.2) and using Monte Carlo to simulate the process (3.3). Let us see in details the different techniques.

3.1 Fast Fourier Transform

We notice that the integral in Formula 6 is the Fourier transform $\hat{f}(x)$ of the function

$$f(\xi) = \frac{1}{2\pi}\phi\left(-\xi - \frac{i}{2}\right)\frac{1}{\xi^2 + \frac{1}{4}}$$

$\hat{f}(x)$ can be computed numerically using the Fast Fourier Transform algorithm. To do this we create a grid by discretizing the logmoneyness (that from now on we will indicate as z) and the variable of integration (x). Then we can fix two possible couples of parameters, (M, x_1) or (M, dz), and define consequently the other 5 parameters (N, dx, z_1, z_N, x_N). We can now approximate $\hat{f}$ for all logmoneyness z_k at the same time:

$$\hat{f}(z_k) = dxe^{-ix_1 z_k} FFT(k)$$

where the input of FFT is $f_j = e^{-iz_1(j-1)dx} f(x_j)$.
The choice of the optimal parameters are discussed in subsection 3.4.

3.2 Quadrature

The integral can also be computed directly using a quadrature formula. We use the Matlab function `integral`, which exploits the global adaptive quadrature method. We set 10^{-16} as threshold of both the absolute and relative errors; the function satisfies the less strict constraint.

3.3 Monte Carlo

The third way consists in simulating the dynamics of the forward (Formula 5) and to apply the Monte Carlo approach. In order to do this we have to generate:

- g, a standard normal random variable
- G, an inverse Gaussian random variable (since $\alpha = 1/2$) with unitary mean and $\kappa/(t - t_0)$ variance. Its theoretical parameters are μ (the scale parameter) and λ (the shape parameter). In order to match the first and second theoretical moments, that are $\mathbb{E}[G] = \mu$ and $Var[G] = \mu^3/\lambda$, we impose $\mu = 1$ and $\lambda = (t - t_0)/\kappa$. We can see the convergence of the simulated moments to the theoretical ones in Figure 2. To guarantee the matching we choose to perform 10^5 simulations.

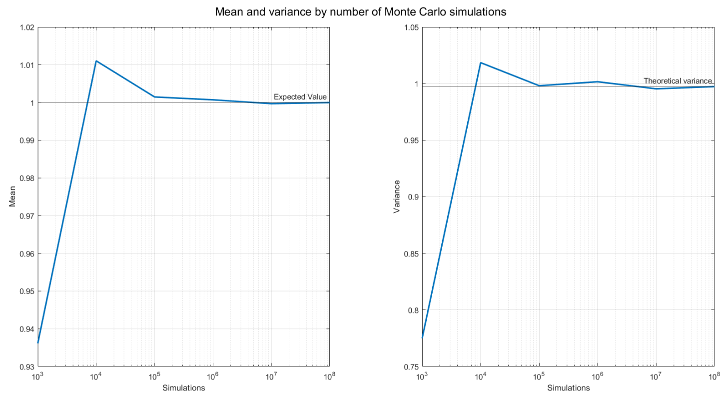

Figure 2: Mean and Variance of the simulated IG r.v. by number of Monte Carlo simulations

As before, this technique can be optimized by applying an antithetic variables approach: we induce a negative correlation between two sets of simulations by generating opposite standard normal random variables.
The plot of the prices as a function of the logmoneyness with the 95% confidence interval is in Figure 3

Figure 3: Prices varying the logmoneyness with MC (with confidence interval)

3.4 Comparison between the three methods

For the 3 techniques described before, the graphs of the prices varying the logmoneyness are shown in Figure 4. The time spent in average for the computation are: 0.11 seconds for FFT, 1.71 seconds for the quadrature, 38.10 seconds for Monte Carlo ($N_{sim} = 10^5$). The Fast Fourier Transform is the fastest approximation (its complexity is $O(N \log N)$) and for this reason it is the chosen one for the calibration in Section 4. In order to pick the optimal values for the parameters we fix the quadrature result (which is slower) as price benchmark and we look for the values of (M, x_1) or (M, dz) that minimizes the L^∞-norm of the distance between the two prices. We choose this metric

