Politecnico di Milano

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Financial Engineering

Assignment 1 - Group 1

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1 Pricing a European Call Option

Our task is to price an ATM-spot European Call Option with time-to-maturity 3 months. We use three different methods:

1. via `blkprice` Matlab function: the result is computed through a closed formula, given by the Black '76 model. Its declaration is:

   ```
   [Call,Put] = blkprice(Price,Strike,Rate,Time,Volatility)
   ```

2. with a CRR tree approach: it is a backward algorithm, from the maturity to the value date, through M steps. In order to have an equivalence with the Black dynamics, we first have to impose that the upward and downward symmetric movements of the fwd-price return in a single time step Δt are $\Delta x = \sigma\sqrt{\Delta t}$. Hence the multiplicative coefficients are $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u$.

   Starting from the first tree level prior the maturity date, where we have $M + 1$ nodes, at each step we compute the value of the option by discounting the expectation of the value in the following step. Doing this iteratively for M times, we get the value in the seed, which is the value date.

3. with Monte Carlo. By exploiting the Law of Large Numbers and the Central Limit Theorem, we can derive a formula for the approximation of the Call price:

   $$C(t_0, t) = B(t_0, t)\frac{1}{M}\sum_{k=1}^{M}\left\{F_0 \exp\left[-\frac{\sigma^2}{2}(t - t_0) + \sigma\sqrt{(t - t_0)}g_i\right] - K\right\}^+ \tag{1}$$

   where $\{g_i\}$ are independent samplings of a Standard Normal distribution.

The results for the three methods are shown in Table 1.

blkprice	**CRR**	**Monte Carlo**
0.039764	0.039849	0.037006

Table 1: European Call Option price

In this case we impose $M = 100$. In the next section the focus is on the choice of the optimal M.

The reported values, expressed in Euro, are for a single contract. Since we have 1 Mln contracts, the final results must be multiplied for 10^6.

2 Choosing M

Since CRR and Monte Carlo are numerical methods, there is an approximation error. We first define a measure of the error, then we choose an appropriate threshold, in order to pick the M which makes the error smaller than our cut-off level.

In the CRR method the error measure is the absolute value of the difference between the calculated value and the exact value (the Black '76 one).

Instead, in the Monte Carlo method we define the error as $\sqrt{\frac{v_N}{N}}$, where $\sqrt{v_N}$, the squared root of the sample variance, is the unbiased estimator of the standard deviation of the Monte Carlo price.

We take the threshold as the bid-ask spread. In real life, especially for a market maker who needs to have very precise prices, it would be better to take as threshold a value lower than the bid-ask (for example one-tenth of it).

CRR	**Monte Carlo**
16	524 288

Table 2: Selected M

3 Plot of the errors

Having the errors of the CRR and the Monte Carlo approaches, we now want to study their trend, varying the value of M. The plots of the error in a log-log scale is shown in Figure 1.

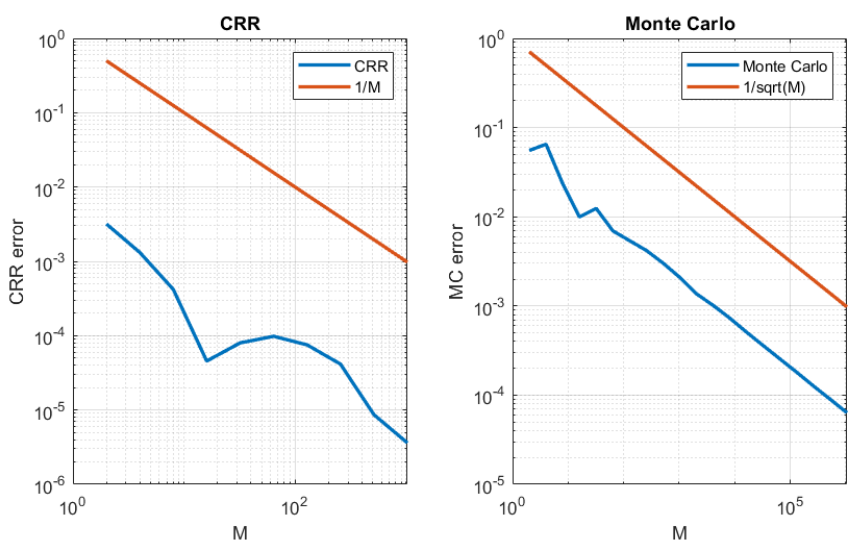


Figure 1: On the left, the error of CRR; on the right the error of Monte Carlo

As we can see, the CRR method error rescales as $1/M$, while the Monte Carlo method rescales as $1/\sqrt{M}$, especially for large values of M.

