Problem Set 2

Problem 1: "$\Rightarrow$" Let $\emptyset \neq A \subseteq X$ be an irreducible subspace and assume A is not a singleton. Since X is a Hausdorff space, so A is it too. Let $x \neq y \in A$ and $x \in U$, $y \in V$ open subsets of A with $U \cap V = \emptyset$, such subsets exist since A is a Hausdorff space. It follows, that $A = A \setminus U \cap V = A \setminus U \cup A \setminus V$ $\lightning$, because A is irreducible.

"$\Leftarrow$" Assume $A \subseteq X$ is a singleton and $U \neq V$ closed subsets of A with $U \cup V = A$. Since A is a singleton, we get $U = A$ and $V = \emptyset$ or $U = \emptyset$ and $V = A$. It follows, that A is irreducible.

Problem 2: We assume, that X is reducible. Therefore we have closed subsets $U, V \subseteq X$ with $U \cup V = X$. Since f is open, we get $f(X \setminus U)$, $f(X \setminus V)$ are open in Y and the intersection of those two images is non-trivially, because Y is irreducible. So let be $y \in f(X \setminus U) \cap f(X \setminus V)$, which implies $f^{-1}(y) \cap X \setminus U \neq \emptyset$ and $f^{-1}(y) \cap X \setminus V \neq \emptyset$. So we have two open sets in $f^{-1}(y)$, which have a trivial intersection, that implies $f^{-1}(y)$ is reducible $\lightning$

Problem 3. For the sake of clarity we use without proof, that a subset $U \subseteq X$ is irreducible iff $\bar{U}$ is irreducible.

We prove by contradiction and assume

$\dim Y > \dim X =: n$.

We take a chain $\emptyset \neq Y_0 \subsetneq \dots \subsetneq Y_m \subset Y$ with $m > n$.

By taking the closure we get

$\emptyset \neq \bar{Y}_0 \subsetneq \dots \subsetneq \bar{Y}_m \subset X$. The inclusion is still strict, because $Y_i = Y \cap \bar{Y}_i$ in X and Y_i are closed in Y.

Since $\bar{Y}_i$ are still irreducible, we found a bigger chain in X than $\dim X$. $\square$

Problem 4:

"$\Rightarrow$" Assume $f^{-1}(m_S) = m_R$, then $\underbrace{f(f^{-1}(m_S))}_{= f(m_R)} \subseteq m_S$ property for every map.

"$\Leftarrow$" By $f(m_R) \subseteq m_S$, we get

$m_R \subseteq f^{-1}(f(m_R)) \subseteq f^{-1}(m_S)$.

Since m_S is an ideal, therefore $f^{-1}(m_S)$ is still an ideal and it is a proper ideal, because if $1 \in f^{-1}(m_S)$, then $1 = f(1) \in m_S$ and that can't be.

Since m_R is a maximum ideal, it follows $m_R = f^{-1}(m_S)$. $\square$