# BIRDS-ON-WIRE MODEL

**Luca Barriviera, Wenbin Yan**

## 1 Introduction

In this project, we will model and study the behavior of $N$ birds sitting on a line on a telephone wire. In the model, each bird will try to adjust to find a comfortable standing position. The inspiration came from watching the Pixar short movie *For the Birds*.

## 2 Formulation of Model

We first consider the situation when there are $L-1$ points, $\mathcal{X}_L := \{\frac{1}{L}, \frac{2}{L} \cdots, \frac{L-1}{L}\}$, on a wire $[0,1]$. Let the positions of the $N$ birds at time $t$ defined by $\{(X_t^n)_{t\geq 0}\}_{1\leq n\leq N}$, with $n$ describing the $n^{th}$ bird and $(X_t^n)_{t\geq 0}$ taking values in $\mathcal{X}_L$. From now on, we take $X_t^0 := X^0 = x^0 = 0$, $X_t^{N+1} := X^{N+1} = x^{N+1} = 1$, and $0 = X^0 < X_0^1 < \cdots X_0^N < X^{N+1} = 1$ for convenience, which respectively represents the two edges of the wire. We also let $X_t$ as $(X_t^1, X_t^2, \cdots, X_t^N)^T$ which takes value in $E_L := \{X \in \mathcal{X}_L{}^N : X^1 < X^2 < \cdots < X^N\}$ to show the whole graph of the distribution of birds on the wire. We also let $L \gg N$ so that we can always have enough positions for birds.

We suppose that birds will feel uncomfortable because of the birds around them. Hence we define the *uncomfortable penalty* by a decreasing function $\varphi_L : \mathcal{X}_L \to [0, +\infty)$, to show that the smaller the distance between two birds, the larger the uncomfortable penalty. Specially, we would let $\varphi_L(x) \to 0$ as $x \to \infty$ to show that birds will not influence each other anymore if they are far enough from each other.

### 2.1 Definition of the First Jump

Because of feeling uncomfortable with the birds around it, a bird will try to move to adjust its position. To formalize this idea, we should keep some key principles in mind when constructing the model:

1. The birds should always avoid collisions, if there is no possibility of collision the birds should be allowed to move in both directions.
2. For each bird, the total uncomfortability (penalty) should only depend on the distance from other birds.
3. For each bird we need to calculate the *left uncomfortability* and the *right uncomfortability* to understand in which direction each bird is more willing to move.

Considering these statements, let us now describe the first jump.
For each bird we consider 2 exponential random variables, one related to the *left uncomfortability* and one related to the *right uncomfortability*, all defined from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ with parameter $\lambda_1^{n,i} := \lambda_L^{n,i}(X_0)$, where $(\Omega, \mathcal{F}, \mathbb{P})$

is a probability space and the willing degree $\{\lambda_L^{n,i} : E_L \to \mathbb{R}^+\}_{1\leq n\leq N, i\in\{1,2\}}$ are defined as follows.

$$\begin{cases} \dfrac{1}{\lambda_L^{n,1}(x)} & := \dfrac{1}{L} \cdot \dfrac{\Phi_L^{n,1}(x)}{|\Phi_L^{n,1}(x) - \Phi_L^{n,2}(x)|} + \eta_L(x^n, x^{n-1}), \\ \dfrac{1}{\lambda_L^{n,2}(x)} & := \dfrac{1}{L} \cdot \dfrac{\Phi_L^{n,2}(x)}{|\Phi_L^{n,1}(x) - \Phi_L^{n,2}(x)|} + \eta_L(x^{n+1}, x^n). \end{cases}$$

where the total uncomfortable penalty function of the left and right sides $\Phi_L^{n,1}, \Phi_L^{n,2} : E_L \to \mathbb{R}^+$ are defined as follows:

$$\begin{cases} \Phi_L^{n,1}(x) & := \sum_{k=0}^{n-1} \varphi_L(x^n - x^k), \\ \Phi_L^{n,2}(x) & := \sum_{k=n+1}^{N+1} \varphi_L(x^k - x^n). \end{cases}$$

and the collision penalty function $\eta_L(x, y) := \infty \cdot \mathbb{1}_{x-y=\frac{1}{L}}$ is used to avoid the collision of birds.
We then take into consideration the random variable that attains the minimum. The bird related to that random variable will move towards the related direction. In other words, we define the position of the $n^{th}$ bird at time $t$ as follows:

$$\begin{cases} X_t^n := X_0^n & , \text{ when } t \in [0, \xi_1), \\ X_t^n := X_0^n + \frac{1}{L} \cdot (-1)^{i_1} \mathbb{1}_{n=n_1} & , \text{ when } t = \xi_1. \end{cases}$$