The reason of the MC error slope is the following: the Monte Carlo method is based on the Central Limit Theorem, which tells us that the error of the estimate is approximately normally distributed with mean 0 and standard deviation $\sigma/\sqrt{M}$, where σ in this case represents the standard deviation of the MC price. Thus, the Monte Carlo method has a $O(M^{-1/2})$ convergence rate.

4 Pricing a European Call Option with European barrier

A European Call Option with European barrier KI is an exotic option with the following payoff: $(F(t,t) - K)\mathbb{1}_{F(t,t)\geq KI}$. In our case $KI > K$. In practise, we have a positive payoff only if the underlying is above the barrier at maturity.

We use CRR and Monte Carlo methods as before, changing the payoff with the one mentioned above.

Also in this case, there exists a closed formula. The proof is similar to the Black's one for the European Call Option. We start by finding an equivalent formulation for the inequality $F(t,t) \geq KI$: if $g \sim N(0,1)$ we have that

$$\begin{aligned} F(t,t) \geq KI &\iff F_0 \exp\left(-\frac{\sigma^2}{2}(t-t_0) + \sigma\sqrt{(t-t_0)}g\right) \geq KI \\ &\iff g \leq \frac{\log\left(F_0/KI\right)}{\sigma\sqrt{t-t_0}} - \frac{\sigma}{2}\sqrt{t-t_0} \end{aligned} \tag{2}$$

Proceeding in the same way as in the Black's proof we obtain the closed formula:

$$\begin{aligned} C(t_0,t) &= B(t_0,t)\mathbb{E}_0\left[(F(t,t)-K)\mathbb{1}_{F(t,t)\geq KI}\right] \\ &= B(t_0,t)\left\{F_0 N(d_1) - K N(d_2)\right\} \end{aligned} \tag{3}$$

where $d_2 = \frac{\log\left(F_0/KI\right)}{\sigma\sqrt{t-t_0}} - \frac{\sigma}{2}\sqrt{t-t_0}$ and $d_1 = d_2 + \sigma\sqrt{t-t_0}$.

The prices obtained with the three different procedures are in Table 3.

closed formula	**CRR**	**Monte Carlo**
0.002093	0.002182	0.002105

Table 3: Barrier Option price

5 Vega of the Barrier Option

The objective is to compute the $\upsilon = \frac{\partial C}{\partial \sigma}$, where C is the price of the European Call Option with European Barrier KI. We want to plot its trend with respect to the underlying: given a variation of S_t from 0.70 to 1.50, choosing a 0.01 step length, we get a resulting variation in the forward and consequently in the value of our option. Considering this discretization, we have 81 different futures prices to analyze.

In the case of the exact formula we derive directly the expression of the υ:

$$\upsilon = B\left\{F_0 \cdot f(d_1)\left(-\frac{\log\left(\frac{F_0}{K_I}\right)}{\sigma^2\sqrt{t-t_0}} + \frac{\sqrt{t-t_0}}{2}\right) - K \cdot f(d_2)\left(-\frac{\log\left(\frac{F_0}{K_I}\right)}{\sigma^2\sqrt{t-t_0}} - \frac{\sqrt{t-t_0}}{2}\right)\right\}\Delta\sigma \tag{4}$$

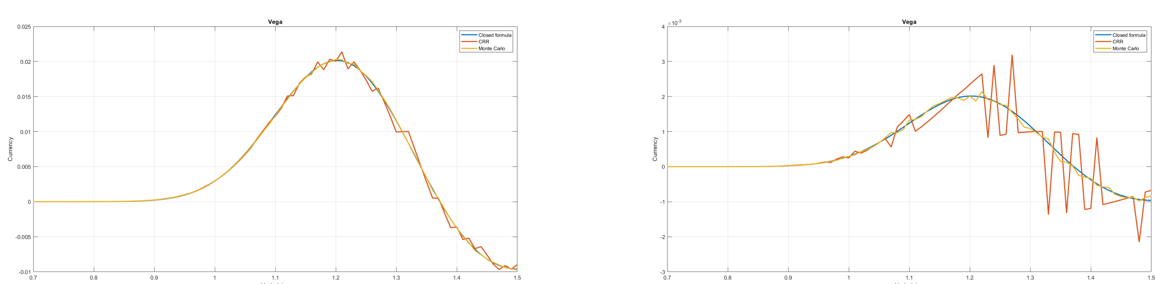
where f is the probability density function of a Standard Normal random variable. We multiply for $\Delta\sigma$ in order to have the υ expressed in Euro.

