OPTIMIZATION METHODS AND GAME THEORY

EXAMS SOLUTIONS

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EXERCISE 1

DOES A GLOBAL SOLUTION EXIST? WHY?

- **If the feasible region is closed and bounded and the objective function is continuous, then we can apply the Weierstrass theorem and conclude that a global solution exists.**
- If the feasible region is closed but not bounded, we can check if the function is continuous and coercive. If it's true, then a global solution exists. Coercive means that
$$\lim_{||x||\to\infty} f(x) = +\infty$$
- Usually, the objective function is a **quadratic function** and we will have to say if it is convex or no. To do that, we must compute the **Hessian**. If the Hessian is positive definite then we can assume that a global minimum exists (Eaves conditions are surely satisfied). Look at the second point of this exercise.

To draw the feasible region in Matlab:

```
v = -20:0.05:20;
[x1,x2] = meshgrid(v);
conditions = (x1.^2+x2.^2-2*x1-4*x2+3<=0)&(x1.^2+x2.^2+2*x1-4*x2+3<=0);
cond = zeros(length(v));
cond(conditions) = NaN;
surf(x1, x2, cond)
```

```
view(0,90)
```

The “v” variable allows to specify the range of values in which we are going to draw the feasible region. The “conditions” variable allows to specify the list of constraint functions.

To draw the feasible region manually:

- If the constraint function is
$$x^2 + y^2 + ax + by + c = 0$$
it is a circumference. So, the centre is
$$C = (-\frac{a}{2}, -\frac{b}{2})$$
and the radius is
$$r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - c}$$
- If the constraint function is
$$\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} = 1$$
it is an ellipse. So, the centre is
$$C = (x_c, y_c)$$
and the vertices are
$$V_1 = (x_c - a, y_c)\ V_2 = (x_c + a, y_c)\ V_3 = (x_c, y_c - b)\ V_4 = (x_c, y_c + b)$$
- To recognize a quadratic function defined as
$$ax^2 + bxy + cy^2 + 2dx + 2ey + f = 0$$
we can look at $b^2 - 4ac$. If it is 0, the function is a parabola. If it is greater than 0, then it is a hyperbola. If it is lower than 0, with b≠0 and/or a≠c, then it is an ellipse. Finally, if a=c and b=0, then it can be a circumference.

IS IT A CONVEX PROBLEM? WHY?

- If the objective function is a **quadratic function**, then it needs a positive semidefinite or positive definite **Hessian** to be convex. A quadratic function is defined as
$$\frac{1}{2}x^TQx + c^Tx$$
and the Hessian is Q. To check if it is positive semidefinite, we must compute the **eigenvalues** of Q. If they are both positive, then it is positive definite. If one is positive and the other one is zero, then it is positive semidefinite. To compute the eigenvalues, given
$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
we have to solve $(a - h)(b - h) - bc = 0$. The results h_1 and h_2 are the eigenvalues. If the Hessian is positive definite then the quadratic function is also strongly convex.
- If the objective function is a **linear function**, it is convex but not strongly convex.

To specify the quadratic function in Matlab and find the eigenvalues, suppose a function

$$f(x) = \frac{1}{2}x_1^2 + x_2^2 - 2x_1x_2 - 2x_1 - 2x_2$$

then, it is defined by

```
Q = [1 -1; -1 2];
c = [-2; -6];
eigenvalues = eig(Q)
```

DOES THE ABADIE CONSTRAINTS QUALIFICATION HOLD IN ANY FEASIBLE POINT?

Consider a problem in the form

$$\min f(x)$$
$$g(x) \leq 0$$
$$h(x) = 0$$

- If all the constraint functions are **linear**, then the Abadie constraints qualification holds at any feasible point.
- **Slater condition**: if the g(x) constraints are **convex**, the h(x) constraints are **linear** and there exists a point $\bar{x}$ such that $g(\bar{x}) < 0$ (so, there exists an **internal point** in which the g(x) constraints are not active and the h(x) constraints are active), then the Abadie constraints qualification holds at any feasible point.
- If all **the gradients of the active constraints** (a constraint function $g_i(x)$ is active in $\bar{x}$ if $g_i(\bar{x}) = 0$ and the h(x) constraints are always active) in each point of the feasible region are **linear independent**, then the Abadie constraints qualification holds at any feasible point. When we have combinations of linear and quadratic functions, this condition becomes a problem in the points in which multiple constraints are active.