### 2.2 Definition of Other Jumps

Suppose that $M - 1$ jumps already happened, let us consider the $M^{th}$ jump. We know that after the $(M-1)^{th}$ jump, the positions of the $N$ birds are now $\{X_{\xi_{M-1}}^n\}_{1\leq n\leq N}$. Then, similar to what we have done for the first jump, we define $2N$ independent random variables $\{\xi_M^{n,i}\}_{1\leq n\leq N, i\in\{1,2\}}$, independent of $\{\xi_m^{n,i}\}$ for $i \in \{1, 2\}$, $1 \leq n \leq N$, and $1 \leq m \leq M - 1$, defined from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, respectively following the exponential distribution with parameter $\lambda_M^{n,i} := \lambda_L^{n,i}(X_{\xi_{M-1}})$.

Choosing the smallest one among $\{\xi_M^{n,i}\}_{1\leq n\leq N, i\in\{1,2\}}$, that is $\xi_M := \xi_M^{n_M, i_M}$, we can define:

$$\begin{cases} X_t^n := X_0^n & , \text{ when } t \in [\xi_{M-1}, \xi_M), \\ X_t^n := X_0^n + \frac{1}{L} \cdot (-1)^{i_M} \mathbb{1}_{n=n_M} & , \text{ when } t = \xi_M. \end{cases}$$

From now on, we call such a process *Birds-on-Wire* process.

### 2.3 Markov Property of Birds-on-Wire Process

To make the problem easier to discuss, we will transfer the definition above for the *Birds-on-Wire* process, which is a definition that makes sense, but not an easy-to-deal one. We now want to simplify the process and show that the *Birds-on-Wire* process is indeed a Markov jump process.

Recall that for a jump process defined on a finite state space $E_L$, we need to define a family of non-negative numbers $(\lambda(x))_{x\in E_L}$, giving the inverse of the mean sojourn time in state $x$, and a Markov transition matrix $P = (P(x, y))_{x,y\in E_L}$, so that for any $x \in E_L$ we have,

$$P(x, y) \geq 0, \sum_{y\in E_L} P(x, y), \ P(x, x) = 0.$$

**Definition 2.1 (Birds-on-Wire Process)** *We define the inverse of the mean sojourn time $\lambda$ in state $x \in E_L$ and the Markov transition matrix respectively to be,*

$$\lambda(x) := \sum_{i=1}^{2} \sum_{n=1}^{N} \lambda_L^{n,i}(x),$$

$$P(x, y) := \begin{cases} \lambda_L^{n,1}(x)/\lambda(x) & , \textit{ if } x^n - y^n = 1/L = \|x - y\|_{l_1}, \\ \lambda_L^{n,2}(x)/\lambda(x) & , \textit{ if } y^n - x^n = 1/L = \|x - y\|_{l_1}, \\ 0 & , \textit{ if } 1/L \neq \|x - y\|_{l_1}. \end{cases}$$

It is easy to prove that such a process agrees with the process defined in sections 2.1 and 2.2. Hence, it defines a Markov jump process describing the movement of birds on a wire.

# 3 Time-asymptotic Stability of Model

We want to prove mainly two theorems in this section. First, the Markov jump process defined above has a unique distribution in $\mathcal{X}^N$, as time $t$ tends to $\infty$. Moreover, this unique distribution will be more and more centered at a point in $E$ as $L$ tends to $\infty$. To prove the first result, we would take the usual way, that is, by proving Doeblin's condition to prove the existence and uniqueness of the distribution.

