Politecnico di Milano

Financial Engineering

Assignment 6 - Group 1

Authors:
Luca Amadori
Nicola Baldoni
Greta Belgeri

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1 Computation of the upfront

The objective is to compute the upfront $X\%$ in a structured bond, such that its Net Present Value (NPV) is equal to zero. In particular, since the upfront is involved only in the swap part, the analysis starts with this instrument.
In this case study the bank (counterparty A) pays quarterly the 3m Euribor with the adjoint of $s_{spolA} = 2.00\%$. On the other hand, the investment bank (counterparty B) pays the upfront X% of the principal amount and a quarterly coupon: the first one is 3%, while after that it is 3m Euribor plus $s_{spolB} = 1.10\%$, capped at a value $\hat{K}$ that varies every 5 years. The expression of the coupon (that resembles the payoff of a protected call) can be written in this way

$$L(t_0; t_i, t_{i+1}) + s_{spolB} - (L(t_0; t_i, t_{i+1}) - (\hat{K} - s_{spolB}))^+$$

where $L(t_0; t_i, t_{i+1})$ is the 3m Euribor fixed on the 20th February 2024 (t_0), with calculation start in t_i and calculation end in t_{i+1}. We can notice that the second term is the payoff of a caplet, hence in the end counterparty B receives (due to the minus sign) three caps with different starting date and maturities.

The first thing to do for the computation of the upfront is to price the caplet. For this pricing problem we use the normal Libor Market Model, which we calibrate using the flat volatilities available in the market. In order to find the spot volatilities $\{\sigma_i\}$ (4 for every year, except the first year where there are only 3), we impose that:

$$\begin{cases} Cap(T_\beta) - Cap(T_\alpha) = \sum_{i;T_\alpha < T_i \leq T_\beta} caplet_i(\sigma_i) \\ \sigma_i = \sigma_\alpha + \frac{T_i - T_\alpha}{T_\beta - T_\alpha}(\sigma_\beta - \sigma_\alpha) \end{cases} \tag{1}$$

where $Cap(T_\alpha)$ is the price of a cap whose last caplet expires in T_α, $caplet_i$ is the caplet with expiry in T_i and σ_i is the spot volatility of $caplet_i$. All the prices in Formula 1 are computed using Bachelier formula for the pricing of a caplet (2), but the left hand side uses the flat volatility, whereas the right hand side is a function of the spot volatility.

$$\begin{aligned} \text{caplet}_i(\sigma_i) &= B(T_0, T_{i+1})\delta(T_i, T_{i+1}) \left\{ [L(T_0; T_0, T_i) - K]\, N(d^n) + \sigma_i \sqrt{T_i - T_0}\phi(d^n) \right\} \\ d^n &= \frac{L(T_0; T_0, T_i) - K}{\sigma_i \sqrt{T_i - T_0}} \end{aligned} \tag{2}$$

The surface of the spot volatilities is in Figure 1. As expected, the volatility has the shape of a smile and its edges steepen when the maturity decreases. We can compare it with the plot of the flat volatilities (Figure 2): we notice that the surface of the flat is always above the spot.

Once the spot volatilities are computed, it is enough to compute the NPV of the payments of the two counterparties in order to find the upfront. More in detail, thanks to the known spot volatilities

