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Algebraic geometry 1

problem set 3

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Problem 1

Let k be a field, and let $K = k(x, y)$ be the field of rational functions in two variables.
Let $G = \mathbb{Z}^2$ with the lexicographic order ($(a, b) \leq (c, d)$ if and only if $a < c$, or $a = c$ and $b \leq d$).

1. Let $v(x^a y^b) = (a, b)$, $v(\sum c_{a,b} x^a y^b) = \min\{(a, b) | c_{a,b} \neq 0\}$, and $v(\frac{f}{g}) = v(f) - v(g)$. Show that v defines a valuation on K with value group G.

2. Show that the valuation ring $R = \{f \in K | v(f) \geq 0\}$ is not Noetherian.

3. Show that $\Gamma = \{c \in G | c \geq 0\}$ is a submonoid of G. *Ideals* of Γ are subsets I of Γ such that for all $\alpha \in I, \gamma \in \Gamma$ we have $\alpha + \gamma \in I$. An ideal I is called *prime* if $c_1 + c_2 \in I \Rightarrow c_1 \in I$ or $c_2 \in I$. Show that (prime) ideals of R are in bijection with (prime) ideals of Γ.

   Describe the topological space $\mathrm{Spec}(R)$.

Problem 2

Let X be a scheme. Show that points of X are in bijection with equivalence classes of morphisms from fields spectra $f\colon \mathrm{Spec}(F) \to X$, where $f_1 \sim f_2$ if there is a common field extension $\iota_i\colon F_i \hookrightarrow \Omega,\ i = 1, 2$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \xrightarrow{\iota_1^\#} & \mathrm{Spec}(F_1) \\ \downarrow{\scriptstyle \iota_2^\#} & & \downarrow{\scriptstyle f_1} \\ \mathrm{Spec}(F_2) & \xrightarrow{f_2} & X \end{array}$$

Problem 3

Let $f\colon X \to Y$ be a continuous map of topological spaces, let $\mathcal{G}$ be a presheaf of Abelian groups on Y. Consider the following association:

$$U \subseteq_{\text{open}} X \mapsto f^{-1}G(U) = \varinjlim_{f(U) \subseteq V \subseteq_{\text{open}} Y} \mathcal{G}(V).$$

Show that there are canonically defined restriction homomorphisms making this into a presheaf of Abelian groups on X. Show that the following adjunction property holds: for every presheaf of Abelian groups $\mathcal{F}$ on X, one has

$$\mathrm{Hom}_{(\mathrm{PreSh}_X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{(\mathrm{PreSh}_Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Notice that, when f is the inclusion of a point p in Y, $f^{-1}\mathcal{G} = \mathcal{G}_p$ (the stalk of $\mathcal{G}$ at p).
In general, with notations as above, for every $p \in X$ we have

$$(f^{-1}\mathcal{G})_p = \mathcal{G}_{f(p)}.$$

Show that in general $f^{-1}\mathcal{G}$ is not a sheaf even if $\mathcal{G}$ is.

Problem 4

Let R be a ring (commutative with 1). Show that the following are equivalent:

1. every prime ideal is intersection of maximals;
2. every radical ideal is intersection of maximals;
3. $V_m(I) = V_m(J)$ (where $V_m(I) = V(I) \cap \mathrm{Spec}_m(R)$) implies $\sqrt{I} = \sqrt{J}$;
4. $\mathrm{Spec}_m(R) \subseteq \mathrm{Spec}(R)$ is dense in every closed subset of $\mathrm{Spec}(R)$;
5. the association $Z \mapsto Z \cap \mathrm{Spec}_m(R)$ induces a bijection between the closed subsets of $\mathrm{Spec}(R)$ and those of $\mathrm{Spec}_m(R)$.

A ring satisfying (any one of) the above properties is called *Jacobson*.
Show that a local ring is Jacobson if and only if it has Krull dimension 0.