

# ALGEBRAIC GEOMETRY 1

## PROBLEM SET 10

**Keywords:** projectivities, linear subspaces, conics, and Hilbert polynomials

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**Problem 1** (3 pt)

Linear subspaces and projectivities.

1. Let  $r$  and  $s$  be disjoint lines in  $\mathbb{P}^3(k)$ . For every point  $p \in \mathbb{P}^3(k) \setminus (r \cup s)$ , there exists a unique line  $l_p$  passing through  $p$  and intersecting both  $r$  and  $s$ .
2. Let  $\phi$  be a projectivity of  $\mathbb{P}^n(k)$ . Show that the locus of fixed points of  $\phi$  is a union of linear subspaces  $L_1, \dots, L_r$ , such that  $L_i$  does not meet the linear span of  $(L_j)_{j \neq i}$ .
3. Suppose that the fixed locus of  $\phi \in PGL_4(k)$  consists of two lines  $r$  and  $s$ . For every point  $p \in \mathbb{P}^3(k) \setminus (r \cup s)$ , the line joining  $p$  to  $f(p)$  meets both  $r$  and  $s$ .

**Problem 2** (2 pt)

Let  $r$  be a line in  $\mathbb{P}^2(\mathbb{C})$ . Show that  $r$  contains either one or infinitely many real points. (Hint: use conjugation.)

**Problem 3** (3 pt)

Let  $k$  be algebraically closed. Let  $p_1, \dots, p_5$  be distinct points of  $\mathbb{P}^2(k)$ .

1. Show that there exists at least a conic  $C$  passing through  $p_1, \dots, p_5$ .
2.  $C$  is non-degenerate (reduced and irreducible) if and only if any three of the points are not collinear. Moreover, in this case  $C$  is unique.
3. If  $C$  is degenerate, then it is unique if and only if any four of the points are not collinear.

**Problem 4** (3 pt)

Compute the degrees of the Segre and Veronese embeddings.