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Algebraic geometry 1

problem set 12

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Problem 1 *(3 pt)*

Let X ba a k-scheme. Describe a natural bijection between $\mathrm{Mor}_k(\operatorname{Spec} k[\epsilon]/(\epsilon^2), X)$ and the set of k-points p of X with the additional datum of a tangent vector at p. In particular, the subset of morphisms mapping $\operatorname{Spec}(k) = \operatorname{Spec}(k[\epsilon]/(\epsilon^2))_{red}$ to p admits the structure of a k-vector space. Describe the differential of a morphism $f\colon X \to Y$ at p in these terms when $k(p) = k(f(p))$.

Problem 2 *(3 pt)*

Let X be a k-scheme of finite type, $D \subseteq X$ an effective Cartier divisor (a hypersurface which is locally cut out by a non-zero divisor in X). If D is regular at a point p, then so is X.
Is this still true if D is the vanishing locus of a zero divisor?

Problem 3 *(3 pt)*

Let $C = V(y^2 - x^3) \subseteq \mathbb{A}^2_k$ with $k = \bar{k}$ and $\mathrm{char}(k) \neq 2, 3$ be an "ordinary cusp".

1. Compute the singular locus of C.
2. Let $\tilde{A}$ denote the blow-up of $\mathbb{A}^2$ in C^{sing}, and let C' denote the strict transform of C. Compute the intersection of C' with the exceptional divisor in $\tilde{A}$, and the tangent cone of C at the origin of $\mathbb{A}^2$.
3. Show that C is not isomorphic to $\mathbb{A}^1$, but C' is, and show that the morphism $C' \to C$ is finite and birational.

Problem 4 *(3 pt)*

Let us work over $k = \bar{k}$ with $\mathrm{char}(k) \neq 2, 3$.

1. Let X be the cone over the twisted cubic. Compute X^{sing}. What is the exceptional divisor of the blow-up of X at X^{sing}?
2. Let Y be the blow-up of $\mathbb{A}^3$ along the union of two intersecting lines $V(xy, z)$. Compute Y^{sing}. Show that Y is not isomorphic to what one gets by blowing up first one line, and then the strict transform of the other.
3. Let Z be the surface $V(x^2 - yz^2)$ in $\mathbb{A}^3$ (this "pinch point" is a higher-dimensional analogue of the singularity encountered in the previous problem). Compute Z^{sing}. Compute the blow-up of Z at the origin.