Politecnico di Milano

Financial Engineering

Assignment 7 - Group 1

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1 Certificate Pricing

The goal is to compute the upfront of the hedging swap of an interest rate structured product. In particular, the swap involves two parties: a bank (party A) and an Investment Bank (party B). Party A pays quarterly 3 months Euribor plus 1.30%, subjected to the Early Redemption and Final Coupon clauses. On the other hand, party B pays the upfront at start date and a yearly coupon: 6% in the first year, conditionally to the fact that the value of Stoxx50 on the coupon reset date (2 business days prior to the respective coupon payment date) is less than 3200, and 2%, paid in the second year, conditionally to the fact that the Early Redemption clause was not applied in the first date. In this case, the Early Redemption clause is triggered if on a Coupon Reset Date the Cumulative Coupon Accrual of the coupon paid by party B is equal or above 6% . The principal amount is 100mln €.

In the valuation we use a NIG model, whose calibrated parameters are in Table 1. The Mean Squared Error of the calibration is 1.080.

σ	η	κ
0.1048	12.7725	1.3156

Table 1: Calibrated NIG parameters

σ represents the average volatility, η the skew and κ is the volatility of the volatility.

For the computation of the upfront we need the discounted payments of the two parties, which depend on whether the Early Redemption clause in the first year is triggered or not. The probability of this event is 1-Π and it can be computed exploiting a closed formula or via Monte Carlo simulations.

In the first case Π derives from Lewis (2001) [**1**]:

$$\Pi = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\Big[\frac{e^{iuk}\Phi_T(u)}{iu}\Big]du$$

where Φ is the characteristic function of the NIG model. In order to compute this integral we use the Matlab function `integral` (with absolute and relative error tolerance 10^{-16}).

In the second case we compute the probability Π by simulating the value of the underlying in the reset date. Additionally, we exploit also the Antithetic Variables technique to have a more precise result.

Finally, we calculate the upfront imposing the NPV of the swap equal to zero:

$$X_x = NPV_A - NPV_B \tag{1}$$

$$NPV_x = \Pi \cdot NPV_{x,NONSTOP} + (1-\Pi) \cdot NPV_{x,STOP}$$

considering $NPV_{x,STOP}$ as the sum of the discounted payments of A in case the Early Redemption clause applies in the first year and the opposite for $NPV_{x,NONSTOP}$. The resulting upfront (with 10^6 simulations for Monte Carlo) is

$X_{NIG}\%$	$X_{MC}\%$	$X_{MC_AV}\%$
2.4894%	2.4886%	2.4890%

Table 2: Upfront (expressed as a percentage of the notional value)

The 95% confidence interval for the Monte Carlo method is [2.4805%,2.4967%], while the one for the Antithetic Variables version is [2.4814%,2.4966%]. Their lengths are respectively $1.6187 \cdot 10^{-4}\%$ and $1.5163 \cdot 10^{-4}\%$.

1.1 Model selection

A different approach for the computation of the upfront is the use of Black model for the evolution of the underlying, that affects the computation of the probability that the trigger event happens. In particular, we consider

$$\Pi = N(d_2)$$

with

$$d_2 = \frac{log\left(\frac{F_0}{K}\right)}{\sigma\sqrt{t-t_0}} - \frac{1}{2}\sigma\sqrt{t-t_0}$$

and $N(.)$ the cumulative distribution function of a standard normal random variable. In the previous equation, t is the Coupon Reset Date (2 business days prior to the respective Coupon Payment Date), the Coupon Payment Date is one year after the start date, F_0 is the value of the forward at time t, K is the strike price (in this case 3200) and σ is the implied volatility for the strike of interest, obtained by spline interpolating the volatility surface. The upfront is computed as in (1).

However, this approach is not totally correct since it does not take into account the digital risk; this can be noticed by the fact that the difference between the upfront obtained in this way and the one computed with the closed formula is 0.9507%.

