Remark. We use throughout that a topological space X is irreducible if and only if any two non-empty open subsets of X intersect non-trivially.

Problem 1. Let X be a Hausdorff topological space. If there are distinct $x, y \in X$, then there are disjoint open subsets $U, V \subseteq X$, such that $x \in U$ and $y \in V$ by the Hausdorff assumption. These are two non-empty open subsets of X that intersect trivially, hence X is reducible. It follows that if X is an irreducible Hausdorff topological space, it contains at most one point. Since the empty set is not irreducible by convention, it follows that X is a singleton. Conversely, the singleton is trivially irreducible for its only proper subspace is empty. (Since subspaces of Hausdorff spaces are Hausdorff, this characterization also applies to subspaces of Hausdorff spaces.)

Problem 2. Assume that X is reducible then there exist $\emptyset \neq U_1, U_2 \subseteq_{op} X$ so that $U_1 \cap U_2 = \emptyset$. f is an open map of topological spaces which implies that $f(U_1)$, $f(U_2)$ and by that $f(U_1) \cap f(U_2)$ are open subspaces of Y. Furthermore $f(U_1) \cap f(U_2)$ is not empty because of the irreducibility of Y which means there exists a $y \in f(U_1) \cap f(U_2)$. We now look at the fibre $f^{-1}(y)$ of f over y. $f^{-1}(y)$ is irreducible and since $y \in f(U_1) \cap f(U_2)$ it follows that $f^{-1}(y) \cap U_i \neq \emptyset$ for $i = 1, 2$ and both intersections are open in $f^{-1}(y)$. This means $f^{-1}(y)$ has two non-overlapping, non-empty open supspaces which is a contradiction to $f^{-1}(y)$ being irreducible. □

Lemma 1. Let X be a topological space and $S \subseteq Y \subseteq X$. Then, $\mathrm{Cl}_Y(S) = \mathrm{Cl}_X(S) \cap Y$.

Proof. This is immediate from the characterization of the closure of a subset as the intersection of all closed sets in the ambient space containing it and the fact that the closed sets in the subspace Y are precisely the intersections of closed sets in X with Y. □

Lemma 2. Let X be a topological space and $S \subseteq X$. Then S is irreducible if and only if $\overline{S}$ is irreducible.

Proof. Assume that S is irreducible. If $\overline{S} = X_1 \cup X_2$ with closed subsets $X_1, X_2 \subset \overline{S}$ (since $\overline{S}$ is closed in X, these are even closed in X), we have $S = (X_1 \cap S) \cup (X_2 \cap S)$ and $X_1 \cap S, X_2 \cap S \subseteq S$ are closed. Since S is irreducible, we may WLOG assume that $S = X_1 \cap S \subseteq X_1$, hence $\overline{S} \subseteq X_1 \subseteq \overline{S}$, i.e. $\overline{S} = X_1$. Thus, $\overline{S}$ is irreducible.

Conversely, assume that $\overline{S}$ is irreducible. If $S = X_1 \cup X_2$ with closed subsets $X_1, X_2 \subseteq S$, we have $X_i = \overline{X_i} \cap S$ and $\overline{X_i} \subseteq \overline{S}$ is closed for $i = 0, 1$ by Lemma 1. Now, $\overline{S} = \overline{X_1 \cup X_2} = \overline{X_1} \cup \overline{X_2}$, so we may WLOG assume that $\overline{S} = \overline{X_1}$ since $\overline{S}$ is irreducible, hence $S = \overline{X_1} \cap S = X_1$ by Lemma 1. Thus, S is irreducible. □

Problem 3. Let X be a topological space (the further hypotheses are not necessary) and $Y \subseteq X$. Assume $n \leq \dim(Y)$ and let $\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n$ be a chain of closed, irreducible subspaces of Y witnessing this. Then, consider the chain $\emptyset \neq \overline{Y_0} \subseteq \overline{Y_1} \subseteq \ldots \subseteq \overline{Y_n}$ of closed, irreducible (by Lemma 2) subspaces of X. The inclusions are in fact strict for if $\overline{Y_i} = \overline{Y_j}$, then $Y_i = \overline{Y_i} \cap Y = \overline{Y_j} \cap Y = Y_j$ by Lemma 1 and $i = j$ since the inclusions in the original chain are strict. Thus, this chain witnesses that $n \leq \dim(X)$. Since n was arbitrary, it follows that $\dim(Y) \leq \dim(X)$.

Problem 4. Let $\varphi\colon R \to S$ be a ring homomorphism, $\mathfrak{m}_R$ a maximal ideal of R and $\mathfrak{m}_S$ a proper ideal of S (it need not be maximal for the argument to work). If $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$, then $\varphi(\mathfrak{m}_R) = \varphi(\varphi^{-1}(\mathfrak{m}_S)) \subseteq \mathfrak{m}_S$. Conversely, if $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$, then $\mathfrak{m}_R \subseteq \varphi^{-1}(\varphi(\mathfrak{m}_R)) \subseteq \varphi^{-1}(\mathfrak{m}_S)$, but $\mathfrak{m}_R$ is a maximal ideal and $\varphi^{-1}(\mathfrak{m}_S)$ is a proper ideal, because $\varphi^{-1}(\mathfrak{m}_S) = R \ni 1_R$ would imply $1_S = \varphi(1_R) \in \varphi\left(\varphi^{-1}(\mathfrak{m}_S)\right) \subseteq \mathfrak{m}_S$ contrary to the assumption that $\mathfrak{m}_S$ is a proper ideal of S, hence $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$ by maximality.

Problem 5. We use the following propositions:

Proposition. (4.17) Let X be an affine variety and let $f \in k[X]$. Then the distinguished open subset $D(f)$ is an affine variety. The isomorphism is $D(f) \cong V(tf(x) - 1)$.

Proposition. (2.25 c) Let X be a non-empty irreducible affine variety. If $f \in k[X]$ is non-zero every irreducible component of $V(f)$ has codimension 1 in X (and hence dimension $\dim(X) - 1$).

We identify the space of all $n \times n$ matrices $k^{n\times n}$ with $\mathbb{A}^{n^2}(k)$. We see that the vanishing locus of the determinant function det are the non-invertible $n \times n$ matrices over k. Proposition 4.17 states that $D(\det) = GL_n(k)$ is a affine algebraic variety.

$D(\det)$ is isomorph to $V(t \cdot \det(x) - 1)$ with $(t \cdot \det(x) - 1) \in k[\mathbb{A}^{n^2+1}(k)]$, which is a function in $n^2 + 1$ variables. By proposition (2.25 c) the dimension of $V(t \cdot \det(x) - 1)$ is $\dim(\mathbb{A}^{n^2+1}(k)) - 1 = n^2$. □