IS THE POINT (A; B) A LOCAL MINIMUM? WHY?

To check if the point (a;b) is a local minimum, we have to use the **KKT system**, defined as

$$\begin{cases} \nabla f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(\bar{x}) \\ \bar{\lambda}_i g_i(\bar{x}) = 0 \quad \forall i = 1 \dots m \\ \bar{\lambda} \geq 0 \\ g(\bar{x}) \leq 0 \\ h(\bar{x}) = 0 \end{cases}$$

If the point is a solution of the KKT system, then it is a local minimum.

Notice that in the KKT system the first line produces n equations and the second line produces m equations.

FIND ALL THE SOLUTIONS OF THE KKT SYSTEM

The **KKT system** is the one defined in the previous point. We don't know a priori how many solutions are available because it is a non-linear system. A solution is given by $\bar{x}$, usually two variables, $\bar{\lambda}$, m variables, and $\bar{\mu}$, p variables.

FIND ALL LOCAL MINIMA AND GLOBAL MINIMA

To decide if a solution of the **KKT system** is a global minimum, we have to insert that solution in the **objective function**. When we have all the values of the objective function for each solution of the KKT system, and we are sure that there is a global minimum, then we can say that the lowest value/values is/are the global minimum/minima.

If the objective function is convex, then we have that each solution of the KKT system is a local minimum. Otherwise, we need the critical cone and restrictions method. First of all, we need the **Lagrangian function**, defined as

$$L(x, \lambda, \mu) = \mathrm{f(x)} + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

and we need to find its **Hessian** (usually, in these exercises, the Lagrangian is a quadratic form, so the Hessian is Q). Then, for each solution of the KKT system, we need to compute the **critical cone**, defined as

$$C(\bar{x}, \bar{\lambda}, \bar{\mu}) = \left\{ d \in R^n : \begin{matrix} d^T \nabla g_i(\bar{x}) = 0 \quad \forall i \in Active(\bar{x}) \quad with\, \lambda_i > 0 \\ d^T \nabla g_i(\bar{x}) \leq 0 \quad \forall i \in Active(\bar{x}) \quad with\, \lambda_i = 0 \\ d^T \nabla h_j(\bar{x}) = 0 \quad \forall j = 1..p \end{matrix} \right\}$$

So, the critical cone is given by the **direction d** (in two variables it is a generic vector (d_1, d_2)) that can assume some values specified by the given formula applied on the **active constraints** in the point $\bar{x}$ of that KKT solution. After that, we have to check

$$d^T \nabla_{xx} L(x, \lambda, \mu) d \quad \forall d \in C(\bar{x}, \bar{\lambda}, \bar{\mu}), d \neq 0$$

If this value is positive, then we can assume that this solution of the KKT system is a local minimum. If this value is negative, it is not a local minimum. If the value is zero or the direction d can only be zero, we need to proceed with the **restrictions**. For each active constraint in that solution of the KKT system, we find the function that connects x_1 and x_2 on this constraint and then we rewrite the objective function only in x_1 or x_2: that will give us a function in a single variable that it's easy to draw. Then, considering the interval given by the other constraints, we look on that function what this point is: a local minimum or a local maximum. If for all the restrictions this point is a local minimum, then we can conclude that this solution of the KKT system is a local minimum. Otherwise, if there is almost a restriction in which this point is a local maximum, then we can conclude that this solution of the KKT system is not a local minimum.

To draw the restriction in Matlab:

```
x1 = [0:5:100];
x2 = x1.^2+5;
plot(x1, x2)
```

To solve the quadratic optimization problem in Matlab with linear constraints and to check if the global minimum and its Lagrange multipliers (λ) are correct:

```
[x,fval,exitflag,output,lambda] = quadprog(Q,c,A,b,Aeq,beq,lb,ub);
```

The inputs are, respectively, the matrix Q and the vector c that define the quadratic objective function, the matrix A and the vector b that define the inequality constraints, the matrix Aeq and the vector beq that define the equality constraints, the vector lb that defines the lower bounds and the vector ub that defines the upper bounds. The outputs are, respectively, the global minimum, the objective function computed on the global minimum, a flag that indicates if there is solution, information about the optimization process and the Lagrange multipliers computed in the global minimum.