## 3.1 Ergodic Behavior

Recall the following result for Markov jump processes. If $E$ is a finite state space while $(X_t)_{t\geq 0}$ is a Markov jump process defined on it. We consider the case where one point of $E$ is accessible from any other point:

$$\text{(DC)} \quad \exists x_0 \in E \text{ and } t_0 > 0 \text{ such that } \forall x \in E, \mathbb{P}\left(X_{t_0} = x_0 \mid X_0 = x\right) > 0.$$

When Doeblin's condition is true, we can find a unique $\mu_S \in \mathcal{P}(E)$, and two constants $C, \lambda > 0$ such that for any $t \geq 0$ and any initial condition $X_0$,

$$\sum_{i \in E} \left| \mathbb{P}\left(X_t = i\right) - \left(\mu_S\right)_i \right| \leq C e^{-\lambda t}$$

Hence it is only left to prove the Doeblin condition for the *Birds-on-Wire* process.

**Theorem 3.1** *The Birds-on-Wire process satisfies the Doeblin condition.*

To prove this theorem, we first define the stable point in $E_L$ corresponding to $\varphi_L$ and not-too-close states in $E_L$, and their related lemmas.

**Definition 3.1** *Given function $\varphi_L$ on $E_L$, if we can find $\bar{x}_L \in E_L$ such that it satisfies the following stable condition in the discrete case:*

$$\textit{(SCD)} \quad \Phi_L^{n,1}(\bar{x}_L) = \Phi_L^{n,2}(\bar{x}_L),\ \forall 1 \leq n \leq N.$$

*we then call $\bar{x}_L$ a stable point in $E_L$ corresponding to $\varphi_L$.*

*Likely, if there is a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ and a point $\bar{x} \in E := \{x \in (0,1)^N : \bar{x}^1 < \bar{x}^2 < \cdots < \bar{x}^N\}$ such that it satisfies the following stable condition in the continuous case:*

$$\textit{(SCC)} \quad \Phi^{n,1}(\bar{x}) = \Phi^{n,2}(\bar{x}),\ \forall 1 \leq n \leq N.$$

*where*

$$\begin{cases} \Phi^{n,1}(x) & := \sum_{k=0}^{n-1} \phi(x^n - x^k), \\ \Phi^{n,2}(x) & := \sum_{k=n+1}^{N+1} \phi(x^k - x^n). \end{cases}$$

*Then we call $x$ a stable point in $E$ corresponding to $\phi$.*

**Definition 3.2** *Consider a subset of $E_L$, that is $E_L^1 := \{x \in E_L : \min_{0 \leq n \leq N}\{|x^n - x^{n+1}|\} > 1/L\}$, which means for any state in $E_L^1$, we cannot find two birds standing just beside each other. We call $E_L^1$ the not-too-close-states set and $E_L^0 := E_L \setminus E_L^1$ too-close-states set. Moreover, we call the elements of them respectively not-too-close state and too-close state. We also set $E_L^l := \{x \in E_L : \min_{0 \leq n \leq N}\{|x^n - x^{n+1}|\} > l/L\}$ for $l \geq 1$ in $\mathbb{N}$.*

Regarding the stable point, we can prove the following lemma.

**Lemma 3.1** *If $\phi$ is a decreasing bijective function on $\mathbb{R}^+$, then there exists a unique stable point $\bar{x}$ in $E$ corresponding to $\phi$ such that $\min_{n \neq m}\{|\bar{x}^n - \bar{x}^m|\} \geq \phi^{-1}(N! \cdot \phi(\frac{1}{2N}))$. Hence, for a decreasing function $\varphi_L$ on $E_L$, there exists at most a stable point $\bar{x}_L$ in $E_L$ corresponding to $\varphi_L$. Moreover, we would have that $\bar{x}_L \in E_L^1$ when $L \gg N$.*

**Proof:** We first prove the first part of the lemma. Let the function $\psi : (0,1] \to \mathbb{R}^+$ being defined as follows:

$$\psi(x) := \int_x^1 \phi(t)dt, \ \forall\, x \in (0,1].$$

It is clear that function $\psi$ is a differentiable strictly convex function on $(0,1]$ and that it tends to either $\infty$ or a positive number when $x$ tends to 0. Now we define the pseudo-potential function $\Psi : E \to \mathbb{R}^+$ as follows:

$$\Psi(x) := \sum_{n=1}^{N+1} \sum_{m=0}^{n-1} \psi(x^n - x^m)$$

We should note that the pseudo-potential function $\Psi$ is a strictly convex function on $E$ since it is a linear combination of convex functions and strictly convex functions, while $E$ is a convex and bounded open set in $\mathbb{R}^N$. Hence, $\Psi$ will guarantee a unique one minimizer $\bar{x} \in E$, while at such a point, we would find the first-order sufficient condition for a minimizer to be true, that is,

