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Algebraic geometry 1

problem set 11

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Problem 1

Consider the conic $C = V_+(x_0x_1 - x_2^2) \subseteq \mathbb{P}^2$ and the lines $L_1 = V_+(x_1),\ L_2 = V_+(x_0 - x_1)$. Compute the intersection of C with $L_i,\ i = 1, 2$ (with multiplicities).

Problem 2

An irreducible curve of degree d in $\mathbb{P}^n$ is contained in a linear subspace of dimension at most d.

Problem 3

A dimension 0 Noetherian local ring is regular if and only if it is a field.
A dimension 1 Noetherian local ring is regular if and only if it is a PID. (You can take for granted this version of the Artin-Rees lemma: if $(A, \mathfrak{m})$ is a local Noetherian ring, then $\bigcap_{i=0}^{\infty} \mathfrak{m}^i = 0$.)

Problem 4

The corank of the Jacobian matrix does not depend on the presentation.
Let $R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ and $\underline{a} \in k^n$ such that $f_i(\underline{a}) = 0$ for all $i = 1, \ldots, r$.
Show that the corank of $\mathrm{Jac}(F)(\underline{a})$ is unaffected by:

1. adding to $(f_1, \ldots, f_r)$ another polynomial from the ideal they generate;
2. adding to the generators of R a new variable x_{n+1}, and to the ideal a new relation of the form $x_{n+1} - g(x_1, \ldots, x_n)$.

Problem 5

Projective Jacobian criterion.

1. Let $f \in k[x_0, \ldots, x_d]$ be a homogeneous polynomial of degree d. Then $\sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f$.
2. Let $\underline{a} \in X \subseteq \mathbb{P}^n(k)$, with $I(X) = \langle f_1, \ldots, f_r \rangle$. Prove that X is smooth at $\underline{a}$ if and only if the rank of the $r \times (n+1)$ matrix $\mathrm{Jac}(F)(\underline{a}) = \left(\frac{\partial f_i}{\partial x_j}(\underline{a}) \right)$ is at least $n - \dim_a X$ (note that the matrix is only well-defined up to scalar, which does not affect the rank).
3. Let $X \subseteq \mathbb{P}^3(k)$ be the twisted cubic $V_+(x_0x_2 - x_1^3, x_0x_3 - x_1x_2, x_1x_3 - x_2^2)$. Use the previous point to show that X is smooth.