FIND THE OBJECTIVE FUNCTION AND CONSTRAINTS OF THE LAGRANGIAN DUAL PROBLEM

The **Lagrangian function** is defined as

$$L(x, \lambda, \mu) = \mathrm{f(x)} + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

The **Lagrangian relaxation** is defined as

$$\min_{x \in D} L(x, \lambda, \mu) \quad with\, \lambda \geq 0$$

The first thing to do is to **find the Lagrangian function** and then to **minimize it on x**. It means that we can isolate the terms that don't contain x. If we can isolate the terms that contain only, for example, x_1, then we can work on them in single variable. So, minimizing it means that we put its first derivative equal to zero and so on. The solution is, probably, in function of λ.
Then, we **replace x with the value of the solution in the Lagrangian**. This is what we are going to call $\varphi(\lambda, \mu)$.

The **Lagrangian dual function** is defined as

$$\varphi(\lambda, \mu) = \inf_{x \in D} L(x, \lambda, \mu) \quad with\, \lambda \geq 0$$

The **Lagrangian dual problem** is defined as

$$\begin{cases} \max \varphi(\lambda, \mu) \\ \lambda \geq 0 \end{cases}$$

IS (A, B) AN OPTIMAL SOLUTION OF THE LAGRANGIAN DUAL PROBLEM? WHY?

The best thing to do here is to find the **value of λ when (x_1, x_2) is equal to (a,b)**, look at the value of $\varphi(\lambda, \mu)$ with that λ and evaluate the same function with the optimal solution of the primal problem. If the obtained values are the same, then (a,b) is an optimal solution of the Lagrangian dual problem. Otherwise, it is not.

SOLVE THE LAGRANGIAN DUAL PROBLEM. IS THE STRONG DUALITY PROPERTY SATISFIED?

We know that the optimal value of the dual problem is always lower or equal to the optimal value of the primal problem, that can be written as

$$v(D) \leq v(P)$$

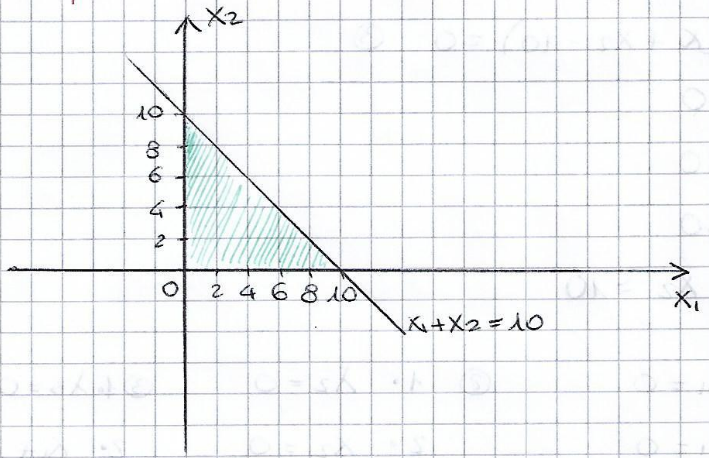
So, we have to evaluate $\varphi(\lambda, \mu)$ in the optimal solution of the primal problem (we know $\bar{\lambda}$ and $\bar{\mu}$). If v(D) is equal to v(P), then we can conclude that there is strong duality. Otherwise, we can conclude that there is weak duality.

FAC-SIMILE 2019 - EXERCISE 1

$$\begin{cases} \min -x_1^2 - x_2^2 + 12x_1 + 4x_2 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_1 + x_2 \leq 10 \end{cases}$$

a) Do global optimal solutions exist? Why?

(Feasible region):



- CLOSED and BOUNDED feasible region
- CONTINUOUS objective function

⇒ A global optimal solution exists for the WEIERSTRASS THEOREM.

b) Is it a convex problem? Why?

The objective function can be written as

$$\frac{1}{2} x^T Q x + c^T x = \frac{1}{2}(x_1, x_2)\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (12, 4)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The Hessian is $Q = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

The eigenvalues are given by

$$\begin{pmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{pmatrix} \Rightarrow (-2-\lambda)(-2-\lambda) = 0 \Rightarrow \begin{array}{l} \lambda_1 = -2 \\ \lambda_2 = -2 \end{array}$$

⇒ The Hessian is NEGATIVE DEFINITE. The objective function is NOT CONVEX.

c) Does the Abadie constraints qualification hold in any feasible point?