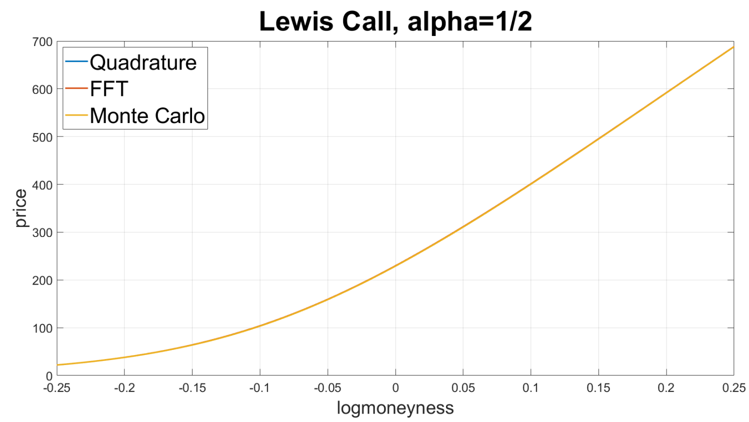

Figure 4: Prices computed

because it considers the worst case scenario. The trend of the error varying (M, x_1) is in Figure 5, while the the case of (M, dz) is in Figure 6.

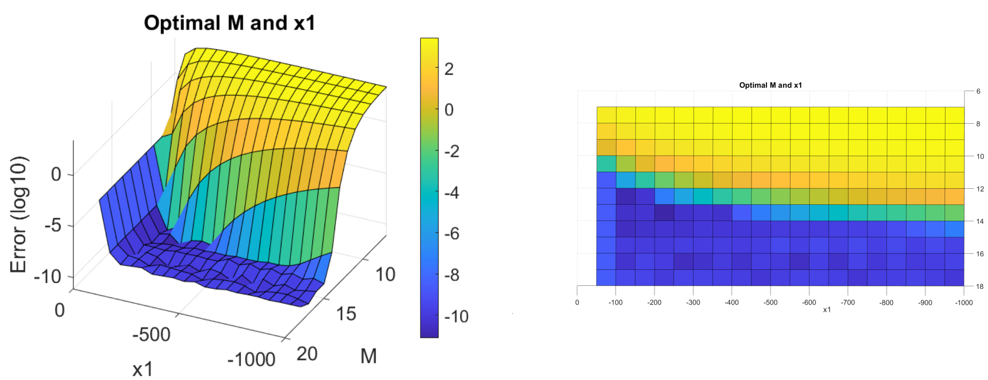

Figure 5: Price error varying (M, x_1)

We decide to take the parameters that reach the plateau of the error graph, after which the slope of the decreasing trend gets almost flat. The values chosen in this way are $(14, -400)$ in the first case and $(14, \frac{1}{64})$ in the second case.

3.5 Modifying α

If we change the distribution of G, choosing $\alpha = 2/3$, we do not notice significant changes in the precision of the results. With the choice of parameters discussed in subsection 3.4 the L^∞-norm of the difference between FFT and quadrature is $1.5928 \cdot 10^{-10}$ for $\alpha = 1/2$ and $8.1197 \cdot 10^{-11}$ for $\alpha = 2/3$.

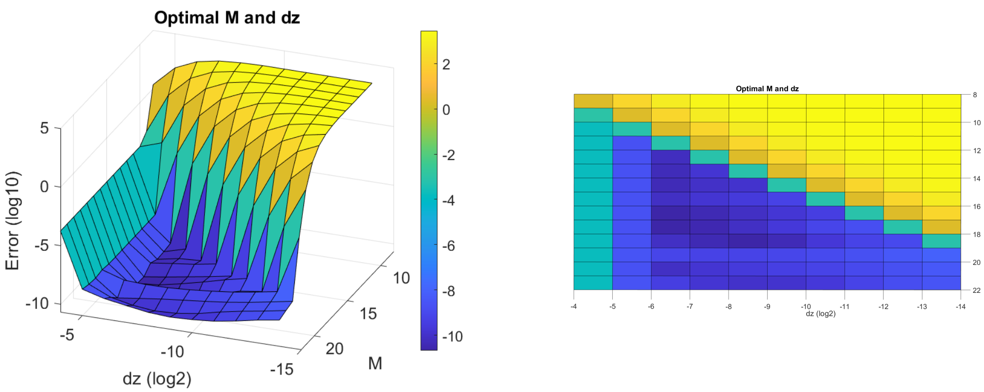

Figure 6: Price error varying (M, dz)

4 Volatility Surface calibration

In this section we perform a calibration of the NMVM model ($\alpha = 1/3$) on the S&P500 volatility surface. We remark that the options on this index are among the most liquid ones in the equity market at world level.