$X_{NIG}\%$	$X_{Black}\%$
2.4894%	1.5388%

Table 3: Upfront with closed formula and Black model

We can also consider other models for the underlying by varying the value of α (getting different models of Mean Variance Mixture). In Figure 1 we can see the trend of the price. The red dot is the NIG price.

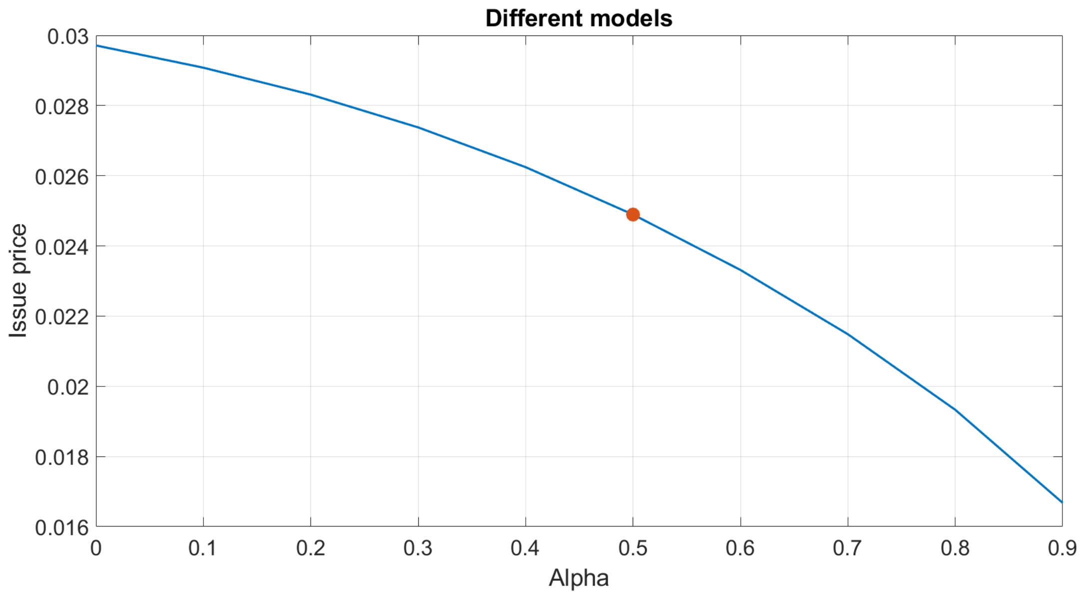

Figure 1: Prices with respect to various alpha

1.2 Three years expiry case

We analyze the case in which the structured bond has a three-year expiry, with the same payoff for the first year repeated also on the second one and Early redemption option at the end of the first two years. In this framework there are four possible scenarios:

- getting the coupon at the first year
- getting the coupon at the second year
- getting the coupon at both first and second years
- getting the coupon at the thid year

In this case a closed formula cannot be used since the pricing process is path-dependent: it is impossible to compute the conditional probabilities of the scenarios mentioned above, that are necessary to compute the discounted payments of the two parties. For this reason we can only use the Monte Carlo approach. The obtained upfront calculated with 10^6 simulations is 3.6408%, with a 95% confidence interval of [3.6294%,3.6522%].

1.3 Black model error

In Subsection 1.1 it is explained that the Black model is not a correct approach since it does not takes into account the digital risk. For this reason we correct the Black approach by adding this kind of risk, considering the case of two-years expiry. We know that

$$dc_B(K) = -\frac{\partial c_B(K, \sigma(K))}{\partial K} - \frac{\partial \sigma(K)}{\partial K}\upsilon$$

where

$$dc_B(K) = B(t_0, t)N(d_2) = -\frac{\partial c_B(K)}{\partial K}$$

$$\upsilon = B(t_0, t)F_0\sqrt{t - t_0}\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}$$

