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Date out: *Nov 29*, Date in: –

# Algebraic geometry 1

## problem set 7

**Keywords:** closed subvarieties, conics, twisted cubics, and some topology

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### Problem 1

Let $Y$ be an algebraic prevariety, and let $X \subseteq Y$ be a closed subset. Show that $X$ admits a unique structure of prevariety making $\iota\colon X \hookrightarrow Y$ into a closed embedding (a morphism of locally ringed spaces such that $\mathcal{O}_Y \twoheadrightarrow \iota_*\mathcal{O}_X$).

### Problem 2

Let $k$ be an algebraically closed field of characteristic different from 2.

1. Let $X_2 \subseteq \mathbb{A}^2(k)$ be the vanishing locus of an irreducible polynomial of degree 2. Show that after an affine (linear + translation) change of coordinates, $X_2$ can be identified with the vanishing locus of either $f_1 = x_1^2 - x_2$ or $f_2 = x_1x_2 - 1$.
2. Classify all conics (hypersurfaces of degree 2) in $\mathbb{P}^2(k)$.
3. Show that an integral (irreducible and reduced) conic is isomorphic to $\mathbb{P}^1(k)$.
4. Bonus: find a field $k$ and an integral conic in $\mathbb{P}^2_k$ that is not isomorphic to $\mathbb{P}^1_k$.

### Problem 3

Let $X = V(x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2) \subseteq \mathbb{P}^3$. Show that $X$ is isomorphic to $\mathbb{P}^1$. Show that no two of the three equations above are enough to cut out $X$.

### Problem 4

Let $f\colon X \to Y$ be a continuous map of topological spaces.

1. Show that $X$ is Hausdorff if and only if $\Delta_X \subseteq X \times X$ is closed.
2. Some topologists would call $f$ *proper* if the preimage of any compact subset of $Y$ is compact (some algebraic geometers would call this a quasi-compact morphism). Show that $f$ is proper and closed if and only if $f$ is universally closed, in the sense that for every topological space $Z$ and continuous map $g\colon Z \to Y$, the fibre product $X \times_Y Z \to Z$ is a closed map.

Note: in this exercise, products are endowed with the product topology.