Problem set 2

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1) X Hausdorff worked with: Tobias, Simon

$A \subseteq X$ irreducible $\Leftrightarrow$ A is a singleton

Proof:

$\Leftarrow$: obvious

$\Rightarrow$):

Assume $x, y \in A$ $x \neq y$

$\Rightarrow \exists\ U, V \underset{\text{open}}{\subseteq} X$. $U \cap V = \emptyset$ $x \in U$, $y \in V$

$\Rightarrow U^c \cap A$ and $V^c \cap A$ are closed in A

$\Rightarrow (U^c \cap A) \cup (V^c \cap A) = ((U \cap A) \cap (V \cap A))^c = (\emptyset)^c = A$

$x \notin (U^c \cap A) \Rightarrow U^c \cap A \neq A$

$y \notin (V^c \cap A) \Rightarrow V^c \cap A \neq A$

$\lightning \Rightarrow$ A is a singleton

2) $f: X \to Y$ continuous.

Y and every fibre of f are irred. $\Rightarrow$ X is irred.

Proof:

Assume X reducible

$\Rightarrow \exists\ U_1, U_2$ open, $U_1 \cap U_2 = \emptyset$

$\Rightarrow f(U_1), f(U_2)$ open (if $f(U_1) \cap f(U_2) = \emptyset$ $\lightning$ to Y irred)

if $f(U_1) \cap f(U_2) \neq \emptyset$

take $y \in f(U_1) \cap f(U_2)$

$\Rightarrow$ $f^{-1}(\{s\}) \cap U_1$ open in $f^{-1}(\{s\})$ and $\neq \emptyset$

$f^{-1}(\{s\}) \cap U_2$ open in $f^{-1}(\{s\})$ and $\neq \emptyset$

$\Rightarrow$ $(f^{-1}(\{s\}) \cap U_1) \cap (f^{-1}(\{s\}) \cap U_2) = \emptyset$ (because $U_1 \cap U_2 = \emptyset$)

↯ to $f^{-1}(\{s\})$ irred $\square$

3) X irred top. space of finite dimension n.

$Y \subseteq X$ show $\dim(Y) \leq \dim(X)$

Proof:

Let $\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_k \subseteq Y$, Y_i closed, irred.

$\Rightarrow$ $\emptyset \neq \overline{Y_0} \subseteq \overline{Y_1} \subseteq \dots \subseteq \overline{Y_k} \subseteq X$, closed, irred

~~show $\overline{Y_i} \subsetneq \overline{Y_{i+1}}$ because~~

$Y_i = \overline{Y_i} \cap Y \neq \overline{Y_{i+1}} \cap Y = Y_{i+1} \Rightarrow \overline{Y_i} \subsetneq \overline{Y_{i+1}}$

$\Rightarrow \dim(X) \geq \dim(Y)$ $\square$

4) $\varphi: R \to S$ hom. of local rings

Show $\varphi^{-1}(m_S) = m_R \Leftrightarrow \varphi(m_R) \subseteq m_S$

Proof:

$\Rightarrow$: $\varphi(m_R)$ is ~~proper~~ proper ideal $\Rightarrow \varphi(m_R) \subseteq m_S$

$\Leftarrow$: $m_R \subseteq \varphi^{-1}(m_S)$ obvious

$\varphi^{-1}(m_S)$ is proper ideal $\Rightarrow \varphi^{-1}(m_S) \subseteq m_R$

$\square$

5) $GL_n(k)$ is an affine algebraic variety of dimension n^2.

Proof

~~$Mat_n(k)$ is isomorph to k^{n^2}~~

$Mat_n(k)$ is isomorph to $k^{n^2} \hookrightarrow \mathbb{A}^{n^2}$

$det : \mathbb{A}^{n^2} \to k \quad \in k[x_1, \dots, x_{n^2}]$

$V(det) = \mathbb{A}^{n^2} \setminus GL_n(k) \Rightarrow D(det) = GL_n(k)$

this is an affine variety after Proposition 4.17

analogue to the proof of Prop. 4.17

$x_{n^2+1} \cdot det(x) - 1 \quad \in k[x_1, \dots, x_{n^2+1}]$

$V(x_{n^2+1} \cdot det(x) - 1) = D(det(x)) = GL_n(k)$

and $\dim V(x_{n^2+1} \cdot det - 1) = (n^2+1) - 1$ after Prop. 2.25(c)

$\Rightarrow \dim(GL_n(k)) = n^2$

$\square$