$$d_1 = \frac{\log(F_0/K)}{\sigma\sqrt{t - t_0}} + \frac{1}{2}\sigma\sqrt{t - t_0}$$

$$d_2 = d_1 - \sigma\sqrt{t - t_0}$$

We approximate the slope impact of the surface as

$$\frac{\partial \sigma(K)}{\partial K} = \frac{\sigma_2 - \sigma_1}{K_2 - K_1}$$

where K_2 is the given strike (3200), K_1 is the previous strike price in the set of the strikes and σ_1, σ_2 are the corresponding volatility.
We can compute the probability that the coupon in the first year is not paid as

$$\Pi = N(d_2) - \frac{\partial \sigma(K)}{\partial K}\frac{\upsilon}{B(t_0, t)}$$

and then calculate the upfront as in 1.

$X_{NIG}\%$	$X_{DIG}\%$
2.4894%	2.3866%

Table 4: Upfront with closed formula and corrected Black model

The difference between the closed formula using the NIG model and the corrected Black model is 0.1028%.

2 Bermudian Swaption Pricing via Hull-White

2.1 Trinomial Tree

We consider a 1-factor Hull-White model, with a = 11% and $\sigma = 0.8\%$ and a single curve framework. The model dynamic is the following:

$$\begin{cases} dx_t = -ax_t dt + \sigma dW_t \\ x_{t_0} = 0 \end{cases}$$

Under this condition we are asked to price a 10y Bermudan yearly Payer Swaption Strike 5% non-call 2 i.e. an option for which each year, starting from the second, you can choose to enter in a payer swap with fixed leg 5% and maturity ten years after t_0. We consider a discretized dynamic dividing the ten years in n nodes (we impose $n \equiv 1 \bmod 10$ to have a node in each exercise date)

$$\Delta x_{i+1} = x_{i+1} - x_i = -e^{a\Delta t}\hat{\mu}x_{i+1} + e^{a\Delta t}\hat{\sigma}g_i$$
$$\text{where } g_i \sim \mathcal{N}(0,1),\ \hat{\mu} = 1 - e^{-a\Delta t},\ \Delta t = \frac{10}{n} \text{ and } \hat{\sigma} = \sigma\sqrt{\frac{1-e^{-2a\Delta t}}{2a}}.$$

We represent the discretized dynamic using a trinomial tree: starting in t_0 with x=0, in each date we can move to a superior node with probability p_u, move to a inferior node with probability p_d or remain on the same level with probability p_m (see figure 2).We get p_u, p_d and p_m imposing the equality of the theoretical mean and variance of the model with the tree's ones. We choose $\Delta x = \hat{\sigma}\sqrt{3}$. We consider a vertical index of integers l such that the first node has l=0.

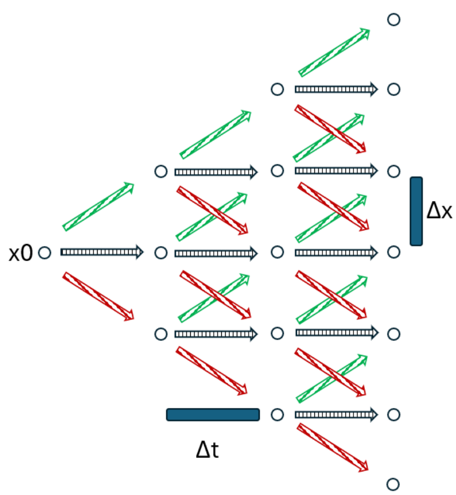

Figure 2: Trinomial Tree

The l^{th} node has $x = l\Delta x$. To have p_u, p_d and p_m positive we have to impose a l_{max} such that $1 - \sqrt{\frac{2}{3}} < l_{max}\hat{\mu} < \sqrt{\frac{2}{3}}$. To have a simpler grid we take the smallest l_{max} possible i.e. $l_{max} = \left\lceil \frac{1-\sqrt{\frac{2}{3}}}{\hat{\mu}} \right\rceil$. The same constraint is applied to the negative side as $-l_{max}$. We have to consider differently this extreme cases: in the top you can remain on the same level, move down of one step of move down of two step (and vice versa in the bottom case)(see Figure 3). Also the formulas to compute probabilities are different.