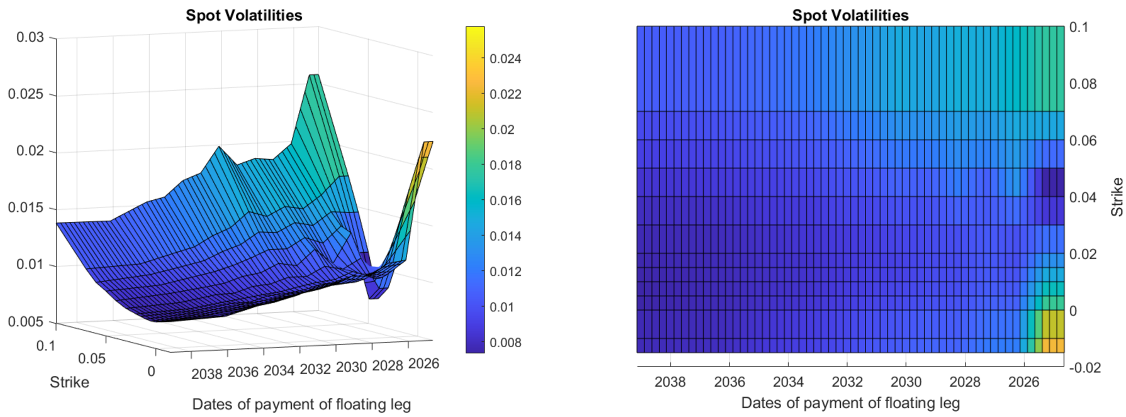

Figure 1: Spot volatilities surface

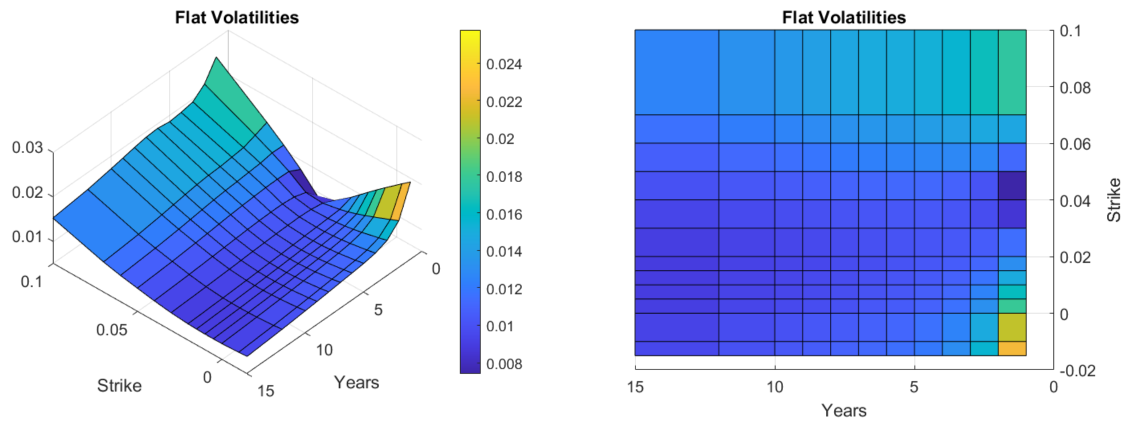

Figure 2: Flat volatilities surface

it is possible to compute the volatilities related to the strike rates of the three coupons by spline interpolation. These values are then used in the Bachelier formula to compute the price of the caplets that compose the three caps. Moreover, the computation gets easier noticing that both counterparties pay the same Libor in the same dates, so these payments erases in the computation of the NPV. The resulting upfront is 18.8690%.

2 Delta-bucket sensitivities

The main sensitivity for an interest rate product is the bucket-DV01. It quantifies the variation in Net Present Value attributable to a 1 basis point increase in a single bucket utilized during the bootstrap process. In our case there are 4 buckets related to the deposits, 7 related to the futures and 17 for the swaps, since the first swap is not used in the bootstrap.
We obtain the results in Tables 1,2,3.

The barplots of the bucket DV01 are in Figures 3, 4, 5. Their order of magnitude is shown in Figure 6 (it is the logarithm scale, hence the shorter bars correspond to the higher values). We can

# Bucket	DV01
1	0
2	0
3	$5.1533 \cdot 10^{-6}$
4	$2.4708 \cdot 10^{-6}$

Table 1: Bucket-DV01 for deposits

Bucket	DV01
1	$1.6557 \cdot 10^{-5}$
2	$1.1757 \cdot 10^{-5}$
3	$5.1031 \cdot 10^{-6}$
4	$1.2717 \cdot 10^{-6}$
5	$-3.3329 \cdot 10^{-8}$
6	$-2.2723 \cdot 10^{-7}$
7	$-6.4469 \cdot 10^{-8}$