Yes, because all the constraints are linear.

d) Is the point (4, 4) a local minimum? Why?

KKT system:

$$\begin{cases} -2x_1 + 12 - \lambda_1 + \lambda_3 = 0 \\ -2x_2 + 4 - \lambda_2 + \lambda_3 = 0 \\ -\lambda_1 x_1 = 0 & ① \\ -\lambda_2 x_2 = 0 & ② \\ \lambda_3 (x_1 + x_2 - 10) = 0 & ③ \\ \lambda \geq 0 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_1 + x_2 \leq 10 \end{cases}$$

① 1. $\lambda_1 = 0$ ② 1. $\lambda_2 = 0$ ③ 1. $\lambda_3 = 0$

2. $x_1 = 0$ 2. $x_2 = 0$ 2. $x_1 + x_2 - 10 = 0$

1.1.1. $\lambda = (0, 0, 0)$

$$\begin{cases} -2x_1 + 12 = 0 \\ -2x_2 + 4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 6 \\ x_2 = 2 \end{cases}$$

1.1.2. $\lambda_1 = 0, \ \lambda_2 = 0, \ x_1 + x_2 - 10 = 0$

$$\begin{cases} -2x_1 + 12 + \lambda_3 = 0 \\ -2x_2 + 4 + \lambda_3 = 0 \\ x_1 + x_2 - 10 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 7 \\ x_2 = 3 \\ \lambda_3 = 2 \end{cases}$$

1.2.1. $\lambda_1 = 0, \ x_2 = 0, \ \lambda_3 = 0$

$$\begin{cases} -2x_1 + 12 = 0 \\ 4 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 6 \\ \lambda_2 = 4 \end{cases}$$

1.2.2. $\lambda_1 = 0, \ x_2 = 0, \ x_1 + x_2 - 10 = 0$

$\hookrightarrow x_1 = 10$

$$\begin{cases} 4 - \lambda_2 + \lambda_3 = 0 \\ -20 + 12 + \lambda_3 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_2 = 12 \\ \lambda_3 = 8 \end{cases}$$

2.1.1 $x_1=0,\ \lambda_2=0,\ \lambda_3=0$

$$\begin{cases} 12-\lambda_1=0 \\ -2x_2+4=0 \end{cases} \Rightarrow \begin{cases} \lambda_1=12 \\ x_2=2 \end{cases}$$

2.1.2. $x_1=0,\ \lambda_2=0,\ x_1+x_2-10=0$

$\hookrightarrow x_2=10$

$$\begin{cases} 12-\lambda_1+\lambda_3=0 \\ -20+4+\lambda_3=0 \end{cases} \Rightarrow \begin{cases} \lambda_1=28 \\ \lambda_3=16 \end{cases}$$

2.2.1. $x_1=0,\ x_2=0,\ \lambda_3=0$

$$\begin{cases} 12-\lambda_1=0 \\ 4-\lambda_2=0 \end{cases} \Rightarrow \begin{cases} \lambda_1=12 \\ \lambda_2=4 \end{cases}$$

2.2.2. $x_1=0,\ x_2=0,\ x_1+x_2-10=0$

$\hookrightarrow -10=0$ IMPOSSIBLE, this solution is not suitable

So, (4,4) is not a solution of the KKT system.

e) Find all the solutions of the KKT system

The solutions are the ones computed in the point d. Let's compute the objective function in each of them.

1.1.1. $x=(6,2)\ \lambda=(0,0,0)$ $f(x)=-36-4+72+8=$
$=40$

1.1.2. $x=(7,3)\ \lambda=(0,0,2)$ $f(x)=-49-9+84+12=38$

1.2.1 $x=(6,0)\ \lambda=(0,4,0)$ $f(x)=-36+72=36$

1.2.2. $x=(10,0)\ \lambda=(0,12,8)$ $f(x)=-100+120=20$

2.1.1. $x=(0,2)\ \lambda=(12,0,0)$ $f(x)=-4+8=4$

2.1.2. $x=(0,10)\ \lambda=(28,0,16)$ $f(x)=-100+40=-60$

2.2.1 $x=(0,0)\ \lambda=(12,4,0)$ $f(x)=0$

f) Find all the local minima and the global minima

The global minimum is $x=(0,10)$ because it has the lowest value of the objective function in the KKT solutions.

