LOCALISATION AND QUASIMAP COHOMOLOGY

Contents

1. Localisation formula

1.1. **Notation from toric geometry.** Let X_Σ be a smooth complete toric variety, for $\Sigma \subseteq N$ a rational polyhedral fan and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Let us denote by r the Picard rank of X, by n its dimension, and by $N = n + r$ the number of rays in Σ. Let v_ρ denote the primitive generator of the ray ρ, and assume that $\Sigma(1)$ is an ordered set.

The (non-effective) action of the big torus $T = \mathbb{G}_{\mathrm{m}}^N$ on X induces an action on $\mathcal{Q}_{g,n}(X, \beta)$, by scaling the sections. Let us denote by $\lambda_1, \ldots, \lambda_N$ the corresponding weights.

Let us denote by $\{\sigma_i\}_{i \in \Sigma^{\max}}$ the T-fixed points on X (corresponding to maximal cones of Σ) and by $\{\tau_{i,j}\}_{i,j \in \Sigma^{\max}}$ the 1-dimensional orbits (corresponding to facets of the maximal cones; $\tau_{i,j}$ -if it exists- connects σ_i and σ_j).

Let σ_i and σ_j be two adjacent maximal cones; since X is smooth, $\{v_\rho\}_{\rho \prec \tau_{i,j}} \cup \{v_n\}$ is a $\mathbb{Z}$-basis of N (where v_n is the only ray in σ_i, but not in $\tau_{i,j}$), so we can find the dual basis $\{m_1, \ldots, m_n\}$ of M. Define $\lambda_{\sigma_j}^{\sigma_i} = \sum_{\rho \in \Sigma(1)} \langle m_n, v_\rho \rangle \lambda_\rho$. Compare with [Spi00, §§6.4 and 7.3].

Lemma 1.1. Let σ_i be a T-fixed point on X and $\tau_{i,j}$ be a 1-dimensional orbit through it, furthermore let D_ρ be a toric divisor. Then the weight of the T-action on $\mathcal{O}(D_\rho)_{\sigma_i}$ is

$$\begin{cases} \lambda_{\sigma_j}^{\sigma_i}, & \text{if } \rho \prec \sigma_i \text{ and } \tau_{i,j} \cup \{v_\rho\} = \sigma_i \\ 0, & \text{otherwise.} \end{cases}$$

The weight of the T-action on $T(\tau_{i,j})_{\sigma_i}$ is $\lambda_{\sigma_j}^{\sigma_i}$.

Proof. Let σ_i be spanned by $\{v_{i_1}, \ldots, v_{i_n}\}$. If $[z_1 : \ldots : z_N]$ are homogeneous coordinates on X, then local coordinates around σ_i are given by

$$\left(x_{i_1} = z_{i_1} \prod_{j \neq i_1} z_j^{\langle m_{i_1}, v_j \rangle}, \ldots, x_{i_n} = z_{i_n} \prod_{j \neq i_n} z_j^{\langle m_{i_n}, v_j \rangle} \right),$$

where $\{m_{i_1}, \dots, m_{i_n}\}$ is the dual basis of $\{v_{i_1}, \dots, v_{i_n}\}$.

If $\rho \not< \sigma_i$ then the weight is 0 because we can find a divisor representing $\mathcal{O}(D_\rho)$ that does not pass through σ_i. Otherwise $\rho = i_j$ for some $j \in \{1, \dots, n\}$, so D_ρ has local equation $x_{i_j} = 0$ near σ_i, which makes the first statement clear.

The second part follows from the exact sequence

$$0 \to T\tau_{i,j} \to TX_{|\tau_{i,j}} \to \bigoplus_{\rho < \tau_{i,j}} \mathcal{O}_{\tau_{i,j}}(D_\rho) \to 0$$

together with the Euler exact sequence for TX and the first part. □

2. T-fixed loci

The following discussion is inspired by [MOP11, §7.3]. T-fixed loci for $\mathcal{Q}_{g,n}(X, \beta)$ are indexed by decorated graphs

$$(\Gamma, v, \gamma, b, \varepsilon, \delta, \mu)$$

where:

(1) $\Gamma = (V, E)$ is a graph with vertex set V and edge set E (no self-edges allowed);
(2) $v \colon V \to \{\sigma_i\}_{i \in \Sigma^{\max}}$ assigns a fix point to each vertex;
(3) $\gamma \colon V \to \mathbb{Z}_{\geq 0}$ is a genus assignment;
(4) $b \colon V \to H_2^+(X, \mathbb{Z})$ assigns an effective curve class to each vertex;
(5) $\varepsilon \colon E \to \{\tau_{i,j}\}_{i,j \in \Sigma^{\max}}$ assigns a one-dimensional orbit to each edge;
(6) $\delta \colon E \to \mathbb{Z}_{\geq 1}$ specifies the degree of the covering map;
(7) $\mu \colon V \to 2^{\{1,\dots,n\}}$ is a distribution of the markings to the vertices V.

