Problem set 2

Algebraic Geometry I

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Problem 1

„$\Rightarrow$": Let X be irreducible and Hausdorff. Assume for contradiction that X contains elements $x \neq y$. Then there exit open neighborhoods $x \in U$, $y \in V$ with $U \cap V = \emptyset \Rightarrow U^c \cup V^c = X$ and $x \notin U^c$, $y \notin V^c$ $(\Rightarrow U, V \neq X) \Rightarrow X$ is reducible $\lightning$.

„$\Leftarrow$": The only strict subset of a singleton is $\emptyset$. Thus it is irreducible, since $\{a\}$ cannot be written as a union of strict subsets.

Problem 2

Assume that X is not irreducible so there exit open nonempty sets $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \emptyset$.

Since f is an open map, $f(U_1)$ and $f(U_2)$ are open in Y.

Since Y is irred., there exits $y \in f(U_1) \cap f(U_2)$.

We have that $f^{-1}(\{y\})$ is irred. and therefore

$$\underbrace{\left(f^{-1}(\{y\}) \cap U_1\right)}_{\substack{\neq \emptyset \text{ open} \\ (\text{since } y \in f(U_1))}} \cap \underbrace{\left(f^{-1}(\{y\}) \cap U_1\right)}_{\neq \emptyset \text{ open}} \neq \emptyset, \text{ which contradicts}$$

$U_1 \cap U_2 = \emptyset$.

Problem 3

Let $\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_k \subseteq Y$ be a chain of irreducible closed subsets of Y. Since Y_i are closed in Y, we have $Y_i = \overline{Y_i} \cap Y$ and thus $\overline{Y_i} \neq \overline{Y_{i+1}}$ (they don't even agree on Y).

Thus, $\emptyset \neq \overline{Y_0} \subsetneq \overline{Y_1} \subsetneq \cdots \subsetneq \overline{Y_k} \subseteq X$ is a chain of closed and irreducible (since $A \subseteq X$ is irred. $\Leftrightarrow \overline{A}$ irred.) subsets and thus $k \leq n$.

Problem 4

"$\Rightarrow$": If $\varphi^{-1}(m_S) = m_R$, then $\varphi(m_R) = \varphi(\varphi^{-1}(m_S)) \subseteq m_S$.

"$\Leftarrow$": If $\varphi(m_R) \subseteq m_S$ then $m_R \subseteq \varphi^{-1}(\varphi(m_R)) \subseteq \varphi^{-1}(m_S) \neq R$.

(doesn't contain 1, since $\varphi(1) = 1 \notin m_S$)

Since m_R is maximal it follows $m_R = \varphi^{-1}(m_S)$.

Problem 5

Identify $\mathrm{Mat}_n(k)$ with $\mathbb{A}_k^{n^2}$ and consider the determinant polynomial $\det \in A(\mathbb{A}_k^{n^2})$. We know that $GL_n(k) = D(\det)$ is an affine variety isomorphic to

$\{(x,t) \in \mathbb{A}_k^{n^2} \times \mathbb{A}_k^1;\ t \cdot \det(x) - 1 = 0\} = V_{n^2+1}(t \cdot \det(x) - 1)$.

By Gathmann 2.25 (c) it follows that $\dim GL_n(k) = n^2$.

Proposition 2.25 (Properties of dimension). *Let X and Y be non-empty irreducible affine varieties.*

(a) *We have* $\dim(X \times Y) = \dim X + \dim Y$. *In particular,* $\dim \mathbb{A}^n = n$.

(b) *If $Y \subset X$ we have* $\dim X = \dim Y + \operatorname{codim}_X Y$. *In particular,* $\operatorname{codim}_X\{a\} = \dim X$ *for every point $a \in X$.*

(c) *If $f \in A(X)$ is non-zero every irreducible component of $V(f)$ has codimension 1 in X (and hence dimension* $\dim X - 1$ *by (b)).*