Problem 1

Let X be a Hausdorff topological space. A subspace of X is irreducible if and only if it is a singleton.

Proof. $\underline{\implies}$ Let $\emptyset \neq A \subseteq X$ be an irreducible subspace. Assume there are 2 points $x \neq y \in A$. Then there are open neighborhoods $x \in U_X, y \in U_y \subseteq X$ of x and y such that $U_x \cap U_y = \emptyset$.

Regard $V_x := X \setminus U_x$ and $V_y := X \setminus U_y$. We have

$$V_x \cup V_y = X \setminus U_x \cup X \setminus U_y = X \setminus (U_x \cap U_y) = X$$

and the two sets are closed in X and therefore $A \cap V_x$, $A \cap V_y$ are closed in A with the subspace topology. But then

$$(A \cap V_x) \cup (A \cap V_y) = A \cap X = A$$

is a covering of A with closed proper subsets of A and A is reducible.

$\underline{\impliedby:}$ Let $\{x\} = A \subseteq X$ be a singleton. Then the only proper subset of A is the empty set. But then A can never be written as the union of proper subsets and is irreducible. $\square$

Problem 2

Let $f\colon X \to Y$ be a continuous and open map of topological spaces. Suppose that Y is irreducible, and so is every fiber of f. Then X is irreducible as well.

Proof. Suppose $X = X_1 \cup X_2$ with closed sets X_i. Consider $U_1 := X_1 \setminus X_2 = X \setminus X_2$ and $U_1 := X_2 \setminus X_1 = X \setminus X_1$. Both sets are open in X. Assume for contradiction that U_1 and U_2 are both nonempty.

Then because f is open, the $f(U_i)$ are open as well. Hence, because Y is irreducible,

$$(Y \setminus f(U_1)) \cup (Y \setminus f(U_2)) \subsetneq Y,$$

so $f(U_1) \cap f(U_2) \neq \emptyset$. Choose $p \in f(U_1) \cap f(U_2)$. Then

$$(f^{-1}(p) \cap U_1) \cup (f^{-1}(p) \cap U_2) = f^{-1}(p) \cap (U_1 \cup U_2) = f^{-1}(p)$$

is a closed covering of $f^{-1}(p)$ by nonempty closed proper subsets. This is a contradiction to the irreducibility of the fibers of f. $\square$

Problem 3

Let X be an irreducible topological space of finite dimension n. For any subspace $Y \subseteq X$, $\dim Y \leq \dim X$.

Proof. $\dim X$ is the length of the longest chain

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subseteq X$$

of irreducible subspaces X_i. The same is true for $\dim Y$. Now assume $m := \dim Y > n$. Then there is a chain

$$\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_m \subseteq Y \tag{1}$$

of irreducible subspaces Y_i (in the subspace topology of $Y \subseteq X$).

Let $A \subseteq Y$. If A is irreducible in Y, it is irreducible in X because: Let $A = A_1 \cup A_2$ be a covering of A with closed subsets $A_i \subsetneq X$. Then $(Y \cap A_1) \cup (Y \cap A_2)$ is a closed covering of A in Y by proper subsets, so A is reducible in Y.

Hence, the chain (1) is also a chain of irreducible subspaces in X and $m \leq n$. A contradiction. $\square$

Problem 4

Let $\varphi\colon R \to S$ be a homomorphism of local rings. Show that $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$ if and only if $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$.

Proof. $\underline{\implies:}$ Suppose $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$. Then $\varphi(\mathfrak{m}_R) = \varphi(\varphi^{-1}(\mathfrak{m}_S)) \subseteq \mathfrak{m}_S$ follows.

$\underline{\impliedby:}$ Suppose $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$. Then obviously we have $\mathfrak{m}_R \subseteq \varphi^{-1}(\mathfrak{m}_S)$. Now assume there is an element $x \in \varphi^{-1}(\mathfrak{m}_S) \setminus \mathfrak{m}_R$. Then $1_R \in (\mathfrak{m}_R, x)$ and therefore $1_S \in \varphi(\mathfrak{m}_R, x)$. But because $\varphi(x) \in \mathfrak{m}_S$,

$$1_S = \varphi(1_S) \in \varphi(\mathfrak{m}_R, x) = (\varphi(\mathfrak{m}_R), \varphi(x)) \subseteq \mathfrak{m}_S.$$

A contradiction. $\square$

Problem 5

Let $GL_n(k)$ denote the set of invertible $n \times n$ matrices over k. Show that it is an affine algebraic variety of dimension n^2.

Proof. Identify all $n \times n$ matrices by $\mathbb{A}^{n^2}(k)$. Then Proposition 4.17 shows that $D(\det) = GL_n(k)$ is an affine algebraic variety.

The proof shows that $D(\det) \cong X := V(t \cdot \det(x) - 1) \subseteq \mathbb{A}^{n^2+1}(k)$. Then X is irreducible, so the only irreducible component of X. Since $t \cdot \det(x) - 1$ is non-zero in $k[\mathbb{A}^{n^2+1}]$, Proposition 2.25c asserts that $\dim(X) = \dim(\mathbb{A}^{n^2+1}) - 1 = n^2$. $\square$