These data are required to satisfy a number of compatibility conditions:

- Γ must be connected;
- if $e \colon v_1 \to v_2$ then $\varepsilon(e) = \tau_{i,j}$ with $v(v_1) = \sigma_i$ and $v(v_2) = \sigma_j$ (or viceversa);
- $h^1(\Gamma) + \sum_{v \in V} \gamma(v) = g$;
- b is compatible with v;
- $\sum_{v \in V} b(v) + \sum_{e \in E} \delta(e)[\varepsilon(e)] = \beta$.

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We are going to denote by $\mathrm{val} \colon V \to \mathbb{Z}_{\geq 1}$ the number of edges adjacent to a vertex, and by $\deg \colon V \to \mathbb{Z}_{\geq 2}$ the sum of val with the number of marked points associated to each vertex.

The corresponding T-fixed locus is isomorphic, up to a finite map, to:

$$\prod_{v \in V} \overline{\mathcal{M}}_{\gamma(v), \deg(v) | \sum_{\rho \not< v(v)} b(v) \cdot D_\rho}$$

The moduli spaces corresponding to degenerate vertices (where $\deg(v) = 2$, $\gamma(v) = 0$, and $b(v) = 0$) are treated as points in this product. Notice that $\overline{\mathcal{M}}_{g,n|d}/S_d \simeq \mathcal{Q}_{g,n}(\mathbb{A}^1 /\!\!/ \mathbb{G}_{\mathrm{m}}, d)$. Hence the finite map has degree

$$|\mathbf{A}| \cdot \prod_{v \in V} (b(v) \cdot D_\rho)!$$

where $|\mathbf{A}|$ can be extrapolated from

$$0 \to \prod_{e \in E} \mathbb{Z}/\delta(e)\mathbb{Z} \to \mathbf{A} \to \mathrm{Aut}(\Gamma) \to 0.$$

Notice here that, for every maximal cone σ_i, the collection $\{D_\rho\}_{\rho \not< \sigma_i}$ constitutes a basis of $\mathrm{Pic}(X)$ (since every support function can be made into vanishing on every $\rho \not< \sigma_i$ by subtracting an appropriate $m \in M$).

The corresponding quasimap can be described as follows: edges correspond to maps (without basepoints) from $\mathbb{P}^1$ to the corresponding 1-dimensional T-orbit $\varepsilon(e)$, of degree $\delta(e)$ and totally ramified at the two T-fixed points. Pick instead a vertex $v \in V$: according to $v(v) = \sigma_i$, we may write $\mathcal{O}(D_{i_j}) = \bigotimes_{\rho \not< \sigma_i} \mathcal{O}(D_\rho)^{\otimes a_{i_j,\rho}}$ for each i_j, $j = 1, \ldots, n$ such that the corresponding ray belongs to σ_i. For a marked curve C_v in the mixed moduli space $\overline{\mathcal{M}}_{\gamma(v), \deg(v)|\sum_{\rho \not< v(v)} b(v) \cdot D_\rho}$ with markings

$$\{p_1, \ldots, p_{\deg(v)}\} \cup \bigcup_{\rho \not< \sigma_i} \{q_{\rho,1}, \ldots, q_{\rho, b(v) \cdot D_\rho}\}$$

the corresponding quasimap is given by:

$$\left((C_v, \{p_1, \ldots, p_{\deg(v)}\}), (\mathcal{O}_{C_v} \hookrightarrow \mathcal{O}_{C_v}(\sum_{j=1}^{b(v)\cdot D_\rho} q_{\rho,j}) =: L_\rho)_{\rho \not< \sigma_i}, (\mathcal{O}_{C_v} \xrightarrow{0} \bigotimes_{\rho \not< \sigma_i} L_\rho^{\otimes a_{i_j,\rho}} =: L_{i_j})_{j=1,\ldots,n} \right)$$

Gluing along flags $F = (e, v)$ is made possible by the required compatibilities.

References

[MOP11] Alina Marian, Dragos Oprea, and Rahul Pandharipande. The moduli space of stable quotients. *Geom. Topol.*, 15(3):1651–1706, 2011.

[Spi00] H. Spielberg. The Gromov-Witten invariants of symplectic manifolds. *ArXiv Mathematics e-prints*, June 2000.