To compute the NPV of the swaps and the stochastic discount factors we need the discount factors, computed in each node with the following formula.

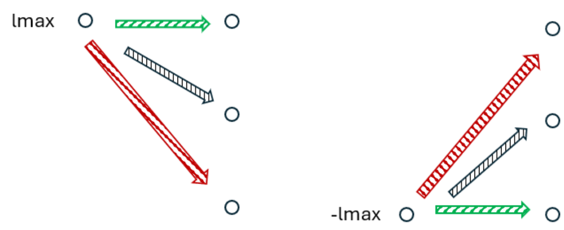

Figure 3: Top and Bottom cases

$$B(t_i, t_i + \tau) = B(t_0; t_i, t_i + \tau) \left\{ -x_i \frac{\sigma(0, \tau)}{\sigma} - \frac{1}{2} \int_{t_0}^{t_i} \left[\sigma(u, t_i + \tau)^2 - \sigma(u, t_i)^2 \right] du \right\} \tag{2}$$

where

$$\sigma(t, s) = \tfrac{\sigma}{a} \left[1 - e^{-a(t-s)} \right]$$

and the integral is equal to

$$\left(\tfrac{\sigma}{a}\right)^2 \left[\tfrac{2}{a} \left(1 - e^{-a\tau}\right) \left(1 - e^{-at_i}\right) - \tfrac{1}{2a} \left(1 - e^{-2a\tau}\right) \left(1 - e^{-2at_i}\right) \right]$$

Climbing backwards the tree, in each node we compute the continuation value of the swaption as the expected value of the discounted values that the swaption can assume in the previous (then following in times) node. The stochastic discount factors depend on the dynamic of the curve (by x_{i+1} and Δx_{i+1}) and are different in the nodes.

$$D(t_i, t_{i+1}) = B(t_i, t_{i+1}) * \exp \left\{ -\frac{1}{2} \left(\hat{\sigma}^*\right)^2 - \frac{\hat{\sigma}^*}{\hat{\sigma}} \left[e^{-adt} \Delta x_{i+1} + \hat{\mu} x_{i+1} \right] \right\} \tag{3}$$

where

$$\hat{\sigma} := \tfrac{\sigma}{a} \sqrt{dt - 2 \tfrac{1 - e^{-adt}}{a} + \tfrac{1 - e^{-2adt}}{2a}}$$

Then in the exercise dates we confront the continuation value with the present value of the swap in such date (and with such x) and we keep the highest. Finally, the continuation value in the first node is the swaption price i.e. €0.0218 (with 1000 nodes). In Figure 4 we can see the trend of the Bermudan prices with respect to the number of the tree's nodes.

2.2 Control of the result

The obtained tree can also be used to price an European Swaption with expiry T_α and maturity T_ω by avoiding the yearly comparison between intrinsic and continuation value. This permits to