To determine if a solution of the KKT system is a local minima, we need the Lagrangian function.

$$L(x,\lambda,\mu)=-x_1^2-x_2^2+12x_1+4x_2+\lambda_1x_1+\lambda_2x_2+\lambda_3(x_1+x_2-10)$$

$$\nabla_{xx}L(x,\lambda,\mu)=\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}$$

$$d^T \nabla_{xx} L(x,\lambda,\mu)\, d = (d_1, d_2)\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}d_1\\ d_2\end{pmatrix} = -2d_1^2 - 2d_2^2$$

Now we have to evaluate it for each critical cone computed on the KKT solutions.
Notice that $-2d_1^2 - 2d_2^2$ can't be positive. So this method can tell us which solution are surely not a local minimum, but it can't tell us which solutions are local minimum.

- $x=(6,2)$ $\lambda=(0,0,0)$

There are no active constraints, so (d_1, d_2) hasn't constraints.

$$\Rightarrow -2d_1^2 - 2d_2^2 < 0 \quad \forall d \neq 0$$

- $x=(7,3)$ $\lambda=(0,0,2)$

The third constraint is active.

$$(d_1, d_2)\begin{pmatrix}1\\1\end{pmatrix} = 0 \Rightarrow d_1 + d_2 = 0 \Rightarrow d_1 = -d_2$$

$$\Rightarrow -2d_1^2 - 2d_2^2 = -4d_1^2 < 0 \quad \forall d \neq 0$$

- $x=(6,0)$ $\lambda=(0,4,0)$

The second constraint is active.

$$(d_1, d_2)\begin{pmatrix}0\\-1\end{pmatrix} = 0 \Rightarrow -d_2 = 0$$

$$\Rightarrow -2d_1^2 - 2d_2^2 = -2d_1^2 < 0 \quad \forall d \neq 0$$

- $x=(10,0)$ $\lambda=(0,12,8)$

The second and the third constraints are active.

$$\begin{cases}(d_1, d_2)\begin{pmatrix}0\\-1\end{pmatrix} = 0\\ (d_1, d_2)\begin{pmatrix}1\\1\end{pmatrix} = 0\end{cases} \Rightarrow \begin{cases}-d_2 = 0\\ d_1 = -d_2\end{cases} \Rightarrow d_1 = d_2 = 0$$

$\Rightarrow -2d_1^2 - 2d_2^2 = 0$ This point can be a local minimum.

- $x=(0,2)$ $\lambda=(12,0,0)$

The first constraint is active.

$$(d_1, d_2)\begin{pmatrix}-1\\0\end{pmatrix} = 0 \Rightarrow -d_1 = 0$$

$$\Rightarrow -2d_1^2 - 2d_2^2 = -2d_2^2 < 0 \quad \forall d \neq 0$$

- $x=(0,10)$ $\lambda=(28,0,16)$

We know that this is the global optimum so it's useless to evaluate.

- $x=(0,0)$ $\lambda=(12,4,0)$

The first and the second constraints are active.

$$\begin{cases} (d_1, d_2)\begin{pmatrix}-1\\0\end{pmatrix}=0 \Rightarrow -d_1=0 \\ (d_1, d_2)\begin{pmatrix}0\\-1\end{pmatrix}=0 \Rightarrow -d_2=0 \end{cases}$$

$\Rightarrow \underbrace{-2d_1^2-2d_2^2}=0$ This point can be a local minimum.

↳ This can't be considered for $d\neq0$ because we have only $d=0$ as possible value. The same is happened for $x=(10,0)$.

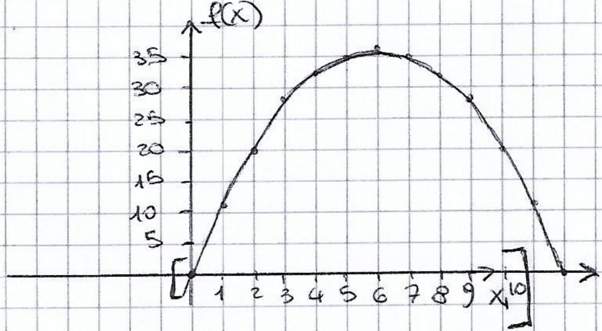
~~Now we can only proceed empirically~~

Now we can proceed using the restrictions.

