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Algebraic geometry 1

problem set 6

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Problem 1 *(4 pt)*

Show that the following properties satisfy the assumptions of the Affine Communication Lemma [Vakil, Prop. 5.3.2].

1. locally of finite type over a base ring R;
2. reduced.

Problem 2 *(3 pt)*

Let $R = \bigoplus R_d$ be a $\mathbb{Z}$-graded ring. Prove the following facts.

1. A homogeneous ideal $\mathfrak{p} \subsetneq R$ is prime if and only if, for any pair of *homogeneous* elements a, b in R, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
2. Let $S \subseteq R$ be a multiplicative set consisting of homogeneous elements. Then $S^{-1}R$ admits a natural structure of $\mathbb{Z}$-graded ring making $R \to S^{-1}R$ into a graded homomorphism (of degree 0).
3. Every maximal homogeneous ideal of R is of the form:

$$\ldots \oplus R_{-1} \oplus \mathfrak{m} \oplus R_1 \oplus \ldots$$

for some maximal ideal $\mathfrak{m}$ of R_0.

Problem 3 *(3 pt)*

Let $R = \bigoplus R_d$ be a $\mathbb{Z}$-graded ring, and $f \in R$ an invertible homogeneous element of positive degree e. Then there is a bijection between prime ideals of R_0 and homogeneous prime ideals of R. (Hint: given $\mathfrak{p} \in \mathrm{Spec}(R_0)$, define $P = \bigoplus P_d$ where $g \in P_d \subseteq R_d$ if and only if

$$\frac{g^e}{f^d} \in \mathfrak{p}.$$

Show that P is a homogeneous prime ideal of R, and that $P \cap R_0 = \mathfrak{p}$.)