$$0 = \partial_{x^n}\Psi(\bar{x}) = -\sum_{m=0}^{n-1} \phi(\bar{x}^n - \bar{x}^m) + \sum_{m=n+1}^{N+1} \phi(\bar{x}^m - \bar{x}^n), \ \forall\, 1 \le n \le N.$$

and it agrees with the condition (SCC). By the uniqueness of the minimizer of $\Psi$ and the fact that (SCC) is a sufficient condition for the minimizer of $\Psi$, we have proved the first part of the lemma.

The fact, that there exists at most one stable point $\bar{x}_L$ in $E_L$ corresponding to $\varphi_L$, is easy to prove once we have noticed that it is always possible to construct a decreasing bijective function $\phi$ on $\mathbb{R}^+$ such that $\varphi_L(x) = \phi(x), \ \forall\, x \in \mathcal{X}_L$ for any decreasing function $\varphi_L$ on $E_L$. It is only left to prove that once $L$ is large enough, such a stable point can only be a not-too-close state.

To prove this, we can consider again the continuous situation. We can try to prove that there exists a constant $\epsilon > 0$, only related to $\phi$ and $N$, such that if there exists a stable state $\bar{x}$ corresponding to $\phi$ in $E$, then $\min_{n \neq m}\{|\bar{x}^n - \bar{x}^m|\} \ge \epsilon$. Once this is true, we can set $\phi$ to be an extension of $\varphi_L$ and then set $L > 2/\epsilon$.

Indeed, when $N = 1$, it is easy to be proved. When $N > 1$, let $\epsilon = \phi^{-1}(N! \cdot \phi(\frac{1}{2N}))$. If $\bar{x}^1 < \epsilon$ is true, since for any $1 \le n \le N$, we have

$$\sum_{k=0}^{n-1} \phi(\bar{x}^n - \bar{x}^k) = \Phi^{n,1}(\bar{x}) = \Phi^{n,2}(\bar{x}) = \sum_{k=n+1}^{N+1} \phi(\bar{x}^k - \bar{x}^n),$$

the following inequality must be true,

$$\phi(\bar{x}^{n+1} - \bar{x}^n) \ge \frac{\phi(\bar{x}^n - \bar{x}^{n-1})}{N-n+1}.$$

Hence, by induction and the fact that $\phi$ is a decreasing function, we would have for $1 \le n \le N-1$,

$$\phi(\bar{x}^{n+1} - \bar{x}^n) \ge \frac{\phi(\bar{x}^1)}{N \cdot (N-1) \cdot ... \cdot (N-n+1)} \ge \frac{\phi(\bar{x}^1)}{N!},$$

which implies that

$$\bar{x}^N = \bar{x}^N - \bar{x}^{N-1} + \bar{x}^{N-1} - \bar{x}^{N-2} + \cdots + \bar{x}^1 - \bar{x}^0 < N \cdot \phi^{-1}(N! \cdot \phi(\bar{x}^1)) \le \frac{1}{2}$$

Hence,

$$\phi(\frac{1}{2}) < \frac{N! \cdot \phi(\frac{1}{2N})}{N!} \le \phi^{-1}(N! \cdot \phi(\bar{x}^1)) \le \phi(\bar{x}^N - \bar{x}^{N-1}) \le \Phi^{N,1}(\bar{x}) = \Phi^{N,2}(\bar{x}) = \phi(1 - \bar{x}^N) \le \phi(\frac{1}{2})$$

which makes a contradiction. Hence we have $\bar{x}^1 \ge \epsilon$. Out of the symmetry of the problem, we also have $1 - \bar{x}^N \ge \epsilon$. Holding these two inequalities, we then can do the same thing for $\bar{x}^2 - \bar{x}^1$ and $\bar{x}^N - \bar{x}^{N-1}$ and by induction for all the $|\bar{x}^n - \bar{x}^m|, \ 1 \le n \neq m \le N+1$, which proves what we want. □