Table 2: Bucket-DV01 for futures

Bucket	DV01
1	$-2.8433 \cdot 10^{-6}$
2	$-1.0957 \cdot 10^{-5}$
3	$-1.0281 \cdot 10^{-5}$
4	$7.2665 \cdot 10^{-6}$
5	$-1.0825 \cdot 10^{-5}$
6	$-1.3037 \cdot 10^{-5}$
7	$-9.2029 \cdot 10^{-6}$
8	$-1.4195 \cdot 10^{-5}$
9	$4.1612 \cdot 10^{-5}$
10	$1.5327 \cdot 10^{-5}$
11	$-2.8943 \cdot 10^{-5}$
12	$4.1696 \cdot 10^{-4}$
13	$1.2310 \cdot 10^{-6}$
14	$-3.0456 \cdot 10^{-7}$
15	$6.5123 \cdot 10^{-8}$
16	$-5.7023 \cdot 10^{-9}$
17	$7.1267 \cdot 10^{-10}$

Table 3: Bucket-DV01 for swaps

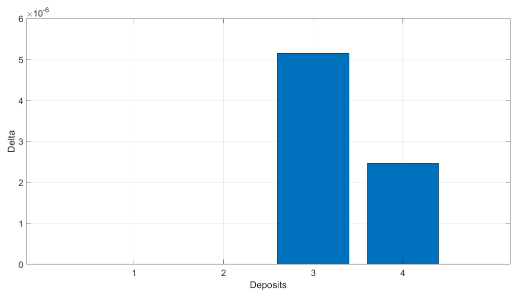

Figure 3: Bucket-DV01 for deposits (maximum y axis value: $6 \cdot 10^{-6}$)

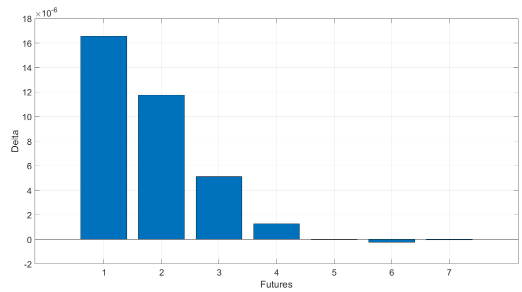

Figure 4: Bucket-DV01 for futures (maximum y axis value: $18 \cdot 10^{-6}$)

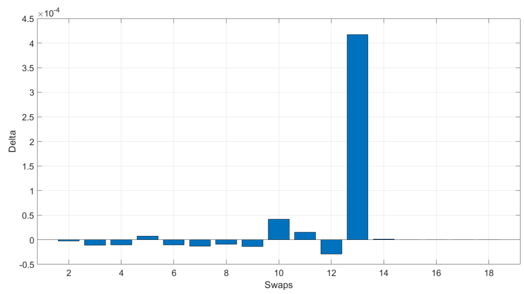

Figure 5: Bucket-DV01 for swaps (maximum y axis value: $4.5 \cdot 10^{-4}$)

see that the larger Deltas are the ones regarding the short term buckets (around 6 months). For the buckets related to the mid term swaps we get some negative values: counterparty B's gain is negatively influenced by an increasing of this rates. There is an isolated positive peak for 15 years.