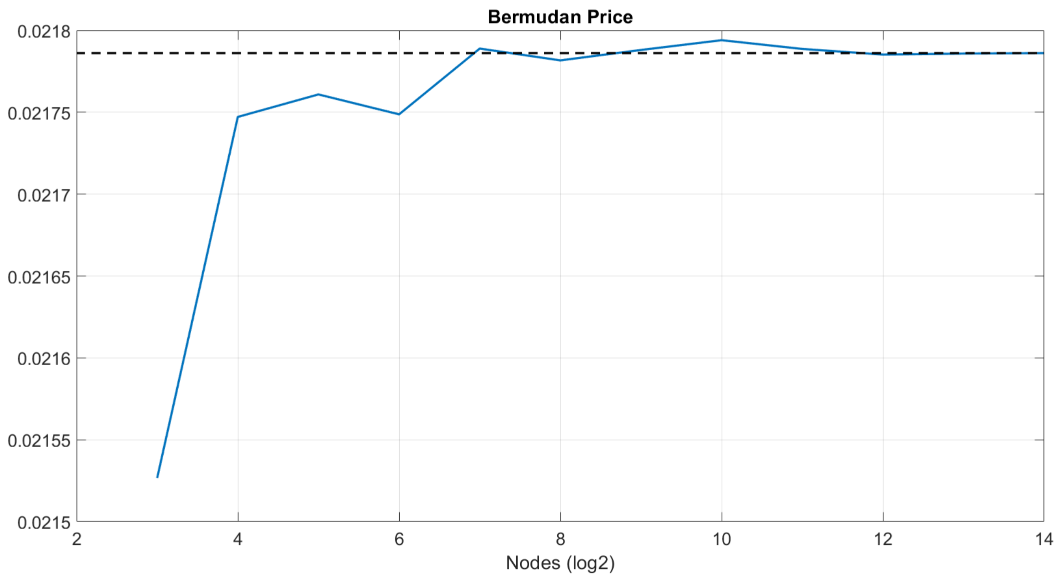

Figure 4: Bermudan price with respect to number of tree's nodes

check the functioning of the tree, since we have a closed formula to price the European version and through which we can compare the obtained values.

We know that the price of an European Swaption with strike c corresponds to the price of a coupon bond Put option with strike 1 and annual coupon

$$c_i = \begin{cases} c\delta_i & \text{if } i < \omega \\ 1 + c\delta_i & \text{if } i = \omega \end{cases}$$

We first compute the price of a coupon bond Call option through the Jamshidian formula (1989):

$$C_{\alpha\omega}(t_0) = \sum_{i=\alpha+1}^{\omega} c_i \mathbb{E}_0 \left[D(T_0, T_\alpha)(B_{\alpha i}(T_\alpha, x_\alpha) - K_i)^+ \right] \tag{4}$$

The strikes K_i can be found by imposing

$$P_{\alpha\omega}(T_\alpha, x^*) = \sum_{i=\alpha+1}^{\omega} c_i B_{\alpha\omega}(T_\alpha, x^*) = 1$$

to find x^* and then we compute $K_i = B_{\alpha\omega}(T_\alpha, x^*)$. $B(T_\alpha; T_\alpha, T_i)$ is calculated using Formula (2).

The Jamshidian formula (4) is equal to a weighted sum of prices of Call options on zero coupon bonds $B_{\alpha i}(T_\alpha, x_\alpha)$, which, since we are in a Gaussian HJM framework, corresponds to:

$$\begin{aligned} C_i(t_0) &= B(t_0, T_\alpha)\{B(t_0; T_\alpha, T_i)N(d_1) - K_i N(d_2)\} \\ d_{1,2} &= \frac{\log B(t_0; T_\alpha, T_i)/K_i}{\Sigma_i \sqrt{T_\alpha - t_0}} \pm \frac{1}{2}\Sigma_i \sqrt{T_\alpha - t_0} \end{aligned}$$

where

$$\begin{aligned} \Sigma_i^2 &= \frac{1}{T_\alpha - t_0}\int_{t_0}^{T_\alpha} v(t)\, dt = \frac{1}{T_\alpha - t_0}\int_{t_0}^{T_\alpha} \sigma(t, T_i) - \sigma(t, T_\alpha)\, dt \\ &= \left(\frac{\sigma}{a}\right)^2 \left(e^{-a(T_i - T_\alpha)}\right)^2 \left(1 - e^{-2aT_\alpha}\right) \end{aligned}$$

Once we get the price of the coupon bond Call option with Formula (4) we derive the price of the corresponding put using the put-call parity relation:

$$C_{\alpha\omega}(t_0) - P_{\alpha\omega}(t_0) = B(t_0, T_\alpha)\left(\sum_{i=\alpha+1}^{\omega} c_i B(t_0; T_\alpha, T_i) - 1\right)$$

We can now price the set of co-terminal ($T_\omega = 10y$) European Swaptions (T_α from 2y to 9y) both with the tree (10^3 nodes) and the exact formula. The results are in Table 5.