- $x=(10,0)$

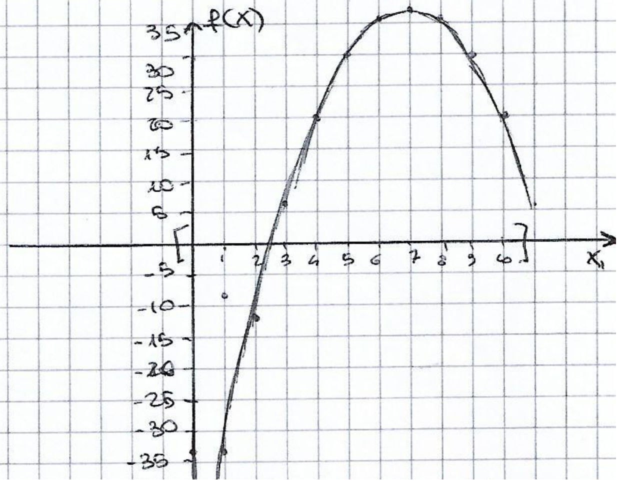
$$\begin{cases} -x_1^2-x_2^2+12x_1+4x_2\Big|_{x_2=0} = -x_1^2+12x_1 \\ x_1\geq 0 \\ x_1\leq 10 \end{cases}$$

$\Rightarrow$ Looking toward the second constraint, it is a local minimum.



$$\begin{cases} -x_1^2-x_2^2+12x_1+4x_2\Big|_{x_1+x_2=10} = \\ = -x_1^2-(-x_1+10)^2+12x_1+4(-x_1+10) = -x_1^2-x_1^2-100+20x_1+12x_1-4x_1+40 \\ = -2x_1^2+28x_1-60 \\ x_1\geq 0 \\ x_2=-x_1+10\geq 0 \Rightarrow x_1\leq 10 \end{cases}$$

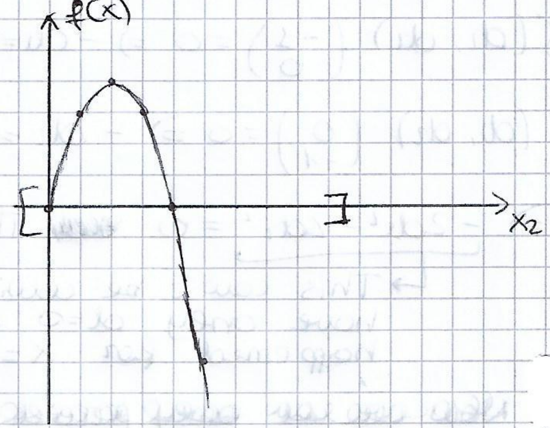
$\Rightarrow$ Also looking toward the third constraint, it is a local minimum.
So, $x=(10,0)$ is a LOCAL MINIMUM.



• $X = (0,0)$

The restriction ~~along~~ on the ~~first~~ second constraint is the same that we've seen for $X = (10,0)$ and $X = (0,0)$ is a local minimum on it.

$$\left\{\begin{array}{l} -x_1^2 - x_2^2 + 12x_1 + 4x_2 \Big|_{x_1} = -x_2^2 + 4x_2 \\ x_2 \geq 0 \\ x_2 \leq 10 \end{array}\right.$$



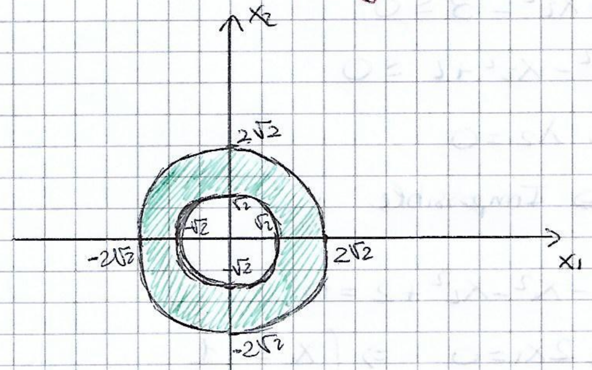
⇒ looking toward the first ~~restriction~~ constraint it is a local minimum.
So, $X = (0,0)$ is a LOCAL MINIMUM.

