Midterm Review APMA

1.3 Classification of Differential Equations

ODE

dydx = x^2
y = y(x)

dpdt = -p
p = p(t)

PDE

∂u∂x + ∂u∂y = x + y
u = u(x,y)

Linear vs. NonLinear

Linear = ay, by'', y''', c #'

NonLinear = sin(y), ln(y)', yy', e^y, y^2 #'

Special Case = { 1/y dydt = t^2 }
{ y dydt = y^2 }

2.1 The First-Order Differential Equations

Special Cases:

Type 1:
dydt = p(t) # when p(t) is isolated
dy = p(t)*dt
∫dy = ∫p(t)*dt
y(t) = ∫p(t)*dt + C

Type 2:
dydt = p(t)*y # when p(t) not isolated
# if y != 0:

(1/y)*dydt = p(t)

∫1/y dy = ∫p(t) dt

ln|y| = ∫p(t)*dt + C1

$y = e^{\int p(t) dt + c_1}$
$y = c_1 e^{\int p(t) dt}$

General Case of 1st-Order Linear ODEs

$y' + p(t)y = g(t)$
$y(t_o) = y_o$
introduce an integrating factor $\Omega(t)$ that satisfies
$\Omega'(t) = p(t)\Omega(t)$
$\Omega(t) = e^{\int p(t)dt}$
then we can set
$(\Omega y)' = g(t)\Omega(t)$

########### IVP:
$y(t) = 1/\Omega(t) * [\int_{t_o}^{t} g(s)\Omega(s) + y_o]$

2.2 Separable Equations

An ODE is called separable if it can form:

$N(y) * \frac{dy}{dx} + M(x) = 0$

$\Rightarrow N(y)dy + M(x)dx = 0$

$\Rightarrow \int N(y) + \int M(x) = C$

Summary So Far:

1. $\frac{dy}{dt} = p(t)$ # direct integration
2. $\frac{dy}{dt} = p(t)y$ # divide then integrate
3. $\frac{dy}{dt} + p(t)y = g(t)$ # Integrating factor
4. $N(y) \frac{dy}{dx} + M(x) = 0$ # Separable equation
5. $N(x,y) \frac{dy}{dx} + M(x,y) = 0$ # Exact equation

2.6 Exact Equations

1st method
$M(x,y) + N(x,y)y' = 0$

1. Check Exactness

$My = Nx$

2. Construct $\Omega(x,y)$

$\Omega x = M(x,y)$

$\Rightarrow \Omega(x,y) = \int M(x,y)dx + h(y)$
Take ∂y
$\Rightarrow \Omega y = (\int M(x,y)dx) * dy + h'(y)$

But Ωy equals $N(x,y)$ so solve for $h'(y)$ with difference in 2 equations

$\Omega(x,y) = C$

2nd method

construct $\Omega(x,y)$

$\Omega y = N(x,y)$

$\Rightarrow \Omega = \int N(x,y)dy + g(x)$
Take ∂x
$\Rightarrow \Omega x = (\int N(x,y)dy) * dx + g'(x)$

But Ωx equals $M(x,y)$ so solve for $g'(x)$ with difference in 2 equations

3rd method

$\Omega = \int Mdx + h(y)$

$\Omega = \int Ndy + g(x)$

Compare Ωs to gather value of $h(y)$ & $g(x)$

4th method - Integrating Factor

multiply eqn by $O(x) = x$

$OM + ONy' = 0$

then $OyM + OMy = OxN + ONx$

$\Rightarrow OyM - OxN = (Nx - My)O$
#PDE for O - hard to solve

Case1:
if O = O(x) is a function of x only

dOdx = −(Nx − My)/N ∗ O

if RHS is a function of only x

dOdx = f(x)O

1/O dO = f(x) dx

Solved

Case2:

if O = O(y) is a function of y only

dOdy = (Nx − My)/M ∗ O

if RHS is a function of only y

dOdy = f(y)O

1/O dO = f(y) dy

Solved

Case3:

if O = O(z) and z is a function of x and y

OM + ONy' = 0

OMy = ONx # exactness condition

OyM + OMy = OxN + ONx

(dOdz)(dzdy) M + OMy = (dOdz)(dzdx) N + ONx

=> dOdz = [Nx − My / (dzdy)M − (dzdx)N] ∗ O

if RHS is a function of z only

dOdz = f(z)O

3.1 Homogeneous Constant Co.

ay'' + by' + cy = 0

try e^rt, then

ar^2 + br + c = 0

r1 = quadForm
r2 = quadForm

r1 != r2, real

y(t) = c1e^r1t + c2e^r2t

Now with complex roots:

e^it = cos(t) + isin(t)

#e^-it = cos(t) - i sin(t)

r1 = lambda + i mew
r2 = lambda + i mew

y(t) = e^lambda*t * (cos(mew t) + i sin(mew t))

Now with repeated roots:

1. get characteristic equation

y(t) = c1e^rt + c2te^rt

Reduction of Order

At^2 y'' + Bt y' - y = 0. (E1)

y1(t) = t^-1

Variation of Parameters
y(t) = v(t)y1(t)
y' = v'y1 + vy1'
y'' = v''y1 + 2v'y1' + vy1''

plug y, y', y'' into equation (E1)

get first order eqn for v (Av'' - v') #something like this

u = v' then u' = v''

Create separable equation with u, (Au' - u) #something like this

Find v' then $v(t)$

Plug back into $y(t) = v(t)\, y_1(t)$

Euler's Eqn

$$At^2 y'' + Bt y' + c y = 0$$

try
$y = t^r$
$y' = rt^{r-1}$
$y'' = r(r-1)t^{r-2}$

get characteristic, if $r_1 \neq r_2$, real distinct roots

$$y(t) = c_1 t^{r_1} + c_2 t^{r_2}$$

Method for Undetermined Coeffecients

E1 == $y'' + p(t)y' + q(t)y = g(t)$
E2 == $y'' + p(t)y' + q(t)y = 0$

1. Homogeneous solution
2. Guess the particular solution
3. add them together

Gen Soln. is
$y(t) = y_h(t) + y_p(t)$

Ex__

$$y'' + 3y' - 4y = 3e^{2t}$$

1. homo =
$y_h(t) = c_1 e^{-4t} + c_2 e^{t}$

2. Guess particular =
$y_p(t) = Ae^{2t}$

$y_p'(t) = 2Ae^{2t}$
$y_p''(t) = 4Ae^{2t}$

plug back in to E1 -->

$$4Ae^{2t} + 3 \cdot 2Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$A = 1/2$

$y_p(t) = 1/2\, e^{2t}$

== c1e^−4t + c2e^t + 1/2 e^2t

Rules for Guessing are derived from RHS similarities