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Algebraic geometry 1

problem set 4

Keywords: Schemes - topology and morphisms, gluing, fibre products

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Problem 1 *(1 pt)*

Show that every affine scheme with the Zariski topology is quasi-compact.

Problem 2 *(2 pt)*

1. Show that every scheme admits a unique morphism to $\mathrm{Spec}(\mathbb{Z})$.
2. Show that global regular functions on a scheme X are in bijection with morphisms
$$X \to \mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x]).$$
3. Consider the ring homomorphisms $\alpha\colon \mathbb{Z}[x] \to \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x]$ mapping x to $x \otimes 1 + 1 \otimes x$, and $\mu\colon \mathbb{Z}[x] \to \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x]$ mapping x to $x \otimes x$. Can you interpret them in view of the previous point?

Problem 3 *(2 pt)*

1. Consider the scheme $U = \mathbb{A}^2_{\mathbb{Z}} \setminus V(x_1, x_2)$. Show that it is not affine. (Hint: compute global regular functions.)
2. Produce an example of a scheme with an affine cover, such that the intersection of two affine opens is not affine. (Hint: gluing.)

Problem 4 *(3 pt)*

Let $(f, f^{\#}) : X \to Y$ be a morphism of affine algebraic varieties over $\mathbf{k}$. Prove or disprove:

1. f is surjective if and only if $f^{\#}$ is injective.
2. f is injective if and only if $f^{\#}$ is surjective.

Would anything be different if we considered a morphism of affine schemes?

Problem 5 *(3 pt)*

1. Show that $\mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ consists of two points.
2. Let X and Y be algebraic varieties over an algebraically closed field $\mathbf{k}$. Show that $X \times Y$ is an algebraic variety, whose underlying set $\underline{X \times Y}$ is $\underline{X} \times \underline{Y}$.
3. Show that if X and Y are irreducible, then so is $X \times Y$.