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Algebraic geometry 1

problem set 1

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Problem 1

Show that every affine algebraic set over $\mathbb{R}$ is the vanishing locus of a single polynomial.

Problem 2

Let k be a field. Show that there exist infinitely many irreducible polynomials in $k[x]$.

Problem 3

Let k be an algebraically closed field. Show that the following formulations of Hilbert's Nullstellensatz are equivalent.

1. If I is a proper ideal of $k[x_1, \dots, x_n]$, then $V(I) \neq \emptyset$.
2. Every maximal ideal of $k[x_1, \dots, x_n]$ is of the form

$$\mathfrak{m}_{\underline{a}} = (x_1 - a_1, \dots, x_n - a_n).$$

Problem 4

1. Let $f\colon X \to Y$ be a continuous map of topological spaces. Show that $f(X)$ is connected (resp. irreducible) if X is.
2. Let X be an affine algebraic set over $\mathbb{C}$. Show that $\mathrm{id}\colon X_{Eucl} \to X_{Zar}$ is continuous.
3. Show that $\mathbb{A}^1(\mathbb{C})$ is connected.

Problem 5

Show that the vanishing locus of $(x_2 - x_1^2, x_3 - x_1^3)$ is irreducible. Can you find a parametric presentation of it? This is called a twisted cubic.