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Algebraic geometry 1

problem set 8

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Problem 1 *(5 pts)*

Automorphisms of $\mathbb{P}^n(k)$ induced by linear automorphisms of k^{n+1} are called *projectivities*.

1. Show that every automorphism of $\mathbb{P}^1(k)$ is a *projectivity*, which can be written as
$$z \mapsto \frac{az+b}{cz+d}.$$
2. Compute the stabiliser $\mathrm{Stab}_G([1:0])$ in $G = PGL_2(k)$ of the point $[1:0] \in \mathbb{P}^1(k)$, and $\mathrm{Stab}_G([1:0]) \cap \mathrm{Stab}_G([0:1])$.
3. Show that the action of $PGL_2(k)$ on $\mathbb{P}^1(k)$ is sharply 3-transitive (for every two triples of distinct points there is exactly one projectivity moving one to the other).
4. The *cross-ratio* of four distinct points (z_1, z_2, z_3, z_4) of $\mathbb{P}^1(k) \setminus \{\infty\}$ is defined as:
$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in k.$$
Extend this definition in case one of the points is ∞. Show that projectivities preserve the cross-ratio. Conclude that the action of $PGL_2(k)$ on $\mathbb{P}_1(k)$ is not 4-transitive in general. (Bonus: what are the possible values of the cross-ratio? What happens in the limit when two points come together (over $\mathbb{C}$)?)

Problem 2 *(4 pts)*

More on projectivities.

1. Show that the action of $PGL_{n+1}(k)$ on $\mathbb{P}^n(k)$ is 2-transitive, but not 3-transitive in general.
2. An $(n+2)$-tuple of points in $\mathbb{P}^n$ is said to be *in general (linear) position* if no $n+1$ of them are contained in a hyperplane. Show that $PGL_{n+1}(k)$ acts transitively on the set of $(n+2)$-tuples in general position.
3. Let $U = D_p(f) \subset \mathbb{P}^n(k)$ be the complement of a hypersurface of degree d. Identify U with an affine algebraic variety.
4. Show that $PGL_{n+1}(k)$ is an affine algebraic variety.

Problem 3 *(5 pts)*

1. Let X be an irreducible projective variety of dimension n. Show that the affine cone C_X is irreducible of dimension $n+1$.

2. Let X and Y be projective subvarieties of $\mathbb{P}^n(k)$ such that $\dim(X) + \dim(Y) \geq n$. Then the intersection of X and Y is not empty.
3. Show that $\mathbb{P}^m \times \mathbb{P}^n$ is not isomorphic to $\mathbb{P}^{m+n}$ (unless m or n is 0).
4. Show that there is no morphism $\mathbb{P}^N \to \mathbb{P}^n$ for $N > n$ (unless $n = 0$).

Problem 4 *(2 pts)*

Show that the diagonal of a scheme is always a locally closed immersion (closed in an open).