# ALGEBRAIC GEOMETRY 1

PROBLEM SET 7

**Keywords:** closed subvarieties, conics, twisted cubics, and some topology

#### Problem 1

Let Y be an algebraic prevariety, and let  $X \subseteq Y$  be a closed subset. Show that X admits a unique structure of prevariety making  $\iota \colon X \hookrightarrow Y$  into a closed embedding (a morphism of locally ringed spaces such that  $\mathcal{O}_Y \twoheadrightarrow \iota_* \mathcal{O}_X$ ).

#### Problem 2

Let k be an algebraically closed field of characteristic different from 2.

- 1. Let  $X_2 \subseteq \mathbb{A}^2(k)$  be the vanishing locus of an irreducible polynomial of degree 2. Show that after an affine (linear + translation) change of coordinates,  $X_2$  can be identified with the vanishing locus of either  $f_1 = x_1^2 x_2$  or  $f_2 = x_1x_2 1$ .
- 2. Classify all conics (hypersurfaces of degree 2) in  $\mathbb{P}^2(k)$ .
- 3. Show that an integral (irreducible and reduced) conic is isomorphic to  $\mathbb{P}^1(k)$ .
- 4. Bonus: find a field k and an integral conic in  $\mathbb{P}^2_k$  that is not isomorphic to  $\mathbb{P}^1_k$ .

### Problem 3

Let  $X = V(x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2) \subseteq \mathbb{P}^3$ . Show that X is isomorphic to  $\mathbb{P}^1$ . Show that no two of the three equations above are enough to cut out X.

## Problem 4

Let  $f: X \to Y$  be a continuous map of topological spaces.

- 1. Show that X is Hausdorff if and only if  $\Delta_X \subseteq X \times X$  is closed.
- 2. Some topologists would call f proper if the preimage of any compact subset of Y is compact (some algebraic geometers would call this a quasi-compact morphism). Show that f is proper and closed if and only if f is universally closed, in the sense that for every topological space Z and continuous map  $g: Z \to Y$ , the fibre product  $X \times_Y Z \to Z$  is a closed map.

Note: in this exercise, products are endowed with the product topology.