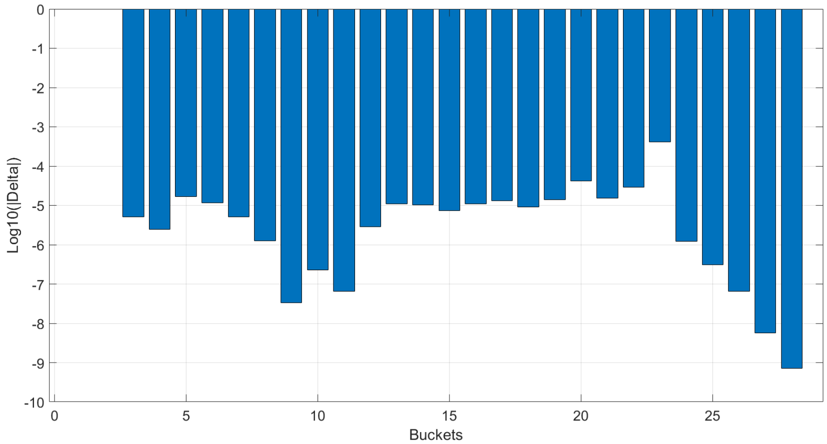

Figure 6: Delta-buckets in logarithm scale

3 Vega

3.1 Total Vega

Another important sensitivity for interest rates products is the total Vega, which is the change of the NPV related to an increasing of 1bp of all the flat volatilities given by the broker.
The difference between the NPV after the shift and the original NPV is the Vega, that results to be equal to €$5.5832 \cdot 10^5$. This mean that counterparty B gains from the increasing of the flat volatilities, since the values of the caps that it receives increases (this can be derived from the Bachelier formula).

3.2 Vega-bucket sensitivities

It is possible to have a more detailed description of the change of the NPV due to an increase of the flat volatility by looking at the Vega-bucket sensitivities. It is obtained by subtracting the NPV related to the initial flat volatilities from the NPV computed using the flat volatility of a certain bucket increased by 1bp. As in case of total Vega, the new flat volatilities determine new spot volatilities, that should be computed again since they enter into the computation of the NPV.
The buckets we are interested in are: 1 year, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15 years. The results are in Table 4. The largest value of Vega are for buckets 5, 10 and 15 years, the ones that correspond to the expiries of the caps. We notice that we have some negative values: this can because some spot volatilities decrease due to the spline interpolation.

Bucket	Vega-bucket
1	$1.7894 \cdot 10^{-6}$
2	$-4.1941 \cdot 10{-}7$
3	$3.3642 \cdot 10{-}7$
4	$-2.194510 \cdot 10{-}7$
5	$1.9145 \cdot 10^{-5}$
6	$-2.9862 \cdot 10^{-7}$
7	$4.0141 \cdot 10^{-7}$
8	$-6.3893 \cdot 10^{-8}$
9	$-1.5668 \cdot 10^{-7}$
10	$6.8606 \cdot 10^{-5}$
12	0
15	0.0010

Table 4: Vega-bucket sensitivities (for a unitary notional)

4 Coarse-Grained Delta Bucket

In order to have the Delta bucket sensitivity we need to perform the bootstrap 28 times (having 28 quoted instruments for the first 15 years). Considering the sensitivities for macro-buckets of dates is more efficient. In our case we have 4 intervals, with peaks in 2,5,10 and 15 years. To aggregate the measures within each set we assign a weight w to each date and we compute a weighted sum of the corresponding sensitivities. The weights are made such that the sum within each bucket is 1 and they are linear between the peaks. The computed DV01 for each CG bucket is in Table 5.

# CG Bucket	Delta
2	$2.8433 \cdot 10^{-5}$
5	$-2.6261 \cdot 10^{-5}$
10	$1.2238 \cdot 10^{-5}$
15	$4.0846 \cdot 10^{-4}$

Table 5: Coarse-Grained Delta bucket sensitivities

The DV01 of each CG bucket can be hedged using 4 swaps, with expiries 2, 5, 10, 15 years and unknown notionals. We compute the Coarse-Grained bucket sensitivities of every swap and we impose that the total Delta of the portfolio for all CG buckets is null. We thus obtain a linear system of 4 equations with 4 unknowns, that can be written as

$$\Delta_{swaps} N_{swaps} + \Delta_{sp} N_{sp} = 0$$

where Δ_{swaps} is a (4×4) matrix containing the sensitivities of the swaps for every bucket

$$\begin{bmatrix} & \text{Swap 2y} & \text{Swap 5y} & \text{Swap 10y} & \text{Swap 15y} \\ \text{CG Bucket 2} & 1.9071 \cdot 10^{-4} & 0 & 0 & 0 \\ \text{CG Bucket 5} & 0 & 4.6007 \cdot 10^{-4} & 0 & 0 \\ \text{CG Bucket 10} & 0 & 8.6496 \cdot 10^{-4} & 0.0009 & 0 \\ \text{CG Bucket 15} & 0 & 0 & 0 & 0.0012 \end{bmatrix}$$