**Lemma 3.2** *If $\varphi_L$ does not have a stable point in $E_L$, then the Birds-on-Wire process is limited on jump points, that is $\{X_{\xi_m}\}_{m \ge 1}$, is an irreducible Markov chain, and hence by the fact that $E_L$ is finite, the Birds-on-Wire process satisfies the condition (DC) and hence the Doeblin condition. If $\varphi_L$ has a unique stable point in $E_L$, then the corresponding Birds-on-Wire process still satisfies the Doeblin condition with $x_0$ to be the stable point.*

We will prove lemma 3.1 by four steps. Firstly, we would prove that if there is no stable point in $E_L$ corresponding to $\varphi_L$ and $L \gg N$, we can not find a state $\bar{x}$ in $E_L$ such that $\mathbb{P}(X_t \neq \bar{x}|X_0 = \bar{x})$. Then, we prove that for any too-close state, it is always possible to find a path to a not-too-close state. Furthermore, it is always possible to find a path between two not-too-close states. With the first three steps, we can prove the first part of the lemma. In the fourth step, we would prove that the first three steps can be extended and hence the second part of the lemma is also true.

**Proof:**

Step 1: It is easy to see that if we can find a state $\bar{x}$ in $E_L$ such that $\mathbb{P}(X_t \neq \bar{x}|X_0 = \bar{x})$. Then we would have $\lambda_L^{n,i}(\bar{x})$ all to be zero, which can only be true when for any $1 \leq n \leq N$, we have one of the following three conditions to be true:

$$\text{(a) } \Phi_L^{n,1}(\bar{x}) = \Phi_L^{n,2}(\bar{x}) \quad \text{(b) } \eta_L(\bar{x}^n, \bar{x}^{n-1}) = \infty \quad \text{(c) } \eta_L(\bar{x}^{n+1}, \bar{x}^n) = \infty.$$

Since there is no stable state, we cannot hold (a) to be true for all $n$ at state $\bar{x}$. Hence, at least one of the $n$ would hold (b) or (c). Then to promise this bird not to move, we would have to let both (b) and (c) be true. By induction and the fact that $L \gg N$, we would know that one of the first birds and the last bird can move. Hence, we can not find a state $\bar{x}$ in $E_L$ such that $\mathbb{P}(X_t \neq \bar{x}|X_0 = \bar{x})$.

Step 2: We first define what a path is. Let us consider $x$ and $x'$, both in $E_L^1$, if $\{x_i\}_{0 \leq i \leq k+1}$ satisfies that:

$$x_0 = x,\ x_{k+1} = x',\ \|x_i - x_{i+1}\|_{l_1} \leq 1/L,\ \mathbb{P}(X_{\xi_1} = x_{i+1}|X_0 = x_i) > 0\ (\ \forall\, 0 \leq i \leq k\ ),$$

we call it a path from $x$ to $x'$. If the last inequality above is not satisfied, we call it a pseudo-path. In fact, for any too-close state $x$, we can always find a path from it to a not-too-close state. We can prove this by showing that it is always possible to find a path to decrease $BTC(x) := \#\{1 \leq n \leq N : \eta_L(x^n, x^{n-1}) + \eta_L(x^{n+1}, x^n) \neq 0\}$, that is, the total number of birds which are too close to each other. Set $l_n = (x^{n+1} - x^n) \wedge (x^n - x^{n-1})$ for $1 \leq n \leq N$, $l_0 = x^1$, and $l_{N+1} = 1 - x^N + 1$. If bird $n$ has a bird next to it, then we would have $l_n = 0$. Consider the pair $(n_0, n_1)$ in $\{1 \leq n \leq N : l_n = 0\} \times \{0 \leq n \leq N+1 : l_n \geq 2\}$ such that $|n_1 - n_0|$ reaches the minimum. Since $L \gg N$, we know this minimum is not $\infty$. When $n_0 < n_1$, for all the $n_0 < n < n_1$, we would know that $l_n = 1$. Otherwise, it contradicts the fact that $|n_1 - n_0|$ reaches the minimum. Hence, we can move all the birds $n_0 \leq n < n_1$, one by one, left to right, from $x^n$ to $x^n + 1$. It is easy to check this creates a path from $x$ to $x'$, where $BTC(x) > BTC(x')$. Similarly, when $n_0 > n_1$, we can also find such a path. In total, we can always find a path from a too-close state that reaches a not-too-close state.