Exact price (%)	Difference
0.8189	$4.3635 \cdot 10^{-6}$
1.2044	$7.5478 \cdot 10^{-6}$
1.4239	$13.2535 \cdot 10^{-6}$
1.4808	$9.6707 \cdot 10^{-6}$
1.4010	$0.0289 \cdot 10^{-6}$
1.4922	$-4.3254 \cdot 10^{-6}$
0.8751	$2.8442 \cdot 10^{-6}$
0.4741	$-0.0286 \cdot 10^{-6}$

Table 5: Exact price (expressed as a percentage of the notional) and its difference with respect to the tree price

Since the discrepancies between the two prices are small, we can say that the tree has been correctly implemented. We also look at the trend of the L^1-distance between the two prices among the co-terminal set varying the number of nodes N in the tree, obtaining the plot in Figure 5. We notice that the error rescales with N as $1/N$, until it reaches a plateau with $N = 10^{11}$. With this number of nodes we have the price of the Bermudan equal to 2.1789%.

2.3 Upper and lower bound

Regarding the 10y Bermudan yearly Payer Swaption, an interval for its price can be given by:

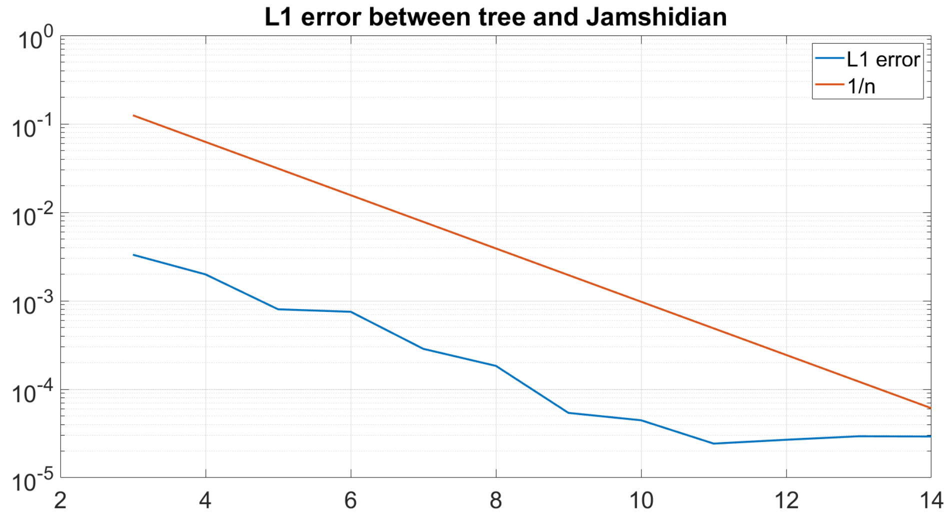

Figure 5: Error between the tree price and the exact price of the European Swaption, varying N

- lower bound: the maximum price of the set of co-terminal European Swaptions, 1.4808%.
- upper bound: the sum of the prices in the whole set, 8.8706%.

The prices of the European options are computed through the closed formula.

References

[1] Lewis, Alan L. (2001), *A simple option formula for general jump-diffusion and other exponential Lévy processes*, Envision Financial Systems and OptionCity.net

[2] Jamshidian, Farshid (1989), *An exact Bond Option Formula*, Journal of Finance, Volume 44, Issue 1

[3] Hull, John & White, Alan (1996), *Using Hull-White Interest-Rate Trees*, Journal of Derivatives