10/06/2019 – EXERCISE 1

$$\begin{cases} \min \; x_1 - x_2 \\ x_1^2 + x_2^2 - 8 \leq 0 \\ -x_1^2 - x_2^2 + 2 \leq 0 \end{cases}$$

a) Do a global solution exist? Why?

Feasible region:



- The feasible region is closed and bounded
- The objective function is continuous

⇒ According to the Weierstrass theorem, there exists a global minimum.

b) Is it a convex problem? Why?

The linear function are always convex, g_1 is convex but g_2 is not convex, so it is not a CONVEX PROBLEM. We can see that from the Hessians

$$H_{g_1}(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \text{POSITIVE DEFINITE} \Rightarrow \text{CONVEX} \quad (\lambda_{1,2} = 2)$$

$$H_{g_2}(x) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \text{NEGATIVE DEFINITE} \Rightarrow \text{CONCAVE} \quad (\lambda_{1,2} = -2)$$

c) Does the Abadie constraints qualification hold in any feasible point?

We need to use the third condition. We have active constraints only on the circumferences and there aren't multiple active constraints.

$$\nabla g_1(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \text{ on } g_1(x) = 0$$

$$\nabla g_2(x) = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix} \text{ on } g_2(x) = 0$$

They are always linear independent, so Abadie constraints qualification holds in any feasible point.

d) Find all the solutions of the KKT system

$$\begin{cases} 1+\lambda_1 2x_1 - \lambda_2 2x_1 = 0 \\ -1+\lambda_1 2x_2 - \lambda_2 2x_2 = 0 \\ \lambda_1 (x_1^2+x_2^2-8) = 0 & (1) \\ \lambda_2 (-x_1^2-x_2^2+2) = 0 & (2) \\ \lambda \geq 0 \\ x_1^2+x_2^2-8 \leq 0 \\ -x_1^2-x_2^2+2 \leq 0 \end{cases}$$

1.1 $\lambda_1 = 0$, $\lambda_2 = 0$

$$\begin{cases} 1=0 \\ -1=0 \end{cases} \Rightarrow \text{Impossible}$$

1.2 $\lambda_1 = 0$, $-x_1^2-x_2^2+2=0$

$$\begin{cases} 1-\lambda_2 2x_1 = 0 \\ -1-\lambda_2 2x_2 = 0 \\ -x_1^2-x_2^2+2=0 \end{cases} \Rightarrow \begin{cases} x_1 = \dfrac{1}{2\lambda_2} \\ x_2 = -\dfrac{1}{2\lambda_2} \\ -\dfrac{1}{4\lambda_2^2} - \dfrac{1}{4\lambda_2^2} + 2 = 0 \end{cases}$$

$$\Rightarrow \frac{1}{2\lambda_2^2} = 2 \Rightarrow \lambda_2^2 = \frac{1}{4} \Rightarrow \lambda_2 = \pm\frac{1}{2}$$

If $\lambda_2 = \frac{1}{2}$, then $x_1 = 1$ and $x_2 = -1$

The solution with $\lambda_2 = -\frac{1}{2}$ is not good.

2.1 $x_1^2+x_2^2=8$, $\lambda_2 = 0$

$$\begin{cases} 1+\lambda_1 2x_1 = 0 \\ -1+\lambda_1 2x_2 = 0 \\ x_1^2+x_2^2 = 8 \end{cases} \Rightarrow \quad x_1 = -\frac{1}{2\lambda_1} \quad x_2 = \frac{1}{2\lambda_1}$$

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda_1^2} = 8 \Rightarrow \lambda_1^2 = \frac{1}{16} \Rightarrow \lambda_1 = \pm\frac{1}{4}$$

If $\lambda_1 = \frac{1}{4}$, then $x_1 = -2$ and $x_2 = 2$

The solution with $\lambda_1 = -\frac{1}{4}$ is not good.

2.2 $x_1^2+x_2^2=8$, $x_1^2+x_2^2=2$

This is impossible.