Step 3: Now, let us prove that it is always possible to find a path for two not-too-close states, that is $x_1$ and $x_2$, with $\|x_1 - x_2\|_{l_1} = 1/L$. Note that this can be true when only one of the $N$ birds has a different position, which we will denote as $n_0$. If $\Phi^{n_0,1}(x_1) \neq \Phi^{n_0,2}(x_1)$, then the problem has been solved. If it's not the case, since neither $x_1$ nor $x_2$ is a stable state or a too-close state, we can find another bird allowed to move both to the left or right as $n_1$, since the left total uncomfortable penalty is not equal to the right one and there is no bird around it. Now we will construct a path from $x_1$ to $x_2$ by only moving bird $n_0$ and $n_1$. Since $\Phi^{n_1,1}(x_1) \neq \Phi^{n_1,2}(x_1)$, one should be larger than another. If $\Phi^{n_1,1}(x_1) < \Phi^{n_1,2}(x_1)$, we firstly move bird $n_1$ to the right position just next to its current position, then move bird $n_0$ to the position $x_2^{n_0}$, and finally move bird $n_1$ back to its original position. If $\Phi^{n_1,1}(x_1) > \Phi^{n_1,2}(x_1)$, we can do the same thing, but only the first step is replaced with moving bird $n_1$ to the left position. One can check in any situation, such a way to move birds is always a path from $x_1$ to $x_2$.

Now we can prove that for any two not-too-close states $x_1$ and $x_2$, we can find such a path. If it is not true, then we can find all the states in $E_L^1$ such that $x_1$ can reach, which are all contained in a set we call $F_1$, note that by the conclusion before, for any two states in $F_1$, there is a path between them. Similarly, set such a set for $x_2$ and that is $F_2$. Firstly, we know that $\min\{\|y_1 - y_2\|_{l_1} :\ \forall\, y_1 \in F_1,\ \forall\, y_2 \in F_2\} \geq 2/L$. Otherwise, there is indeed such a path which leads to a contradiction. Since we can always find a pseudo-path between two not-too-close states, we can find a point in $E_L^1 \setminus (F_1 \cup F_2)$ and set it to be $x_3$. We then define a subset in $E_L^1$ containing any state such that there is a path between itself and $x_3$, that is $F_3$. It's easy to see again that $\min_{1 \leq i < j \leq 3}\{\|y_i - y_j\|_{l_1} : y_i \in F_i,\ y_j \in F_j\} \geq 2/L$. By induction, it would be possible to construct a sequence $\{x_i\}_{i \geq 1}$ in $E_L^1$, all different from each other, which leads to a contradiction with the fact that $E_L^1$ is a finite set.

Hence, in total, we would know that it is always possible to find a path for two not-too-close states. Hence it is easy to prove the condition (DC) for any not-too-close state.

Step 4: To complete the proof of the lemma, we just need to note that the proof above can also be true if there is only a unique stable state since such a process will only stop moving when it reaches the stable state. Hence we can prove the condition (DC) for the unique stable state.

□

It is easy to see that theorem 3.1 is true by lemma 3.1. Now we know that there exists a stable distribution in $\mathcal{P}(E)$ and the distribution of $X_t$ converges to it in total variation with an exponential speed.

### 3.2 Centrally Distributed Behavior

To show that the *Birds-on-Wire* process $X_t$ can be influenced by $L$, we write $X_t$ as $X_t^L$ to stress this fact.

**Guess 3.1** *If there is a decreasing bijective function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\varphi_L(x) = \phi(x), (\ \forall\, x \in \mathcal{X}_L,\ \forall\, L \geq N+1\ )$$

*there exists a point $\bar{x}$ in $E$ such that given $L$ large enough, we have the following inequality:*

$$\mathbb{P}\left(\|X_t^L - \bar{x}\|_{l_1} \leq \omega(1/L)\right) \geq 0.95,\ \textit{as}\ t \to \infty.$$

*where $\omega$ is a modulus of continuity related to $\phi$ and $N$.*

It is hard for us to prove it rigorously. Hence we implement the code to check this idea. The Python implementation gives us some interesting numerical results that make us think that the initial Guess might be correct. The numerical final statement of the guess can be as follows:

$$\mathbb{P}\left(\|X_t^L - \bar{x}\|_{l_1} \leq 0.165 \cdot L^{0.488}\right) \geq 0.95,\ \text{as}\ t \to \infty.$$