In the case of CRR and Monte Carlo, instead, we need to use a discretization of the derivative with respect to the volatility. We choose the central difference scheme:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} \tag{5}$$

In this way we obtain an error of $O(h^2)$.

We try different values of the discretization parameters: in Figure 2a $\Delta\sigma = \sigma/5$, while in Figure 2b we reduce $\Delta\sigma$ to $\sigma/50$, expecting the error to decrease, due to the theory. Instead, in practise we notice that there are instabilities: this is because we are using a numerical method (difference scheme) over other numerical methods (CRR and Monte Carlo). Indeed, if we apply the discretization of the derivative to the exact formula, we have a very precise result. To solve the problem we have to increase the precision of CRR and Monte Carlo: we run a final simulation with $\Delta\sigma = \sigma/50$, $M = 10^5$ for CRR and $M = 10^7$ for MC (see Figure 3) using "Parallel Computing Toolbox" to reduce runtime (in the uploaded script, we have the sequential version).



(a) Plot of the υ with $\Delta\sigma = \sigma/5$

(b) Plot of the υ with $\Delta\sigma = \sigma/50$

Figure 2: Plot of the υ

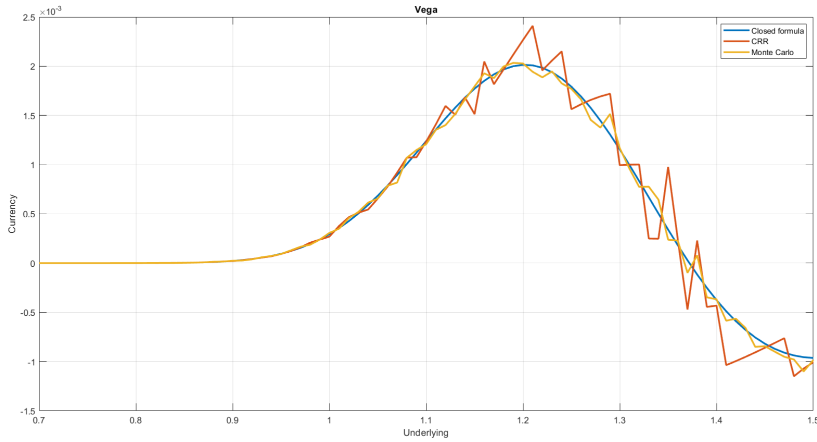


Figure 3: Plot of the υ with $\Delta\sigma = \sigma/50$ and large M

Looking at the curve, we notice that, unlike the European case where the υ has the shape of a positive bell, for an U&I Barrier Option with European Barrier, the υ assumes also negative values. This happens for $S_0 \geq 1.38$ Euro: when the underlying is sufficiently beyond the barrier (1.3 Euro), we can only have a profit if we remain above KI, where the option is activated in-the-money. A higher volatility increases the risk of the underlying going under the barrier at maturity: even if S_t is above the strike K, the option is not activated and it expires worthless.

6 Antithetic Variables technique

In the Monte Carlo method, in order to reduce the error, we should use a very large number of samplings. However, in this way the procedure becomes expensive in term of computation time. Variance reduction techniques have the aim to reduce the variance (thus to have a better accuracy), still having the same number of trials.

We sample $M/2$ standard normal random variables $\{g_i\}$ and we obtain other $M/2$ random variables by changing the sign of all the g_i's. In this way we have two sets of negative correlated r.v.'s. We use the first $M/2$ ones to compute f_1 in the usual MC way, and the second $M/2$ ones to compute f_2. In the end, the sample value of the price is the average of f_1 and f_2: since f_1 and f_2 are negative correlated, the standard deviation of the MC price (and consequently the error) will be less than the one computed with M random trials.

We can observe this in Figure 4.

