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Algebraic geometry 1

problem set 7

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Problem 1

Let Y be an algebraic prevariety, and let $X \subseteq Y$ be a closed subset. Show that X admits a unique structure of prevariety making $\iota\colon X \hookrightarrow Y$ into a closed embedding (a morphism of locally ringed spaces such that $\mathcal{O}_Y \twoheadrightarrow \iota_*\mathcal{O}_X$).

Problem 2

Automorphisms of $\mathbb{P}^n(k)$ induced by linear automorphisms of k^{n+1} are called *projectivities*.

1. Show that every automorphism of $\mathbb{P}^1(k)$ is a *projectivity*, which can be written as
$$z \mapsto \frac{az+b}{cz+d}.$$
2. Compute the stabiliser $\mathrm{Stab}_G([1:0])$ in $G = PGL_2(k)$ of the point $[1:0] \in \mathbb{P}^1(k)$, and $\mathrm{Stab}_G([1:0]) \cap \mathrm{Stab}_G([0:1])$.
3. Show that the action of $PGL_2(k)$ on $\mathbb{P}^1(k)$ is sharply 3-transitive (for every two triples of distinct points there is exactly one projectivity moving one to the other).
4. The *cross-ratio* of four distinct points (z_1, z_2, z_3, z_4) of $\mathbb{P}^1(k) \setminus \{\infty\}$ is defined as:
$$\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} \in k.$$
Extend this definition in case one of the points is ∞. Show that projectivities preserve the cross-ratio. Conclude that the action of $PGL_2(k)$ on $\mathbb{P}_1(k)$ is not 4-transitive in general. (Bonus: what are the possible values of the cross-ratio? What happens in the limit when two points come together (over $\mathbb{C}$)?)
5. Show that the action of $PGL_{n+1}(k)$ on $\mathbb{P}^n(k)$ is 2-transitive, but not 3-transitive in general.
6. An $(n+2)$-tuple of points in $\mathbb{P}^n$ is said to be *in general (linear) position* if no $n+1$ of them are contained in a hyperplane. Show that $PGL_{n+1}(k)$ acts transitively on the set of $(n+2)$-tuples in general position.

Problem 3

Let $X = V(x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2) \subseteq \mathbb{P}^3$. Show that X is isomorphic to $\mathbb{P}^1$. Show that no two of the three equations above are enough to cut out X.

Problem 4

Let k be an algebraically closed field of characteristic different from 2.

1. Let $X_2 \subseteq \mathbb{A}^2(k)$ be the vanishing locus of an irreducible polynomial of degree 2. Show that after an affine (linear + translation) change of coordinates, X_2 can be identified with the vanishing locus of either $f_1 = x_1^2 - x_2$ or $f_2 = x_1 x_2 - 1$.

2. Classify all conics (hypersurfaces of degree 2) in $\mathbb{P}^2(k)$.

3. Show that an integral conic is isomorphic to $\mathbb{P}^1(k)$.

4. Bonus: find a field k and an integral conic in $\mathbb{P}^2(k)$ that is not isomorphic to $\mathbb{P}^1(k)$.