LOCALISATION AND QUASIMAP COHOMOLOGY

Contents

1. Localisation formula

1.1. **Notation from toric geometry.** Let X_Σ be a smooth complete toric variety, for $\Sigma \subseteq N$ a rational polyhedral fan and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Let us denote by r the Picard rank of X, by n its dimension, and by $N = n + r$ the number of rays in Σ. Let v_ρ denote the primitive generator of the ray ρ, and assume that $\Sigma(1)$ is an ordered set.

The (non-effective) action of the big torus $T = \mathbb{G}_{\mathrm{m}}^N$ on X induces an action on $Q_{g,n}(X, \beta)$, by scaling the sections. Let us denote by $\lambda_1, \ldots, \lambda_N$ the corresponding weights.

Let us denote by $\{\sigma_i\}_{i\in\Sigma^{\max}}$ the T-fixed points on X (corresponding to maximal cones of Σ) and by $\{\tau_{i,j}\}_{i,j\in\Sigma^{\max}}$ the 1-dimensional orbits (corresponding to facets of the maximal cones; $\tau_{i,j}$ -if it exists- connects σ_i and σ_j).

Let σ_i and σ_j be two adjacent maximal cones; since X is smooth, $\{v_\rho\}_{\rho\prec\tau_{i,j}} \cup \{v_n\}$ is a $\mathbb{Z}$-basis of N (where v_n is the only ray in σ_i, but not in $\tau_{i,j}$), so we can find the dual basis $\{m_1, \ldots, m_n\}$ of M. Define $\lambda_{\sigma_j}^{\sigma_i} = \sum_{\rho\in\Sigma(1)} \langle m_n, v_\rho\rangle \lambda_\rho$. Compare with [Spi00, §§6.4 and 7.3].

Lemma 1.1. Let σ_i be a T-fixed point on X and $\tau_{i,j}$ be a 1-dimensional orbit through it, furthermore let D_ρ be a toric divisor. Then the weight of the T-action on $\mathcal{O}(D_\rho)_{\sigma_i}$ is

$$\begin{cases} \lambda_{\sigma_j}^{\sigma_i}, & \text{if } \rho \prec \sigma_i \text{ and } \tau_{i,j} \cup \{v_\rho\} = \sigma_i \\ 0, & \text{otherwise.} \end{cases}$$

The weight of the T-action on $T(\tau_{i,j})_{\sigma_i}$ is $\lambda_{\sigma_j}^{\sigma_i}$.

Proof. Let σ_i be spanned by $\{v_{i_1}, \ldots, v_{i_n}\}$. If $[z_1 : \ldots : z_N]$ are homogeneous coordinates on X, then local coordinates around σ_i are given by

$$\left(x_{i_1} = z_{i_1} \prod_{j\neq i_1} z_j^{\langle m_{i_1}, v_j\rangle}, \ldots, x_{i_n} = z_{i_n} \prod_{j\neq i_n} z_j^{\langle m_{i_n}, v_j\rangle} \right),$$

where $\{m_{i_1}, \dots, m_{i_n}\}$ is the dual basis of $\{v_{i_1}, \dots, v_{i_n}\}$.

If $\rho \not\prec \sigma_i$ then the weight is 0 because we can find a divisor representing $\mathcal{O}(D_\rho)$ that does not pass through σ_i. Otherwise $\rho = i_j$ for some $j \in \{1, \dots, n\}$, so D_ρ has local equation $x_{i_j} = 0$ near σ_i, which makes the first statement clear.

The second part follows from the exact sequence

$$0 \to T\tau_{i,j} \to TX_{|\tau_{i,j}} \to \bigoplus_{\rho \prec \tau_{i,j}} \mathcal{O}_{\tau_{i,j}}(D_\rho) \to 0$$

together with the Euler exact sequence for TX and the first part. □

2. T-fixed points

The following discussion is inspired by [MOP11, §7.3].

References

[MOP11] Alina Marian, Dragos Oprea, and Rahul Pandharipande. The moduli space of stable quotients. *Geom. Topol.*, 15(3):1651–1706, 2011.

[Spi00] H. Spielberg. The Gromov-Witten invariants of symplectic manifolds. *ArXiv Mathematics e-prints*, June 2000.