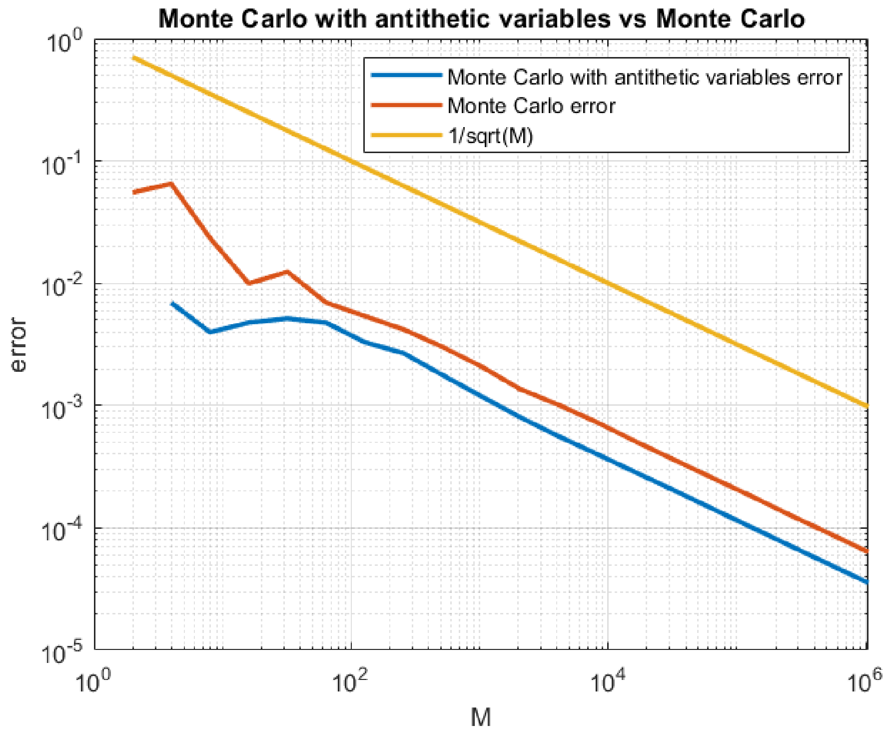


Figure 4: Reduction of the MC error thanks to the antithetic variables technique

7 Pricing a Bermudan Call Option

Our goal is to price a Bermudan call option using the CRR tree. We have the possibility of exercising the option every 30 days, assuming 30/360 day count convention, starting from the value date. The option time to maturity is 3 months, so we have 2 possible exercise dates: the first is March 15 and the second is April 15.

The pricing tree algorithm is equal to the one used for the plain vanilla call option, with the exception of the steps where we have the possibility to exercise. In these particular steps for computing the correct value of the Bermudan option we need to take the maximum between the intrinsic value, i.e. how much the option is in the money in that instant of time, and the continuation value.

The obtained price in Euro is: 0.041474.

8 Comparison between Bermudan and European Call Options varying the dividend yield

We now compare the price of the Bermudan Option with the one of the corresponding European Option, varying the dividend yield between 0% and 6%. The plot is shown in Figure 5.

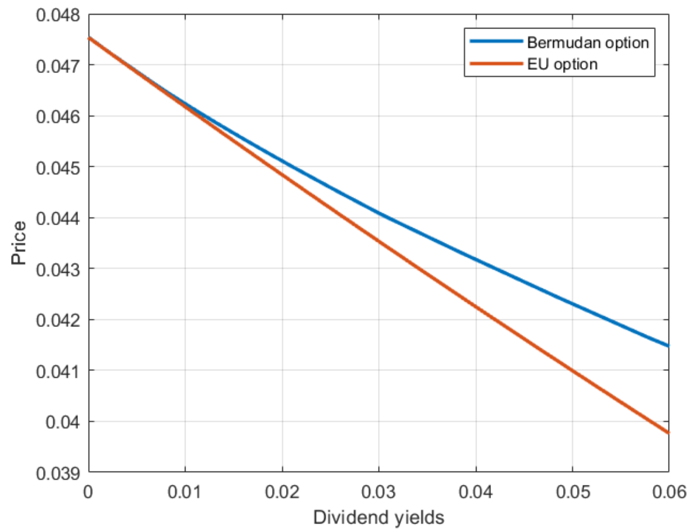


Figure 5: Comparison between EU call price and Bermudan call price

As we expect, the two values when there are no dividends coincide: this is because, by theory, a Bermudan Call Option with no dividends is never early exercised and thus is equal to a European Call Option. For larger values of the dividend yield, the price of the Bermudan is greater than the European, since we have the opportunity of early exercise.