Having the implied volatility surface, we start by extracting the market prices using the Black model for call options

$$c_B(K) = B(t_0, t)\{F_0 N(d_1) - K N(d_2)\}$$

where d_1 and d_2 are in Formula 3.
On the other hand, model prices are computed using the Lewis formula (exploiting the FFT method with $M = 14$ and $x_1 = -400$) as a function of the three model parameters: σ, η, κ.

We implement the global calibration (with unitary weights), by minimizing the Euclidean distance between model and market prices (exploiting the Matlab function `lsqnonlin`). We also impose condition (9) that guarantees that the Laplace transform is well defined.

The results are in Table 2.

σ	η	κ	ERROR
0.1241	7.0588	1.6724	18.3844

Table 2: NMVM parameters

We can plot the implied volatility obtained with the model against the market's one (Figure 7). σ represents the average implied volatility, κ its convexity and η its asymmetry (the value is large, indeed we have a skewed surface).
We can also plot how the Euclidean distance varies with the strikes (Figure 8). We notice a sinusoidal behavior, that maybe can be due to the use of FFT.

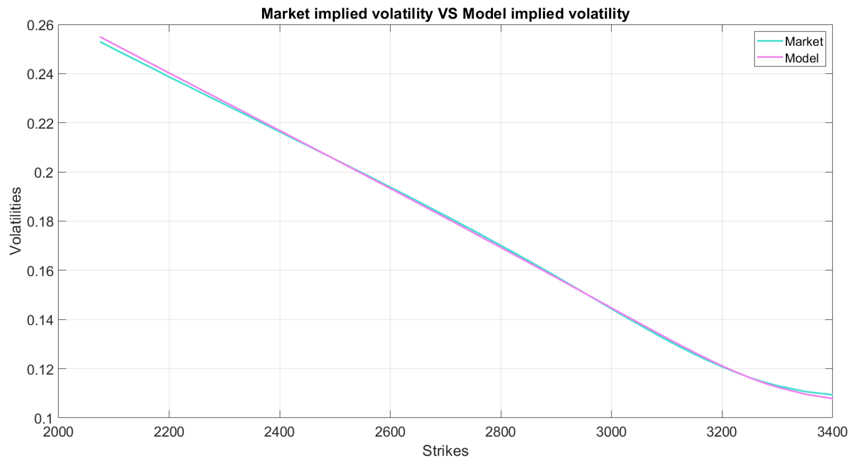

Figure 7: Comparison between market and model implied volatilities

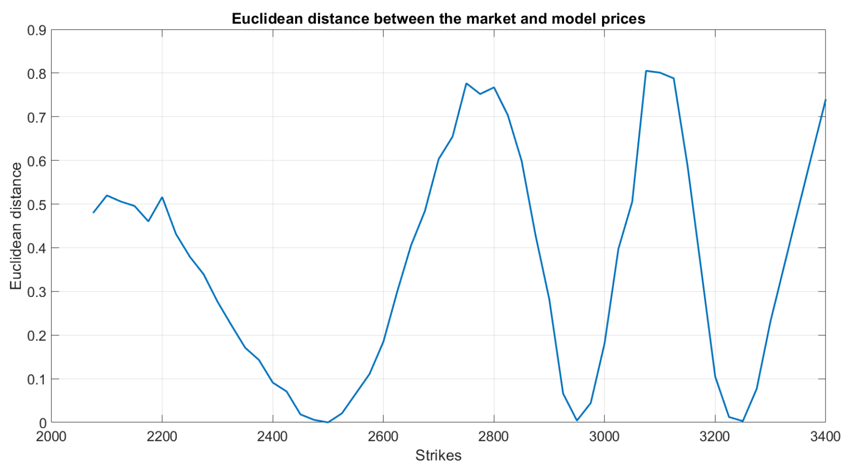

Figure 8: Euclidean distance between the market and model prices

In conclusion, with the NMVM model we obtain good results for the calibration and at the same time we have the property of parsimony, having only 3 parameters, while other used models have more (e.g. Heston has 5).