So, the solutions of the KKT system are

$$x=(1,-1) \quad \lambda=(0,\tfrac{1}{2}) \rightarrow f(x)=2$$

$$x=(-2,2) \quad \lambda=(\tfrac{1}{4},0) \rightarrow f(x)=-4$$

e) Find all local minima and global minima

The global minimum is $x=(-2,2)$, because we know that a global minimum exists and that it is a solution of the KKT.

$$L(x,\lambda,\mu) = f(x)+\lambda_1 g_1(x)+\lambda_2 g_2(x) =$$

$$= x_1-x_2+\lambda_1(x_1^2+x_2^2-8)+\lambda_2(-x_1^2-x_2^2+2) =$$

$$= (\lambda_1-\lambda_2)x_1^2+(\lambda_1-\lambda_2)x_2^2+x_1-x_2-8\lambda_1+2\lambda_2$$

$$\nabla_{xx}L(x,\lambda,\mu) = \begin{pmatrix} \lambda_1-\lambda_2 & 0 \\ 0 & \lambda_1-\lambda_2 \end{pmatrix}$$

We need to evaluate

$$d^T \nabla_{xx} L(x,\lambda,\mu)\, d = (\lambda_1-\lambda_2)d_1^2+(\lambda_1-\lambda_2)d_2^2$$

on the solution $x=(1,-1)$, $\lambda=(0,\frac{1}{2})$ of the KKT.

$$C(x,\lambda,\mu) = \{ d\in\mathbb{R}^m : -d_1\cdot 2\bar{x}_1 - d_2\cdot 2\bar{x}_2 = -2d_1+2d_2=0 \}$$

$$\Rightarrow 2d_1=2d_2 \Rightarrow d_1=d_2$$

$$d^T\nabla_{xx}L(\bar{x},\lambda)\, d = -\frac{1}{2}d_1-\frac{1}{2}d_2 = -d_1$$

$-d_1$ is positive only for $d_1<0$, so this is not a local minimum.

f) Find the objective function and constraints of the Lagrangian dual problem

$$L(x,\lambda,\mu) = (\lambda_1-\lambda_2)x_1^2+(\lambda_1-\lambda_2)x_2^2+x_1-x_2-8\lambda_1+2\lambda_2$$

$$\min_x L(x,\lambda,\mu) = \min_x \left[(\lambda_1-\lambda_2)x_1^2+(\lambda_1-\lambda_2)x_2^2+x_1-x_2-8\lambda_1+2\lambda_2\right]$$

$$= 2\lambda_2-8\lambda_1+\min_{x_1}\left[(\lambda_1-\lambda_2)x_1^2+x_1\right]+\min_{x_2}\left[(\lambda_1-\lambda_2)x_2^2-x_2\right]$$

$$x_1: \begin{cases} (\lambda_1-\lambda_2)2x_1+1=0 \\ (\lambda_1-\lambda_2)>0 \end{cases} \Rightarrow \begin{cases} x_1=-\frac{1}{2(\lambda_1-\lambda_2)} \\ (\lambda_1-\lambda_2)>0 \end{cases}$$

$$x_2: \begin{cases} (\lambda_1-\lambda_2)2x_2-1=0 \\ (\lambda_1-\lambda_2)>0 \end{cases} \Rightarrow \begin{cases} x_2=\frac{1}{2(\lambda_1-\lambda_2)} \\ (\lambda_1-\lambda_2)>0 \end{cases}$$

$$\varphi(\lambda) = (\lambda_1-\lambda_2)\cdot\frac{1}{4(\lambda_1-\lambda_2)^2} + (\lambda_1-\lambda_2)\frac{1}{4(\lambda_1-\lambda_2)^2} - \frac{1}{2(\lambda_1-\lambda_2)} - \frac{1}{2(\lambda_1-\lambda_2)} -$$

$$- 8\lambda_1 + 2\lambda_2$$

$$= -\frac{1}{2(\lambda_1-\lambda_2)} - 8\lambda_1 + 2\lambda_2$$

Remember
$\lambda_1 - \lambda_2 > 0$

So, the lagrangian dual problem is

$$\begin{cases} \max\ \varphi(\lambda) = -\frac{1}{2(\lambda_1-\lambda_2)} - 8\lambda_1 + 2\lambda_2 \\ \lambda_1 - \lambda_2 > 0 \\ \lambda \geq 0 \end{cases}$$

Evaluate $\varphi(\lambda)$ in the optimal solution of the primal, so in $\lambda = (\frac{1}{4}, 0)$.

$$\varphi(\bar{\lambda}) = -\frac{1}{8\cdot\frac{1}{4}} - 8\cdot\frac{1}{4} = -4$$

This is the same value that we found in the primal.

$\Downarrow$

STRONG DUALITY