N_{swaps} is a (4×1) vector with the notionals of the 4 (payer) swaps, Δ_{sp} a (4×1) vector with the values contained in Table 5 and $N_{sp} = 50 \cdot 10^6$ is the corresponding notional. As expected, Δ_{swaps} is an upper triangular matrix (actually it is a diagonal matrix where the elements out of the diagonal are of the order of 10^{-16}), since swaps are not influenced by translation in a part of the curve after their expiries. By solving this linear system we obtain the results in Table 6. Since

N swap 2y	N swap 5y	N swap 10y	N swap 15y
$-7.4545 \cdot 10^6$	$2.8540 \cdot 10^6$	$-7.0741 \cdot 10^5$	$-1.6807 \cdot 10^7$

Table 6: Notional of the four swaps (€)

swaps has always a positive Delta, we get notionals with opposite sign compared to the CG-bucket sensitivities of the structure product.

5 Vega-Delta hedging

The structured bond has a Vega of €$5.5832 \cdot 10^5$, as explained in Subsection 3.1. We can hedge it using an ATM Cap with expiry 5 years (when we say ATM regarding to a Cap we mean that its strike is equal to the ATM Swap rate with the same expiry).
First, we compute υ_{cap} by shifting the flat vols and by recomputing the value of the Cap. We obtain $\upsilon_{cap} = 2.5371 \cdot 10^{-4}$. Then we impose that

$$\upsilon_{sp} N_{sp} + \upsilon_{cap} N_{cap} = 0$$

Thus, we can make the Vega of the portfolio null with a 5y Cap with notional $N_{cap} = -2.2006 \cdot 10^8$€. As expected, we get a negative value, since caps has positive Vega.

Once we have Vega-hedged the portfolio, we can make it also Delta-hedged by introducing a 5y Swap. We choose this instrument because it has no Vega (having no dependencies with the flat vols), hence it does not impact the computations already made.

By imposing

$$\Delta_{sp} N_{sp} + \Delta_{cap} N_{cap} + \Delta_{swap} N_{swap} = 0$$

we find that $N_{swap} = 1.3172 \cdot 10^8$€.

6 Coarse-Grained Vega Bucket

As we did in Section 4 for the Delta, we now proceed in the same way with the Vega: we divide the set of dates in Coarse-Grained buckets, we apply the weights and we obtain the Vega for every bucket by shifting the flat vols and recomputing the NPV. The chosen macro-buckets are 5 and 15 years. We obtain the results in Table 7. Now we can Vega-hedge the portfolio containing the

# CG Bucket	Vega
5	$5.4848 \cdot 10^{-5}$
15	$1.0626 \cdot 10^{-3}$

Table 7: Coarse-Grained Vega bucket sensitivities

structured bond with a 5y and a 15y ATM Caps (similarly to what we did with the total Vega in Section 5) by imposing the linear system (2 equations, 2 unknowns)

$$\upsilon_{caps} N_{caps} + \upsilon_{sp} N_{sp} = 0$$

where υ_{caps} is a (2×2) matrix with the Vega of the caps for every bucket

$$\begin{bmatrix} & \text{Cap 5y} & \text{Cap 15y} \\ \text{CG Bucket 1} & 2.5353 \cdot 10^{-4} & -1.7845 \cdot 10^{-6} \\ \text{CG Bucket 2} & 0 & 1.1955 \cdot 10^{-3} \end{bmatrix}$$

N_{caps} is a (2×1) vector with the notionals of the two caps, υ_{sp} a (2×1) vector with the values contained in Table 7. The values of the notionals needed for the hedging are in Table 8.

N cap 5y	N cap 15y
$-1.1130 \cdot 10^{7}$	$-4.4443 \cdot 10^{7}$

Table 8: Notionals of the two Caps (€)

Finally, we can hedge also the CG-bucketed Delta by introducing two swaps (which has null Vega) with expiries 5 and 15 years. We proceed in the same way as in Section 5 and we obtain the swap notionals in Table 9.

N swap 5y	N swap 15y
$3.3964 \cdot 10^{6}$	$5.2322 \cdot 10^{6}$

Table 9: Notionals of the two swaps (€)